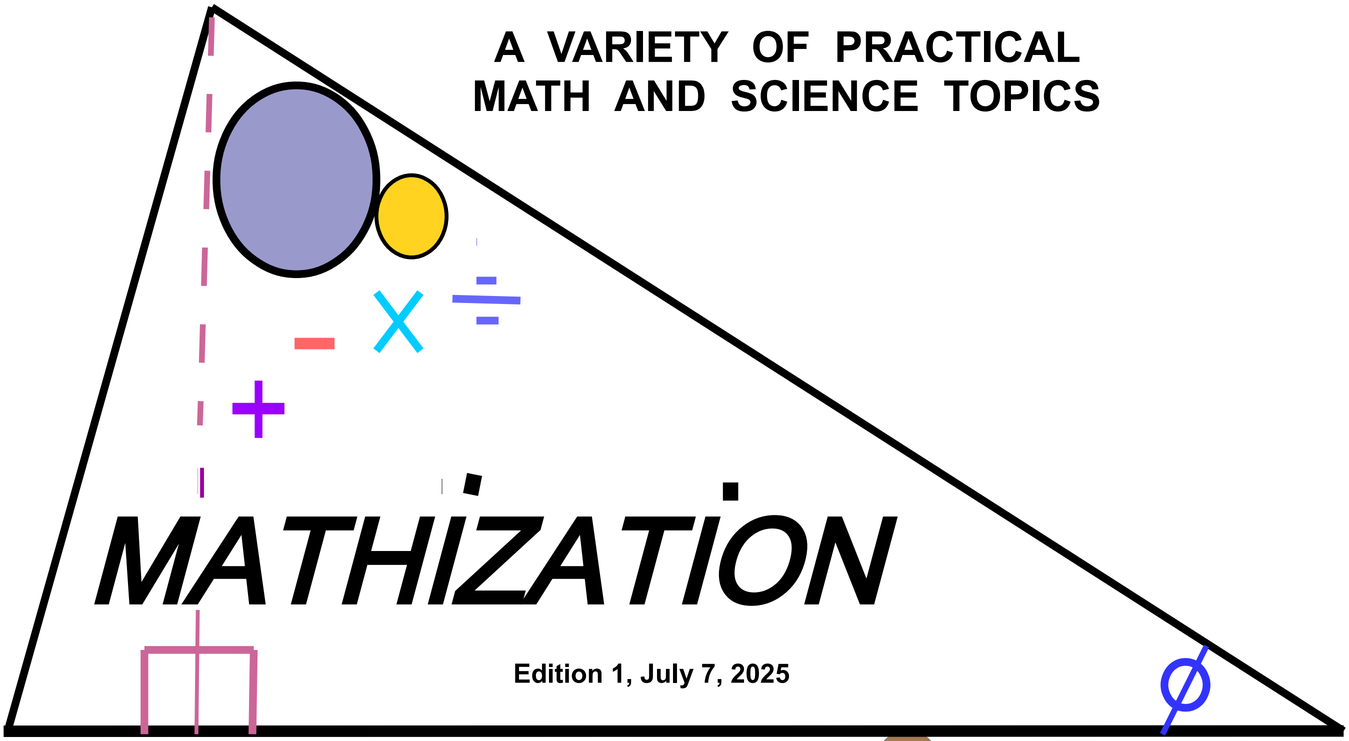
A VARIETY OF PRACTICAL
MATH AND SCIENCE TOPICS



MATHIZATION

Edition 1, July 7, 2025

ARITHMETIC
ALGEBRA
GEOMETRY
TRIGONOMETRY
SCIENCE
C COMPUTER PROGRAMS

A Solution For Math

JP ALBERTSON

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FORWARD

This ebook is about learning some basic and essential mathematics from the author's perspective and considerations. Mathematics is the study of measurements, numerical relationships, quantifying, solving, and where arithmetic is the general study of counting, which can be called basic or fundamental mathematics. I have considered this book to be practical for the average person who just needs some basic knowledge on a particular subject and-or concept and then goes and does something useful with it. Each other math book is usually a trade-off between the maximum number of topics and an extensive and-or complicated view of a few topics. This book may be just right for those seeking a way to finally learn math, especially for its unique overview of topics and an informal approach, rather than leaving the casual reader wondering if they should have taken some kind of advanced calculus and-or perhaps being born with exceptional mathematical abilities and thought processes. There is already many other math books available, and for me to write something new would mean writing something that is hopefully different in some ways. This book will hopefully make you feel comfortable about mathematics to the point where you actually want to keep reading this and other learning material and applying what you know when needed. Most books, inducing this book, are also lacking in some way and are therefore never completely finished, and at some point, they are still presented to the readers at its current edit or version so as some hopeful and timely use can be made of what is expressed. Editing this large book like this is a lengthy process and it has become increasingly difficult, but what is presented here is reasonable, and more than I originally intended since many concepts then needed to be included for the readers sake. This book does not discuss any particular branch of mathematics in great depth, due to the huge number or pages that would be required, and then not having the particular or simpler style of this book. Still, this book does show you the essentials and power of mathematics. To acquire that power, the reader must give a reasonable effort. Be patient since math and science books are usually not known for being "fast" or "lite" reading material, and then we sometimes need to learn how to learn new things. The decimal number symbols and the language of mathematics is necessary, standardized and used world-wide.

Some form of mathematics is found almost everywhere these days, even within some home cooking recipes, and you need to have some knowledge of math to help feel comfortable when you do encounter it. If you like modern technology and science (to study so as to find and understand knowledge) and want to learn how it ticks, then a good place to begin is with some math. Each page or topic, when understood, and even applied, can be a potential achievement in itself, especially for the casual user or reader who may only need to learn a few things on occasion from this book. You can even have and store this book as an available and useful "knowledge resource" for when it is promptly needed rather than as an information overload, burden to complete, and easy to forget, hence there is no need, requirement or pressure to read this entire book just to have some basic and useful knowledge on just a few specific topics in need. There is probably enough material within this book for a course and-or a lifetime of usefulness if you are willing. Some may only have a partial need for this book, while others may consider all within. You should first skim through the table of contents and book pages so as to have a general idea of what is within it.

The homemade "grassroots effort" and thoughtful style of this book was especially written with compassion for the readers having some difficulty learning math and-or science, and as an effort to help prevent those who may otherwise fail or leave an education because they can't yet manage to understand something that has to do with math. It was also written for those who desire to reeducate themselves, and-or for those who lack the resources for other ways to learn, and yet still hope, desire and need to have something available to reach for and explore so as to move a step forward. From having just occasional interest, a beginner, master, or even a practical computer programmer at any level can benefit from at least some of the material within. All books have their own style, methods, material, and available knowledge within, and so if this book currently seems excessive to the reader, or perhaps not even sufficiently detailed enough for some, there are other fine, well written and printed on paper, math books available to consider at some point in your learning journey. Please check the books available at book seller websites such as **Amazon.com** and-or other websites that focus on a specific topic that you are interested in, and which may also offer more than books about a topic. Please take a visit to your local bookstore shops for there may be books there that are unavailable elsewhere. This relatively basic ebook may sometimes be considered as somewhat less than the higher standards set by some rigorous (strict, formalized, "pure mathematical science" - the study of math itself) books in several ways, and yet it still excels in simplified, practical ways for the average person. Due to the number of pages in this book, it is currently too costly to print into a paper book(s).

This book contains much, perhaps enough for several books (practical math, basic science, basic electronics, and some

C language example math programs) on their own, or a collective volume of books, and **you are in no way expected to read it all and-or to try to memorize it**, but rather at least have a minimal or basic reasonable idea of a topic in order to begin someplace when needed, and so enjoy this book at your own pace, interest and needs - even if its just one topic or page now and then for **this book is not a formal course** having specific educational goals and with many practice problems, and therfore, generally not having much of a variety as a book such as this. Placing these writing together into one book or volume also helps with the distribution of them, for it is easier to distribute just one book in one instance, rather than three (ex., math, science, computer programs) books and three instances over time and availability. This book started out to be a basic three chapter math book with many new helpful ideas, yet I still had to somehow make this book different enough to be even be considered as something to obtain, and so I then included many practical science topics and computer programs.

As for trying to learn and-or understand things, it is surely much better to learn slowly and easily, than not at all. Consider math as a helpful tool to learn, build and apply other knowledge, rather than just as a temporary goal to complete some kind of study or course. This book can be viewed before, during, and after any course (tutor, private or public course such as from a local college) as a helpful supplement to learning. One thing to always remember is that it is not how much that you know of a seemingly infinite or endless topic, but how you can apply what little you do know so as to make some type of practical and beneficial use of it for yourself and-or others. To learn more about the infinite subject of mathematics, just apply what you already know as the first step to begin leaning new concepts. This process could be called the "fundamental math learning concept" where each new concept is built upon previous concepts. It's like a logical chain, steps or progression of concepts from one to the next following one. Some concepts can be fascinating and an achievement to learn, but there is nothing within this book that is only for the magically gifted to attempt. Concepts are rather given in successive steps so as anyone could reach and learn them. Even the most highly advanced math topics are built upon and utilize the basics of math such as: **addition, subtraction, multiplication, and division** to a great extent, and therefore knowing just these, you can go a long way in terms of practicality, usefulness and learning new things.

This book also includes some example plain text, readable and editable, **computer source code programs** that are mostly based on the math material found within this book. The computer programming language used is the popular **C programming language**. Even if you may never program a computer, you are still encouraged to take a glance at them when your able for the sake of familiarization. A computer can quickly calculate many mathematical values that would otherwise be very difficult and-or time consuming to do. Computer programming is an interesting, joyful, artful, and rewarding hobby that could even become beneficial to others when making even relatively simple computer programs that can easily calculate values that others such as scientists, hobbyists and businesses could benefit from. Always remember that there is "strength in numbers", "numbers matter", and that nobody is magically born knowing how to do math, but they can learn the knowledge step by step at their own pace and needs.

Some history and facts were included in some topics so as to help provide a more practical understanding of what has previously taken place in our world so as to get to this modern level of understanding, and also to give some readers a "good read" of many things to consider further.

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This book effort is still an ongoing process, and there is some room to always correct and-or improve it. I apologize in advance for any possible logical errors, typos (spelling mistakes), omissions, amateur articles (usually topics beyond the full scope of this limited, practical book), and occasional sentences that could be rewritten. This book has become difficult to edit as of the year 2024, and new edit versions wilil take longer. I also have some health issues, and second-hand cigarrette smoke has made my situation worse throughout my life and living places, and even as I write this now - "challenging" my health, particularly my circulation, and "peace of mind", and it makes this book creation difficult. Please donate. This book does does not compete with the educational books commissioned and-or professionally made by others, and it was rather made to be an interesting and helpful educational supplement for any book, course, hobby, household needs, work, etc. Each writer and-or book has a style to appreciate in some way.

What is given in this book is worthwhile and can be enjoyed through the years and needs as they arise. This book slowly grew from being a basic math book to be now of a very large size, and therefore, it is very time consuming to edit, and it is the main reason to publish what is available in it as of now. In the current state of this book, I feel it is acceptable enough to help others in some way, even as supplementary material for any course, book or needs they may have. I believe this book can inspire others just like many other books can, and it might also put some helpful wind into the sails of some various educational assistants. You are also encouraged to take a course, public or private, in what you feel is needed or required, and regardless if you have this book or not as an additional source of helpful information and ideas. Many articles or topics in this practical book are not meant to be a full rigorous study of it, but rather just a practical introduction with just enough knowledge for the average person to consider and-or be satisfied with, and without it then being overwhelming. Some articles may contain a mention of previous topics and-or values, and so as to provide some assistance and convenience to the reader, although it does increase the number of pages in this book. There are many other nice math and science topics and details which could of been placed in this book, however, I know that it already has more than enough material for the average reader to consider from just one single book, and that all books of this nature can be lacking or incomplete in some way. I will continue to work on this book effort so as to improve it in a variety of ways just as any book already written may be improved.

Math books of all types, well written and presented by other authors are often available in some stores. This book is a helpful supplement or educational support for any book, course, hobby, work or home-learning that you already have or may then consider. This book is not a replacement for a formal and-or required education in math, and in fact, it probably falls short in many areas. This book is not to express or showcase any special mathematical topics, ability or knowledge so as to impress people or challenge their previously learned mathematical abilities with various unlearned, random and clever mathematical puzzles that perhaps would be generally difficult to solve at first, but once you know the solution it make you question your mental abilities. This book is rather another way to express the most common and practical math topics. This book was also written due to the mention by others on various media outlets that the average student math grades and worker abilities in the United States were lower than those of other countries. Some employers may require a training certificate and-or graduation certificate in some field of study, or perhaps from just a single course or lecture, and this book can not formally offer that, but it rather offers some additional education so as to help facilitate or supplement a more formal book, course, home study, hobby, work, or for inspiration to understand many things of math and science.

This book can also be placed on a modern phone so as to have it immediatly available for use if your phone or device has enough power remaining in it, or where the internet reception is not available such as during some hiking or camping trip.

This book is also very helpful for both new or experienced computer programmers regardless of the specific programming language of choice, and both the programming concepts of a computer programming language, and a program's source code (ie., the plain text, readable version of it) can be translated and applied wherever needed.

"Thank-You", from the author.

LINKS AND OFFERS

The following are some fairly stable links of many years of availability to some offers, causes, programs or apps, and websites I found, and thought I would then share them as an effort to help each other and give the reader somethings to go check if they would like to. Since this book may take a while to view and-or read, they were arbitrarily placed upfront here as part of the book introduction area. These links, which are like readable electronic addresses to (internet) websites and-or web pages are subject to possible change beyond the knowledge and control of the author of this book, and if so, you may sometimes need to do an internet search such as by using an internet web browser (access and viewer program or website such as Google.com) when considering a link. Simply type and-or copy and paste any links into your internet browser and-or email program to begin.

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C Language Compiler **TinyC** is a fast C-language compiler that will take a plain text (.txt) file of C language, computer programming text source code, and creates an executable (.exe) program file from it. This works in a DOS or text computer command-line environment, even if you have a Windows ® computer operating system. The text source code programs in this ebook were compiled into computer programs with TinyC. For advanced programmers, there may also be some TinyC functions or code libraries available for basic Windows system accessing and programming. C++ is an advanced form of the C programming language that includes
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ELECTRONICS **Talking Electronics** is a popular electronics website for the electronics hobbyist. There are many educational articles, circuits and kits available: Their website is: talkingelectronics.com

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Basic Supplementary Income (BSI), or Basic Supplementary Assistance (BSA) is only concept created by the author of this book, and it is for the destitute in financial need, and so as to improve health and well-being, reduce physical and financial stress, potentially reduce crime and help prevent very high medical costs via a relatively low cost prevention method. This can create a better quality of life and social dignity. Perhaps $5 to $10 USD (2025), and depending on the specific location and its local costs for items, added per day on a debit card, and so as to help prevent misuse and to promote financial responsibility. This money will also directly improve many businesses in need, and this is very similar to how the "Food Stamp" program that helps maintain and improve society, and it indirectly helps the entire food industry, jobs and other businesses. Perhaps higher valued items that are in need such as shoes and clothing can be saved up for day by day. Surely, this value wont even come close to paying someones rent, but it can still do much, and it should then also be easier to pass into legislation. **BSI** is not the same concept as the **UBI** - Universal Basic Income concept which basically costs several times more, and it is usually then not made into legislation to help people, and then nothing is getting done in terms of UBI. The BSI concept should be more passable into legislation.

Animal Wellbeing - Support animal wellbeing and the banning of animal testing. Animals are innocent beings who can not speak as we do or hold fairness rallies against such matters. Try to not eat meat, and try to eat more vegetables, nuts and a variety of plant matter. There are many new alternatives to using animal testing. Peanut butter is said as being a complete protein. Nuts and-or nut butters are also high in certain minerals (ie., elements). Vegetables, etc, can be shredded using a grater, ground up in a machine and-or mashed in a soup, and so as to be easier to eat and digest, and then have more surface area for nutrient release and absorption into the body. This helps prevent overeating so as to get enough nutrients. A **vegetarian** may eat some animal products like milk, cheese and eggs, but a **vegan** does not eat any animal products, and both avoid animal meat for their health and-or conscience, nonetheless, everyone should have a goal of eating less or no meat. There is no joy in taking the life of a living being, and-or having it suffer its life in poor living conditions.. **PETA** (People For The Ethical Treatment Of Animals) is a nice organization.

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Math and science are knowledge and-or educational fields of growth which have taken many hundreds of years of previous growth of new ideas and discoveries by many other people, and so as others can access, learn, improve and build upon them. No one should ever feel that they should have been somehow born with knowledge and-or special abilities to do something which can only be learned over time and-or with some helpful course. A person who appears as "smart" is often just someone who has taken the time to memorized a few things already known, perhaps just a specific subject and probably just a niche in that, and to then simply recall them when needed. This book itself might even be one that a niche of people are also looking for and need.

This book is dedicated to my wife Elizabeth, and to our relatives, helpers, animal lovers and friends. Special mention to **Mr. Norman Borofski** for helping to raise me and my brothers, and also to **Mr. Dana Schweitzer** for his academic inspiration to many, both from Pennsylvania. This book is also dedicated to the future thinkers and tinkerers who may help the future of humanity in some way.

This Edition is 1.0 as of July 7, 2025. All previous beta versions are still very useful, but there are some new writings and edits available in the newer versions of this book. For the latest version number or date, etc., search the popular **https://www.youtube.com** video website for: **Mathization ebook or @mathization**

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INTRODUCTION

This book is intended for readers who are just beginning with arithmetic, algebra, and trigonometry, or for those who always wanted or needed another grasp on these subjects. Knowing how to work with numbers can enlighten you and give you confidence in problem solving, reading comprehension and discussions. This book is composed of four general mathematical topics and a generous science and appendix section, and each is given an entire section of this book. This book is intended for a rapid understanding of these topics by basically using a less formal or less critical approach to mathematics. Due to the style of this book, it can also be used as a basic reference, a helpful supplement or complement to another book or course you are currently taking, an aid for a teacher or those being educated at home, or as a step to a more formal or advanced mathematics course. If a topic or example seems of no certain need or interest, or perhaps too complicated at the moment, please skip over it to another useful topic or example and consider that which was skipped over to be reviewed at a later time. If a topic needs to be referenced at another time, please write down the topic and the page number(s) and-or formula(s). Some of the concepts are not explicitly noted in the table of contents, and are rather given "on the fly" and-or briefly, that is, they are given as you read and come upon them as necessary or helpful, but out of the general book order, and so you may need to write the topic and page number on a piece of paper.

With some available liberties of creating an e-book, such as having an unfixed length or number of pages, style, content, an effort to be different so as to be more desired, etc., this book contains many pages needed to convey many thoughts that can be roughly grouped into the following categories: main text and knowledge, examples, figures or graphics (many in a simple homemade form), computer programs, and numeric data tables. There is about 40 pages worth of "blank space" is either due to the leftover space of unfilled pages and-or pages intentionally left blank to help assist with the editing (ie., corrections and-or additions) of this ebook. The many pages helps to give the reader much material to have available and learn from just obtaining one book, especially if your access to other materials is limited and-or having a limited amount of money to purchase books, etc. There is a saying that can be adapted here, as that there is sometimes a certain quality to be found in having a large quantity or volume of material to access immediately when needed. For writers, if just two pages a day are written, a person could create about a 600 page book during just one year or less if the book contains many images or graphics.

Many topics that are frequently reserved only for algebra studies are shown in the first section: ARITHMETIC. Here, instead of using algebraic variables to show a concept, actual numeric values are utilized instead. By doing this, understanding basic algebra becomes easier since many of its concepts were already effectively shown using actual numbers instead of variables. Likewise, the transition from BASIC ALGEBRA to the BASIC TRIGONOMETRY section was made to be as smooth as possible. The concepts of trigonometry should also not be overlooked since they seem to arise everywhere in mathematics, and in the least expected places elsewhere in life.

A special section of some ADVANCED TOPICS is included at the end of this book. It is intended to mainly show you how to make some very important calculations when a scientific calculator (a preprogrammed, multifunctional calculator) is unavailable. An inexpensive regular "home" calculator can also be a great assistance throughout this book, especially for performing otherwise tedious and time consuming computations, such as finding the square-root of a number, and for checking results easily and quickly so as you can move on to other things without loosing your train of thought due to lengthy calculations. There are many calculator programs for computers, and apps (applications, programs) for phones such as RealCalc. The basic versions of these programs are often free to download, and you often will have a generous option of purchasing a fuller version and so as to fairly support the creators of it. If you already have a scientific calculator, or not, you will effectively learn throughout this book what the common buttons, commands or functions on most scientific calculators actually do. It should be noted that this section can still be considered as a continuation of the topics covered in this book. Don't forget the SCIENCE AND APPENDIX section, for it also contains much very useful information and derivations related to the other book sections, and is itself a good mathematical read.

At the beginning of the ARITHMETIC section, their is a discussion about the decimal number system. Here, you might find out that there is more to "plain old numbers" than you might think. After that topic, discussions about expressions and units (of reference or measurement) are given. To get through these discussions, you should (hopefully) already of had some initial thought, education, or understanding of the basic mathematical operations of addition, subtraction, multiplication and division. Those operations are given a more general discussion after these more fundamental concepts are first used to build upon.

SECTION 1: ARITHMETIC

THE DECIMAL NUMBER SYSTEM

The decimal number system is the common number system we are all taught to use as the accepted standard numeric or number system in the modern world. For many people, many of the fundamental concepts of numbers are either unknown, weak, and-or taken for granted. Because of a lack of a firm understanding of the basic concepts and fundamentals of numbers in general, this is where some problems may start in learning math and-or using math in a more practical manner with confidence.

We are first taught to count things, items or objects which may, or may not even be similar. To count (as a verb) things or items is a repetitive (to repeat, "do again, and again") process of steps of a measurement so as to numerically quantify (to express as a numerical representation) the total (all, entire) amount, "number of" or quantity of things or items being considered. Here, to measure is the act or process to find (often by comparing to some standard or reference) and give a certain type of "size" or "size of it all (things or items)", and that this measurement (the result of measuring) is called the quantity or "the count (as a noun, and a single number result)", sum or total of the items or things. The measurement result, or simply, "the count", is expressed or represented numerically as a single, symbolic and representative number. In a way, a number is much like a special type of word, and a unique word which has a unique meaning. With the decimal numeric system, a number can be a single symbol (0 through 9) called a single digit number or numeral, or a grouping of several of these symbols so as to represent larger counts, values or results of a calculation such as combining or summing two counts together so as to have another representative count (measurement) of things and a number to represent that count.

We measure a quantity or amount of items or things by considering (identifying or reckoning) each individual item or thing and then (numerically) summing each item as an increment of 1 to the total or previous count of items that were already considered. We will consider 1 thing or item can be numerically considered and represented as equal (=) to a count of 1 numerically. Summing (summation) is to take this numeric value or representation of 1 and mathematically or numerically successively combine it to or increment (increase, here by 1) the current count or number value so as to make an updated, larger and more accurate numerical (number) representation. Before the final or true total or sum is found of all the things, the active (in the process), or "running" sum is called an "intermediate", or incomplete or partial sum of the entire quantity being considered. We see that to count is a process to successively (ie., summing, addition), and numerically quantify (quantize, quantization, account for, calculate) so as to represent any quantity (several, many) or amount of items or things so as to produce a single numerical (number) representation of the total (including all, complete, sum, result) amount, "(final) count" or quantity of them. Each item, whether the same or not, can be assigned and represented as a numeric value of 1. For example, a count of several, here two items can be expressed as: "one and one" in written or spoken form, or (1 + 1) in numeric representation, and the result or sum is written or expressed as:
1 + 1 = 2, and is spoken as: "one plus one, equals two".

In Europe (or the "old world"), the decimal (based on 10) number system has only been in practical use since the 1300's AD, hence about 700 years ago. With their invention of the decimal (base 10, based on a count of 10), number system and positional notation in about the 5th century AD, about 1500 years ago, the people of the country of India ("Hindu's") made many initial advances in mathematical calculations. This superior (positional) counting and recording system is the main model and influence of the Arab (Arabian, middle-east area) numbering system with its more refined and standardized number symbols, and the decimal system eventually became known as the Hindu-Arabic (decimal and positional) numbering system. This system eventually replaced most other numerical systems in use throughout the world, such as the Roman Numeral system (with symbols: I=1, X=10, V=5, etc.) which could represent any number (after a possible calculation, hence not always explicitly, for example: IX=9, VII=7, XV=15, XVI=16) but was generally good for only very simple calculations. The Roman Numeral system is lacking and insufficient for these modern times, and is rarely used. For the decimal system, the name "decimal" is given due to the "deci"or "ten" (10) counting symbols it utilizes. Many people believe this system was brought about from the fact that we have ten fingers, and surely people counted with their fingers as placeholders (ie., as a memory) and-or to quickly express a value to someone via sight. Before writing as we know it was invented, ancient mankind in ancient Africa was already aware of the use and need for mathematics and was already using objects such as fingers, stones or notches (a primitive digital-like, binary system with something=set=1 and-or nothing=not-set=0) for a long lasting artificial memory or record, and a primitive form of writing to (symbolically) represent, store (record), and communicate things such as a quantity of something. For example, notches on a stick or stone, or perhaps dots on a wall were used for representing quantities or amounts of things like the number (quantity) of apples, people, and-or days. Each finger (often called a **digit** in ancient times) or notch symbolically (with a visual symbol such as | , a recording) represented a quantity of one. By using two hands, a single and unique quantity of one through ten could be represented by using that many fingers. Thereafter, such as in India, other simple or compact written (for expression and record) symbols were invented so as to easily represent these and larger quantities. Three items (| | |) and-or fingers could be expressed as 3. Four items and-or fingers (| | | |) could be represented and-or expressed as the (more compact) symbol 4. The initial seeds of math were planted long ago in Africa, and the usefulness of it are still being used ever since, and it was only a matter of time till mankind created machines like calculators and computers to quickly calculate values. It is interesting to note that computer and-or digital binary numbers are essentially composed of a series of notch and-or its absence of symbols. For example, a count of ten in the computer, binary is 1010.

Once writing was developed in ancient times before the decimal system, various written symbols (ex. such as an "X" for ten) could now represent and-or store (in writing as a written record or memory for later use) the 10 possible "finger" or "digit" representations of a quantity. When the decimal system was developed, the symbols were simplified as these incrementing numerical values and symbols of: 1, 2, 3, 4, 5, 6, 7, 8, and 9. A quantity of "no quantity at all" or "none" had to be considered and was assigned and represented as the a numeric symbol of: 0. We call this the "zero" symbol, and it was created by the Mayan people in the "New World", commonly known as the Americas. It was quickly adopted by the rest of the world as a standard decimal symbol, often replacing just a blank or empty space previously used for 0. Note that today, a value of ten (or larger) is not considered as one of the basic decimal (ten) counting symbols or numbers in the decimal number system. The ten symbols and values, typically called digits or the digits of a decimal number, used in the decimal system, where each next symbol represents a count or value of one (1) greater than the preceding value are: 0, 1, 2, 3, 4, 5, 6, 7, 8, and 9. 0 will also sometimes be utilized to create numbers that represent quantities equal to or larger than ten (10) which is the next value after an increment (of 1) is applied to 9. 10 ("ten") is therefore now using two decimal symbols and places to represent one unique numerical value or number. A single symbol or even a large group of these numeric symbols placed together, so as to represent a unique count (number of items) and its corresponding numeric value, is sometimes called a (unique) decimal, or (decimal represented, or representational) number. Each particular number, displayed or written, mathematically represents a particular quantity (often just spoken as simply a "number" due to the common numeric or mathematical symbolic representation of quantities) of something. Numbers used for common counting are simply called as the "counting" (quantifying or measuring) or "natural" (individual units (ex. an apple) that often occur in nature) numbers. When the symbol 0 ("zero", none or nothing) is also considered along with the counting numbers, numbers of this type are classified as the "whole" numbers. For example, if you had 5 whole or individual things and gave them all away, you now have 0 whole or individual things remaining or left. Here, 0 (as a "logical" or "thought" number) is the valid numerical symbol and the representation for the quantity left or remaining. Including (ie., adding , combining, grouping) another whole object with 9 whole objects will create 10 whole objects.

A good method to understand and use these ten (10) basic numbers is to both physically and-or mentally visualize, think, memorize, compare, and draw each number as a group of similar objects (perhaps coins, fingers ("digits" is the equivalent old Roman word), a certain graphic shape, or simple pencil dots. Here below, arranged in arbitrary groups of 4 so as to have a nice orderly appearance and arrangement for faster recognition, counting, and so as to also reduce the possibility of error while summing. An advantage of this method below is that it is pure and without a number system needed.

See the following graphic:

zero	one	two	three	four	five	six	seven	eight	nine
0	1	2	3	4	5	6	7	8	9
	*	* *	* * *	* * * *	* * * * *	* * * * * *	* * * * * * *	* * * * * * * *	* * * * * * * * *

: English words for the first ten numbers. These are generally a form of the North European, Germanic area words. Sometimes, "zero" is spoken as "not".

The above dot numerical representation system can also be thought of and-or counted as:

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1  2
3  4

5  6
7  8

  9
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A common question then arises: How then are larger quantities or values greater than 9 numerically or symbolically represented using the decimal numeric system? The answer is to use what is called positional notation. To use positional notation, the symbols are properly (standardized) arranged and set next to each other in new locations or positions of the indicated number, These positions are also called "digit" positions. A digit of a number is composed of a one-symbol number (0 to 9) of the decimal system, and its (digit) position within the overall number. The quantity or value that this arrangement of digits of the number represents (being equivalent or equal to) is not simply the sum of its digits:

Ex.   35 does not represent, and is not equal to the addition of 3 an 5,   or = 3 plus 5,   or = (3 + 5) = 8

First consider that you have a count of 8 and add 1 to it. We know that the symbol to represent this new count is the nine symbol (9):

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                         8          +  1   =          9
   8          : * * * *  * * * *   +  *   =   * * * *  * * * *  *   =  8 + 1 = 9 ,  "eight plus one is nine"
+ _1_                                                                              "eight and one equals nine"
   9                          "eight combined with one is nine"   ,    "one added to eight is nine"
```

If we add 1 to this value of 9, we will either need a new symbol (which doesn't exist in the decimal number system) or a combination of the existing symbols to represent this new value. What symbols that we already have should we use? The apparent choice is to let 9 "roll-over" (as if all the (incremented) symbols were all on a rotating and repetitive whee) or "reset" back to the lowest symbol of 0 (so this position can possibly be reused for representing higher quantities and for further additions) and increase the next leftward digit (considered 0 if not indicated or shown) or decimal number and-or position, by a count of 1:

```
   08                              : Leading 0's are alright to express some viewpoint about something
+ __1_        or        1            since it will not add any value to the total
   09          =       09
+ __1_               +_01_              9        +  1   =        10
   10                  10          : * * * * * * * * *  +  *   =   * * * * * * * * * *   =  9 + 1  =  10
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The number 10, spoken as "ten", or sometimes "one, zero", represents the quantity of "nine and (plus) one", (9+1). It can also be said that for the decimal (based on 10) number system, each 1 or count in the next leftward position represents (9+1) = 10 ("ten") of the previous position - of which even for itself represents 10 of any previous existing rightward position in the number. This concept is called positional notation. Notation is an accepted or standard way to indicate or express something.

The specific value of representing 10, or "ten times", more of the previous position is called the positional weight, the digits weight, or simply the weight of that position. Note that weight has nothing to do with the specific digit value (0 to 9) or count in that single position or location. The product (multiplication) of the positional count (the digit shown at that position in the number) and its corresponding positional weight yields (produces) a positional product. The total value or quantity being represented by a series of digits of a number is simply the sum of their positional products, and which produces a positional sum which represents the entire number being considered. Don't forget to add in the first digit's positional product which has a positional weight of one (or (single) "units") only. For example, in the number 275, starting with 2, which is in the position or column having a positional weight of 100 ("one-hundred", or (a) hundred), the positional products are: 2 times 100, which is 200, and 7 times 10, which is 70, and 5 times 1, which is 5. The positional sum is therefore: 200, plus 70, plus 5, is or equals (=) 275, and this can be spoken of a in terms of its digits from left to right: "two, seven, five", or more often as: "two-hundred, and seventy, and five", or even more simply as: "two-hundred and seventy five".

Digits leftward in a number are called more significant digits since they represent a larger part of the value and positional sum. This is so since their positional weights are larger. The most leftward digit that has a value greater than zero is called the more or most significant digit (MSD). Likewise, digits more rightward are called less or least significant since they represent the least or smallest part of the positional sum. The most rightward digit having a value greater than zero is called the least significant digit (LSD). For approximations of a number, the least significant digits often become unnecessary and are not used or indicated since they only represent a small (as an insignificant small percentage or fractional part of the whole value) value and part of the total value or number. For example, if you have or counted 559 or 561 things, you could say something like "I have about, or approximately, five-hundred and sixty (=560) things". Here, a value of 1 was omitted from the sum, quantity or number, and this value of 1 is a very small value as compared to 560.

To be proficient in addition, a student must first know how to add (and subtract) any two of the basic (0 through 9) decimal numbers. Since addition is normally performed column by column (starting from the least significant digits), and the highest value for each digit is 9, the highest possible column sum is 9 plus 9 which equals 18, plus 1 possible count as a carry or "carry over" from the previous column making a total of 19. The series of whole numbers from 0 to 19 where the next number is a count of 1 higher is: 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19

A good method to illustrate the concept of the addition of two numbers is to display or draw an object or group for each number, and then count or mathematically quantify and express them by successively recognizing (ie., "reckoning") the quantity that each number represents, and then increasing (repeated incrementing) the intermediate sum being found so as to find the total quantity or sum of the two numbers and-or the quantity of all of the objects those numbers represent.

```
Ex.     1    +    2     =     3      =     3
        *    +   * *        *  * *       * * *
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As an aid to adding, or rather than use your fingers, you can discretely use dots on paper or in your mind as previously indicated for the basic decimal numbers. Usually a 4-dot system that has a square layout is a practical method.

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*  *
*  *
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For higher values, you can simply reuse the same 4-dot system locations, or start another one. For example:

5 would be something like this when expressed as items or dots:

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*  *    *
*  *
```

Let's say you need to add 4 and 3. You can first put 4 (the initial running total) in your mind, and then tap or draw out 3 dots (of the 4-dot system) while adding one (1) to the running or incrementing (increasing) sum or total in your mind for each new dot considered. For this example we would not even need all 4 dots of the 4-dot system, but only 3:

3

* * : You can see that a "visualization memorization" method of knowing
* how many dots to make, and-or how many you are viewing needs to
available when needed.

The steps in your mind would be something like: You first have 4 in your mind, and you need to make 3 more dots to add to that value. Then make a dot and add 1, making a (updated, modified, intermediate or partial) total of 5. Then make another dot and add 1 more making a total of 6. Then make another dot (the final third dot) and add 1 more in your mind, making a total of 7. The result can be said as: "four plus three, equals seven" or perhaps written as: "4 plus 3, equals 7". This "dot system" might seem very primitive and almost silly to some, but I think many people, even advanced mathematicians, still utilize it to some extent. It's an alternative to counting with the help of your fingers, but that matter is up to you, so as to use whatever works best for you. Another tip is that you can switch the two values being summed and the result will still be the same. This method often helps to perform an addition. Consider adding 1 and 9. It is easier to start with 9 in your mind than then simply add 1 to it so as to have 10, rather than starting with 1 and then having to add or increment 9 to it.

The result of adding any number with 0 is that same number since nothing was effectively added to that number and the number should remain the same:

Ex. 7 + 0 = 7 Ex. 0 + 4.1 = 4.1

The basic "addition table" is a "look-up" table. With a table, you can view, seek and find corresponding values so as to find the answers or results already solved for. The addition table or chart is a good aid for learning the addition of small numbers. It can even be extended as needed to included larger numbers and their sums.

ADDITION TABLE

0	1	2	3	4	5	6	7	8	9	. . .
1	2	3	4	5	6	7	8	9	10	
2	3	4	5	6	7	8	9	10	11	
3	4	5	6	7	8	9	10	11	12	
4	5	6	7	8	9	10	11	12	13	
5	6	7	8	9	10	11	12	13	14	
6	7	8	9	10	11	12	13	14	15	
7	8	9	10	11	12	13	14	15	16	
8	9	10	11	12	13	14	15	16	17	
9	10	11	12	13	14	15	16	17	18	
.									.	
.									.	

To use the table, the sum of any two numbers is at the intersection (meeting, crossing or joining) of the horizontal (left and right direction) row, and vertical (up and down direction) column. For example, if you were to add 3 and 2 you would look for a row or column of 3, and then find the corresponding (here, this means the only other, single and unique part of the row and column pair) intersection of the 2 row or column, and the result (sum) seen at their intersection will be 5. Just the same, you could of also started by taking the 2 row or column, and then finding the intersection of the 3 row or column. The result will be the same value of 5. To be proficient in addition and subtraction, all the possible sums (for the basic (0 through 9) decimal number values) above should be memorized or available without too much difficulty. If you can't memorize all of it, then the "dot method" to help add two values, as described above, can be utilized as an aid to counting and-or summing to higher values.

If you are given a number from the table, you can then find two numbers (a corresponding pair) that were, or can be, summed up together to make that given number, hence the table above can also be used in a backwards or reverse type of manner for subtraction purposes. That is, given a sum (the total) and you have or take away either the row or column value part, you can arrive at the other corresponding value part required to make that given sum. Given one number (here the row or column number), you can find the other or "remaining part" corresponding row or column number that was part of that given sum or number. If you take a number in the table and subtract (or "reverse count", "reverse increment" which is called decrement) either the row or column value, you can then see the other corresponding row or column value (technically called the "difference" when subtraction is being considered) needed to make that number total (or sum). For example, take value 10, and you want to subtract or remove 6 from it. Look up 10 in the addition table, and then look for either the row or column of 6. The other corresponding row or column (required to make that specific pair with an intersection at 10) will be 4. You could say that subtraction is the "backwards" or "reverse" of addition since you are first given the result and are to find the other (corresponding) number of a pair (two) of numbers that will sum to that value or quantity.

Some examples of how to say multi-digit numbers, and the word "and" is often used for the word "plus" or "added to or with":

10 = "ten" , 10 + 1 = 11 = "ten plus, and or added to one"
11 = "eleven" , 12 = "twelve" , 13 = "thirteen" , 14 = "fourteen" , 15 = "fifteen" , 16 = "sixteen" , 17 = "seventeen" ,
18 = "eighteen" , 19.57 = "nineteen point fifty-seven" or "nineteen point (and) fifty-seven hundredths" ,
20 = "twenty" , 21 = "twenty-one" , 30 = "thirty" , 32 = "thirty-two" , 40 = "forty" , 43 = "forty-three" ,
50 = "fifty" , 54 = "fifty-four" , 60 = "sixty" , 66 = "sixty-six" , 70 = "seventy" , 77 = "seventy-seven" ,
80 = "eighty" , 89 = "eighty-nine" , 90 = "ninety" , 98 = "ninety-eight" ,
100 = "one-hundred" , 105 = "one-hundred and five" , 123 = "one-hundred and twenty-three"
1,000 = "one-thousand" , 1123 = "one-thousand , (and or plus) one-hundred and twenty-three"
10,437 = "ten-thousand, (**"and" or "plus"**) four-hundred and thirty-seven"
100,987 = "one-hundred-thousand, (and) nine-hundred and eighty-seven"
1,531,251 = "one-million, (and) five-hundred and thirty-one thousand, (and) two-hundred and fifty-one"
1,907,000,000 = "one-billion, (and) nine-hundred and seven million"

More examples of how to say various numbers:

0 = zero , 1 = one , 2 = two , 3 = three , 4 = four , 5 = five , 6 = six , 7 = seven , 8 = eight , 9 = nine

10 = ten , 11 = eleven = 10 + 1 , 12 = twelve , 13 = thirteen , 14 = fourteen , 15 = fifteen = 10 + 5
16 = sixteen , 17 = seventeen , 18 = eighteen , 19 = nineteen = 10 + 9

20 = twenty , 21 = twenty one = twenty plus one = twenty and one , 22 = twenty two = 20 + 2
30 = thirty
40 = forty
50 = fifty , 53 = fifty three = 50 + 3
60 = sixty
70 = seventy
80 = eighty
90 = ninety , 95 = ninety and five = ninety plus five = 90 + 5 = ninety-five

100 = one-hundred
125 = one-hundred and twenty and five or= one-hundred and twenty-five = 100 + 25 = 100 + 20 + 5

Spanish words for the ten number symbols:

0 = cero , 1 = uno , 2 = dos , 3 = tres , 4 = cuatro , 5 = cinco , 6 = seis , 7 = siete , 8 = ocho , 9 = nueve , 10 diez

French words for the ten number symbols:

0 = zero , 1 = un , 2 = deux , 3 = trois , 4 = quatre , 5 = cinq , 6 = six , 7 = sept , 8 = huit , 9 = neuf , 10 = dix

German words for the ten number symbols:

0 = null , 1 = eins , 2 = zwei , 3 = drei , 4 = vier , 5 = funf , 6 = sechs , 7 sieben , 8 = acht , 9 = neun , 10 = zehn

WHAT DOES MEASURE MEAN?

To measure is to find the size (length, weight or mass [amount of matter or material], volume or amount) of something, or the quantity of several things or objects in a group or collection of objects. Today, numbers are the most common way of (numerically and symbolically) representing a quantity or the result of a measurement. Other than that, it is possible to use words such as: "small", "medium", and "big" as measurements, but these can sometimes be ambiguous (unclear, uncertain) to someone else. To measure is done by the act of comparing, or the comparison of one thing to another (often similar) thing that is used as a reference for the comparison. The word of "meter" is similar to the word of "measurement" and "metrics", and is also the name given to the basic unit of length or distance in the metric (measurement, reference) system. A measurement value or number is the numeric result, count or sum of comparing. If the thing was a stone, or stick of a certain dimensions or qualities (ie. width, length, weight), that thing could be used as the basis (reference) of comparison or measurement for all other similar things. A common scaled ruler, having one to several unit sized markings (indications, steps) is basically a replica (a similar representation or copy) of something, such as another standardized or agreed upon reference ruler. The ruler or (measurement) reference gauge is then used as the "rule" or "standard (standard reference)" to consider and use with all similar types of measurements such as lengths, weights, etc. The simplest (non-numeric) kinds of comparison is to state that the things being compared are not the same or are equal, or that something is simply more or less than something else without giving a numeric quantity of some specific size or the difference in sizes and-or amounts. To measure quantitatively, numbers and some form of a standardized measurement unit which is one specific instance and-or identification of the reference and comparison value used for a measurement. More about units or "units of measurement" will also be discussed further ahead in this book. For measuring the weight of stones, the units could be some small reference stone, and this "reference-stone" would then be the units (of reference for the measurement) of weight and for measuring the weight of any other thing, including other stone weights and so as to also have a weight reference or unit that is twice or other multiples as heavy. Even the numbers in the decimal number (counting, quantifying) system that we use for everyday counting are in reference to something, and that something is number 1, the numeric unit. Each count greater, higher, or the "next higher or larger number", is based on this reference of 1. Given a number, to find the next higher number, add or combine this 1 to that given number. To measure something quantitatively is to express how many (ie., the number of) units (the standardized thing of reference, comparison to, and measurement of) that something has, or can be compared to something considered as the reference. [FIG 1]

Letting this length be the standard (reference) unit of measurement for all other measurements.

This will be the "rule" of measurement. It is a "unit ruler" or ruler that is only one unit long.

Joining several together will help us measure larger or longer lengths more quickly and prevent error:

The points indicate the start and end of each reference unit. Since they are the same, the ruler consists of many, or a multiple, of that unit.

Indicating how many in an increasing or incremental (integer) order:

: a ruler

The branch of science for the study of measurements, standards and references is called **meterology.**

The above discussion was how units of distance or length can be added so as to make a larger distance. In a similar way, this is also true for any other unit of measrment such as weight, mass, area, volume, time, or even for similar objects.

EXPRESSIONS

A mathematical expression is essentially another form or representation of a number, a process (such as a single formula [ie., a "mathematical recipe"]), some mathematical manipulations (such as to use the square root of a number), or steps (explicitly indicated or understood) so as to find a single number or result that represents the given expression for it. That is, the expression or representation, however it is written or contains, and the actual number that it represents, have exactly the same value, hence they are equal (equivalent, =) and either can actually take the place or be (mathematically) substituted for of the other. For example, 5, and the expression of (3 + 2) are equal, and either could take the place or represent the other. Even a representation of just a number can consist of mathematical operations (processes) such as addition (+, "plus" is the common symbol or instruction for addition), subtraction (-, "minus"), multiplication (x, "times") , division (÷ or /, "divided by"), and operands (the numbers or values to be acted upon by the operations), which when taken as one single group, it represents a single value (the given number). Some of this was previously indicated with the basic concepts of positional products and sums, hence, even a number can be "viewed", "thought of", or considered as being an (special) expression and vice-versa since they are equivalent.

operand operation operand : A basic technical format of a basic expression. The (mathematical) operation is to be applied to, or in reference to the operands (numbers). Some expressions might only have or need one operand.

The 4 most common mathematical operations and their corresponding mathematical symbol(s) are:

Operation	Symbol(s)	Example	Some Common Wording
addition	+	1 + 2 = 3	"one plus two, equals three"
subtraction	-	3 - 1 = 2	"three minus or take away one, equals two"
multiplication	x , ( ) , * , (dot)	5 x 2 = 10	"five times two, equals ten"
division	/ , -----	10 / 5 = 2	"ten divided by five, equals two" or "five goes into ten, two times, or twice" "5 can be subtracted from 10, a total of 2 times".

Ex. 30 + 5 This is an expression containing one operation and two operands. The operation is the addition operation, indicated by the "add", "plus", "join" or "combine" symbol of (+), and the operands are 30 and 5. The symbol usually chosen to indicate an addition (operation, "add") of two values is called the addition or "plus" symbol, and was made popular by **Michael Stiple** (~ 1554) and **Robert Recorde** (~ 1557), although the concept of addition, etc., was already known for thousands of years, and the (+) symbol was beginning to be used in the late 1400's as meaning "and". Recorde is also credited to the subtraction (-) operation symbol, and the equals (=) or equality symbol. This Math book uses the tilde symbol: **~** to mean **"about"**. **~=** means "about equal", and **=~** means "equals about", hence these mean nearly or approximately.

In a reverse type of manner, it is very possible for an entire expression to be considered as a single operand.

Ex. Considering the last example and having any expression that represents a value of 30 as an operand:

$$\begin{aligned} (20 + 10) + 5 &= 35 \\ 30 + 5 &= 35 \\ 35 &= 35 \end{aligned}$$

: (The entirety or grouping of: (20 + 10) can be considered as a single operand (to be first "simplified", or solved, to a value of 30) to be added to the operand of 5.

$$\begin{aligned} (31 - 1) + 5 &= 35 \\ 30 + 5 &= 35 \\ 35 &= 35 \end{aligned}$$

We see in both cases that the expression within the parenthesis is equal to (an operand of) 30 when it is "simplified" (by performing or processing the mathematical operations) or represented as a single value. More is said about this below.

TO REDUCE OR SIMPLIFY AN EXPRESSION

To reduce an expression or to "simplify" an expression is to perform the indicated operations within the expression, or arranging it in another manner to effectively produce less or no further operations on the remaining operands. This process is also known as "simplification". As shown above, the simplified expression is 35 after performing the additions. In a natural conversation, you would normally say something like "I have thirty-five of something", rather than say something like "I have thirty, and then I have five more of that something" - which is sort of an indirect or roundabout way of saying a total and single value of 35. So then, we often see the need to simplify things such as mathematical expressions by solving them or making them uncomplicated, simpler, and nicer (easier to work with).

Ex. Simplify the expression: 2 + 3 + 4

The simplified form of this expression is 9. That is, the entire value of : (2 + 3 + 4) = 9.

As previously indicated, even basic numbers can be expressed (ie. in a mathematical expression) as a positional sum (of the numbers' digits, or digits of that number). The positional sum is an expression containing the sum of each digit times the positional (decimal system) weight corresponding to that digit or (column) position within the given number.

Ex. Show 35 as a positional sum of its digits.

Here, we are to write an expression for 35. Use the positional notation concepts summarized below:

Positional Weight = Is 10 times the previous (next rightward) position's weight.
Positional Product = The value in that position times its corresponding positional weight.
You can consider this expression or its value as an operand in the final sum.
Positional Sum = Sum of all positional products. This expression represents and equals
the total value represented by the number.

Taking 5 as the first digit in question, the weight for this position is understood as 1. This is known as the (basic) "units" (of measurement) or "ones" position. Note that even this weight too is 10 times the previous (next rightward) digits weight which is 0.1. That is, 10 times 0.1 is 1. The second digit will also have a weight of 10 times the previous positions weight. Hence , its weight is 10 times 1, which is 10. This is known as the "tens of one" or simply the "tens" position. Breaking each digit into its' positional product (separated by the addition symbol needed for the positional sum):

35
3 x (10) + 5 x (1) : here, showing the expression for the sum of the positional products, or:
30 + 5 : here, after multiplying for some simplification, showing a positional sum
(of the positional products) expression

Note, we now know the second digit does not represent 3, but it actually represents 30, or 30 units, and in strict mathematical or numerical terms, this would be 30 ones.

Ex. Show the number 3215 as a positional sum.

Taking 5 as the first digit, the weight of the first digit is one (1). The weight of the second digit is ten times that of the previous (first digit's) weight or 1x10=10. The third digits positional weight is 10 times that of the second digits positional weight or 10x10=100. This is known as the "tens of ten", "hundreds of one" or simply "hundreds" position. Likewise, the fourth digits positional weight is 10 times that of the third digits positional weight or 100x10=1000. This is known as the "tens of hundred", "thousands of one" or simply the "thousands" position. Expressing this information as a sum of positional products of 3215, we now have:

3215

3 2 1 5

(3 x 1000) + (2 x 100) + (1 x 10) + (5 x 1) : an expression for the positional products. After some simplification:
3000 + 200 + 10 + 5 : expression for the positional sum of 3215
: "three thousand, two hundred and fifteen"

POSITIONAL WEIGHTS IN THE DECIMAL NUMBER SYSTEM

The discussion above about expressing a number as an expression has lead us to this. The names for the commonly used weights in the decimal system are summarized below. The commas used when writing the numbers, being a visual aid, are optional. Here, the value of one (1) is used only to indicate the position in question since the value in any position can be between and including 0 through 9.

Whole Portion Weights (In Increasing Weights)		Fractional (values smaller than one) Portion Weights (In Decreasing Weights)		
ones or units	1			
tens	10	tenths	0.1	(or 1/10 = 1/10^1 = 1 (10^-1)
hundreds	100	hundredths	0.01	(or 1/100) = 1/10^2 = 1 (10^-2)
thousands	1,000	thousandths	0.001	(or 1/1000) = 1/10^3 = 1 (10^-3)
ten-thousands	10,000	tenth-thousandths	0.000,1	
hundred-thousands	100,000	hundredth-thousandths	0.000,01	
millions	1,000,000	millionths	0.000,001	
ten-millions	10,000,000	tenth-millionths	0.000,000,1	
hundred-millions	100,000,000	hundredth-millionths	0.000,000,01	
billions	1,000,000,000	billionths	0.000,000,001	
etc.		etc.		

Ex. 5.2 = "five point two" := 5 + 0.2 , "point" is formally "decimal point" or "(decimal) number point"
Ex. 0.0015 or with helpful commas as: 0.001,5

This value above would be commonly spoken as: "one, point, five-thousandths". It is also possible for the value to be described in many other ways such as:
"one-thousandth and (plus) five ten-thousandths"
"one-thousandth and five-hundred millionths"

Considering just digits, rather than digits and their positional weights, this can be spoken or thought as::
"zero, (decimal) point, zero, zero, one, five".

The decimal point (written as a period character = .) is a separator that separates the number of whole units from the number or value that numerically represents some part, fraction or portion of only 1 single unit. Before the decimal point was introduced by John Napier, and who studied logarithms, other symbols and methods were used. A portion or fraction (part of) is only a part of 1 whole unit. A fractional part of a number is that part less than 1. The decimal point also means to the effect: to add this part or portion of 1 unit also to the total. A number that contains a decimal point, and a fractional value, is sometimes loosely spoken as being a "decimal number". The parts of a decimal number can be expressed or considered as:

whole_value decimal_point fractional_value
whole_value **.** fractional_value : Ex: 100.5 = "one hundred point five"
whole_value and fractional value
whole_value plus fractional value
whole_value + fractional value

A value can be a complete fractional value where that value is less than 1. That is, the value does not even have a whole portion except the value of 0.

Ex. 0.5 : The whole portion is 0, and the fractional portion is 0.5 = "zero point five" = "five tenths (of one)"

More advanced mathematics (that will be shown) shows that each positional weight can be represented as an indicated power of 10. In an expression form, this is a base of 10 with an exponent. The exponent increases by one for each digit position leftward and likewise decreases by one for each digit position rightward. The weights in the decimal number system using exponential notation, and the common decimal equivalent form are:

. . . ,10,000 , 1000, 100, 10, 1 . 0.1, 0.01, 0.001, 0.000,1, . . .
. . . , 10^4 , 10^3, 10^2, 10^1, 10^0 . 10^-1, 10^-2, 10^-3, 10^-4, . . .

Note for example : 10^2 = 10 x 10 = 100 : basically, the exponent means repeated multiplication, here twice (2)
10^3 = 10 x 10 x 10 = (10 x 10) x 10 = 100 x 10 = 1,000
10^4 = 10 x 10 x 10 x 10 = (10 x 10 x 10) x 10 = 1,000 x 10 = 10,000

Note also that 10^0 is the weight of the ones position. That is, 10^0 = 1. Notice that for the whole units weights, that each greater weight essentially has "another zero" placed onto the end of the previous weight. In fact, this is a simple method to multiply any value by 10; simply put a zero at the "end" (in the "ones position") of the number (also move the decimal point one position rightward of course). Ex. 7x10 = 7.0x10 = 70.0 = 70, Ex. 3.52x10 = 3.520x10 = 35.20 = 35.2
Notice also that when 10 is the base of a power, that the exponent indicates the number of zeros in the decimal equivalent. For example 10^4 equals one with four zeros after it or 10,000.

Ex. Here is the positional sum of 0.0015

0 (1) + 0 (10^-1) + 0(10^-2) + 1(10^-3) + 5 (10-4) or perhaps more commonly:

$0 + \frac{0}{10} + \frac{0}{100} + \frac{1}{1000} + \frac{5}{10,000}$: here, the first 0, is for the "ones" whole units position

For an extra note, the first three values in the sum are equal to 0 after simplification (ie, here solving or simplifying the fraction; no or 0 parts of anything is always 0. Also, anything times 0 is always 0), and for the sum, this then leaves us with just:

$\frac{1}{1000} + \frac{5}{10,000}$ combining (summing) these fractions to a single (fraction) value - the method which will be shown further in this book, we have:

$\frac{15}{10,000}$: "fifteen ten-thousandths" (= 0.0015 after performing the division expression)

Here is some more explanation of the weights that are less than 1. Let's begin at the weight of 1000. Since any weight is a product (result of multiplication) of and factor of 10 times more than the previous (lesser) weight, previous weights are always a factor of 10 times less, therefore divide this factor of 10 out of a weight to find the previous weight. Dividing 1000 by 10 results in 100. Dividing 100 by 10 results in 10. Dividing 10 by 10 results in 1 which is the weight of the "ones" column or position. Dividing 1 by 10 and expressing the weight in various forms, we have:

$10\overline{)1} = \frac{1}{10}$ or = 0.1 or = $10^{\wedge}\text{-}1$: weight of the tenths column
: a "tenth of one"

$\frac{1}{10}$ = " $\frac{\text{one}}{\text{ten}}$ " or simply: "one-tenth" or a "tenth" (of one)

Dividing the weight of the tenths column by 10 we have the weight of the hundredths column:

$\frac{\frac{1}{10}}{10} = \frac{\frac{1}{10}}{\frac{10}{1}} = \frac{1}{100}$ = 0.01 or = 10^-2 : weight of the hundredths column
: a "tenth of a tenth"
:Note, 0.1/10 = 0.01 = 1/100

$\frac{1}{100}$ = "$\frac{\text{one}}{\text{one-hundred}}$" or simply: "one-hundredth" or a "hundredth" (of one)

Many other counting systems such as hexadecimal (has a base 12 instead of the common 10 as used in the decimal system, and often used with machine language/computer programming), octal (base 8), and the binary (base 2) number system that is used for understanding computers and other digital electronic circuits, use the same concepts of positional notation as the decimal (base 10) number system. The basic difference among these systems is that they use more or less counting symbols. The ADVANCED TOPIC section of this book contains a short discussion about the binary number system.

UNITS OF MEASUREMENT

To add properly, you must add things that are similar. These similar things are mathematically referred to as having "similar" or "like" (ie. same or common) units (of measuring, measurement or reference). Consider the numeric part of a value as: "how many", and the units part of the value as: "of what". Units are said as being that to which the numeric quantity or value (expressed and represented mathematically as a number) is in reference or measurement to. For example, it could be an inch, a mile or meter of which a measurement is based upon or in reference to. For example, you can add (mathematically, or even physically combine or group together **similar items**, such as into a container) a quantity of oranges to another quantity of oranges, and get a larger net (final sum) quantity of oranges. That is, here the like units (or "identifiers" or "type") of the items or objects being added are oranges. When you see the word "units" used in mathematics, it is a short version of the phrase: "units of measurement".

Ex. 5 oranges + 5 oranges = (5 + 5) oranges = 10 oranges

This mathematical writing or expression also indicates that you can simply add the numerical values or quantities of "like things" that have "like" or similar or units, and the resulting sum will have those same units.

You cannot add a quantity of oranges to a quantity of apples or anything else, since their units are unlike or not similar, and get a larger quantity of either apples or oranges. Their will be more fruit, but the quantity of either apples or oranges will still be the same even when placed in the same basket.

Many times you will be working with "plain" or unitless numeric values. You can consider their units as being "units" or "wholes" as in an entirety or completeness of something, or as "ones" for a more numeric sense.

Ex. 5 + 2 = 5 ones + 2 ones = 7 ones
Ex. 20 + 10 = 20 ones + 10 ones = 30 ones

When the quantities or values being added are measurements, their units are correctly called units of measurement. Again, to add properly, their units, or the unit, of measurement must be the same. If they are not the same, such as when adding "apples and oranges" (unlike units), it is possible to convert one value (its numeric part) to an equivalent representation that will have the same units of measurement which is needed for their combining (addition) and to express the result as having just one unit of measurement or reference.

Ex. 5 ft. + 10 ft. = 15 ft. : ft. = feet, which is a specific length unit, or unit of length
Ex. 3 ft. + 12 in. = : in. = inches which is another length measurement unit :

This expression above cannot be simplified as is since the units of feet (ft.) and inches (in.) are unlike. One of the units must be changed to that of the other. For example, the quantity of inches can be converted to its equivalent measurement with units of feet, or the quantity of feet can be converted to its equivalent measurement with units of inches.

Since 1 ft. = 12 in., or likewise, 12 in. = 1 ft. , and substituting this into the expression above:
3 ft. + 12 in. = 3 ft. + 1 ft = (3 + 1) ft. = 4 ft.

Note, that which is said about having similar or like units for addition, also holds true for subtraction and other mathematical operations.

Occasionally, a values units must be the combination of unlike units that were used during the mathematical operation or definition. The concept of the (necessary) use of several units (as like one new unit) is sometimes called "derived units", and may or may not be identified as a single unit name. For example, speed (or average velocity) equals distance, which has units of measurement of length, divided by a time value with its own units of measurement of time.

$$\text{speed (length units/time units)} = \frac{\text{Distance (length units)}}{\text{Time (time units)}}$$

: ex. of a value that can have a combination of units or **derived units**. Mathematically: $D = s\ t = v\ t$

Hence, the units for speed is a length unit divided by a time unit which are unlike units of measurement.

Ex. $50 \text{ mi./hr.} = \frac{100 \text{ mi.}}{2 \text{ hr.}}$: perhaps you went 100 miles in 2 hours, your average speed is then 50mi/hr Why is the word average used? Because usually, or normally, you sometimes will go slower, and perhaps even a bit faster than 50mi/hr during your travel.

The units for speed in this example is (miles/hour), and the speed in this example is said to be "fifty miles per (1) hour".

Sometimes, combinations of units may be assigned a single unit name. For example, in physics (physical science) and electronic studies, a quantity or number of charged particles (Q), typically electrons from atoms, is measured in reference to a specific amount of charged particles called a coulomb (C) of charge. Hence, the units for charge are coulombs (C). Current (I) is defined as the amount of this charge passing a point per second (s) of time (T):

$$I = \frac{Q_c}{T_s}$$

Hence, the units of measurement for current is coulombs/second or (c/s). This quotient of c/s, or 1 coulomb per 1 second, is commonly assigned the new and single (derived) unit name, or identifier, of amperes (A), that is (c/s) = A.

Ex. $\frac{10c}{5s} = \frac{2c}{1s} = 2 \text{ c/s} = 2A$: Amps are a derived unit consisting of the units of coulombs, and time or seconds.

When dividing quantities or values that have like or similar units, the units are effectively "canceled out" (and-or reduced to 1, since anything times 1 does not change that value) as if they were some value or algebraic variable (an alphabetical symbol(s) representing, and as a "placeholder", a numeric value or quantity) by themselves. The result is said to be unitless or a ("plain", literal, "strict" and specific value only) numeric value.

Ex. The ratio (which essentially is, and resolves to a quotient, which is a division result) of 10 ft. to 5 ft. is 2, and not 2 ft.:

$\frac{10 \text{ ft.}}{5 \text{ ft.}} = 2$ since 5 ft. x 2 = 10 ft. also, = (5 x 2)ft. = 10ft.

The result is not indicated as 2 ft. since 5 ft. x 2 ft. = 10 feet x feet = 10 ft.^2 : "ten feet squared" or more commonly as "ten square feet"

Note: $\frac{10 \text{ ft.}}{5} = 2 \text{ ft.}$, since 2 ft. x 5 = 10 ft.

That is, the units of a result of a multiplication or division operation where only one operand has units, will have those indicated or expressed units, and for the division operation, the units must be in the "top number" or dividend operand.

Even the plain, unit-less numbers that we use are in actually reference to the value of 0. If given the number 5, it is a value that is in reference to 0, and is a count of 5 counting numbers and-or positions greater than 0 on the continuous, unbroken "string" or "line" of numbers called the number line.

BASIC MATHEMATICAL OPERATIONS

ADDITION

When we "add" quantities, we are performing a mathematical operation called addition which is essentially joining, mixing-in, summing, or combining; usually making something greater or larger. The standard symbol for the addition operation is the "plus" symbol (+). But remember, the quantities being added must be "like" or "similar" quantities. That is, they must have the same units or units of measurement to be added, otherwise, there will be an awkward "apples and oranges" situation where they can't be combined as needed to produce one numeric result having just one of those units of either apples or oranges. Formally (some accepted standard, though somewhat rarely used in everyday conversations), the operands or values of an addition operation are called the augend and addend. These both are infrequently called summands since they compose, or are part of their sum (result of a summation or addition process) value. The result of an addition operation is called the sum which means the total value of all the combined numbers.

number + another number = sum : This can be read or spoken as: "number plus another number equals (the) sum". "sum" is sometimes called the "total"

Ex. 5 + 2 = 7 : "five plus two equals seven" or "the sum of 5 and 2 is seven"

augend + addend = sum : formal or fundamental addition (symbolic) expression, expressed horizontally, or:

$$\begin{array}{r} \text{augend} \\ + \ \underline{\text{addend}} \\ \text{sum} \end{array}$$

: addition expressed vertically

: augend is a word somewhat similar to augmentation, and other similar words, most containing the letters aug, and has the basic meaning of increase or change. The words augend and addend are not heavily used today. Sum has its origins in the word summit as in the (noun, thing) peak (highest part) of a mountain, and (verb, action, "summation") to go to the top or peak.

When the units are different, as in feet and miles, a conversion of one of the quantities can be made so that the units are identical for the addition.

Ex. Add 5280 feet and 2 miles.

5280 ft. + 2 mi. : The above placed into or expressed as a mathematical expression

Since the values being added have different or unlike units, a direct addition cannot be performed such as would incorrectly result in either 5282 feet or 5282 miles. By converting (representing) 5280 feet as (equivalent to) 1 mile, we will have values with like units, and the addition can then be properly performed:

5280 ft. + 2 mi. After converting or expressing 5,280 feet to it's equivalent value in miles (with miles units), we have:
1 mi. + 2 mi. : The operands now have like units and can therefore be combined or added.
(1 + 2) mi. : An equivalent expression that indicates that the numeric parts (of the similar units) will be added.
3 mi. : After adding the numeric part of the values with like units, and keeping and expressing the same, common, identical or like units in the total.

Below is an example that shows mathematical expressions for both horizontal (left and right) and vertical (up and down) addition. Vertical addition, because of its practicality, is usually the preferred method to calculate the sum of any two numbers since corresponding digit positions can be aligned into columns having the same weight value. Adding the digit values of each column produces a sum having the same weight as the columns weight. The addition begins in the rightmost column. Showing horizontal addition, or other mathematical operations, is a good way to get comfortable with writing or presenting mathematical expressions or formulas (standardized or commonly accepted (known) expressions).

Ex. Add 35 and 24

Horizontally: 35 + 24
(3 x 10) + (5 x 1) + (2 x 10) + (4 x 1) : expression showing the sum of the positional products
30 + 5 + 20 + 4 : after some simplification we have this positional sum
59 : since: 30 + 5 = 35 , 35 + 20 = 55 , and 55 + 4 = 59

Vertically:
$$\begin{array}{r} 35 \\ \underline{+\ 24} \\ 59 \end{array}$$

checking (to help verify, locate any errors, and accept the result):

sum = (sum of tens columns positional products) + (sum of ones columns positional products)

((3 x 10) + (2 x 10)) + ((5 x 1) + (4 x 1))
(3+2) x (10) + (5+4) x (1)
5 x 10 + 9 x 1
50 + 9
59

Or: (30 + 20) + (5 + 4)
50 + 9
59

A carry or carry-over is produced when the sum of corresponding (same column) digit positions exceeds 9 as in the decimal system. This was also mentioned during the discussion of positional notation.

Ex.
$$\begin{array}{r} 35 \\ \underline{+\ \ 9} \end{array}$$

Here, by first adding the rightmost or "ones" column, we get 14 which is two digits, hence greater than the maximum digit value of 9. Simply "carry" or move the second digit (the 1), which has a positional weight of 10, to the second column which has a (corresponding) positional weight of 10, or 10 more than the previous, and continue to add.

$$\begin{array}{r} 1 \\ 35 \\ \underline{+\ \ 9} \\ 44 \end{array}$$

: this 1 could be shown as 10, but the zero is often not written since it does not contribute to/affect the sum.

Or the rarely used non-carry method: (summing corresponding positional products):

$$\begin{array}{r} 35 \\ \underline{+\ 09} \\ 14 \\ \underline{+\ 30} \\ 44 \end{array}$$

: In this method, you need not start adding at the ones column.
: (5 x 1) + (9 x 1) = 5 + 9 = 14
: (3 x 10) + (0 x 10) = 30 + 0 = 30

Again, each unit (1) of carry is 10 of the previous digits positional weight.

checking:

35 + 9
((3 x 10) + (5 x 1)) + (9 x 1) with some simplification of the expressions:
(30 + 5) + 9
30 + 5 + 9 : expressed as a sum of the positional products
30 + 14
44

Note above, in the expression (30 + 14), that this is an intermediate simplified expression showing that the 5 ones and 9 ones were combined (added). Often intermediate expressions like this are not shown, but it can assist you in understanding or remembering the steps you took to simplify an expression or to solve an equation (two different expressions that are equal each other).

Another, more popular, method to check an addition problem is with subtraction. Subtracting either operand from the sum should yield the other operand of the addition operation. This was mentioned during the discussion about the "addition table". Using the last example:

$$\begin{array}{r} 44 \\ -\underline{\ 35\ } \\ 9 \end{array} \quad \text{or:} \quad \begin{array}{r} 44 \\ -\underline{\ \ \ 9\ } \\ 35 \end{array}$$

: instead of "carries" (of 10) as used with addition, we now have "borrows" used with subtraction. Essentially, these are reverse concepts of each other. In common speak, people may say "a borrow of 1", or "a carry of 1", however, this 1 is actually 10 of the smaller weight, next rightward position that receives it.

Extra, an advanced technical note: The "(calculation and result comparison) checking process" for an addition operation can be formally expressed as subtracting either operand from both sides of the addition (operation) expression or formula:

augend + addend = sum   subtracting the addend value from each side:

augend + addend - addend = sum - addend   since subtracting or removing a value from itself (the same value) results in, and is equal to, 0:
value - value = 0, therefore, addend - addend = 0 :

augend + 0 = sum - addend   since adding 0 to any value or sum does not change it, we can express this as simply:

augend = sum - addend   by "switching sides" (for clarity) this can be expressed as the examples indicated:

sum - addend = augend   in more generalized or representative expressions:

From: sum = value1 + value2   therefore,   sum - value1 = value2   and   sum - value2 = value1

About the fancy words used in mathematics:

The fancy word terms used above, like augend, are for some formality and knowledge, and in case you read some rare math text. In general, you will not encounter many of these rare-like words in your mathematical journey. In short, you do not have to worry about memorizing them, but at least have a basic idea about them and what they may mean. On occasion, it helps to know what a math word or concept is and means, and so as to express it in writing and-or verbally. Many English math and science words are based on old Greek and Roman words of which the English audience is not exactly familiar with, but they can eventually become familarized to and accept them.

SOLVING FOR AN UNKNOWN VALUE OF AN ADDITION PROBLEM

This discussion is about solving addition problems where the sum is not necessarily being found, but one of the numbers (ie. operands; formally the augend or addend) that is part of the given sum. As you will see, the method may seem trivial at first, but has far reaching potential as the basis for solving equations ("equa" is from "equate" and "equals" meaning that which is equivalent or the same value) as in math and-or algebra. To equate is to make or state an equivalence to, usually with the "equals" (=) mathematical symbol, and it can be interpreted as "is", "is this", "is equals or equivalent to this", "is the same as this". Some of this discussion was mentioned in the last topic.

If you add 2 to 3, you get 5:

3 + 2 = 5 : the (entire) expression of: (3 + 2) is being equated to 5, and vice-versa.
Commonly spoken as: "three plus two equals five", or
"three plus two is the same as five"

Hence, if you take or remove that 2 back out of the total sum of 5, by subtraction, you should arrive at the other number that is part of the sum (here, 5), which is 3. Or, if you take 3 out of the 5, you should arrive at the other number which is 2.

5 - 2 = 3 and 5 - 3 = 2

Therefore, we see a method of solving problems such as the following that is identical to 3 + 2 = 5 that was first given above , except that one operand here is now unknown and is to be found:

3 + ____ = 5 or as: ____ + 3 = 5 : after simply switching the operands being summed, and:
2 + ____ = 5 or as: ____ + 2 = 5

Instead of ____ which means "blank", "unknown", or "find this value", you might also have the word "number", or a letter symbol such as used in algebra for a symbolic (ie. a placeholder or representative) number, such as ?, n , x, or whatever happens to be used to represent and express a value that is to be found.

Ex. Solve this equation : ____ + 4 = 7 : To solve this equation, we are to find the number which makes the equation true, hence we are to find the correct "unknown" value that makes the equation true so as to have a balance (equivalence) on both sides of the equals symbol. To have this balance, both sides must always be kept equal.

To find the other specific or corresponding number that is part of the sum, remove or subtract the known or given (operand) number, the 4, which is the known other part (operand) that made that sum, from 7 to see what is the remaining or other part that made up or is part of that sum of 7:

7 - 4 = 3 , hence, ____ = 3 : we find that the "unknown" or "unknown value" equals 3

Checking by substituting the value of 3 into the "unknown values" placeholder or position:

____ + 4 = 7

3 + 4 = 7 : this equation is correct since both sides, or expressions, of the equality sign, equal 7:
7 = 7

In the formal methods of equation solving, which will be describe further ahead in this book, 4 will actually be subtracted from the total values on both sides of the equals signs, and not just from 7, during the process of solving for the unknown value. This formally keeps both sides of the equation always in "balance" (equivalent or equivalence) when they are both mathematically acted upon or processed together:

left-side = right-side : a simple description of an equation. Each side can be any valid expression, and that both expressions equal the same value or number.

____ + 4 = 7 : the (incomplete, unsolved) equation in question

____ + 4 - 4 = 7 - 4 : expressing the subtracting of 4 from both sides of the equation

____ + 0 = 3 since any value, known or unknown, plus (+) zero is itself or still the same value, this can be expressed as:

____ = 3 : this indicates that the unknown value is equal to 3, and it is the same result as shown above

If 4 was not also subtracted from the left-side, and only subtracted from the right side, we would of created an unacceptable "inbalance" of the equation, and which would lead to an incorrect result than the true or actual result such as when given:

____ + 4 = 7

____ + 4 = 7 - 4 : an incorrect method to solving this equation for an unknown, because 4 is being subtracted from just one side of the equation, rather than both:

____ + 4 = 3 : more advanced math (where signed numbers are considered) will show that the solution here is: (3 - 4)= -1. and -1 is obviously not equal to 3, the true result being found. In short, an unbalance (non-equivalence, error) and incorrect result was created by not applying the same mathematical operation to or upon each, and entire, side or expression. The application and incorrect process used here could be said as being wrong since it was only "one-sided"; applied to just one side of the equation, and not correctly to both sides of the equation.

ADDITION WITH FRACTIONAL PARTS

The decimal point (.) within a number is used to visually separate the number of "whole" (complete) or unit values from that which is a fraction ("part of", or "portion of", you can also think of the word fracture [break/shatter into pieces or smaller parts]) of one single "whole" or unit. That is, (true) fractional parts of 1 are always less than 1. A piece of a whole thing is always less than the whole or entirety of which it came from. If you like, you can consider the decimal point as meaning "add this (fraction part) also", or as a small hidden plus (+) symbol. For example, 5.2 can be considered as 5 + 0.2, or 5.0 + 0.2. This is also the positional sum expression for 5.2. Addition (or subtraction) of values that may contain fractional parts is performed in exactly the same manner as those without. Just be sure to add corresponding columns, that is, they must have the same weight (as when units must be identical) to added properly. A quick and simple method to do this is to align the decimal points of the two or more values (and of the resulting sum), and then all the corresponding columns will be in alignment and easier to work (process, add, combine) with.

Note that even if one or both numbers do not have a decimal point, you can still place one there. Adding leading or trailing zeros to a number will not change the value of a number.

4 + 2 = 4.0 + 2.0 , 4.3 + 2 = 4.3 + 2.0

Ex. Add nine-tenths (0.9) and one-tenth (0.1)

```
  1
  0.9        : nine-tenths, plus
+ 0.1        : one-tenths, equals
  ---
  1.0        : or= (nine + one) tenths = (9/10) + (1/10) = (9 + 1) / 10 = (10/10) = ten tenths = one = 1
```

Ex. Add 1043 and 2.1005

```
   1043.0000   : Any leading zeros in the whole number part, or trailing zeros in the fractional part
+     2.1005     are insignificant and will not change the numbers or the result after using them in
   ---------
   1045.1005     mathematical operations.
```

SUBTRACTION

Subtraction, or to "subtract", is to reduce, remove, deduct, take-away, take-out, or decrease a value, quantity or its representative ("quantized", "quantization") numerical value or number by some amount. The accepted symbol for a subtraction operation is the "minus" (-) symbol. The word minus is based up or "rooted" in the words and meanings of: "tract" (to pull, remove or take away from), "smaller", "subject", "piece" ("sub" word prefix), "less" or "lesser", "minimal", "minimum", "minor" (for "minus"), and "diminish" (to "lessen", "reduce"). The general "formula" for a subtraction operation is:

minuend - subtrahend = difference : The difference is sometimes called the remaining or value "left over". In monetary use, such as at a store, it is often called the "change" value. Change has another more important or technical meaning such as how much did a value, such as the minuend value, change in value so as to be equal to that difference value. The answer here is that the subtrahend value is equal to that change value of the minuend value. The word difference is rooted in the words "different" and "diverse".

Minuend and subtrahend represent the operands of a subtraction operation or problem, and the difference represents the result or output of the subtraction operation. You can think of the subtrahend as that which is being subtracted. Actually, a difference or difference value is the simplest, and perhaps the greatest mathematical measure or representation of the comparison of two values. Given the values of the minuend and subtrahend, we can say that the minuend is difference larger than the subtrahend, and the subtrahend is difference smaller than the minuend. The mathematical measure of the comparison of two similar values by division is called a ratio which yields a ratio value or result, and this will be discussed further ahead in this book, and it basically indicates how many times larger one value is than the other value, or how one value changes when another value changes by some amount (usually by 1, so as to have a fundamental rate or ratio).

Note that by "switching (the left and right) sides" of the equality sign, or equation, that this could also be correctly expressed or written as:

difference = minuend - subtrahend

To advance your math skills, and in particular, for solving for some unknown value in an equation, you should try to feel comfortable with switching both sides of an equation.

A good method to illustrate the concept of subtraction is to display a group of objects that represents the larger number (the minuend) and remove (perhaps circle or cross-out the objects if paper is being used) the number that is to be subtracted, the answer, formally called the difference, will be that which is "left over" or remaining.

Ex. Show 3 minus 2 = 3 take away 2 = ~~*~~ ~~*~~ * = 1 , and-or:

3 - 2 = 1 : "three, take-away two, is one"
* * * - * * = *

To be good at subtraction, you first need to know the sum of any two of the basic (0 through 9) decimal values. These values may represent groups of objects. For example if you were to add a group of 3 and a group of 4 together, you could create a new and larger group of 7 objects:

3 + 4 = 7 : (* * *) + (* * * *) = (* * * * * * *)

If you had a basket containing 7 apples, you could say, for example, that in the basket there is a group of 3 apples, and a group of 4 apples, and a total of 7 apples if you added those two groups together, perhaps to make just 1 group. If you took out, or mathematically subtracted, either group of apples, you would be left with the other group of apples. To illustrate this point, "grouping symbols" (here, parenthesis) are used around each group of which can be regrouped into two smaller groups, of which a group can be removed or taken away during the subtraction:

7 - 3 = 4 + 3 - 3 = 4 = 4
7 - 3 = 4 : (* * * * * * *) - (* * *) = (* * * *) + (* * *) - (* * *) = (* * * *) = * * * * and:

7 - 4 = 3 + 4 - 4 = 3 = 3
7 - 4 = 3 : (* * * * * * *) - (* * * *) = (* * *) + (* * * *) - (* * * *) = (* * *) = * * *

When both groups (here, groups of apples) are combined together into one new (and larger) group, either physically or mathematically with addition, you will then have the sum of objects of both groups, and the number of objects in the new combined group:

4 + 3 = 7 : (* * * *) + (* * *) = * * * * + * * * = * * * * * * * = (* * * * * * *)

Above, parentheses were used to express each group being considered. Parentheses ( ), and brackets [ ], are common grouping symbols used in math. Grouping symbols are sometimes used to clarify an expression, and-or to treat it as like one value.

Like addition, subtraction problems are often expressed horizontally, but are usually solved vertically "by hand" (manually):

minuend
- subtrahend
difference

To check the result (the difference) of a subtraction problem is to sum (add) the subtrahend to the difference. If this result is equal to the minuend, the result is correct. As done in addition, in subtraction, values or quantities with only like units can be subtracted.

In "vertical subtraction", to borrow is to "reverse carry". That is, the borrow of 1 is equivalent to 10 times the positional weight which receives it.

Ex. 45
- 7

Here, without using signed number concepts that will be discussed ahead, 7 cannot be taken from or out of 5 since "5 does not have enough", or 5 is simply less than 7. We can make enough by borrowing 1 (or 1 tens) from or "out of" the second digit (the 4). We are actually taking 10 ones since the positional weight there is 10. The digit value which gave us these 10 ones must then be reduced by one. These 10 ones borrowed are added to the 5 to make a sum of 15. Now, 7 can be taken from 15, leaving 8.

checking: 15 - 7 = 8 since 8 + 7 = 15

3
4(15)
- 7
3 8

checking: 1
38 : difference
\+ 7 : **+** subtrahend
45 : = minuend

Ex. Subtract 7 from 105

$$\begin{array}{r} 105 \\ -\ \underline{\quad 7} \end{array}$$

Obviously, 5 is less than 7. Since the second digit position has no value other than 0 in it, there is nothing immediately there to borrow from. To place a value there to borrow from, the second digit can borrow from the third digit (the 1 shown). Borrowing 1 from the third digit leaves zero. Since the weight of the third position is 100, it is actually borrowing 100 which is 10 times that of the previous digits weight (the second digits weight) of 10.

$$\begin{array}{rl} & 0 \\ & 1(10)5 \\ - & \underline{\qquad\ \ 7} \end{array}$$

Now that there is a value in the second digit position, the first digit can borrow from it. Borrowing 1 from the 10 leaves 9:

$$\begin{array}{rl} & 0\ \ 9 \\ & 1(10)(15) \\ - & \underline{\qquad\quad\ 7} \\ & \ \ \ 9\quad 8 \end{array}$$

: difference = minuend - subtrahend = difference

checking:

$$\begin{array}{rrl} & 1 & \\ & 98 & :\ \text{difference} \\ + & \underline{\ \ 7\ } & :+\ \underline{\text{subtrahend}} \\ & 105 & :\ \text{minuend} \end{array}$$

: minuend = difference + subtrahend = minuend

When several zeros are "in a row", and you need to borrow, start the borrowing at the first leftward non-zero value.

Ex. Subtract 5 from 3002 Showing this with enumerated (numbered) steps of the calculation process:

1)
$$\begin{array}{rl} & 2 \\ & 3(10)02 \\ - & \underline{\qquad\quad 5} \end{array}$$

2)
$$\begin{array}{rl} & 2\ \ 9 \\ & 3(10)(10)2 \\ - & \underline{\qquad\qquad\ 5} \end{array}$$

: showing some intermediate steps

3)
$$\begin{array}{rl} & 2\ \ 9\quad 9 \\ & 3(10)(10)(12) \\ - & \underline{\qquad\qquad\quad\ \ 5} \\ & \ 2\ \ 9\quad 9\ \ 7 \end{array}$$

checking:
$$\begin{array}{rr} & 111 \\ & 2997 \\ + & \underline{\quad\ 5} \\ & 3002 \end{array}$$

The result of subtracting 0 from any number is that same number since nothing was effectively subtracted or removed:

Ex. 7 - 0 = 7

If any number is subtracted from 0, the results are somewhat similar to that mentioned above, but since the concepts of signed numbers, particularly negative numbers, have not been discussed yet in this book,, examples will not be presented here. We all seen or heard of 0 degrees on the thermometer, and even "negative temperatures" where the temperature is actually less than the reference value of 0. For example, -10 ("negative ten") degrees Celsius or Fahrenheit. Anyway, without these more advanced concepts, it is common sense that nothing can actually be physically (as opposed to logically) subtracted or taken away from nothing or 0.

Extra: The checking process for a subtraction operation can be formally expressed as:

minuend - subtrahend = difference , after adding the subtrahend to each side of this equivalence (same values):

minuend - subtrahend + subtrahend = difference + subtrahend

If you subtract value2 from value1, and then add that value2 back, the net result is that you have added or subtracted nothing or 0:

value1 = value1 + value 2 - value2 = value1 + 0 = value1 = value1 - value2 + value2 = value1 - 0 = value1

minuend = difference + subtrahend or = difference + subtrahend = minuend here is a generalization:

value1 + value2 = value3 : a generalized addition formula, solving for value1 by subtracting value2 from each side:
value3 - value2 = value1 : a generalized subtraction formula, also:
value3 - value1 = value2 solving for value3, here, as the minuend, from either of these two equations:
value3 = value1 + value2 : minuend = difference + subtrahend or = subtrahend + difference

SOLVING FOR AN UNKNOWN VALUE OF A SUBTRACTION PROBLEM

This discussion is about solving subtraction problems where the difference is not necessarily being found, but one of the numbers, either the minuend or subtrahend operand of a subtraction operation or problem. This discussion is similar to the previous discussion of SOLVING FOR AN UNKNOWN VALUE OF AN ADDITION PROBLEM.

For example, if we take 3 away from 7, the value of 7 is reduced to a value of 4:

7 - 3 = 4

If we do the exact reverse and add (rather than subtract) or combine this value of 3 (that was taken out of the 7) back with or to the value of 4 (the difference, or "remainder" if you will, of which 7 was reduced or changed to), we should effectively arrive at the original value in which 3 was subtracted from.

4 + 3 = 7

Now, with this information in mind, if given:

7 - ___ = 4 : "Seven minus some number is equal to 4.". We know that 4 added to some number, that we are to find, will sum to 7 and that this is the value that was subtracted from 7. Expressing this mathematically, we get, and are to effectively solve the new (addition) problem or equation of:

4 + ___ = 7 : "Four plus some number is equal to seven."

Actually, in more formal equation solving, which will be discussed further ahead in this book, this new equation shown above is easily arrived at by taking the original equation given: (7 - ___ = 4) and adding ___ (which represents some unknown value that we are solving for) to each side. If you want to, you can also substitute (replace) the ___ symbol with ?, x, n or something to symbolically (ie. a symbol) represent a number to be found. When any value is both given to or added (even if its 0 or nothing), and then taken away, removed, reduced or subtracted (regardless of the order of events), it results in a value of 0, and effectively eliminating that value. We do this to eliminate a value from one side of an equation and effectively move it to the other side of the equation. Again, that which is done to one side of an equation, must also be done to the other side of the equation so as to keep the equation in mathematical true (equals) balance. Now, simply solve for ____ just like the method shown previously for solving addition problems (hint: subtract 4 from each side):

____ = 7 - 4 = 3

The problem above may have been stated as:

___ - 3 = 4

Usually, we have been told verbally how to solve this when checking a subtraction problem: "add the difference and subtrahend" to find the minuend, hence:

___ = 3 + 4 or: 4 + 3 = : Which is effectively the result of adding 3 to each side of the above previous equation, and eliminating the 3 on the left side since 3 - 3 = 0. Simplifying the right side we have:

___ = 7 or: 7 = ___

Now, let's give some more reasoning to this, and it's based on the above discussion. If we reduce or takeaway a number by 3 and get a smaller number (the difference), we can do the reverse and add or combine that which was taken away back to this new or reduced number to get the original number before any subtracting had taken place.

Before moving on to other topics, it must be mentioned that addition and subtraction are logically, inverse operations of each other. You can think of an inverse operation as an operation that will "undo" or "reverse" the last operation.

Ex. Starting with 10, and then performing an addition operation on it:

$10 + 5 = 15$

To solve for the original value of (10) given this value of (15), you must perform the inverse operation of addition on it, which is a subtraction operation:

$15 - 5 = 10$: Subtracting 5 is the inverse process or operation of adding 5.

If you first started with (15) and then subtracted (5):

$15 - 5 = 10$

To solve for the original value of (15) given this value of (10), you must perform the inverse operation of subtraction on it, which is an addition operation:

$10 + 5 = 15$

This is exactly what we did at the beginning of the example. Notice that the value of (5) was used in both inverse operations. Performing an inverse operation is also a way to check a mathematical problem. Likewise, most other mathematical operations have a corresponding inverse operation:

addition and subtraction,
multiplication and division,
powers and roots,
logarithms (basically an exponent) and antilogarithms (basically a power),
trigonometric functions and inverse, "arc" or "angle" trigonometric functions

A lot of ideas have just been mentioned above in this current topic which may very well seem a bit complex or overwhelming, and perhaps even unnecessary at first. Just give it a look over once in a while, and you will begin to find more and more reasoning, justification and understanding of it.

MULTIPLICATION

Multiplication, or to multiply, is a "short-hand", or quick method of representing, expressing and performing repeated addition. "Multi" is a word prefix that means: many, such as in the word multiple, hence multiplication basically means many or multiple repeated additions of a value or something. Similarly, multiplication means a combination of the words: "multiple" or "many", and "apply" or "application", hence to apply addition multiple times or repeatedly.

The symbols for multiplication are (x), (**.**), or sometimes parentheses or other grouping symbols surrounding the operands. For example: 5 + 5 = (5)(2), here 5 is being added twice, and which is the same as 5 multiplied by two. (5)(2) = (5) x (2) = 5 x 2. The basic, general expression, format or "formula" for multiplication is:

operand1 x operand2 = result : the operands of a multiplication problem are called factors, and the result is called the product, therefore, this can be expressed as:

factor1 x factor2 = product after assigning formal names or identifiers to each factor:

multiplicand x multiplier = product : common multiplication expression or formula

The symbol for the operation of multiplication is often called and spoken as the "times" symbol. It means repetition or the number of times to repeat the addition of the given number, here the multiplicand. The (x) symbol for multiplication is thought to be a plus sign that is turned or turning as if in a repeated process. At about 1700, the dot (.) symbol was sometimes used between two factors being multiplied so as to avoid using the (x) symbol which looks like the variable or "(number) placeholder" (x). This book does not use the dot symbol (**.**) for multiplication very much, but rather uses the parentheses around each factor or operand of the multiplication expression. Computers programs often use the star or asterisk symbol (*) between two factors of a multiplication expression, and this symbol was already sometimes used for multiplication, possibly to look like a modified (x) symbol for multiplication. The multiplicand and multiplier are the operands of a multiplication operation. The product (means something that was "produced") is the result of a multiplication and-or an expression to multiply.

The multiplier is the number of times the multiplicand is to be repeatedly added (summed) to an initial sum or starting value of zero or nothing. The multiplicand and multiplier are also called "(the) factors" of that result or product since these factors will have influence (effect) on the resulting value (the product).

Adding any number just once (to a starting or initial sum of 0) is the same as multiplying that number by 1:

7 + 0 = 7 and,
7 x 1 = 7 : multiplying any number by 1 does not change it's value. 7 is being added to an initial working sum or total of 0, only 1 time, and the result is therefore 7. Note that this is also equivalent to 1 x 7 , and 1 multiplied or increased 7 times, or added to itself 7 times is : 1 + 1 + 1 + 1 + 1 + 1 + 1 = 7

Adding 5, ten times would result in an addition expression or problem (to solve for the result) looking like this:

5 + 5 + 5 + 5 + 5 + 5 + 5 + 5 + 5 + 5 : sum = 50

That is a very long addition problem or expression, and a "running total" of many "intermediate sums" would have to be written down or remembered as you try to simplify or solve it. Since the above expression is of the repeated addition of the same value (5 added to the running (processing) total (sum), or to itself, 10 times), transforming this expression into an equivalent and simpler multiplication expression, we get:

5 + 5 + 5 + 5 + 5 + 5 + 5 + 5 + 5 + 5 = 5 x 10 = (5)(10) : multiplication is repeated addition

Since both expressions are said to be equivalent in value, they are shown as equated (ie. equals) together.

5 x 10 : product = 50, and this is a much "shorter" expression for 5 added to itself 10 times.

Multiplication with a zero operand: The product of any value multiplied by 0 is 0.

Ex. 4 x 0 = 0 : With respect to addition, four added zero (or none) times is equal to zero.
Remember that 4 x 0 is also equal to 0 x 4 = 0 + 0 + 0 + 0 = 0

Ex. 0 x 20 = 0 : With respect to addition, zero added to itself twenty times is zero.
20 multiplied to itself zero (or none) times is equal to 0 since no addition to the initial starting sum of 0 has taken place.

To solve common multiplication problems, you must know from memory, or by using a multiplication table, the products of many factors. Usually, all the products of factors between 1 and 10 are memorized, such as shown in the multiplication table below:

A MULTIPLICATION TABLE

	1	2	3	4	5	6	7	8	9	10
1	1	2	3	4	5	6	7	8	9	10
2	2	4	6	8	10	12	14	16	18	20
3	3	6	9	12	15	18	21	24	27	30
4	4	8	12	16	20	24	28	32	36	40
5	5	10	15	20	25	30	35	40	45	50
6	6	12	18	24	30	36	42	48	54	60
7	7	14	21	28	35	42	49	56	63	70
8	8	16	24	32	40	48	56	64	72	80
9	9	18	27	36	45	54	63	72	81	90
10	10	20	30	40	50	60	70	80	90	100

To use the table, the resulting product of any two factors (indicated at the top row, and at the left column) is the value at the horizontal and vertical intersection. This table can also be used for division where the value at the intersection becomes the dividend. Also notice in the table that there are two methods to find the same product:

multiplicand x multiplier = product
multiplier x multiplicand = product : the product will be the same for both expressions and-or equations

Either value at the top and left side can be considered as either the multiplicand or multiplier. This can be very easily verified by representing the multiplicand and multiplier in a physical or graphical form (such as with objects). For example, 3 x 2 can be represented by 3 objects and 2 groups of those three objects and which is the same as adding three to itself twice: 3 x 2 = 3 + 3. Expressing this physically or graphically, where an object represents 1:

* * * : (3 objects, first group)
* * * : (3 objects, second group)

Since multiplication is repeated addition, the product is the sum of all the objects. Since there are 6 objects, the product or repeated sum is 6.
Without removing or adding any objects, the product of 3 x 2 can be represented by tilting the above objects on to their side, producing 2 objects and 3 groups of those 2 objects, hence 2 x 3 = 2 + 2 + 2 :

* * : (2 objects, first group)
* * : (2 objects, second group)
* * : (2 objects, third group)

The sum is still 6. Hence: 3 x 2 = 2 x 3 = 6 : Verifies that the multiplicand and multiplier factors can be exchanged and still produce the same product or result, just like the numbers (augend and addend) of an addition problem can be exchanged.

Ex. There are 7 piles with 10 stones in each pile. How many total stones are there in all the piles?

Since we are totaling (summing) the number of stones, we are to add each (identical) group, or pile, of stones a total of 7 (piles) times. Since this is a repetitive or repeated addition problem, we can represent and solve this more easily as a multiplication problem:

10 stones x 7 = 70 stones : or more advanced as: (10 stones per [for each] pile) x 7 piles = 70 stones

The are all sorts of rhymes and apparent "tricks" to remember, and-or to check, the basic multiplication table and problems. For example, to remember the 9's products, take that specific multiplier (or multiple of 9) and subtract 1 from it. This will be the first or most significant number of the product. The other digit (if any) of the result will be that number which when combined to the first number found sums up to 9.

Ex. 9 x 4

Subtracting 1 from 4 leaves 3. and the number that when added to 3 that creates the sum of 9 is 6 (you can simply use subtraction to find this next digit value: 9 - 3 = 6).

9 x 4 = 36 : note that for the product, here 36, that its digits sum to 9: 3 + 6 = 9

Ex. 9 x 7

The first digit will be (7-1) = 6, and the second digit will be 9 - (7-1) = 9 - 6 = 3

9 x 7 = 63 : note that the sum of the two digits of the product is 9: 6 + 3 = 9

Perhaps the most simple method to use as an aid to basic multiplication is the fact that if you already know:

multiplicand x multiplier = product

Then you can easily solve for:

multiplicand x (multiplier + 1) = product : Increasing the multiplier by one to find the next product.

Ex. If you already know that 6 x 4 = 24 , and you then want to know what 6 x 5 is, then simply add one more 6 to the value of 24, since after all, multiplication is still repeated addition, and that you can use a previous sum or product as an intermediate sum of another multiplication operation. Expressing this method and example in mathematical statements:

Find 6 x 5

6 x 5 = 6 + 6 + 6 + 6 + 6 : expressing the multiplication as its equivalent addition
6 x 5 = (6 + 6 + 6 + 6) + 6 expressing the first 4 additions as multiplication we get :

6 x 5 = (6 x 4) + 6 : equivalent expressions, and what this discussion is about
6 x 5 = 24 + 6 : If you knew 6 x 4 = 24, you can then find 6 x 5 with just another addition of 6.

6 x 5 = 30 : If you knew 5 x 6, you would have the same result of 30.
: If you knew 5 x 5, then 5 x 6 = (5 x 5) + 5 = 25 + 5 = 30

Multiplication, like addition and subtraction, when done without a calculator, is usually performed vertically. To multiply this way, the positional product of each digit of the multiplicand is multiplied to (or by) the positional product of each digit of the multiplier once, producing an intermediate product (or intermediate sum to be added together, since a product is technically a sum of repeated additions) that will be part of the sum.

Ex. Multiply twenty and seven together.
(This example also shows how to multiply values (factors) greater than 10 that are not listed in the basic table.)

20 x 7 : expressed horizontally

$$\begin{array}{r} 20 \\ \times\ \underline{\ \ 7} \end{array}$$: expressing this vertically

$$\begin{array}{rrl} & 20 & \\ \times & \underline{\ \ 7} & \\ & 0 & : 0 \times 7 = 0\text{ , a positional sum of }0 \\ + & \underline{140} & : 20 \times 7 = (2 \times 10) \times 7 = (2 \times 10 \times 7) = (2 \times 7) \times 10 = 14 \times 10 = 140,\text{ a positional sum of }140 \\ & 140 & : \text{summing up these positional products (or intermediate sums)} \end{array}$$

Ex.

$$\begin{array}{rrl} & 28 & \\ \times & \underline{17} & \\ & 56 & : 8 \times 7 = 56 \\ & 140 & : 20 \times 7 \\ & 80 & : 8 \times 10 \\ + & \underline{200} & : 20 \times 10 \\ & 476 & \end{array}$$

A more popular vertical method is to not use the positional products, but to use just the single digits of the multiplicand and multiplier, and utilize any carry's:

Ex.

$$\begin{array}{rr} & 28 \\ \times & \underline{\ \ 7} \end{array}$$

First, the 8 is multiplied by 7. The resulting product is 56. The 6 is placed into the "ones" digit position of the product (which may be an intermediate product that will be used in the final sum). The second digit of this intermediate result, the 5, is a "carry" to its corresponding "tens" column and is added to the product of the next multiplication. That is: 7x2=14, and then 5 is added to 14, making a sum of 19. This result is placed in the next leftward and significant available position as the product of that multiplier digit is constructed:

$$\begin{array}{rrl} & 5\ \ & \\ & 28 & \\ \times & \underline{\ \ 7} & \\ & 196 & : \text{steps of: } 8 \times 7 = 56,\ \ 7 \times 2 = 14,\ \ 14 + 5 = 19\ \ \text{(or a 9 with a carry of 1 into the next column)} \end{array}$$

The basic reasoning that the carry of 5 was not added to the 2 before performing the multiplication is that, like above, 8 is being multiplied by 7, and 20 (not 70) is to be multiplied by that 7 also giving:

28 x 7 = ((2)(10) x (7) + (7 x 8) = 20 + (8 x 7)   or: (20 + 8)   : "both twenty and eight, multiplied
x 7   by seven, and their products
added together"

(20x7) + (8x7)   : or expressed as: (8 x 7) + (20 x 7)
140 + 56   56 + 140
196   196

Continuing with the above discussion, multiplying by the positional products (such as the 20 indicated above) is not necessary as long as you place an appropriate amount of "trailing" zeros onto each next intermediate product(s) and continue multiplying with just the single value or digit in that digits position. This is perhaps the easiest and most popular method to multiply. The reason behind this method is that when you multiply with just the value in that digits position, instead of the positional product, you are effectively dividing the correct intermediate product by the positional weight of that digit, and placing in the "trailing zeros" will correct this (effectively multiplying, the intermediate product, by the positional weight that was effectively "divided out" using this process). Excluding the first time, each time another digit of the multiplier is used for multiplying, the intermediate product or sum is shifted leftward one digit position which essentially means placing in another trailing 0 and effectively multiplying the result by the positional weight of the multiplier digit. This multiplication by 10 is often thought of as simply shifting the decimal point one digit rightward. That is, when any value is multiplied (or divided) by 10, or some power of 10 such as 10^2 = 100, the resulting digits are always the same, and only the decimal point will moved. Leading or trailing zeros may sometimes be needed, and placed before (such as when dividing by 10) or after (such as when multiplying by 10) the number, and only if needed. For example(s):
3 x 10 = 3.0 x 10 = 30 , 25 x 10 = 250 , 2.5 x 10 = 25 , 2.615 x 10 = 26.15 , 350 / 10 = 35.0 ,
3 / 10 = 3.0 / 10 = 0.3 , 3 / 1000 = 3.0 / 1000 = 0.003 and 0.003 x 1000 = 3.0 = 3

Ex. Multiply 28 by 17

5
28
x 17
196   : 8 x 7=56, carry the 5; 2 x 7=14, 14+5=19
+280   : First, place a trailing zero=0; 8 x 1=8 (instead of 8 x 10 = 80), and 2 x 1=2 (instead of 20 x 10 = 20)
476

also note, 196 = 28 x 7 = (20+8) x 7 = (20 x 7) + (8 x 7) = 140 + 56 = 196
also note, 280 = 28 x 10 = (20 x 10) + (8 x 10) = 200 + 80 = 280

Ex. Solve 107 x 304

107
x 304
428   : 7 x 4 = 28, carry the 2; 0 x 4 = 0 and then that 0 + 2 = 2 and place it ; 4 x 1 = 4 and place it
0000   : place a trailing 0. Here, all three products will also be 0.
+ 32100   : now place another, or 2 trailing zeros here, then continue multiplying
32528

It should also be mentioned that some of the steps presented in the above discussions about multiplication imply a mathematical concept formally known as the distributive property or simply "distribution". It will be described and used further ahead in this book, but here is a simple explanation:

Ex. Given 5 + 5 = 5(1) + 5(1) ,   We know this repeated addition can be expressed using multiplication as:
5 x 2 = 5(1) + 5(1)   and this can then be expressed as:
5 x (1 + 1) = 5 (1) + 5 (1)   hence:

5 (1 + 1) = 5(1) + 5(1) : This has the general format of the concept of distribution, where a product of two factors, where one contains several terms (here as expressed additions or a sum), is equal to the sum of the products of each term and the multiplying factor . Here both sides and expressions equal 10.

Rather than simplify the second indicated factor of (1+1) to a value of 2, with the distributive concept, we can use a reverse type of process to that mentioned above, 5 can be multiplied to each number that is a part of the sum that comprises this factor of (1 + 1):

5 (1 + 1) "distributing" (essentially multiplying) the factor of 5 to each term (ie. a partial sum of the second factor) to create a new expression:
5(1) + 5(1) : this equivalence was also shown above
5 + 5

The main point to remember is that an initial factor multiplied to another factor that is composed or expressed as sum of numbers is equal to the sum of all the products of that initial factor and each member of the factor that is composed of a sum of numbers.

Ex. 7 (1 + 1) = 7(1) + 7(1)
7(2) = 7 + 7
14 = 14 : checks

Ex. 7 (1 + 2) = 7(1) + 7(2)
7(3) = 7 + 14
21 = 21 : checks

Ex. 15 = (10 + 5) = 3 (10 + 5) = 3(10) + 3(5) = (10 x 3) + (5 x 3) = 30 + 15 = 45
x 3 x 3
45 15 : 5 x 3
+ 30 : 10 x 3
45

Here is another helpful discussion about multiplication, especially with multi-digit operands, and it is intended to give more clarification and understanding to using a "left-shift" in the intermediate sums during vertical multiplication solving. A left shit by one digit position is equivalent to multiplying a number by 10. This discussion is somewhat detailed, and you may skip over it till perhaps another time.

Ex. Given 1, if we left-shift this one position to the left we have: 10 , and this is the same as multiplying 1 by 10 and the product is 10. If you were to repeatedly sum 1 a total of 10 times you would have:

1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 = 1 x 10 = 10 : Hence, simply left-shift the digit leftward one position and place a trailing 0 into the ones position or column.

Ex. Given 2 and we left-shit this one position, we get 20, and this is the same as multiplying 2 or= 2**.**0 by 10.

Notice that the result has the same digits used, but they are essentially shifted to the left one position and a trailing 0 is put at the end of the number (in the ones position or column). This new value is 10 times more than the given value. 20 is ten times more than 2. If you were to repeatedly sum 2, a total of 10 times you would have:

2 + 2 + 2 + 2 + 2 + 2 + 2 + 2 + 2 + 2 = 2 x 10 = 20

Hence, simply left-shift the digit(s) leftward one position and place a trailing 0 into the ones position or column.
Extra: 2 can be expressed as (1+1), and, we would have: (1 + 1)10 = 10(1+1) = 2(10) = (10)(2) = (1)(10) + (1)(10)

The right hand expression is an example of what is known as applying the "distributive law" (essentially multiplication) to the left hand expression. The left hand expression is essentially a "factored form" of the right hand expression. The factors are (1 + 1) and 10 which is the common factor in both indicated (sum of) products or terms on the right hand side.

2 x 10 = (1+1) + (1+1) + (1+1) + (1+1) + (1+1) + (1+1) + (1+1) + (1+1) + (1+1) + (1+1) which could be expressed as:
2 x 10 = 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 with some grouping:
2 x 10 = **(**1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1**)** + **(**1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1**)** which could be expressed as:

2 x 10 = (1 x 10) + (1 x 10) : which results in 1 left-shifted once, plus, 1 left shifted once (and not 1 left-shifted twice), and this is: 10+10, and these are a group of "partial (part of) sums" of 20.

Now consider this, let symbol N symbolically represent the value of any basic number that you can think of:

$$\begin{array}{rcrcrcr} N & = & N & \text{or} = & N & = & N \\ \times\ \underline{10} & & \times\ \underline{(10 + 0)} & & \times\ \underline{(10 + 0)} & & \times\ \underline{(1 \times 10)} \\ & & N(10) + N(0) & & N(0) & & \\ & & & & +\ \underline{N(10)} & & \end{array}$$

N x 10 = N (1 x 10) = N ((1)(10)) = N(10)
Note, we **cannot** use distribution above, since the second factor: (1 x 10) is not a sum of values such as: (1 + 10)

These can also be expressed as:

Nx10 = N(10) = N(10 + 0) = N(1 x 10) : note, you cannot use distribution on the right expression to simplify it, since the second factor (1 x 10) is not a sum of terms. N(1x10) = N((1)(10)) = N(1)(10) = N10 = 10N
The result is **NOT**: 1N + 10N = 11N

Considering the indicated product on the right hand side, we essentially have 1 with a left-shift being multiplied to N: That is: N(1x10) = (N)((1)(10)) = N(10). Anything multiplied by 1 is that same number, such as 10(1) = 10. Hence we have: N(1)(10) = N(10), and when we left-shift all the digits of N, we have the product of it and 10. Symbolically using an appended 0 to indicate a left- shift or multiplication by 10:

N(10) = N0 : When N is a plain "integer" number that does not contain any fractional portion past the decimal point, when it is multiplied by 10, N is left-shifted, that is, every digit in the number of N is shifted 1 position leftward, and a trailing 0 is placed rightward of N's last digit. If N does contain a fractional decimal number part, then simply move the decimal point one position rightward when multiplying by 10, for example: If N=3.49, N(10) = Nx10 = (3.49)(10) = 34.9
If N=5 , N(10) = 5(10) = 5.0(10) = 50 = 50.0

Ex. N(11) = N(1 + 10)

We see that N will be summed 11 times, and this is equivalent to summing N once, and then combining that (partial) sum to the summing of N ten times.

N(11) = N + N + N + N + N + N + N + N + N + N + N which can be expressed as:
N(1 + 10) = **(**N**)** + **(**N + N + N + N + N + N + N + N + N + N**)**
N(1 + 10) = (1xN) + (10xN) : this notation is also an example of the "distributive law" that will be discussed

Hence, the result is N, plus N left-shifted once. If N=15, N(11) = N(1 + 10) = N + N0 = 15 + 150 = 165

Ex. To multiply a value by 20. Since 20 = (2 x 10) , this expression indicates that the summing of a value twenty times (ie, repeated addition, which is multiplication) is equivalent to summing up that value twice to first produce a (partial) sum, and then summing up that new (partial) sum a total of ten times to find the final sum or product. We now know that we can left-shift a value one digit position to quickly have the product of it and ten:

N x 20 = N(20) = N(2 x 10) = (Nx2)x(10) = (Nx2)0 : here, the last expression indicates the left-shift of the value of: (Nx2), and putting, or appending, a trailing 0

Ex. N x 23 = N(23) = N(20 + 3) = N((2x10) + 3) = 2(N)(10) + Nx3 = 2(N0) + Nx3

To multiply a value by 100, simply perform two left shifts on that value and place on, or append, two trailing zeros since:

N(100) = N(10 x 10) = (N(10))10 = (N0)10 = N00

Ex. 573(100) = 57300

Clearly, the number of left-shifts when multiplying by a power of 10 is equal to the number of zeros in that value. For example, to multiply a value by 1000, simply perform three left shifts on that value and place on three trailing zeros.

To help multiply (or divide) by or to a large constant (steady, unchanging in value) number that has many digits quickly when not using the assistance of an electronic calculator, it is very helpful to first make a table of values of all the products (or repeated additions) of that constant value and all the single digit decimal values from 0 through 9.

Ex. If 307,629 is to be multiplied by any other value:

Table Of 0 through 9 Multiples Of The Given Value

307,629 x 0 = 0
307,629 x 1 = 307,629
307,629 x 2 = 615,258
307,629 x 3 = 922,887
307,629 x 4 = 1,230,516
307,629 x 5 = 1,538,145
307,629 x 6 = 1,845,774
307,629 x 7 = 2,153,403
307,629 x 8 = 2,461,032
307,629 x 9 = 2,768,661

Or by adding 307629 to each previous product to find the new product **:**
307,629 + 307,629 or = 307,629 x 2 = 615,258 :
615,258 + 307,629 or = 307,629 x 3 = 922,887 :
922,887 + 307,629 or = 307,629 x 4 = 1,230,516 :

```
      307,629
    x   57249
      2768661
     12305160
     61525800
   2153403000
+ 15381450000
  17611452621
```

: write down the corresponding products in the table, and any trailing 0's needed, then sum these intermediate products so as to have the resulting product

The above is also one method to use when the number of digits of the operand(s) and-or the result is excessive for the device and cannot be directly entered into and-or displayed as a result by the computer or calculator.

MULTIPLICATION WITH DECIMAL FRACTIONS

Multiplication by a number with decimal fractional values, with, or without. a whole or fractional part is performed in the exact same manner as when multiplying just whole or "integer" values that do not contain a fractional part. The fractional part of a decimal (system) number numerically represents a part less than one (1), and this is expressed numerically to the right of the decimal point in a decimal number.

Example of a number containing a whole part and a fractional part: 7.32 = 7 + 0.32 = whole part + fractional part =
= whole part **.** fractional part Here, the whole part is 7, and the fractional part (of 1) is 0.32

Simply multiply the values and (initially) ignore any possible decimal points as you process or calculate the result. The decimal point of the result must then be properly placed or positioned. Simply add up the number of digits that were not only indicated, but actually used (during the calculation) rightward of the decimal point in each of the operands, and this is the number of places to move the decimal point leftward in the product. Verification of this procedure is shown below, though it requires some slightly more advanced concepts of fractions. Commonly, the words "decimal digits" or "decimal places" are taken to mean the digits of the fractional part of a decimal number that are to the right side of the decimal point.

Before continuing, you may first want to consider this about the basic concepts of both the decimal number system and fractions or fractional values. This paragraph section may seem a bit complicated at first to some, and they may skip over this till perhaps a later time. In the decimal system, when we count up to 10, each of the ten steps, each being an equal or equivalent step, and here for this specific example,will result in an increment or increase of 1 to the entire number of which the increment is being applied to. Note that the actual value (applied to the entire number that is being incremented) of each step is technically, the (decimal system's) actual or specific value that the increment represents to the entire decimal number, and this value is the numeric weight of that (digit, or column) position in question or reference when just the decimal numeric system itself is being considered. and each step is one of the ten possible steps (before digit or column position "rollover" back to the start or 0), or "one of ten (steps)" for each digit, column or ("decimal") position, and this can therefore be expressed numerically as: 1 of 10. This concept, and each step or one increment out of ten possible increments (per digit position or column), can be mathematically expressed as an indicated fractional part as: 1 of 10 = 1 per 10 = (1/10) or simply: 1/10, and this does equal 0.1 as expressed using the most basic or fundamental decimal numeric form as just one value or number. Notice that the expression (1/10) does look like a common division expression or problem, and that the value of 0.1 is also equal to the result of the indicated division expression. Since each step is only part of the entirety or whole thing, the actual value of each step is said to be only a fractional part. Considering even the number 1, it too, for example, can be, or considered as being composed of ten equal steps. It need not be exactly ten steps, and another example or number system could have more or less steps, but we are now just considering the basics of the decimal system with its common 10 steps in each digit position or column. Ten increments or (fractional) steps, each being (0.1) numerically, will produce the numeric values or sums, starting at 0.0, of 0.1 through 0.9, and 1.0 = 1. Remember, that in the decimal system, each count or step in the next greater or leftward position is equal to ten, "ten more", or "ten of", steps, and therefore, each step or increment in the next lesser or rightward position is "ten less", or "a tenth" of each step in the current or reference position. Some more will be explained about the fundamentals of fractions later, and we will now continue with the current multiplication discussion.

Ex. 1   1.0   : one decimal digit used and,
x <u>1</u> = x <u>1.0</u>   : one decimal digit used
1   00
\+ <u>100</u>
1.00   : decimal point has been moved two places leftward

Ex. (1)(0.1)

Note that multiplying anything by one is always equal to that same value. Here, a tenth of one is being multiplied by one:

(1)(0.1) = (0.1)(1) = 0.1

$$\begin{array}{rl} 0.1 & \quad\text{: one decimal place used} \\ \times\ \underline{\quad 1} & \\ 0.1 & \quad\text{: after moving the decimal place one position leftward} \end{array}$$

The "one-tenth" (of one) description is easily seen when this decimal (fractional) value is converted to a standard fractional form with some help from the concepts of positional notation and-or scientific notation (SN) which will be discussed ahead.

0.1 = (1)(0.1) = $(1)\left(\frac{1}{10}\right)$ = (1)(10^-1) = $\frac{1}{10}$: "a tenth of one", and since any value multiplied by 1 is that value

Ex. (0.1)(0.1)

A good knowledge of the decimal system, which can happen with some routine practice, will quickly give an answer of "a tenth, of a tenth", or "one-tenth, of one-tenth", or simply, "a hundredth" or "a hundredth of".

$$(0.1)(0.1) = \frac{(1)(1)}{(10)(10)} = \frac{1}{100} = 0.01$$

standard method:

$$\begin{array}{r} 0.1 \\ \times\ \underline{0.1} \\ 01 \\ +\ \underline{000} \\ 0.01 \end{array}$$

Ex. (0.5)(2.0)

$$(0.5)(2.0) = \frac{(5)\ (2)}{(10)\ (1)}$$

After simplification (using multiplication) of the indicated factors in the top (the numerator) and bottom (the denominator) values of the fraction or division problem:

$$= \frac{10}{10} = 1$$

standard method:

$$\begin{array}{r} 2.0 \\ \times\ \underline{0.5} \\ 100 \\ +\ \underline{000} \\ 1.00 \end{array}$$

For another check, multiplying any value by 2 is equivalent to adding that value twice:

$$\begin{array}{r} 1 \\ 0.5 \\ +\ \underline{0.5} \\ 1.0 \end{array}$$

Ex. (0.3)(0.5)

$$(0.3)(0.5) = \frac{(3)(5)}{(10)(10)} = \frac{15}{100} = \frac{15}{10^2} = 15(10^{-2}) = 0.15$$

checking:

standard method:

$$\begin{array}{r} 1 \\ 0.3 \\ \times\ \underline{0.5} \\ 15 \\ +\ \underline{000} \\ 0.15 \end{array}$$

Ex. (1.5)(0.2)

(1+ 0.5)(0.2)

First note that any number divided by itself is equal to 1 or 1/1. Below, 10/10 = 1/1 = 1 will be used. Also, if you have "ten-tenths" of something and add "five-tenths" of something, then it's obvious that you have "ten plus five tenths" of that something, hence "fifteen-tenths" as shown below. The tenths are used here as the same or "like" (or common) units, and the numeric parts can then be combined.

$$\left(\frac{1}{1}+\frac{5}{10}\right)\left(\frac{2}{10}\right) = \left(\frac{10}{10}+\frac{5}{10}\right)\left(\frac{2}{10}\right) = \frac{(15)(2)}{(10)(10)} = \frac{30}{100} = 0.30$$

standard method:

$$\begin{array}{r} 1 \\ 1.5 \\ \underline{0.2} \\ 30 \\ +\ \underline{000} \\ 0.30 \end{array}$$

DIVISION

Division is to divide (separate or split into groups) something into two or more smaller parts. The symbols to indicate or express a division operation are shown below. The separate parts may possibly not even be the same value. For example, you could divide a bag containing 6 apples among 2 people. Perhaps you will give one person 4 apples, and give the other person 2 apples. Here, the apples would be completely (but not balanced or evenly (equally)) divided out among the people. Formally, in mathematics, division is to separate that something into smaller parts all having the same size and-or numeric value, hence the result of a division operation is generally thought to be "even" (equally, the same, or in balance).

For the example noted above, to have an even division, each person would receive the exact same value of 3 apples. With 6 apples for 2 people, we can divide the number of apples by the number of people to see how many apples each person will receive. 6apples / 2 people = 3 apples / 1person = "3 apples per person". (6/2)=3. If there were 3 people, for an even division (or "distribution" of and-or to) of the 6 apples, each would receive 2 apples since 6apples / 3people = 2apples /1people = "2 apples per person". (6/3) = 2. The apples are then said to have been "evenly divided (of-and) out" to the people. We see that the word "even", or "evenly", actually means that each part or portion is in balance to any other parts, that is, each part and-or its numeric value is the same or equal to all the other resulting parts after the division is completed. For example, if something was divided into 3 parts, each part will have the same numeric value, and for example, if 30 was divided into 3 parts, numerically expressed as: 30/3, each part of the three total parts will have the same numeric value of: (30/3) = 10 = "ten". The sum of each of each smaller parts (same) value will equal the original or total value before division was performed on it: (10 + 10 + 10) = (10)(3) = 30, and this multiplication is actually a way to check a division problem, and here it's: (30/3) = 10.

Mathematically, division is the "inverse" of multiplication (repeated addition), hence, dividing is essentially performing repeated subtraction. Essentially then, the "starting minuend" is called the dividend, and the divisor is effectively the subtrahend. A quotient is the result or output of a division operation, and it is how many times the divisor can be subtracted from (or "go into") the dividend.

In relation to division, you will often hear the question that is similar to: "How many times can value1 (the dividend) be divided by value2 (the divisor)?", or in other words: "How many times can value2 (the divisor) go into value1 (the dividend)?" A way to state the division process (that is somewhat like a factoring process) and result (quotient), is that it is how many times that value2 (the divisor) must be increased or multiplied so that it equals value1 (the dividend). This is the same as saying how many times bigger that value2 (the dividend) is than value1 (the divisor).

Consider: let: value1 = factor1 and value2 = factor2

(factor1)(factor2) = product

$\frac{\text{product}}{\text{factor1}}$ = factor2 : "product is factor2 times bigger than factor1" , product = (factor1)(factor2)
: "factor1 can go or divide into product a total of factor2 times"

$\frac{\text{product}}{\text{factor2}}$ = factor1 : "product is factor1 times bigger than factor2" , product = (factor2)(factor1)
"factor2 can go or divide into product a total of factor1 times"
: product = (factor1)(factor2) = (factor2)(factor1)

The symbols for division along with the general "division formulas" or expression formats are:

dividend ÷ divisor = quotient : quotient is a word derived from quot and quota which basically means the current numerical value or size as in "how many (times)" or "how much"
dividend / divisor = quotient You may sometimes hear something like "Today's quota is 7 sales", which means to the effect: "Today's goal or output of selling is to make 7 sales".

$\frac{\text{dividend}}{\text{divisor}} = \text{quotient}$: In this "fractional like" form or expression, the quotient is and represents how many times greater (or sometimes, less) that the dividend is compared to the divisor.

$\begin{array}{r|l} & \underline{\text{quotient}} \\ \text{divisor} & \text{dividend} \end{array}$: this is often the form to use when manually calculating division by hand

Before the concept of division by using repeated subtraction is shown, perhaps the simplest way to understand the process of division or how to divide, is by observing and comparing. For example, if you had 6 objects (such as coins, or dots on paper), how many times (ie. comparisons) can 1 of those objects correspond (ie. match) to all of those objects? In other words, what is 6 divided by 1, hence how many times can 1 be subtracted or taken away from 6? The question may also be stated as something like: How many times can 1 thing "go into" (or compare to) 6 things?

6 divided by 1, graphically:

* * * * * * : 6 now repeatedly compare to, or matching to 1 (or "1 to 1"):

* : 1 time (1 comparison or correspondence, there are now 5 dots left)
* : 2 times (2 comparisons or correspondences, there are 4 dots left)
* : 3 times
* : 4 times
* : 5 times
* : 6 times (there are now 0 dots left, the division process must stop)

We see that 6 divided by 1 is 6. It could be said that "1 can go into 6, a total of 6 times". Any value divided by 1 is always equal to that same value. If you are using objects such as coins, you can physically remove the necessary coins, or if you are using objects drawn on paper, you can circle or cross-out the objects after each comparison to indicate that they have already been used or considered.

Ex. 6 divided by 2:

* * * * * * : 6 repeatedly comparing 2 (the divisor) to each 2 new and remaining parts to be considered of the 6 total parts:

* * : 1 time
* * : 2 times
* * : 3 times

6 divided by 2 is 3. "2 can go into 6, a total of 3 times" or "2 can be subtracted from 6 a total of 3 times"
or perhaps something like: "6 can be divided by a group containing 2, a total of 3 times",
, or "there are 3 groups of 2, within, or out from, 6"

Ex. 6 divided by 3:

* * * * * * : 6 repeatedly comparing to 3 (comparing 3 to each new 3 and remaining parts of the 6):

* * * : 1 time
* * * : 2 times

6 divided by 3 is 2. "3 can go into 6, a total of 2 times"

Ex. 6 divided by 6

* * * * * * : 6 repeatedly comparing to 6 (or the 6 smaller parts to the 6 total parts):

* * * * * * : 1 time

6 divided by 6 is 1. Any value divided by itself (the same value) is always equal to 1. Notice in the examples that the larger the divisor is for a given division problem, the smaller the result or quotient, hence it could be said that the divisor and quotient are (mathematically) inversely related, that is, as one value goes larger in value, the other or corresponding value goes smaller in value, and vice-versa.

6/2 = 3 As the divisor goes larger, the quotient goes smaller. This is an (mathematical) inverse relationship.
6/3 = 2 A direct (mathematical) relationship is where if a value grows, a corresponding value also grows.
6/6 = 1

If you had a quantity of 30 and kept removing or subtracting 5 from it, you will make 6 "even subtractions". The word "even" here essentially means a full or complete division and that there was no part, difference or remainder (r), of the 30, left over after all the subtractions.

1.	2.	3.	4.	5.	6.
30	25	20	15	10	5
- 5	- 5	- 5	- 5	- 5	- 5
25	20	15	10	5	0 = r

Or: (((30 - 5) - 5) - 5 - . . .) (. . . means "and so on") : expressing 30 divided by 5 as a repeated subtraction expression

Or: 30 - 5 - 5 - 5 - . . .

An easier way to express this subtraction problem is with division. The problem could be written as:

$5\overline{)30}$: "thirty divided by five", or = 30 divided by 5

You can check the result (the quotient = 6) for correctness by repeatedly subtracting it from the dividend (30). If you get the original divisor (by getting 5 even subtractions = 5), the result is correct. Since division is essentially repeated subtraction, you can check a division problem with another division problem where the dividend is divided by the quotient. If this quotient value is equal to the original divisor, the division was correct and the quotient (result) is correct.

Ex. $5\overset{\underline{\ 6\ }}{\overline{)30}}$ is correct since $6\overset{\underline{\ 5\ }}{\overline{)30}}$ This is so since:

$$30 = 30$$

$5 + 5 + 5 + 5 + 5 + 5 = 6 + 6 + 6 + 6 + 6$: "sum of six, fives" = "sum of five, sixes"

$5 \times 6 = 6 \times 5$: the repeated addition, expressed as a multiplication

Given just the left hand side of the equation, 5 can be removed (ie. subtracted) from 30, a total of 6 times.
Given just the right hand side of the equation, 6 can be removed from 30, a total of 5 times.

To perform and check the results of division without using the slower and more tedious method of repeated subtraction or performing another division, you can use multiplication (which is repeated addition as seen above). To check if the quotient is correct, multiply the divisor and quotient, and if the result (a product) is equal to the dividend, the division problem was performed correctly:

Since $\frac{\text{dividend}}{\text{divisor}} = \text{quotient}$ We mathematically have after multiplying both sides by the same value (so as to keep both sides of the equation in balance, or the same value), here, the divisor:

$\frac{\text{dividend}}{\text{divisor}} \frac{(\text{divisor})}{(1)} = \text{quotient (divisor)}$ After some canceling, this can be simplified or reduced to just:

dividend = (quotient)(divisor) : CHECKING A DIVISION PROBLEM (using multiplication). We see that the quotient and divisor are essentially two (of the many possible) corresponding factors of the dividend. If one of these factors is wrong, their product will not match or check (be equal to) with that dividend.

Since 5 can divide into, "go into", "sum into", "be taken from", or be part of the total sum of 30, a total of 6 times, repeatedly adding 5, to an initial sum or start of 0, six total times can be represented by a simple multiplication problem:

$0 + 5 + 5 + 5 + 5 + 5 + 5 = 30$ or simply:
$5 + 5 + 5 + 5 + 5 + 5 = 30$ which can be (mathematically) written or expressed in a simplified form as:

5 x 6 = 30 : Note, both 5 and 6 are factors of their product, or 30. Dividing 30 (as a dividend) by either factor will yield (produce, or result to) the other factor. Multiplying these factors together will yield that dividend.

As noted above, dividing 30 by 5, we get 6, and if we divide 30 by 6, we get the other factor which is 5. So if then given a multiplication problem of the form:

____ x 6 = 30 : or = 6 x ____ = 30 or:

____ x 5 = 30 : or = 5 x ____ = 30

To solve these for the unknown factor, simply divide 30 (the product of factors) by the given factor, so as to find the other (corresponding, correct) factor.

Division with a zero as an operand: 0 divided by any value, results in a quotient that is same value of 0.

Ex. $\frac{0}{3} = 0$: "zero or no thirds, equals zero thirds, or simply zero". In reference to subtraction, 3 cannot be subtracted from 0 even once, since there is nothing there to subtract from that dividend. Zero, or no parts of 3 parts is 0.

This checks when multiplying the quotient and divisor together, yielding the dividend of 0.

3 x 0 = 0

The quotient of any value divided by 0 is not definable, hence this form of a division operation is often called "undefined". This comes from the reasoning that their is no quotient, that when multiplied by a 0 divisor, that will yield a product equal to the given dividend. Also, all values multiplied by 0 result in a product of 0.

Ex. $\frac{10}{0}$ = undefined since: $0 \times \mathbf{?} = 10$:here, ? does not represent a specific value, but rather indicates an endless question, or one that can't be answered, hence an unsolvable or undefined value, and equation.

Note that when checking the product of a multiplication by or with a 0 factor, that when using 0 as the dividing factor, that other non-zero factor of that product cannot be confirmed:

Ex. $4 \times 0 = 0$

Using what was mentioned previously: product / factor = other factor of that product:

checking: $\frac{0}{4} = 0$: checks and $\frac{0}{0} = ?$ (not 4) : here, ? means undefined

When using a modern electronic computer or hand-held electronic calculator, division by zero is usually noted as the symbol (E) displayed, and which means "Error" because division by 0 as a dividend or factor does not have any result.

How would you solve for the unknown, here a divisor (or denominator), represented as $\underline{\quad\quad}$, of a problem of the form:

$\frac{10}{?} = 5$: 10 divided by what number is 5 ?
(here, 10 is the dividend, and 5 is the quotient)

First, we know that to check a division problem, we multiply the quotient and divisor to get the dividend, (that is, the quotient and divisor are both (corresponding) factors of the dividend) hence:

$5 \times \underline{\quad\quad} = 10$

And by the above discussion, dividing a product of two factors by either factor will give the other corresponding or correct factor. Therefore, to find the value of the other factor (here, being equal to the divisor in question), we divide 10 by 5:

$\frac{10}{5} = \underline{\quad\quad}$

$\frac{10}{5} = 2$

checking: $\frac{10}{2} = 5$: checks since $5 \times 2 = 10$

Let us now estimate the value of a quotient. Consider this problem:

$6\overline{)15}$

From the multiplication table or our memory, we know that the quotient is greater than 2 and less than 3 since $6 \times 2 = 12$, $6 \times 3 = 18$, and 15 is numerically located someplace between 12 and 18. 2 will be chosen as a start since 6 can go into 15 at least 2 times (but not 3 times) and there will now be a remainder (the value or difference "left over" or unprocessed amount of the dividend, here 15) of the quotient or "starting minuend" since the division has not been completed yet. Hence, the result or quotient will be 2, plus some fractional value of the divisor, here 6. That is,

quotient = number of times the entire divisor can be used in the division, or evenly divide into the dividend
+ any portion (fraction) of a possible one single division by the divisor to account for any remainder

or quotient = number of even (whole or entire) divisions + any portion (fraction) of one division ,and for the example:

quotient = (2 + some fractional value of 6) : and for checking: (this quotient value) x 6 = 15

> "Some multiple (a value between 2 and 3) of 6, equals 15". Since it's a value between 2 and 3, its not one of the whole or "counting numbers" (sometimes called "integers" for incrementing or incremental "steps" of numbers). Here, the quotient value will have both a whole part and a fractional part, and this kind of a decimal number is sometimes called a "real number". Consider that values are not always incremented by 1, but could be incremented or changed by smaller values such as a part or fraction of 1, and hence the need for real numbers to both represent these increment values and results (such as a sum).

Here, the whole part (the value of 2) of the quotient, can be interpreted as being 2 whole or entire sixes, hence "twice" or occurring 2 times in that dividend, and therefore, at least 2 even divisions are possible. Since there is a remainder "left over" to account for the entire dividend, added (or "plus") to this quotient value is some numeric (fractional, less than 1) value that represents this fractional part of a partial division, and not a complete division. This division is applied to the remainder or what is left of the dividend. The numeric representation, using decimal positional notation, of a fractional part of anything or any value is always less than the entirety or whole thing which can be represented numerically as 100% ("one-hundred percent") or just 1.00 = 1, and hence any fractional value or part will be less than (< is the "less than" mathematical symbol) 1.0

Here is how much of the dividend (here for this example, it's 15) is "left-over" or remaining after all the "whole divisions" by 6:

15 - (6 x 2)
15 - 12
3 : remainder of the dividend

or:

$$\begin{array}{r} 15 \\ -\underline{\ 6} \\ 9 \\ -\underline{\ 6} \\ 3 \end{array}$$

: here, optionally using a repeated subtraction method

: 3 is the remainder of the dividend of 15, and 3 is only a part of one whole or entire value of 6. It could also be said that since only 3 is remaining, that only 3 of the parts, of the 6 parts of the divisor, can evenly divide or "go into" this value of 3. Hence, there will not be a (1) full or complete division upon the remainder.

This remainder (here, it's 3) of the dividend (here, it's 15) is clearly less than 6 and it is therefore numerically only some part or fractional value of 6. Even though 3 is not less than 1, it is still only some fractional (part) value of 6 since 6 is being considered as a single (1) "whole quantity" of reference now and 3 is only a part (portion, or a fraction) of it. The fractional part of the quotient, which represents the numeric value of an incomplete, fractional or partial of one complete possible division, can be said as (and equivalent to) "3 parts of the 6" are left over or remain. The whole part of the quotient is 2.

The quotient can be said to be:

"Two plus three parts of six" = $2 + \frac{3}{6}$: a quotient is expressed in terms of the divisor, that is, how many times the divisor can go into that quotient.

That is, $\frac{15}{6} = 2 + \frac{3}{6}$: note also that 3 is equal to half of 6
The quotient could be expressed as "Two and a half" of that divisor value, hence two and (plus) half of a (one) complete division by the divisor.

.

The quotient is always considered as being in reference to the specific divisor that divided the dividend. The fractional part of the quotient is, and numerically represents 3/6 = 1/2 = 0.5, and is not, for example: 3/5 or 3/7, or 3/15 or any other value. That is, the remainder of one complete division is expressed as a fractional part (or remainder left) of the divisor used for the division, and is not expressed as a fractional part that the remainder of the dividend is to, or of, the entire dividend, (ie. here, not as 3/15 = 0.2).

The quotient can be expressed as equal to:

Since: $\frac{\text{dividend}}{\text{divisor}} = \text{quotient}$ and dividend = (divisor)(quotient)

$$\frac{15}{6} = 2 + \frac{3}{6} \qquad\qquad 15 = (6)\left(2 + \frac{3}{6}\right) \quad \text{or=} \quad \left(\frac{6}{1}\right)\left(\frac{2}{1} + \frac{3}{6}\right)$$

quotient = 2 whole divisions (or subtractions) + numerically expressed or indicated part of a one division by the divisor since a complete division is no longer possible. In short, this value will be and represents a fraction of 1 division.

quotient = 2 whole divisions (or subtractions) + 3/6 = 1/2 = 0.5 = "half" of a 1 complete division by the denominator

The representation (halfs, thirds, fourths, tenths, thousandth, etc.) of the dividend is often not so apparent given a strict numeric decimal value that has a fractional value, and then again, just how do you numerically represent half or any other portion anyway, especially at this stage in the book? The simple trick to overcoming this problem is to increase the remainder, and any further remainders, so that the divisor can divide into it at least once, rather than some fractional portion. At each stage of this "continued division", one extra (lesser weight) digit will be placed in the quotient. This is the basis of "long division" which will be discussed next. Observe the actual quotient of the division problem above, expressed in various forms or expressions:

$$2 + 6\overline{)3} = 2 + 6\overline{)3.0} \quad , \text{ or } = 2 + \frac{3}{6} \quad , \text{ or } = 2 + \frac{1}{2} \quad , \text{ or } = 2 + 0.5 = 2.5$$

Now, another example and more analysis will be shown below so as to give a fuller understanding of the basics of division.

Ex. 31 divided by 5

$5\overline{)31}$ Since 5 x 6 is 30, and 5 x 7 is 35, we will start with 6 times that the divisor of 5 can go into, or be subtracted completely from this quotient of 31.

$$\begin{array}{r} 6 \\ 5\overline{)31} \\ -\underline{30} \\ 1 \end{array}$$

: 5 x 6 = 30

: 1 is the remaining (or remainder) portion of the dividend which cannot be evenly divided by 5. The division cannot continue normally, and is therefore an incomplete division remains. This 1, by itself, is NOT part of the quotient (the result of the division). A quotient is a value that is how many times the divisor can go into, or divide into, the dividend. Hence, a quotient is how many times the dividend can be divided by the divisor. Having said this, the remainder of the quotient is NOT be added to, or expressed as to be added to, the quotient. Here, for this example, the quotient is NOT: 6 + 1, 6.1, or "6 with a remainder of 1 (one more division?)", since 1 is part of the dividend, and is not part of the quotient. Hence the quotient is also NOT: 6 + 1/31 , or 6 + 1/6.

Sometimes the quotient might be expressed as: Quotient and Remainder of the dividend left over
Q and R , and for the above example:
Q=6 and R=1
R is technically, not part of the Quotient

Here, 6 is the number of ("multiple") times the divisor can "go into (wholly, completely, entirely or evenly)" the dividend. Since there is a remaining part (the 1) of the dividend left, and 5 cannot go into 1, this "last", incomplete or "partial" division is simply expressed or indicated and added to the quotient (as a fraction or part of a (one) complete division). The quotient is therefore:

$$6 + 5\overline{)1} \quad = \quad 6 \text{ and } \frac{1}{5} \quad = \quad 6 + 1/5$$

: Note that the added in, division of 1 by 5, to the quotient, is essentially adding in a fractional part of the divisor value. One-fifth of a whole (one) division by the divisor is one-fifth of the 5. One-fifth of five = ((1/5) of 5) = 1 which does equal the remainder of the dividend left over due to an incomplete division.

And for an even deeper understanding:

$5\overline{)\ 31}$ The dividend can be expressed in many ways such as:

$5\overline{)\ 30 + 1}$ which can be expressed as:

$\frac{30 + 1}{5}$:a division expression, and it is also considered a fraction (here, of 5) which is equivalent to: (Knowledge, understanding about fractions may be needed here, and is discussed further ahead.)

$\frac{30}{5} + \frac{1}{5}$ simplifying the first division in this sum, we have the same quotient as shown previously:

$6 + 1/5 = 6 + 0.2 = 6.2$. Checking: $(5)(6.2) = 31$

It's easier to learn manual (by "hand") division by example, so as to solve (find, arrive at) the result (the quotient), rather than by "wordy" and seemingly complex explanations. An example is shown below. This example is sometimes referred to as "long division". Even though the dividend, divisor, or quotient values may contain both an integer (basically a whole part) and fractional part (a value less than 1), the process of performing the operation with "long division" is usually done completely with (single digit) integers (basically whole numbers or the "counting numbers"). The fractional part of the dividend is of course still considered and utilized during the division, but it will be first included as part of the dividend expressed as only a whole number. If the divisor has some number of "decimal places", the decimal point in both the divisor and dividend is moved the same number of decimal places rightward, essentially ridding the fractional part of the divisor and possibly the dividend. Doing this to the divisor, and then the dividend, the decimal point in the quotient will also be adjusted correctly to its proper position. The actual division of the dividend by the divisor begins at the most significant (leading) digits of the dividend and continues to the least significant (trailing) non-zero digit. If the divisor cannot divide at least once into the remaining portion of the dividend yet to be processed, the result or remaining part of the division process is called the remainder (of one division) and is expressed as being part (a fraction of 1 to be added into the quotient, and not a full increment of 1) of the quotient.

About moving the decimal point in both the divisor and dividend, and the same number of places:

Since: $\text{quotient} = \frac{\text{dividend}}{\text{divisor}}$

Mathematically, the same quotient will result if you multiply both the divisor and dividend, or both numerator and denominator of a fraction - creating an "equivalent (same quotient value) fraction", by just the same value, for example, 10 or some other power of 10, such as 100, 1000, etc. When you move a decimal point rightward in a value, each time you move it rightward, you are essentially multiplying each new value by 10, and-or multiplying the original value by another (incremental, integer) higher power of 10. First consider that any value divided by itself is equal to 1. Ex. (10/10) = 1, and that multiplying any value by 1 is that same value:

$$\text{quotient} = \frac{\text{dividend}}{\text{divisor}} = \frac{(\text{dividend } (10))}{(\text{divisor } (10))} = \frac{\text{dividend}}{\text{divisor}}\ \frac{(10)}{(10)} = \frac{(\text{dividend})}{(\text{ divisor })} \times 1 = \frac{\text{dividend}}{\text{divisor}}$$

As an introductory example of the actual working method of dividing "by hand", first consider any value divided by 1. The result, the quotient, is surely and always that same value as the dividend:

1 | 123 : one-hundred and twenty three divided by one

_123_
1 | 123 : the divider of 1 can go into the first digit of the dividend once, put a 1 in the quotient
- _1_ : 1 x 1 = 1, then subtract this, and then "bring down" a remaining digit of the dividend, and
02 : continue to divide, here, 2 divided by 1, is 2, put a 2 in the quotient
- __2 : 2 x 1 = 2, subtract this, and then bring down a remaining digit of the quotient
03 : continue to divide, here, 3 divided by 1 is equal to 3, so put a 3 in the quotient
- __3 : 3 x 1 = 3, subtract this, and the remainder of the dividend is 0, therefore, the division is complete,
0 and it was an even or perfect division with no remainder of the dividend left over, and therefore,
no partial or incomplete division will be expressed as part (summed to) of the quotient.

Ex. Divide 172 by 5

First, 5 cannot divide into the first leading, significant digit (the 1), so then check if it can divide into the first two leading digits (17):

__3_._ __34._ +
5 | 172.0 or 5 | 172.0 5 | 20.0
15 170 : (34x5=170)
22 20

17 - 15 is 2. If this difference was greater than or equal to 5 (the divisor), an error has obviously been performed. The error would be that the corresponding digit in the quotient is too small, yielding a product that is too small, and then a difference that is too large and being greater than the divisor itself. Since 2 is less than 5, we can continue the division by first "bringing down" the next digit of the dividend. Here, 22 is an intermediate difference or remainder (of the dividend). To verify this, we note that the 3 in the second digit (from the decimal point) of the quotient actually represents 30, and 30 x 5=150, and 172 - 150 = 22. Since 22 can be divided by 5, it is not the final remainder of the dividend, and that must always be less than the divisor. The (integer or whole number) division continues:

__34.__
5 | 172.
- 15
22
- 20 : 5 x 4 = 20
2 = r : r = remainder of the dividend left (here, r or R = 2,of the dividend of 172)

Since any integer multiple of 5 cannot divide "evenly" (at least once) into 2, then 2 is the remainder (r) left of the 172. 2 is obviously less than 5. The remainder is always less than the divisor.

When checking a division problem where there is a remainder (of the dividend) left over, the remainder is added to the product of the quotient and divisor, so as the total or sum will equal that dividend. For the last example:

$$\begin{array}{rl} 2 & \\ 34 & : \quad \text{quotient} \\ \times\ \underline{\ \ 5} & : \times\ \text{divisor} \\ 170 & \\ +\ \underline{\ \ 2} & : +\underline{\text{ remainder }} \\ 172 & : =\ \text{dividend} \qquad : \text{checks} \end{array}$$

The quotient in the last example may also be written or noted as:

$34 \text{ and } \frac{2}{5}$ or = 34 and 2/5 or = 34 + 2/5 or more simply and commonly as: 34 2/5 = 34.(2/5) = 34.4

We see that a quotient is sometimes explicitly expressed as:

quotient = (even, whole or entire divisions) + (remainder/divisor) which equals:
quotient = (even, whole or entire divisions) + (partial or fraction of 1 division) : as an expression, or as:
quotient = (even, whole or entire divisions) **.** (partial or fraction of 1 division) : as one (numeric, decimal) value

Using this notation, checking to see if the product of the quotient and divisor equals the dividend requires knowledge of more advanced mathematics (that will be shown ahead), specifically the use of adding or combining fractions. Note that the remainder is expressed in the notation above as part or fraction of the divisor (not as a part of the dividend or quotient). The value of this fraction though is equal to the value of the fractional part of the quotient when it is expressed as just a single standard decimal number.

The division does not have to stop at the remainder of the dividend left. If you "bring down trailing zeros" and continue to divide, you will automatically get the decimal equivalent of the remainder as part of (and added directly to) the quotient.

$$\begin{array}{rl} \underline{\ \ 34.4} & : \text{Note, 0.4 is the fractional part of the quotient, and this part equals 2/5 as indicated above.} \\ 5 \mid 172.0 & \quad \text{Also, } 5 \times 0.4 = 2 = \text{whole value remainder of the dividend, and } 2/5 = 0.4 \\ \underline{15\ \ \ } & \quad \text{Again, at this point, 2 is the remainder "left over" of the dividend, and is NOT the remainder of the} \\ 22\ \ & \quad \text{quotient. 2/5 (or 0.4) can be considered as the numeric expression for the remainder or final part of} \\ \underline{20\ \ } & \quad \text{a one (incomplete) division, and which is to then be added to as (a fractional) part of the quotient.} \\ 20 & : \text{bringing down a "trailing" 0} \\ \underline{20} & \\ 0 & : \text{here, 0 indicates the division is complete when there are no more digits of the quotient left.} \\ & \quad \text{172 has been fully divided by 5.} \end{array}$$

checking :

$$\begin{array}{r} 22\ \ \ \\ 34.4 \\ \times\ \underline{\ \ \ \ \ 5} \\ 172.0 \end{array}$$

Ex. Divide 1 by 0.2

$\frac{1.0 \text{ units}}{0.2 \text{ units}}$ or = $0.2 \;\big|\overline{\ 1.0\ }$: here, strict numeric values with unexpressed units can be understood and-or treated as having units of "ones" or "wholes".

Here, we can multiply both the numerator and denominator by 10 to clear or rid any of them of a fractional portion, or simply by moving the decimal point in both the divisor and dividend one position rightward: Note, that 10/10 = 1, and multiplying anything by 1 is still that same value numerically, and is without issue or worry of obtaining an incorrect result:

$$\frac{1.0 \text{ units } (10)}{0.2 \text{ units } (10)} = \frac{10 \text{ units}}{2 \text{ units}} = 5 \quad \text{or commonly as: } = \quad 2 \,\overset{\underline{\;\;5.0\;}}{\big)\,10.0}$$

As seen above, when dividing one quantity by another quantity whose units are the same, the result is a simple or "unit-less", plain and "strict", numeric value only. The problem could also have been expressed as:

$$\frac{1.0 \text{ units}}{2 \text{ tenths of a unit}}$$

Since there are 10 tenths of a unit in each 1 unit, after multiplying the numerator by 10 for the (converted to) equivalent number of tenths of a unit in that value given (here it's 1 only) , we will have like or similar units to properly work with:

$$\frac{10 \text{ tenths of a unit}}{2 \text{ tenths of a unit}} = \frac{10 \text{ tenths}}{2 \text{ tenths}} = 5 \quad , \quad \text{checking: } (5)(2 \text{ tenths}) = (5 \times 2) \text{ tenths} = 10 \text{ tenths}$$

This new (derived) units of: "tenths of a unit", or simply "tenths" is verified below:

$$1 \text{ unit} = \frac{10 \text{ units}}{10} = \frac{10\,(1 \text{ units})}{10} = \frac{10\,(\text{units})}{(10)}$$

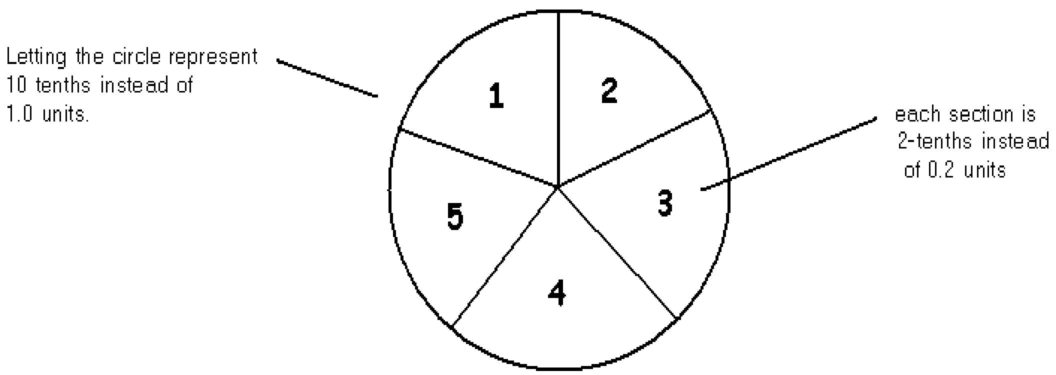
The value of : $\frac{(\text{units})}{10} = \left(\frac{1}{10}\right)$ units , is commonly called: "one-tenth of a unit" , a "tenth of a unit", or simply "tenths". (1/10) unit = 0.1 unit

1 unit = 10 tenths of a unit or simply :
1 unit = 10 tenths

This unit of tenths could have been called something else, but they are kept as is to reflect that they are derived from a whole (1) unit, and that the decimal system is based on ten = 10.

For the above example, it is perhaps easier to also visualize the division when the dividing is by a whole or integer number of units (like 2 tenths) rather than some fractional (like 0.2 units) or mixed (whole + fractional) number of units: [FIG 2]

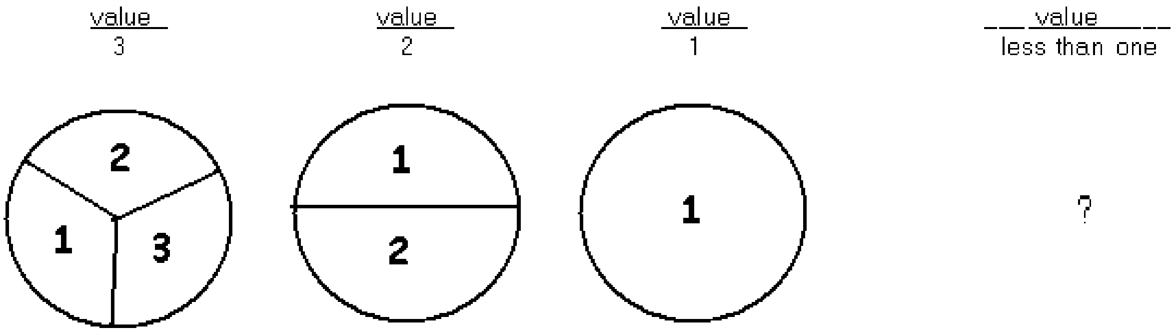
(1/ 5) graphically:



1 is the numeric representation of the whole or entire thing in question, and is graphically represented as one complete or whole circle. Note here that 2-tenths is equal to or composed of 1 tenth and 1 tenth. Each section of the circle divided into 5 parts or sections, has a numeric value or representation as (1/5). (1/5)=0.2 and this can be considered as:

$$\text{quotient} = \frac{\text{dividend}}{\text{divisor}} = \frac{1}{5} = 0.2$$

Consider these general graphics or visualization representations below of the division of any specific value greater than or equal to 1. The entire circle area represents the specific value used as the dividend, the number of segments of the circle represents the divisor value, and each equivalent segment size or (area) portion of the circle represents the quotient value: [FIG 3]



For the first 3 values divided, the quotient (represented as a portion or segment area of the circle, and is equal to the size of each portion or segment of that circle) is clearly less than the value in the dividend or numerator, but for the circle or value on the right, the quotient or result is actually a bigger circle or value, that is, the quotient is actually larger than the dividend value. This is a reason why we changed the divisor numbers to be greater than or equal to 1 for the divisions in the above examples, such as when the dividend and divisor are first multiplied by 10 before the division starts.

A very helpful method to perform "long-division" (a manual division computation, by pen and hand) is to take the worthwhile effort to create a table of all products of the divisor and integers from 1 to 9 before performing the division calculation by hand. Some of these products may not be needed due to the specific values used in the division expression or problem, but they will be there (already calculated) if needed. This will eliminate a lot of possible guesswork, and therefore it can save some time, and may even prevent some errors.

Ex. Divide 5.2 by 741,279

Below, rather than use multiplication to create a useful table for division, repeated addition is utilized since it (addition) is easier in itself and requires one less repetitive step to create the table. When the division cannot be complete (as like it with an "even division" that would be complete at some point, and not have an "endless" quotient composted of unending digits), there is a good chance all the multipliers (0 to 9) will be needed and used in the quotient. Sometimes, one or several specific digits repeat over and over in a quotient:

```
     741279    (1)       :  741,279 x 1
+    741279
    1482558    (2)
+    741279
    2223837    (3)
+    741279
    2965116    (4)
+    741279
    3706395    (5)       : 741,279 x 5
+    741279
    4447674    (6)
+    741279
    5188953    (7)
+    741279
    5930232    (8)
+    741279
    6671511    (9)       : 741,279 x 9
```

Using this created table shown above of the single digit integer products (or multiples) of the divisor:

```
               0.0000070149
741279 |  5.2000000000
        -  5188953         : 7    , optionally indicating each new digit of the quotient during this process.
             110470             : each difference here should always be less than the divisor,  then "bringing down"
          -        0        : 0      a zero will essentially multiply this difference by 10 so the division can continue
             1104700
           -  741279        : 1
              3634210
            -  2965116       : 4
                6690940
              -  6671511     : 9
                   19429
```

For practicality, here, the quotient is "rounded" (basically means limited to, and approximated) to only 9 decimal places is: $0.000,007,01\underline{5}$ The underlined 5 means that this value was rounded at that decimal place and precision.

Division can be used to check the result (a product) of a multiplication operation. Given the product of two factors, divide the product by one of the given factors, and if the result (a quotient) is equal to the other factor, the product is correct.

Ex. 5 x 4 = 20 now checking with: 20/4 = 5 and, or 20/5 = 4 : we get the other factor or operand

Some Devices To Help Make Numerical Computations

Before fast electronic calculators and-or computers there were mechanical computers of all sorts, with some just being adding and subtraction machines. The first programmable mechanical computer design is credited to **Charles Babbage** (1791-1871), from England, when he proposed the **Analytical Engine** machine in 1837. Some of the ideas Babbage incorporated into his machine were based on the punch cards for the **Jacquard loom** (essentially a machine for weaving or inter-twining strings of fabric so as to a larger fabric or cloth) which were cards with holes in them, and where each hole represented some data for a desired (ie., programmable) pattern for the loom to automatically make with fabric. The cards were essentially or effectively the memory (ie., storage, record) medium, and were also used as steps (or statements) to then instruct the machine via the data (ie., coded information [ex. numbers] and-or instructions) on it. This machine had input, memory, logic (ie., truth statements), (conditional, ex: "IF" and "DO") program flow and-or control, looping (ie., repetition) and output ability such as with a mechanical printing machine. With his computer machine, Babbage wanted to make more accurate (ie., correct) and precise (ie., mathematical precision, fineness, resolution) mathematical tables for many others to utilize. It is of note that people who made calculations since the days of antiquity were sometimes called as "computers" or "calculators". Babbage invented a type of user programmable computer, and due to the holes (ie., a binary value of 0 or 1) and no-holes (ie., opposite binary 1 or 0) actually being data and or-an part of an instruction (ie., a step in the computer or calculating program), the computer functioned as a binary functioning computer.

The mechanical power and "timing" (ie., for individual steps, processes, calculations, and the speed of it mechanically functioning and-or computer processing) for this machine came from human power via a physical or mechanical force (technically, the application of energy, and which can be indirectly calculated by an equation such as: Force = mass times acceleration) applied to a mechanical hand-crank or turning handle. The faster this could be turned, the faster the computer could perform its internal steps and calculations to produce a result. In theory, his machine is much like a modern computer and would calculate a binary result. Babbage also created a printer with numbers 0 through 9 (in reverse image for printing) on the edge of each wheel, and many wheels were placed side by side so as to create and print large valued numbers having many columns and-or digits. The machine converted its binary number and-or notated result to its decimal system equivalent, and then set the corresponding wheel with the number to be printed. The number and-or result was then printed onto paper after applying ink to the numbers on the wheels. Most likely, this machine could be built so as to also display an immediate visual (ie., non-printed) result like a modern day calculator does. His assistant, lady **Ada Lovelace** (1815-1852), from England, is often noted as being the first computer programmer. Lovelace had a good mathematical education by various tutors, reasoning skills, and a thoughtful imagination that was helpful for new problem solving and progress. She also created some of the algorithms **(**steps and-or procedures, a guide or plan, "program formula", and of which resulted in instructions for input, output, calculations (math functions such as add, subtract, etc,), logic (ex. compare two values), and program flow and control such as "jumping" or starting at another location in the program**)** for the computer to follow so as to utilize and process the specific program and-or calculation in it.

It is interesting to note that electronic computers can perform multiplication by repeated addition. Subtraction is performed using a special form of addition known as complement (value(s)) addition. Division can be performed by repeated subtraction. Hence, it is possible for a computer to perform all the basic mathematical operations by using addition, and Charles Babbage knew this, and this would simplify the design of the machine. Electron computers could then be designed to contain just the electronic circuitry to perform one main mathematical operation such as addition, and therefore eliminate or reduce some circuitry and complexity needed for the other mathematical operations, and making the computer less expensive.

It is also of note that a computer program user does not need to understand all about computer science and-or computer programming (ie., a specific computer language for that machine), and a computer programmer does not need to understand the internals (ex., gears, circuits) of how the machine functions, and the machine maker does not need to know how the metal was mined and manufactured.

USING MULTIPLICATION AND DIVISION TO MEASURE OR COUNT SMALL THINGS

Here is an example of using division to find that which is normally not practical. Something small, such as a grain of sand or rice, cannot be weighed using many weight measuring ("scales") devices. A method will now be shown as to how to weigh these small objects. This method utilizes the fact that if you magnify (increase) something either physically or mathematically by a multiplication, that you can find that original something by a corresponding "demagnification" (a decrease, factorized) using division.

Given 1 grain of hard, dried rice, if you multiply the number of grains of rice by 500, you essentially have also multiplied its' weight by the same value. Now, this total weight of 500 grains of rice can be measured by using a commonly available fine scale. If these 500 grains of rice weigh 1 ounce, you can use division to find 1 grain of rice and the weight of 1 common, "average", typical, or usual grain of the type of rice being considered.

$1\text{ grain} \times 500 = 500\text{ grains}$

mathematically ("solving for 1 grain") :
(dividing the product by one factor, to find the other factor)

$$\frac{500\text{ grains}}{500} = 1\text{ grain}$$

Considering the weight of each grain of rice:

$\text{weight of 1 grain} \times 500 = \text{weight of 500 grains}$

Mathematically solving for the weight of 1 grain:

$$\text{weight of 1 grain} = \frac{\text{weight of 500 grains}}{500} =$$

Now substituting the numeric values given or measured:
(weight of 500 grains = 0.333 oz, [depends on the type of rice])

Note: **This book uses ~= or =~ to mean "about equal to" and-or "equals about".**

$$\frac{0.333\text{ oz.}}{500\text{ grains}} = \frac{0.000667\text{ oz}}{(1)\text{ grain}}$$

: Weight of 1 grain **=~** seven ten-thousandths of an (1) ounce".
: After dividing both numerator and denominator by 500 we get less than a thousandth of an ounce per (one) grain. Later in this book, it will be shown that the entire numerical equation shown is called a "proportion" (for a given problem or system) and can be re-used, for example, so as to now find the weights of any number of grains of rice. Though the known reference weight (of either 500 grains, or 1 grain of rice) and the weight you will be trying to find will be different; not in balance, mathematically, both sides will still be in balance, and that balance is that each side has the same exact quotient value. Here, each side is equal to (0.000667 oz/grain). 0.000667oz = about 0.0189 grams = about 0.019g = 19 milligrams.

Also, given:

500 grains of rice = 0.333 ounces

If we divided both sides of this equation by 0.333 we can find the number of grains of rice per ounce of weight:

$$\frac{0.333\text{ ounces}}{0.333} = \frac{500\text{ grains of rice}}{0.333}$$

1 ounce = (about, ~) 1500 grains of hard, dry, uncooked rice

: For the type actually measured, it was a white rice variety, that was almost 1/4 inch long, and about 3/32 of an inch wide at the thickest part in the middle. It appeared to be a "medium grain" sized of rice. The rice was also hard, dry, and uncooked.

As a check of calculation correctness, and a possible improved value, the above process should be repeated with another set of near randomly selected rice grains so as to have an average value will be as correct as possible for the weight of a rice grain, for most future circumstances. Scales to measure low weights, perhaps a hundredth of a gram, are available, and it is possible to construct your own. If you multiply the number of grains per ounce by 16, you can calculate an estimate of the number of grains per pound (16 ounces). Doing this, we find that there is about 24000 grains of rice in a pound of rice.

To weigh something such as a grain of sand, the basics of the above method can also be utilized. First, you would have to count many more grains of sand just to have a weight of usable value for a scale. This is a very impractical method for most people. To overcome this, you can first find out how many grains are in a certain volume (a measure of 3 dimensional space) of sand, say one cubic centimeter or inch. A cube as shown in this book is a solid with all it's 3 dimensions, length, width, and height all being the same value of length, and for our purposes here, they will have the same length of 1 inch. A cube shape is like that of a square area, an (bounded or specific) area of plane, hence a planar area which has then been moved upwards a height equal to the length of a side of that square shape. This creates a volume or (3 dimensional) spacial area called a cube. A cube is the basic or standard unit of measurement for volumes or spacial areas. For this example, if you find that there are 50 grains of sand lined-up per ("in a", "in one", "for", or "equivalent to") half and inch = 0.5in, then there are (50 x 2) = 100 grains of sand lined-up per 1 inch. There are then (100 x 100) = 10,000 grains of sand per square inch, of course, this assumes that the each grain of sand is roughly the same (average) size in the length and width dimensions, if they were not, you would have to first count and then estimate or average how many are actually in the length and-or width dimensions. Continuing, there are (100 x 100 x 100) = 1,000,000 grains of sand per cubic inch, again, assuming each grain of sand is roughly the same height.

weight of 1 grain x 1,000,000 = weight of 1,000,000 grains mathematically:

$$\text{weight of 1 grain of sand} = \frac{\text{weight of 1,000,000 grains}}{1{,}000{,}000}$$

:after dividing (each side of the equation) by 1,000,000

If this cubic inch of sand weighs 3 ounces, that is:

weight of 1,000,000 grains of sand = 3 oz: again, dividing each side by 1,000,000:

$$\text{weight of 1 grain of sand} = \frac{3\text{ oz.}}{1{,}000{,}000} = 0.000{,}003\text{ oz.}$$

: 3 millionths of an ounce

A note on how similar atoms of a given element are typically counted. Since there are so many atoms in a given mass (a measurement or measure of actual, physical matter) of atoms, and which therefore cannot even be seen so as to be counted by a person, the number of atoms in a given mass are counted by the weight of that given mass. This is because, for each gram of mass of a certain given element, there is a known corresponding weight and amount of atoms.

Summarizing this discussion, many identical things of small size and-or weight can be easily counted by using a calculation with their total weight and-or their total volume. Shown here is a more advanced note due to its relevance with this topic: A quantity of atoms of a certain element can be counted by knowing their total weight, and which is directly proportional to the total amount of mass of each atom, hence total mass of a quantity of atoms can be calculated by knowing the total weight: At twice the weight there will be twice the mass, and twice the number of similar objects such as rice, and twice the volume.

Weight = force = (mass)(acceleration = gravity = g). Mathematically: mass = weight / g.

The number of atoms can be calculated as: (total mass of the atoms) / (mass / 1 atom) = atoms

SOMETHING ABOUT INFINITY

Infinity (ie. "non-finite", "non-ending", "endlessly", "without end") does not, and cannot, have a specific value, and is rooted in a concept of something that is, and can be or conceived (but perhaps not necessarily easily provable) to be as possibly and probably unending. Outer space is not a process, but rather a thing, and is said to be infinite in size, "unending" and can therefore be without any specific measurement or value of a size or distance. A process can be repeated without end, hence that process is said to be an infinite process or simply infinite. If there will always be a "next second" of time, or at least in thought or theory, time can therefore be said as being something that is infinite. Something, even when counting or using numbers, can be infinitely large (without bounds) or infinitely small (almost 0, or almost nothing). The digits of Pi (3.14**...**), that is mathematically part of all circles, are said to be unending and therefore infinite. Some conceivably infinite processes may have just recently began, such as some machine that just started counting higher from an initial value of 0, and hence an infinite process may have not always been in existence, and hence it was not always infinite in some way, but perhaps infinite in only one kind of way, direction or ability now.

Ex. Since 1 can always be added to a number to create or find the next larger number, this is an infinite process, and here, the sum is also infinite in value:

0 + 1 + 2 + 3 + 4 + 5 + **. . .** : an infinite process of summing, and here the sum is not any specific value, but is an unknown value that could be described as infinitely high and-or unending value

Below are some more notes and ideas about the topic of infinity to perhaps consider and inspire some further thoughts, and perhaps may flare the temper of a few strict mathematical purists. If you are new to math, you may simply skip over this topic without any problem, and return when you are able.

When the divisor of a division problem gets smaller and smaller (said as "approaching 0", "becoming near to 0", "nearing 0", "approaching a limit of 0", but never actually being equal to 0 because that would mean the divisor and a real valid division problem no longer exists, and also that division by 0 is undefined or an error) you could conceive that the quotient can possibly have an aspect of infinity (its mathematical symbol is: ∞; which somewhat looks like the decimal number 8 symbol tilted on its side), and rather than speak in terms of theory, its' (largest possible, or conceivable) numeric decimal system integer value can be practically conceived or imagined as a never ending series of 9's or: 9999 **. . .** , even though infinity is a unending process and not a specific numeric value. The **. . .** ("ellipses" = "and so forth") symbol essentially indicates to repeat the process unendingly. Sometimes, when a result in an unending process is sufficient or acceptable, the process can be stopped.

There is also no limit to how small something can be. Things and-or numbers such as this are said to be "endlessly or infinitely" small. The smallest numeric value can be conceivably expressed as a process of an endless number of zero's between the decimal point and the number 1 such as: 0.00000 **. . .** 001 , or using other expressions such as 0.1^n, or 1^-n, as integer (n) approaches infinity. Consider when the divisor of a division problem gets larger and larger, and-or the dividend gets smaller and smaller, the quotient will become (infinitely or endlessly) smaller and smaller.

It is widely accepted now that 0.9999... = 1 since 1/3 + 1/3 + 1/3 = 0.3333**...** + 0.3333**...** + 0.333**...** = 0.999**...** = 3/3 = 1. If it can be (maybe controversially) conceived that 0.3333**...** can be represented as 0.3333**...**3, then it should also be conceived that 0.0000**....**1 can be thought of as that infinitely small number, something or part (here represented with common decimal, base 10, notation) that when summed to 0.9999**...** will mathematically express and cause that value to "finally roll over" and sum to 1. Note also that whereas 0.9999**...** = 1, we now likewise conceive that 0.0000**....**1 = 0 since it will approach and effectively be a value of 0. Some will argue that 0.333...3 and 0.000...1 have or express a "last value or digit", and hence, the notation does not represent or express an infinite value, but by using the **...** symbol, the infinity concept still exists, and therefore, the "last digit" argument does not hold since there isn't a last digit due to the infinite process. Though 0.9999... will approach 1 which is a finite value, 0.9999... represents an unending a process to approach and reach that finite value

For some extra ideas on the above concepts of infinity, there are two types of infinity, and it's best to think of them as a non-ending process without any specific numerical value. The largest (decimal system) number or the number that could represent the infinitely big could be written as: (9999...9) The number that could represent the infinitely small could be written as: (0.0000...1) Why use the 1 digit at the end? Because in our number system we begin counting with 1, and therefore, the conceivable number, expression or notation that represents the infinitely small should include this first digit that is at least greater than 0 or nothing, and using 1 indicates that there is "something" and numerically represented. The two closest numbers next to 1, one being smaller, and the other one being bigger can be conceived or imagined as: (0.9999...9) and (1.000...1). Their conceived difference is (0.0000...2) which is larger than the conceivably smallest possible decimal (infinite) number (0.0000...1) greater than 0. Some would argue that (0.9999...9) is the same as 1, but that is like saying (1.0000...1) is the same as 1, and then that is like saying (0.9999...9) is then equal to (1.0000...1) which, as just mentioned above, has a difference between them of (0.0000...2), and not 0.0 as when they are actually equal. Still, you could argue that (0.0000...2) is an infinite process and that it will also approach a value of 0. It is fair to equate 0.9999....9 to 1.0 since in theory, there is to be no empty space between two adjacent points and no other number between two adjacent numbers, and even though 0.999....9 is expressed as (an infinite) process and not a specific decimal number, and which rather "zeros in" or converges to 1.

THE ORDER OF OPERATIONS

The order of mathematical operations is a commonly accepted and basic plan for simplifying and solving mathematical equations. It is accepted and used so that there is a formal-like consistency and less chance of misunderstandings and error. The plan states which operations of an expression should be performed first or have precedence (ie., more importance or priority). Due to this, it is also a consideration and guide for expressing equations so as they may be correctly solved by someone who is observing the order of operations plan. The 4-step plan is given below in steps of precedence.

1. Simplify powers, roots, and logarithms.

Ex. 5^2 = 25 : a power example, , here: "(a base of) 5, "raised" to the second (2) power" or "5 square"

Ex. $\sqrt{25} = 5$: a root example, , here: "the second (2) or square root of 25"

Ex. $\log_{5} 25 = 2$: a logarithm example, here: "the log of 25, using a base value of 5, is 2" , 5^2 = 25

The concepts of powers, roots, and logarithms will be covered further ahead in this book. A logarithm expression is simply an expression for the mathematical operation to find or solve for the exponent of the base value of an indicated power value.

2. Simplify expressions that are in grouping symbols.

Grouping symbols surrounding an expression indicate that the expression is to be possibly simplified first before any other mathematical operation can be performed on the resulting (and usually simpler) expression or value. Grouping symbols are often used to force precedence and-or to bound and clarify an expression so as to prevent a misunderstanding and error. If there are other grouping symbols contained within or between grouping symbols, sometimes this situation or expression is said as having "nested" or "inner" grouping symbols, the expressions within them are simplified first and from the innermost to the outermost (ie. inner to outer). That is, each corresponding pair of grouping symbols will be successively cleared or canceled (canceled-out or removed) if possible. Hence, this step may also be called: **Cancel or clear grouping symbols.**

Some examples of grouping symbols are:

( ) : parentheses and other similar shapes such as brackets: **[ ]**

---- : division symbol or fraction symbol
Here, the dividend or numerator represents one value, and the divisor or denominator represents one value. That is, either or both might be expressions that can be simplified first, and then the division operation can be performed so as to produce a quotient.

3. Perform multiplication and division.

This is done (for conformity to avoid any confusion or ambiguity) from the leftward side to the rightward side of the expression. Usually, performing division has slightly more precedence than multiplication since it contains the division grouping symbol. Performing the division will effectively clear that grouping symbol, producing a quotient value which can then be possibly used as an operand for another operation.

4. Perform additions and subtractions.

Like multiplication and division, these are also done from leftward to rightward in the expression.

Note that for the second step, simplify expressions in grouping symbols or clear grouping symbols, that steps 1,3, or 4 may be needed first, during, or "along with" for this process.

It is possible to express the concept of the order of operations using just a few words such as: "perform the most advanced operations first". Another plan to simplify expressions is what I call the CDC plan which is closely related to the plan given above, and will be presented further ahead in this book. Below are some basic working examples of using the order of operations.

Ex. 5 +1 x 7

Without following the order of operations method, and simplifying or solving from left to right, 5 would first be added to 1, and then this sum would then be multiplied by 7. This is how some "chain-logic" (evaluated left to right, as written, entered or processed) "home-use" or "4-function" calculators, and even some outdated or simple computers, would evaluate the expression, but for conformity, we are encouraged to use the order of operations from now on that considers the (order of) precedence or importance of the mathematical operations, rather than their (left to right expressed) order.

Without following the order of operations:

5 + 1 x 7 When explicitly expressed and evaluated as: (5 + 1) x 7 using "chain-logic" :
6 x 7
42

Following the order of operations:

The 1 would first be multiplied by 7, and then this product will be added to 5. The result is very different than the last result even though both original expressions are the same. This method is the preferred method used by "algebraic logic" or "scientific" calculators and computers to evaluate an entered expression.

5 + 1 x 7 is evaluated as, using the order of operations: = 5 + (1 x 7)
5 + 7
12

To achieve this same result using a chain-logic calculator, evaluate the higher-ordered or most advanced operations of the given expression first. If you want, you can also rewrite the expression with some grouping symbols to clarify how it is to be evaluated (processed, simplified or solved):

5 + 1 x 7 by using some grouping symbols:
5 + (1 x 7) or perhaps by switching the two operands of the addition operation:
(1 x 7) + 5
7 + 5
12

When an expression inside grouping symbols is simplified to a single value, it is possible and desirable to remove (clear or rid) the grouping symbols that surround it as long as it does not change the intended order or operations and therefore, the result.

Ex. 5 + (2 + 3)

5 + (5) : after using step 4 and performed the addition operation expressed within the grouping symbols
5 + 5 : step 2, basically cleared/rid the grouping symbol that are not needed now
If you want, you can consider 5 + (5) = 5 + 1(5) = 5 + 5 , so as to remove the grouping symbol
10 : step 4, addition

Ex. 5 + 3(3(7x4)) + 2

5 + 3(3(28)) + 2 : step 3, after simplifying the innermost grouping symbols, here (7x4) = 28
5 + 3(84) +2 : step 3 and 2, multiplication (to clear grouping symbols), here 3(28) = 84
5 + 252 + 2 : step 3 and 2, using multiplication which then clears the grouping symbol
259 : step 4, using addition, here, the order of the summing of the values, does not matter

Ex. Simplify 5 + 3 x 10

Since multiplication has more precedence than addition, this expression is usually written or expressed, and simplified as:
5 + (3x10) : for some basic home use calculators you might have to use these steps in the order of: (3 x 10) + 5

Here, the grouping symbols clearly indicate or "force" a precedence of performing the operations. That is, here 5 is to be added to the entire value enclosed in the grouping symbols. This value within the parentheses is an expression which must be simplified first whether you are using a calculator or not:

5 + (30) : after performing step 3
5 + 30 : after performing step 2
35 : after performing step 4

Ex. $5 \times \frac{10}{2} + 3$

or $\left(\frac{5}{1} \times \frac{10}{2} \right) + 3$: due to the precedence of multiplication, a grouping symbol is shown here for visual clarification

(5 x 5) + 3 : after performing step 3, (division, here, 10 divided by 2)
25 + 3 : after performing step 3, (multiplication), then step 2
28 : after performing step 4, addition

The more you practice with the simplification or "simplifying" expressions, and the order of operations, the more "natural" it will feel and become to you.

COMMUTATIVE AND ASSOCIATIVE LAWS

Multiplication and addition are said to be commutative (based on the word "(to) commute" which means "moveable", "changeable", "transportable"). That is, if you exchange or "switch" the operands (values the operators are to be applied to) of the operation, the result will still be the same. This may be done to help clarify, express, or evaluate an expression.

5 + 2 = 2 + 5
Ex. 5 + 2 = 2 + 5 : * * * * * + * * = * * + * * * * * : somewhat like a mirror image.
7 = 7 * * * * * * * : both have the same sum.
7

Ex. 4 x 9 = 9 x 4 : maybe you don't know 4 x 9 offhand, but perhaps you have already memorized 9 x 4
36 = 36

Multiplication and addition are also said to be associative (to associate, join, pair, connect, align or group with). That is, you can associate any two or more operands together by grouping (usually with parenthesis; ( ) , or backets; [ ]) and the result will be the same.

Ex. 5 + 2 + 3 = (5 + 2) + 3 = 5 + (2 + 3)
10 = 7 + 3 = 5 + 5
10 = 10 = 10

Ex. 2 x 4 x 3 = [2x4] x 3 = 2 x [4x3] = (3 x 2)(4)
24 = 8 x 3 = 2 x 12 (6)(4) = 6 x 4
24 = 24 = 24 24

FRACTIONS

A part is less-than the whole of which it comes from. The whole thing is "100%", all, complete, entirety or "one" (oneness and numerically 100%=1, or the undivided something). When only a part or parts of a whole are to be expressed mathematically, fraction or fractional notation (simply called "fractions") is used to numerically represent this part or fractional value of the whole. A fraction of something is a piece, portion, segment or part of that whole something that was essentially broken or divided into parts or pieces. The part(s) in question is said to be a fraction, or fractional part of the whole. A fraction or part is numerically represented as a value that is likewise less than the whole (or entire) thing or amount expressed numerically. If we let 1=1.0=100% numerically represent the entire thing, then a value less than 1 will represent a lesser or smaller part(s), portion, or fraction of that whole thing.

Symbolically: (fractional part(s) of the whole part) < (whole part) : A fractional part(s) is less than (<) the whole part of which it was from.

Numerically: (value(s) or sum of fractional or part values is less than 1) <= 1.0

Ex: $0.5 < 1.0$: Here, a part numerically represented as 0.5 is less than the whole (1)
Ex: $0.5 + 0.5 = 1$: Here, the sum of two parts numerical representations sum to 1.
Ex: $0.3 < 1.0$
Ex: $0.3 + 0.5 = 0.8 < 1$

We can often recognize that a given part is only a fractional part of something. For example, 5 is only a part of 8. 5 is the part or portion of the entire, all or whole value of 8. 5 is different and less than 8, and this can be mathematically verified by subtracting the two values to show that there is a difference in values: 8-5 = 3. 3 is the difference or amount of separation between those two values of 5 and 8. For example, 200 is only part of 203. 200 is a fraction of 203, and their difference is 3. Rather than express a numerical relationship of two values as a difference, a better way to quantify and express their numerical relationship is to show how many times bigger or smaller one value is to the other (reference) value. The resulting value will show the decimal equivalent value that the part is with respect to the whole which can be expressed numerically as 1.0 (= 100% = all, entire, whole). 5 of, from, or out of 8 and expressed as a fractional expression is: 5/8 , and this value = 0.625 after division is performed , 200 out of 203 = 200/203 = 0.985 Even though the difference was 3 for each example, this difference value of 3 does not numerically represent how much the part or portion of the whole is numerically. 3 doesn't quantify the portion, but rather only the difference. The values of 0 .625 and 0.985 just shown do. The value of 0.985 is much more than 0.625, and is therefore a larger fractional part. 200 is therefore a larger part or portion of 203, than as 5 is to 8. Since 0.985 is close in value to 1, it indicates that the part or portion in question is almost the entire value in question. 8 or 8 parts = 8/8 , or 203 of 203 parts = 203/203, would result that the fractional value in question can be represented with and has a value of 1, and this indicates that the fractional value in question is equal to the entire, whole part or value.

We have seen that quantifying and expressing the numerical relationship of a part to the whole by using the difference value does not numerically represent how much that the fractional portion is of, represents, or how much it is of the entirety or whole part, but it rather just expresses or indicates how much different, and not the similar, the two values are. As just seen in the two examples above, the difference may be the same, and yet the portions of the whole are much different. A difference value does not quantify or numerically express a portion value. A quotient value can quantify or numerically express a portion. The mathematical operation will be same as a division problem, but the concept and terminology is different since we are not actually trying to divide any parts up into even smaller parts, but rather expressing just the numeric relationship of one part in relation to, or with respect to, another part or the whole part.

The basic mathematical (standard) "formula" or expression for a fractional value is a numerator value divided by a denominator value.

$\frac{\text{numerator}}{\text{denominator}}$: The word numerator is from the word numerate, meaning to count or assign a number to or of.
: The word denominator is from the word denominate. Nominate means to assign a name to, and here it's a number name or identifier. To denominate is to completely assign a name,

hence denominator means the complete number to or of. The numerator number used is in reference to the denominator number used.

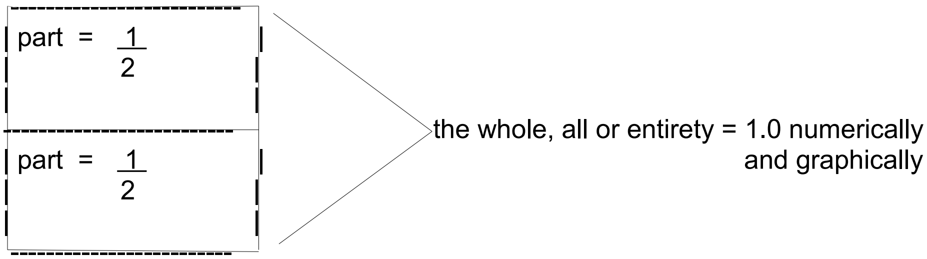
You can roughly think of this as: $\frac{\text{parts}}{\text{total parts}}$: "parts to total parts" or "parts of the total number of parts"

We see that this is essentially the same as a division expression and problem, except for the names used. Comparing the two types of expressions, the numerator is similar to the dividend, and the denominator is similar to the divisor. When something is divided, such as into (fractional) parts, it is somewhat obvious that a division type of expression will be used to express this.

If a whole (1) thing or quantity is evenly split or divided (a division problem) to 2 equal parts, we can represent this numerically as this division expression:

$\frac{1}{2}$: this also expresses the numerical representation or value of each similar (fractional) part. This numeric expression is therefore called a fraction. After dividing, the quotient is called the fractional (in decimal number form) value. Each identical fractional part is equal in value, here, (1/2) = 0.5

The numerical value of, or given to, each separate (and identical) part is the value of this fraction (but only when a whole or 1 is divided). Each part is said to be "one-half" of the whole. Numerically, each part has a value of (1/2), or the equivalent decimal value of 0.5, when the 1 is divided by the 2. 0.5 is clearly less than the whole of 1. [FIG 4]



The sum (mathematical, and-or physical combining) of all the fractional parts will always equal that which was divided into those fractional parts. For the example just shown:

part + part = whole, all or entirety that was divided up into two parts
(1/2) + (1/2) = 1/2 + 1/2 = 0.5 + 0.5 = 1.0

Here is how to express the whole (1) of something divided into 4 equal sized parts: $\frac{1}{4}$

Each part will also have this numeric value of (1/4) or= 0.25 Considering just two of these four parts taken, put or summed together as a group, then what fractional part of the whole does this represent? Clearly, since two parts is twice as much as one part, the fractional value must be twice as much:

one-fourth and one-fourth equals two-fourths or: 2 x one-fourth = $2 \times \frac{(1)}{(4)} = \frac{2(1)}{4} = \frac{2}{4}$: "two-fourths"

$\frac{1}{4} + \frac{1}{4} = \frac{2}{4} = 0.50$

0.25 + 0.25 = 0.50 : or= 0.25(2) = 0.50 since repeated addition can be represented as multiplication

Though it is possible, do not interpret the value of 2 in (2/4) as meaning 2 wholes being divided up into 4 parts, since for this problem we are dealing with only one whole, or entirety of a thing, numerically expressed as 1, that was divided up. So therefore, a numerator can be defined as the number of (all similar) parts in question or being considered of the whole, and a denominator is said to be the total number of parts that the whole was evenly divided up to. You can visually read the mathematical notation of the fraction symbol as meaning "of", "out of" or "from".

Ex. $\frac{1}{2}$: "one, out of two" or "one part, out of two parts" or simply as "one, out of two" or "one, of two" , or "one-half" = "one-half of the whole" = (1/2) of (1) = (1/2) of (1/1) = (1/2) (1/1) = (1/1) (1/2) = (1) 1/2 = 1/2

Ex. $\frac{2}{6}$: "2, out of 6" or "2 parts, of something divided or separated into 6 total parts" , This can be considered as equal to: (1/6) + (1/6) = 2 (1/6) = (2/1) (1/6) = 2 / 6 and-or = 1/3 = 0.333...

If given a fraction and you perform the indicated division, the resulting quotient is the decimal equivalent of that fraction.

Ex. $$\frac{1}{2} = \begin{array}{r} 0.5 \\ 2\overline{)1.0} \\ -\underline{1\;0} \\ 0 \end{array} = 0.5$$

Below is another ex. where the numerator is half the denominator, and that the total fractional part(s) in question is also numerically 0.5 = 50% of the whole or denominator:

Ex. $$\frac{3}{6} = \begin{array}{r} 0.5 \\ 6\overline{)3.0} \\ \underline{30} \\ 0 \end{array} = 0.5$$

: checking, 6 x 0.5 = 3

Ex. $$\frac{4.5}{6} = \begin{array}{r} 0.75 \\ 6\overline{)4.50} \\ \underline{42} \\ 30 \\ \underline{30} \\ 0 \end{array} = 0.75$$

: Compared to the last example, this larger quotient indicates that the portion or fraction is larger. As the numerator comes closer and closer in value to the denominator, with little difference, the fractional value will approach (become nearer and nearer in value to) that of the whole or value of 1=100%. The closer this value is to 1, the fractional part in question becomes greater and nears the whole part. For this example, the part(s) in question could be said to be 0.75, or 75% of the whole (or entirety, entire) part. The portion of the whole part is 0.75.

Ex. $\frac{5.4}{6} = 0.9$: 0.9 is close in value to 1.0, and it indicates that 5.4 is a large part or portion of the whole (100% = 1) or entirety of 6 parts.

Ex. Of the following fractions, which fraction is larger, that is, which represents a larger value and hence a larger portion of the whole, all, or entire thing?

$\frac{2}{3}$ or $\frac{3}{5}$ after performing division to find their decimal equivalents we can often compare them more easily:

0.66666... and 0.60

Therefore, the part, portion or fraction of 2/3 is (slightly) larger than 3/5. 0.666666 > 0.60 and (2/3) > (3/5)

If the denominators of the fractions are the same (or mathematically made to be the same as done when adding fractions), you can then easily find the largest fraction simply by observing which numerator is the largest. Actually, one quick method of adding fractions is to first convert each of them to their decimal equivalents value, and then perform the addition. Though the addition of fractions has not been covered yet, here is the above example solved with the "addition of fractions with the same (or "like") denominator method" so as to find which fraction is larger:

$\frac{2}{3}$ and $\frac{3}{5}$ Converting each fraction to an equivalent (valued) fraction (numerically, it will still have the same value) where the denominators of both fractions are identical or the same:

$\frac{2(5)}{15}$ and $\frac{3(3)}{15}$: both denominators changed to 15, hence the fractions can be visually compared easily

$\frac{10}{15}$ and $\frac{9}{15}$: Note 2/3 = 10/15 = 0.66666... and 3/5 = 9/15 = 0.6
These are the same values as shown above, and this is because these new fractions created from or using each are (mathematically, in value) equivalent fractions. It is also incorrect to think that a fraction can only have one equivalent fraction associated with it since there is actually an infinite number of equivalent fractions possible for any fraction or portion of the whole (1).

Now by just observing the two numerators in the fractions above, since 10 is obviously greater than 9, then 2/3 is the larger fraction.

Given any two fractions and knowing the process of converting each of those fractions to an equivalent valued fraction, and both with like denominators, observe that you can quickly see which is larger by what is called "cross multiplying" the numerators and denominators of the two fractions. For the give example, since (2)(5) = 10 is larger than (3)(3) = 9, and both of these values will have the same denominator (here 15 which is the product of both denominators), the fraction of 2/3 is therefore larger than the fraction of 3/5. Another method, perhaps not so obvious, to find which fraction is larger, is to subtract one fraction from the other. You can first convert both fractions to their decimal equivalent, or to fractions that have identical denominators so they can then be properly subtracted. Assuming both fractions are greater than 0, if the resulting value (the difference) is greater than 0, then that minuend fraction is larger than the subtrahend fraction.

This book tries to avoid the (weak, but practical) concept of "cross multiplying" because, it is often taught, accepted and-or taken for granted, without first knowing the mathematical verification as to why it can be used. The "just do this and you will have it" attitude, should in many cases, be unacceptable when you want to have a firm grasp or understanding of math and what you are doing with it, and so as to then understand more advanced concepts that are built upon the previous concepts.

The whole part, all, or the entirety of something can be thought of as, or represented by, a complete circle which can represent and be noted as equivalent to and expressed as any numeric value. Given a certain or specific fractional value of any entire value (entirety, all, whole), each part, portion, or fraction is numerically the same regardless of the specific whole or entire value being considered. For example, given a fraction of (1/3), or "one-third" of any value, (1/3) = 0.33333... =33.3% of that whole=1=100% value or circle. The actual numeric value corresponding to that indicated fractional value of (1/3) depends on what the whole value actually is. A portion or fraction of something is always less than the whole or complete value, however, the actual numeric value of that portion may be less than or greater than 1. For example, consider this:

Whole, all, entirety or complete value	Indicated fractional value.		Actual numeric value of the indicated (here, (1/3)) fractional value of the whole.
1.0	(1/3) of 1	=	1/3 = 0.33333... = whole part divided by 3 = (whole part)/3
1.5	(1/3) of 1.5	=	1.5/3 = 0.5 : an example showing that the "whole part or entirety, may not always be an integer or counting number value, and may be greater than 1

2	(1/3) of 2	= 2/3 = 0.66666...	: note that this is the same as having (1/3) and (1/3), or (1/3) + (1/3) which is (2/3) = ""two-thirds" when summed: (1/3) + (1/3) = (0.33333...) + (0.33333...) = 0.66666... = (2/3)
3	(1/3) of 3	= 3/3 = 1.0	
4	(1/3) of 4	= 4/3 = 1.3333...	: here we see that the actual "fractional value" or portion of 4 is actually larger than 1
5	(1/3) of 5	= 5/3 = 1.66666...	
6	(1/3) of 6	= 6/3 = 2.0 = 2 parts of 6 , which can be expressed as: 2/6 = 1/3 2 parts of 6 equals 1 part of 3 = 0.33333**...**	

MIXED NUMBERS

A "mixed number" is an expression of a numerical value that (explicitly) indicates both the number of complete or whole parts (hence a multiple of a whole, completes or entirety), and a fraction value that expresses the parts or portion of just one whole value.

mixed number = whole part + fractional part

Mixed numbers are often found in food recipes, such as for example, to indicate "two and a half cups of flour" or= "two cups plus a half of a cup", and the numeric representation of this using a mixed number method is:

2 1/2 cups of flour, or perhaps just as: 2 1/2 cups flour : 2 is the whole number of filled or complete cups or cup units, and 1/2 is a fraction of 1 cup full.

If the decimal value of a fraction is less than one, the fraction is said to be a proper fraction. When the numerator is larger in value than the denominator of a fraction, the fraction is said to be an improper fraction since it represents a value greater than one, (1), since at the fundamental level, fractional numerical values are usually meant as, or understood to be less than the whole, or 100% of something = 1 numerically. A fraction is (usually) meant to be only a (smaller) part or portion of something. A mixed number has both a whole part and a fractional part, hence it has a mix or several values.

First consider values such as 0.1 and 0.95. Any value that numerically represents a part that is less than the whole (entirety, all, 100% = 1.0) is only a fractional value, part or fraction. 10 by itself is not, or does not represent, a fractional part, however when you (only) have 10 of 100 entire things, which can be expressed numerically as 10/100, that value of 10 is now only a fraction or part of the whole (considered as 1=100%) or entire 100 things. Expressing this in the most basic decimal form, we see that 10/100 = 0.10, which clearly is less than 1, and hence it is only a fractional part of the whole. It can be said that "10 is (only) a fractional part of 100".

Ex. $\frac{9}{4} = 2$ with a remainder (of the dividend) of 1 : 9/4 is an improper fraction

Expressing the above quotient, of the indicated division, as a mixed number (containing both a whole value and a fractional value of the divisor), this is written as:

$2\,\frac{1}{4}$ which actually means: $2 + \frac{1}{4}$:"two and one fourths". This is a mixed number. 2 is the whole part, and 1/4 is the fractional part. The addition of both parts, is not always explicitly indicated and is "understood" as being so.

That is, 2 is not being multiplied, but added to $\frac{1}{4}$: If 2 was being multiplied to 1/4, the multiplication would have to be explicitly indicated such as: 2(1/4), or = (2)(1/4), or = 2 x (1/4)

Also note that when division is performed on the fractional part of a mixed number, we get the decimal equivalent of the mixed number. The value will also equal the improper fraction:

$\frac{9}{4} = 2\,\frac{1}{4} = 2 + \frac{1}{4} = 2 + 0.25 = 2.25$: 2.25 is like a "mixed number" that contains a whole part and a fractional part.

Now that improper fractions can be converted to mixed numbers, mixed numbers can also be converted back to improper fractions. To do this, multiply the whole number part of the mixed number by the denominator of the fractional part of the mixed number, and then add the resulting product to the numerator of the fractional part of the mixed number. The resulting sum is a numerator of a fraction whose denominator is the same as used for the fractional part of the mixed number.

Ex. $3\frac{2}{4} = \frac{(3 \times 4) + 2}{4} = \frac{12 + 2}{4} = \frac{14}{4}$

The conversion process shown above is actually a "short-hand" method of adding fractions correctly of which could verify or check the method and example shown above. For the last example, the fractions to be added, and their sum, would be:

$$\frac{3}{1} + \frac{2}{4} = \frac{14}{4}$$

Hence, a mixed number represents an expression that is the sum of the number of whole (or entire) parts, and fractional parts. Note that 3/1=3. Any value can be divided by or "placed over" or divided by one, and the value will not be changed. But as mentioned previously, the fractions cannot be summed (added or "combined") until their denominators are equivalent. When the denominators of the fractions are the same, the fractions are said to have like or common denominators, and this is much like that which is necessary when adding things such as measurements that need to have the same units (of measurement or reference) so as to produce a result having those same units. Fractions with the same denominators can be said to be "like fractions". More on the addition of fractions will be shown later.

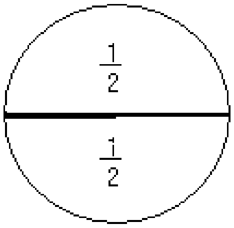
As another check to the above example, first consider that any value divided by itself it 1: (3/3) = 1

$$3 + \frac{2}{4} = (1 + 1 + 1) + \frac{2}{4} = \left(\frac{4}{4} + \frac{4}{4} + \frac{4}{4}\right) + \frac{2}{4} = \frac{12}{4} + \frac{2}{4} = \frac{12 + 2}{4} = \frac{14}{4} = 3.5$$

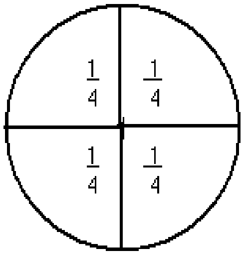
Note also, and previously discussed, that: $3 + \frac{2}{4} = \frac{(3 \times 4) + 2}{4} = \frac{14}{4} = 3.5$

EQUIVALENT FRACTIONS

If you divide a given circle into two equal parts that are the same size or portion, say an upper and lower half, each part is said to be one-half, or numerically as (1/2), of the entire 1 circle. Each part and fraction that numerically represents these parts are equal and therefore they are called equivalent fractions. Here, each part is numerically: (1/2). [FIG 5]



The 1 in the numerator position of the fraction represents one part in question or being considered, and the 2 in the denominator position indicates that the whole (here a circle) was divided into 2 equal parts. If you divide a whole (1) circle into 4 equal parts, each part is said to be one-fourth of that whole (1) circle. Numerically, each part is: (1/4). [FIG 6]



Clearly, by observation, two parts (one-fourth and one-fourth, or= two-fourths) of the circle that was divided into 4 equal parts is equivalent to the portion of one part (here, one-half) of the circle that was divided into 2 equal parts. Numerically, that is:

$\frac{2}{4} = \frac{1}{2}$: both fractions are equal in value, here it's 0.5 .
: "2 is to 4, as is 1 is to 2"

Each fraction has a value of 0.5 which means its portion is 50% of the (entire 100%) circle. Each part is half (= 1/2) of the circle, and even if the circles are of different sizes, because the portions of the circle(s) being referred to are mathematically the same fractional value of 1 circle, regardless of its actual size. A portion or percentage value is a basic generalization that is in reference to any whole or entirety regardless of its actual specific sizes or values involved. Equivalent fractions represent the same portion or fraction of something.

2/4 comes from the fact that there is 2 parts in question of the circle that was divided into 4 equal parts. Actually,
1/4 + 1/4 or (1/4) x 2 = 2/4 = 0.5 of 1 = (0.5)1 = 0.5 Also: (1/4) + (1/4) = 0.25 + 0.25 = 0.5

The fractions of 2/4 and 1/2 are said to be equivalent fractions since they represent the same quantity or value (here it's half or 0.5 of the circle, hence the same portion). The fraction of 2/4 has larger values for both the numerator and denominator, and it is said to be of "higher terms", that is, it has a higher numerator and denominator. Likewise, the fraction of 1/2 is said to be of "lower terms" since its numerator and denominator are lower in value than the other fraction. Still, the numerical values of both fractions represent an equal part of the circle. Actually, to make any equivalent fraction, simply multiply or divide both the numerator and denominator of a fraction by the same value. For example, a quick and simple way to find the next higher term equivalent fraction of 2/4 can be found by multiplying the numerator and denominator by the same value, such as for example, 2:

$$\frac{2}{4} = \frac{2 \times 2}{4 \times 2} = \frac{2(2)}{4(2)} = \frac{4}{8}$$

hence, $\frac{4}{8} = \frac{2}{4} = \frac{1}{2}$: = 0.5 , equivalent fractions, each represents 0.5 of the whole (1)

Given these numeric values, it can be said that 4 parts of a circle divided into 8 equal parts represents the same portion of a circle as 2 parts of a circle that was divided into 4 equal parts.

The fraction of 1/2 for the group of all possible equivalent fractions shown is said to be the fraction with the lowest terms. Also for this fraction of 1/2, since both the numerator and denominator cannot be evenly divided by any whole number greater than or equal to one, it is in lowest possible terms of all other equivalent fractions.

As a simple verification that they are equivalent fractions, consider that multiplying any value (including fractional values also) by 1 will not change that value. If any fraction is multiplied by say (2/2)=1, that is, the numerator and denominator are both multiplied by the same value, it is equivalent to multiplying by 1, and we know that multiplying any value by 1 does not change that value.

since $\frac{2}{2} = 1 = \frac{1}{1}$, multiplying this to any value, the value will not change:

$\frac{2}{4} = \frac{2\ (2)}{4\ (2)} = \frac{4}{8}$: the fractions look different (in the numerators and denominators), but they are numerically equivalent in value, here it's 0.5

Reversing this process, if we divide the same value from both the numerator (num.) and denominator (den.), the resulting fraction is still an equivalent fraction. For example:

$\frac{2}{4} = \frac{2/2}{4/2} = \dfrac{\frac{2}{2}}{\frac{4}{2}} = \frac{1}{2}$: after simplifying the num. and den. fractions

Again, a way to show or verify that these are equivalent (equal) fractions (of the whole) is to convert (represent) each to its decimal form by performing the indicated division.

$\frac{4}{8} = 0.5$, $\frac{2}{4} = 0.5$, $\frac{1}{2} = 0.5$:The fractions must be equivalent since each represents a value of (0.5).

Ex. Show that 2.00564 / 7 and 114.32148 / 399 are equivalent fractions or portions.

By using a calculator to perform the indicated division so as to find their quotient values:

$\frac{2.00564}{7} = 0.28652$ and $\frac{114.32148}{399} = 0.28652$: they are shown to be equivalent fractions numerically Each fraction represents a portion or total (sum of portions) that is about 28.7% of the whole (100%) or entirety.

There is another, less used method to create equivalent fractions described in the BASIC ALGEBRA section of this book.

Sometimes it could be helpful to express a given fraction as an equivalent part or fraction of 1, or as an equivalent fraction of 1 part out of a total number of parts. Doing this can give a basic grasp of the fraction when expressed as a single (1) portion of all the portions, or as a portion of 1.

Ex. 0.5 to 7 $= \frac{0.5}{7}$: 0.5 parts out of 7 parts $= 0.0714\underline{3}$ when expressed as a single decimal value

Equivalent fractions are usually solved by finding the multiplier needed to apply to both the num. and den., but since a value of 1 is used as a value in the equivalent traction being created, it become easier to complete that equivalent fraction.

0.5 is to 7, as 1 is to what? $\frac{0.5}{7} = \frac{1}{x}$: here, x is used in place of the word: "what"

Notice that if you divide the numerator here, by itself, that the result is 1. To keep the fraction the same value, this must also be done to the denominator:

$\frac{0.5}{7} = \frac{(0.5 / 0.5)}{(7 / 0.5)} = \frac{1}{14}$: 0.5 parts out of 7 parts can be numerically expressed as 1 part out of 14, or 1 part of 14

OR:

0.5 is to 7, as what is to 1? $\frac{0.5}{7} = \frac{x}{1}$

In a similar manner to the above, if we divide the denominator by itself, we will have a value of 1, and to keep the fraction as having the same value, we must do the same to the numerator:

$\frac{0.5}{7} = \frac{(0.5/7)}{(7/7)} = \frac{0.07143}{1}$: 0.5 parts out of 7 parts can be expressed as $0.0714\underline{3}$ parts to 1
: This is essentially the same result of simply dividing the num. by the den.

Ex. If a fast runner can run 1 mile in 4 minutes, how many miles an hour did they, or could they potentially run?

First, so as to have like units in the denominators, lets convert the units of hours to minutes:
1 hour = 1 hr = 60 min

$\frac{1\text{ mi}}{4\text{ min}} = \frac{x\text{ mi}}{60\text{ min}}$, since: 60 min / 4 min = 15, and also multiplying the numerator by this value:

$x\text{ mi} = (1\text{mi})(15) = 15\text{mi}$

1 mile per 4 minutes = 15 mi per 60 minutes = 15 mi / 60 min = 15 mi / 1 hour = 15 mi / hour

Another way is to mathematically solve for x mi:

$$x\text{ mi} = \frac{(1\text{mi})(60\text{ min})}{(4\text{ min})} = \frac{(1\text{mi})(\overset{15}{\cancel{60\text{ min}}})}{\underset{1}{\cancel{(4\text{ min})}}} = (1\text{mi})(15) = 15\text{ mi}$$

Ex. A basic and "generalized" (with non-specific units) recipe for hard candy is shown below. It also shows how to make an "equivalent proportions" recipe so as to make a bigger or smaller batch (amount, quantity) of the candy. Even though a batch can be a different size, weight or volume, the percentage, portion or fraction of each ingredient in the new batch will still be the same percentage, portion or fraction as that of each ingredient in the original recipe. It is sometimes noted that since the ingredient amounts used in a new sized batch are the same portion as that used in the original recipe, that the new batch and ingredients are said as being "proportional" (same portions, or "within proportion(s)") to the original recipe and-or batch. This ensures that the new sized batch will still taste the same as that of the original recipe and-or batch.

Main Ingredients: 2 units sugar : here "units" could be cups, or some other possible container
3/4 units corn syrup : 3/4 = 0.75 units used for the measurement(s)
2/3 units water
Other Ingredients: A few drops of flavor oil, and a few drops food color.

Mix the main ingredients in a bowl and then pour the mixed ingredients into a cooking pot. Cook (without stirring) till all the water is evaporated and to a temperature of 300°. Do not let the mixture turn dark yellow-brown and burn. Remove from heat, mix in flavor and color. Pour mixture onto a greased cookie sheet and let it cool. For the purpose of this mathematical example, the actual units of cups for the main ingredients is not explicitly indicated since you should use or expect any possible units for these measurements; perhaps you'll have some homemade units of and for your measurements such as: a jar, or spoon full, etc. Let's say you want to make a a different quantity (more or less) of candy. To do so, you need to adjust all the measurements (amounts of units used) for each ingredient equally by the same (magnification) factor value. Let's say you also want to make a more basic or general ("generalized") recipe so as to make batches of any quantity of candy that is based on a (1) unit of sugar, then for each additional unit of sugar added in, you will simply add in the new "general recipe's" measurement for all the other ingredients. Since there are 2 units of sugar used in the original recipe, you can divide these 2 units by 2 to have a recipe based on just using 1 unit of sugar. To keep the same ratio (ie. relative size amts.), or numerical proportions (consistent portions) of the amounts used for the ingredients in the recipe, you must then also divide all the other ingredient measurements by the same value, here, 2, and this will ensure that they are, and have, the same proportion (proper (same) and consistent fractional or portion value) as that of the original recipe, and that the ratios (essentially the quotient value of their division) of each two ingredients amounts is the same as that of the original Even though the specific quantity of an ingredient is now more or less, it is still numerically the same recipe. fractional or portion value as that of the original recipe. Expressing the basic concept of this mathematically:

$$\frac{\text{units of ingredient A}}{\text{units of ingredient B}} = \frac{\frac{\text{units of ingredient A}}{2}}{\frac{\text{units of ingredient B}}{2}}$$

: still has the same ratio value among , of, or
between these two amounts of ingredients.
Essentially created an equivalent fraction.
Divide the amount of all ingredients by 2.

The ratios, or relative (relational) sizes of one ingredient value to another, will still be the same as that in the original recipe. The total mixture or completed item being made, here candy, will also be divided in half since all of its parts (quantities) were divided in half, or vice-versa.

Dividing 3/4 by 2 we have 3/8 (division of fractions will be explained later), and dividing 2/3 by 2 we have 2/6 = 1/3, and the main ingredients and measurements of the new generalized recipe will become:

Main Ingredients: 1 units sugar For each 1 unit of sugar used, use:
3/8 units corn syrup : 3/8 = 0.375 units which is half of 0.75 = 3/4 units
1/3 units water

Since 1/2 = 0.50, you could have multiplied each ingredient amount by 0.50 instead of multiplying by 1/2 which is also equivalent to dividing by 2 as seen above. For some extra verification of all this, consider:

$$\frac{3}{4} \times \frac{1}{2} = \frac{(3)(1)}{(4)(2)} = \frac{3}{8}$$

: note that 3/4 and 3/8 are not equivalent fractions since the numerator and denominator were multiplied by different values.

$$0.75 \times 0.5 = 0.375$$ also:

If we divide a fraction, say 3/4, by 2 (same as 2/1), it is the same as multiplying both the entire new num. and den. by (1/2) as seen below. This procedure also shows why they are no longer equivalent fractions, since then num. (3), and den. (4) of 3/4 will not actually be multiplied by the same value. In short, dividing or multiplying a fraction by a value does not create an equivalent fraction, however, dividing both the num. and den. of a fraction by the same value does. For example if we have 4/8 = 0.5, and multiply this fraction by 2 or doubling (adding itself) it, we have (4/8)(2/1) = 8/8 = 1 which indicates that this is not an equivalent fraction (0.5 and 1 are not equal) but it is a magnification (ie. growth) of the fraction to where it does not represent the same part of the whole any longer, and is a completely new (different) fractional part or value.

same as multiplying the entire num. and den. by 0.5 = 1/2

$$\frac{\frac{3}{4}}{\frac{2}{1}} = \frac{\frac{3}{4}\;\frac{(1)}{(2)}}{\frac{2}{1}\;\frac{(1)}{(2)}} = \frac{\frac{3}{8}}{\frac{2}{2}} = \frac{\frac{3}{8}}{1} = \frac{3}{8} \qquad : \; = 0.375$$

Checking (note that 3/4 = 0.75):

$$\frac{0.75}{2} = 0.375$$

: as seen above, 3/4 = 0.75, and 3/8 = 0.375 are not equivalent fractions

We also see above, that a quick way to divide one fraction by another is to simply invert ("turn upside down") the denominator fraction and then simply multiply both the numerator and denominator by this value. Here, 2/1 inverted is 1/2. As another example, if you were to divide a fraction by 5/6, you could simply multiply both the num. and dem. of that fraction by 6/5 for the resulting quotient value. This process is also a good way to "clear" or "rid" the fractions, where possible, during a simplification (of an expression) process so as to have just one value (the quotient) to continue working with.

For completeness of the new recipe, note also that (2/3)unit / 2 = 2/6 unit. Dividing this num. and den. by 2, we have 1/3 unit , such as 1/3 of a cup, and which is usually easier to understand and measure in the kitchen, than 2/6 units or "two - sixths" of a cup.

When two or more things are said as being linear in value, they are also said as being proportional in value. That is, when one thing increase by some factor, the other thing will increase by the same rate or factor, and this is much like a magnification and-or a demagnification. This is also very much like an equivalent fraction concept, and can be solved as such. This book will sometimes mention that things such as this can be solved using a "proportional equation" or "equivalent fraction equation". Expressing all this in a pseudo-mathematical analysis and expression for when thing1 and thing2 are said as being proportional, or linear in value(s) and-or magnifications:

$$\frac{\text{thing1}}{\text{thing2}} = \mathbf{n} = \text{magnification}$$

, mathematically: thing1 = (n) thing2
and: thing1 / n = thing2

, $$\frac{\text{thing1}\ (\text{factor1})}{\text{thing2}\ (\text{factor1})} = \frac{\text{thing1magnified}}{\text{thing2magnified}} = \mathbf{n}$$

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ADDING FRACTIONS

Adding fractions is not as difficult as you might think. Actually, we can think that we have always been working with fractions when we were just using whole values. For example, we know that 4+3=7, but we also know that we can make fractions by placing these values over one, which we know will not change their values.

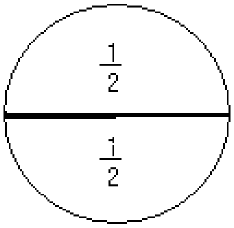
$4 + 3 = \frac{4}{1} + \frac{3}{1} = \frac{7}{1} = 7$: Add the numerators (the 4, and 3) of fractions with like denominators, and keep the denominator. The like denominators here are 1.

Checking:

$4 + 3 = \frac{(4 + 3)}{1} = \frac{7}{1} = 7$: checks

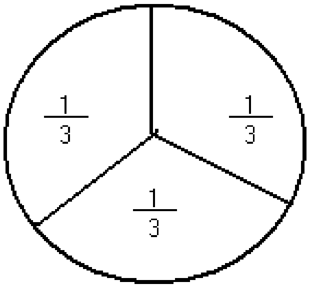
This leads to a general rule for combining (adding) fractions: Add the numerators of fractions with like, same or common denominators, and keep that same denominator. This is commonly known as the process of "adding like fractions".

If a circle (or 1 whole circle) is divided into two equal (ie. evenly divided) parts, each and every single part will numerically be (1/2), and we know that the sum of these parts (all the fractions or fractional parts) should equal to one (the whole circle) or the entire value of which they were taken out from and are in reference to. Since the fractions in question have like denominators (here, 2 or "halfs"), which makes them "like-fractions", simply add the numerators and keep the (same) denominator. [FIG 7]

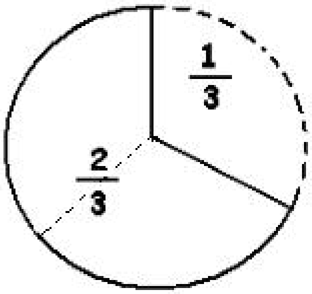


$\frac{1}{2} + \frac{1}{2} = \frac{(1 + 1)}{2} = \frac{2}{2} = 1$: A half (of a circle), plus another half, equals one (1) full circle.

If a circle or some quantity was evenly divided into 3 parts, each part is numerically (1/3). [FIG 8]



Summing up just two of those fractional parts or values, we have: [FIG 9]



$\frac{1}{3} + \frac{1}{3} = \frac{1 + 1}{3} = \frac{2}{3}$: "two thirds of a circle"

If we were to then add in one more of the parts, so as to have all 3 parts in the sum, we should then have one whole circle:

$\frac{2}{3} + \frac{1}{3} = \frac{2 + 1}{3} = \frac{3}{3} = 1$

As a check to this, expressing the sum of all three of the parts:

$\frac{1}{3} + \frac{1}{3} + \frac{1}{3} = \frac{1 + 1 + 1}{3} = \frac{3}{3} = 1$: checks

Also, expressing and performing the repeated addition above , with multiplication:

$\frac{1}{3} \times 3 = \left(\frac{1}{3}\right)\left(\frac{3}{1}\right) = \frac{(1)(3)}{(3)(1)} = \frac{3}{3} = 1$: checks

Ex. Add 4 tenths, 2 tenths, and 1 tenth

Notice that all the values have units of tenths. Hence, there is no "apples and oranges" problem with unlike units. Simply add the numerical values (or variables as in algebra) of quantities with like units for the sum, and keep the same units.

4 tenths + 2 tenths + 1 tenth = (4+2+1) tenths = 7 tenths

We can also do this problem using the addition of fractions method. Here, the fractions will be "like fractions" which have the common or like denominator of 10 or "tenths" (tenths of one) and therefore can be added.

First: 4 tenths = 1 tenth x 4 = 4 x 1 tenth = 4 (0.1) = 0.4

4 tenths $= \frac{1}{10} + \frac{1}{10} + \frac{1}{10} + \frac{1}{10} = 4\left(\frac{1}{(10)}\right) = \frac{4}{10} = 0.4$

And for the specific problem mentioned:

$\frac{4}{10} + \frac{2}{10} + \frac{1}{10} = \frac{(4+2+1)}{10} = \frac{7}{10}$: "seven tenths" , 7/10 = 0.7 as a basic decimal number.
A check by converting each fraction to its decimal value:

0.4 + 0.2 + 0.1 = 0.7 : "seven tenths of one (1)"

Ex. You have a certain recipe that you only want to make one-half of its resulting amount. How much of each ingredient or part must you then use to make that recipe?

Here is the original recipe:

amount_of_ingredient1 + amount_of_ingredient2 + amount_of_ingredient3 = resulting_recipe

If we divide the right hand side of the equation by 2, so as to make half of the resulting_recipe, the equation will be unbalanced. If we have done nothing to the amount of the ingredients used, surely the resulting_recipe will still be the same, and yet we would be expecting half the results. What we need to do is divide both side of this equation by the same value of 2 so as this equation, recipe or formula, is still correct and in balance or equivalence (equivalent, equal):

$$\frac{\text{amount\_of\_ingredient1} + \text{amount\_of\_ingredient2} + \text{amount\_of\_ingredient3}}{2} = \frac{\text{resulting\_recipe}}{2}$$

Now, what exactly is the sum of all the ingredients combined divided by 2? The sum, of all the ingredients or parts of the result, divided by 2 is equal to the sum of the amount of each ingredient or part divided by 2:

$$\frac{\text{amount\_of\_ingredient 1}}{2} + \frac{\text{amount\_of\_ingredient2}}{2} + \frac{\text{amount\_of\_ingredient3}}{2} = \text{resulting\_recipe}$$

Since $\frac{1}{2} = 0.5$, you can optionally multiply the amount of each ingredient by 0.5.

In simple words, the above can be described as: To make half the recipe, use half of all the ingredients, and therefore half of each individual ingredient. To make a certain or specific fraction of the recipe, use the same fraction of the amount of each ingredient.

Ex. If amount_of_ingredient1 is 1 cup: 1 cup , and to make half the amount of the original recipe, use:

$$\frac{1\text{cup}}{2} = \frac{1}{2}\text{ cup} = \text{"one-half cup"} = 0.5\text{ cups}$$

Ex. If amount_of_ingredient2 is 7oz, and to make half (50% = 0.5 = 1/2) the amount of the original recipe, use::

$$\frac{7\text{ oz}}{2} = 3.5\text{ oz}$$

ADDING MONEY AS FRACTIONS, AND A SHORT DISCUSSION ABOUT PERCENTS

When adding fractions which do not have a common denominator among them, one (or several) of them must be converted into an equivalent fraction so that all the fractions will then have the same (common) denominator. For example, fourths (ex. 1/4) cannot be added to thirds (ex. 2/3) unless there is some conversion of one of these fractions into an equivalent valued fraction and which will have the same denominator as the other fraction.

Here is a practical illustration of why the denominators must be the same for the addition (and subtraction) of fractions, and it is very similar to the concept of having like units or quantities for proper addition:

Given a number of pennies and nickels, you can't add the number of pennies (which is a certain fraction of a dollar, actually one-hundredth = 1/100 = 0.01 of a (1) dollar = "1 cent" = "1 percent") and the number of nickels (which is another certain fraction of a dollar, actually one-twentieth = 1/20 = 0.05 of a (1) dollar) together to find out how many total (or the total monetary value of) pennies you have, or to find out how many total nickels you have. A nickel is a different coin and monetary value than a penny. Before adding nickels and pennies, you must first convert the number or quantity of pennies to an equivalent number of nickels, so as to express the sum in terms of nickels, or convert the number of nickels to an equivalent number of pennies, so as to express the sum in terms of pennies or cents.

For the conversions: 1 nickel = 5 pennies :the known basic conversion reference, or, after switching sides:
5 pennies = 1 nickel :as an extra note, since 5 pennies is the same monetary value as a nickel, or 0.05. It is sometimes said that "there are 5 pennies in a nickel". A penny is also a fraction of a nickel:
1 penny =1 cent = 1/5 nickel = "one-fifth of a nickel". If you divide both sides of the conversion expression by 5 you can verify this.

If given a (1) nickel, you can multiply it by 5 to find the corresponding number of pennies. If given twice (2) as many nickels, the total quantity of pennies will also be twice (2 times) as much:

1 nickel = 5 pennies : basic conversion facts expressed as an equation
1 nickels + 1 nickel = 5 pennies + 5 pennies since multiplication is repeated addition :
2 (1 nickels) = 2 (5 pennies) multiplying or distributing the multiplying value:
2 x 1nickels = 2 x 5 pennies
2 nickels = 10 pennies

And in general, if given some quantity (Q) or number of nickels, the total quantity of pennies will also therefore increase by that times as much:
From: 1 nickel = 5 pennies expressing the multi. of both sides by the same value, here Q:
Q (1 nickel) = Q (5 pennies) or:
(Q x 1) nickel = (Q x 5) pennies after some simplification:
Q nickels = Q x 5 pennies

This might be spoken as: "multiply the quantity or number of nickels by 5 to find the equivalent number of pennies", or "The quantity of pennies is 5 times more than the quantity of nickels". In a reverse type of manner, we now know that if given a quantity of pennies, we can divide it by 5 to find the corresponding number of nickels.

Ex. 10 pennies + 2 nickels : converting nickels to pennies , first, it is known that: 1 nickel = 5 pennies
10 pennies + 2 x 5 pennies
10 pennies + 10 pennies performing the indicated addition of similar things, simply add their
(10 + 10) pennies numerical quantities and keep the same units, and this is much like
20 pennies adding the numerical coefficients (ie. factors) of variables as in algebra. Algebra is basically symbolic math where text letters represent, or are "placeholders", for (known or unknown) numeric values of mathematical expressions.

Ex. 10 pennies + 2 nickels converting the number of pennies to its equivalent quantity of nickels, by dividing the number of pennies by 5:

(10 pennies / 5 pennies per nickel) + 2 nickels or:
(10 pennies/1) / (5 pennies/1 nickel) + 2 nickels or:

$$\frac{\frac{10\ \text{pennies}}{1}}{\frac{5\ \text{pennies}}{1\ \text{nickel}}} = \frac{\frac{10\ \cancel{\text{pennies}}}{1}\left(\frac{1\ \text{nickel}}{5\ \cancel{\text{pennies}}}\right)}{\frac{\cancel{5\ \text{pennies}}}{\cancel{1\ \text{nickel}}}\left(\frac{\cancel{1\ \text{nickel}}}{\cancel{5\ \text{pennies}}}\right)} = \frac{10\ \text{nickels}}{5} = 2\ \text{nickels}$$

: or can use: 10 pennies/5pennies = 2

2 nickels + 2 nickels performing the indicated addition:
(2 + 2) nickels : or (2 x 2) nickels , since 2 was added to itself twice and
4 nickels that multiplication can represent repeated addition

By using the fractional values (of a dollar) that a penny and nickel represent, and using letter Q for the quantity of each coin, we have expressions for the total monetary (in dollar units) values of a quantity of either pennies or nickels. Remember, multiplication is repeated addition. First, since each penny has a value of 0.01 of a dollar, for each next penny, add 0.01 to the total sum or monetary value. This repeated addition, can be represented and expressed with multiplication; here by Q which is the number or quantity of pennies, or nickels, and times it was added into the total sum:

Pennies	Nickels
$Q\left(\frac{1}{100}\right) = Q(0.01)$	$Q\left(\frac{1}{20}\right) = Q(0.05)$

If P was used to represent the Quantity of pennies, and N was used to represent the Quantity of nickels, the above could be expressed as:

Total pennies monetary value = P(0.01) and Total nickels monetary value = N(0.05)

For converting the monetary value of a quantity of nickels to the corresponding monetary value of pennies, we will multiply the corresponding monetary value of pennies by 5 since a nickel is defined as equivalent to the monetary value of 5 pennies = (0.01 x 5 = 0.05). Note that the fractional value of 1/100, or "one cent", of 1 dollar assigned to a penny coin is also assigned as the pennies monetary value of 1 dollar. The monetary value of a penny is often called a "cent", meaning one-hundredth of. The word percent mathematically means "per hundredths of", or "parts per hundredth of", for example 5 percent means 5 equal parts of something (considered as 1 whole or entirety) that was divided into 100 equal parts (each being 1/100), hence 5 one-hundredths = 1/100 + 1/100 + 1/100 + 1/100 + 1/100 = (5/1)(1/100) = 5/100 = 0.05 = 5% = "five percent". More will be said about this later in the topic of percents. Since a nickel is (monetarily) equal to 5 pennies, it therefore also represents the monetary value of 5 pennies which is 5 cents = 0.05 = 5/100 = "five hundredths of 1 dollar" = 1/20 of a dollar when 5/100 is reduced to a lower termed equivalent fraction. A nickel is 5/100 = 1/20 = one-twentieth of a dollar. Coins less than a dollar in monetary value are fractions of a (1) dollar's monetary value.

Converting the monetary value of a quantity (Q) of nickels to that of pennies.

Pennies		Nickels	
$(Q \times 5)\left(\frac{1}{100}\right)$	=	$Q\left(\frac{1}{20}\right)$	: here, we let Q equal the quantity of nickels Perhaps you might use N instead of Q as mentioned previously. Remember that multiplication is essentially repeated addition.
$Q \times 5\,(0.01)$	=	$Q\ (0.05)$	: or

$Q \times 0.05 \quad = \quad Q \times 0.05$: checks

Ex. Find the total monetary value of 2 nickels and 4 pennies.

2 nickels + 4 pennies

$2\left(\frac{1}{20}\right) + 4\left(\frac{1}{100}\right)$: denominators and not identical, and "unlike fractions" cannot be added or "combined". Can use: 2 (0.05) + 4 (0.01) = 0.10 + 0.04 = 0.14

$\frac{2}{20} + \frac{4}{100}$: denominators are not identical, so we have unlike fractions that cannot be easily added or combined as currently expressed so as to have just one fraction

Converting the monetary value of nickels to its' equivalent monetary value in pennies so that an addition (of fractions with a common, similar (same) or "like" denominator of 100) can take place, we have:

$2 \times \frac{5\,(1)}{5\,(20)} + \frac{4}{100}$: the numerator and denominator in the first fraction were both multiplied by 5 to create an equivalent (valued) fraction of which has a denominator value of 100.

$\left(\frac{2}{1}\right)\left(\frac{5}{100}\right) + \frac{4}{100}$

$\frac{10}{100} + \frac{4}{100}$: "ten-hundredths plus 4-hundredths" equals:
: We now have "like fractions" with the same denominators that can be added.

$\frac{14}{100} = 0.14$: "fourteen-hundredths" or= 14 cents

If you wanted to express a quantity of coins in terms (ie. with similar variables or units) of dimes (which is a certain fraction of a dollar, actually one-tenth = 1/10 = 0.10 of a dollar), you would have to convert the number of pennies and nickels to their corresponding number of dimes. Again, this is a common "apples and oranges" type of problem, except here, it is possible to make numeric conversions. Since 1 dime = 2 nickels = 10 pennies, divide the number of pennies by 10 to find the corresponding number of dimes, and divide the number of nickels by 2 to find the corresponding number of dimes.

Ex. 10 pennies + 4 nickels after dividing the quantity or number pennies by 10, and nickels by 2 for the conversion to it's (monetary) equivalent quantity of dimes:
1 dime + 2 dimes : we now have "like units" that can be combined
(1 + 2) dimes : expressing the sum of the quantities of the like or similar units
3 dimes : 3 dimes = 3 (tenths of 1 dollar) = 3 (0.10) = 3/10 = 0.30
This value can be spoken as: "thirty-hundredths" or "thirty-cents".

Also note in the above discussions, that when converting a quantity with larger units (of measurement) to a quantity with smaller units, that a multiplication is performed on the quantity with larger units. This is because there will be a greater quantity of these smaller units per each larger unit. For example, when converting (a quantity of) nickels to a corresponding number (or quantity) of pennies, multiply by 5, and when converting (a quantity of) miles to feet, multiply by 5280. Likewise, in a inverse type of manner, when converting to a larger units, a division is performed on the quantity that has the smaller units (of measurement or reference) since there will be a fewer quantity of the larger units.

Here is an extra side note of some **fractions of a quantity** that are given a unit-like name, that you may occasionally encounter: First, here are some common **fractions of a unit**: sixteenth = one-sixteenth of 1 inch = (1/16) in. and 1 centimeter = one-hundredth of a meter (1/100) meter. Similar to fractions of a unit are **multiples of a unit** given a unit-like name. For example, 1 ft x 3 = 3 feet = 1 yard , and (1000) (1m) = 1000 meters = 1 kilometer = 1 km

A simple example of the concepts: If there are 10 balls in a bag, and 9 are blue in color, and 1 is yellow in color, it can be said that the number of yellow balls in this bag of balls is 1 part of 10 parts. Mathematically 1 out of 10 is: 1 / 10 = one-tenth = 0.1 = 10%. The number of blue balls in the bag is 9 parts in or of 10 total parts = 9 / 10 = 0.90 = 90%

Ex. What is ppm and ppb? **ppm** = **parts per million parts of something** = parts / 1,000,000 parts
1ppm is numerically: 1part / 1,000,000parts = 0.000,000,1
10,000 ppm = 10000 parts / 1000000 parts = 0.01 = 1%

For example, and a discussion of the example:

Seawater is about 10560 ppm sodium = ~ 1% , and about 19000 ppm chloride = ~ 2%
Seawater is often mentioned as being 3% salt. **(food or edible) Salt is sodium-chloride (NaCl) molecules** and is commonly known as table salt that we can put on foods to enhance their flavors if need be. The scientific and-or chemical formula for a water molecule is H201 = **H20** since it has 2 atoms of hydrogen and 1 atom of oxygen. Seawater or ocean water is sometimes said as being about 35 **parts per thousand (ppt)**, hence 35 parts / 1000 parts = 0.035 = 3.5% salt (sodium-chloride,table salt). Note that if the total parts was 1000 cc, this value is equivalent to 1 Liter = 1 L of volume, and 3.5% would correspond to 3.5cc of volume.
1000cc of water has a mass of 1000 grams = 1000g
A thin depth of seawater can be evaporated (ex., in the sunlight and heat) so as to obtain its dissolved minerals such as salt which is considered a necessary nutrient to the human body.

Salt, and here specifically as sodium-chloride, is also a vital nutrient for some body processes. The recommended daily amount of salt for a moderately active adult of average size and-or weight is about but about 2000 mg = 2g of sodium which can be obtained from 5 to 6 grams of table salt, about 1 level teaspoon (about 5 mL= 5cc in volume) , of table salt has about 2000 mg = 2g of sodium and 3000 mg = 3g of chloride.

Chloride (cl^- , a healthy ion form of the more dangerous chlorine atom) is also a vital nutrient for some body process. Sodium-chloride or common table salt is generally not found in plants, but chloride is due to that it is vital for the photosynthesis process so as to make food such as a carbohydrate [carbon and water, and water is hydrogen and oxygen] type of sugar. The sunlight energy will first splitting water molecules and releasing electrons for further chemical reactions to take place. An **ion** is a atom or molecule that is not neutral or balanced in charge, generally due to it loosing or gaining an electron(s).

Where does all the wood from a tree come from? Besides some minerals from the soil, its generally not much mass from the soil, but it comes from the carbon dioxide in the air being converted to carbon during photosynthesis. A living plant has about 65% of its mass and-or weight as water, and has abut 15% of its mass as carbon. As the plant dries due to water evaporation, such as when becoming dried wood, that percentage will increase to about 50% of the mass and-or corresponding weight of that wood.

A (photo-voltaic, light energy to electrical energy) solarcell (or solar cell) that creates free electrons from the light energy from the Sun is like an electrical analogy of photosynthesis which plants do.

ppb = parts of something per billion parts of something = parts / 1,000,000,000 parts
ppt = **parts of something per thousand total parts of something**
ppt, ppm, ppb, can be considered as the number of parts of a certain mass and-or weight, or volume.

COMMON DENOMINATOR FOR ADDING FRACTIONS

The common, same or "like" denominator found for the proper addition of fractions will be some multiple (including 1) of the largest denominator value of all the fractions in question. Hence, the common denominator will be greater than or equal to the highest denominator value. It will be shown that a common denominator of two or more fractions can simply be the product of all their denominators, and that it is therefore evenly divisible by each of those denominators since it will contain a multiple of each denominator.

Ex. Add 1/4 and 4/8 : "add one-fourth, and four-eighths"

First, notice that the highest denominator value of these two fractions is 8, and that the fraction of 1/4 can be converted to an equivalent fraction of 2/8 if we multiply both the numerator and denominator by 2. Both fractions will then have the same denominator value and can be added as fractions:

$$\frac{1(2)}{4(2)} = \frac{2}{8}$$

We now have "like fractions" (with a common denominator) which can be added.

$$\frac{2}{8} + \frac{4}{8} = \frac{6}{8}$$

Note that this example may have been stated as: How many eighths are 1/4 and 4/8? The answer is six eighths.

For the two fractions being summed in the last example, the common denominator derived at from the two given and "unlike" denominators is called the lowest common denominator (sometimes and simply indicated as LCD, as in this book). The name comes from the fact that it is the lowest possible value for the common denominator that can be evenly divided by each denominator of the fractions in question. Note, if all the fractions to be summed are first reduced to their lowest or "simplest terms" (ie. Made into equivalent fractions - with a lower numerator and denominator) before the LCD is sought, that this resulting LCD of those equivalent fractions may even be lower in value.

If the numerator and denominator, each being considered as a product value, have factors that are common to both, then the fraction can be reduced or changed to an equivalent fraction of lower terms by dividing out, or "canceling" (ie. removing) these common factors. If you divide both the numerator and denominator by the highest or greatest common factor of both the numerator and denominator, the fraction will be reduced to its lowest terms. Remember, this processes is allowed since the fractions will still be equivalent in value. The greatest common factor can be found as the product of the smallest power of each different prime factor of the two numbers; prime numbers will be discussed ahead. Whereas multiplying both the numerator and denominator creates a "magnified (increased)" equivalent fraction that is said to be of "higher terms (of both num. and den. values)", dividing both the numerator and denominator creates a "demagnified (decreased, reduced)" equivalent fraction of "lower terms (of both the num. and den. values").

Ex. For the fractions below, reduce each to an equivalent fraction of lowest terms before the addition.

$$\frac{2}{8} + \frac{4}{8} = \frac{6}{8}$$

The fraction of 2/8 can be reduced to lower terms by dividing both the numerator and denominator by 2. The value of 2 was chosen since it is the highest factor common to both the numerator and denominator. This is seen when the numerator and denominator of the fraction are first factored:

$\frac{2}{8} = \frac{(2)(1)}{(4)(2)}$: factored numerator and denominator, showing the common factor(s) of 2

Dividing both the numerator and denominator by the highest common factor of 2, or simply "cross canceling" (crossing out, to reduce and-or possibly eliminate to just a factor of 1) these factors with a line or hash mark. You can also write a small one near these values, indicating the result of a division, here by 2, and that the result was 1. After canceling common factors, the result in an equivalent fraction of lowest terms.

$$\frac{2}{8} = \frac{\frac{(2)}{(2)}}{\frac{(8)}{(2)}} = \frac{1}{4} \qquad : 2/2 = 1 \quad \text{and} \quad 8/2 = 4$$

or: $\frac{2}{8} = \frac{\cancel{(2)}(1)}{(4)\cancel{(2)}} = \frac{1}{4}$: showing the "cross canceling", "canceling out" or "dividing out". Also,
: remember that 2/2 = 1 and anything times 1 is that same value. A multiplier or factor of 1 need not be shown always, but it can always be implied and-or removed so as to simplify equations as much as possible.

Likewise, for the fraction of 4/8:

$\frac{4}{8} = \frac{\cancel{(4)}(1)}{\cancel{(4)}(2)}$: showing the factoring of both the numerator and denominator, and the highest common factor of 4

Note that both the numerator and denominator also have a common factor of 2, but this is not the highest common factor since 4 is, and this will help create a fraction of lowest terms.

$$\frac{4}{8} = \frac{\frac{4}{4}}{\frac{8}{4}} = \frac{1}{2}$$

hence, $\frac{2}{4} + \frac{4}{8} = \frac{1}{4} + \frac{1}{2}$: results in an equivalent expression, after we reduced each fraction to lower terms.

After converting all the fractions to their corresponding equivalent fractions (with lower terms), their sum should still be equivalent to the sum of 6/8 as shown above. First consider that we need "like fractions" (have the same denominator) for proper summing:

$\frac{6}{8} = \frac{\frac{6}{2}}{\frac{8}{2}} = \frac{3}{4}$: "six-eights" reduced or expressed as "three-fourths".

Now, a common denominator possible between the two fractions is 4, which is also the LCD. By multiplying both the numerator and denominator of the fraction (1/2) by 2, we can get "like fractions" which have a common denominator for their addition.

$\frac{1}{4} + \frac{1}{2} = \frac{1}{4} + \frac{1(2)}{2(2)} = \frac{1}{4} + \frac{2}{4} = \frac{3}{4}$: which checks

As mentioned previously, the easiest way to find a common denominator when adding or subtracting fractions is to use the product of all the denominators. This new and common denominator will then contain a factor of each single denominator. This denominator is usually larger than the LCD possible. The LCD is more desirable since reducing the result to lowest terms is already or more easily done. A common way to start finding the LCD of all the fractions is to factor the denominators into prime factors. Prime numbers are discussed next as a helpful part of this current topic.

PRIME NUMBERS

Prime or primary numbers are numbers that cannot be evenly divided by any other (whole, integer or counting) number except itself and 1, in other words, a prime number being considered as a product or not, cannot be factored into two other factors that are whole numbers. For example, 2 is a prime number since it is only divisible by 2 (itself) and 1. 4 is not a prime number since it is evenly divisible by 2. 5 is a prime number. 6 is not a prime number since it has factors of 2 and 3. A composite number, such as 6, is composed of factors that are either composite and-or prime. The first 10, and perhaps the most commonly used prime numbers are:

1, 2, 3, 5, 7, 11, 13, 17, 19, 23 , . . .

Note that 2 is the only even prime number. An even number will have its most rightward digit being either: 0, 2, 4, 6, or 8, and the entire number is therefore divisible by 2, that is, the number can be divided in half or evenly into two identical whole or integer numbers. An odd number cannot be divided in half evenly and expressed as two identical integers. Any even number will contain 2 as one of its factors, hence even numbers other than 2 cannot be a prime or "unfactorable number". In terms of a (algebra) formula, 2n is an even number, and 2n + 1 and-or 2n - 1 is an odd number.

To factor (factorization, factorize) is where you are given a product value (perhaps a numerator or denominator as is the case of working with fractions) and you are to find its factors (multiplicands and multipliers) that compose or construct that given product value. You could say that these factors are "unseen" or "hidden" because they are now effectively "contained within", "composing", "making up", or "part of" that product value.

Some numbers or product values may have more than one set of possible factors, for example:

18 = 6 x 3 = 9 x 2

Some of these factors are composite numbers that can be factored further to only prime numbers. After doing this, we see that the given value (here 18) has only one corresponding and specific set of prime number factors:

18 = 6 x 3 = 9 x 2
18 = (3 x 2) x 3 = (3 x 3) x 2 by rearranging or regrouping we can see that they are the same value(s):
18 = 3 x 3 x 2 = 3 x 3 x 2 :18 completely factored to prime number factors, or simply: "primes"
: The different prime factors of 18 are 2 and 3.

Factoring a factor further can be considered and expressed mathematically as:

product = (factor1)(factor2) If anyone of these factors is considered as a product that can be factored, such as (factor2) which can be factored into (factor3) and (factor4):
product = (factor1)(factor3)(factor4) . . .

When adding or subtracting fractions: Considering all the completely factored (to "primes") denominators, each different valued prime factor is chosen only once, and specifically the highest multiple or power it occurs in a given denominator in the entire group of denominators, and these selected values are then multiplied together, and this product is the LCD of all the fractions.

Ex. Given the denominators of 4, 8 and 3, what would be the LCD?

The factors of 4 are 2 and 2, and both are prime. The factors of 8 are 4 and 2. 4 is not prime since it has factors of 2 and 2. Hence, the prime factors of 8, when completely factored to prime factors, are 2, 2 and 2. The factors of 3 are 3 and 1, hence it is already a prime number. The different prime factors are 2 and 3. The prime factor of 1 is always understood. Since 2 appears a maximum of three times in a single denominator, and 3 appears only once, the LCD is:

(2 x 2 x 2) x (3)
8 x 3
24 : 24 is the lowest common denominator (LDC) that each denominator can wholly ("evenly", entirely, completely) divided into without a remainder left over.

Again, notice that the highest power or multiple of each prime factor must be chosen so that each denominator, more specifically, the largest of the group, can (evenly) divide into the LCD at least once, and hence the LCD must be at least equal to or greater than the largest of the denominators.

One might now ask: Given a fraction, how is the numerator of an equivalent fraction found once the denominator (such as a common denominator for adding or subtracting fractions) of the equivalent fraction is found? The answer is found by dividing the "new" denominator by the "old" denominator. This effectively finds the multiplier (factor of the product) value that the "old" denominator was multiplied by to get the "new" denominator of the equivalent fraction. This is the case when the equivalent fraction being created is of higher terms. It is possible that a multiplier value can be less than one, such as is the case if the equivalent fraction is of lower terms. Once the multiplier is found, the same value must also be multiplied to the numerator so as to make an equivalent fraction that still represents the same numeric value, part or portion, that the original fraction does.

Ex. $\frac{2}{3} + \frac{1}{4}$

A common denominator to add these fractions is simply the product of those denominators.

$(3)(4) = 12$ substituting 12 for all the denominators of the equivalent fractions being created:

$\frac{\_\_}{12} + \frac{\_\_}{12}$ Now we are to find the proper or correct numerators of these equivalent fractions being created:

Since the first fraction's denominator was multiplied by 4 (from 12/3 = 4), we must also multiply the numerator of that fraction by 4 to keep it an equivalent fraction. Likewise, since the second fraction's denominator was multiplied by 3 (from 12/4 = 3), we must also multiply the numerator of that fraction by 3 to keep the fraction equivalent in value:

$\frac{2\,(4)}{3\,(4)} + \frac{1\,(3)}{4\,(3)}$ performing the indicated multiplications:

$\frac{8}{12} + \frac{3}{12} = \frac{11}{12}$: we have "like fractions" (have the same denominator), and can therefore be added, and keeping that same or common denominator

Ex. $\frac{10}{5 + 2} + \frac{4}{7 + 1}$

This is an example of adding or combining fractions where the denominators are actual expressions themselves. Notice that each denominator can easily be simplified first, but here, the order of operations will not be strictly followed in order to show another method. Frequently, there are times in algebra (where letters are used to represent unknown values or numbers yet to be found) where simplification is not preferred or possible. The fractions can still be combined (added) by considering each entire denominator expression as just like one single or unique value to work with:

$\frac{\text{numerator1}}{\text{denominator1}} + \frac{\text{numerator2}}{\text{denominator2}}$: expressing a generalized or "formula" form, here, of adding fractions

A common denominator, and the lowest common denominator if a denominator is not evenly factorable (divisible) or composite is simply the product of those denominators. The fractions are combined in exactly the same manner as the last example:

Here, a common denominator, expressed in a factored form, is: common denominator = (denominator1)(denominator2).

$$\frac{\text{numerator1 (denominator2)}}{\text{denominator1 (denominator2)}} + \frac{\text{numerator 2 (denominator1)}}{\text{denominator 2 (denominator1)}}$$: expressed as the sum of the equivalent fractions

When combining (adding) like fractions, add the numerators over the same denominator:

$$\frac{(\text{numerator1})(\text{denominator2}) + (\text{numerator2})(\text{denominator1})}{(\text{denominator1})(\text{denominator2})}$$

Using the example given:

$$\frac{10}{(5+2)} + \frac{4}{(7+1)}$$ creating equivalent (to each fraction given) and like denominator fractions: Using the product of (5+2) and (7+1) as a common denominator:

$$\frac{10 \quad (7+1)}{(5+2)(7+1)} + \frac{4 \quad (5+2)}{(7+1)(5+2)}$$ combining (adding) these like fractions we have:

$$\frac{10(7+1) + 4(5+2)}{(5+2)(7+1)}$$: a sum of several fractions, expressed as one fraction

Checking: $$\frac{10}{(5+2)} + \frac{4}{(7+1)} = \frac{10}{7} + \frac{4}{8} = 1.42857 + 0.5 = 1.92857$$:approximately

$$\frac{10(7+1) + 4(5+2)}{(5+2)(7+1)} = \frac{80 + 28}{(7)(8)} = \frac{108}{56} = 1.92857$$: checks

Sometimes when working with fractions, an equivalent fraction of (expressed as) lower terms or values of the given numerator or denominator must be created. To find out what the common divisor to both the numerator and denominator is, divide the "old" numerator or denominator by the corresponding "new" numerator or denominator. Note that the method indicated above for finding an equivalent higher term fraction will still work as a method for finding an equivalent lower term fraction. If that method is continued to be utilized, the effective multiplier will always be less than one.

Ex. $\frac{2}{3} = \frac{\_\_\_\_}{12}$:Create an equivalent fraction of higher terms (because the denominator of 12 of the fraction being created is higher than 3) by finding the numerator that correctly corresponds to this denominator so as the fractions are equivalent in value.

The "new" numerator to be found will also be higher. By dividing 12 by 3 we get the value of 4. Since the "old" denominator was essentially multiplied by 4 to get this "new" denominator. The "old" numerator must then also be multiplied by 4 to create the "new" numerator of the equivalent fraction:

$\frac{2}{3} = \frac{2(4)}{3(4)} = \frac{8}{12}$: It may help in the understanding of this by remembering that (4/4) = 1, and multiplying anything, such as a fraction or quotient, by 1 will not change its value, and here, the quotient or value of both (equivalent) fractions is the same: 2/3 = 8/12 = 0.666666... = about 67% of the whole or entirety (= 100% = 1).

Ex. $\frac{8}{20} = \frac{2}{}$

Here, the equivalent fraction being found will be of lower terms since the "new" numerator is less than the "old" numerator. 8 divided by 2 is 4 (that is, 2 and 4 are two factors of 8). Therefore, a factor of 4 was divided out of 8. To make an equivalent fraction, the same must also be done to the denominator.

$$\frac{8}{20} = \frac{\frac{8}{4}}{\frac{20}{4}} = \frac{\frac{2}{1}}{\frac{5}{1}} = \frac{2}{5}$$

hence, $\frac{8}{20} = \frac{2}{5}$: checking, after dividing the numerators by the denominators, both fractions equal 0.4

Here is the same example as above, but the method for finding an equivalent higher term fraction is utilized:

$\frac{8}{20} = \frac{2}{\ }$

Finding the effective multiplier of the numerator 8 that equals a product of 2: $\frac{2}{8} = 0.25$

Checking: $(8)(0.25) = 2$

Now multiplying this same value to the denominator to create the equivalent fraction: $20 \times 0.25 = 5$

Hence: $\frac{8}{20} = \frac{2}{5}$

The example above, for creating an equivalent fraction, can also be solved by the basic facts about division and multiplication:

$\frac{8}{20} = \frac{2}{\ }$ simplifying by performing the indicated division on the left side of this equation:

$0.4 = \frac{2}{\text{denominator}}$ Here we can consider 2 as the dividend of a division operation, and the factors of that dividend are equal to the quotient (here 0.4), and denominator or divisor (to be found).

From: $\frac{\text{dividend}}{\text{divisor}} = \text{quotient}$ we have:

dividend = divisor x quotient : the dividend as a product. Dividing a product by either of it's factors (the divisor or quotient) will yield the other corresponding factor of that product:

$\frac{\text{dividend}}{\text{quotient}} = \text{divisor} = \frac{2}{0.4} = 5 =$ the denominator being found of the equivalent fraction

ADDITION OF MIXED NUMBERS

When adding mixed numbers, you can first change them to (improper) fractions and then add those fractions. You can also add the whole (integer) parts, and then add this sum to the sum of the fractional parts. This is the common method taught, and this method produces another mixed number.

Ex. $5\frac{2}{3} + 2\frac{1}{4}$, This can also be expressed as: $\frac{5}{1} + \frac{2}{3} + \frac{2}{1} + \frac{1}{4}$

Using the first method discussed:

$5\frac{2}{3} + 2\frac{1}{4} = \frac{17}{3} + \frac{9}{4} = \frac{68}{12} + \frac{27}{12} = \frac{95}{12}$ or $= 7 + \frac{11}{12}$ when you divided 95 by 12, the result is 7 with a dividend remainder of 11

Note that for example: $5\frac{2}{3} = 5 + \frac{2}{3}$: "five and (+) two-thirds". Expressing 5 as a sum of 1's:

$1 + 1 + 1 + 1 + 1 + \frac{2}{3}$ since any number divided by itself equals 1, this can be expressed as this when considering the denominator value of 3, so as to make "like fractions", that have "like" denominators, which can then be added:

$\frac{3}{3} + \frac{3}{3} + \frac{3}{3} + \frac{3}{3} + \frac{3}{3} + \frac{2}{3}$ combining or adding the first 5 fractions:

$\frac{15}{3} + \frac{2}{3} = \frac{17}{3}$: after combining these two fractions, and : this fraction was indicated above

Using the second method discussed:

$$\begin{array}{r} 5\ \frac{2}{3} \\ +\ 2\ \frac{1}{4} \\ \hline \end{array}$$

OR :

$$\begin{array}{rl} 5\ \frac{2}{3} & = \frac{8}{12} \\ +\ 2\ \frac{1}{4} & = \frac{3}{12} \\ \hline \end{array}$$

$(5+2) + \left(\frac{2}{3} + \frac{1}{4}\right)$: horizontally

$7 \quad \frac{11}{12}$: vertically

$7 + \left(\frac{8}{12} + \frac{3}{12}\right)$

$7 + \frac{11}{12}$ or simply : $7\frac{11}{12}$: checks

Another way to add (or subtract) improper fractions is to convert the fractional parts to their decimal equivalents. This was also mentioned previously.

Ex. $5\frac{2}{3} + 2\frac{1}{4}$

```
first,       0.66             0.25
          3| 2.00         4| 1.00
             18               8
              20             20
              18             20
               2              0
```

Note that 2/3 produces a repeating decimal. You may "round-off" (truncate or shorten) the quotient where needed.

$$5\frac{2}{3} + 2\frac{1}{4} = (5 + 0.66..) + (2 + 0.25) = 5.66 + 2.25$$

```
     1
   5.66              basic format is:       whole . fractional
+  2.25                                 +   whole . fractional
   7.91                                     whole . fractional
                                                 sum
```

Comparing this to the previous example: $7\frac{11}{12}$

```
           0.916
7  +  12| 11.000          ~  7.91 (not rounded)  ,   ~ means about or approximately equal to the following value.
          108              Note also that 11/12 also produces a repeating decimal as seen above.
             20
             12
              80
              72
               8
```

SUBTRACTION OF FRACTIONS

The rules and concepts regarding basic addition and subtraction, and for the addition of fractions, also hold true for the subtraction of fractions. The main issue is to have like denominators so the subtraction of fractions can take place.

Ex. You have 1 whole circle divided into 3 parts, hence each part is a fraction of that circle and has the numeric value or representation of (1/3). The sum of all of these 3 parts is therefore: (1/3) + (1/3) + (1/3) = 3 (1/3) = 3/3 = "three thirds" = 1. If you take away (ie. remove or subtract) one of these parts, what value will remain of the entire circle?

$\frac{3}{3} - \frac{1}{3} = \frac{(3-1)}{3} = \frac{2}{3}$: "two-thirds" of the circle remains

Below is an example using mixed numbers, and then a discussion about the procedure will follow.

Ex. Subtract 2 6/7 from 3 5/7

$$\begin{array}{r} 3\,\frac{5}{7} \\ -\ 2\,\frac{6}{7} \\ \hline \end{array} \quad = \quad \begin{array}{r} \overset{2}{\cancel{3}} + \left(\frac{7}{7} + \frac{5}{7}\right) \\ -\ 2 + \frac{6}{7} \qquad\quad \\ \hline \end{array} \quad = \quad \begin{array}{r} 2\,\frac{12}{7} \\ -\ 2\,\frac{6}{7} \\ \hline 0 + \frac{6}{7} = \frac{6}{7} \end{array}$$

Considering the fractional parts of the mixed numbers, 6 cannot be taken from 5, hence we need to borrow. One might ask: where did the 7/7 come from? The answer is that borrowing 1 from the 3 is essentially borrowing 7/7 since 7/7=1. The denominator of 7 was chosen since the other fraction used in the subtraction operation also has a denominator of 7. This makes subtracting (or adding) the fractions easy since they will all have common or "like" denominators.

Ex. $\frac{10}{3} - \frac{4}{7}$

We must first find a common denominator for the subtraction. It is usually best to find the LCD. Here, the LCD has a value of 21 since 3 and 7 are both prime and they can simply be multiplied together. Now convert each of the fractions to equivalent fractions that have the same denominator of 21 :

$\frac{10}{3} = \frac{70}{21}$ since 21/3 = 7 , and then $\frac{10(7)}{3(7)} = \frac{70}{21}$

$\frac{4}{7} = \frac{12}{21}$ since $\frac{4(3)}{7(3)} = \frac{12}{21}$

$\frac{70}{21} - \frac{12}{21} = \frac{58}{21}$: Since both of these operands were expressed in a fractional form, the result is arbitrarily expressed in a fractional form rather than as a mixed number.

As a mixed number:

$\frac{58}{21} = 2\,\frac{16}{21}$: after performing the indicated division of: 58/21 and expressing the result as a mixed number. Remember that you can check a subtraction problem by using addition.

DECIMAL VALUES AND FRACTIONS

The basic way to change a decimal value that is less than 1 to its equivalent fractional value is to first find the positional weight of the least significant digit, and then remove the decimal point and make a product of this value and weight. From the discussion of positional weights, this weight will be a power of 10 with a negative (basically, conceivable numbers that are less than 0) number as the exponent. More advanced mathematics (besides what has already been discussed in this book, and that which will be shown ahead) will show that multiplying by a value with a negative exponent is actually dividing by that value with a regular (positive, even if it's not explicitly indicated) exponent of the same value.

Ex. $0.5 = 5(10\text{^}{-1}) = \frac{5}{10\text{^}1} = \frac{5}{10}$: "five-tenths of one". Note that the last or "least significant" digit, the 5, is located in the "tenths" position.

Note that 0.5 was made larger to a value 5, so it should be clear, as seen above, that to equal the original value (here 0.5), a division would then be necessary to reduce it to it's original value. We also know that a division by 10 will essentially moves the decimal point one position to the left toward the more significant digits.

Checking by performing the indicated division:

```
      0.5_
 10 | 5.0
      5 0
        0
```

You may also consider the decimal point being moved leftward in any value without using division, for example:

$5 = 0.5\,(10^1) = 0.5\,(10)$: one place leftward , and the indicated exponent of 10 is 1, and 10 has 1 zero
$0.5 = 0.05\,(10^2) = 0.05\,(100)$: two places leftward, and the indicated exponent of 10 is 2, and 100 has 2 zeros

Note here that 5 was made smaller to 0.5, so it should be clear that for it to equal the original value (here 5), a multiplication applied to 0.5, to increase this value, would then be necessary.

Ex. Represent 0.47 as a fraction

$0.47 = 47(10\text{^}{-2}) = \frac{47}{10\text{^}2} = \frac{47}{100}$: "forty-seven hundredths". Note that the last or "least significant" digit, the 7, is located in the "hundredths" position.

Checking by using the concepts of positional notation, and the sum of positional products:

$$0.47 = 4(10\text{^}{-1}) + 7(10\text{^}{-2}) = \frac{4}{10\text{^}1} + \frac{7}{10\text{^}2} = \frac{4}{10} + \frac{7}{100} = \frac{40}{100} + \frac{7}{100} = \frac{47}{100}$$

The last fraction is "forty-seven hundredths".

Another question about decimals you might ask is: What is the reasoning of moving the decimal point during "long" division? In decimal division, to have the nice ability to work (divide) with an integer divisor (no fractional part) and to automatically adjust the decimal point of the result, the decimal point of the divisor is moved completely to the right of the least significant digit (note: include all trailing zero's in the whole portion) and then the decimal point of the dividend is moved to the right the same number of digit positions (including any possible new trailing zero's placed as least significant digits). This process is essentially creating an equivalent fraction by multiplying the divisor (denominator) and dividend (numerator) by a common (same) value such as when creating an equivalent fraction. Specifically, whenever the decimal point is moved rightward, this resulting value is, and was caused by, some multiple or power of 10. Some of these concepts have been previously mentioned, and in particular for "long" or manual division.

Ex. $1/0.5 = \frac{1}{0.5} = 0.5\overline{)\,1.0} = \begin{array}{r} 2.0 \\ 5\overline{)\,10.0} \\ \underline{10} \\ 0 \end{array} = 2.0$: 5/10 is where the decimal point is moved 1 position rightward in both the numerator and denominator

Here, since the decimal point was "shifted" right one digit position, in the divisor (or denominator), the common multiplier to both the dividend and divisor (or numerator and denominator) was essentially 10.

checking: $\frac{1}{0.5} = \frac{1\ (10)}{0.5(\ 10)} = \frac{10}{5} = 2$: checks

MULTIPLICATION OF FRACTIONS

As like with the addition of fractions, we can essentially say that we have always been multiplying fractions if we just place each whole (entire) value over 1, and the relatively simple process of multiplying fractions becomes apparent.

Ex. $5 \times 3 = \frac{5}{1} \times \frac{3}{1} = \frac{(5)}{(1)}\frac{(3)}{(1)} = \frac{15}{1} = 15$: division by 1 does not change any value

The basic rule for multiplying fractions is to simply multiply the numerators (N) together, and multiply the denominators (D) together. Unlike for the addition and subtraction of fractions, the denominators when multiplying fractions need not be the same. The product of all the numerators being multiplied is the resulting numerator of the product. The product of all the denominators being multiplied is the resulting denominator of the product.

Symbolically: $\frac{(N1)}{(D1)} \times \frac{(N2)}{(D2)} = \frac{(N1)(N2)}{(D1)(D2)} = \frac{(N3)}{(D3)}$: a symbolic form of the multiplication of fractions

Ex. If you had half a circle (1/2) and added this to the other half (1/2) of that circle, which is the same as multiplying the first half by 2 since multiplication is repeated addition, the result should be one (1) whole circle. This example also verifies the rules for multiplying fractions.

$\frac{1}{2} + \frac{1}{2} = \frac{2}{2} = 1$ Since this is repeated addition of the same value, we can express this with multiplication :

$\frac{1}{2} \times \frac{2}{1} = \frac{(1)(2)}{(2)(1)} = \frac{2}{2} = 1$

Consider if you have the fraction:

$\frac{3}{5}$ which, by factoring the numerator and denominator, can be expressed as:

$\frac{(3)(1)}{(1)(5)}$ by grouping fractions, this can be expressed as the multiplication of two fractions:

$\frac{(3)}{(1)}\frac{(1)}{(5)}$: with the understanding that these fractions can be multiplied.

Checking: $\left(\frac{3}{1}\right)\left(\frac{1}{5}\right) = 3 \times \left(\frac{1}{5}\right) = \left(\frac{1}{5}\right) \times 3 = \frac{1}{5} + \frac{1}{5} + \frac{1}{5} = \frac{1+1+1}{5} = \frac{3}{5}$

For more verification of this multiplication of fractions rule, first consider that there is nothing intuitive about this rule. Why should the following be true:

$\frac{2}{3} \times \frac{4}{5} = \frac{(2)(4)}{(3)(5)}$ or if this was (somewhat oddly) explicitly expressed in terms of "hand or long division":

$$3\overline{)\,2} \quad \times \quad 5\overline{)\,4} \quad = \quad (3)(5)\overline{)\,(2)(4)}$$

To make this mathematical discussion "less wordy", let's label the first numerator and denominator as N_1 and D_1, and the second numerator and denominator as N_2 and D_2 (Note, this does not mean Nx2, Dx2, etc, but these are unique or different (symbolic) identifiers or names assigned to a value that is either known, unknown, or being solved for).

First consider this where like fractions are combined, and the numerator (2) is repeatedly added (ie. multiplication):

$$\frac{2}{3} + \frac{2}{3} + \frac{2}{3} + \frac{2}{3} = \frac{2 + 2 + 2 + 2}{3} = \frac{2(4)}{3} = \frac{2\,(4)}{3\,(1)} = \frac{8}{3}$$

Or by repeatedly adding the same fraction 4 times, and expressing this with multiplication:

$$\frac{2}{3} \times 4 = \frac{2}{3} \times \frac{4}{1} = \frac{8}{3}$$

The two must therefore be equal, and expressing this we get: $\frac{2}{3} \times \frac{4}{1} = \frac{(2)(4)}{(3)(1)} = \frac{8}{3}$

In general, we have $\frac{N_1}{D_1}$ to be repeatedly added (ie. multiplication) to itself say N_2 times.

Then we have: $\frac{N_1}{D1} \times N_2 = \frac{N_1N_2}{D_1}$

If N_2 was shown as initially divided by some value (including 1 as shown above), say D_2, this can be expressed as:

$$\frac{N_1\left(\frac{N_2}{D_2}\right)}{D_1} = \frac{\frac{N_1\,N_2}{D_2}}{\frac{D_1}{1}}$$

: think of this numerator as the result of repeatedly adding (N_2/D_2) a total of N_1 times. Remember, in adding fractions, the like or common denominator is also expressed in the sum.

By multiplying both the numerator and denominator by D_2 (creating an equivalent fraction):

$$\frac{\frac{N_1\,N_2\,D_2}{D_2}}{D_1\,D_2} = \frac{N_1\,N_2}{D_1\,D_2}$$

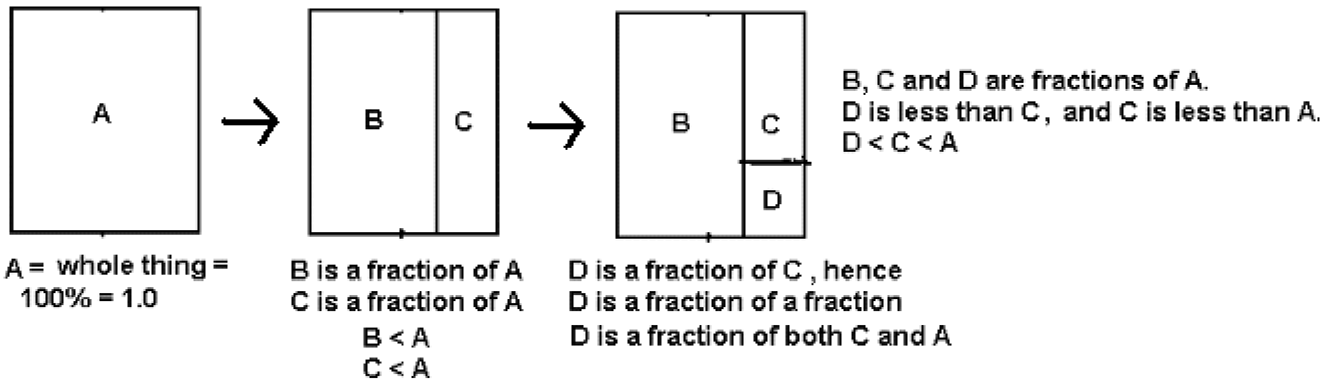
: after canceling, and therefore:

This same result could also be had by multiplying both the num. and den. by (1/D1). In conclusion we have:

$$\frac{N_1}{D_1} \times \frac{N_2}{D_2} = \frac{N_1\,N_2}{D_1\,D_2}$$

A FRACTION OF A FRACTION

Below is a visual or graphical example of having a fraction of a fraction, or having only a part of a part of something. As you might imagine, a part, of only part of something, is a smaller part and is represented by a smaller numeric value. The method to solve for the numerical result of a fraction of a fraction is to simply multiply the fractions together. The example below will demonstrate why, but first a graphical illustration of a fraction of a fraction. [FIG 9A]



C < B

Ex. What is one-fourth of one-half? This may also be mathematically expressed as:

1/4 of 1/2 = ? : 1/2 is a fraction, and 1/4 is a fraction, hence we are to find a fractional value of another fractional value.

The question in general is: "What is the value that represents this fraction of another fraction?" At first, you might be tempted to do something like divide, but then the question would have been stated as something like: "How many times can 1/4 go into (or divide into) 1/2?", or "How many times can 1/2 be divided by 1/4?" So solving the original question by division is not to be used. Can multiplication be the answer? Yes. A keyword to look for is the word "of". as indicated in 1/4 of 1/2. The word "of" is often associated with multiplication problems. You can also consider reading the word "of" used here as meaning: "of this", or "of this value".

Let's start at a basic example. If you have half of 10, you have 5 of the 10. Whenever you want to find half of some value, you divide that value by 2 to find out the numeric value.

$\frac{10}{2}$ half, or one-half of 10. This can be expressed as:

$\frac{10 \times 1}{2 \times 1}$ using the commutative rule in the denominator:

$\frac{10 \times 1}{1 \times 2}$ which can be expressed as:

$10 \times \frac{1}{2}$: a whole number times a fraction, which can be expressed as:

$\frac{10}{1} \times \frac{1}{2}$: clearly, a fraction times a fraction. Since this is "one-half of ten", the times symbol (x) is clearly associated with the word "of". For possible clarity and for how "one-half of ten" is spoken, this can also be mathematically expressed (with the commutative rule) as:

$\frac{1}{2} \times \frac{10}{1} = \frac{1}{2}$ of $\frac{10}{1} = \frac{10}{2}$: "one-half of ten"

If the whole number (10 as in this ex.) was divided by a number, say 4, we would then have a fraction times (multiplied by) another fraction:

$\frac{10}{4} \times \frac{1}{2}$: this could be expressed, read or spoken as: "one-half , of one-fourth of ten". Here is some more verification of this:

$\left(\frac{10}{1} \times \frac{1}{4}\right) \times \frac{1}{2}$: "one-half, of one-fourth of ten". For clarity, this could be expressed as:

$\frac{1}{2} \times \left(\frac{1}{4} \times \frac{10}{1}\right)$: "one-half, of one-fourth of ten".

So to solve for 1/4 of 1/2, "one-fourth, of one-half", simply multiply the two fractions:

$\frac{1}{2} \times \frac{1}{4} = \frac{1}{8}$: "one-fourth, of one-half is one-eighth" or "one-half, of one-fourth is one-eighth". If you know how to divide fractions, you can divide this product by either factor so as to find the other (associated or corresponding) factor as a check.

Here is a check using pure decimal numeric values one-fourth of one-half is:

one-fourth of one-half is = 1/4 of 1/2 = (0.25)(0.5) = 0.125 = 1/8

Note also that one-half of one-fourth has the same result, particularly because the factors are the same:

one-half of one-fourth is = 1/2 of 1/4 = (0.5)(0.25) = 0.125 = 1/8

Here are some other basic examples for more verification to this process:

Ex. What is one-half of one?

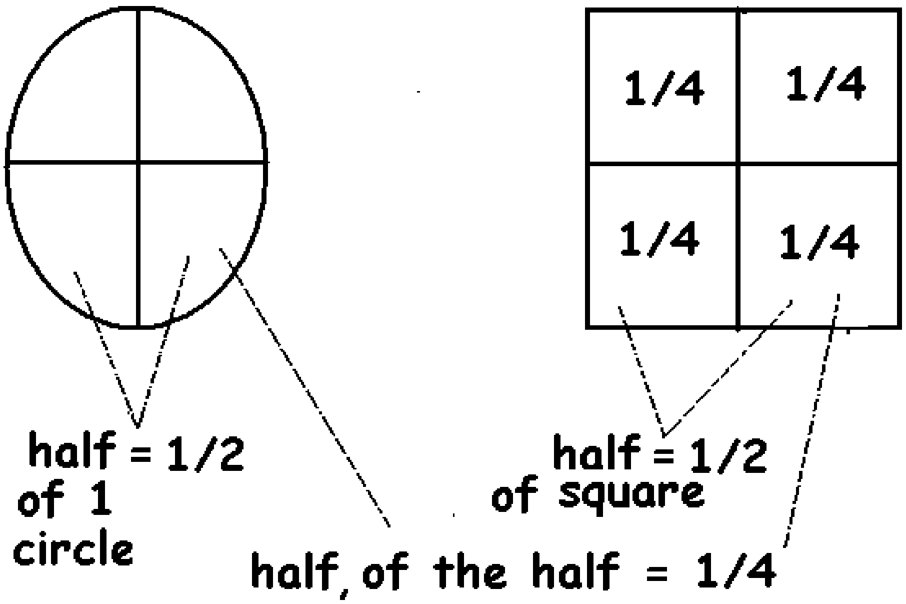
$\frac{1}{1} \times \frac{1}{2} = \frac{(1)(1)}{(1)(2)} = \frac{(1)(1)}{(1)(2)} = \frac{1}{2}$: "one-half", also you can divide any value, including a fraction, by 2 to find the half-value of it.

Ex. What is one-half, of one-half?

$\frac{1}{2} \times \frac{1}{2} = \frac{(1)(1)}{(2)(2)} = \frac{(1)(1)}{(2)(2)} = \frac{1}{4}$: "one-fourth". As mentioned above, this can also be solved by dividing the value by 2. (1 / 2) / 2 = (1 / 4)
Dividing by 2 is the same as multiplying by (1 / 2) = 0.5

Here is a graphical representation of this example (FIG 10): The circle or square was first divided into two equal parts (ie. in half), and then each of these parts was divided into two equal parts (ie. in half). The entire (1) square or circle has effectively been divided into 4 equal parts, and each part will therefore have a numeric value of (1/ 4).

[FIG 10]



If you were to then divide each quarter section (which are already existing fractions, of the entirety (=1)) into two equal parts, the square or circle would then contain a total of 8 equal parts or sections (fractions). Each of these new sections is a half (1 / 2) of any give quarter (1 / 4) section, hence mathematically: (1 / 2 of 1 / 4), and this new section only represents a smaller value, of (1 / 8) of the entire whole structure since it is a fraction (part) of another fractional (part less than 1 whole) value, and can therefore only be less in value:

$\frac{1}{2} \times \frac{1}{4} = \frac{1}{4} \times \frac{1}{2} = \frac{1}{8}$: one-half of one-fourth or = one-fourth of one-half = one-eighth

Before moving on to the next topic, the concept and wording of a fraction of a fraction may need to be stressed a little more carefully. Sometimes what is actually meant or being discussed can be subtle and different. Consider this:

"five-tenths of ten" and "five-tenths out of ten" : the key difference is the words "of" (= x), and "out of" (= /)

$\frac{5}{10} \times 10$!= $\frac{(5/10)}{10}$!= or $\neq$ or /=/ means: "is not equal to"

0.5×10 != $\frac{0.5}{10}$

5 != 0.05

DIVISION OF FRACTIONS

A fraction divided by another fractional or whole value is called a complex or compound fraction. That is, both the numerator and denominator of these fractions will also be fractions. A compound fraction is basically a "nested fraction" where the numerator and denominator fractions are the innermost fractions of the larger compound fraction. Some examples of compound fractions have already been seen in this book.

$$\text{A compound fraction:}\qquad \cfrac{\;\frac{2}{3}\;}{\frac{4}{8}} \qquad \begin{array}{l} :\text{ numerator} \\ \\ :\text{ denominator} \end{array}$$

Actually, you can think that you have always been working with compound fractions when working with just simple fractions by dividing both the numerator and denominator by the same value (as used when creating equivalent fractions) of 1:

$$\frac{2}{4} = \cfrac{\;\frac{2}{1}\;}{\frac{4}{1}}$$

Clearly, by simplifying both the numerator fraction and denominator fraction, of the compound fraction, the compound fraction is simplified to a simple fraction. Often, the simplified form of a compound fraction is expressed in terms of the numerators and denominators. Before a working example is shown, you should know more about canceling, or "to cancel", and what a reciprocal means.

Canceling is a process to possibly eliminate fractions to a single numeric value or number, essentially ridding the fraction completely. Remember, this was also stated in the order of operations second rule: clear/cancel grouping symbols. To cancel is to "divide out", "cancel out" or remove factors that are within and-or common to both the numerator and denominator. The result is either an equivalent fraction of lower or reduced terms, or a single numeric value or number. Canceling is basically like a "pre-division" and simplification step before the order of operations is followed more correctly where factors, if possible, are first multiplied together before the division. Sometimes a "hash mark" (/) type of symbol is shown when some division or canceling has been performed. A good approach for simplifying a compound fraction is to first try to rid the compound fraction of its denominator fraction. The method to do this will be shown below, and it is essentially done using the equivalent fraction concepts.

The reciprocal or "inverse" of a number is that number divided into one (1). Sometimes this is said as "one over the number (or value)". The word reciprocal is related to the word reciprocate which means "to move or change back and forth". The reciprocal of 5 or 5/1 is 1/5 as a fraction, or 0.2 when the indicated division is performed, which is its decimal equivalent. Likewise, the reciprocal of 0.2 is 1/0.2 = 5 which is the same value we started with. A reciprocal value of any integer (ie. counting) number is unique, that is, it can never be equal to the reciprocal of some other value. The product of two corresponding reciprocals is always 1. For example (1/4)(4/1) = 4/4 = 1 = (0.25)(4). One might ask: What is the reciprocal of a fraction? The answer is to simply "invert", or "turn upside down" the fraction. The numerator will become the denominator, and the denominator will become the numerator. As indicated in this discussion, the reciprocal of 5 or 5/1 is 1/5. The reciprocal of 2/3 is 3/2, and this is verified below:

The reciprocal of (2/3) is mathematically noted or expressed as: $\cfrac{1}{\frac{(2)}{(3)}} = \cfrac{\;\frac{1}{1}\;}{\frac{2}{3}}$: these are compound fraction expressions

Given this compound fraction, we note that if we multiply the denominator's fraction by its reciprocal value of 3/2, that we can effectively cancel it out.. The denominator will then be equal to one (1) after the canceling ("out", or "ridding") of each common factor of the entire numerator and entire denominator has been performed. Even though it has a value of 1, the denominator is still effectively eliminated since any value (here the numerator) divided by, or "placed over", 1 or not divided by any indicated value, is still equal to that value. Also, in order not to change the value of the (compound) fraction, that which is done to the denominator (fraction) must also be done to the numerator (fraction). This is necessary to create a (numerically) equivalent (compound) fraction.

$$\frac{\frac{1}{1}}{\frac{2}{3}} = \frac{\frac{1(3)}{1(2)}}{\frac{2(3)}{3(2)}} = \frac{\frac{3}{2}}{\frac{6}{6}} = \frac{\frac{3}{2}}{1} = \frac{3}{2}$$

: verification that 3/2 is the reciprocal of 2/3
The product of two corresponding reciprocals is always 1.

Note, it is best to attempt to cancel-out factors common to both the (entire) numerator and (entire) denominator before performing the indicated multiplication. This was also mentioned in step 2 of the order of operations.

$$\frac{\frac{1}{1}}{\frac{2}{3}} = \frac{\frac{1(3)}{1(2)}}{\frac{\cancel{2}(\cancel{3})}{\cancel{3}(\cancel{2})}} = \frac{\frac{3}{2}}{1} = \frac{3}{2}$$

: expressing the canceling with hash marks. If the values are not the same, such as for this example, a 2 in both the num. and den., you can reduce them both by dividing each by a common factor or value and indicate the quotients next to those factors that were reduced.

Hence, multiplying the denominator of a compound fraction by its reciprocal will easily cancel out or rid it of its denominator, leaving only a "simple" fraction in the numerator to work with. Here is a simple verification: How many times will one-half (1/2) of a circle divide into one (1=1/1) full circle? Clearly, the answer is 2, let's check:

$$\frac{\frac{1}{1}}{\frac{1}{2}} = \frac{\frac{1(2)}{1(1)}}{\frac{1(2)}{2(1)}} = \frac{\frac{2}{1}}{\frac{2}{2}} = \frac{\frac{2}{1}}{1} = \frac{2}{1} = 2$$

: note also: 1 / (1/2) = 1 / 0.5 = 2

Ex. $$\frac{\frac{1}{2}}{\frac{1}{4}} = \frac{\frac{1(\overset{2}{\cancel{4}})}{\cancel{2}(1)}}{\frac{1(4)}{4(1)}} = \frac{\frac{2}{1}}{1} = \frac{2}{1} = 2$$

: one-fourth will go into one-half, twice

Checking using the decimal equivalents of the fractions:

$$\frac{\frac{1}{2}}{\frac{1}{4}} = \frac{2\,|\overline{1.0}}{4\,|\,1.0} = \frac{0.5}{0.25} = 2$$

: checks

Ex. $$\frac{\frac{1}{4}}{\frac{5}{8}} = \frac{\frac{1(\overset{2}{\cancel{8}})}{\cancel{4}(5)}}{\frac{5(8)}{8(5)}} = \frac{2}{5}$$

Checking, by multiplying the denominator (divisor) by the resulting fraction (the quotient) to see if the product is equal to the numerator (dividend):

$$\frac{(5)(2)}{(8)(5)} = \frac{10}{40} = \frac{\overset{1}{\cancel{10}}}{\underset{4}{\cancel{40}}} = \frac{1}{4} \qquad \text{: checks}$$

Ex. A certain cooking recipe requires 1/3 of a cup of water, however you decide to only use enough of each ingredient to make half of the total quantity the recipe will make, how much water will you then need?

The answer is to use one-half (half = the whole or given part evenly divided by 2) of each ingredient.

$$\frac{\ \frac{1}{3}\text{ cup}\ }{\frac{2}{1}} = \frac{\ \frac{1(1)}{3(2)}\ }{\frac{\cancel{2}(1)}{1(\cancel{2})}} = \frac{1}{6}\text{ cup} \qquad \text{: one-sixth of a cup of water.}$$

As mentioned previously, product of a number and its corresponding reciprocal is always equal to 1. Reciprocals are also the reciprocals of each other.

$$5 \times \frac{1}{5} = \frac{(5)(1)}{(1)(5)} = \frac{5}{5} = 1 \qquad \text{or since } (1/5) = 0.2 :$$

$$5 \times 0.2 = 1.0$$

We see that in the above expressions for 1, that in one factor, the number is a multiplier, and in the other factor, that same number is a divisor. Since multiplication and division are inverse operations of each other, you may sometimes see reciprocals being mentioned as a "multiplicative inverse".

Most scientific calculators have a reciprocal-key or "calculator function" for the reciprocal operation. This key or button is usually labeled as: [1/x] . If your calculator does not have this convenient key, then simply divide 1 by the number in question that you want to find the reciprocal of.

Note: Canceling can only be done on one factor at a time of both the entire numerator or denominator. Once a value is reduced to 1, it is therefore essentially eliminated (canceled out) and what is left of its initial value cannot be used for any more possible canceling.

Ex. $\frac{(8)(4)}{2} = \frac{(\overset{4}{\cancel{8}})(4)}{\cancel{2}}$ or $= \frac{(8)(\overset{2}{\cancel{4}})}{\cancel{2}} = 16 \neq \frac{(\overset{4}{\cancel{8}})(\overset{2}{\cancel{4}})}{\cancel{2}} = 8$: once an initial factor is canceled or factored, it cannot be reused as that same initial factor value , but any factor remaining of an initial factor can be used for further canceling.

Checking: $\frac{(8)(4)}{2} = \frac{32}{2} = 16$

CANCELING ENTIRE EXPRESSIONS

Since an expression essentially, or eventually, represents one specific quantity or value in question, often indicated within a surrounding grouping symbol for clarity, it can be canceled-out just like one single numeric values if it's a (common) factor to both the entire numerator and denominator of the fraction.

Ex.

$$\frac{5(3+2)}{7(3+2)} = \frac{5\overset{1}{\cancel{(3+2)}}}{7\underset{1}{\cancel{(3+2)}}} = \frac{5}{7}$$

Checking by varying the order of operations, which can be performed wherever possible as long as the result is not caused to be made incorrect. Here, for some simplification of the expressions, addition is first performed within the grouping symbols as part of the effort to clear the grouping symbols in the numerator and denominator of the fraction. Canceling (division or dividing out) common factors is then chosen over multiplication, so as to first simplify the numerator and denominator) and help prevent the need to reduce the resulting fraction to lower terms:

$$\frac{5(3+2)}{7(3+2)} = \frac{5\overset{1}{\cancel{(5)}}}{7\underset{1}{\cancel{(5)}}} = \frac{5}{7}$$

: or = 25/35 when "unreduced" (to simpler terms), and if both the numerator and denominator are divided by 5, the equivalent fraction created is 5/7

PERCENTAGES

The word cent (or centi) in the word percentage means "one-hundredth of" which is 1/100 or 0.01 numerically. Therefore, percent means how many "(per) hundredths of something" or "parts per hundred of something, that was (physically, and-or mathematically, numerically) divided into one-hundred parts". The concept of percent was briefly mentioned in the topic of: A Short Discussion About Percentages.

A percentage, or more correctly, a percentage rate is simply a numerical measure of a part, fraction or portion of something, and is in reference and-or comparison to that whole or entire something. Symbolically:

$\frac{\text{Part}}{\text{Whole}}$ = numeric representation of how much Part and-or its numeric portion or representation is in reference to, or with respect to, the Whole thing. We see that this expression or mathematical representation of a percentage rate is the same as that of an expression for a fraction, and this is so since we are numerically finding out what is the relative size (ie., between 0 to 1, or 0% to 100% = "zero percent to one-hundred percent") that this portion or part is in reference to the entire or whole thing. The value of this expression is a rate or comparison of two values, and is called the portion rate or percentage rate, and the actual numeric value of Part is technically the actual percentage value (without the rate or comparison meaning) in question that was used to find the (percentage) rate:

Part / Whole = percentage value / total = percentage rate therefore, mathematically:
(Whole) (percentage rate) = percentage value = Part

Below are some examples to help demonstrate as to how we may use the concepts of percentages.

1. The monetary value of a common penny coin is one cent, or "one-hundredth" of a dollar, hence 1/100 = 0.01 of a dollar. To find a percentage is to find the total number of hundredths in question of any value or quantity that was divided (or imagined or conceived as divided) into 100 equal parts or fractions. The accepted symbol that indicates the amount of hundredths in question, or that hundredths are being considered, is %. It is called the percent symbol. Percent values are mostly used to easily generalize or indicate the basic comprehensible, relative value or size of a portion or fractional value of something, rather than use exact specific values, especially when the size or value of that something can change or vary, yet the (percentage) rate of the portion value in reference or respect to the whole value, remains constant, say for example, as being always 50% of any current value or total at hand or being considered. For some simplified examples of relative values and their percentages: "none" - such as 0 percent, "tiny" - such as 1 percent, "small" - such as 10 percent, "medium" - such as 50 percent, "big" - such as 70 percent, "large" - such as 90%, all, "entire" or "whole" - such as 100%.

2. When using a fixed or constant percentage value, the actual or specific result value automatically adjusts for any changes in the overall size of that something. For example, a store may indicate that the selling cost of any item(s) is 10% cheaper or less than its previous selling price rather than list the actual amount(s) reduced and-or the previous selling price of the item(s). A store may even have a "store-wide (all items) sale for a day", where everything in the store is 20% cheaper than the indicated price on the price-tag, rather than make many new and temporary price-tags for all of the store items.

3. By some contract or agreement, a person might own or be allotted a specific percentage, say 10%, of all (entire) the grain produced for the year at a farm. rather than say the allotment is to be specifically 500 pounds only, or 500 pounds maximum. In a good growing year, 10% of all the grain might be 600 pounds, and in a bad growing year, 10% of all the grain might only be 200 pounds. This way, all the people who were allotted or assigned to that grain produced at that farm will also share in the gains and losses in the actual total amount of grain produced during that growing season. Clearly, if the farm only produced 200 pounds of grain, and if a person was to be allotted a specific value of 500 pounds, it is no even possible for that person to receive that specific amount of grain.

4. Sales tax, often some low fixed percent, say 5 or 6 percent (5%, or 6%), at a store to purchase a specific item(s) will be

based on the selling price of the item(s). The higher the price of the item(s), the more the total sales tax money value will be, and yet the indicated percentage or sales tax rate is a constant or fixed value. At this point in this book, it could be obvious or intuitive that if the cost of the items doubles (2), or goes up by some factor, then the sales tax money needed for that item will also double or go up by that same exact factor (here 2), and yet the indicated percentage or sales tax rate is still the same constant or fixed value. When changes correspond in such a direct manner as this, the relationship of the resulting values, or the change values themselves, is said to be a direct, constant, straight, "one to one",or "linear" (line-like) mathematical relationship.

5. In a hospital, sometimes the amount or dosage of a certain medication to give a patient will be a specific (fixed, constant) percentage (ie. a fractional value or portion) of a persons weight. The more someone weighs, the more the specific amount of medicine they will need and receive, and yet the percentage value remains the same or "fixed" for patients that have various weights.

6. If a certain drink mixture is described as containing 10% fruit juice, regardless of the volume of that fruit juice, and hence for any specific volume (such as a spoonful, glassful, or a liter) of the mixture considered to be put into several glassfuls to drink, each glassful will contain 10% fruit juice. If you know the specific volume of the entire mixture used, and after solving for 10% of that volume, you have found the specific volume or amount of fruit juice in that specific volume of the mixture.

If you have 5% (commonly spoken as "five percent" or= a "five percent rate") of something, you have 5 one-hundredths of it:

$$\frac{1}{100} + \frac{1}{100} + \frac{1}{100} + \frac{1}{100} + \frac{1}{100} = \frac{5}{100}$$: "five one-hundredths" or simply: "five hundredths"

Or, since we have a repeated addition of adding the same value 5 times, we can express this with multiplication as:

$$\frac{(5)}{(1)} \times \frac{(1)}{(100)} = \frac{(5)(1)}{(1)(100)} = \frac{5}{100} = 0.05$$ in decimal form. or:

5(1/100) = 5(0.01) = 0.05

This is often indicated as: 5 hundredths = 5 percent = 5% when percents are considered. The percent symbol and the value preceding it are sometimes, and more formally, called the rate. The word rate is rooted in the words of: reason, rational (reasoning, logic), and ration (an allotment, fraction, or portion of something, and in the numerical sense, it similar to the word ratio). In this example, the (percent) rate is 5 percent = 5% = (5/100) = 0.05.

Ex. What is five percent of three hundred apples? This can be expressed as: 5% of 300 apples is how many apples? According to the definition of percent, 5% of 300 apples is = 5 hundredths of 300 apples = (5/100) of 300 apples. 300 represents the whole (entire, all = 100% = 1.0) lot, amount or quantity that is to be divided into 100 even (or equal, similar) parts when percents are being considered.

First of all, 1% (= one-hundredth = 1/100 = 0.01) of 300 apples is 300 apples divided into 100 equal parts:

$$\frac{300 \text{ apples}}{100} = 3 \text{ apples}$$: (a group of) 3 apples physically compose, and is mathematically, 1% of 300 apples. Checking by multiplying the quotient and divisor, the dividend is: (3 apples)(100) = 300 apples, and this expression also indicates that 300 apples is 100 times bigger or more than 3 apples. Likewise, it also expresses that if you have 3 apples, and increase, magnify (mathematically, a multiplication = multiples or many of) or multiply it by 100 (or 100 times, or multiples of), you will have 300 apples.

If we mathematically compare 3 apples to 300 apples, perhaps to find the "magnification size or factor", "how big one is compared to the other", or "how many times bigger one is to the other", we can get this expression below, which can also be mathematically derived from the above equation by dividing both sides by (300 apples).

$\frac{3 \text{ apples}}{300 \text{ apples}} = (1/100) = 0.01 = 1\%$:This shows why a percent is really a rate or plain unitless numeric ratio of two values. Extra: the reciprocal of this expressed fraction is: (300 apples) / (3 apples) = 100. This value of 100 indicates that 300 apples is 100 times more or bigger than 3 apples.

This indicates that three apples out of, or compared to 300 apples, is (only) 1% of them. Three apples is a portion (part, or fraction (less than all = the entirety = 100% = 1.0)) of the 300 apples. Mathematically, this portion is 0.01 = 1/100 = "one-hundredth" = 1%. The rate can be said as: "3 out of 300", or "3 per 300", or more commonly, basically or fundamentally (in terms of hundredths) as: "1 out of 100", or "1 per 100", and numerically, this is expressed as: 1/100 = 0.01 = 1%.

["Technically", the indicated rate is equal to the indicated percentage, and here, its 0.01 or 1%, and that the actual or specific value that this percentage rate represents (of the entirety) for this example is (3 apples). (3 apples) is therefore called the (actual or specific) percentage or percentage value. This may seem a bit confusing at first, but it is something to consider. It would help if the mathematical (rate) value (in terms of hundredths of the entirety) of a portion is called the percentage rate, or the indicated percentage, rather than simply and commonly as just the percentage. As we have seen, we can have a fixed percentage rate of say 6% sales tax, but the actual tax value or money needed, or the actual percentage value that this 6% rate indicates or is associated with, can vary widely since it depends on, and is mathematically in reference to, or based on, and calculated using the "total" (all, entire) cost or money required during a purchase or sale. Some more will be said about this further below. **]**

1% of anything or value is a hundredth of it, this can be mathematically expressed as: (value /100) or (total/100). 1% of 300 can therefore be expressed as (300/100) = 3 if we perform this indicated division. This leads to the conclusion that 3 apples is 1% of 300 apples . If we multiply this specific 1% value by 5, we will then know the number of apples equal to 5% of 300 apples, since 5 percent is obviously 5 times more than 1 percent. (5 percent/1 percent = 5). This could be stated in a question as something like: "If 3 apples = 1% of all the apples, how many apples are 5% of all the apples?".

(1% of apples) x 5 = (5 x 1%) of apples = 5% of apples :

$$\frac{300\text{ apples}}{100}\,\frac{(5)}{(1)} = \frac{\overset{3}{\cancel{300}}\text{ apples }(5)}{\underset{1}{\cancel{100}}\quad(1)} = 15\text{ apples}$$

or: 3 apples (5) = 3 apples x 5 = (3 x 5) apples = 15 apples

checking: $\frac{15\text{ apples}}{300\text{ apples}} = (15/300) = (5/100) = 0.05 = 5\%$

We showed that 5% = 5/100 = 0.05 = 5 (1/100) = 5 x 0.01 = 0.01 x 5 = 1% x 5 , and note that:

$\frac{300\text{ apples}}{100}\,\frac{(5)}{(1)}$ can be written or expressed as as:

$\frac{300\text{ apples}}{1}\,\frac{(\ 5\ )}{(100)}$: due to multiplication being commutative. This is equal to:

(300 apples)(0.05) = 15 apples = (300 apples)(5/100) = (300 apples)(5%) = 5% of 300 apples

Therefore, to simply get the actual (rather than the rate or indicated) percentage value (ie., the specific or actual quantity or value that this rate actually represents), multiply the total value or quantity given by the indicated rate expressed in decimal or fractional form. The general "formula" for finding percentages is usually written as:

(total)(rate) = percentage : the common and basic "percentage formula"

This percentage formula stems from (ie. is derived from) the basic form of any ratio. Here, rate is a ratio value (a mathematical representation of the comparison of two values which is essentially a numeric (and unitless) quotient) value of the actual percentage value (typically only some amount of the total amount) in question divided by the total or Whole value in question. Note the similarity of the word rate and ratio. Whereas two numbers can be mathematically compared by their resulting (linear) distance separation or difference (ie. a simple change, or mathematical measure of the separation), two numbers can also be mathematically compared by their resulting ratio value (ie. a multiplier or magnification change of one value [as a numerator] as compared to, or in reference to, the other value [such as a denominator]). Ratios will be given a more general discussion further ahead in this book.

Remember, to find the unknown factor in a product, you simply divide by the other given factor. Considering this using the formula above, this results to the following which is also equivalent to dividing both sides of the equation by (total), and then canceling out common factors in the entire numerators and denominators.

$$\frac{\text{percentage}}{\text{total}} = \frac{\text{Part}}{\text{Whole}} = \text{rate}$$

: Showing that the rate or "percentage rate" value is the mathematically or numerically "indicated percent" of the actual percentage value with respect to the total value. Likewise,the given formula can be expressed as:

$$\frac{\text{percentage}}{\text{rate}} = \text{total}$$

: total is sometimes called the base, base value, or working value being considered

From: (total) (rate) = percentage value

$$(300\text{ apples})(5\%) = (300\text{ apples})\left(\frac{5}{(100)}\right) = (300\text{ apples})(0.05) = 15\text{ apples}$$

The largest possible part or fraction of something is numerically: 100% = 100 parts out of, or of, 100 parts = 100/100 = 1 of something which is the entire, "all of it", or whole thing. It is possible to have percentages, or more correctly, rates, that are higher than 100% as when the that something is now larger than before, hence that higher than 100% value is in reference to a previous value, since the largest possible part or fraction of any value is still only 100%. If something is claimed to be 100% larger, or a 100% increase or bigger, it generally means that it has increased or been magnified (multiplied) by a factor of: (original value + increase) / original value = (100% + 100%) /100% = 200% / 100% = 2, and may be stated as being two times more or greater than the previous or original (reference) value, or now being twice as big.

Ex. What is 3.75% of $1530.35 to the nearest penny?

This can also be expressed as: What is 3.75% of $1530.35 rounded to the nearest penny.

"To the nearest" means **rounded** or mathematically adjusted and truncated (reduced or limited to) at that decimal position or "precision" to reflect if the next least significant digit (possibly rounded itself) was 5 or greater, and if so, it will be increased or "rounded up", by 1. Any and all digits having lesser numerical significance will be discarded as unneeded. Hence, the result, when rounding was used or considered, can only be assumed to be an approximation (not exact) or close value due to this process. **Rounding** is used to eliminate often unnecessary (lengthy. least significant or practically meaningless in value, that are of low effect or concern) digits in a number and to make further calculations shorter in length and easier to process, and to have nicer looking result values. The small error and effect of, and after discarding even some small (fractional) part of a value(s) can eventually "add up", or multiply in significance and therefore have growing influence on the results of further calculations, and in such a manner that the error or the difference from the actual or true result can be significant enough to consider and not ignore and the result should then be considered and explicitly stated as being an approximation.

To use a percentage value as an operand of an expression, first convert it to its equivalent decimal form. This can be done visually by moving the decimal point two digit positions leftward. For example:

5% = 0.05 : "five percent" equals 0.05 in decimal, or strict numeric form.

3.75% = 0.0375 : since 3.75% = 3.75/100 = 0.0375
The percent symbol, %, numerically means: (1/100).

Using a calculator: $1530.35 x 0.0375 = $57.388125

Rounding to the nearest penny (or hundredth of a dollar) we get: $\$57.3\underline{9}$ The underlined number, here for 9, is an expression or symbol to express that the actual, true or entire number given was rounded to and truncated this this position or precision ("precise", smallest part of).

Ex. In a coupon you received, a store claims that a certain items selling price of $7.99 will be reduced (decreased, discounted) by 10% (or "10% off") when you give them your coupon. What will be the new selling price?

From: (total)(rate) = percentage

($7.99)(10%) = percentage Often spoken as: "10 percent of $7.99" is:
($7.99)(0.10) = $0.799 : Which is $\$0.\underline{80}$ when rounded to the nearest penny. In rounding, as with addition, you must still consider carries. The cost of the item will be 80 cents less than the previous value.

New price = Old price - Reduction
New price = Old price - (10% of Old price)
New price = $7.99 - $0.80
New price = $7.19 : new selling price. Any sales tax should now be considered on this new and lesser value.

This problem may have been solved knowing that if 10% is removed, (100% - 10%) = 90% of the price remains, and 90% of $7.99 is:

($7.99)(90%) = $7.19

You can easily find 10% of a numeric decimal value if you move the decimal point one place to the left. This effectively multiplies that value by (1/10) = 0.10, or divides that value by 10.

Ex. 10% of 4.57 = 0.457

Just the same, to find 1% = 0.01 of a value, simply move the decimal point two places leftward, which is the same as multiplying that value by 0.01, or dividing it by 100.

Ex. 1% of 103 = (103)(0.01) = 103/100 = 1.03

Ex. 1% of 47 = 0.47

If you double this 1% value of 0.47, you then can find 2% of 47. Showing some helpful intermediate steps:

2% of 47 = (1% of 47) + (1% of 47) = 2 x (1% of 47) = 2 (0.47) = 0.47 + 0.47 = 0.94

The last equation was probably the easiest or quickest way to find the result, but it may not have been so (initially) obvious of just adding the 1% value twice so as to have the 2% value : 0.47 + 0.47 = 0.94

If you know 10% of a value, you can double that percentage value to find 20% of that same value.

Ex. If 10% of 100 is 10 , 20% of 100 = (10% of 100) + (10% of 100) = (10% of 100)(2) = 2(10) = 10 + 10 = 20

It is somewhat easy to find 50% of a value since 50% = 0.50 = 50/100 = 1/2, that is, you can simply divide a value by 2 to find 50% or "one half" (or simply: "half") of it.

Ex. 50% of 60 = (60/1)(1/2) = 60/2 = 30 : Likewise, if you divide a value by 2 or "in half", you have the 50% portion and value of it.

Ex. An item is on sale for \$5.20 and this value is indicated as being 30% cheaper. What was the original selling price?

There is a straight-forward algebraic approach to solving this, but since this book has not yet covered much of algebra yet, a simple mathematical method will be shown first:

If 30% was taken away (the reduction) from the 100% value (the original/starting or previous value), this leaves us with a value equivalent to:

100% - 30% = 70% of the original value (price). Hence, the ratio (fractional or percentage value if you will) of the current selling price (with respect) to the previous price is 70% or 0.70 in decimal form:

$$\frac{\text{current-price}}{\text{previous-price}} = 0.70$$

: "the current-price is 70 percent of the previous-price".
Also, 100% - 30% = 70% = 1.0 - 0.30 = 0.70

$$\frac{5.20}{\text{previous-price}} = 0.70$$

therefore, mathematically, as when checking a division problem we multiply the quotient and the divisor together to find the dividend, or an expression for it:

$$5.20 = \text{previous-price} \times 0.70$$

and therefore, dividing the product by one factor to find the other factor. as when solving or checking a multiplication problem:

$$\frac{5.20}{0.70} = \text{previous-price}$$

simplifying this,we find that:

previous-price = \$7.43 : original selling price

You may also consider this check:

current-price = previous-price - reduction in price
current-price = previous-price - 30% of the previous-price
current-price = 7.43 - (7.43)(0.30)
current-price = 7.43 - 2.23
current-price = 5.20

Since: previous or original price x percent reduced = reduction , we can get:

$$\frac{\text{reduction}}{\text{previous-price}} = \text{percent reduced}$$

$$\frac{2.23}{7.43} = 0.30 = 30\%$$

: "2.23 is 30% of 7.43" or: 2.23 = 30% x 7.43 or: 2.23 = (30%)(7.43)
In these above expressions, you may also use 0.30 for 30%, such as for actually doing or calculating those mathematical equations.

It could be stated, as indicated in the equations, that:

30% of the previous or original price is the reduction in price.

The current price is 70% of the previous or original price.

A similar, but a more algebraic approach, using the concepts of signed numbers that are yet to be discussed, to solve this problem would be:

previous-price - reduction = current-price   using the concept of substitution to substitute the values we know:
(1.0) previous-price - (0.30) previous-price = 5.20   Combining the like terms of: previous-price:
(0.70) previous-price = 5.20   After dividing both sides by 0.70 and canceling:

previous-price = $\frac{5.20}{0.70}$ = 7.43

The most straight-forward way to solve this problem above is to use the "percentage formula". Assigning \$5.20 as the actual or specific value of the percentage (left or remaining) of the Total value, and expressing that (100% - 30%) = 70% of the (unknown) Total is \$5.20:

Total x Rate = Percentage
? x 70% = \$5.20   Solving for Total = ? , dividing the product by one factor to find the other factor:

Total = $\frac{\$5.20}{0.70}$ = \$7.43

Ex. If you were to receive 50% of only 1/4 of something, how much of that something would you then receive?

First, note that what we are actually trying to find is a fractional value (here, expressed as a percentage; 50%) of another fraction (here 1/4), in short, we are to find a fraction of a fraction. This topic has been discussed previously in this book. (1/4) of something is 0.25 = 25 percent of something. You are to receive 50% of 25% of something.

Since 50% is 0.50 in decimal form, and we know that we can multiply this decimal form to the total value in question so as to find the actual percentage value:

$\frac{(0.50)(1)}{1(4)} = \frac{0.50}{4} = 0.125 = 12.5\%$ of something

Or, since 1/4 = 0.25 (or 25%) in decimal form:

(0.25 of something) (0.50) = (0.25)(0.50) of something = 0.125 of something , or = 12.5% of something

Note that since fractions and indicated percentages of a value are only relative to a specific value in question, "something", as in this example, can actually be any thing or value, and is considered as 100% = entirety or Whole.

50% of 25% = $\frac{(50)}{(100)}\frac{(25)}{(100)} = \left(\frac{1}{2}\right)$ of $\left(\frac{1}{4}\right). = \left(\frac{1}{2}\right)\left(\frac{1}{4}\right). = \frac{1}{8} = 0.125 = 12.5\%$

Ex. If 4 is 20% of something, (or same as: "if 20% of something is 4"), what number (ie. the actual percentage value) corresponds to 30% of that something?

We can easily solve this by setting up a proportion type of problem. Even though the concept of proportions has not been covered up to this point, let's first consider the common (formula) percentage type of equations that can be created with the information stated in the problem:

From: total x percentage rate = percentage

Equation 1: (total)(0.20) = 4
Equation 2: (total)(0.30) = (number)

Since the indicated percentage increases from 20% to 30%, the (number) to be found will be greater than 4.

Taking each equation and considering the two factors (that make the product) in each, and solving for (total):

$$\text{total} = \frac{4}{0.20}$$

: Note, this here could also be solved by division, and then substituting this quotient value into the following, derived from the second equation:

$$\text{total} = \frac{\text{number}}{0.30}$$

Since the right-hand side of both of these equations is equal to: total , they can both be equated as being equal. The resulting equation is an example of what is known as a proportion (same or equivalent portions, parts, or fraction of the whole) type of expression:

$$\frac{4}{0.20} = \frac{\text{number}}{0.30}$$

: a proportion type of problem, solving for number :
: This might be read or spoken as "4 is to 20%, as (equal to) number is to 30%".
: Note also that the left hand side could of been simplified here to: 4 / 0.20 = 20

If may not be apparent to you at this point, but notice that since each side is a fraction, and that both sides are equal, then the fractions can only be equivalent fractions and the problem could be solved as such. Much more will also be said about proportions later in this book. The two equations above for: total , are also known as a "system of equations" or as "simultaneous equations" since the value for "total" will solve both equations simultaneously (at the same time) or together, or "at once" when it is found just once, and since it has the same value for both equations.

Considering the entire left hand side of the equation as a quotient value of a division problem, such as on the right hand side, we know that to check a division problem, you multiply the quotient and divisor together and check if it equals the dividend. Indicating (expressing) this with the stated values:

$$\text{number} = (0.30) \times \left(\frac{4}{(0.20)}\right) = \frac{(0.30)(4)}{0.20}$$

after simplifying the right-hand side :

number = 6

Extra: Here is an example of how the value of: "total" can be found which could then be used to solve any of the original two equations:

From: $\frac{4}{\text{total}} = 0.20 = 20\%$: mathematically expressing that "four is (only) twenty percent of "total"

Multiplying the quotient and divisor together should yield the dividend:

4 = 0.20(total) : as was stated in the problem, and the equations: (Eq. 1)

Dividing 4 by one factor should yield the other factor, hence solving for "total", and this is also the result of dividing both sides of the equation by the same value of 0.20:

$$\text{total} = \frac{4}{0.20}$$

total = 20

Checking the value of number: (total) (0.30) = (number)
20 x 0.30 = 6.0 : checks

Here are some more examples of using percentages:

Ex. If 26% of something (perhaps an amount of money) is given to a person A, and then person A promises 35% of his share to person B, what percent of the original total (the 100% = 1, entire, all, full value, of the money) will person B actually receive?

First, note that this is a problem pertaining to a fractional value of another fraction. Person B will receive a fraction of Person A's fraction of the total.

Let's start by converting the percentages to decimal values:

26% = 0.26

and 35% of this is:

$$\begin{array}{r} 0.26 \\ \times\ \underline{0.35} \\ 130 \\ +\ \underline{780} \\ 0.0910 \end{array} = 9.1\%$$

: (0.26)(0.35) , a fraction, of another fraction

: Person B will receive 9.1% of the total money

Notice that the problem above is a relative or relative values (ie. with respect to one another, and not something else) type of problem and that the values are all relative (ie., not specific in actual values which could be any values conceivable) in the form of percentages. No actual amounts such as the total money value was used. The total money value could be any value, say from 1 dollar up to 1000 dollars, and the actual value does not matter for this type of problem, and the same (indicated) percentages (more specifically, the percentage rates) would result regardless of the actual values being considered.

Ex. If you are given $1.25 from a total of $7, what percent of this total did you receive?

First, consider that you received only a part or fraction of $7 since $1.25 is less than $7. This problem is solved by placing the part you received over the total, hence making a proper fraction:

$$\frac{1.25}{7.00} = 0.17857 = 17.857\%$$

: You received $1.25 which is 17.857% of $7.
Again, formally, $1.25 is the (resulting) percentage value
due to applying the percentage rate of 17.857%

CONVERTING A FRACTION TO A PERCENTAGE

If you have 5/10, or "five-tenths" of something, what percentage of that something do you have? Here, we have 5 parts of something divided into only 10 equal parts. Since a percentage is the result wanted, we really want to know how many parts we would have if we divided that something into 100 equal parts instead of 10. Considering both of these fractions, the denominator of 100 is ten times more than the other denominator of 10. Hence, we can make an equivalent fraction where the numerator and denominator are both ten times larger:

$\frac{5}{10} = \frac{\quad}{100}$ creating an equivalent fraction:

$$\frac{5}{10} = \frac{5(10)}{10(10)} = \frac{50}{100}$$

$\frac{50}{100}$ can be written as: $50\left(\frac{1}{100}\right) = 50\%$: If you have five-tenths of something, you have fifty-hundredths or 50% of something

If we perform the indicated division and multiplication, we get:

$$\frac{50(1)}{1(100)} = 50(0.01) = 0.50 = 50\%$$

If we perform the indicated division on the original fraction, we should also get this result without the need for creating the equivalent fraction. After all, equivalent fractions represent the same value, and if you find the value of one fraction, you have found the value of the other equivalent fraction:

$$\frac{5}{10} = \begin{array}{r} 0.5 \\ 10\,\overline{)\,5.0} \\ \underline{50} \\ 0 \end{array} = 0.50 = 50\%$$

This method can also be verified using factorization, and then canceling out common factors (to both the entire numerator and entire denominator):

$$\frac{50(1)}{1(100)} = \frac{5\overset{1}{\cancel{(10)}}(1)}{1\underset{10}{\cancel{(100)}}} = \frac{5}{10} = 0.50 = 50\% \qquad \text{or:} \quad (50/100) = (\overset{1}{\cancel{50}}/\underset{2}{\cancel{100}}) = 1/2 = 0.5 = 50\%$$

We see that the decimal equivalent of a fraction is equal to the fractional value or percentage (rate) that the numerator is (in reference, or respect) to the denominator, or it can be stated as the decimal equivalent of a fraction is equal to the rate that the numerator is to the denominator. The decimal equivalent of a fraction is also equal to the "magnification factor" or "multiplication, multiplier, or factor" of how many times bigger (or even smaller, if its less than 1, hence it's only a fractional part) that the numerator is in reference to the denominator. Consider:

numerator x magnification = denominator therefore: $\frac{\text{numerator}}{\text{denominator}} =$ magnification or fractional value

factor1 x factor2 = product therefore: $\frac{\text{factor1}}{\text{product}} = \text{factor2} = \frac{\text{dividend}}{\text{divisor}} = \text{quotient}$

Given: $\frac{5}{10} = 0.50 = 50\%$, we can then say:

"5 is 50% of 10" or = : "50% of 10 is 5" : If you substitute the equals symbol for "is", you will see that these two statements are the same, but the expressions on both sides of the equals sign were switched.

Ex. 7 is what percent of 8 ?

Assuming there are 7 parts in question of something divided up into 8 equal parts, and expressing this as a fraction, we get:

$\frac{7}{8} = 0.875$.

1

0.875 is the decimal equivalent of the percentage

Note, if we divide both the numerator and denominator by the denominator, we get: (7/8) / (8/8) = (7/8) / 1 = $\frac{0.875}{1}$ = 0.875

To show this value with the percent symbol, multiply it by 100 or simply move the decimal point 2 digit positions rightward:

0.875 = "eighty-seven point five, hundredths" = 87.5/100 = 87.5 (1/100) = 87.5%

We can then say that: "7 is 87.5% of 8" or=: "87.5% of 8 is 7"

To convert 87.5% back to its full decimal equivalent, show that 87.5% is 87.5 parts of 100 equal parts. This is easily done remembering that % = (1/100):

$$87.5\% = \frac{87.5}{1}\left(\frac{1}{100}\right) = \frac{87.5}{100} = 0.875$$

Hence, to convert an indicated percent to a strict, single numeric decimal value, divide it by 100 or simply move the decimal point 2 digit places leftward and remove the percent symbol.

Ex. A certain mixture contains 4 ingredients. The units of measurement for each ingredient could be ounces, teaspoons, tablespoons, cups, ounces, liters, milli-liters, and for this example, they will simply be indicated as (generic, generalization, unspecific) units so as to have a non-specific aspect. Regardless of the specific units (of measurement or reference) used, the percentage values would be the same. Here is the list of ingredients for the recipe (ingredients or substances, and instructions) of the mixture:

3 units of ingredient A : for example, all the same units of measurement might be cups, ounces or grams
2 units of ingredient B
1 unit of ingredient C
0.75 units of ingredient D

What percent of the entire mixture is ingredient B?

Summing up all the units included in the entire mixture:

3 units + 2 units + 1 unit + 0.75 unit = 6.75 units : total units of ingredients included in the entire mixture

$\frac{\text{amount of ingredient B}}{\text{total amount of all ingredients}} = \frac{2 \text{ units}}{6.75 \text{ units}} = 0.296\underline{3} = 29.63\%$: Ingredient B is about 30% of the entire mixture.

If you were to calculate each ingredients percentage of the total mixture, the sum of these percentages will be 100% = 1.0

Ex. An expected or ideal value of 10.0 is typical for a certain thing, but its measurement is found to be slightly over at 10.81. Express this difference in values as a percentage.

Difference = measured value - ideal value = 10.81 - 10.0 = +0.81

Expressing the difference as a percentage from the ideal value: $\frac{+0.81}{10.0} = 0.081 = 8.1\%$ over or increased

Note also for relative values: $\frac{10.81}{10.0} = 1.081 = 1.0 + 0.081 = 100\% + 8.1\%$ and $1.081 - 1.0 = 0.081 = 8.1\%$

Ex. Given 5, and if it is increased by 1, what percent, and multiplying factor (n) did 5 increase by:

$\frac{1}{5} = 0.2 = \mathbf{20\%}$, 1 is 20% of 5 , adding 1 to 5 is adding 20% of 5, increase to 5 , : Checking: $5\,(0.2) = 1$

$5 + 1 = 5 + (5)(0.20) = 5\,(1 + 0.20) = 5\,(\mathbf{1.20}) = 6$

$\frac{(5+1)}{5} = \frac{6}{5} = 1.2 = 1.00 + 0.20 = 100\% + 20\% = 120\%$, : Checking: $5\,(1.2) = 6$

We found that 1 is 20% of 5, and the multiplying factor to 5 is then 1.2 , and we can also find that:

1 out of the total of 6 = $\frac{1}{6} = 0.1667 = 16.6\underline{7}\%$ of the total, and $\frac{5}{6} = 0.8333 = 83.33\%$ of the total

$5/6 + 1/6 = 6/6$
$0.8333 + 0.1667 = 1.0$
$83.33\% + 15.67\% = 100\%$

FINDING A VALUE OR ITS' PERCENTAGE BETWEEN A RANGE OF VALUES

Given a range of values from 7 to 15, what percentage (%) from 7 (ie., seven will be considered as the start, or 0% value within the range specified) is 10, in other words, 10 is what percent of the "total numeric distance = 100% range of values" between 7 and 15? Also, if given a certain percentage (rate) between a range of values, what value between that range corresponds to that percentage (rate)?

Expressing some of this in number form, we are NOT finding this: $\frac{10}{15} = 66.\underline{7} = 66.\underline{7}\%$:Though this is mathematically right, this is not what we are solving for.

This last concept was mentioned because the denominator would then actually represents a range of 0 to 15, that is 15 is in reference to, and is the distance from, 0 and not some other value such as 10 as first mentioned.

We can use line segments (a small portion of a line that extends to infinity) an-or a number (ie., numbered, indicated) line as a graphic aid for analyzing these types of problems since a length value is essentially the difference between the start and end values on a line.

Line Segment 1: _____
Line Segment 2: ____________________

Line segment 1 is obviously less in length than line segment 2, and is therefore only a certain fraction or percentage of line segment 2.

Given two values, say 2 and 4, 2 is 0.50 = 50% of 4 since: 2/4=0.5 and 4 x 0.5 = 2 or:

$\% = \frac{\text{number1}}{\text{number2}}$: percentage that number 1 is with respect, or in reference to, number 2

$\frac{2}{4} = 0.5 = 50\%$: number1 is equivalent to a 0.50=50% portion of number2

These values can be thought of as the lengths of line segments:

```
----------------          : length1
       2

--------------------------------     : length2
               4
```

$\% = \frac{\text{length1}}{\text{length2}}$: percentage that length1 is with respect to length2

Each line segment can also be considered as a part of the same or different number lines where the difference between the maximum and minimum values is the length of that line segment.

Since: max. = min. + length : You can also think of length as distance. Algebraically:
length = max. - min.

```
     ----|----------------|----          : length = max. - min. = 10 - 8 = 2
 . . .   8               10    . . .      max. = maximum, here the maximum value
                                          min. = minimum, here the minimum value
```

We see that length and-or the difference or change between two lengths or values is actually calculated as a difference of the two value or locations. This difference or length between two values is sometimes called their "range" or "extent".

```
         ----|--------------------------------|---
. . .        3                                7    . . .
```

: length = max. - min. = 7 - 3 = 4
We also see that length is actually a "difference" which is a measure of separation. This difference could also be called the "change" from one value (here the min. value) to another (here, the max. value).

Below, a smaller line segment (line1) with a length of 0.5 is superimposed (placed on or over) on a larger line segment (line2) with a length of 2.0 that is within (bounded by) the range from 3 to 5.

```
        ----|----------|-----------------------------|---      : numbered line
. . .       3         3.5                            5    . . .
            |— 0.5 —|                                |
            |———————————————— 2 —————————————————————|
```

Using the above drawing, 3.5 is what percent (%) of this range from 3 to 5? This is very similar to the original problem stated above. Again, considering the numbers as line segments is a good method for such problems.

$$\% = \frac{\text{length1}}{\text{length2}} = \frac{(\text{max.1} - \text{min.1})}{(\text{max.2} - \text{min 2})} = \frac{\text{distance in question}}{\text{total distance}}$$: CORRESPONDING PERCENTAGE A VALUE IS WITHIN TWO OTHER VALUES

Note that here in this specific example, that min.1 of line1, and min.2 of line2 are actually the same value of 3.

$$\% = \frac{3.5 - 3.0}{5.0 - 3.0} = \frac{0.5}{2.0}$$

% = 25% : It could be said that 3.5 is 25% of and-or along the way of the distance or length from 3.5 to 5.

Now solving the original problem stated:

$$\frac{(10 - 7)}{(15 - 7)} = \frac{3}{8} = 0.375 \quad \text{or} \quad 37.5\%$$: This could be stated as "10 is at the 37.5% point in the range from 7 to 15". Or, "37.5% of the range from 7 to 15, is 10".

Likewise, given a certain indicated percentage (ie. the percentage rate or simply the rate) between some range of values, the corresponding value to that percentage can be found:

Solving for max.1 that is in the above formula:

max.1 = min.1 + (%) (max.2 - min.2) : CORRESPONDING VALUE TO A PERCENTAGE RATE WITHIN TWO VALUES (The minimum values are also the same.)

You can think of max. 1 as the number that is somewhere between the range of values of min. 2 and max. 2.

```
             0%                                                100%
        --------|--------------|--------------------------------------------|--------
. . .         min.2                                                       max.2      . . .
              min.1          max.1
```

Ex. Using values from the above example, what value corresponds to 25% (0.25 = 1/4 = "one-fourth") of the range from 3 to 5?

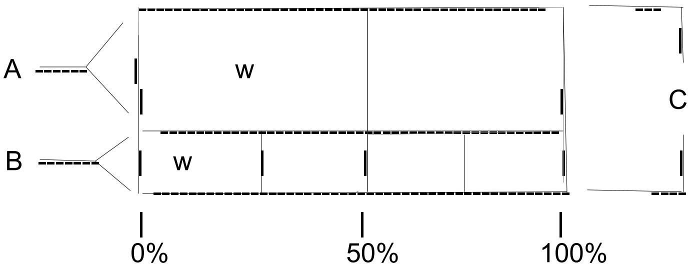
$\text{value} = \text{max.1} = 3 + (0.25)(5 - 3)$
$\text{value} = \text{max.1} = 3 + (0.25)\ 2$
$\text{value} = 3 + 0.5$
$\text{value} = 3.5$

Note also that if the min. values, or the "starting points", were at 0, then the equation simply reduces to:

$\text{max.1} = \text{min.1} + (\%)(\text{max.2} - \text{min.2})$
$\text{max 1} = 0 + (\%)\ (\text{max. 2} - 0)$
$\text{max 1} = (\%)\ (\text{max.2})$

A NOTE ON COMBINING PERCENTAGES

The ingredients for some manufactured products identified as Product A and Product B are to be combined to produce Product C. If 50% of Product A is water, and 25% of Product B is water, what percent of Product C is water (w)? [FIG 11]



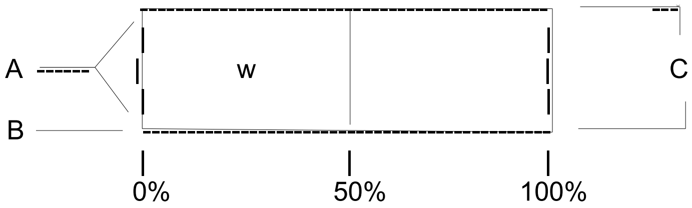
To begin with, the answer is **not** the sum of the percentages: 50% + 25% = 75% : a wrong method and solution

In general, the answer is also **not** the average percentage: $\frac{50\% + 25\%}{2} = \frac{75\%}{2} = 37.5\%$

This is so since the percent values are in reference, or with respect to, two different things, here Product A, and Product B, rather than just one thing in question.

That is, in general, relative values (ie. percentage values such as 50% and 25% here) cannot be combined by addition or averaging to produce another relative value. The only exception is when the units (perhaps weight or volume) of both (or more) products being combined have the same actual value or measurement.

Now consider product B getting smaller and smaller in weight, but still, 25% of it is water, even if its an negligible amount. The entire amount of units of Product C, including the amount of water, will also get smaller and smaller due to Product B getting smaller and smaller. As the actual units or amount of water in Product B becomes very small and meaningless, the percent of water in Product C will approach nearer and nearer to that of Product A. Hence, Product C will then contain only the water from Product A which is 50% water during that condition, and it is obvious that the percentage values cannot be simply combined to find the total percentage value for this kind of problem: [FIG 12]



: If Product B was very small negligible amount, it would still consist of 25% water.

The following is correct, but it is unsolvable given just the values mentioned:

Product C = product A + product B

This is also correct, but it is unsolvable given just the values mentioned, because we need their specific amounts.

Product C water /=/ 50% of product A + 25% of product B

To find the actual percentage of water in Product C, you must first know the actual amount (not just a relative value(s)) of water in both Product A and Product B. If Product A weighs 200grams, then 50%, or 200g x 0.50 = 100g is water. If Product B weighs 60grams, then 25% or 60g x 0.25 = 15g is water. When combined, Product C will contain:

Amount of water in Product C = Amt. of water in Product A + Amt. of water in Product B
Amount of water in Product C = 100g + 15g
Amount of water in Product C = 115g

The total weight of Product C is:

Weight of Product C = Weight of Product A + Weight of Product B
Weight of Product C = 200g + 60g
Weight of Product C = 260g

Technically, grams is a measurement of the mass (ie. a measurement of the amount of physical matter, atoms) of a substance and not the weight (a force due to gravity) of that substance. However, mass and weight are physically directly related, and mathematically proportional to each other. The ratios of two values with units of mass (such as grams), will be equal in numeric value to their corresponding ratio of their weights (such as units called Newtons).

If 115g of Product C is water, then 115g is: $\frac{115\text{g}}{260\text{g}} = 0.4423 = 44.23\%$ of the total weight of Product C is water.

If it was initially indicated that the amount of water in Product B is equivalent to 30% of the water in Product A, an equation for the relative amount of water in Product C in relation to only Product A, can be derived without knowing any specific values (amount of water, or weight) for either Product A or Product B. The amount of water in Product B is now indicated in reference to the amount of water in Product A, hence the amount of water in Product B is a fraction (here 30% = 0.30) of the amount of water in Product A and which is a fraction (0.50 = 50%) of Product A. Hence, the amount of water in Product B is a fraction of another fraction. The basic solution here is to substitute the mathematical equivalent of the amount of water in Product B with a value mathematically expressed in relation or respect to, or in reference of (ie. "in terms of") Product A. This will help create a simpler and-or a solvable equation:

water in Product C = water in Product A + water in Product B
water in Product C = (50% of Product A) + (30% of the water in Product A)
water in Product C = 50% Product A + 30% of (50% of Product A)
water in Product C = 0.50 Product A + (0.30 x 0.50) of Product A
water in Product C = 0.50 Product A + 0.15 Product A combining like values with the same units or "terms":
water in Product C = (0.50 + 0.15) Product A
water in Product C = 65% Product A : note, Product C still contains the amount of water in Product B, even though Product B is not included in this last expression. Also, the original description, only 50% of Product A is water, but in this expression here, the amount of water in Product C is expressed as being 65% of the water in Product A since this value includes, or takes account of, the percentage of water in Product B.

Ex. A mixture of 8 ounces weight (rather than volume or space measurement) contains 10% sugar by weight. If 1 more ounce of sugar is added into the mixture, what total percent of the mixture is now sugar? Let's first find out the number of ounces of sugar that were originally in the mixture, and then add 1 more ounce to that so as to find the total weight of sugar in the new or modified mixture:

total amount of sugar in the original mixture = (8 ounces)(10%) = (0.10) 8oz = 0.8oz

total amount of sugar in the new mixture = original amount + added in amount = 0.8oz + 1oz = 1.8oz

$$\text{percent of sugar in the new mixture} = \frac{\text{total amount of sugar}}{\text{total amount of mixture}} = \frac{1.8\text{oz}}{8\text{oz} + 1\text{ oz}} = \frac{1.8\text{oz}}{9\text{oz}} = 0.2 = 20\%$$

PERFORMING OR EXPRESSING DIVISION WITH MULTIPLICATION

The concept and process can be useful for re-expressing or simplifying expressions, and is as follows:

$\frac{\text{dividend}}{\text{divisor}}$ or = $\frac{\text{numerator}}{\text{denominator}}$ equals these expressions:

$$\left(\frac{\text{dividend}}{1}\right)\left(\frac{1}{\text{divisor}}\right) = \text{(dividend) (reciprocal of the divisor)}$$

Hence an alternative to division is to multiply by the reciprocal value of the divisor.

The process of finding the reciprocal is somewhat defeating to the entire process since it in itself normally requires division. However, the concept is important since it shows how to express or convert a division problem to one of a multiplication problem. Most people also find it easier to see and-or to perform multiplication than division, hence it is helpful to express an equation that uses some multiplication rather than division.

Ex. $\frac{1234}{5} = \frac{1235}{1}\left(\frac{1}{5}\right) = 1234\,(0.2)$: = 246.8

If you make a table of the reciprocals of the prime numbers, you can then solve for (non-prime) composite number reciprocals (which may or may not contain a fractional portion) by factoring the number into prime factors whose reciprocals are already known. Solving a reciprocal is therefore converted to a problem of multiplication. For example:

number	reciprocal of number = 1/number	
1	1	
2	0.5	
3	0.333333333	
5	0.2	
7	0.142857142	: clearly, the larger the number, the smaller it's corresponding reciprocal is

Ex. Find $\frac{5}{6}$

$$\frac{5}{6} = \frac{5}{1}\left(\frac{1}{6}\right) = \frac{5\,(1)(1)}{(2)(3)} = \frac{5}{1}\left(\frac{1}{2}\right)\left(\frac{1}{3}\right) = 5\,(0.5)\,(0.333333333) = 0.833333333$$

Note above, that the prime factors of 6 are 2 and 3. If you multiply the reciprocals of 2 and 3, you will have the reciprocal of 6:

$$\frac{1}{6} = \frac{1}{(2)(3)} = \frac{(1)(1)}{(2)(3)} = \frac{1}{2} \times \frac{1}{3} = 0.5 \times 0.333333.. = 0.166666666\underline{7}$$

Ex. Find $\frac{1}{0.6}$

First using an equivalent fraction method:

$$\frac{1}{0.6} = \frac{1\,(10)}{0.6(10)} = \left(\frac{1}{6}\right)10 = \left(\frac{1}{3}\right)\left(\frac{1}{2}\right)10 = (0.333333333)\,(0.5)\,(10) = 1.666666666$$

Now using the concepts of "scientific" and exponential notation:

$$\frac{1}{0.6} = \frac{1}{(6)(10^{-1})} = \left(\frac{1}{6}\right)(10^{+1}) = \left(\frac{1}{3}\right)\left(\frac{1}{2}\right)(10) = (0.333333333)(0.5)(10) = 1.666666666$$

AVERAGE

An average is a single numerical value that can be used to represent an entire set of group numerical values. The average will always be a value between the minimum and maximum values of the specific set of data values given or used, and may, or may not, be equal to any member of that set of values. The mathematical definition of average is (initially, or loosely) based upon the "half-way" or "center-point" point between two values of which you can imagine as the "end-points" on a line-segment (a small part or section of an infinite line) which bound that half-way or "middle" point:

```
<-------------------|-----------------------------------------|-----------------------------------------|--------------->
                min. value                               mid. value                                  max. value
                                                       "half-way" value
```

$$\text{"half-way" value} = \frac{\text{sum of two values}}{2}$$

:Halfway (or "mid-point" between two values) Value Formula
The two values in question can be called the extreme
values or "end points".

The value or distance from both of the two end points (ie. the minimum and maximum extreme values) to the "half-way" point is exactly the same. The total distance or value between those defining, bounding, or end points is equal to the length or difference between those two points or values:

difference = maximum value - minimum value : a (numeric) difference is the amount of (numeric) separation, and is also known as the change in one value with respect to, or in reference to, another value.

The halfway-value formula is only slightly intuitive and may need some further explanation. First, the formula is based on the half (or half-value) of any given number, and that half of a value is equal to the given value divided by 2:

$$\text{half-value of a number} = \frac{\text{number}}{2}$$

: here number is actually equal to the max. value, and 0 is equal to the min. value in relation to the "half-way" value formula. This could be expressed as:

half-value of a number
or "halfway" from 0 to number

$$= \frac{0 + \text{number}}{2} = \frac{0}{2} + \frac{\text{number}}{2} = 0 + \frac{\text{number}}{2} = \frac{\text{number}}{2}$$

The value of half of a number, is also (equivalent to) the "halfway-value" between 0 and that given number. This "halfway-value" is half of the "distance" or amount from 0 to number. This value is exactly between those two given values, and is therefore said as being "equidistant" (equal, or equivalent, in distance) from them. Here, 0 can be considered as the starting or minimum value of this distance (or range of values), and number can be considered as the ending or maximum value of the distance (or range of values).

Note, for ex., if given 6 things, it is incorrect to consider the average as being a value between the third and fourth thing:

```
                        Between 3 and 4 , is  3.5 , =  the value between  3 and 4, and is not the average of 6.
                                                  3.5 is actually the average value of 3 and 4 = (3 + 4)/7 = 7/2 = 3.5
      ( 6 things)                       ↓
    * * * * * *           * * *    * * *
                          1 2 3    4  5 6          : showing each thing or item enumerated (assigned with an
                                                     increasing  number, with the last number being the count or
                                                     total of all the items)
```

The average (point or location, the "halfway-point") is a numerical value (formally defined as) between two values or numbers is considered as being between a min. value, and a max. value, and for here, it is: 0 and 6. That is, 0 is actually the (unspecified) min., or "start value" (to consider, or include in the calculation, so as to consider the distance or values between 0 and 1), and it is actually the reference value of what 6 is in reference to, and therefore, 0 must also be considered for the average. (6 + 0) / 2 = (6/2) = 3 (and not 3.5) . 3 is technically the equidistant value from 0 and the max value 6. Consider that length or distance values start at 0, and not 1, that is, there is still some distance to consider between 0 and 1 which cannot be ignored. This will also account for the average of fractional values (ie. less than 1). A single value, such as 6, technically, does not have an average among several values to consider. Still, you can divided any number or value into two equal valued parts by dividing it by 2. Consider that If you had some data values all being near (about, close to), or even the same value of 8, and therefore, the average, common or typical value would be said as being about 8, and not (8+0)/2 = 8/2=4. In short, the average value of a set of data or values, may not even be the "half-way" or "mid value" as when 0 is considered, and therefore, it is generally incorrect to consider an average or representative value as the same thing as the "half-way" value.

Sometimes the word "range" or "range of values" is used as a generalized indication of the distance or separation (difference) between the min. and max. values. This may be spoken as something like, for example: "the range of the data values or numbers start at 300, and end at 400", or "the (range of) values go from 300 to 400". The specific size of this specific range (of data values) is the difference between the maximum and minimum values, and here, it is:
(400 - 300) = 100. If you had several various data values, of say having values that are between or within 500 and 600 in value, the (entire) range of values is still 100 since (600-500 = 100). It could also be said that the rage of values extends from, and-or, are within the values of, or from, 500 to 600.

Don't confuse the actual "half-way" point value with half of the distance or physical length (ie., a difference) between the two points. For example: Given values of 103 and 105, the half-way point or value is 104, and the difference or length between these defining, bounding, or extreme points is only 105 - 103 = 2, and half of this distance is just 1. Clearly, the "half-way" point or (data) value (of all the data values), and the half-length (ie., half of the difference or (range)) distance between the max. and min. extreme values, or "end points", is not the same concept and-or value.

Given 0 and 10, what is the "half-way" or "middle" value?

$$\frac{10 + 0}{2} = \frac{10}{2} = 5.0$$

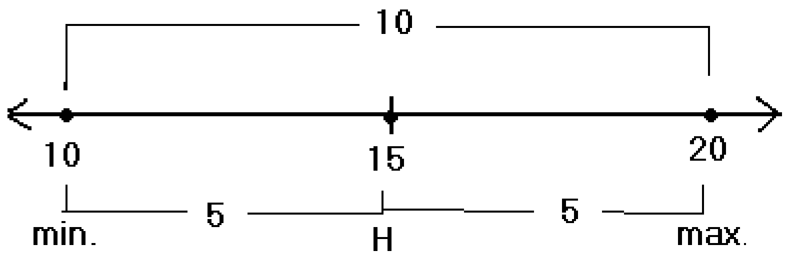
In the example.

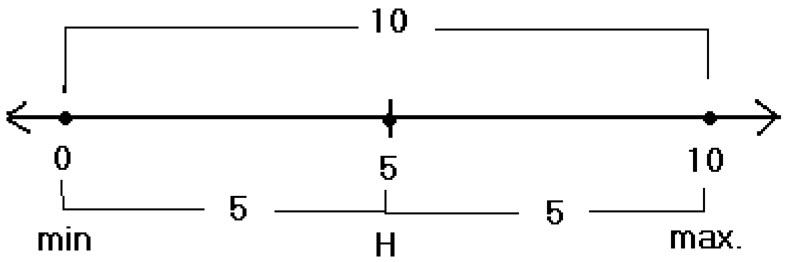
difference = max. - min = 10 - 0 = 10. This also verifies the "halfway-point", or "point-halfway", value formula above. 5 is both the halfway value (between 0 and 10) and is also equal to the difference (here, 5) that this halfway values is from both of the extreme or end values, here 0 and 10. Clearly, as expressed, 5 equal to just half of the difference (here indicated as: max - min. = 10 - 0 = 10) in the two given values, as seen expressed as: 10/2 = 5. The distance from 5 to the two "end points" is the same, so therefore, twice (2) of the "halfway" value" (here 5) that is between the two given extreme or end values is equal to the sum of those two values (here, 0 and 10 = 10).

(halfway-value) x 2 = sum of the two values. Therefore, mathematically:

halfway-value = sum of the two values / 2 :as expressed in the initial formula.

At this point in the discussion, we can consider this illustration given these two sets containing just two numbers:
(10 and 20), and (0 and 10): [FIG 13]





In both cases, the difference (D) from, or between, the minimum value to the maximum value is (max. - min) = 10, and half of this difference is 5. H is the halfway point and value between the minimum and maximum values. Also:

H - min. = max - H : = D/2 = (max. - min.)/2
Adding min. to both sides of this equation:

H - min + min = max - H + min since a value added (here, min), and then subtracted is equal to adding 0:
H - 0 = max. + min - H : adding H to both sides:
H + H = max. + min = - H + H combining 1H + 1H, we get (1 + 1)H = 2H
2H = max. + min. expressing the division of both sides by 2, and canceling common factors:

$H = \frac{\text{max + min.}}{2}$: formula for the halfway point or value between two values

Also: H = min. + (D/2) = max. - (D/2) : since (H-min) = (D/2) and (max-H) = (D/2)

However, even with all that has been mentioned, an average value is only based on the "half-way" value concept, and it is incorrect to use this as a general way or formula to find the average of any set of values, especially where there is more than two values in the set to consider. An average is only equal to the half-way value if there are only two values involved. When more than two values are involved, another more generalized formula for an average must be used, and this will be derived (developed) in the discussion that follows.

Given a set that contains (data) values which are identical, it is common sense that the representative or average value of those values is equal to any one member of that set.

Ex. A set contains 100 and 100, what is the average?

average = 100 : since a value of 100 is said to, or can represent this set (all the values) very well.
You could say something also like: "The typical or representative value is 100".

Considering the last example, if one or more of the values is only slightly different, it is fair to say that the average will still be about the same value of 100. For example, if this new set contains values of 100 and 101, what is the average? You can make an estimate of the average here at being slightly higher than 100. The actual average is 100.5

Ex. A set contains 5, 5, and 5, what is the average?

average = 5 :5 is said to represent this set very well.

Even though the average of the above sets is quite obvious, what is a numerical formula for this average given any set which contains all the same values? Notice that the sum of the values ("elements", "members", or "data") of the set (list or group of things related) divided by the number of individual values or members (of that set) is equal to the average value of that set of values:

$$\frac{100 + 100}{2} = \frac{200}{2} = 100$$

$$\frac{5 + 5 + 5}{3} = \frac{15}{3} = 5$$ now expressing this into a general formula:

$$\text{average} = \frac{\text{sum of values}}{\text{number of values}}$$: AVERAGE FORMULA (when all the data values are the same)

Given values that are not identical, say 2 and 4, what is their average? Does the formula above hold true for different values? It should be obvious that the average should not be either 2 or 4, since the values are not the same and are different to each other. As you may have guessed, after viewing the previous discussions, that the average is somewhere or a value between those two values.

Here is a representation of all the values from 0 up to any value that we can call or indicate as B. Letter (or mathematical variable) A represents the half-way point and average (A) of these two values:

0--------------------A--------------------B

Here, the value of $A = \frac{B+0}{2} = \frac{B}{2}$ since there is only two values involved and one value is 0.

If we assign the variable of C to represent the value of 0 above, we get:

C--------------------A--------------------B

$A = \frac{B + C}{2}$: formula for average of any two values

If we let both C and B increase by 1, you would expect their half-way point and-or average to increase by 1 also, and so that it is still "centered" or "midway" between them.

The average of 0 and 2 is 1. If the members of this set are increase by 1, the average (ie, the "center" of the two values) of this set would naturally be expected to also increase by 1. If the values increased by (1), such as values (0) and (2) increased by (1): (0+1) and (2+1), or= the new values of (1) and (3), their average is now equal in value to the average of (0) and (2), and also increased by (1). The new average value is: (old average + 1) = (1 + 1) = 2.

Let's verify this concept a bit more mathematically in equation forms:

Let's see if this equation is correct: $(A + 1) = \frac{(B+1) + (C+1)}{2}$ simplifying the right side:

$A + 1 = = \frac{B + C + 2}{2}$

Since we know that $A = \frac{B + C}{2}$ we can substitute this value for A into the previous equation:

$\frac{B + C}{2} + \frac{1}{1} = \frac{B + C + 2}{2}$ creating like fractions, as needed for their addition, on the left side:
(: note 1 = 1/1 = 2/2, and then combining fractions of the left side:)

$\frac{B + C}{2} + \frac{2}{2} = \frac{B + C + 2}{2}$ combining fractions on the left side:

$\frac{B + C + 2}{2} = \frac{B + C + 2}{2}$:checks

Ex. The average of 2 and 4 is: $\frac{2 + 4}{2} = \frac{6}{2} = 3$, now consider:

If one member increased by 1, and the other member decreased by 1, you would expect the average to remain the same in order to still remain "centered" between the two. This is verified below, where C is decreased by 1, and B is increased by 1:

$A = \frac{(C - 1) + (B + 1)}{2} = \frac{C - 1 + B + 1}{2}$ simplifying using the commutative law of addition:
Also note that: $-1 + 1 = +1 - 1 = 0$

$A = \frac{B + C}{2}$: checks , the average remained the same

Ex. Using the last examples values: $\frac{2 + 4}{2}$

$$\frac{(2 - 1) + (4 + 1)}{2} = \frac{1 + 5}{2} = \frac{6}{2} = 3$$

Now, let's give some verification to general average formula when the average of more than two different values is to be found:

Average 100 and 100. According the above discussions:

$\frac{100 + 100}{2} = \frac{200}{2} = 100$

If we average 99 and 101, the average should be, and somewhat obviously a value between those two values: = 100

$\frac{99 + 101}{2} = \frac{200}{2} = 100$

Likewise, now average 98 and 102, and we should expect the result to be the same:

$$\frac{98 + 102}{2} = \frac{200}{2} = 100$$

Now with three values: 100, 100, and 100:

$$\frac{100 + 100 + 100}{3} = \frac{300}{3} = 100$$ which can be mathematically written or expressed as:

$$\frac{100}{3} + \frac{100}{3} + \frac{100}{3} = 33.333... + 33.333... + 33.333... = 100$$ combining (adding) the first two fractions we get:

$$\frac{200}{3} + \frac{100}{3} = \frac{300}{3} = 100$$

Now with other values: 99, 100, and 100:

$$\frac{99 + 100 + 101}{3}$$ which can be mathematically written as:

$$\frac{99}{3} + \frac{101}{3} + \frac{100}{3}$$ by combining (adding) the first and second fractions:

$$\frac{200}{3} + \frac{100}{3} = \frac{300}{3} = 100$$: and this is the same equation expressed above

Now, perhaps when it's less obvious:

Average 1, 2 and 297

$$\frac{1 + 2 + 297}{3} = \frac{300}{3} = 100$$ which can be written as:

$$\frac{1}{3} + \frac{2}{3} + \frac{297}{3}$$ which can be written or expressed as:

$$\frac{1}{3} + \frac{2}{3} + \frac{(200 + 97)}{3} = \frac{1}{3} + \frac{2}{3} + \frac{200}{3} + \frac{97}{3}$$ combining some terms:

$$\frac{200}{3} + \frac{100}{3}$$: same fractions as in the previous examples above, and this can even be expressed as:

$$\frac{100}{3} + \frac{100}{3} + \frac{100}{3} = \frac{300}{3} = 100$$ Giving verification to the general average formula, for when the values are not the same.

When finding the average, a data value of extreme value, either large or small, can effect the value of the average in such a manner that this average, if used, does not fairly represent most of the data values. For example:

The average of: 1, 3, 3, 5, 6, 7, 8 is = (1 + 3 + 3 + 5 + 6 + 7 + 8) / 7 = 33/7 = 4.71

Now here is the same set of data, but instead of a data item with a value of 8, there is a value of 300:

The average of: 1, 3, 3, 5, 6, 7, 300 is 46.43 : this new average is not even close in value to most of the values in this set.

Often, "extreme" values, when not to be expected and/or considered, are not to be used when finding the average. These "abnormal" values could be an indication of some type of measurement and-or communication error. Where the average value is concerned, they would be discarded (removed, ignored) and not considered (here added) in the sum of the set members, and in the number of the total members in the set.

The word average is sometimes referred to as the arithmetic mean ("in common with", or "about") or simply "mean". An average value of a set of data values, say a data set represented or identified as x, is frequently shown as an x with a bar-line over it:

$\bar{x}$: expresses and represents the average value a the set of values, here identified as set x

A common use for average is in finding a single value that can represent several values such as test scores or measurements. Average has widespread use in the branch of mathematics called **statistics**. In brief, a statistical value is a single and a particular representative value of many data values of a group ("ie., set) of data. A statistic provides a quick numeric grasp or understanding of a situation and can be used for estimates, expectations and-or predictions. Average will be discussed again further ahead in this book during the topics of average deviation and standard deviation, and due to that the topics of square roots and signed numbers need to be discussed first.

COMMON MONEY PROBLEMS

Money can be described as a medium of exchange or trade, and-or a storage of wealth. The concept of **bartering** is about negotiating the exchange or trade of products between two parties or people. For example, the products might be that one person has wheat, and the other person has pottery. Often one or both people do not have or need what the other party wants or has, and so a concept called "**money**" was created as a standardized exchange medium of which both people would be happy to and-or mandated to exchange and-or accept in place of the actual desired product that is unavailable, and then use that money later and elsewhere so as to purchase the product(s) they need.

Typical monetary (money) problems involve finding the total money value when given some whole dollars and coins (which essentially are fractions of a dollar or dollar unit), and how much change there is after a purchase. Here, change is the difference between the total amount given to pay for a product and the actual price of that product. When a product is purchased, often some extra money, such as paying with an even amount of dollars, is used to pay for the item(s), hence some value (ie., "change" or the difference) needs to be given back to you. Change in the monetary sense is often spoken as something like: "What you will get, or be given back, after a purchase". Some of the concepts of money and fractions, particularly with coins, has been previously discussed in this book during the topic of: adding money as fractions.

amount given - price = change : change essentially means the difference in, or of, values, hence it is mathematically equal to the amount of separation. When a value has increased or decreased, it is said to have changed since it is now a different value. The amount that is different is called the change value. The difference between any two values is also then called the change value, even if none of the values actually changed in value.

A good method to understanding money (particularly coins) problems is to understand the penny. For instance, someone might ask, "How many pennies are in a dime"? In the physical sense, the answer is none, since a penny is a coin, not a monetary value, and a dime is a separate coin unto itself. There is however the equivalent of 10 cents, not pennies, in a dime since a dime's 10 cent monetary value is equivalent to, or can represent, the monetary value of 10 pennies which is also 10 cents. Even though this question is probably "understood" by most, it may of been better to say, "How many pennies are equivalent (in monetary value) to a dime"?

The basic unit in our monetary system is the dollar. A penny's monetary value is defined as one-hundredth (= 1/100) of 1 dollar, and this value or "derived" (and fractional) unit is commonly called a cent. A cent is often symbolized as letter c, or c with a vertical line through it. The numerical representation of 1 cent is therefore:

$1 \text{ cent} = \frac{1 \text{ dollar}}{100} = 0.01 \text{ dollar}$: a cent is one-hundredth of a dollar ($1). Again, dollar is a unit of measurement here, and not a specific numeric value or variable. Cent is the name of a (smaller, fractional) unit of measurement derived from the (base of the monetary system) units of dollars.

A dollars' monetary value is therefore equivalent to 100 cents. Since (0.01)(100)=1.00, or (cent)(number of cents) = (total dollars), given some total number of cents, simply move the decimal point 2 places leftward (the same as dividing by 100) to find the decimal equivalent in dollars.

Ex. 365cents is equivalent to: $\frac{365}{100} = 3.65$dollars.

Likewise, given a total dollar value, simply move the decimal point two places rightward to find the equivalent number of cents.

Ex. $1.79 is equivalent to: 1.79 x 100cents = 179 cents : $ means dollars, and an indicated value is therefore in reference to, and has units called **dollars**
$ is called the "dollar symbol".

The monetary value of a nickel coin is equivalent to that of 5 pennies:

1 cent + 1 cent + 1 cent + 1 cent + 1cent = 1 cent x 5 = 0.01 x 5 = 0.05 : "five cents"

Here are the monetary values of the common coins:

COIN	CENTS (c)	DECIMAL EQUIVALENT (in dollars, $)	FRACTIONAL EQUIVALENT (of a dollar, $)
PENNY	1	0.01	1/100 = "one-hundredth of a dollar"
NICKEL	5	0.05	5/100 = 1/20 = "one-twentieth of a dollar"
DIME	10	0.10	10/100 = 1/10 = "one-tenth of a dollar"
QUARTER	25	0.25	25/100 = 1/4 = "one-quarter of a dollar"

Given several coins, to find the total monetary value, simply sum up the corresponding cent values for all the coins. If some coins are the same, multiplication can be used instead of repeated addition.

Ex. How much money is 1 quarter, 3 dimes, 4 nickels and 2 pennies?

COINS	MONETARY VALUE ($)	CENTS (c)	
1 quarter	0.25	25	: 0.25 x 1 = 0.25 = 25 cents x 1 = 25 cents
3 dimes	0.30	30	: 0.10 x 3 = 0.30 = 10 cents x 3 = 30 cents = 0.30
4 nickels	0.20	20	: 0.05 x 4 = 0.20 = 5 cents x 4 = 20 cents = 0.20
2 pennies	0.02	2	: 0.01 x 2 = 0.02 = 1 cents x 2 = 2 cents = 0.02
	+ ---------	+ ----------	
	$ 0.77 =	77c	

If you were to give 77c to someone, you could use the coins mentioned in the last example, however, to give the least number of coins, first find the highest number (multiple) of the larger (in monetary value) coins first and of which the total monetary value of those coins does not exceed the required change. As a simple example, rather than give 25 pennies for 25c, you could simply give 1 quarter for 25c. For the 77c example above, 25 cents (from 1 quarter) can go into (divide into) or be subtracted from 77 cents a total of 3 times with a 2 cent remainder:

$$\begin{array}{r@{\,}l} & 3 \\ 25 & \overline{)77} \\ & \underline{75} \\ & 2 \end{array}$$

: 3 quarters is (equivalent to) 75 cents = 0.25 x 3 = $0.75 = 75c

Since both dimes and nickels have monetary values larger than 2 cents, the only coin left to consider is the penny, and it takes two pennies to account for two cents monetary value. For this example, we find that the minimum number of coins to give is only 5 coins consisting of 3 quarters, and 2 pennies, instead of the 10 total coins listed above.

Ex. If you gave a 5 dollar bill to purchase an item that cost only $1.99, how much change should you receive?

amount given - price = change
$5.00 - $1.99 = $3.01

At the time of writing this book, pennies, and other coins, are still in widespread use, but they may be obsolete in the future, and hence this topic could then be considered a historical review of the times when they were available. At the time of this writing, a basic loaf of sliced sandwich bread, about 1 pound in weight, can be purchased at a thrift (low item cost, bargain) grocery (foods) store for about the price of: $1.25 . As of the year 2025, about half of the people are still using cash (ie., money) to purchase products, and the other half is using a debit or credit card associated with their bank savings and-or bank account.

Some Currency Exchange Rates

At the time of writing this book edit and edition at May 7, 2020, know that these values can vary by a few percent in just 1 day, and with a average tenancy to rather increase in value. Even if these values change, you can still use it for practical and historic comparison purposes.

1 English or Great Britain Pound (GBP) = $1.24 USD : **USD = United States Dollars , 1 cent = 1 penny = $0.01 USD**
$1 USD = 0.81 GBP or "sterling" = 106.26 Japanese Yen = 0.93 Euro (EU Union) = 7.10 Yuan (Chinese)
$1 USD = C$1.41CAD (Canadian Dollar) = 24.33 Peso (Mexico) = 74.43 Ruble (Russian) = 0.98 Franc (Swiss)
$1 USD = 50.56 Philippine Peso = 5.72 Brazilian Real = 15.73 Egyptian Pound = 7.75 Hong Kong Dollar
$1 USD = 3.52 Israel New Shekel = 1.66 New Zealand Dollar = 4.22 Zloty (Poland) = 18.74 Rand (South Africa)
$1 USD = 1225.54 Won (South Korea) = 6.70 Trinidad And Tobago Dollar = 23445.50 Dong (Vietnam)
$1 USD = 607.69 CFA franc (West Africa) = 33.09 Birr (Ethiopia) = 76.05 Rupee (India) = 42105.00 Rial (Iran)
$1 USD = 1.55 AUD (Australian Dollar) = 4.47 Leu (Romania) = 7.21 Lira (Turkey) = 55.30 Sudanese pound
$1 USD = 582.0 Somali (Somalia) Shilling = 931.79 Rwandan (Rwanda) franc = 159.01 Pakistani Rupee
$1 USD = 12 Cedi (Ghana) = 1 Panamanian Balboa (PAB, Panama and some other countries, often islands)

The unit of money in the United States of America is the dollar. 1 dollar = $1 , A fraction of a dollar is usually expressed in the number of cents (ie., from the word percent) of a dollar. 1 cent is 1percent of a dollar. 1 dollar = $1 = 100 cents = 100c. A price of an item expressed as: $dollars.cents = dollars + cents of a dollar = dollar + cents
Ex. $1.75 = 1 dollar and 75 cents = $1 + 75c

Common United States Coins: quarter = a quarter of 1 dollar = $1/4 = $0.25 = 100c / 4 = 25c
dime = a tenth of 1 dollar = $1/10 = $0.10 = 100c / 10 = 10c
nickel = one-twentieth of 1 dollar = $1 / 20 = $0.05 = 100c / 20 = 5c
penny = one-hundredth of 1 dollar = $1 / 100 = $0.01 = 100c / 100 = 1c = "1 cent"

The **Euro** dollar became available in Europe on January 1, 2022

Before the "decimalization" of the British monetary system on February 15, 1971:
The unit of money in the England is the **pound** (L), and is sometimes called a (slang) "**quid**" meaning "an exchange of".
1**L** = 20 **shillings** = 20**s** , hence 1s = (1/20) L = 0.05 L , A shilling is sometimes called a "**bob**".
1 shilling = 12 pennies (or "pence")= 12**d** , 2 shillings = 1 florin , 5 shillings = 1 crown
1 penny = 2 half-pennys = 4 farthings , hence 1 farthing = (1/4) of a penny
1L = 20 shillings = (20s)(12d / 1s) = 240d
Ex. L5.3s.2d = L5 , 3s , 2d = 5 pounds + 3 shillings + 2 pennies

After the "decimalization" of the British monetary system on February 15, 1971:
1 penny = 1p , 1 shilling = 5p , 1 florin = 10p , 1 crown = 25p , 1 pound = 1L = 100p

Some Metal Prices Per Ounce (On May 7, 2020, silver and gold also show the USD price at February 20, 2023) ,
Note: 1 oz of weight = 1 dry ounce = 1 oz = 28.35 g , 1g = 0.035274 oz. A "dry ounce" means for solids, and not "wet" or fluid or liquid ounces. (General for plain bullion (bulk) and-or scrap (waste, recycled) metals, and not for specific manufactured products):

Gold = $1684 USD / 1 oz and $1845 / 1 oz on February 20, 2023, In terms of bartering and receiving change when
using gold or silver, it is perhaps best to get small weights of it, such as 1/10 oz, 1/20 oz , and-or 1 to 5 gram
sizes.

Silver = $14.81 USD / 1 oz and $21.84 / 1 oz on February 20, 2023 and this is an increase of,
(new / old) = $21.84 / $14.81 = 1.47$\underline{5}$ or about 0.47 = 47% more than 1.0 = 100% , or calculated as:
change or difference = (new - old) = (21.84 - 14.81) = 7.03 and
(change from old / old) = (change from reference value / reference value) = 7.03 / 14.81 = 0.47$\underline{5}$

A 90% = 0.90 silver USD quarter weighs 0.22 oz, and therefore has (0.22 oz)(0.90) = 0.198 oz of silver , and has (price per ounce)(number of ounces) = ($21.84 / 1 oz)(0.198 oz) = $4.32 worth of silver as of Feb., 20, 2023. 5 of these old and worn-out "junk silver" quarters will weigh about: (5 coins) (0.198 oz / coin) =
= 0.99 oz =~ 1 oz.

Platinum = $752 USD / 1 oz , Palladium = $ 1712 USD / 1oz
Copper = $ 0.14 / 1 oz , Brass = $0.13 / 1 oz , Bronze = $0.07 / 1 oz , Aluminum = 0.04\underline{5}$ / 1oz , Lead = $0.05/oz
Nickel = $0.38 / 1 oz , Tin = $0.47 / 1 oz , Cobalt = $0.90 / 1 oz , Zinc = $0.06 / 1 oz
Steel = $0.028 av / 1 oz , Stainless Steel = steel with about 11% chromium or more = $0.038 / 1 oz

It is of note that although an object may weight say 1 ounce due to be constructed of a certain metal(s) for example, that product is usually many times that value due to manufacturing and many other associated costs to produce that product.

A **blacksmith** can take a piece of pre-made steel from a steel factory or scrap (left over, unused, to be recycled) metal, and cut and hammer (ie., "forge") it to produce a product. Before doing this, he may soften or "**anneal**" the steel by heating it hot and then allowed to slowly cool so that it is slightly softer to work with and less brittle, and this allows it to be more easily shaped and is less prone to breaking in the process. Once the steel has been worked into the desired shape, it is usually "**heat-treated**" so as to have a better, more uniform and stronger (atomic, atom) crystal structure that is more resistance to wear, that is, its harder. This process is usually done by heating the steel to a high temperature and then quickly quenching or dipping it in water or oil. Making the steel harder also makes it more brittle, This process may also be followed by what is called "**tempering**" the steel so as to help reduce its brittleness slightly and yet still maintain a fairly hard steel, and this process is to heat it up to a lower temperature, and then let the steel cool slowly.

One helpful tool a blacksmith can use for bending metal, either heated to be softer, more pliable, or even cold thin metal, is called a **flypress**. This heavy machine made of metal, has a large upper wheel where an input force is applied and then amplified to be a very large torque force of which then causes a large screw and its tip to make contact with the metal being worked, and so as to easily bend it, punch holes in it, etc. Pre-shaped metal forms can also be used to help make a consistent product by bending the metal upon its surface.

It is of note that a blacksmith may routinely, mechanically force some extra air (ie., containing oxygen) into the fire and-or coals that are combusting and releasing heat or thermal energy. Extra oxygen helps improve the combustion efficiency and-or rate of burning or combustion. When wind blows onto a hot coal and-or fire it will burn (ie., combust) more and get hotter. A fire-pit dug into the ground and-or surrounding a fire or any other open flame helps prevent unwanted fires. A **fire** can be considered a slow and-or moderate (ie., not an explosion) form of **combustion** which is a (fast) chemical reaction which utilizes oxygen, and in which heat and light are produced as long as there is fuel (ie., source of the energy) which includes and-or requires oxygen to do so. To start or ignite a fire usually requires a hot, heat source applied to the fuel (ex., wood, gasoline, oil, all of which are hydro-carbon, life based, but fuels such methane or natural gas is also made in the Earth). Once ignited, a fire will continue like a self-sustaining chain-reaction without an outside heat source since the fuel being combusted itself is now the heat source.

Metal parts such as screws, nuts, bolts, springs, gears, and washers are often standardized. The main types of screws are wood screes for wood, and machine screws for metal. A typical machine screw will have a length and a thread count per inch (tpi) or centimeter. The type of "head" and-or tool needed to set a screw in place is also important such as the cross-head, straight head or hex-head. Metal parts and tools such as sockets and wrenches are made to either an

English size or Metric size. Metric sized parts and tools are becoming more widespread and common in the U.S.A.

Metals such as gold, silver, platinum, palladium are relatively rare and are known as "precious metals" and which mostly means that they are valuable in some way(s). These metals (except silver which can tarnish easily) are often needed for some modern technology such as plating steel electrical contacts so as to reduce rust (Iron-oxide caused by oxygen and water on iron or steel) corrosion (here, slowly dissolving, like in an acid) and maintain good electrical contact for power and signal conduction. Gold has been used for plating metals(s) to make them look impressive and to prevent rusting and-or corrosion. The "platinum group" of metals consists of: platinum, palladium, iridium, osmium, ruthenium and rhodium. A common use for thee metals is in the medical field since they are not easily reactive (such as oxidation or rust, corrosion) to their environment. Some of these metals are used in catalytic converters that reduce the amount of the pollutants in the exhaust of combustion engines, and this is done by a chemical reaction of the exhaust gasses upon the surface of the metal.

WERE THINGS REALLY CHEAPER "BACK THEN"?

Consider this example so as to find out if things were really cheaper (less expensive) in the "old days" (previously, in the past, years ago, decades ago). This method of analysis will compare, in a relative manner (by using percentages; %), the price of the same product to the minimum labor wage (income) per hour. This method of analysis can also be used for the current situation of one or more locations (cities, states, countries, etc), and which depends upon the current local (supply, demand, etc) economy there.

Price of a product as a % of Minimum Wage = (Price of product / Minimum Wage) x 100

Year	Price	Minimum Wage	Price of a product, as a % of Minimum Wage	
1930	$0.15	$0.75 per hour	20.00%	: 15 cents or hundredths of a dollar or 100 cents = 15/100
2017	$1.25	$6.50 per hour	19.23%	: 125 cents or pennies

This analysis shows that the price of a certain product in the year 1930 was about the same percentage, and in fact, it was slightly higher "back then", and hence, it was slightly cheaper in 2017. 15c is certainly cheaper than $1.50 or 150 cents, but the percentage and-or actual cost of that item for the consumer is about the same as it was "back then". Often, modern production methods and the greater availability (distribution, transportation, selling price, fair competition) of some resources have, or can, effectively reduce the average selling price of an item(s) for consumers (people purchasing the product). This analysis does not mean that all items were effectively cheaper "back then", and the analysis must be considered for each specific item in question.

Associated with the above concept is the topic called **value shopping.** For example given two similar quality products for sale that may have a different quantity and-or different price, how can we value shop so as to find the best value in terms of having the lowest cost? To do this we can find the cost per ounce or pound of each item and then compare them:

$\frac{\text{total cost of item 1}}{\text{total ounces of item 1}}$, and compare this to: $\frac{\text{total cost of item 2}}{\text{total ounces of item 2}}$: both fractions will find the cost per ounce of the associated item, when the division, "reduction" (to lowest terms) or cancellation takes place.

POWERS

Whereas on a fundamental level, multiplication is the sum of the repeated addition of the same number, the **power** of a number is the product of the repeated multiplication of the same number. The number (ie. the factor) used for the repeated multiplication is formally known as the **base** of the power or resulting value of it. You could say that the base of a power is what the power value to be found is based upon, sourced, rooted upon, or derived from. A power of a number is the result of the repeated multiplication of the base value.

The "shorthand" notation used to represent this repeated multiplication is called exponential notation. You can basically view and think of it as the "power operation". This notation or expression of a power uses the base (effectively the multiplicand) value, and the number of times (ie. the multiplier) the repeated multiplication of it by itself is to occur is indicated by a value formally called the **exponent** (or the exponent of the base, and-or indicated (expressed) power). For example, 10 x 10, or = 10^1 x 10^1, can be represented or expressed using an exponential or power form as 10^2. 10 is the base, and 2 is the exponent of the base. 10 x 10 = 10^2 = 100. Sometimes an exponent is called the power indicated or the indicated power of a number. In a more technical definition, the exponent indicates the number of times that the base is to be multiplied to an initial value of 1. Every value has a factor of 1, and multiplying any value by 1 will not affect it.

Also note this similarity to addition:

Since multiplication = repeated addition, and powers = repeated multiplication (a multiple, of a multiplication), in a logical kind of way, powers could be considered as (multiples of) repeated addition. This concept, or the link to addition, is generally not considered much in applied or everyday mathematics, but it is a deeper understanding if you need it.

Here is the formal or general expression and equation for a power value:

$\text{power} = \text{base}^{\text{exponent}}$: GENERAL POWER EXPRESSION OR "FORMULA"
or:

power = base^exponent : GENERAL POWER FORMULA
This book mostly uses the carat symbol = ^
to indicate an exponent, rather than use
superscript values (numbers or letters).

If you were to call the power value as "number", and use N as a shorthand or symbolic identifier for it, and likewise, use (b) for base, and (e) for exponent, we have this:

N = b^e : GENERAL POWER FORMULA
An common alternate notation is to use (x) for (e):
N = b^x

10 = 10^1 : "plain 10" without any exponent indicated is actually "ten to the first power"
10 x 10 = 10^1 x 10^1 = 10^(1+1) = 10^2 : "ten to the second power" = "the second power of 10"
10 x 10 x 10 = 10^1 x 10^1 x 10^1 = 10^(1+1+1) = 10^3 : "ten to the third power" = "the third power of 10"

Due to the associative law, this can be expressed or checked as:

10 x 10 x 10 : an "expanded" form of the expressed power of 10^3
(10^1 x 10^1)(10^1)
(10^2)(10^1) = 100 x 10
10^(2+1) = 10^3 = 1000

Clearly, when the base value (or variable as used in algebra) of the powers are the same, and are being multiplied, you can simply add their exponents and keep that same base for the simplified expression:

10 x 10 x 10 x 10 = 10^(1+1+1+1) = 10^4
10 x 10 x 10 x 10 = 10^(1+1+1) (10^1) = 10^3 10^1 = 10^(3+1) = 10^4

Ex. What is the third, and fourth power of 2 ?

2^3 = 2 x 2 x 2 = 8
2^4 = 2 x 2 x 2 x 2 = 16

Note that the last answer could also be found by multiplying the preceding power of 2 that was 2^3, by 2 = 2^1:

2^4 = 2 x 2 x 2 x 2 = (2 x 2 x 2) x 2 = 2^3 x 2^1 = 8 x 2 = 16

Note, frequently the words "exponents" and "powers" are exchanged for each other. For example, "ten to the second power", or "the second power of 10", really means a base of 10 with an exponent of 2. 2 is not the power, but the exponent or indicated power. (10^2) or 100 is the actual power (value, result) of 10 in question. Some older texts may use the word "index" (meaning the indicated power) instead of the word "exponent".

Given a number raised to a (integer or basic number) power, how many times greater is the next power? The answer is that the next power of a number is that number times greater:

Ex. 10^3 / 10^2 = 1000 / 100 = 10 = 10^1 : "one-thousand is ten times greater than one-hundred"

Ex. How many times greater is 4^3 = 64, than 4^2 = 16: 64/16 = 4 : "sixty-four is four times greater than sixteen"

Later in this book, all this can be generalized, or generally expressed as:
(Number^(n+1)) / Number^n = Number^(n+1 -n) = Number^1 = Number = N , and
N^(n+1) = N^1 N^n
Also, the difference between two successive integer powers of a number can be simply expressed as:
(N^(n+1) - N^n , Ex. 10^3 - 10^2 = 1000 - 100 = 900
A more formal formula for the difference between successive (ie., N^n and N^(n+1)) integer powers of a number is:
(N-1) (N^n) , Ex: With N=base=10, and n=exponent=2, difference = (9)(10^2) = (9)(100) = 900

Powers with any base, and an exponent of 2 are often called "squares". It gets this name from the fact that area is defined and measured (with its units) in reference or with respect to a basic square shape and unit of measurement for areas, which is a two-dimensional (ie., planar (plane) or "flat, level shape", having. a length and width dimension only, with no height or thickness), four sided enclosed figure, where each of those two dimensions is equivalent in length and having the same units of measurement. This is true even if the area being measured is triangular, rectangular, circular or any other possible bounded, limited or defined region of a (infinite) plane. The area of these shapes will be measured with respect to their equivalent number (ie. a multiple or repeated sum) of square units. A "square unit" is one unit long and (by) one unit wide. The units of any particular area shape are therefore that of the units of the two dimensions of a square (or "unit square") of which it (the area) is to be measured in reference (or compared to) to. Hence, an area value will have units of: units x units = units^2 or "square units". For example, an area might be stated simply as one value of 10 square feet, or 10 sq. ft., rather than say something like the area is 5 feet (units) long, by 2 feet (units) wide. Though the values of 5 feet and 2 feet are important measurements for something like a rug or piece of metal that you may need, the store may sell the rug or metal as priced per "square foot", rather than in the particular or specific sections or sizes that only you may need for a certain project.

Note: 10^2 is often referred to as "ten square" : here the (base) value of 10 is squared , 10^2=100 , but

10ft^2 is "ten, square feet" where only the unit of feet is squared, and not the base value of 10, that is, it is not equal to 10^2 ft = 100 feet or 100 square-feet

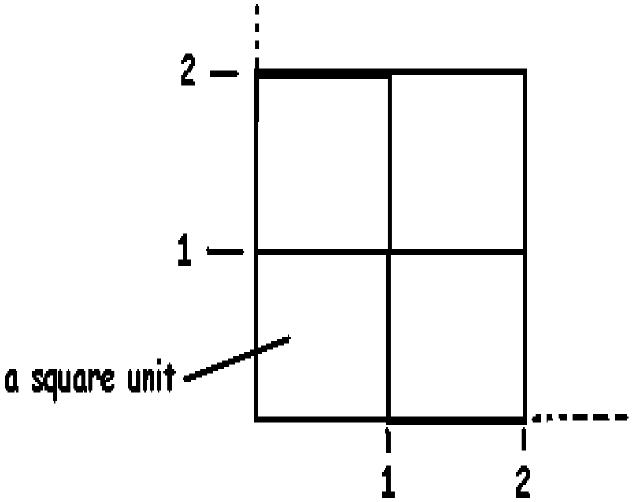
Ex. If you had a rectangular area that was 2 units long, and 2 units wide, the total area is:

square units in row 1 + square units in row 2

2 square units + 2 square units Since multiplication is repeated addition:

2 (2 square units) = 4 square units = 4 units square = 4 units x units = 4 units^2

[FIG 14]



Clearly, the area is 4 square units

Ex. If you had a rectangular area that was 2 units long, and 3 units wide, the total area is:

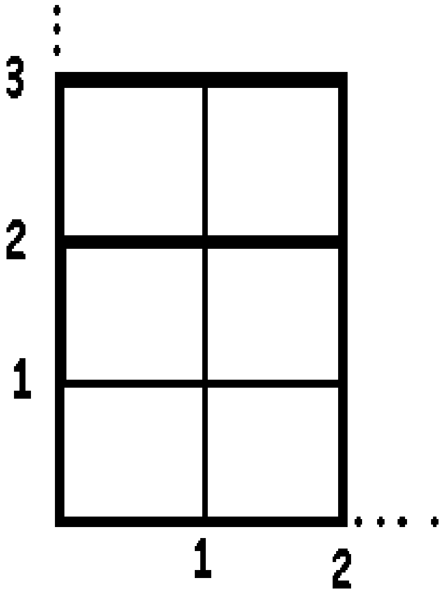
2 square units + 2 square units + 2 square units

3 (2 square units) = 6 square units = 6 units square = 6 units^2

Note also that: 3 units x 2 units = (3) (2) (1 units) (1 units) = 6 (1 units^2) = 6 units^2

We also see that the area of a rectangular shape is simply the product of the size (or measured length if you will) of each dimension (length and width) of the rectangle. This is so since multiplication (ie. for a product) essentially represents repeated addition. Here, the repeated addition is of similar squares or "square units" since each dimension of these squares have the same length and same units of measurement for that length.

[FIG 15]



Clearly, the area is 6 sq. units or 6 units^2

Ex. 5^2 can be called "five to the second power" or "five square" and equal 25.

The third power of a number is often called the "cube" of that number. This is from the fact that a certain (reference of measurement) volume called a cube (ie. a cube of space or volume) has three dimensions, each having the same basic units of measurement The dimensions are called the length, width, and height. A cube is very similar to a square area with its length and width units, but the cube also places a height of 1 unit above that flat planar square, and creates a (3-dimensional) "space" or volume. All possible volumes are measured in reference to a "cube unit" or "cubic unit" of volume where the (sides) dimensions are all 1 unit of length in value.

Note: number^3 = a number raised to the third power is sometimes called the cube of that number.
Ex.: 10^3 is sometimes called "ten cubed" , 10^3 = 10 x 10 x 10 = 1,000

If a value doubles (multiplied by 2), such as the side length of a square shape, it's square (ie. square area) will increase by four times. This is an example of a non-proportional or non-linear relationship of two values, where as when one value increases or changes by a factor, the other value increases or changes by some other constant factor. For the example shown below, the side lengths of a square change in a linear manner, but the corresponding area changes in an exponential manner, and the values of the area increase or change rapidly due to the squared value of its side length.

Ex. 1^2 = 1 if 1 is doubled to 2, its square is:
2^2 = 4 an increase of a factor of (4/1) = 4. If 2 is doubled to 4, its square is:
4^2 = 16 an increase of a factor of (16/4) = 4, if 4 is doubled to 8, its square is:
8^2 = 64 an increase of a factor of (64/16) = 4

Here is a mathematical verification of this increase, or magnification, factor, of 4:

4^2 = (1 x 4)^2 = 4^2 = 4 x 4 = 16
8^2 = (2 x 4)^2 = (2 x 4)(2 x 4) = 4 x 4 x 2 x 2 : this factor of 2 x 2 = 2^2 = 4 will always be present
= 4^2 x 4

If a value doubles (multiplied by 2), such as the length of a cube shape, it's cube (ie. volume space) will increase by a factor of 8.

Ex. 1^3 = 1 : a cube with a side length of 1 = a cube or volume of 1 cubic unit
2^3 = 8 : a cube with a side length of 2 = a cube or volume of 8 cubic units = 8 units^3.
8 is 8 times more than 1
4^3 = 64 : 64 is 8 times more than 8

Here is a mathematical verification of this increase or magnification factor of 8:

2^3 = (1 x 2)^3 = 2^3 = 2 x 2 x 2 = 8

4^3 = (2 x 2)^3 = (2 x 2)(2 x 2)(2 x 2) = (2 x 2 x 2)(2 x 2 x 2) : this factor of 2^3 = 8 will always be present
= 2^3 x 8 when the side length doubles, consider:

Given : area1 = side^3 , if side length doubles:
area2 = (2 side)^3 = (2^3)(side^3) = 8 side^3 = 8 area1

NOTE: It is incorrect to think that a cube is the second power (or numeric square) of an area such as that of a square. Give the side length (s) of a square area, its area is: Areas of the square = As = (s)(s) = s^2 . If this area was the side of a cube, its volume is NOT equal to Vc /=/ (As)^2 = (s^2)^2 = s^4, but is rather equal to Vc = (Vs)(s) = (s)(s)(s) = s^3. The reason for this is that given a square or square area with two dimensions, a three-dimensional cube only has one more physical or real dimension to be considered into the equation for an area.

8 cubic-feet = **8 ft^3** = 8 (1 ft^3) = (3$\sqrt{8}$)^3 (1 ft^3) = (2^3) (1 ft^3) = ((2)(1 ft))^3 = **(2 ft)^3** = **(cube root 8 ft^3)^3**

Be aware of what value is actually being raised to a power, and-or to properly express it as so, perhaps by using parenthesis to clarify the expression:

(0.5 in)^3 does not equal (0.5) in^3 = 0.5 in^3

(0.5 inches) cubed does not equal (0.5) cubic inches.

(0.5 in)^3 = (0.5 in)(0.5 in)(0.5 in) or= (0.5^3)(1 in)^3) = (0.5 in)^3 (1 in^3) = 0.125 in^3

AN OBSERVATION ON SQUARING VALUES

When a value (such as a base of a power) greater than 1 is squared or "raised" to any indicated power (ie. the exponent) greater than 1, the result is always a greater value than the value (the base) that was "raised" or increased to that power value.

Ex. $2^2 = 4$
$2^3 = 8$
$3.5^2 = 12.25$
$1^{1.01} = 1$
$1^{73} = 1$
$1.1^3 = 1.331$
$4^{1.0001} = 4.000554556$

Now note that when a value less than one is squared or "raised" to any other indicated power greater than 1, the result is smaller than the number itself. The greater the indicated power (ie. the exponent that the number is raised to), the smaller the resulting power value. Actually, multiplying any value by a (proper, less than 1) fractional value always results in a lower value. This is similar to the concept to having "a fraction, of a fraction", and the result is smaller than the original fraction.

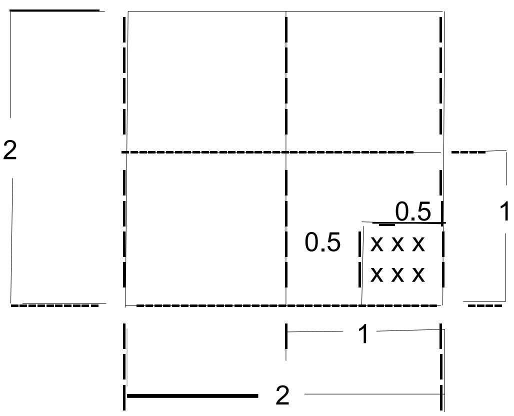
Ex. First, $0.5 = \frac{5}{10} = \frac{1}{2}$: "one-half. or one-half of one (whole)" , squaring this value we have:

$(0.5)^2 = (0.5)(0.5) = \left(\frac{1}{2}\right)\left(\frac{1}{2}\right) = \frac{1}{4} = 0.25$ and:

$0.25 = \frac{25}{100} = \frac{1}{4}$: "one-fourth or one-quarter of one (whole, all or entirely)". 0.25 is less than 0.5 , and sometimes this is indicated or mathematically expressed as: $0.25 < 0.5$, $<$ is the mathematical notation symbol for "less than", or "not equal to, but less than the following number".

Even though 0.5 is half of the length (1st dimension) on 1 unit, this value will only represent one-fourth of the area (a two dimensional concept) of one square, as indicated above when the value is squared.

A geometric representation of this is shown below. The largest square shape represents $2^2 = 4$. The shaded square represents $(0.5)^2 = 0.25 = 1/4$ (= one-fourth of one square. One square $= 1^2 = 1$). A unit square (or a squared unit) is the basic unit of measurement for measuring (2 dimensional) surface or planar (ie.,flat) areas. [FIG 16]



: consider this a square shape, if not displayed as so

0.5 is half (ie., 1/2) of 1, but when 0.5 is squared (ie., 0.5^2), it becomes smaller and is only one-quarter (ie., 1/4) of 1.

The third or cubed power of the value of 0.5 will be considered below. As mentioned previously, a unit cubed is the basic unit of measurement for (3 dimensional) volumes. You can consider a volume as moving a flat area (a plane, or planar area, having a length and width dimension only) through a height distance. Area and volume will be discussed to a greater depth further ahead in this book.

(0.5)^3 = 0.125 = 1/8 : One-eighth of one cube. One cube = 1^3 = 1.
It will take 8 of these cubes, each having a side
length of 0.5, to fill a single cube of 1 unit length.

Even though 0.5 is half of the length (1st dimension) of 1, this value will only represent one-eighth of the volume (a 3 dimensional concept) of one cube. In a reverse type of manner, it could also be said that if a length value (such as 0.5 here) doubles, the volume (or cubed value) will increase by eight times. If the length values is halved (divided by 2), the volume will decrease (a demagnification) by eight times.

Ex. 0.5^3 = 0.125 : 0.5 is half of 1, and its cube or volume is 8 times less than that of 1
1^3 = 1 : 1 is double or twice that of 0.5, and its cube is 8 times more than that of 0.5
2^3 = 8 : 2 is double or twice that of 1, and its cube is 8 times more than that of 1
4^3 = 64 : 4 is double or twice that of 2, and its cube is 8 times more than that of 2

These kind of relationships of the length to the areas and volume is not a constant or linear (mathematical, or numerical) relationship. For example, if the length doubles, the area and volume will increase, but they more than double. Area will increase by 4, and volume will increase by 8. This kind of non-linear (non-corresponding or the same growth or magnification [ie. multiplication of]) relationship is called an exponential relationship. Things that are said to be linear, are linearly related, that is, their mathematical relationship of their values is a linear or "line-like" relationship. Specifically, a mathematical relationship of or between two values is linear when the ratio (or rate) of the change of one variable with respect to the change of another variable is a constant value. Although this is of a topic to be discussed further ahead in this book, this can be mathematically expressing as: (change in variable1) / (change in variable2) = the rate of the changes, or simply the rate of change. For a linear relationship, the rate of changes between the two values or variables is a constant value. It can also be said that In a linear relationship, as one value, the independent variable changes in value by 1, the other corresponding or dependent value or variable will always change by the same constant value that is called the slope or "steepness" of the line. On a graph, a linear mathematical relationship between two variables can be shown as a line. In a non-linear relationship of values or variables, when one values changes by some factor, that the other does not change by some constant factor, but by a factor that is non-constant and is also changing, such as increasing or decreasing in value. The rate of changes between the two values or variables is then not a constant value. On a graph, their relationship can be shown to be some type of curve.

Special note on dividing expressed powers: If given an expressed power value of a number that contains the base (b) and the exponent (e), and if that power value is divided by a value (d), it is incorrect to divide either or both the base and the exponent by that divider. The true result is found only by dividing the actual total or number value of that indicated power by that denominator.

Ex. $$\frac{10\hat{}3}{2} \;/\!=\!/\; \left(\frac{10}{2}\right)\hat{}3 = 5\hat{}3 = 125, \quad \text{or } /\!=\!/ \;\; 10\hat{}(3/2) = 10\hat{}1.5 = 31.62 \;\;, \quad \text{the true result is: } \frac{(10\hat{}3)}{2} = \frac{1000}{2} = 500$$

A CHART OF THE VALUES OF A POWER WITH RESPECT TO THE BASE AND EXPONENT

First, N = 1^exponent = 1 : base is 1, and exponent is any value

N = base^1 = base : N=number=actual resulting (power) value , and base is any value

Considering only positive (greater than zero , > 0) valued exponents:

When the base is greater than 1: (base > 1): Examples:

base^(exponent < 1) 3^(0.5) = 1.7321 : (N < base)
base^(exponent = 1) 3^1 = 3 : (N = base)
base^(exponent > 1) 3^2 = 9 : (N > base)

When the base is less than 1: (base < 1): Examples:

base^(exponent < 1) 0.5^0.5 = 0.70711 : (N > base)
base^(exponent = 1) 0.5^1 = 0.5 : (N = base)
base^(exponent > 1) 0.5^2 = 0.25 : (N < base) , and here due to a fractional value times a fractional value results in a smaller fractional value.
A fraction of a another fraction =
(fraction2) (fraction1) = fraction3
Ex: 0.5^2 = (0.5)(0.5) = 0.25

MULTIPLYING LIKE VALUES

When multiplying like or similar values, variables (which can represent any value) or terms as in algebra, the result can be expressed as a power of that value; where the value becomes a base and whose exponent is the sum of the exponents of that value.

Note that a value without an exponent is considered or understood to have an exponent of one (1) For example: 3 = 3^1. The explanation of this is that a value to the first power is the first multiple by itself to an initial product of 1, and this is shown directly below, but may be skipped over if appears somewhat complicated for now:

For 3^1: 1 x 3 = 3 x 1 = 3 = 3^1 , : the first power of a number is equal to the first multiple or instance of that number which is the same as multiplying that number by 1. The first power of a number is equal to itself.

Note that 1 x 3 is technically equal to 1 repeatedly added to itself 3 times: 1 + 1 + 1 = (1)(3) = 3 = (3)(1) = 3 added to itself 1 time. It could be said that the first multiple of a number is equal to itself.

For 3^2: 1 x 3 x 3 = (1 x 3)(3) = (3^1)(3) = (3^1)(3^1) = 3 x 3 = 9 = 3^2 = 3^(1+1)

Ex. 5 x 5 = 5^1 x 5^1 = 5^(1+1) = 5^2 : Note: 5^1 x 5^1 /=/ 2(5^1) = 5^1 + 5^1 = 10
That is, you do not add the numerical coefficients of the same or like values when multiplying, but rather add their exponents.

5^1 + 5^1 = ((1)(5^1)) + ((1)(5^1) = ((1+1)(5^1) = 2 (5^1)
This is repeated addition of the like values. Add their numerical coefficients or factors for the resulting coefficient or factor of that value.

Remember that multiplication is associated with repeated addition, and powers are associated with repeated multiplication.

Ex. (4^1)(4^2) = 4^(1+2) = 4^3

checking: (4^1) (4^2) Or: (4^1)(4^2)
(4^1) (4^1)(4^1) (4)(16)
4^(1+1+1) 64
4^3 = 64 : 4 x 4 = 16, and 16 x 4 = 64

Ex . 2^2 + 2^2

Since this is repeated addition of a value (here, 2^2) twice, we can express this with multiplication:

2^2 + 2^2 = 2 x 2^2 = (2^1)(2^2) = 2^(1+2) = 2^3 : add the exponents of like factors to simplify

Checking: 2^2 + 2^2 = 4 + 4 = 8 = 2^3

As previously mentioned, try not to confuse the concept of adding like values with the concept of multiplying like values. When adding like values, their numerical coefficients are added and the variable is kept the same. When multiplying like values, their numerical coefficients are multiplied, and their exponents are added.

Ex. First add, and then multiply these two values: $5x$ and $2x$

Note, variables, such as x, are used here. If you are somewhat worried or unfamiliar with using variables, you can replace (or substitute) it with a (constant, unchanging, known, fixed or solid) numeric value. For example, if we let $x=3$, we would have: $5x = 5(3)$ and $2x = 2(3)$. Their sum would be:

$5(3) + 2(3) = (5+2)(3) = 7(3)$: adding the values. Checking:

$3(5) + 3(2) = (3+3+3+3+3)+(3+3) = 3+3+3+3+3+3+3$

Using the concepts and expression for repeated multiplication, the result is: $3(7) = 7(3) = 21$

$5x + 2x = (5 + 2)x = 7x$: adding the like terms.
Here, you can see (x) is a common factor to all the terms (here 5x and 2x), hence they are called "like-terms".

Checking: $5x + 2x$ Expressing the terms using repeated addition:
$(1x + 1x + 1x + 1x + 1x) + (1x + 1x)$ Which can be expressed simply as:
$(x + x + x + x + x) + (x + x)$ Now using the associative law of addition:
$(x + x + x + x + x + x + x)$ Expressing this repeated addition, of x, using multiplication:
$7x$

Note that for x and x^2, although being the same variable, they are not like terms since their exponents are different and therefore these (powers of the variable) represent two entirely different values. For example: $5x + 2x^2$!=, does not equal: $(5 + 2)x$, or $(5 + 2)x^2$, or $(5+2)(x + x^2)$

$(5x)(2x) = (5)(2)(x\,x) = 10x^2$: multiplying the like terms. Everything, both the numerical coefficients, and variables are to be multiplied together. This example is verified by the concepts of the associative law of multiplication that states that grouping (ie. associating) does not matter when multiplying, and the result will be the same.

Before moving on to another topic, another good point must be made. If given an (indicated) power value, that value can essentially be factored by using the same base with exponents values that sum up to that given exponent of the power. Using an example previously given:

Ex. Since: $(4^1)(4^2) = 4^{(1+2)} = 4^3$ we see that (4^1) and (4^2) are factors of the product 4^3, hence 4^3 can be factored to:

$4^3 = (4^1)(4^2)$: the factors of 4^3 can be any indicated (with an exponent) power of that base of 4, just as long as the sum of the exponents of all the "like factors" sum up to 3

$4^3 = 4^{(1.5 + 1.5)} = (4^{1.5})(4^{1.5})$

A POWER TO A POWER

If an indicated power of a value (base), or variable as in algebra, is itself raised to an indicated power, simply multiply the exponents for the resulting exponent (of that base) to simplify this type of expression. A simple verification of this is shown below:

First note: (5)^2 = (5^1)(5^1) = (5^1)^2 = 5^(1x2) = 5^2

Ex. (5^2)^2 = 5^(2x2) = 5^4

checking: (5^2)^2 = (5^2)(5^2) = 5^(2+2) = 5^4 : this check uses "extending" or "expanding" the indicated expression to a fuller form

or: (5^2)^2 = (5^2)(5^2) = (5^1)(5^1)(5^1)(5^1) = 5^(1+1+1+1) = 5^4

or: (5^2)^2 = (25)^2 = 625 = 5^4

To help understand this, we can remember that multiplication is a "shorthand" expression for a repeated addition expression.

Ex. (4^2)^3 = (4^2) (4^2) (4^2) = 4^(2+2+2) = 4^6 : showing the repeated addition of the exponents

(4^2)^3 = 4^(2x3) = 4^6 : showing using multiplication, instead of repeated addition, of the exponents

Ex. (0.7^2)^3 (0.7^2)^2 = (0.7^2)^(3+2) = (0.7^2)^5 = 0.7^10

checking: (0.7^2)^3 (0.7^2)^2
(0.7^6)(0.7^4)
0.7^(6+4)
0.7^10

A PRODUCT TO A POWER

This is very similar to a "power to a power" mentioned above, but the power that this product is being raised to will be "distributed" (shared or applied among, essentially multiplied) to the exponent of each factor of the indicated product when simplifying the expression.

Ex. (4 x 2)^2 = ((4^1)(2^1))^2 = (4^(1x2))(2^(1x2)) : Here, showing that the exponent was "distributed" (or applied) to each factor and its corresponding exponent..
= (4^2)(2^2)
= (16) (4) = 64

checking: (4 x 2)^2 = (8)^2 = 64 : Here, the expression in the grouping symbols was simplified first.

Ex. (2^2 x 3^2)^2 = (2^(2x2)) (3^(2x2)) = (2^4)(3^4) = 16 x 81 = 1296

checking: (2^2 x 3^2)^2 = (4 x 9)^2 = (36)^2 = 1296

We also see a rule can be developed here due to the above example. First, given a product of two (expressed) powers, and if the exponent is not indicated, it is understood as 1. The product of two numbers having the same exponent is the product of those numbers and whose exponent is that exponent. For example:

(2^2 x 3^2) = (2 x 3)^2 here is a verification using the above discussion, distributing the exponent:
(2^2 x 3^2) = (2^(1x2))(3^(1x2))
(2^2 x 3^2) = (2^2 x 3^2) : the result is the original expression on the left side, hence they are equivalent

Checking:

(2^2 x 3^2) = (2 x 3)^2
(2^2 x 3^2) = 6^2 : another, somewhat obvious, rule to observe here is that you can factor a product (here 6) to where each factor (here, 2 and 3) has the same indicated power
4 x 9 = 6^2
36 = 36

Ex. (9^0.5)(4^0.5) = [(9)(4)]^0.5 = [36]^0.5 = 36^0.5 : doing the reverse, you can factor the base of an indicated power, and each factor will have the same exponent

A SIMPLE EXPRESSION TO A POWER

The word simple in the title refers to addition and subtraction. First, a very important word of caution: You cannot simply distribute the power to the values being added or subtracted in an expression that is raised to a power.

Ex. (5 + 2)^2 /=/ (does not equal) ((5^2) + (2^2)) = 25 + 4 = 29

The correct result can only be found (when not using the order of operations directly) by performing the indicated operation. This is sometimes referred to as "extending" or "fully expressing" (the given expression) to the indicated power.

(5 + 2)^2 = (5 + 2) (5 +2)

To simplify this, especially when algebraic variables are used, requires you knowing about the distributive law of multiplication which will be shown later. Since only numeric values are used here, by using the order of operations, we can get the correct answer by simplifying the original expression which represents a value that could be simplified and-or expressed as a single number that is contained within the grouping symbols.

(5 + 2)^2 = (7)^2 = 7^2 = (7)(7) = 49 : basic simplification within the grouping symbols first, or by:

(5 + 2)^2 = (5+2)^1 (5+2)^1 = (5 + 2)(5 +2) = (7)(7) = 49 : "extended method", then simplifying

DIVIDING LIKE VALUES THAT HAVE EXPONENTS

When dividing indicated or expressed powers that have the same base value, or variables as in algebra, simply subtract their exponents so as to have the exponent of the resulting quotient and which will also have that same base value.

Ex. $\frac{5^3}{5^1} = \frac{5^{(3-1)}}{1} = 5^2 = 25$:here, the same base value is 5

checking: $\frac{5^3}{5^1} = \frac{125}{5} = 25$ or: $\frac{5^3}{5^1} = \frac{\overset{1}{\cancel{(5)}}(5)(5)}{\underset{1}{\cancel{(5)}}} = 5^2 = 25$

Hence, subtracting (to reduce, make less) exponents is essentially equal to canceling some factors common to both the entire numerator and denominator. As another check, the product of the quotient and the divisor (or "divider") should equal the dividend. Using the example above:

$(5^2)(5^1) = 5^{(2+1)} = 5^3$: checks

Ex. $\frac{7^9}{7^3} = \frac{7^{(9-3)}}{1} = 7^6$: $7^9 = 7^3 \times 7^{(9-3)} = 7^3 \times 7^6 = 7^{(6+3)}$

Checking: Here, the numerator is factored by using the inverse of the rule (add the exponents) when multiplying like values, or variables as in algebra. It is factored to create like values in both the numerator and denominator which can then be canceled. Since 7^3 is the smallest power of the value, or variable as in algebra, in question, there is no reason in this example to factor it. 7^9 will be factored so that there will be a highest common factor (HCF) of 7^3 for both the numerator and denominator which can then be canceled out:

$\frac{7^9}{7^3} = \frac{\overset{1}{\cancel{(7^3)}}(7^6)}{\underset{1}{\cancel{7^3}}} = \frac{7^6}{1} = 7^6$: A way to factor 7^9 is: $7^9 = 7^3 \times 7^{(9-3)} = 7^3 \times 7^{(6)}$

What if the exponent of the same base in the denominator is larger? One answer is to subtract the least value exponent from the largest exponent of the like or similar base values, and this ensures a result with a positive valued exponent rather than a negative valued exponent (which requires you knowing about signed values or numbers that will be shown further in this book).

Ex. $\frac{5^1}{5^2} = \frac{1}{5^{(2-1)}} = \frac{1}{5^1}$

checking: $\frac{5^1}{5^2} = \frac{\cancel{(5^1)}}{\cancel{(5^1)}(5^1)} = \frac{1}{5^1}$: after canceling a common factor of (5^1).

also: $\frac{5^1}{5^2} = \frac{\overset{1}{\cancel{5}}}{\underset{5}{\cancel{25}}} = \frac{1}{5} = \frac{1}{5^1}$: equivalent fractions, and both equal 0.2

AN EXPONENT OF 0?

Any value, or variable as in algebra, with an exponent of zero (0) is equal to one (1). Besides being logically or mathematically correct, using 0 as an exponent is sometimes (explicitly) expressed or used to help clarify a simplification or a derivation process.

Ex. 10^0 = 1 : This is easily verified when the exponent in the numerator and denominator are equal. Also, any value divided by itself is equal to 1:

$$\frac{10}{10} = \frac{10^1}{10^1} = \frac{10^{(1-1)}}{1} = 10^0 = 1$$

In general, or symbolically expressed: number^0 = 1

(0) indicates that the given base is to be repeatedly multiplied to a value of 1, a total of 0 (ie. none) times. Since there won't be any multiplication performed on this value of 1, the result is that this 1 will remain unchanged. Also note that 10^0 is the weight of the "ones" digit position (in the decimal number system) expressed with exponential notation.

10^0 = 1 = 1 : weight of the "ones" digit position in the decimal number system
10^1 = 1 x 10 = 10
10^2 = 1 x 10 x 10 = 100 : with 10 as the base, each new value is 10 times more, and therefore, each preceding value is 10 times less (ie. a division by 10)

Also note:

3^0 = 1 and this is also the same value as for example:

5^0 = 1

Hence: 3^0 = 5^0 = 1

4.7^0 + 3.2^0 = 1 + 1 = 2

If you had 3^2 / 3^2 , the result is 1 since any value divided by itself it 1. Hence both the base (here it's 3) and the exponent (here it's 2) is essentially eliminated and the result is just 1 as it would be for any other base and exponent (including 1 also) divided by itself. Mathematically, shown in an expression form, you can eliminate (set to 0) the exponent by subtracting the exponents. Subtracting exponents is a common form for simplification of the division of like values (or variables) that have the same base:

3^2 / 3^2 = 1 = 3^(2-2) = 3^0 = 1 = n^0 : n = any other number

A FRACTION TO A POWER

A fraction to an indicated power is equal to both the numerator and denominator "raised" to that indicated power. The exponent (ie. the indicated power) of the fraction is essentially "distributed" (applied to, and essentially multiplied) to the numerator and denominator exponents.

Ex. $\left[\frac{(5)}{(1)}\right]^2 = \left[\frac{(5^1)}{(1^1)}\right]^2 = \frac{(5^1)^2}{(1^1)^2} = \frac{5^{(1\text{x}2)}}{1^{(1\text{x}2)}} = \frac{5^2}{1^2} = \frac{25}{1} = 25$: for this book, large brackets will be utilized, but large parentheses can also be used

Checking this by performing the indicated operation:

$\left[\frac{(5)}{(1)}\right]^2 = \frac{(5)(5)}{(1)(1)} = \frac{25}{1} = 25$: this expressed the extending (showing multiplication) of the indicated power

Ex. $\left[\frac{(2)}{(3^2)}\right]^3 = \frac{(2)^3}{(3^2)^3} = \frac{2^3}{3^6}$

Likewise, in a reverse-like manner, the division of two values, or variables as in algebra, having the same exponent or raised to the same indicated power is equal to the fraction (division) of those variables raised to that indicated power.

Ex. $\frac{2^2}{3^2} = \left[\frac{(2)}{(3)}\right]^2$

checking: $\frac{2^2}{3^2} = \frac{4}{9}$ and $\left[\frac{(2)}{(3)}\right]^2 = \frac{(2)}{(3)}\frac{(2)}{(3)} = \frac{4}{9}$

Ex. $\frac{5^3}{7^3} = \left[\frac{(5)}{(7)}\right]^3$

Ex. $\frac{(3^4)(3^3)}{5^7} = \frac{3^{(4+3)}}{5^7} = \frac{3^7}{5^7} = \left[\frac{(3)}{(5)}\right]^7$

Ex. $\frac{3^{10}}{5^7} = \frac{3^{(3+7)}}{5^7} = \frac{(3^3)(3^7)}{5^7} = (3^3)\frac{(3^7)}{(5^7)} = (3^3)\left[\frac{(3)}{(5)}\right]^7$

ROOTS AND RADICALS

Finding the root of a number is mathematically the opposite or "inverse" of finding a power of a number. The following description may seem complex to the new student, however, after viewing a few examples below, the concept becomes an understandable and necessary part of mathematics.

Here, you are essentially given the power of some number (ie. a base), and the indicated power (the base's exponent), and the base (now called a root for this type of mathematical operation) of that power value is to be found. The symbol for a root operation is the radical symbol ($\sqrt{\quad}$). The value, essentially the power value, underneath the radical symbol is now formally called the radicand. The root to be found is indicated by a value correspondingly called an index. This index means "take or find this specific indicated root of the radicand". The index is essentially the exponent that the root (when considered as a base) will need to be raised to equal that radicand (when considered as a power value or number). Some older ways would refer to a radical as a "surd", and an index as the "root extraction number".

Here is the general "root" expression and formula:

root = $\sqrt[\text{index}]{\text{radicand}}$: GENERAL ROOT EXPRESSION OR "FORMULA"

This can be rearranged mathematically, to solve for the radicand as:

root ^index = radicand this is mathematically equivalent to a power expression:

base^exponent = power hence, comparing this to the general root expression above:

base = $\sqrt[\text{exponent}]{\text{power}}$: showing that the base of a power is found using the root of a power, and therefore base=root

Mathematical operation or expression to solve for the index (the indicated root of a radical, or the indicated (as an exponent number) power of a base value) value is called a logarithm, and it will be discussed more further ahead in this book. A logarithm is basically equivalent to the index of a root and-or exponent value of a base. As expressed above, to solve for a value of these root and power expressions that have, contain or include these three values will require having or knowing the other two values used. Likewise, to solve for the specific value of the logarithm equation will require the two other specific values (the radicand and corresponding root value, or the base and corresponding power value) of which it is in reference to.

The radical symbol may look somewhat like a division symbol, and the only casual relationship among theses operations and the symbol used is that the result, either a quotient or a root, is usually a smaller value. When the index or "indicated root to be taken" is not explicitly given, it is to be understood as being 2 or the second or "square-root".

It is easy to imagine any of the various or different possible factors of a number, for example:

36 = (12)(3) = (4)(3)(3) = (2)(2)(3)(3) = (18)(2) = (9)(4) = (6)(6)

Here, the only factors of the number that are all identical, and have a product of 36, are 6. 6 is the only factor that the product of itself (here twice, or 2) several times equals the given value of 36. The repeated product of 6, twice, is mathematically expressed as 6 x 6 = 6^2 = 36 . The expression 6^2 is called the square (or second (2) indicated power) of 6. This single factor value of 6 is therefore called the square (or second = 2) root of 36. A very common root of a number to "take" (solve for) is a square root. Squared = second = 2 as a mathematical value. When solving for the square root of a number, power or product value, you are effectively solving for a single factor (or base) that when it is squared or "raised" (increased in value) to the second (2) indicated power, that it will equal that given value.

For special note of a roots similarity and-or its relationship to division: A root can be thought of as a special instance of division where if we divide (and sometimes repeatedly) a value by a certain or specific dividend value (called the root), we will have a final result or quotient that is equal to that same dividend or root value used. For example, the square root of 25 is 5. If you were to divide 25 by 5, the result is 5 which is the same value as the divisor. Checking 5^2 = 25. 5/5=1 shows a complete and pure division (after 2 divisions) with no remainder. For example, the cube root of 8 is 2. If you were to repeatedly (here 3 times) divide 8, and all the following results by 2 repeatedly, you would have: 8/2 = 4, 4/2 = 2 and 2/2=1. Checking 2^3 = 8. For example, the fourth root of 16 is 2. 2^4=16. 16/2 = 8, and 8/2 = 4, 4/2 = 2, and 2/2=1. Mathematically, since: radicand = root^index , we then have: radicand / (root^index) = 1. Checking: 16 / 2^4 = 16/16 = 1

base^exponent = power = 6^2 = 36 = radicand = root^index

root = $\sqrt[\text{index}]{\text{radicand}}$ = 6 = $\sqrt[2]{36}$ = base = $\sqrt[\text{exponent}]{\text{power}}$

Given a number or value of 9. The two identical factors of 9 are 3 and 3. 9 = (3)(3) = 3^2. We see that this has a form of the expression: base^exponent Here the base is 3, and the exponent is 2. The expression also has the form of: root^index , 3 is the root, and 2 is the index. 3 is equivalent to the second (2) root of the radicand of 9. The first root of a value it itself.

3^2 = 9 and 3 = $\sqrt[2]{9}$: "the square root of 9 is 3" and "3 squared is 9"
: "the second root of 9 is 3" and "3 raised to the second power is 9"
or "the second power of 3 is 9"

Ex. What is the square root of 16 ?

$\sqrt[2]{16}$: 16 is the radicand, the index is 2 and is common and understood as the (indicated) root value if it is not explicitly expressed or indicated:

$\sqrt[2]{16}$ = $\sqrt{16}$ = 4 : since 4 x 4 = 4^2 = 16

Ex. 10 = $\sqrt{100}$: since 10^2 = 10 x 10 = 100

Ex. What is the third (3) or "cube" root of 75 ?

$\sqrt[3]{75}$ = 5 : since 5^3 = 75

checking: 5^3 = 5 x 5 x 5 = 25 x 5 = 75

Extra: $\log_5 75 = 3$, since 5^3=75 : "log" is an abbreviation of the word "logarithm" which is essentially the inverse (or reverse) mathematical operation to that of a power.
"The log or exponent of, or for a base of 5 necessary to produce or be equal to its power value of 75 is 3".

The word cube comes from concepts of the measurement of volume. Volume is a 3-dimensional (length, width, height) measurement of spacial (ie., space area). A planar area is measured with units based on only 2 dimension (length and width). The units of measurement of volume have 3-dimensions (length, width, and height) and the reference unit of volume also has the same unit of length for each dimension, and is therefore a ("geometric") cube shaped unit of reference or measurement. A cube shape is essentially the shape of a square box with equal side lengths. Here is how cube or "cubic" units of measurement are expressed:

units x units x units or simply: unit^3 : a cubic unit , ex. a cubic inch = in^3 , a cubic centimeter = cm^3 = cc

For general mathematical expressions and purposes, such as here for powers and roots, that are not necessarily related to the specific topic of cubic volumes, the value of 3 , instead of the word cube, should be utilized.

Frequently, calculators are utilized for finding roots since they have preprogrammed **algorithms** (steps and procedures to solve a problem) that operate or function very fast. Most basic "home" or "4- function calculators" now also include a generous square root function. These calculators and others are as inexpensive as $1USD as of 2020, and are very important due to their helpfulness. There are also calculator programs and "apps" (application = programs) available for computers and-or phones.

Ex. $2\sqrt{2}$ ~ (about) 1.414213562 : using a 10 digit calculator. Any least significant digits which cannot be be entered, processed and-or displayed are truncated (removed), and are therefore not available for further (and more exact) calculation. What is displayed is sufficient for 99% of all types of people and professions. Some values have an endless number of digits, and only a limited number can be processed either by man and-or machine.

The actual value is often expressed as: 1.414213562**. . .** **: . . .** indicates further, and possibly, endless digits

Several methods will be presented in this book for calculating any indicated root. The appendix section of this book includes a method for calculating integer (basically a number without any fractional part) roots. Many methods for calculating roots, particularly square roots, have been developed and the one shown in the appendix is perhaps the most straight-forward and simplest to remember. The appendix also contains many pre-calculated tables.

Probably the oldest method to calculate a square-root of a number is by using an initial good guess at the square-root. If the guess is correct, the product of the root and the root (ie. root^2) as two factors should equal that number (radicand) of which the corresponding square-root is to be found. If the product or "square" of the guessed or assumed root is not equal to the radicand, the guess is typically adjusted by increasing or decreasing a specific digit of it and trying again. Here, a specific digit usually means the current digit in question or the next least significant digit when you are getting "closer" to the true root.

Ex. What is the square-root of 9.61?

Since 3 squared equals 9, and 4 squared equals 16, the square root of 9.61 is somewhere between 3 and 4, and that the square-root of 9.61 appears to be much closer to 3 than 4. Since 9.61 is slightly more than 9, we will adjust the root estimate value of 3 to a slightly larger value:

Trying 3.1 as the new root estimate and then checking it, we get 3.1 x 3.1 = 9.61, hence 3.1 is the square-root of 9.61:

$$3.1 = \sqrt{9.61}$$

Occasionally, the root of a number is exact. Sometimes this root would be called a rational, or rationalized, root. For example, the square root of 9 is 3, the square root of 64 is 8. Often, the square root of values, such as 2, and even though it is a constant value (always the same, and never changing) it is irrational or not-resolvable to an exact, specific or completed value since it will have non-ending and non-repeating decimal digits. Only "in theory" or possibly "conceivable in the mind" can a number like the square root of 2 have an exact or specific value, and rather than express or write the approximate answer out to a certain number of decimal places, it is simply expressed or given as just $\sqrt{2}$.

Irrational numbers cannot be expressed as the division of two integers (ie. plain rational numbers). Just because a calculator displays 8, 9 or 10 digits, don't automatically assume that this is the exact square root, it is however a very close approximation for nearly all practical purposes. If your calculator can display 9 digits and if the displayed result is less than 9 digits, only then can you assume this is the exact square root.

Ex. An area of land is said as being 100 square feet, how long is a side (assuming the land is square shaped, or if it could be represented as simple to conceive square shape)?

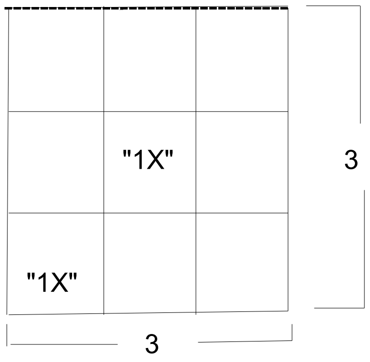
$\sqrt{100}$ = 10 :since 10 x 10 = 100 , and "technically correct" by using the given units:

$\sqrt{100\text{ ft}^2}$ = 10ft^1 :since 10ft x 10ft = (10)(10)(1ft)(1ft) = 100 square feet = 100ft^2

Ex. A small hand-held telescope is noted by the manufacturer as being 9X ("nine times" or "9 power") magnification, that is, the eye with the aid of the telescope will appear (ie., apparent) to see an object, or its area, as 9 times larger than the object as seen (ie. 1x or 1 time, no magnification, true size seen) without the telescope. How many times will the length and width of an object appear to be magnified through this telescope? Since area can be thought of and measured as a two dimensional "square" concept, calculated by multiplying the length and width dimensions which have the same numeric values and units of measurement, then doing the inverse by taking the square root of the given area (of the object as seen magnified) will yield the length (= width for that area, =side) dimension of that apparent area magnification.

$\sqrt{9}$ = 3 since: 3 x 3 = 9

Therefore, an object as viewed through the telescope will appear about 3 times longer and wider (not 9). The total (magnified) area appears as (3 x 3) = 9 times more or bigger. Below is a graphical representation of this example , the "1x" = "1 times" = "1 power" indicated square is a representation of what the unaided eye without a telescope will see. [FIG 17]



: 9X = "nine times" = "nine power"
Nine of one area. An area of one increased, magnified, or spread out over to the same size as an area of nine.
The "**linear magnification**" of 9 power is $\sqrt{9}$ = 3

You might have a thought that for an 18 power, "18x", telescope that the length and width, or linear (basically means straight), dimensions seen should double (from 3 to 6) since 18 is twice , or double of 9. Let's check:

$\sqrt{18}$ = 4.243 : $\sqrt{\text{"power"}}$ = $\sqrt{\text{apparent, area magnification}}$ = (apparent) linear magnification = (apparent) increases in length and width

The result is less than predicted due to the non-linear nature of powers and roots as mentioned previously in this book. For a "100 power" telescope, the length and width will only appear to be this many times longer;

$\sqrt{100}$ = 10

That is, as the "power of the telescope" (shown as a radicand here), increases or "grows" very large, the dimensions of the object now seen (ie, the root) does not increase or "grow" as fast, but appears to grow less and less. Technically, the rate of change between the linear magnification and the power or area magnification is decreasing or slowing. A large telescope with a large lens or mirror may not have as much magnification as most would think it would. Today, digital image sensors with a very higher density of individual elements (ie. basically pixels) per unit area are helping to effectively increase the **resolution** (ie., "resolve", "fineness", "smallest amount", "precision", "detail") of the image, and which can then even be "digitally magnified", hence effectively magnifying the initial image that the telescope itself can produce. The smallest resolution of a digital image is 1 pixel, and if it represent 1 foot of the actual subject of that image, and the combination of image and subject resolutions can be stated as 1 pixel per foot.

If an already magnified image of an object or area is then magnified again, the resulting magnification,"power" or "times" = "X", is simply the product of those two magnifications. Ex. If an object is magnified by 100 power, and if that magnified image (perhaps even considered as an initial, 1x, image) is then magnified by a power of 4, the resulting magnification of the object is: (100)(4)power = 400 power.

In a telescope or microscope, the image is linear magnified equally in both the horizontal and vertical (or width) directions. If the linear dimensions of an image or area (A1) are both or equally magnified, say by (n) times longer, the area is magnified or increased by a (magnification or "power") factor of n^2, and results in A2.

A1 = (horizontal length) x (vertical length) or = (length) x (width) = LW

A2 = (n) horizontal length x (n) vertical length = n^2 (LW) = n^2 A1

The magnification ratio, factor or "power" is: A2/ A1 = n^2, and the linear magnification is: $\sqrt{n^2} = n$

As an extra note, you can make a crude, low-power telescope using two magnifying glasses separated by a foot or two, but the image will be "upside down". For the larger objective (initial) lens, a low power lens is preferred because a longer focus distance (ie., focal length) actually increases the magnification. For the eyepiece lens which is used to view the image from the objective lens, the shorter the focal length of it, the greater the magnification of the entire telescope lens system. **Do not look at the Sun with your eyes, or with a telescope, because the concentrated and-or focused light it is too "bright" and can burn and damage your eyes. Remind people.**

When a value less than 1 is raised to a higher power, such as squared, the result is a smaller value since a fraction of a fraction is a smaller fraction. Now, in a reverse type of manner, if an indicated root larger than 1, such as a square root, is then taken on this smaller value, the result is a larger value.

Ex. 0.5^2 = (0.5)(0.5) = 0.25 : A fraction of a value (here, another fraction) is a smaller value.
Ex. A fraction squared is a smaller value

$\sqrt{0.25} = 0.5$: here, the root is larger than the radicand itself.
The square root of a fraction (a proper fraction, less than 1) will be greater than that given fraction.

NEGATIVE POWERS OF 10

10^0 = 1
10^-1 = 1/10^1 = 0.1 : 10^0 / 10^1 = 1/10^1 = 1(10^0) / 10^1 = 10^(0-1) = 10^-1 = 0.1 = "a (1) tenth" = "a tenth"

10^-2 = 1/10^2 = 1/100 = 0.01 : 1/100 = "1/one-hundred" = "one/one-hundred" = "one-hundredth"
1/100 = 1 / (10)(10) = (1/10)(1/10) = a tenth of a tenth = one-hundredth
= (0.1)(0.1) = 0.01
= (1/10^-1)(1/10^-1) = 1/10^-2
10^-3 = 1/10^3 = 1/1,000 = 0.001 : one-thousandth , extra: 0.123 = 123 thousandths of a (1) unit
10^-4 = 1/10^4 = 1/10,000 = 0.000,1 : one-tenth of one thousandth = 100 millionths
10^-5 = 1/10^5 = 1/100,000 = 0.000,01 : one-hundredth of one thousandth
10^-6 = 1/10^6 = 1/1,000,000 = 0.000,001 : one-millionth = one-thousandth of one thousandth

The square roots of negative powers of 10 will be shown further ahead.

CHART EXAMPLE THAT SHOWS THE SIMILARITIES BETWEEN POWER AND RADICAL EXPRESSIONS

Except for the names used, the operands and results of a given power expression and corresponding root expression are identical in value due to that powers are roots are mathematically the inverse operations of one another.

Operands of a power operation:

Base	Exponent	Power	Power Expression (mathematical notation)	
2	1	2	2^1	: two to the first power
2	2	4	2^2	: two to the second power
2	3	8	2^3	: two to the third power
2	4	16	2^4	: two to the fourth power
2	5	32	2^5	: two to the fifth power
etc.				

Operands of a radical operation:

Root	Index	Radicand	Radical Expression (notation)	
2	1	2	$\sqrt[1]{2}$	: the first root of two
2	2	4	$\sqrt[2]{4}$	: the second root of four
2	3	8	$\sqrt[3]{8}$	: the third root of eight
2	4	16	$\sqrt[4]{16}$	: the fourth root of sixteen
2	5	32	$\sqrt[5]{32}$	: the fifth root of thirty-two
etc.				

When these values are equal: power = radicand, base = root, exponent = index , we have:

power = base^exponent = radicand = root^index, and:

$\text{base} = \sqrt[\text{exponent}]{\text{power}} = \text{root} = \sqrt[\text{index}]{\text{radicand}}$: a root is essentially the base of a power value

We now know the notation of the expressions to solve for two of the parts (base and power, or root and radicand) of either a power or root expression, and we know the similarities of the power and root expressions. To solve for the exponent or index, a notation called logarithm notation will be used, and this is covered further ahead in this book.

MULTIPLYING RADICALS

Radicals that have the same index (indicated root) can be multiplied. When they are multiplied, the product is a radical with the same index, and the radicand is the product of the radicands. It could also be said that these radicands and radicals that were multiplied together are factors of this new (product) radicand and radical.

Ex. $(2\sqrt{9})(2\sqrt{4}) = 2\sqrt{9}\ 2\sqrt{4} = 2\sqrt{(9)(4)} = 2\sqrt{36} = 6$

Checking by rationalizing the radicals first as indicated in step 1 of the order of operations. Rationalizing a radical expression means ridding or clearing the radical of the radical symbol by solving for the indicated root or expressing it in another form such as with exponential notation which will be discussed further ahead in this book:

$\sqrt{9}\ \sqrt{4} = (3)(2) = 6$

Ex. $3\sqrt{7}\ 3\sqrt{2}\ \sqrt{5}$
$3\sqrt{(7)(2)}\ \sqrt{5}$
$3\sqrt{14}\ \ 2\sqrt{5}$

FACTORING RADICALS

Since radicals with the same index can be multiplied, producing a new radical, we can do the reverse and factor a radical for possible simplification. To factor a radical, the radicand value is factored. These factors can then become radicands of radicals which have the same index as the radical from which they were both factored from.

Ex. $\sqrt{16} = 2\sqrt{(4)(4)} = (2\sqrt{4})(2\sqrt{4}) = 2\sqrt{4}\ 2\sqrt{4}$: The factored radical, where each has the same index as the source radical.

$\sqrt{16} = (2) \quad (2) = 4$

When factoring the radicand for simplification purposes, it is usually best to find factors of it that are indicated powers whose (indicated) exponent is equivalent to the index of the radicand, such as perfect squares (2), cubes (3), fourths (4) , etc., since these radicals are easily rationalized. The indicated root of a value raised to the same indicated power is simply equal to that given value.

Ex. $2\sqrt{18}$

Since the index is 2, factor the radicand for powers that are perfect squares which are values raised to the second power. Note that 18 has factors of 9 and 2, or 6 and 3. Choose the factors of 9 and 2 since 9 is a perfect square of 3. That is, 3^2 = 9, and this radicand can be easily rationalized if the indicated root or index is also 2:

$2\sqrt{18} = 2\sqrt{(9)(2)} = 2\sqrt{9}\ \ 2\sqrt{2}$: factored form. simplifying further, we will factor the radicand of 9 to an indicated (with an exponent) perfect square (2) since the index is 2:

$2\sqrt{3^\wedge 2}\ \ 2\sqrt{2}$ In short, for example, the square root of any value squared (particularly when simply indicated as such) is that value:

$2\sqrt{3^\wedge 2} = \sqrt{3^\wedge 1}\ \sqrt{3^\wedge 1} = 3$: Using the concept that the square-root of any value multiplied by itself (ie squared) is that value, here 3.

$3\ \ 2\sqrt{2}$: Since this is a representation for the square root of 18, this value when squared or raised to the second power should therefore equal 18:

checking: (3 2√2)^2 = (3)^2 (2√2)^2 : after distributing the exponent

= 9 2√2 2√2 : after extending, expressing the indicated square as a multiplication

= 9 2√(2)(2) : multiplying the radicands of like radicals with the same index

= 9 2√4

= 9 (2) = 18 : the original radicand used in this example

Ex. 2√147

2√147 = 2√(49)(3) = 2√(49) 2√(3) = 2√7^2 2√3 = 7 2√3

In short, and as an example to this discussion, the square root of any value squared (especially when indicated as being squared with an exponent of 2), is that same value.

Ex. √ 4^2 = 4 checking:

√ 16 = 4 :It will be shown further ahead in this book that:

indexv√radicand^exponent = radicand^(exponent/index) : even if the exponent = 1

Ex. 3√ 5^3 = √(5)(5)(5) = 5 checking:

3√125 = 5

Since a radicand that is a whole number can easily be factored, this leads to another method to calculate the root of a radicand. You can factor a radicand into radicands that you already know the roots of. You can also completely factor a radicand to its prime factors and use a small table for the (previously solved, calculated) roots of prime factors. Here are the approximate roots of the first 16 prime numbers or factors:

√ 2 = 1.414213562
√ 3 = 1.732050808
√ 5 = 2.236067977
√ 7 = 2.645751311
√ 11 = 3.31662479
√ 13 = 3.605551275
√ 17 = 4.123105626
√ 19 = 4.358898944
√ 23 = 4.795831523
√ 29 = 5.385164807
√ 31 = 5.567764363
√ 37 = 6.08276253
√ 41 = 6.403124237
√ 43 = 6.557438524
√ 47 = 6.8556546
√ 51 = 7.141428429

Ex. √ 105 factoring the radicand:

√ (7)(5)(3) factoring the radical:

√ 7 √ 5 √ 3 using the above table for the square roots of these values:

(2.645751311)(2.236067977)(1.732050808) = 10.24695077

The square roots of negative powers of 10 are also very helpful when the radicand contains a fractional part.

$\sqrt{10^{-1}} = \sqrt{1/10} = \sqrt{10^{-1}} = 0.316227766$: $\sqrt{0.1}$

$\sqrt{100^{-1}} = \sqrt{1/100} = \sqrt{10^{-2}} = 0.1$: $\sqrt{0.01} = \sqrt{0.1}\ \sqrt{0.1} = \sqrt{(10^{-1})(10^{-1})}$

$\sqrt{1000^{-1}} = \sqrt{1/1000} = \sqrt{10^{-3}} = 0.0316227766$: $\sqrt{100^{-2}}\ \ \sqrt{10^{-1}} = \sqrt{0.001}$

$\sqrt{10000^{-1}} = \sqrt{1/10000} = \sqrt{10^{-4}} = 0.01$: $\sqrt{0.0001}$

Ex. $\sqrt{1.05}$: compare this to the last example shown above which has the same digits

$\sqrt{1.05} = \sqrt{(105)(10^{-2})}$: $\sqrt{(105)(1/100)} = \sqrt{(105/100)}$
$\sqrt{1.05} = \sqrt{105}\ \ \sqrt{10^{-2}}$
$\sqrt{1.05} = (10.24695077)(0.1)$: 0.1 = 1/10 = division by 10, so simply move the decimal point 1 place leftward so as to make a smaller number
$\sqrt{1.05} = 1.024695077$

This can also be solved as:

$\sqrt{1.05} = \sqrt{(105)(0.01)} = \sqrt{(5)\,(21)\,(0.01)}$: factored to "primes"
$= \sqrt{5}\ \sqrt{21}\ \sqrt{0.01}$: see the table shown previously for these roots, and then multiply them together.

We see that a radical that has a radicand that is not whole value, but a value that contains a whole part and a fractional part, can essentially be changed or reduced to one of solving the product of radicals or roots with radicands that are whole values.

A RADICAND RAISED TO A POWER EQUAL TO THE INDEX

The root of a radicand that is raised to an indicated power equal to the index of the radical is equal to the base of that radicand.

Ex. $\sqrt[2]{5^2} = 5$: checking: $\sqrt[2]{5^2} = \sqrt[2]{25} = 5$

: also, since $5^2 = 5^1 \times 5^1$, $\sqrt{5^2} = \sqrt{5^1 \times 5^1} = \sqrt{5}\ \sqrt{5} = 5$

Ex. Given the square root of 10, and multiplying it by the square root of 10, we should have 10:

$\sqrt{100} = \sqrt[2]{10^2} = \sqrt{(10)(10)} = \sqrt[2]{10}\ \sqrt[2]{10} = 10$

Ex. $\sqrt[3]{4^3} = 4$: checking: $\sqrt[3]{4^3} = \sqrt[3]{64} = 4$ since $(4)(4)(4) = 64 = 4^3$

Ex. $\sqrt[4]{2^4} = 2$: checking: $\sqrt[4]{2^4} = \sqrt[4]{16} = 2$

Ex. $\sqrt[3]{(2^2)^3} = (2^2) = 4$

checking: $\sqrt[3]{(2^2)^3} = \sqrt[3]{2^{(2\text{x}3)}} = \sqrt[3]{2^6} = \sqrt[3]{64} = 4$

checking: $\sqrt[3]{(2^2)^3} = \sqrt[3]{4^3} = 4$ since $(4^1)(4^1)(4^1) = 4^3$

By observing the examples given, a basic way to rationalize radicals with a radicand raised to an indicated power equal to the index, is to place (or "convert") the radical into an exponential form. This can be accomplished by simply removing the radical symbol, and divide the radicands exponent by the index of the radical.

Ex. $\sqrt[2]{5^2} = 5^{(2/2)} = 5^1$: $5^{(2/2)}$ is the exponential form of $\sqrt[2]{5^2}$

Ex. $\sqrt[3]{4^3} = 4^{(3/3)} = 4^1$

Ex. $\sqrt[4]{2^4} = 2^{(4/4)} = 2^1$

Ex. $\sqrt[3]{(2^2)^3} = (2^2)^{(3/3)} = (2^2)^1 = 2^2 = 4$

If the radicand's exponent is evenly divisible by the index, then the radicand is a "perfect (integer) power" of the index. For example, given the radicands below:

5^4 is a perfect square (2nd power) of $5^{(4/2)} = 5^2$, since $(5^2)^2 = 5^4$.

Therefore, the square root of 5^4 should be equal to 5^2:

$\sqrt[2]{5^4} = 5^{(4/2)} = 5^2$

5^4 is also a perfect fourth (power) of $5^{(4/4)} = 5^1$ since $(5^1)^4 = 5^{(1\text{x}4)} = 5^4$

Hence, 4√ 5^4 = 5^(4/4) = 5^1 = 5 checking:

4√ 625 = 5

Ex. 3√ 7^6 = 7^(6/3) = 7^2 = 49 : 7^6 is a perfect third power or "cube" of 7^2: (7^2)^3 = 7^6

If the radicand's exponent is not evenly divisible by the index when rationalizing, the radicand can be factored into factors having exponents equal to the index. This factorization is essentially the reverse of multiplying like values, or variables as in algebra, where the product is that value and its exponent is the sum of the exponents. Below are some examples with their intermediate steps.

Ex. 2√ 5^3

2√ 5^3 = 2√ (5^2) (5^1) = 2√ 5^2 2√ 5 = 5^(2/2) 2√ 5 = 5 2√ 5

Note that even though the indicated power of the radicand is not evenly divisible by a value equal to the index, still, the result can be correctly indicated as such. This is shown below using the last example, and more will be said about this further in this book.

2 √ 5^3 = 5^(3/2) or 5^1.5 and as shown above, this = 5 2√ 5 , and: (5^1.5)^2 = 5^(1.5 x 2) = 5^3

Ex. 3√ (7^2)^4

3√ (7^2)^4 = 3√ (7^2)^3 (7^2)^1 = 3√ (7^2)^3 3√ 7^2 = (7^2)^(3/3) 3√ 7^2

= 7^2 3√ 7^2 = 49 3√ 7^2 = 49 3√ 49

OR: 3√ (7^2)^4 = 3√ 7^(2x4) = 3√ 7^8 = 3√ (7^3) (7^3) (7^2) =

= 3√ 7^3 3√ 7^3 3√ 7^2 = 7^(3/3) 7^(3/3) 3√ 7^2 = (7^1) (7^1) 3√ 7^2 =

= 7^2 3√ 7^2 = 49 3√ 7^2 = 49 3√ 49

RADICALS WHOSE FACTORS ARE ALL "PERFECT POWERS" OF THE INDEX

The above title is short-hand for saying radicals whose factors are all raised to the same indicated power that is also equal to the index of the radical. This radical can be easily rationalized or simplified by multiplying the base of each power in the radicand's factors.

Ex. $\sqrt[2]{(4^2)(3^2)}$: here, the index is 2, and each base is raised to the 2nd power.
$(4)\ (3)$
12

checking: $\sqrt[2]{(4^2)(3^2)}$
$\sqrt[2]{(16)(9)}$: 16 and 9 are "perfect powers", here, perfect squares which are due to values raised to the second power. Since the index is also 2, these radicand values, when expressed as powers of two, can be used to easily rationalize (ie. remove) the radical expression.
$\sqrt[2]{144}$
12

You can also perform a check by "distributing" the index to each factor of the radicand when converting them to their equivalent or corresponding exponential form so as to rationalize the radical expressions.

$\sqrt[2]{(4^2)(3^2)}$
$\sqrt[2]{4^2}\quad \sqrt[2]{3^2}$
$(4^{(2/2)})\ (3^{(2/2)})$
$(4)\quad (3)$
12

or: $\sqrt[2]{4 \times 4}\quad \sqrt[2]{3 \times 3}$
$\sqrt{4}\ \sqrt{4}\quad \sqrt{3}\ \sqrt{3}$
$4\qquad 3$
12

Another check is this:

$\sqrt[2]{(4^2)(3^2)}$
$\sqrt[2]{(4 \times 3)^2}$
$\sqrt[2]{12^2} = \sqrt{144}$
$12^{(2/2)}$
12

EXPONENTS THAT ARE LESS THAN ONE OR ARE FRACTIONS

Exponents can be any value, they need not be only integers (essentially the familiar basic counting values). This process of dividing the exponent of the radicand by the index leads us directly to exponents that are fractions. This is also a method of rationalizing radicals such as shown on the right hand side of the following equation:

index√ base^exponent = base^(exponent/index) : a general format for converting between radical and exponential expressions.
Note that a power or exponential expression can be a radicand of a radical.
Note also that both sides represent a root.

Ex. 2√ 5^2 = 2√25 = 5^(2/2) = 5^1 Ex. 3√1000 = 3√10^3 = 10^(3/3) = 10^1 = 10

Since a number (ie. base) raised to the first power is still that number, it should be of no surprise that when a number is raised to a value less than one (ie. a fraction), that the resulting (power) value is smaller than that number, hence we find it useful as another way to express some root of a number.

Ex. 2√ 5^1 = 5^(1/2) : "five to the one-half (1/2 = 0.5) power is equal to the square root of five"

2√5 = 5^0.5

Hence, we have found a way to convert a radical form of a root to it's equivalent exponential form and vice-versa.

Ex. 2√25^1 = 25^(1/2) or= 25^0.5 = 5

Ex. 27^(1/3) = 3√27^1 : = 3

Ex. 27^(2/3) = 3√27^2 : = 9

Ex. 2√ 5^3 = 5^(3/2) : = 5^1.5

Ex. 81^(1/4) = 4√ 81 : = 3

Ex. 3√ (5^3)^4 = (5^3)^(4/3) also, by using the "power to a power" rule:

(5^3)^(4/3) = 5^((3/1)(4/3)) = 5^12/3 = 5^4

In general, if the index is less than the indicated power of the radicand, the result is a power of the radicand, and if the index is greater than the indicated power of the radicand, the result is a root of the radicand.

Ex. 0.5√10^2 = 10^(2/0.5) = 10^4 = 10000

2√10^2 = 10^(2/2) = 10^1 = 10

3√10^2 = 10^(2/3) = 10^0.66$\underline{7}$ = 4.64$\underline{2}$

4√10^2 = 10^(2/4) = 10^0.5 = 3.16227766... : also = 10^(2/4) = 10^(1/2) = 2√10^1

A RADICAL TO A POWER

One way to rationalize a radical (ie. an expression for a root) which is raised to a power, is to first convert the radical to its equivalent exponential form. Then use the "power to a power" rule where the exponents are distributed (ie., multiplied).

Ex. (2√ 3)^5

(2√ 3^1)^5 converting the radical to exponential form:

(3^(1/2))^5 "power to a power", "distribute" (multiply) the exponents:

3^((1/2)(5/1))

3^(5/2)

Also, converting 3^(5/2) back to radical form:

3^(5/2) = 2√ 3^5

Hence, (2√ 3)^5 = 2√ 3^5 this equivalence leads to this following statement:

A radical and radicand raised to the same power are equivalent in value. The base of the radicands, and the indexes of the radicals must also be the same. The reasoning behind this equivalence is due to the fact that multiplication is commutative. This can be seen in their exponential (power) form before the exponents are multiplied:

(3^1/2)^(5/1) is the same as: (3^5/1)^(1/2) , both expressions are equal to: 3^(5/2)

Using the above equivalence, this result can also be seen if the exponents are distributed:

3^((1/2)(5/1)) = 3^((5/1)(1/2)) = 3^(5/2)

Before leaving this discussion, it must be pointed out that converting an exponent that is an improper fraction to a mixed number can be useful.

Ex. First: 7^3/2 = 2√ 7^3 = (2√ 7)^3 = 7^1.5

Now, by converting the exponent to a mixed number:

Since 3/2 = 1.5 = 1 + 0.5 = 1 + 1/2 , we have:

7^3/2 = 7^(1 + 1/2) = (7^1) (7^1/2)

7^3/2 = 7 √ 7 = 7^1.5

A RADICAL RAISED TO A POWER EQUAL TO THE INDEX

The root of a radical that is raised to a power equal to the index is equal to the radicand value of that radical.

Ex. $(\sqrt[2]{5})^2 = 5$

checking: $(\sqrt[2]{5})^2$ placing the radical into exponential form :
$(5^{(1/2)})^2$ with a "power to a power", distribute the exponent :
$5^{(2/2)}$
$5^1 = 5$

Or: $(\sqrt[2]{5})^2$
$(\sqrt[2]{5})(\sqrt[2]{5})$: "extended method" (it should also be obvious here that the square root of the same value, is that value, here 5)
$\sqrt[2]{(5)(5)}$: from multiplying "like" radicals (have same index), simply multiply their radicands

$\sqrt[2]{5^2} = \sqrt[2]{25} = 5$ or:
$\sqrt[2]{5^2} = 5^{(2/2)} = 5$

Note that this is very similar to where the root of a radicand in which the radicand is raised to an indicated power equal to the index:

$(\sqrt[2]{5})^2 = \sqrt[2]{5^2} = 5$: Note also for example, that the square (2nd power) of the square-root of number (here a radicand) is equal to that number. If we let letter (r) represent or equal the square root of a number (N), then (r x r) or r^2 will equal that number: This is verified here mathematically:

$r = \sqrt{N}$ then

$r \times r = \sqrt{N}\ \sqrt{N}$: Multiplying both sides by r or $\sqrt{N}$, or either (since they are equal in value), still keeps both sides of the equation in balance. Note that this is therefore also the same as simply squaring each side:

$r^2 = (\sqrt{N})^2$ "extending" the right hand side:

$r^2 = \sqrt{N}\ \sqrt{N} = N^{1/2}\ N^{1/2} = N^{(1/2 + 1/2)} = N^{2/2}$

or $= N^{(0.5 + 0.5)} = N^1$

$r^2 = N$

If r was the cube root of N:

$r = \sqrt[3]{N}$ raising each side to the third power:

$r^3 = \sqrt[3]{N}\ \sqrt[3]{N}\ \sqrt[3]{N} = (\sqrt[3]{N})^3 = N^{1/3}\ N^{1/3}\ N^{1/3} = N^{(3/3)}$

$r^3 = N$

THE ROOT OF A FRACTION

The root of a fraction is equal to the root of the numerator divided by the root of the denominator. The indicated root or index of the roots will also be the same.

Ex. $\sqrt[2]{\frac{36}{4}} = \frac{\sqrt[2]{36}}{\sqrt[2]{4}} = \frac{6}{2} = 3$

Checking using the order of operations more closely where the value (ie. radicand) underneath the radical is simplified first since the division symbol is an inner grouping symbol for this example, and the radical symbol is an outer grouping symbol:

$\sqrt[2]{\frac{36}{4}} = \sqrt[2]{9} = 3$

Here is another check by first converting the radical to exponential notation:

$\sqrt[2]{\frac{36}{4}} = \left[\frac{36}{4}\right]^{(1/2)} = \frac{36^{(1/2)}}{4^{(1/2)}}$: After using the fraction to a power rule of:
: (Distributing the exponent to both the num. and den.)

$= \frac{\sqrt{36}}{\sqrt{4}} = \frac{6}{2} = 3$

RATIONALIZING FRACTIONS THAT CONTAIN RADICALS

There are two basic methods to rationalize (to make sense of) or rid fractions that contain radicals: The equivalent fraction method, and the equivalent radicand method. They are somewhat similar, but different names are used to distinguish them. This discussion also shows how to rationalize a radicand that is a fraction. The main goal in this process is to create a "perfect (indicated) power" of the index , which can then be easily rationalized.

Equivalent Fraction Method:

First, convert both the numerator and denominator to radicals. This creates a fraction of two radicals. Then an equivalent fraction is made by multiplying the fractions numerator and denominator by the denominator raised to a power such that the resulting radical in the denominator will be raised to an indicated power that is some multiple of the index. This denominator, when simplified, is equal to the radicand.

Ex. $\sqrt[3]{\frac{2}{3}} = \frac{\sqrt[3]{2}}{\sqrt[3]{3}} = \frac{\sqrt[3]{2}}{(\sqrt[3]{3})^1}$ Multiplying the numerator and denominator by $(\sqrt[3]{3})^2$:

$\frac{\sqrt[2]{2}\,(\sqrt[3]{3})^2}{(\sqrt[3]{3})^1\,(\sqrt[3]{3})^2} = \frac{\sqrt[3]{2}\,(\sqrt[3]{3})^2}{(\sqrt[3]{3})^3}$ After converting the radical in the denominator to its equivalent exponential form and simplifying. $(\sqrt[3]{3})^3 = 3^{(1/3)3} = 3^{(3/3)} = 3^1$:

$\frac{\sqrt[3]{2}\,(\sqrt[3]{3})\,(\sqrt[3]{3})}{3^{(3/3)}} = \frac{\sqrt[3]{18}}{3}$: In the numerator, the radicands were multiplied since the index (indicated root) is the same for each radical.

Equivalent Radicand Method:

$$\sqrt[3]{\frac{2}{3}} = \sqrt[3]{\frac{2\,(3^2)}{(3^1)(3^2)}} = \sqrt[3]{\frac{(2)(9)}{3^3}} = \frac{\sqrt[3]{18}}{\sqrt[3]{3^3}} = \frac{\sqrt[3]{18}}{3^{(3/3)}} = \frac{\sqrt[3]{18}}{3}$$

The above concepts should be taken into general account and especially when rationalizing (essentially simplifying) fractions containing radicals, or rationalizing radicals whose radicand is a fraction.

A CAUTIONARY NOTE WHEN USING POWERS AND ROOTS WITH FRACTIONS

When creating an equivalent fraction, it is incorrect to "raise" the numerator and denominator to the same power, perhaps in an attempt at creating an equivalent fraction.

Ex. $\frac{4}{8}$ /=/ $\frac{4^2}{8^2}$ or $\left(\frac{4}{8}\right)^2$: /=/ or != are symbols meaning "is not equal to" or "not equals"
Often, this symbol is used: $\neq$

Checking by taking the decimal form of both sides:

0.5 /=/ 0.25 : This is so since the numerator and denominator are not actually being multiplied by the same identical value when you raise them to the same power. For this example, the numerator was multiplied by 4, and the denominator by 8.

Likewise, when creating an equivalent fraction, it is incorrect to take the same indicated root of both the numerator and denominator.

Ex. $\frac{16}{64}$ /=/ $\frac{\sqrt{16}}{\sqrt{64}}$ or $\sqrt{\frac{16}{64}}$: A fraction, and the root of a fraction are two different values, and not equal. This is similar to above example, but the indicated power that the numerator and denominator values would be raised to would be 0.5 or= 1/2.

Checking by taking the decimal form of both sides:

0.25 /=/ 0.5

ADDING RADICALS

Radicals that have the same index and radicand are said to be "like" or similar radicals. Since the radicals represent the same value or quantity, simply add their numerical coefficients (the multiplying factor(s)). If a numerical coefficient is not shown, it is considered as being one (1), and we know that multiplying anything by one is alright since it does not change it's value.

Ex. $\sqrt{9} + \sqrt{9} \quad = \quad (1)\ 2\sqrt{9} + (1)\ 2\sqrt{9}$

$(1 + 1)\ 2\sqrt{9}$
$2\ \ 2\sqrt{9}$: This result should be easily seen since multiplication is really repeated addition of the same value. With further simplification:

$2\ (3)$
6

Checking:

$2\sqrt{9} + 2\sqrt{9}$ Order of operations, step 1 (simplify powers and roots) :
$3 + 3$ Order of operations, step 4 (additions and subtractions; to combine)
6

Ex. $3\sqrt{7} + 2\sqrt{7}$
$(3 + 2)\sqrt{7}$
$5\sqrt{7}$: in general, $\sqrt{7}$ is to be added to itself (or to an initial sum of 0 if you will) 3 times, and then 2 more times for a total of 5 times, hence multiply by 5 since it is essentially being summed (to a starting reference value of 0) a total of 5 times.

Checking:

$3\sqrt{7} + 2\sqrt{7}$ since multiplication is repeated addition:
$(\sqrt{7} + \sqrt{7} + \sqrt{7}) + (\sqrt{7} + \sqrt{7})$
$(1 + 1 + 1 + 1 + 1)\ \sqrt{7}$ After clearing grouping symbols, and expressing
$5\sqrt{7}$ the sum of similar things or values.

MULTIPLYING RADICALS WHICH HAVE UNLIKE INDEXES

It is possible to multiply radicals that have unlike indexes. First, convert and express each radical into an equivalent exponential form, then the exponents can be changed to equivalent fractions with like denominators, and this is essentially creating like indexes for their corresponding radical forms.

Ex. $\sqrt[2]{2}\ \sqrt[3]{4}$
(2^(1/2)) (4^(1/3)) : exponential forms of the radicals

Below, equivalent fractions of the exponents are created in a manner similar to that of adding fractions. A common denominator of 6 was chosen for the fractions and it is also the lowest common denominator (LCD) of the two.

$\frac{1}{2} = \frac{3}{6}$ and $\frac{1}{3} = \frac{2}{6}$

Hence, 2^(1/2) = 2^(3/6) and 4^(1/3) = 4^(2/6) : 2^0.5 and 4^0.3333**...**

Converting each back to its radical form and multiplying:

(2^(3/6)) (4^(2/6))

$\sqrt[6]{2^3}\ \sqrt[6]{4^2}$: created radicals with like indexes

$\sqrt[6]{8}\ \ \sqrt[6]{16} = \sqrt[6]{(8)(16)} =$: multiplying radicands of radicals which have the same index

$\sqrt[6]{128}$

THE ROOT OF A ROOT

The root of another indicated root is equal to a radical whose index is the product (ie., multiplication) of the index of each radical.

Ex. $\sqrt[3]{\sqrt[2]{64}}$: note here that the inner radicand is actually another radical (indicated root to be taken)

$\sqrt[3]{\sqrt[2]{64}} = \sqrt[(3)(2)]{64} = \sqrt[6]{64}$

This is verified below:

Converting the inner radical to exponential form:

$\sqrt[3]{(64^{(1/2)})}$

Then this entire radical is converted to exponential form:

$(64^{(1/2)})^{(1/3)}$

By using the "(indicated) power to a power" rule of multiplying exponents, we distribute the exponents:

$(64^{(1/2)})^{(1/3)} = 64^{(\,(1/2)(1/3)\,)} = 64^{(1/6)}$

Converting this back to radical form, we get:

$64^{(1/6)} = \sqrt[6]{64}$: checks

Due to multiplication being commutative, the exponents of a "power to a power" can be exchanged.

Ex. $(64^{(1/2)})^{(1/3)} = 64^{(\,(1/2)(1/3)\,)} = 64^{(1/6)}$

$(64^{(1/3)})^{(1/2)} = 64^{(\,(1/3)(1/2)\,)} = 64^{(1/6)}$

Converting both expressions on the above left sides to radical form, we get:

$$\sqrt[3]{\sqrt[2]{64}} = \sqrt[2]{\sqrt[3]{64}}$$

checking: first simplifying the "inner radical" radicand:

$$\begin{array}{ccc} \sqrt[3]{\sqrt[2]{64}} & = & \sqrt[2]{\sqrt[3]{64}} \\ \sqrt[3]{8} & = & \sqrt[2]{4} \\ 2 & = & 2 \end{array}$$

This concept can also be used to find some higher integer roots using only square or cube roots. For example, you can find the fourth (4) root of a number (N) by taking the square root of the square root of the number:

$\sqrt[4]{N} = \sqrt[2]{\sqrt[2]{N}}$: the right hand expression is like a "factored index" form of a radical expression

The APPENDIX section has a similar method of how to calculate odd roots, such as cube (3rd) roots, using square roots.

The following discussion is somewhat of an advanced concept, and perhaps not used much in practice, but for completeness and familiarity, it will be presented since it is based on, and is a natural progression stemming from the previous discussions.

A radical that has an indicated power of a radicand, is equal to a radical having that radicand base value, and the index is divided by that indicated power. Expressing this using words as symbolic identifiers or variables:

index\/$\overline{\text{radicand^exponent}}$ = (index/exponent)\/$\overline{\text{radicand}}$

Also consider in this discussion:

index\/$\overline{\text{radicand^exponent}}$ = radicand^(exponent/index) : the exponential form of the radical form

Here is the derivation of the first shown equation:

For simplicity, let: i = index, r = radicand, e = exponent , z = 1/e , therefore e = 1/z

i\/$\overline{\text{r^e}}$ = i\/$\overline{\text{r^(1/z)}}$ = i\/$\overline{\text{z\/r}}$ = iz\/$\overline{\text{r}}$ = i(1/e)\/$\overline{\text{r}}$ = (i/e)\/$\overline{\text{r}}$

Ex. 3\/$\overline{\text{10^2}}$ = (3/2)\/$\overline{\text{10}}$ = 1.5\/$\overline{\text{10}}$ = 10^(2/3) = 10^0.666$\underline{7}$ = 3\/$\overline{\text{100}}$ = 4.641$\underline{6}$

Checking, and a verification using what we know already:

(3/2)\/$\overline{\text{10^1}}$ = 10^(1 / (3/2)) = 10^(2/3) = 3\/$\overline{\text{10^2}}$

EXPRESSING RADICANDS LESS THAN ONE WITH INTEGER RADICANDS

This is an infrequently used method of calculating roots, but if you only have a table of the roots of some integers, it will be very useful. If you plan to calculate roots using other methods, such as those presented in this book, you will at least get another good lesson in mathematical manipulations here. First, represent the radicand in an equivalent proper fraction form, then represent this radical form as the division of two radicals. Placing a value less than one into a (proper) fractional form is discussed ahead in the topic of scientific notation (SN) if you need help doing so.

Ex. $\sqrt{0.5} = \sqrt{5(10^{-1})} = \sqrt{\frac{5}{10}} = \frac{\sqrt{5}}{\sqrt{10}}$: calculate the square root of 0.5 using the square roots of 5 and 10

Or: $\sqrt{0.5} = \sqrt{\frac{50}{100}} = \frac{\sqrt{50}}{\sqrt{100}} = \frac{\sqrt{50}}{10} = \frac{\sqrt{5}\ \sqrt{10}}{10}$

Here is another method to calculate this:

$$\sqrt{0.5} = \sqrt{\frac{0.5^2}{0.5^1}} = \frac{\sqrt{0.5^2}}{\sqrt{0.5}} = \frac{0.5}{\sqrt{0.5}} = \frac{0.5\sqrt{1}}{\sqrt{0.5}} = 0.5\sqrt{\frac{1}{0.5}} = 0.5\sqrt{2}$$

By observing the procedure for the last method above, a formula can be written. Letting N represent (be equivalent to) the radicand in question:

$\sqrt{N} = N\sqrt{\frac{1}{N}}$:The square root of a value is equal to that value times the square root of its reciprocal.
From this we also have: $1/N = (1/N)(N/N) = N/N^2$ = the reciprocal of a number

A check on this formula can be made by converting the radicals to their equivalent exponential form:

$$\sqrt{N} = N^{0.5} \quad \text{and} \quad N\sqrt{\frac{1}{N}} = N\frac{\sqrt{1}}{\sqrt{N}} = \frac{N}{N^{0.5}} = \frac{N^{1-0.5}}{1} = N^{0.5} = N^{(1/2)} = \sqrt{N}$$

Here is a formula, that is somewhat similar to the above formula, that can be used to calculate roots. It is a good mathematical lesson, but is perhaps somewhat advanced or excessive, and the reader may simply skip over it.

$\sqrt{N} = \dfrac{1}{\sqrt{\frac{1}{N}}}$: The square root of N is equal to the reciprocal,
of the square root of the reciprocal of N.

The derivation of this is as follows: The reciprocal of N is 1/N. To do the "reverse", we will express the reciprocal of 1/N to have N:

$\dfrac{\frac{1}{1}}{\frac{1}{N}} = N$: The reciprocal of the reciprocal of N is equal to N. After taking
the square root of both sides, we have the equation above.

The same should be true for any value, including square root values. The reciprocal of: $(1/\sqrt{N})$, or the reciprocal of the reciprocal of the square root of N is:

$$\frac{\frac{1}{1}}{\frac{1}{\sqrt{N}}} = \frac{\frac{1}{1}}{\frac{\sqrt{1}}{\sqrt{N}}} = \frac{\frac{1}{1}}{\sqrt{\frac{1}{N}}} = \sqrt{N}$$

: this can also of been solved by inverting the denominator, and creating an equivalent fraction.

Ex. $\sqrt{5} = \frac{1}{\sqrt{0.2}} \sim 2.23607$

: the "wavy" equals symbol: ~ or ~= means "approximately" or "about" equal to. The reciprocal of 5 is 0.2

RADICANDS THAT VARY BY A CONSTANT FACTOR

This is a method of evaluating radicals where the radicands differ by a common factor. The result is basically the same as that of where a radical is equal to a product of radicals whose radicands are factors of the original radicand. If the many radicands have a common factor, this common factor only needs to be evaluated once. This concept can be of great use if you are creating a table of roots.

Ex. Here, a constant factor between the radicands is 10, and there are others such as 2 and 5:

Since $\sqrt{10} = 3.16227766$ approximately. Now observe this radicand of 100 which is 10 times greater than 10:

$$\sqrt{100} = \sqrt{(10)(10)} = \sqrt{10}\ \sqrt{10} = 3.16227766\ \sqrt{10} = (3.16227766)(3.16227766) = 10$$

$$\sqrt{90} = \sqrt{(10)\,(9)} = \sqrt{10}\ \sqrt{9} = 3.16227766\ \sqrt{9} = (3,16227766)(3) = 9.48683298$$

$$\sqrt{20} = \sqrt{(10)(2)} = \sqrt{10}\ \sqrt{2} = 3.16227766\ \sqrt{2} = (3.16227766)(1.414213562) = 4.472135955$$

A general "formula" for this discussion can be written or expressed mathematically. Below, R is used to represent the initial radicand, N and F represent factors of the radicand, where F is also a common factor between several radicands that you already know the root of:

$\sqrt{R} = \sqrt{FN} = \sqrt{F}\ \sqrt{N}$: for the above examples, F = 10 and its root is 3.16227766

R, F and N are examples of what are called "variables" in algebra, as you can see, each is simply a (symbolic) mathematical representation or placeholder for an actual number value. The actual symbol(s), letters, or words used to represent a variable is sometimes called the variable identifier or variable name. A variable is a general representation of many possible numeric values and-or a particular value being solved for.

LOGARITHMS (LOGS)

For a logarithm (or log) mathematical operation, you are given in the expression a base value and a number (N) equal to some power of that base, and are to find the (indicated) exponent of the base that would make that base equal to this power value or "Number". **Try to think of a log as simply the operation to find an exponent, and-or that a log is equivalent to an exponent.** The word "logarithm" was from that table values or "entries", such as for exponents and power values of a given base that were often logged (written or entered) into a (quick "lookup", find) reference table (list). Instead of using a fancy symbol such as like how a square root symbol is, the symbol chosen for a logarithm operation is simply the world "log".

$\text{exponent} = \text{Log}_{\text{base}} \text{Number}$: GENERAL LOG EXPRESSION, EQUATION OR "FORMULA"

Note, in the expression above on the right side, number is not the exponent of base, but a power of the base, that is:

base^exponent = Number = the actual power value of the base.
The left side of the equation is the indicated or expressed power value.

A simplified notation for a log expression, using simple variable identifiers or names, can be something like this:

$e = \log_b N$: b=base (of the power value), N=number or power value, e=exponent

If no base value is given, it is understood as being 10. Logarithms of base 10 are called "common logarithms" due to the commonly used decimal system that is based on 10.

If you want to view all the basic algebraic relationship between powers, roots, and exponents or logarithms, then see the paragraph called: Algebraically Expressing The More Advanced Mathematical Operations, that is further ahead in this book

Ex. Since 10^3 = 1000, if given just the base and the number (a, or the, power of the base), the notation used to solve for the necessary exponent, of that base so as it would equal that given power value, is the logarithm notation (expression):

$\text{exponent} = \log_{10} 1000$ or: $\text{exponent} = \log 1000$: when a base value is not indicated, it is understood or considered as base 10, the common log base.

$\text{exponent} = 3 = \log_{10} 1000$: since 10^3 = 1000 : 3 is the logarithm or exponent. "The log of 1000, using a base of 10, is 3".

Frequently, a calculator with a logarithm function is used to find logs. Generally, "scientific" calculators have function keys for finding only common and natural logarithms. If the base is not 10 or (e = about 2.72) which is the base for "natural" logarithms what will be discussed in this book, these calculator functions cannot be immediately utilized. Still, most logarithms to be found in common everyday home and scientific usage have either a base of 10 or (e).

Ex. Solve $\log_2 8$: a sometimes used text alternate notation for this expression may be: log2 8

To solve this is to find what would be the exponent of 2 such that the resulting power value of it is equal to 8:

$\text{exponent} = \log_2 8$: here the base of the logarithm or power is 2, and the number or actual power value is 8

$3 = \log_{2} 8$ since $2^3 = 8$

The examples given were very simple and could have been done simply by inspection. More information on finding exponents will be given ahead in the discussion of: Finding Exponents With Logarithms.

The **Number in a logarithm expression must be a positive value greater than 0**. A logarithm (ie., the exponent value) will be a positive value greater than 0 when the number is greater than 1. First, a logarithm is always zero when the number is 1: $\log 1 = 0$: here n=1, consider: $5^0 = 1$, ex. $7^0=1$. Now consider when (n) is greater than 1: Ex. $\log 100 = 2$: here n=100 is greater than 1, and which can be expressed as n>1. Now consider when (n) is less than 1 but still greater than 0 as required: Ex. $\log 0.5 =$ about -0.30103 : here n=0.5 is less than 1, and which can be expressed as n<1 or (n<1), and the logarithm value is therefore less than 0 and is negative in value.

Considering the above statement, some will say for example that given $(-2)^3 = -8$, and therefore: $\log_{(-2)} -8 = 3 = \log_{2} 8$

, and the problem with this is the both the base and Number ambiguity (ie., uncertainty) of the signs of the base and number. Also consider the base ambiguity of this example: $\log_{-2} 4 = 2 = \log_{2} 4$

Without the aid of a modern **electronic calculator**, logarithm tables or mathematical calculation, a mechanical device or simple mechanical calculator or mechanical computer (that which computes, performs or solves computations) called a **slide-rule** can (only) give a good approximation (typically only a couple of decimal places) of a logarithm or anti-logarithm (finding the number or power value of the base). The slide rule gets its name from that it is similar to two rulers constructed to be adjustable by sliding one next to (ie., parallel) or against the other. Basically, a result is indicated by the corresponding value directly opposite and indicated on the other (logarithm) ruler scale. The slide rule is also capable of other mathematical calculations. The longer the slide rule is, the more precision capable in the result since more subdivisions (smaller fractional parts) can be drawn or fitted on the indicated scales.

An alternative to calculating machines is to use pre-made tables of values. These tables generally offer more precision than that capable on a slide rule or graph, but not as much precision as a modern electronic calculator and-or computer.

The first mechanical adding machine is considered to be that made by **Blaise Pascal** (1623-1662), from France, in about 1642. This machine could also perform subtraction. Gottfried Leibniz built an improved adding machine in the late 1600's. There have been many designs and improvements for adding machines ever since, and of which included other mathematical operations.

SOLVING FOR THE BASE OF A POWER

Given a power value of a base value, and it's indicated exponent, what is the corresponding base? An easy way to solve for the base is to place the values into exponential form.

$4 = \log_{\text{base}} 16$: logarithmic equation form , $\text{exponent} = \log_{\text{base}} \text{power}$: power = number

base^4 = 16 : exponential equation form , base^exponent = power : power = number

Since the base is being raised to the fourth power, by taking its root with an index equal to the indicated power (the exponent), the base can be isolated or solved for. This is clearly seen once the radical is placed into exponential form. This operation must be done to the entire values or expressions on both sides of the equal sign of an equation so as to keep the equivalence or balance of both sides. More will be said about this during the discussion of equations. Since a base is effectively the "origin" or "root (of a power)" that the power value (ie., Number) is created or based upon, it should be reasonably obvious that a root operation on this power value will be needed to solve for the corresponding base.

Base^4 = 16 expressing the taking of the 4th root of both sides:

$\sqrt[4]{\text{base^4}} = \sqrt[4]{16}$ expressing or converting the left side to its equivalent exponential form

base^(4/4) = $\sqrt[4]{16}$: exponential form (a rationalized form)

base = 2

hence, $4 = \log_{2} 16$ or 2^4 = 16

FINDING THE NUMBER (POWER VALUE OF THE BASE)

Ex. Find number or power value given: 4 = log number

Here, the base value is understood as the common base of 10. Expressing this in exponential form:

10^4 = number

Solving for the fourth power of 10 using repeated multiplication, or "extending" the indicated power:

10^4 = (10)(10)(10)(10)
10^4 = 100(10)(10)
10^4 = 1000(10)
10^4 = 10000

number = 10,000

Expressing the indicated power 10^4 in log form:

$4 = \log_{10} 10{,}000$ or = $\log_{10}$ 10^4

Ex. Find number given: $\log_{81} \text{number} = \frac{1}{4}$

Again, placing this information into exponential form is the best way to start:

81^(1/4) = number or: 81^0.25 = number : 1/4 = 0.25

Since the indicated power (the exponent) is not an integer, number (the power) cannot be solved for by repeated multiplication. Converting this "fractional power" to its equivalent radical notation:

81^(1/4) = $\sqrt[4]{81^1}$ = $\sqrt[4]{81}$ taking the fourth root of 84:

$\sqrt[4]{81}$ = 3 = number : root form or by switching sides (formally known as the Symmetric Law):

number = 3

Hence, $\log_{81} 3 = \frac{1}{4}$ = 0.25 :log form and 81^0.25 = 3 : exponential form

THE LOG OF 1

The log (exponent) of any base to equal a number (the power value of the base) with a value of 1, is zero (0). This can be shown from the fact that a base with an exponent of zero is equal to one.

Ex. since 10^0 = 1 : extra, 10^0 is also the weight of the ones (1's) column or digit position

Converting and expressing this exponential form to its equivalent logarithmic form, we get:

$0 = \log_{10} 1$: using 0 or zero is necessary since if 10^1 = 10, 10^0 is needed to express 1
When the number is less than 1, specifically between 0 and 1, the log of it will be negative, hence less than 0.

AN EXAMPLE OF LOGS IN EVERYDAY LIFE

The human ear is said to hear or respond to a sound's intensity or energy level it receives in a logarithmic type of manner, and not a linear or "direct" equivalent manner. Light intensity seen is another example of a logarithmic-like sensing or quantification by the body of outside stimulus or input. For example, the sound energy from someone talking may have hundreds of times more sound energy than someone whispering, however, the human ear will only hear or represent this as only several times louder. Placing some examples in logarithmic form where the number (a power of the base) represents the amount of sound energy (actual physical power) or intensity, and the exponent (or logarithm) numerically represents what energy or volume level that the human ear will apparently or effectively hears, senses or perceives.

Apparent volume or energy heard expressed as the log of the actual sound energy

Number represents actual amount of energy.

(Logarithm) , (Number) : logarithm = exponent of the base, and here, base = 10

pin drop: 1 = log 10
whisper: 2 = log 100
talking: 4 = log 10,000
yelling: 6 = log 1,000,000

Clearly it is much easier to express the large range of number values (that change greatly or exponentially) into a highly "compressed", linear-like, and more manageable form such as the logarithmic (ie. exponent) values and representation presented. For example, note that the number of 100 is 10 times (magnification) more than 10, or an increase of 90 from a value of 10, but the log value (or exponent) only increased only by 1 to represent this increase. It could be said that logarithms are "slow (and get slower and slower) growing", and powers are "fast (and get faster and faster) growing".

Extra: In electronic and energy studies, vast increases, gains or comparing two values can be easily expressed and understood as logarithmic or "compressed" values. The common formula used was initially developed for use with the Alexander Graham Bell's telephone system. Since this is somewhat of an advanced example below, the reader may skip of this and review it later.

Ex. A gain, increase or magnification factor can be expressed as: Gain = (output / input), but as we know, gains can be very large and unwieldy, so Bell made a way to compress these values to be more manageable.

Gain in Bel units: log (output / input) Bels : initial Bel unit , and this after an accepted, standardized revision by others, this becomes a tenth of a Bel , hence a deciBel. **1 Bel =10 deciBells** = (10) (0.1Bel) = 1Bel

The deciBel unit is then redefined as: log (output / input) deciBels = dB , and the Bel unit is then ten times this amount:

1 Bel = 10 (1 dB) = 10 log (output / input) dB : this is equivalent to: log (100)(output/input)) dB , the value of 100 effectively magnifies the gain before the log is applied, and this helps to express small gains less than 1.

The Bel unit is also then defined such that a real gain of 10 would be expressed as a gain of 1 Bel or= 10dB.
A real gain of 2, twice as much or double, is expressed as a gain of 0.3Bel = 3dB

For example: If the output power doubled (ratio=2): 10 Log 2 = 10 (0.30103) = 3.0103 dB =~ 3db in gain
If the output power increased by one-million: 10 log (1000000) = 10 (6) = 60 db in gain

For example: If the output halved, such as Pout / Pin = 0.5 , 10 Log 0.5 db =~ 10 (-0.301) = -3 db : : a loss
For example: If the output is a quarter, such as Pout / Pin = 0.25 ,10 Log 0.25 db =~ 10 (-0.602) = - 6 db : a higher loss

We see above that if the power doubles, the dB values increases by 3, and if the power is halved, the dB value decreases by -3. To find the actual gain and-or loss ratio (ie., out value / in value) given an amount of dB gain:

Gain = 10^(gain in dB / 10 dB) Ex: 10^(3 dB / 10 dB) = 10^0.301 = 2 : > 1 is a gain
Ex. 10^(-3 dB / 10 dB) = 10^-0.301 = 0.5 : < 1 is a loss

Other popular use for logarithms are: earthquake intensity measurement, measuring the relative brightness or "magnitude" of stars, and some types of electronic power measurement. Logarithms can also be used as an excellent mathematical tool in any field of study where calculations need to be made, and especially for solving for an exponent. The word "magnitude" essentially means greatness or the level of , and is related to the word magnification which means an increase, bigger or larger.

FACTORING THE NUMBER IN A LOG

The log of a number can be expressed as a sum of the logs of the factors of that given number. This can be remembered as the "(log) product rule" or "(log) factor rule". The log factors must also have the same base as the source logarithm.

log number = log factor_1 + log factor_2 + . . . : . . . = "and so on" since there may be more than two factors

Ex. $\log_{10} 100 = \log_{10} 10 + \log_{10} 10$: using the log factor or log product rule

checking: $2 = 1 + 1$

Also: log 100 = log (10 x 10) = log 10 + log 10 = 2 : shows 100 in factored form, or something like:

log 100 = log (50 x 2) = log (25 x 4) : showing some other factors of 100 for the Number

log 100 = log 50 + log 2 = log 25 + log 4 = 2

The product rule in a reverse manner can be remembered as the "(log) sum rule". The sum of two logs having the same base is equal to a log with the same base, and a number equal to the product of the numbers.

Ex. $\log_{2} 8 + \log_{2} 4 = \log_{2} (8 \times 4) = \log_{2} 32$: log sum rule

checking: $3 + 2 = 5$: since 2^3 = 8, 2^2 = 4 , and 2^5 = 32

THE LOG OF A FRACTION

The log of a fraction is mathematically equal to the log of the denominator subtracted from the log of the numerator. This can be remembered as the "log quotient rule", or "log of a fraction rule":

$\log\left(\frac{\text{numerator}}{\text{denominator}}\right) = \log \text{numerator} - \log \text{denominator}$: log quotient rule

Ex. $\log\left(\frac{1000}{10}\right) = \log_{10} 1000 - \log_{10} 10 = 3 - 1 = 2$

checking: $\log_{10} 100 = 3 - 1$: simplifying first by considering the order of operations step 3; perform multiplications and divisions

$2 = 2$

Ex. $\log 1 = \log \frac{5}{5}$: this is a particular and special example featuring the log of 1.

$\log 1 = \log 5 - \log 5$ combining like values or terms of log 5:
$\log 1 = 0$: Clearly, whatever the value of an expression such as the log 5 is, if you subtracted it from itself, the difference is always 0. The log (regardless of the base used) of 1 is always equal to 0. This is the reverse of the exponential rule that the power of any base with an exponent of 0 is always equal to 1.

$0 = \log_{\text{base}} 1$ and $\text{base}^0 = 1$: using any base for the log, and any base for the indicated power

A cautionary note about the log quotient rule: This rule can only be used if the number in question is a fraction. If the numerator and denominator are both logarithm expressions, the log quotient rule cannot be used. It could, perhaps, be used separately on the numerator and denominator.

Ex. The log rule cannot be used on this expression:

$\frac{\log 1000}{\log 10}$ /=/ $\log 1000 - \log 10 = 3 - 1 = 2$: (incorrect method)

This is also incorrect:

$\frac{\log \text{numerator}}{\log \text{denominator}}$ /=/ $\log(\log \text{numerator}) - \log(\log \text{denominator})$: (incorrect method) Since this value is actually the logarithm of the result, raising the base used to this indicated power, as for finding an anti-logarithm (or the Number), will yield the result.

This correct result is found if the order of operations is followed more closely where the numerator and denominator are simplified separately:

$\frac{\log 1000}{\log 10} = \frac{3}{1} = 3$: (correct method)

The reverse of the quotient rule can be remembered as the "log difference rule". The difference of two logs having the same base is equal to a log with the same base and whose number is the quotient of the minuend number divided by the subtrahend number.

Ex. $\log_3 27 - \log_3 9 = \log_3 \left(\frac{27}{9}\right) = \log_3 (3) = 1$: log difference rule

checking: $3 - 2 = \log_3 (3) = 1$

THE LOG OF A POWER

The log of a number raised to a (indicated, with an exponent) power is equal to the log of that number times the exponent of that number. This can be remembered as the "log exponent" rule" or "log of a power rule". The result is that the indicated (with an exponent) power of a number (N), of which the log is to be found, becomes a numerical coefficient (ie. a multiple, or multiplying factor) of the log of just the base value of that indicated power.

log Number^exponent = (log Number) (exponent) or = (exponent) (log Number) , expressed symbolically:

$\log N^e = e \log N$: LOG (INDICATED) POWER RULE, or the LOG EXPONENT RULE

Ex. $\log_2 4^3 = 3 \log_2 4$: log (indicated) power rule, or the log exponent rule

checking: $\log_2 64 = 3(2)$

$6 = 6$

Here is a verification of this rule:

$\log 5^3$ "extending" the indicated power:
$\log ((5)(5)(5))$ using the log product rule:
$\log 5 + \log 5 + \log 5$ Since multiplication is repeated addition, adding their numerical coefficients (here, 1, for each) we have:
$(1) \log 5 + (1) \log 5 + (1) \log 5$: like values, add their numerical coefficients:
$(1 + 1 + 1) \log 5 = 3 \log 5$: the left side is also a "factored" form of the previous line, and the common factor to all the values summed is (log 5)

There is a simple rule when dividing two log expressions when the base of the logs are the same, and both bases of the numbers (N), expressed as indicated (with an exponent) powers, are also identical. Below, (x) and (y) represent any possible value for the indicated exponents of both numbers that are expressed, indicated powers and which have the same (power) base (here indicated as A) of that indicated power:

$\frac{\log A^x}{\log A^y} = \frac{x}{y}$ this is easily verified using the log exponent rule:

$\frac{\log A^x}{\log A^y} = \frac{x \log A}{y \log A} = \frac{x}{y}$: a value divided by itself is 1, or after canceling out the common factors (here, log A) to both the numerator and denominator.

Even if the base of the indicated power of N is not identical, if you can convert and express N as a power of the other base, the above method will work. In the following example, the base of the expressed power of 2 is converted to its equivalent value having a base of 4:

Ex. $$\frac{\log 4^3}{\log 2^3} = \frac{\log 4^3}{\log (4^{0.5})^3} = \frac{\log 4^3}{\log 4^{1.5}} = \frac{3}{1.5} = 2$$

Note that $4^{0.5} = \sqrt{4} = 2$

checking:

$$\frac{\log 4^3}{\log 2^3} = \frac{\log 64}{\log 8} = \frac{1.806179974}{0.903089987} = 2$$

Another method to simply the original equations is with the exponential log rule:

$$\frac{\log 4^3}{\log 4^{1.5}} = \frac{3 \log 4}{1.5 \log 4} = \frac{3}{1.5} = 2$$

Actually, since the numerator and denominator are both composed of a logarithm expression only, and with no indicated logarithm base, any logarithm base can then be consistently utilized in the equation for the correct solution. If you happen to use a base of 8, this can be solved easily since both Numbers (here 64 and 8) are a power of 8. There are a few other instances shown in this book where any base can be used. Below is a verification of this "any log base" concept:

Given any value, say expressed or represented as N1 (for Number 1), the division of two logarithms of it which always have the same two and different logarithm bases, always results in the same (constant value) quotient, say F.

Ex. $$\frac{\log_2 64}{\log_4 64} = \frac{6}{3} = 2$$: N = 64, b1=2, b2=4

Ex. $$\frac{\log_2 16}{\log_4 16} = \frac{4}{2} = 2$$: N = 16, b1=2, b2=4
Though the numbers differ in both equations, notice that
the result is always 2 when the same two and different bases
bases are used consistently.

Expressing this into a formula:

$$\frac{\log_{b1} N1}{\log_{b2} N1} = F$$

F is a constant when the Number is changed and when log bases are used consistently (ie., the same used values). Mathematically or algebraically:

$$\log_{b1} N1 = (F) \log_{b2} N1$$

: We see that given N, the logarithms with different bases, will only vary by a constant factor of F (or 1/F) when one is divided by the other.

Given another value, say N2, and using the same bases, the quotient will still be F as shown in the last two examples above. This can easily be verified when N2 is some indicated power of N1 and using the log of an indicated power rule and canceling the common factors:

Let $N2 = N1^x$ since (x) can have any possible value, N2 can be any possible value, then:

$$\frac{\log_{b1} N2}{\log_{b2} N2} = \frac{\log_{b1} N1^x}{\log_{b2} N1^x} = \frac{x \log_{b1} N1}{x \log_{b2} N1} = \frac{\log_{b1} N1}{\log_{b2} N1} = F$$

We see that regardless of the Number used, given any two bases, there is a common or constant factor (ie. here a ratio value) between those bases. Mathematically, in the algebraic sense (for a formula) we have:

$$\log_{b1} N2 = (F) \log_{b2} N2$$ dividing the two results, we have another interesting result:

$$\frac{\log_{b1} N1}{\log_{b1} N2} = \frac{(F) \log_{b2} N1}{(F) \log_{b2} N2} = \frac{\log_{b2} N1}{\log_{b2} N2}$$: Same two numbers (N1 and N2) but using another log base for both, will be equal.

Ex. $\frac{\log 25}{\log 5} = \frac{1.397940009}{0.698970004} = 2$: common base of 10, and:

$\frac{\log_e 25}{\log_e 5} = \frac{3.218875825}{1.609437912} = 2$: base of (e), the "natural log base" (about 2.71)

This is further verification that different bases (when used consistently within a problem) can still be utilized to achieve the correct result. Sometimes a log with a base of 10 is converted into one having a log base of (e), and vice-versa. Here, the value of the Number is adjusted so as the actual proper or true result is maintained.

THE LOG OF A POWER OF A PRODUCT

The logarithm of an indicated (with an exponent) power of an expressed product of two or more factors can be expressed as equal to the sum of logarithms where each number (factor of the original expressed Number or product) of the log is raised to the same indicated power as the expressed product of those factors. This rule is similar to the log product rule.

log Number^exponent = log ((factor_1) (factor_2) **. . .**)^exponent = log factor_1 + log factor_2 + **. . .**

Ex. log (1000 x 10)^2 = log (1000^2 x 10^2) = log 1000^2 + log 10^2
log(1,000,000) + log (100)
6 + 2
8

checking: log (1000 x 10)^2
log (10,000)^2
log 100,000,000
8

OR: log (1000 x 10)^2
log (10,000)^2
2 log 10,000
2 (4)
8

OR= 2 log (1000 x 10)
2 (log 1000 + log 10)
2 (3 + 1)
2 (4)
8 : checks

If you were to write a "formula" for the above procedure it would look something like the expressions on both sides of the equality ("equals") sign below where A and B represent the factors of the Number, and x represents the exponent or indicated power.

log (AB)^x = log(A^x B^x) = log A^x + log B^x

Using the log exponential rule on both of the equivalent expressions (an equation) above, we can arrive at other useful "log formulas". We see that the exponent is "distributed" or applied to the exponent of each factor of the number (N).

log (A^x B^x) = log (AB)^x = x log (AB) = log A^x + log B^x = x log A + x log B = x (log A + log B)

Ex. 5 log a + 7 log b considering the log exponent rule, this can be expressed as:
log a^5 + log b^7 considering the log factor and log product rule, this can be expressed as:
log (a^5 b^7)

"SPECIAL" LOG RULE

When the base of the logarithm, and the base of the number are the same, the result or logarithm (ie., the exponent) is equal to the number's exponent.

Ex. $\log_{10} 10 = \log_{10} 10^1 = 1$: special log rule

checking: $10^1 = 10$

Ex. $\log_{10} 10^3 = 3$

checking: $\log_{10} 1000 = 3$

$3 = 3$

A check can also be made using the (log) exponential rule:

$\log_{10} 10^3 = 3 \log_{10} 10 = 3(1) = 3$

If we were to express this concept symbolically or algebraically, so as to be non-specific in values or showing a more "generalized" formula for any possible values used, we would have something like:

$\log_N N^{\text{exponent}} = \text{exponent}$: here, the log base, and the base of the number are the same value of N

Knowing all of the above, it is easy to reason that when the Number is less than the base, that the exponent or log is less than one. When the Number is more than the base, the exponent or log is greater than one:

$\log_b N = 1$: when N = b , "When the Number (or power of the base) equals the base of the logarithm, the log equals 1".

Ex. $\log_{10} 10 = 1$

$\log_b N > 1$: when N > b , "when N is greater than b", the log is greater than 1

Ex. $\log_{10} 100 = 2$

$\log_b N < 1$: when N < b , "when N is less than b, the log is less than 1".

Ex. $\log_{10} 5 = 0.69897$

$\log_b 1 = 0$: when N = 1

You can now imagine that when the number is less than 1, (N<1), but still greater than zero, (N>0), since 0 is not a power value of any base value, that the log will be less than 0. This requires knowing about "negative or signed numbers" that are discussed further ahead in this book.

Ex. $\log 0.99 = -0.0043648\underline{1}$: When (0<N<1), the log is negative or less than 0. Here, since N is almost or close to 1, the log is close in value to 0.

When the Number is greater than 0, and less than 1, (ie. between 0 and 1), this can be symbolically expressed as :

$\log_b (0<N<1) = (<0)$: a value less than 0 is called a negative value, and is indicated with a negative sign, (-) preceding it, as shown in the example above. This will account for the logarithms of all the (fractional) values between 0 and 1. Negative values are used quite a bit when the temperature is below zero degrees = less than 0 degrees = < 0 degrees = < 0°. Negative value < 0. Ex. -1 < 0

The fundamental log rules described up to now will be verified in the ALGEBRA section of this book.

FINDING EXPONENTS WITH LOGARITHMS

Expressing and finding exponents with logarithms is perhaps the greatest use of logarithms.

Ex. Given 10^x = 1000, find the exponent x.

First, take the logarithm (using any consistent base) of each side of the equation:

$10^x = 1000$

$\log 10^x = \log 1000$ using the "log exponent" rule:

$x \log 10 = \log 1000$ now isolate (solve for) x:

Since log 10 is a factor of the left hand side (of the equation), remove it by dividing it by itself, effectively canceling or ridding it from that side and thereby isolating x so as it can be solved for. Of course, to keep both sides of the equation in equivalence or balance, this process must be done to both sides.

$$\frac{x\,\overset{1}{\cancel{\log 10}}}{\underset{1}{\cancel{\log 10}}} = \frac{\log 1000}{\log 10} = \frac{3}{1} = 3$$

: Note, you cannot use: log 1000 - log 10 = 3 - 1 = 2 , since both the numerator and denominator are logarithms.

Extra note: **The log of a negative numbers (less than 0) are undefined.**

If you were to write a generalized representative or algebraic formula for this procedure, it would look something like the following:

given $b^x = N$: Here (b) represents the base of the indicated power, (x) is the indicated exponent of the base, and N represents the number or power value of that base.

$\log b^x = \log N$: after taking logs of both sides

$x \log b = \log N$: using exponential rule, solving for (x):

$$x = \frac{\log N}{\log b}$$

: GENERAL FORMULA FOR SOLVING FOR AN EXPONENT (given: $N = b^x$)

Ex. Solve for x given:

$$x = \log_3 9$$

$$x = \frac{\log N}{\log b} = \frac{\log 9}{\log 3} = 2$$

: Use any base, but the result of 2 can be found by inspection when you consider that both numbers have a factor of 3 in them, and then using a common or consistent base of 3. Note also that:
$\log 9 = \log (3x3) = \log 3 + \log 3 = \log 3^2 = 2 \log 3$

FINDING POWERS WITH LOGARITHMS

Both of the processes for finding powers and roots with the aid of logs essentially results in the need for evaluating an anti-logarithm which is also known as an inverse logarithm. For anti-logarithms, you are to find the number (ie., power value) given the base and the exponent (ie. the logarithm). You can "think of" an anti-logarithm as the reverse of finding the logarithm of a number.

Ex. Evaluate: N = 10^3

Note that this example given can easily be solved by repeated multiplication, but this is not the case when the exponent is not an integer. This simple example was chosen to make the procedure clear.

N = 10^3 taking logs of each side, here, using a base of 10; the common log base:
log N = log 10^3 this can be expressed as:
log N = 3 log 10

Equating (3 log 10) equal to the required exponent of the base (here 10) to equal N (a power of that base):

N = 10^(3 log 10) = 10^(3x1) = 10^3 = 1000

In general, if you were to write a (algebraic) formula for this procedure, it would look something like:

N = b^x : N = power, b = base, x = exponent , taking the log of both sides of this equation:
log N = log b^x or:
log N = x log b therefore, according to the definition of logarithms:

N = 10^(log N) and:

N = 10^(x log b) or= 10^(log b^x) : A GENERAL FORMULA FOR FINDING POWERS
(where N = b^x)

N = 10^(log N) shown above is the general notation for finding antilogarithms (the number) of common (base 10) logarithms. Another, simpler derivation of this notation is from the log definition itself:

x = $\log_{10}$ N : an expression to find the logarithm of a number using a base of 10, therefore:

N = 10^x since x = $\log_{10}$ N, and using substitution:

N = 10^($\log_{10}$ N) : = 10^x, and if any other log base (b) is considered, this formula would be expressed as:
N = b^($\log_{b}$ N) = b^x

Below are some more examples of evaluating powers with logarithms. Included in the examples is more terminology associated with the concepts of logarithms.

Evaluate 10^4 using logs.

N = 10^4 taking the log of both sides, we can use any log base, here we will use the common log base of 10:
log N = log 10^4
log N = 4 log 10
log N = 4 (1)
log N = 4.0

The 4, or the whole portion of the log of (10^4) is technically called the **characteristic** of the logarithm. The fractional portion of the logarithm (here it's 0) is technically called the **mantissa**. Both the characteristic and mantissa are part of the logarithm or exponent. Expressing this in a pseudo (or algebraic) mathematical statement:

log N = exponent = characteristic + mantissa or: characteristic part of the log + mantissa part of the log
= characteristic**.**mantissa : expressed symbolically as decimal mixed number

Therefore, using the inverse log, or anti-log, expression to find the number::

N = 10^logarithm = 10^exponent : since logarithm = $\log_b N$, b^logarithm = base^exponent = N

N = 10^(characteristic + mantissa) OR:

N = (10^characteristic)(10^mantissa) therefore:

N = (antilog characteristic)(antilog mantissa)

Substituting values for the example just given:

N = (antilog mantissa)(antilog characteristic)
N = (antilog 0)(antilog 4)
N = (10^0) (10^4) or = 10^(0+4) = 10^4
N = (1) (10,000)
N = 10,000

Here is a small table to assist in understanding the characteristic and mantissa values (here, using a base of 10):

10^0.0 = 1	: characteristic = 0	mantissa = 0.0 :	log 1	= 0.0
10^0.5 = 3.16228	: characteristic = 0	mantissa = 0.5 :	log 3.16228	= 0.5
10^1.0 = 10	: characteristic = 1	mantissa = 0.0 :	log 10	= 1.0
10^1.5 = 31.62228	: characteristic = 1	mantissa = 0.5 :	log 31.6228	= 1.5
10^2.0 = 100	: characteristic = 2	mantissa = 0.0 :	log 100	= 2.0

All the values from 10 up to 99.999... are said to have a characteristic of 1 when base 10 is being considered. That is, these values are essentially powers of 10 and the characteristic is the integer (whole number) of the indicated power. For all the values from 10 up to 100, where the characteristic of these values is always 1. A log value having an integer value of 1 is characteristic or common of Numbers between, or in, the range from 10 to 100, and the mantissa value of their logarithms will increase from 0.0 to 0.999999... in this range. In short, the characteristic value of the logarithms remains the same (here 1) for this range of values, but the mantissa value of that logarithm increases from 0 to 1 as like a fraction of a unit does. The (common, base 10) logarithms of values in the range from 100 to 1000 have a characteristic of 2, and in the same manner, the mantissa will increase from 0.0 to 0.999999... in this range.

When you see a logarithm (ie. exponent) that has a characteristic of 1, you will know that the anti-logarithm or Number (the power value of the base raised to that indicated power) has a value between 10^1 to 10^2 = 10 and 100. When you see a logarithm that has a characteristic of 2, you will know that the anti-logarithm or Number has a value between 10^2 to 10^3 =100 to 1000, and so on.

Ex. Find the common (base 10) antilog of 2.17

Here, 2.17 is a logarithm of some number, hence it's an exponent value. We see that the characteristic of this logarithm (or exponent) is 2 and the mantissa is 0.17. Since 2 is the characteristic, the corresponding Number is between 10^2 to 10^3 = 100 to 1000.

According to the definition of logs:

2.17 = Log N   and therefore:

N = 10^2.17   : N is the number or antilog. Factoring this indicated power value:

N = (10^2)(10^0.17)
N = (100) (1.479108388)
N = 147.9108388

When solving for the log of a number, the characteristic portion of the log can easily be found by observation or by expressing the number in standard scientific notation (discussed further ahead in this book under the topic of SCIENTIFIC NOTATION which is a form of exponential notation with a base of 10) where the characteristic is equivalent to the whole part of the indicated (as in the exponent) power of 10, that is, it's equivalent to the log of this power of 10, and the mantissa is equivalent to the log of the other factor to the power of 10. Since this other factor (to the power of 10) is less than 10 in value, and the log of (10^1) is 1, its log, and therefore the mantissa, is always less than 1. Another similar method is to express the number using a slightly modified scientific notation where the number is not between 0 and 10, but always less than 1. This second method eventually requires that you know about adding signed numbers as shown below.

Ex. Log of 50 can be found by:

Log 50 = Log (5)(10^1)   or = Log (0.5)(10^2)
Log 50 = Log 5 + Log 10^1   or = Log 0.5 + Log 10^2
Log 50 = 0.698970004 + 1   or = - 0.301029995 + 2
Log 50 = 1.698970004   = 1.698970004

Again, whenever you see (log N), think of it as being an exponent value:
log N = logarithm = exponent = characteristic part + mantissa part = whole part + fractional part
For this example, on its left side, 1 is the characteristic, and 0.698970004 is the mantissa.

Notice that when a number is factored to any form of scientific notation that the digits remain the same, and only the power of 10 changes. If numbers of various values with the same digits in them are all factored to standard scientific notation, then all of them will have the same mantissa portion in their common logarithm. In respect to the above example:

Log 500 = Log (5)(10^2)
Log 500 = Log 5 + Log 10^2
Log 500 = 0.698970004 + 2
Log 500 = 2.698970004

For comparison, if the digits are the same, the mantissas of the logs are identical:

(antilog)	(integer power of 10)	(fractional power of 10)
N	characteristic	mantissa
5	0	0.698970004
50	1	0.698970004
500	2	0.698970004
5000	3	0.698970004

For illustrative purposes, this same sort of "similar digit results" will occur when multiplying (or dividing) any two numbers with the same digits, for example:

$$\begin{array}{r} 123 \\ \times\ \underline{\ \ 57} \\ 7011 \end{array} \qquad \begin{array}{r} 1.23 \\ \times\ \underline{\ \ 5.7} \\ 7.011 \end{array}$$

: both products have the same digits

In this book, you can find more about this subject in the topic of LOGARITHMS OF SIMILAR DIGITS

Ex. Evaluate 2^1.5 using common logarithms.

This example will show that the indicated power value is converted to an equivalent power value, which has a base of 10 and a correctly adjusted exponent, that can be evaluated using common logarithm and antilogarithm tables, etc. For much of this book, the concepts of the characteristic and mantissa of a logarithm are not used much, but you should have an idea of what they are.

N = 2^1.53

log N = log 2^1.53 : after taking the common (base 10) log of both sides
log N = 1.53 log 2 : log power or exponent rule
log N = (1.53)(0.301029995)
log N = 0.460575893 : characteristic = 0, and mantissa = 0.460575893

Since the characteristic is 0, the antilogarithm or the value of the number must be between 1 and 10 since 10^0 = 1 and 10^1 = 10. That is, 0 is a common "characteristic" for the exponents of the values being represented (with a power of a base of 10) between this range. It can also be stated as that the integer portion of the logarithm in question has this characteristic value. If the characteristic was 1, the number would be within the range of 10 and 100, but not including 100, or higher, since 10^2 = 100, and the characteristic would actually be 2.

N = (antilog mantissa)(antilog characteristic) using common antilogs or powers of 10:
N = (10^0.460575$\underline{9}$)(10^0) multiplying like values or variables, add their exponents:
N = 10^0.460575$\underline{9}$ (1) The underlined 9 means rounded to this position.
N = 10^0.460575$\underline{9}$

Usually, to save space and cost, common antilogarithm (base 10) tables range from 0 to 1 for the exponent, hence the antilog of the characteristic must be performed by inspection. Since the antilog of the characteristic is an integer power of 10, simply adjust (ie., move) the decimal point in the antilog of the mantissa in accordance with the indicated exponent in the antilog of the characteristic. It should also be pointed out that log tables are for, or thought as for, the log of numbers whose values are within the range of 0 to 1, or 0 to 10 (when the number is placed in standard scientific notation). It is also possible to use log tables in a sort of reverse type of process to evaluate antilogs (finding the Number value).

After evaluating 10^0.4605759 :

N = 2^1.53 = 2.887858435 : correct to 6 decimal places due to the previous rounding above

Ex. Evaluate 250^2.5

N = 250^2.5
log N = log 250^2.5
log N = 2.5 log 250

Let us try to express or place the Number into a value that's within a log table that goes from log 0 to log 1:

log N = 2.5 log (1000 x 0.250)
log N = 2.5 (log 1000 + log 0.250)
log N = 2.5 (3 + log 0.250) After solving (or from a table) the log of 0.250:
log N = 2.5 (3 + (- 0.602059991))

Note, logarithms of values between 0 and 1 are always negative in sign. Negative in sign simply means that the value is less than zero. More will be said about negative numbers later. Since log 1 = +0, it is easy to reason that the log of a number less than 1, and between 0 and 1, is less than 0 and is indicated as being negative. Continuing the above example:

log N = 2.5 (2.397940009) : after combining (formally, it means joining, addition, summing) values in the second factor, and the result here was essentially due to subtraction.
log N = 5.994850022 : N = antilog (log N) = antilog 5.994850022 = 10^5.994850022

We see that the characteristic is 5 and the mantissa is 0.994850022. Using the basic concepts of logarithms, expressing the antilogarithm of both sides to solve for the number (N):

N = 10^5.994850022
N = 10^(5 + 0.994850022) = (10^5)(10^0.994850022)

Using a table to find the common antilog of 0.994850022, the result is: 9.882117688. This value is to be multiplied by a power of 10 (here, indicated as the 5th power), and the easiest way to do that is to simply move the decimal point a number of places equal to the indicated exponent. Here, the decimal point will be moved 5 places rightward, and this is the same as multiplying by 10^5 = 100,000.

N = (10^5)(9.882117688)
N = (9.882117688)(100,000)
N = 988,211.7688

Ex. Evaluate 1.74^52

To find the value of this indicated power would be very tedious if performed by hand using repeated multiplication.

N = 10^(x log b) : basic formula, previously derived, for finding powers given the base and exponent.
N = 10^(52 log 1.74) factoring 1.74 to have a number less than 1 for solving with a log table:
N = 10^(52 (log (10)(0.174)))
N = 10^(52 (log 10 + log 0.174))
N = 10^((52)(1 + (- 0.759450751)) = 10^((52)(1 - 0.759450751))
N = 10^((52)(0.240549248))
N = 10^ (12.50856091)
N = (10^0.50856091)(10^12) using a scientific calculator or common anti-logarithm table:

$N = (3.225231628)(10^{12})$: this form itself would be typical of an acceptable answer
$N \sim 3{,}225{,}231{,}628{,}000$: approximate, since the last three least significant digits were dropped or omitted and unused during the calculation using a 10 digit calculator, or by visual inspection moving the decimal point.

Note, as mentioned previously, that any base of a logarithm can be used to evaluate a power as long as it is consistent in the calculations. Below is an example of the concept using a base of (e) instead of 10. (e) is well defined and roughly equal to 2.718282, and is used as the base for "natural" logarithms which are often used to mathematically represent things that happen in nature and expressed scientifically or mathematically. Evaluating powers of 10 usually requires a table or calculator to solve, however, all powers of (e) can be solved directly from a formula which will be presented further ahead in this book.

From: $N = b^x$ taking logs (with any base) of both sides:

$\log N = \log b^x$ and if the log taken had a base of (e) instead of 10, this must be explicitly expressed as:

$\ln_e N = \ln_e b^x$ ln indicates a natural logarithm, and is usually substituted for the word log when the base is understood as being (e), the natural logarithm base. This can be also be simply expressed as:

$\ln N = \ln b^x$ Solving for N considering the concept of antilogs:

$N = e^{(\ln b^x)}$ or from the "log of a power rule":

$N = e^{(x \ln b)}$: Finding a power of a value using natural logarithms (given $N = b^x$).

The appendix contains a general method for finding any power.

FINDING ROOTS WITH LOGARITHMS

The process of finding roots with the aid of logarithms is similar to that of evaluating powers with logarithms.

Ex. Evaluate $N = \sqrt[3]{1000}$: here, N is the root value. You can use another variable identifier for N, like R.

First, let's convert this radical form to its equivalent exponential form.

$N = \sqrt[3]{1000}$
N = 1000^(1/3) taking logs of both sides:

log N = log 1000^(1/3) or:

$$\log N = \frac{1}{3} \log 1000 = \frac{\log 1000}{3}$$

Finding N by taking the antilogarithm:

N = 10^((log 1000) / 3) = 10^(3/3) = 10^1 = 10

If you were to write an algebraic formula for the above procedure, it would look something like this:

$N = \sqrt[x]{b}$: N = root, x = index, b = radicand
N = b^(1/x) : exponential form
log N = log b^(1/x) : taking logs of both sides

or: $\log N = \frac{1}{x} \log b = \frac{\log b}{x}$

$N = \sqrt[x]{b}$ = 10^log N = 10^((log b)/x) : A GENERAL FORMULA FOR FINDING ROOTS using logarithms and antilogarithms

An important observation can be made here. The only difference between the (antilogarithm) notation for solving powers and roots is the placement of the variable(x). For powers, (x), the exponent of the indicated power, becomes a multiplier to the exponent of 10 (or any proper base used). For roots, (x), the root-index of the radical, becomes a divisor to the exponent of 10 (or any proper base used).

For powers: N = 10^(x log b)

Ex. 5^2 = 10^(x log b) = 10^(2 log 5) = 10^((2)(0.698970004)) = 10^1.397940009 = 25

For roots, (x) (the index or the indicated root) becomes a divisor to the exponent of 10 (or the proper base used).

Roots: N = 10^ ((log b)/x)

Ex. $\sqrt{25}$ = 10^(log b)/x = 10^((log 25)/2) = 10^(1.397940009/2) = 10^0.698970004 = 5

As like evaluating powers with any base, roots can be evaluated using any base as long as the base is used consistently.

Ex.. **$N = \sqrt[x]{b}$ = e^((ln b)/x) : Formula for finding a root using a power base of (e), and the natural logarithm base of (e). b = radicand, x = index**

Ex. Find the 10th root of 10 , using logarithms for the solution.

$\sqrt[10]{10}$ = 10^(1/10) = 10^0.1 = **e ^(ln 10 / 10)** = e^(2.302585093 / 10) = e^(0.2302585093 = 1.258925412
=~ 1.259 =~ 1.256 =~ 1.25

A proposal by the author of this book, and for logarithm notation in terms of a simple text editor abilities would be:

x = $\text{Log}_b\ N$ = log (b , N) : this is similar to computer programming, function notation with the required parameters expressed. The actual values used are called the specific arguments.

or:

x = $\log_b N$ = log (b , N) or= (log b , N)

or

x = $\log_b N$ = log (b : N) or= (log b **:** N)

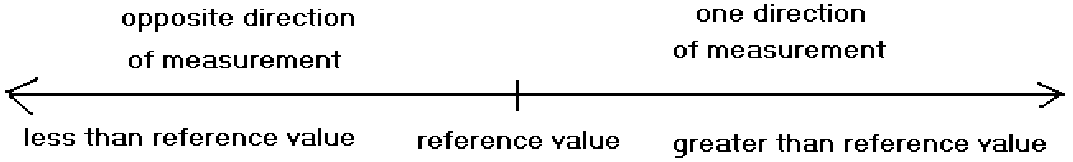
SIGNED NUMBERS

Signed numbers are numbers preceded by a positive (+) or negative (-) mathematical symbol. Numbers preceded by the negative symbol (-) are said to be negative numbers and have a value less than 0 or some other reference value.. Numbers preceded by the positive symbol (+) are said to be positive numbers and have a value greater than or equal to 0 or some other reference value. If a sign does not precede a number, or the sign is not explicitly indicated, the number is to be considered as a positive number. For example:

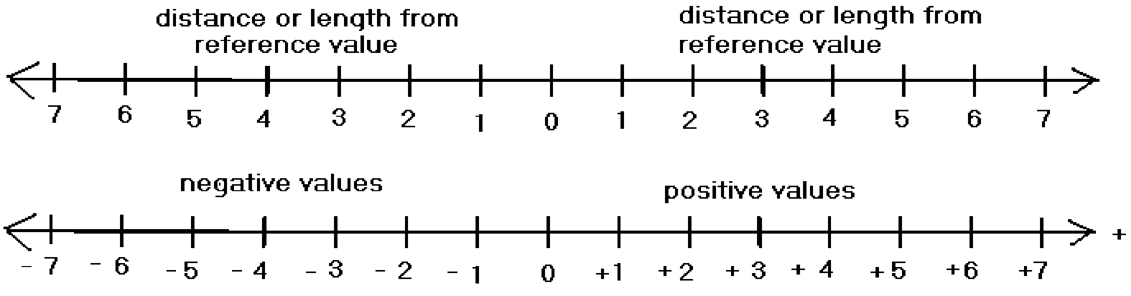
5 = +5 = "positive 5" and -5 = "negative 5"

How can a value be less than 0? A value can be less than 0 when that value is, or is considered as, a negative change, decrease, loss, or reduction from another given value which is considered as the reference value that change value is in reference to. That is, the change is in reference to some other value. So that you can get a good feel or understanding of signed numbers, their are several discussions below.

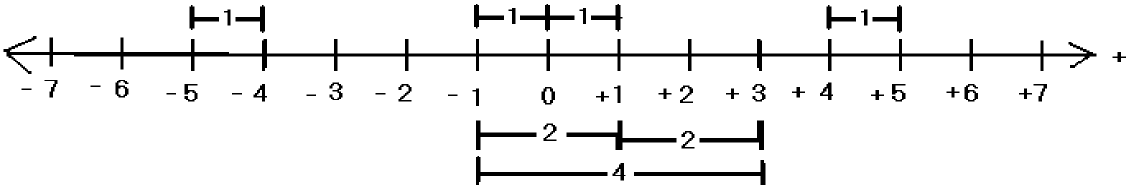
A very common use of both positive and negative numbers is with temperatures. It is common in cold climate areas to have a temperature that is "less than 0 degrees" with either the Fahrenheit or Celsius temperature scales. For example, the temperature may be "negative (or minus) ten degrees", and is indicated as -10°. Since no other reference value was given, this -10 value is in reference to 0 degrees which is the reference value. A negative sign indicates that a value is less than the reference value being considered. Here is a helpful "number-line" that is a graphical or visual reference to consider about signed numbers: [FIG 18]



The reference value can be any value, either positive or negative in sign. Usually when a reference or "starting" value is not given of which a number is in reference to, that number is understood to be in reference to a value of 0.



The length or distance between any two values is calculated as the difference between those two values. For example: The difference between +3 and +1 is: (3 - 1) = 2. The difference between 1 and 0 is: (1 - 0) = 1. The difference between 0 and (-1) is: (0 - (-1) = 1. The difference between +1 and -1 is: (1 - (-1)) = 1 + 1 = 2. The difference between +3 and -1 is: (+3 - (-1)) = +3 + 1 = 4



As you can see in the above drawing, the sign of a value also indicates its physical direction and-or location, and in a mathematical sense, this direction and-or location sign is either: "less than" (ie., - = negative in sign), or "greater than" (ie., + = positive in sign) a reference value.

If you had a starting or reference value of 5 things, and got 2 more things, this value of 2 can be expressed as :
The change of, in, or applied to the value of 5 was an increase of 2 = positive 2 = +2

If you had a starting or reference value of 5 things, and gave away 2 things, this value of 2 can be expressed as :
The change of, in, or applied to the value of 5 was a decrease of 2 = negative 2 = -2

As we can observe above, signed numbers, in particular when they have a mathematical sign (+, -) symbol, give or assign a "direction" or further meaning to that number. This meaning could be that the number represents an increase or gain from an existing or reference value (including 0), and therefore it would be noted with a positive (+) sign or symbol. If the number represents a loss or decrease from any existing or reference value (including 0, or even a negative value), it would be noted with a negative (-) sign. In short, signed numbers are in reference to some existing or reference value, and not necessarily in reference to 0 only.

Since signed numbers represent gains or losses with respect to an existing reference value (0 if none was specified, or if the signed value is a final result of a computation), signed numbers can simply be considered as a "change-value" or simply as the change (as a verb: to modify, or have an affect, and as a noun: the amount of the modification or affect, the numeric change value or value of the change) from the reference value. If a value changed to 5 less, this change would be noted as -5 numerically. Of course, to find 5 less of a value, or a value reduced by 5, this would normally be expressed and processed as a subtraction operation: value - 5 Below, the right hand side of the equations represents expressions for the signed number concepts:

value - 5 = reference_value - 5 : with signed numbers, value becomes a reference value for the change or change-value to be applied to, and have an affect on it

= reference_value with the specific change applied to, added to, or combined (+) with it
= reference _value + change_value

value - 5 = (reference_value) + (-5) : the sign of the change value is still attached to and associated with it

= new_value : the resulting or result value due to a change

Considering this example, note that when signed numbers are considered, expressions such as:

reference_value - 5 , are to be now considered or understood as actually indicating or meaning:

(reference_value) + (-5) = reference_value (+) -5 : reference_value combined (+) with a decrease value

This is so, since when a value changes to a lower value, that value is said to have decreased, or the change in that value is said to be a decrease, and the specific amount of this change or the change value is mathematically said, expressed or represented as negative 5 = -5 , rather than positive 5 = +5 such as if the change was an increase. You can consider -5 as expressing a change or decrease of 5.

What looks like a subtraction expression, is now to be considered or understood to the be combining (as noted with the addition or combine symbol of: +) of signed numbers or values rather than a subtraction expression. The values being combined are called "signed (numeric) values" or "terms". Note that in the original expression above, and in relation or reference to signed numbers, that the combine symbol (+) can be considered as "hidden" or "understood", just as when the positive symbol (+) is considered "understood" when not explicitly indicated for a value greater than 0.

What if you added or combined two decreases or losses? The answer is that the net or total decreases or losses actually gets larger or greater in value when they are combined (essentially signed number addition, which is the addition of signed numbers). The decrease or loss would be said to be greater, a greater loss, hence it would be expressed or represented as a larger negative value. In a notation-like form:

decrease + decrease = larger decrease : the sum of decreases, so as to find the net or total decrease
(-5) + (-5) : add the losses together and keep the (common) sign that they are decreases or losses:
-5 - 5 : here, shown without the "hidden" addition or combine symbol

-(5 + 5) = -(10) = -1(10) = -10 : - (5 + 5) = -1(+5 + 5) = (-1)(5) + (-1)(5) = -5 - 5 = -10
This can also be thought of as:

- (total sum of [signless] decrease values) :And it is also a factored form where (-1) was factored from each term of:
(-5 + -5) = (-1)(5 + 5) = -1(5 + 5) = - (5 + 5) = -(10) = -10

Any positive number has an equivalent negative number counterpart (an (opposite) correspondence or resemblance), and vice-versa. Though the signs are different, they both have the same magnitude (essentially the "plain (signless)" distance or length), value, position or distance from 0. Below is a graphical representation of signed numbers using a "signed number line" graph. From the reference value, here 0, numbers in one direction are assigned (signed) as being positive, and for the opposite direction, they are assigned negative signs and are therefore called negative values, and which could simple be stated or "thought of" as numbers, lengths, distances, changes, etc., in the opposite direction:

```
  <--------|--------|--------|--------|--------|--------|--------|--------|--------|--------|--------|---------->
- ∞ . . . -5      -4      -3      -2      -1       0      +1      +2      +3      +4      +5 . . . +∞
```

∞ is the "infinity" symbol which indicates that there is no limit to the possible values, hence "infinity" has no specific, constant or actual value, and is more of a concept for an unending process. Consider something that is "infinitely big" or "infinitely small". Outer-space is considered as infinitely big or unending.

A change can be either an increase or a decrease. The opposite of an increase is a decrease. The opposite of a decrease is an increase. Increases and decreases are counterparts (or the "negative") of each other. If the change is an increase, the value of that change is noted or expressed with a positive sign: +value. If the change is a decrease, the value of that change is noted or expressed with a negative sign: -value.

You might now ask, how do you mathematically express the opposite of a positive value, and that is to take the negative, or opposite of it:

The opposite or negative of a positive value is: - (positive value) = negative value = -value

Ex. - (increase value) = - (+5) = -5. , a decrease value : this is the same as multiplying the initial value by negative 1 = (-1) = -1, as will be shown later. Consider that if you multiply a value by 1 that the result is still that same (unchanged) value: (value)(+1) = (value) ,and if you multiply that value by (-1), the result is: (-value), or the opposite, whether value itself was actually either positive or negative in sign as a signed value.

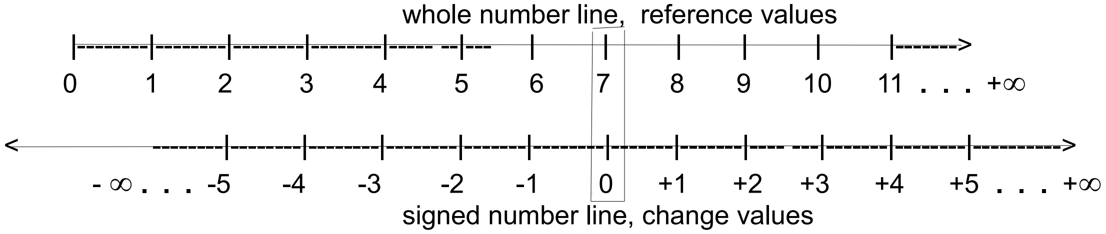
Likewise to mathematically express the opposite of a negative value, you also take the negative or opposite of it, and the result is a positive value:

The opposite or the negative, of a negative value is: - (negative value) = positive value = +value

Ex. - (decrease value) = -(-5) = +5 , an increase value : again, this is also the same as multiplying the value by negative 1 = (-1) = -1, as will be shown later

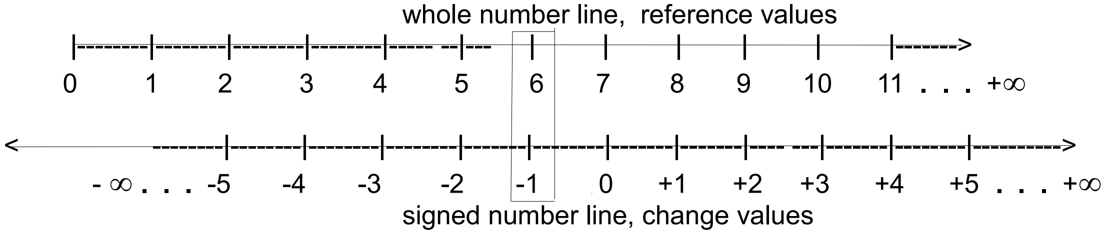
Choosing any signed number value such as graphically expressed on the signed number line, even if it's a negative number, all values rightward of any (reference) value are said to be "more positive" or greater (larger) in value. Likewise, all values leftward, of any value on the line, are said to be "more negative" or less in value (since those values are headed closer to, or towards that direction on the signed number line).

Here is a signed number, or "change value", line positioned or aligned to be in reference to 7 on the common whole number line: [FIG 19]



On the signed number line, or change number line, 0 means no change (yet) applied to the reference value chosen, and here, it's set or indicated as 7. To the left means a decrease or loss. To the right means an increase or gain.

Given a reference value of 7, if the change to 7 is a loss or decrease of 1, move to the left of 7 by 1 position for the result. This process can be indicated and automatically calculated mechanically, by first aligning the 0 change value with the reference value of 7. then move or slide the (amount of) change indicator ("pointer" or "cursor") to the (-1) position. The result is indicated directly across from it on the other ruler or number line, and it is 6.
7 combined with a change of (-1) is 6. 7 + (-1) = 7 -1 = 6 : [FIG 20]



The above concept is also an example of a simple automatic mechanical calculator than can perform addition or subtraction and display the result visually. This construction is a type of "slide rule" system that you can construct with two rulers aligned to be in parallel. To the right of 0 on the signed number or change in values line, you could place a word such as "INCREASE", and to the left of 0, you could place a word such as "DECREASE". To uses this calculator, move or position the zero (or "no change") mark of the signed or changed number line directly across from the reference number in question. Though it may not be apparent, the difference between any two corresponding values on these two number lines as set and shown above, has a value equal to the value in reference (the reference value), and for this example, it is 7:

reference value + change = sum : the change can be either positive or negative.
7 + (-1) = 6 mathematically, if change is subtracted from both sides:

sum - change = reference value : again, the change can be either positive or negative

6 - (-1) = 7 : this should be interpreted as: (+6) + (-(-1)) = (+6) + (+1) = 7

We see that when combining a positive value and a negative value, it results in the difference (or mathematical amount of separation, such as on the number line) between those two values. Considering 7 as the reference value, the difference between 7 and 6 is (7 - 6) = +1 , hence it can be said that 7 is 1more than 6, and-or 6 is 1 less than the reference value of 7. Considering 6 as the reference value, the (signed) difference between it (6) and 7 is (6 - 7) = -1, hence it can be said that 6 is 1 less than 7. The negative value indicates the result is less than the reference value, and this would be so, even if that reference value was also negative in sign.

The common whole number line can also be extended (leftward) to include the negative whole numbers leftward of 0, and so as to make a more functional (mechanical) calculator that can also work with negative (reference, or starting) values.

Where are the sign symbols from? They are a direct result from the addition, adding, or combination (combine, combining, "and", "mixed with" or "joined to or with") symbol of: (+), and the subtraction (remove, take-away, reduce, lessen) symbol (-). However, with signed numbers where the subtraction symbol appears to be utilized, it is rather a negative symbol (-) associated with a value, and is not a mathematical operation symbol.

Ex. If you had zero and added one to it, the expression would be:

0 + 1 = 1 :adding a number to 0, is essentially not adding anything to that number: 1 + 0 = 1, and the result is that same number

Therefore, if you had 1 and subtracted 1 from it, you would have:

1 - 1 = 0 : First, subtracting is applying a decrease or loss. This decrease can be noted mathematically as a negative change, and the actual value of this change will be assigned a negative sign to indicate or express this. Here. the decrease or change is 1, and it is expressed as a signed number as: (-1) or = -1. This change will be applied to the value of 1. 1 combined, added to, or applied with the change = 1 + change = 1 + (-1). Hence: 1 - 1 is to be understood as = 1 + (-1).

1 - 1 = 0, this the same as (+1) - (+1) = (+1) combined with, or applied to (-1) = +1 (+) -1 = 0
Just the same, since the order of addition or combining does not matter, and the sum remains the same, this can be considered a: (-1) combined or applied to or with (+1):
+1 (+) -1 = -1 (+) +1 = 0

Hence when you see: 1 - 1 this can be considered as equal to the sum of two signed values of:
(+1) and (-1) = (+1) + (-1) = +1 -1 = 1 - 1 = (-1) + (+1) = -1 +1 = 0

If you then take this 0 and subtract 1, as in "1 less than 0 (as the reference value)", you would have:

0 - 1 = -1 As a check to this, consider adding 1 to each side, and the equation will still be in balance. Here is where the subtracting operator, operation or symbol becomes the negative symbol or sign, and continuing this concept of notation: 0 + 1 = +1, we have the positive sign.

$0 - 1 + 1 = -1 + 1$: each side can also be expressed as just : $+1 - 1$ or $= -1 +1$ since 0 adds nothing.
$0 = 0$: The combining of an increase and decrease, both having the same value, results as if no (net, total, or final) change has taken place:
(increase) + (decrease) = 0 = (+value) + (-value) = +value - value = 0
Ex. 7 was increased by 3, and then decreased by 3. This can be expressed as:
$(7 + 3) -3 = 10 - 3 = 7 = 7 + 3 - 3 = 7 + 0 = 7$
Mathematically: $7 + 3 - 3 = 7 - 3 + 3 = 4 + 3 = 7 = -3 + 3 + 7 = 7$

OR, it can be said that -1 = "negative one" added to nothing is still that same value of -1:
$(-1) + 0 = 0 + (-1) = -1 = 0\ (+)\ (-1) = 0\ (+)\ -1 = 0 -1$, since when 0 is added to any value, the result is that same value, or that the value is unchanged. So, this is where signs and terms (numbers preceded by a sign) started into use. Rather than say -1 means "subtract one", which is a mathematical operation, people now can even use negative-one (-1) to show that this is a result of a mathematical operation, rather than a mathematical operation that needs to be performed. Writing an expression for the addition of -1 and +1, both being 1 more or less from 0, their combined (+) sum should be 0:

$(+1) + (-1) = 0$: a value summed or combined to its opposite value is 0, or no value.
A value combined or added to 0 is that same value:

$(0 + 1) + (0 - 1) = (1)\ (+)\ (- 1) = 0$ or:

$-1\ (+) +1 = -1 + 1 = 0$: you can either add -1 or +1 to both sides to see that the equation is still in balance.

Rearranging the elements, operands, or addends to be summed or combined , their sum should still be the same, no matter what order they are summed or combined together:

$+1\ (+) -1 = 0$ On the number-line, if given 1, you would go rightward from 0 to +1, then if you were to take away 1, this is indicated as -1, which means to move along the number-line in the opposite direction 1 position leftward. Doing this, we arrive back at 0. We see that signed numbers indicate the new value or change to be added in or combined into the running sum is to reduce (-), or increase (+) the current total of the sum.

The add or combine symbol (+) can be removed if we consider that the sign goes with each operand that is to be combined, and then this combine symbol (+) can be eliminated or hidden, and understood that the signed number operands, or terms, are being added or combined.

Without signed numbers or terms, it would be difficult to solve equations by transposing (essentially eliminating a term, and moving its signed number counterpart) values from one side of the equation to the other so as to keep the equations in balance.

In short, a simple explanation of signed numbers goes something like this:

+5 is five more than something, and -5 is five less than something. or:

+5 is a change that is an increase, and -5 is a change that is a decrease.

If something was 80 degrees and cooled off to 70 degrees, the change in temperature was (80 - 70) = 10 degrees. Since the change in temperature was a decrease in temperature, this would be noted with a negative sign as -10. If the change in temperature was ten degrees higher, this would be noted as +10. -10 means a decrease of 10, or 10 less than something (here its 80 degrees, and for another day it could perhaps mean 10 less than 50 degrees). See, negative numbers don't have to mean less than 0 always, when they are considered as a decrease or a (negative) change from any another value (ie. the reference, initial or starting value, and not necessarily 0).

Ex. Consider we have a "running" (continuously updated) current total or sum, that is, we are to add in all new values as they come along or happen, to the current or accumulated value of the sum. Perhaps we are putting apples into a basket as we pick them. Sometimes there will be a decrease in the number of apples because someone took some apples out of this basket. This essentially causes the sum of apples to change to a lesser or lower value. This decrease or change can be mathematically indicated as a negative sign preceding its (decrease) value. If someone put some apples back into the basket later, this change can be mathematically indicated as a positive sign preceding its (increase) value. With the concepts of signed numbers (numbers that mathematically indicate or represent an increase or decrease), we can mathematically express this running total or sum as a single equation, of running addends or operands, rather than in multiple steps such as something like:

Equation 1: first_sum = number1 + number2 : perhaps various amounts of apples put in the basket
Equation 2: second sum = first_sum - number3 : here, number3 is negative in sign since it is the number of apples that were taken from the basket. This value or term is being summed or combined to: first_sum

These multiple equations can be mathematically expressed (more simply) as:

sum = number1 + number2 - number3 +/- . . .

This expression above is actually a method that would be employed without even knowing the concepts of signed numbers. Since all changes are to be added into, or applied to, the running sum, and if we do not yet know if a change will be an increase of a decrease, we will simply write or indicate this expression as the combining (over time, as changes take place and we know the values of those changes) of known, or yet to be known values.

sum = number1 + number2 + number3 + . . . : here, the word "number " is essentially a formal placeholder (ie. a variable) for any possible value, including its sign, since the value could be either positive or negative,

Using this form of the expression eliminates what appeared to be a subtraction operation in a running summation (sum) and then assuming that the other numbers or operands are always positive in value. With this new notation, it is understood that all terms (a number preceded by its sign) or operands (actual values, constants or variable[any, or many possible values]) or are always considered as signed values that are to be added or combined using the rules (which will be discussed) for signed numbers.

ABSOLUTE VALUE

| | is the absolute value (mathematical) operation symbol. It is used to transform a signed number into its general distance or "magnitude" from zero. The absolute value of a signed number is that number without its sign (hence mathematically equivalent to the positive value).

Ex. $|+5| = 5$: "The absolute value of positive five is five".

Ex. $|-5| = 5$: "The absolute value of negative five is five".

Both of the signed numbers above have an absolute value of 5 since each of the signed numbers has a distance of 5 from 0. This also shows that +5 and -5 are signed number counterparts. Signed number counterparts obviously will sum to 0: value + (-value) = value -value = 0

Consider the difference between 6 and 7. In reference to 6, it could be said that 7 is an increase of 1 and this could be mathematically expressed , noted or indicated as: +1. In reference to 7, it could be said that 6 is a decrease of 1, and this could be mathematically expressed, noted or indicated as: -1. Still, they (6 and 7) both differ from each other by an absolute (or real) value of 1:

$|+1| = |-1| = 1$: if this was a length or distance, it would have to be positive or signless in value since there is no actual physical length or distance that has a negative value.

WHAT ARE THE SIGNS USED FOR?

Values that represent decreases or losses are noted using a negative sign (-) preceding it, and values that represent increases or gains are noted using a positive sign (+) preceding it.

What are terms?

Together, a number and its sign is called a **term**. Terms are the operands of signed number addition or combining. A single term by itself is sometimes called a monomial or single term expression. The word "mono" means "one" (= 1). To solve or "simplify" a multiple (sum of) terms expression, often called a multinomial, is to possibly combine (essentially a summation by signed number addition) the terms into a single term (essentially the sum).

Ex. If the temperature increases by some value, the value of the actual change in temperature would be noted using a positive sign since the temperature increased. If the temperature decreases by some value, this change in temperature would be noted using a negative symbol. Hence, the signs help give meaning to values they precede. In this example, the signs indicate the change in temperature in reference or with respect to an initial temperature of which could also be either positive (pos.) or negative (neg.) in value.

A general expression, statement or formula for the new temperature, in relation to the old temperature, can be:

(old temperature) + (change in temperature) = (new temperature) : each could be either pos. or neg. in sign

For example, if the old temperature was +80° ("eighty degrees") and the change in temperature was 10° cooler, this 10° drop or decrease would be noted as: -10°. The new temperature using the formula above is:

$$(+80^\circ) \quad + \quad (-10^\circ) \quad = \quad +70^\circ$$

Usually, the addition or combine (to add terms mathematically, considering their signs) symbol would be omitted or "hidden", but it still should be understood that these terms are being combined, and the expression would be the familiar looking:

+85° -10° = +70°   showing the hidden or understood combine symbol between terms: +85 (+) -10 = +70

Here, from now on, it will be understood that the terms (+85°) and (-10°), "positive eighty-five, and negative ten", are being added or combined. That is, do not think of the (-) symbol as a subtraction symbol, but rather think of it as the negative symbol that goes with the value of 10, making that term of (-10°). Hence, in a way, there is "no real (formal) subtraction operation" with signed numbers, and there is really just the addition or combining of signed numbers or terms. Differences (or the amount of separation) in values may still need to be found, and then a subtraction operation is used on the signed numbers or terms.

To advance you're mathematical capabilities, it is perhaps better to consider all values as terms or signed values. That is, don't think of signed values or terms as a "special unique feature", but think of them as more like the "main feature", particularly for simplifying and solving equations. With some occasional practice, you will naturally gain more familiarity and confidence using these concepts.

SUBTRACTING A VALUE THAT IS LARGER

With the concepts of signed numbers, it is now mathematically possible to subtract a larger value from a smaller value. As a graphical illustration, observe the drawing below of the number-line: [FIG 21]

```
     <--------|--------|--------|--------|--------|--------|--------|--------|--------|--------|--------|---------->
- ∞ . . . -5      -4      -3      -2      -1       0      +1      +2      +3      +4      +5 . . . +∞
```

If you subtract 1 from 5, you have 4. This can be accomplished using the drawing above and then by moving 1 position leftward (to the lesser values) from the (reference, or "starting") position marked 5. Likewise, if you were to subtract 7 from 5, the result can be found by moving 7 positions leftward from 5. The result is: -2. The mathematical method to get the same results will be shown in the discussions to follow.

ADDING SIGNED NUMBERS

We know that adding two "common or plain" unsigned or positive numbers equals a larger unsigned or positive value. Symbolically:

(positive value or term) + (another positive value or term) = a larger positive value or term

Ex. 5 + 2 = 7 which really means when considering the signs as associated with signed numbers (ie. terms, of an expression):

(+5) + (+2) = +7 : Here, the "hidden" addition operation or combine symbol is explicitly shown between the two terms that are shown parenthesis. The two terms being summed or combined are: positive five: (+5), and positive two: (+2). The result of this expression of a sum of terms is a single term having a value of positive seven, (+7). This may also be expressed sometimes as:

+5 (+) +2 = +7 : here, the implied and understood combine (add) symbol is explicitly indicated or expressed, also:

(+5) (+) (+2) = (+7) : is a possible expression of the above equations

(+5) (+2) = : if the above expressions are expressed this way, it is an error since it now indicates the product of the two values, rather than the addition of the two values, and the result it actually +10, and not +7.

Adding or combining a negative value to another negative value is effectively adding or combining a decrease to another decrease, hence making a larger total or net decrease, and the sign of the sum is therefore negative. Symbolically:

(negative value or term) + (another negative value or term) = a larger negative value or term

Ex. -5 - 2 = -7 which really means:

-5 (+) -2 = -7 : Again, the hidden addition operation or combine symbol is shown here for the understanding that these terms, or a series of terms, are to be mathematically combined (added) to produce a net result or sum term.

If a negative value (or decrease) is added or combined with a positive value, or vice-versa, the resulting (signed) sum is equal to the largest difference of the absolute values of the two, and the sign of this sum is from the operand (here a term) with the largest absolute value. In other words, subtract (take difference) the lowest absolute value from the highest absolute value, and keep the sign of the highest absolute value. On the number-line, the resulting sum will be someplace between those two given values.

(positive value or term) + (negative value or term) = | largest term | - | smallest term | = sign of largest term, difference

Even though the expression is for the sum of two terms, the result is actually found by a difference operation when the signs of those terms differ (are not the same). The result will neither be a larger positive value, or a larger negative value, but a less positive value and-or a less of a negative value.

Ex. +5 -2 = +3 :"positive five added (or combined) to negative two is positive three"
OR.= -2 +5 = +3 : since addition is commutative (commutative law), you can rearrange the terms or (signed) operands being combined, added or summed.

Ex. -5 +2 = -3 : | -5 | - | +2 | = 5 - 2 = 3 applied with the sign of the value that has largest absolute value,
OR. +2 -5 = -3 and since 5 is greater than 3, the proper sign will be from its corresponding term. Here, the term of which 5 was from is: (-5), so therefore, the sign to apply to the resulting sum of the combining of those two terms will be the negative sign. This is so, since although the result is less negative, there is still some of its negative value still remaining after the combine operation, and it must therefore be indicated or expressed as part of the result. Here, the result is negative three = (-3).

If you want, you can easily verify all these results on the number lines shown previously. It is interesting to note that the difference or separation between -5 and +2 on the number line is actually +7, and not -3, and this can be shown by subtracting those two values from each other:

-5 - (+2) = -5 -2 = -7 = 7 in absolute value , or:
+2 - (-5) = +2 +5 = +7 = 7 in absolute, signless value

Again, if given a negative value and you then add a positive value to it, the sum may still be negative, as when the largest of the two signless values was negative, but the sum or result can be said as being "less negative" in reference to its previous and larger negative value. For example, when +2 is added to -5, the sum is -3. -3 is said to be less negative than -5.

Ex. If the temperature outside was -20 and then it "warmed up" to -3, this change is usually said to be better or positive even though it is still below zero (ie. negative) degrees outside. The temperature is headed or getting closer to being a positive value.

If you are performing addition of signed numbers with a standard "home" calculator which does not have a "sign" or "change of sign" key, below is a somewhat "wordy", but simple example of how to add or combine signed numbers in such a situation. First, some calculator that do not have a "sign" key actually allow you to enter negative numbers by first pressing the "minus" or "subtract" key before entering the number (which is essentially the same as entering: 0 - number = -number, and then press = or the next operation key, or by pressing: number - and then = , where the calculator assumes the first operand is 0. For a simple check to see if your calculator can display negative numbers, subtract a larger positive (or signless) value from any smaller positive value, and the result should be negative in sign (indicated before, or sometimes after the result). For example, enter: 2 - 3, and the result displayed should look something like -1. If your calculator does not work with signed values (particularly negative values) you must establish and record the sign of each operand and result mentally or manually on paper.

Add positive five and negative three using a simple "home" calculator.

(+5) + (-3)

Since subtraction of a number is really equal to the addition (combining to make a sum) of the negative, "opposite" or "reverse" of that of that number, we can transform this addition problem back into a common subtraction problem by subtracting the negative, "opposite" or "reverse" of that number.

(+5) + (-1)(-3) : Multiplying by (-1) to have the reverse or counterpart of (-3).

(+5) + (+3) We must change the sign of the operation since we are now adding two positive numbers for an increase, however we want to combine (ie. add) a decrease to (+5) as shown in the original problem, so we must therefore use the common subtraction operation instead of the addition operation:

(+5) - (+3) : as mentioned, this subtraction of a value is equal to the addition of that negative of that value. (+5) - (+3) = (+5) (+) -(+3) = +5 (+) (-3) as first given. And this will be performed on the calculator as:

5 (-) 3 : five minus three
+2

Summarizing the results: +5 (+) -3 in common, non-signed number form = +5 (-) +3

Again, the conceived subtracting a number is really adding or combining the negative of that number when working with signed numbers.

If your calculator does have a (change or enter) sign-key or button (often indicate as: [+/-]), to enter a negative number, first enter the absolute (signless) value of that negative number and then press the "sign key" to display the negative symbol with it. If you press the sign-key again, the value displayed should change back to a positive value.

As an extra note which could be helpful to understanding the difference or separation of absolute values: If the signs of two values being added or combined are exchanged, the result is opposite in sign, hence both sums have the same absolute value:

First note that: |-7| + |+2| = 7 + 2 = 9 , and |-7| - |+2| = 7 -2 = 5

Now consider addition with the operands signs reversed:

+7 -2 = +5 and -7 +2 = -5 , Hence: |+7 - 2 | = |+5| = 5 and |-7 +2| = |-5| = 5 : A difference of 5 for both.

GETTING LESS AFTER ADDITION

Here is a very understandable descriptive analogy of how something can be less after adding or combining.

There is an empty container, its volume or space is first filled halfway with hot water, then the container is filled (ie. combined, or added, (+)) to the top with cold water. Surely, the amount of water has increased, but this is not the case for the temperature. The net (final total or sum) result of the water temperature is that it is not hotter (higher in temperature), but is now only a warm temperature, and that the temperature of the water is now less and is said as being "cooler", "lower" or "less" in value as compared to the original, starting, or reference temperature.. This is commonly thought of as:

"hot plus cold, equals warm" : you can view warm as "less hot", or "not as hot" in temperature, and that
or "cold added to hot, equals warm" is similar to saying a value is less positive than before

Note that the cold water temperature is really in reference to the warm water temperature, and vice-versa. For example, cool 34 degree water can still be called cooler water than 35 degree water, even though its barely noticeable, and when both are actually cold to most people. To account for all possible temperatures, and all possible changes in temperature, a (more generalized) equation such as this is written:

temperature + change in temperature = new or resulting temperature.

Ex. Without using any specific units of measurement such as pounds or kilograms, let's say that a force of 10 is being applied to an object in one direction, and another force of 3 is being applied to the object in the opposite direction. What is the net , effective, or resulting force?

The force in the opposite direction will be assigned a negative sign. The net or effective sum of the forces is:

force_1 + force_2 = new or resulting force or:
force + change in force = new or resulting force

+10 + (-3) = +10 -3 = +7 : The right hand side of the equation shows the expression without the hidden combine symbol since it should be understood by now that the individual terms (+10, and -3) of this expression are being combined.

If the force being applied in the opposite direction had a value of 10, the resulting force would be:

+10 + (-10) = +10 -10 = 0 : Objects usually move when a force is applied to it, and here, the object would not be moving since the net or total force acting upon it is 0.

Ex. Suppose we give an algebraic sign of (+) for the distance an object goes upwards, and a (-) sign for the distance an object goes in the opposite or downward direction. A ball is thrown straight upwards for 50 units of distance, and due to gravity, it will fall downwards 50 units of distance. How far (as in the "signed number" or "algebraic" change in distance from its starting or reference position) did the ball travel?

Total Distance = Sum Of Distances

+50 units (+) -50 units : shown with the implied or hidden combine symbol
+50 units - 50 units combining terms (to simplify):
0 units :the balls ending location is the same as its home, initial or start location

This result might appear to be very odd. You could argue that it went 50 units up, and then turned around and went another 50 units as it fell back, and hence a total of 100 units of distance. This is of course absolutely true if the distances are considered as undirected (essentially the signless or absolute values) or non-signed, but because of the directional mathematics with signed values, the net distance moved is 0 units, as can be clearly seen since the ball is in the same position or location as where it started from. The change in distance of it's ending position and its starting position is essentially 0, whether the ball moved or not.

Although much of this has already been discussed here and there previously in this book, here is another helpful discussion and an example that shows that the sign of a number indicates the difference or change from another (reference) value in such a manor so as indicate (ie. with a sign) if the change was an increase or decrease from the starting or reference value. If no reference value is indicated or known, it is understood that the signed number value is a simple result of an equation, or a basic measurement (which has a reference value of and for that measurement, usually considered or understood as 0, such as on a ruler (a reference scale to measure distance or length) or a weighing scale).

If the value is not considered as a difference, or in reference to some value other than 0, but is a final or a resulting value of some expression, the value is understood at being in reference to a value of 0:

Ex. +10 , "positive ten", without any reference value, is understood as being 10 greater than 0
-10 , "negative ten", without any reference value, is understood as being 10 less than 0

If the value is in fact a difference of two values, it indicates the amount of change in the first value so as to be equal to the second value. This difference can be expressed in two ways:

1. A signless or "magnitude, absolute or plain common numeric value": Ex. 10 , "ten"

2. A signed numeric value: Ex. +10 = "positive ten" , or -10 = "negative ten"

Now consider two values, let v1 = value_1 , and v2 = value_2

Ex. v1 = +10 and v2 = +7 : or if just signless values were given: v1 = 10 and v2 = 7

Lets consider v1 was the initial value and v2 is its final, current, new or resulting value after some change in or of v1. Since these values are different (not the same or identical), a change has occurred to the first, starting or initial value (here, considered as v1). This change (as a measure of the specific or actual difference (a measure of the separation of two values)) or differences in values can be calculated with a subtraction (or "difference operation", since the result of a subtraction operation is a difference) operation.

v1 + change in v1 = v2 or simply:
v1 + change = v2 solving for change:

change = v2 - v1 since the change (in v1) is actually expressed here as a difference (d) value:

change = d = v2 - v1 = ending value - starting value Note, for an unsigned number system, you must use:
d = highest absolute value - lowest absolute value

d = v2 - v1
d = +7 - (+10)
d = +7 - 10
d = -3 : the difference or change was negative, or "negative in value", indicating
the new value is a decrease or less than (<) the starting value. +7 < +10

Knowing there is a difference and its value, it is used to indicate a "numerical comparison or relationship" of one value with respect (in reference) to another. It will indicate if a value is greater than or less than another, and by how much.

If the specific value of d is positive in sign, it indicates an increase or gain, and that the new or ending value is greater (>) than the starting value: new value > starting value.

If the specific value of d is negative in sign, it indicates a decrease, lessening, reduction or loss, and that the new or ending value is less-than (<) the starting value: new value < starting value

For the example, variable (d), the difference or change value is -3.

ending value < starting value : likewise, mathematically: starting value > ending value
$+7 < +10$ $\qquad$ $+10 > +7$

Ex. If $v1 = +7$ and $v2 = +10$

From: $\text{change} = d = v2 - v1$

$\text{change} = d = (+10) - (+7)$
$\text{change} = d = +10 - 7$
$\text{change} = d = +3$: a positive change value or difference indicates a gain or increase has taken place or occurred.
here, ending value > starting value
$+10 > +7$

If you were to apply a change of (-3) to a value, such as (+10), you would express this operation as an addition or combining operation of two or more signed values. Here the combining operation is applied to or upon the starting value of (+10):

starting value + change = ending value , or by switching sides:

ending value = starting value + change
$+7 = +10 + (-3)$
$+7 = +10 \;\; -3$ this is the same as: $+7 = +10\ (+)\ -3$
$+7 = +7$: checks

We see how useful and important the sign is with its corresponding value, as in a signed number system. If the change was only indicated as 3 and without the negative sign or an indication of being "3 less", the ending value would commonly (in an unsigned number system) be calculated to be: $+10 + 3 = 13$ which would be the wrong result and which has a difference of +6 (ie, 6 higher, off or away) from the true result of (+7). In a sign number system, the sign is always kept with the value, and generally not to be discarded, so as it will be used in any further calculations so as to achieve the correct results. The sign is now part of the number or value. (+7) and (-7) are two different numbers. One value, (+7) is 7 more than 0, and the other, (-7) is 7 less than 0 (or possibly some other reference value). (+7) and (-7) are quite different, and expressing this difference mathematically or quantitatively, the difference between them is a relatively huge value of: $(+7) - (-7) = +7 +7 = 14$. The value or number of (-7) is just as important and useful as $(+7) = 7$.

MATHEMATICALLY EXPRESSING SIGNED NUMBER COUNTERPARTS

The mathematical counterpart (corresponding, and "opposite") of a signed value or number is equal to that same value, but with a sign change. Mathematical counterparts of signed values or numbers have the same (signless, or unsigned) absolute value, "magnitude" or distance from 0. To change the sign of a value, simply subtract it from 0, or just take the negative of it by multiplying it by (-1).

The negative (-), "reverse", mathematical or number-line counterpart of a positive value is a negative value. Expressing this mathematically:

Ex. -(+5) = -5 : the negative of positive five is negative five. This is a single term and expression.
If you add 0 to this term (to create a two term expression and to help understand the result), it wont change its value: value + 0 = 0 + value = value
-(+5) + 0 = 0 (+) (-(+5)) = 0 (+) -(+5) = 0 -5
0 - 5 = (0) + (-5) = -5 since 0, or no change, added or applied to any value is equal to that same value. Likewise any value added or applied to 0 is still equal to that same value.

The negative (-), "reverse", or counterpart, of a negative value is a positive value: Expressing this mathematically:

Ex. -(-5) = +5 , or= -(-5) = -1(-5) = +5 = (-1)(-5) = +5 : the negative of negative five is positive five

The expressions on both sides of the equations in the above examples will be mathematically verified in the discussions to follow, such as the multiplication of signed numbers. Technically, the set of whole (common "counting") numbers and their corresponding negative counterparts are classified as the "integer" numbers.

MULTIPLICATION OF SIGNED NUMBERS

The product of two positive terms is a positive term, as expected since it is much like the multiplication of two unsigned values. The product of a negative term and a positive term is a negative term. The product of a negative term and a negative term is a positive term. All these rules can be verified using repeated addition.

Ex. (+2)(+3) = +6 Essentially, this is an example of "repeated increments (addition)". Here two will be added to zero (think of the starting point, a sum that is nothing =0) a total of three times. In other words, three, twos will be summed, or that two will be summed three times.
Note also, (+2)(+3) = (+3)(+2) since the order of multiplying does not matter.

since +2+2+2 or= (+2) + (+2) + (+2) : shown on the right side with the "hidden"
+6 addition or combine symbols.

Ex. (-2)(+1) = (-2)(1) = -2 : any value times one is still that value. If you were to start at the 0 or "reference" position of a number line and reduce this by 2, by going in the leftward or the negative direction (essentially a subtraction), you will arrive at -2. (+0)+(-2) = +0 -2 = -2

Ex. (-2)(+3) = -6 :This is an example of "repeated decrements (subtraction)", so the result should then be less, or more negative. Negative two will be added to 0 a total of three times. This might be more easily seen when rewritten as: (+3)(-2) using the commutative law. Since the second factor is negative, and not positive, the result cannot be positive. Instead of repeatedly adding (+3) a total of (2) times, this is effectively repeatedly subtracting (+3), a total of (2) times, from 0.

since -2-2-2 or (-2)+(-2)+(-2) = (-4) + (-2) = -6

Ex. $(-2)(-3) = +6$ This is verified below:

First, it can be shown that:

$(-3) = (-1)(+3) =$ the negative or counterpart of $+3$ is -3. : adding -1 three times is $(-1)+(-1)+(-1) = -1 -1 -1 = -3$

That is, $(-1)(+3) = -(+3) = -3$: or $= (+3)(-1)$

Then:

$(-2)(-3) = (-2)\ (-1)(+3) = (-1)(-2)(+3) = -\ (-2)(+3)$

Essentially then, the negative or counterpart of (-2) will be used as a multiplicand , and for the repeated addition:

$-(-2) = +2$: the negative or counterpart of (-2) is $-(-2) = +2$
"the negative of negative two, is positive two"
$-(-2)(+3)$
$(+2)(+3)$
$+2+2+2$
$+6$

An alternate verification is:

$(-2)(-3)$
$(-2)(-1)(+3)$ using the commutative law:
$(-1)(-2)(+3)$
$(-1)\ (-6) = -(-6)$ and the negative or counterpart of (-6) is:
$+6 = +6$

A negative value times a positive or signless value results in a larger negative value, and is due to the repeated addition of that negative value. It is not too difficult to then understand that a negative value times a negative value should be of the opposite sign, and cannot have a negative value. The opposite of a negative value is a positive value. Summarizing the signs of the results or products of signed number multiplication:

(positive value) x (positive value) = positive value
(positive value) x (negative value) = negative value
(negative value) x (negative value) = positive value : (negative) (negative value) = - (negative value) = + value

DIVISION OF SIGNED NUMBERS

If the signs of the dividend and divisor are the same, either both positive or both negative, the sign of the quotient will be positive, otherwise, it will be negative. All quotients can be verified using the signed number multiplication rules when multiplying the divisor and quotient together and comparing their product with the dividend.

Ex. $\frac{+10}{+5} = +2$ since $(+5)(+2) = +10$

Ex. $\frac{-10}{-5} = +2$ since $(-5)(+2) = -10$

Ex. $\frac{-10}{+5} = -2$ since $(+5)(-2) = -10$

Ex. $\frac{+10}{-5} = -2$ since $(-5)(-2) = +10 = (-1)(5)(-1)(2)$

$$\begin{gathered}(-1)(-1)(5)(2)\\ -(-1)(10)\\ +1(10)\\ +10\end{gathered}$$

Summarizing the signs of the quotient of signed number division:

$\frac{\textbf{positive value}}{\textbf{positive value}}$ **= positive value** **:** division of values having the same sign

$\frac{\textbf{negative value}}{\textbf{negative value}}$ **= positive value**

$\frac{\textbf{positive value}}{\textbf{negative value}}$ **= negative value** **:** division of values having opposite signs

$\frac{\textbf{negative value}}{\textbf{positive value}}$ **= negative value**

OBSERVATION OF SIGNED NUMBER COUNTERPARTS

The negative (-) or counterpart of a positive value is a negative value.

Ex. -(+5) = -5

Note that the negative symbol preceding (+5) can be interpreted as meaning;

(-1)(+5) = -5

The negative or counter part of a negative value is a positive value.

Ex. -(-5) = +5

Note again that this negative symbol can then be interpreted as being multiplied by (-1).

(-1)(-5) = +5

Also note, -(-5) is actually a single term containing the negative sign that precedes(-5).

Written in this form: (-1)(-5), it is also a single term but it is understood as having a positive sign:

(-1)(-5) = +(-1)(-5) or: +((-1)(-5))
+(+5)
+5

So, the apparent subtraction of a term is really adding the negative of that term. Though differences sometimes need to be found, there is "no real subtraction" operation performed with signed numbers (terms), and there is only their summing or combining.

Ex. +5 -(-2)

This expression is the addition of two terms: (+5) and -(-2) or (-(-2)). The (+5) term is being combined (added) to the negative of the expression inside the grouping symbol which is the term (-2).

That is, +5 -(-2)
(+5) + (-(-2)) : shown with the "hidden" addition (combine) symbol
OR, (+5) + ((-1)(-2))
+5 + (+2)
+5 +2 : now shown without the addition symbol
+7

After practice, many of the intermediate steps will not have to be performed. For the above example:

+5 -(-2)
+5 +2
+7

Noting the above discussion, the same can also be said about the positive symbol. These rules are very helpful when clearing or ridding grouping symbols when simplifying expressions.

+(+5) = +1(+5) = (+1)(+5) = +5
+(-5) = +1(-5) = (+1)(-5) = -5

Subtracting a negative number is essentially removing a negativity or loss from the sum, because the result is a positive value to be added into the sum. Therefore, the result is more positive. Since we are working with signed numbers, it is best to think of subtracting a negative number as the addition of a positive number:

Ex. First: 2-5 : combining with a negative term
2 (+) -5
-3

Now with the combining of the negative of a negative term which can be thought of as subtracting a negative number, and we know this is essentially the addition of a positive number:

2 - (- 5) showing the "hidden" addition symbol (+) between the terms:

2 (+) -(-5) : - (-5) = (-1)(-5) = +5 :
2 +5 : the expression resolved to the addition of a positive number
7

Checking, the difference added to the subtrahend (when the original expression is considered as an expression of subtraction) should equal the minuend:

7 + (-5) removing or hiding the addition symbol:
7 - 5
2 : checks

Ex. -2 - (-5)
-2 +5
+3

Checking: +3 + (-5)
+3 -5
-2 : checks

WATCH THOSE SIGNS

Here is a common example where by not observing the signs, an error could result.

Simplify: $-3 - \frac{-20 + 7}{5}$: The term -3 can be understood to have a denominator of 1

This is a two term expression, it is a sum of terms.

The first term is: -3 The second term is: $-\frac{-20 + 7}{5}$

The lowest common denominator of both terms is (1)(5) = 5. After creating an equivalent fraction of the first term:

$\frac{(5)(-3)}{5} - \frac{(-20 + 7)}{5}$ combine the numerators over the like denominators:

$\frac{-15 - (-20 + 7)}{5}$ distributing (-1) to clear grouping symbols, this will essentially reverse the sign of each term in the grouping symbol. You can think of this as multiplying the entirety of what is in the grouping symbols by (-1), and here, each term in the grouping symbols.
$-(-20 + 7) = -(-13) = +13 = -(-20 + 7) = (-1)(-20) + (-1)(+7) = (+20) + (-7) = +13$

$\frac{-15 + 20 - 7}{5}$ combining the negative terms in the numerator:

$\frac{-22 + 20}{5}$

$\frac{-2}{5}$ an alternate method of displaying this is:

$$\frac{-2}{5} = \frac{(-1)(2)}{(1)(5)} = \frac{(-1)}{(1)}\frac{(2)}{(5)} = -1\left(\frac{2}{5}\right) = -\left(\frac{2}{5}\right) = -\frac{2}{5}$$

Checking: $-3 - \frac{-20 + 7}{5}$

$-3 - \frac{-13}{5}$ showing with a "hidden" addition symbol:

$$-3 \; (+) - \frac{(-13)}{(5)} = -3 \; (+) \frac{(-1)(-13)}{(+1)(5)}$$

$$-3 + \frac{(+13)}{(5)} = -3 + (+2.6) = -3 + 2.6 = -0.4$$

$-0.4 = \frac{-4}{10} = -\frac{2}{5}$: made an equivalent fraction (divided both the num. and den. by 2)

Checking: $-3 - \frac{-20 + 7}{5}$

$$-3 - \left(\frac{-13}{5}\right) = -3 - (-2.6) = -3\ (+)\ (-1)(-2.6) =$$

$$-3 + (+2.6) = -3 + 2.6 = -0.4$$

$-0.4 = -\frac{2}{5}$: -0.4 is the decimal number equivalent of the fraction on the right hand side

"MINUS" OR "NEGATIVE"?

People frequently use the words minus and negative as interchangeable or meaning the same thing. but as a good rule, think of minus or the minus symbol (-) as the operation to find a difference (subtraction) of two values, and think of negative, which happens to have the same sign (-), for values (operands) or results of an operations(s). For example, if you see (-5) by itself, then it is best to not say that it is "minus five" since if this is perhaps the result of some operation(s), what are you subtracting (minus is the symbol for a subtraction operation, and not its result) it from if this is the answer or result you are looking for? It is perhaps more correct to say one of the following when you see something like (-5) by itself: "five less than 0", or simply: "negative five".

NEGATIVE EXPONENTS

Negative exponents can be created when you are canceling or reducing fractions using the exponent rules when dividing like values, or variables as in algebra. Typically, a negative exponent is the result when a larger value exponent in the denominator is subtracted from a smaller value exponent in the numerator.

Ex. $\frac{5^1}{5^2} = \frac{5^{(+1(-)+2)}}{1} = \frac{5^{(+1-2)}}{1} = 5^{-1}$

Checking by multiplying the divisor and quotient to have a product equal to the dividend:

$(5^2)(5^{-1}) = 5^{(2+(-1))} = 5^1$: checks

Note: $\frac{5^1}{5^2} = \frac{\overset{1}{\cancel{(5^1)}}}{\underset{1}{\cancel{(5^1)}}(5^1)} = \frac{1}{5^1}$: if canceling is used. Also : $(5^1) / (5^2) = 5/25 = 1/5$

Since $5^0 = 1$, we have:

$\frac{1}{5^1} = \frac{5^0}{5^1} = \frac{5^{(0-1)}}{1} = 5^{-1}$

Hence, $\frac{5^{-1}}{1} = \frac{1}{5^{+1}}$: We see that a value with a negative exponent really means, or is equal to, a division by that value with a positive exponent. The inverse or "negative" of multiplication (repeated addition) is division (repeated subtraction). As an extra note, the inverse or "negative" of a power (which includes both the base and the exponent; either expressed or simplified to a value) is a root.

Also note that the exponents can also be subtracted in the denominator instead of the numerator:

$\frac{5^1}{5^2} = \frac{1}{5^{(2-1)}} = \frac{1}{5^1}$

Hence, a value with a negative exponent in the numerator is really a division by that value with a positive exponent in the denominator, or in simpler words, multiplying by a negative power is essentially division by its corresponding positive power.

Ex. Simplify $+100(5^{-2})$

$\frac{+100(5^{-2})}{1(\;1\;)} = \frac{+100}{1}\left(\frac{1}{5^2}\right) = \frac{+100}{5^2} = \frac{\overset{+4}{\cancel{+100}}}{\underset{+1}{\cancel{25}}} = \frac{+4}{1} = +4$

Negative exponent form is sometimes desired when simplifying the product of multiplying like values (or variables in algebra), where to simplify is to add the values exponents, and keeping the same value.

Ex. $(3^3)\left(\frac{1}{3^1}\right)$ without using direct multiplication to simplify:

$(3^3)(3^{-1})$ multiplying like values or variables, add their exponents to simplify, keeping that base:
$3^{(3-1)}$ and this can be simplified to:

3^2

Or by representing 1 as = 3^0, like values in both the numerator and denominator can be shown:

$$(3^3)\frac{1}{(3^1)} = \frac{(3^3)\,(3^0)}{(\;1\;)(3^1)} = \frac{(3^3)(3^{(0-1)})}{1} = (3^3)(3^{-1}) = (3^{(3-1)}) \quad \text{combining the exponents:}$$

3^2 = 9 checking: 3^3 / 3^1 = 27 / 3 = 9

Another method to convert a power which has a negative exponent to an equivalent value which has the same exponent which is positive in value comes from the concept of reciprocals, a fraction to a power, and the previous discussion on negative exponents.

Ex. 3^-4 = 1/3^4 = 1^4 / 3^4 = [1/3]^4 = 0.333333333^4 These steps are also verified more below:

1/3^4 = 1 / 81 = 0.012345679 = 0.3333333333...^4 = 0.012345679

$$3^{-4} = \frac{1}{3^4} = \frac{1^4}{3^4} = \left(\frac{1}{3}\right)^4$$

In general, if you were to write a (algebraic) formula for this, you would create something like:

$$N^{-x} = \frac{1}{N^x} = (1/N)^x$$

: (1/N) is the reciprocal of N and (1/N)^x is the reciprocal of N^x , By observation, it could be said that the reciprocal of an expressed power (such as N^x) is equal to that same indicated power (such as x) of the reciprocal of the base of that power.

A reciprocal of some value, say N is sometimes expressed as N^(-1), and this is verified here with several intermediate steps which can be omitted during simplification of expressions:

The reciprocal of N is usually expressed as:

$$\frac{1}{N} = \frac{(1)(1)}{N} = \frac{(1)(N^0)}{N^1} = (1)\,N^{(0-1)} = (1)\,N^{(-1)} = N^{-1} = \frac{1}{N}$$

Some formula express reciprocal values using an exponent of (-1), and it may be odd for example to see something such as (pi) to the negative 1 power:

$$(\text{pi})^{-1} = \frac{1}{(\text{pi})} = (3.14\ldots)^{-1} = \frac{1}{3.14\ldots}$$

THE ROOT OF A PERFECT SQUARE

The square root of a positive perfect square (of a value) is both positive and negative in sign, that is, a positive perfect square has two roots, one is positive, and one is negative.

Ex. What is the square root of 25?

$\sqrt{+25}$ = ±5 : ± means both positive and negative

That is, $\sqrt{+25}$ is +5 and -5, but not both. 25 is said to be a perfect square (value) of both of its square roots.

Checking by squaring (raising to the second power) both roots:

(+5)^2 = (+5)(+5) = +25
(-5)^2 = (-5)(-5) = +25 , symbolically or algebraically expressed: (-n)^2 = n^2
checking: (-n)^2 = (-n)(-n) = (-1)(n)(-1)(n) = (-1)(-1)(n)(n) = +1n^2 = n^2

Ex. $\sqrt{20^2}$ = ±20^(2/2) = ±20^1 = ±20 : to put this radical in exponential form, and at the same time, solving for the square root, divide the exponent of the radicand by 2

Ex. $\sqrt{4^8}$ = ±4^(8/2) = ±4^4

Ex. $\sqrt{50.9796}$ = ±7.14 : usually, a calculator would be used to find this root

Generally, the "principle" (main and positive) roots have more practical or useful importance, and therefore it is usually the intended and formally defined root of any radical. Negative roots are usually meaningless, such as negative lengths or distances, but don't completely ignore the possibilities of negative roots, for example, they are very important when drawing curves that are symmetrical to either an axis of that curve, or to an axis-line of a coordinate system - where the points [ie. a tiny part of a line or curve] have corresponding or counterpart positive and negative value coordinates (corresponding locations, or simply the location or "address", of a point [drawn as a dot on paper]).

Here is a note on the difference between a negative valued base being squared, and the negative of a value squared.

Consider this: (-5)^2 /=/ - 5^2

This above is read as: "Negative five square is not equal to the negative of five square.":

(-5)^2 /=/ - 5^2
(-5)(-5) /=/ -(5^2)
+25 /=/ -(+25)
+25 /=/ -25

For -5^2, due to powers and roots having a higher precedence than combining of terms (signed values, and-or partial or intermediate sums of the sum), this expression really indicates and is to be evaluated as: -(5^2). To indicate or "force" the square of negative five, parentheses must be used as shown above as: (-5)^2. You could also say: "the square of negative five is not equal to the negative of the square of five".

Also consider these possibilities: -(-5^2) /=/ - (-5)^2 or = - ((-5)^2)
-(-25) /=/ -(+25)
25 /=/ -25

[This space for book edits.]

THE DISTRIBUTIVE PROPERTY , DISTRIBUTION

The distributive property means that the product of two factors, in which one or both is a sum of terms (ie., an expression), is equal to the sum of all possible products of each of the terms of one factor, with all of the terms of the other factor. This should be easily seen, as presented below, since multiplication is really repeated addition. Distribution was also briefly mentioned previously in this book. The word distributive comes from the fact that the terms of one factor are "distributed", as in a distribution of something, to be used (here multiplied) with all the terms of the other factor. Frequently, the words "distribute" and "multiplication" are exchanged or used in place of each other due to their similarity or process.

Ex. $+5(+1+2)$ So simplify this expression, distribute the +5 factor to each member of the other factor in the grouping symbols:

$+5(+1)\ (+)\ +5(+2)$: an intermediate step frequently omitted, showing the "hidden" addition or combine symbol shown here as (+).
$+5(+1)\ +5(+2)$ distribute or multiply to clear the grouping symbols in each term:
$+5 \quad +10$ add (combine terms):
$+15$

Checking, using the order of operations more closely where the expression within the grouping symbols is simplified first:

$+5(+1+2) \quad = \quad +5(+3) \quad = \quad +15$

Ex. $2(10) \ = \ 10(2)$ expressing multiplication as repeated addition on the left side, and a factor composed of, or reduced to, partial sums (here of 2) on the right:
$10 + 10 \ = \ 10(1 + 1)$
$10(1) + 10(1) \ = \ 10(1 + 1)$: this expression is of the basic format of the distributive property, and the expression on the right side is essentially a factored form of the left side where 10 is a common factor to each term, and is then factored out of each term and expressed as a product on the right side.

For more verification of the distributive property, here is a slightly different way of expressing the above example:

$2(10) \ = \ 10(2)$ expressing each side using repeated addition:
$2+2+2+2+2+2+2+2+2+2 \ = \ 10 + 10$ expressing 2 as partial sums:
$(1+1)+(1+1)+(1+1)+(1+1)+(1+1)+(1+1)+(1+1)+(1+1)+(1+1)+(1+1) \ = \ (1)(10) + (1)(10)$
$10(1+1) \ = \ 10(1) + 10(1)$: this expression is of the basic format of the distributive property

You may also consider this:

$10 + 10 = 20$ dividing both sides by 10 will keep the equation in balance, that is, both expressions will still be equal:

$\frac{10 + 10}{10} = \frac{20}{10}$ the fraction on the left hand side can be considered as the sum of two "like-fractions" that have the same denominator of 10:

$\frac{10}{10} + \frac{10}{10} = 2$

$1 + 1 = 2$

Since the original equation was divided by 10, we can multiply both sides of this new equation by 10 to undo what was taken out. A multiplication by a number, and then a division by that same number, or vice-versa, will still keep things in balance since that whole process is essentially multiplying by 1 which does not change a value:

number **/** number = 1, or , = (number)(1**/**number) = 1. : a number divided by itself equals 1 which equals the number times its reciprocal.

10(1 + 1) = 2(10)
10(1 + 1) = 20 hence, in relation to original equation, we can see another verification of the distributive law:

10(1 + 1) = 10(1) + 10(1) = 10 + 10 = 20

If you were to create a minimal formula to represent this distributive property concept, letting A, B and C to represent (or be "placeholders for") any values (not yet specifically known), it would be:

A (B + C) = AB + AC **: a symbolic or algebraic formula for the distributive law**
Note that AB = (A)(B) = A x B = "A times B", and AC = (A)(C), etc.
Since letter "x" used as the symbol for multiplication might be considered as a variable, it's use is therefore problematic, and parenthesis or brackets are recommended where needed, such as to clarify an expression.

This expresses that A times the quantity of B + C, is equal to A times B, plus A times C. The right side of the equation is known as a simpler form of the left side. This is so since it's a simple sum of terms rather than an indicated multiplication (which itself represents another mathematical operation). The left side of the equation is also known as the factored form of the right hand equation (where both terms contain a (common) factor of A), and more about this will be said ahead.

A single term is called a monomial, and a sum of two terms is called a binomial. The expressed product of: A(B+C) is technically a single term product which consists of a monomial (A) times a binomial (B+C). After distributing and multiplying the factor of A to each term in its binomial factor, the simplified or "simpler" result is a binomial expression that is a sum of terms.

A(B + C) = AB + AC : distributed form, showing all possible products of the two factors
AB + AC = A(B + C) : factored form , each term on the left side has a factor of (A) in it

The above is just the basic representation of distribution, and the factors A or (B + C) could actually be composed of more terms and the basic concepts can still be applied to those just the same.

For additional verification of the distributive property, consider this where it is expressed like a common (vertical) multiplication problem:

A (B + C) = (B + C) A : due to the commutative law, the expressed order can be switched

$$\begin{array}{r} B + C \\ \times\ \underline{\quad\quad\quad A\ } \\ (AB) + (AC) \end{array}$$

For example:

$$2\,(10 + 8) \;=\; \begin{array}{r} 10 + 8 \\ \times\ \underline{\quad\quad 2} \\ 20 + 16 \\ 36\quad \end{array} \;=\; \begin{array}{r} 18 \\ \times\ \underline{\ 2\ } \\ 36 \end{array}$$

: this right side is the result simplifying the expression in the grouping symbols first, and then multiplying

2(10 + 8) = 2(10) + 2(8) : 18 multiples of 2, is the same as 10 multiples of 2, plus 8 multiples of 2
20 + 16 = 36

Note also that the second (from the left) vertical multiplication expression above essentially shows the right hand multiplication as a multiplication of its positional sums in the decimal system. It could also be said that 10 and 8 are "partial sums" of 18. In short, each "partial sum" (whatever values they may be) of the multiplicand is to be multiplied by the multiplier once. Below, integer numbers 9 and 10 are expressed as the sum of two integer "partial sums". This will help give a deeper understanding of partial sums that are integers, and of the concepts of distribution such as that distribution can be "thought of" as either one or all factors converted or expressed as a sum of partial sums of it. After some intermediate multiplications, the result is essentially that of repeated addition, as expected since multiplication is essentially repeated addition. Though in these examples, it is easier to simplify the expressions in the grouping symbols first before using distribution, it also helps verify using the concepts of distribution with values, such as variables, in grouping symbols that cannot be simplified first or further. Here in a pseudo expression or formula:

(factor1) (factor2)
(factor1) (sum of partial sums of factor2)
(factor1) (partialsum1 + partialsum2) Now expressing the distribution of factor1 to each partial sum in factor2:
(factor1) (partialsum1) + (factor1)(partialsum2)

Ex. (5) (2) : This example verifies the concepts of distribution.
(5) (1 + 1) The multiplication of 5 and 2, expressed as a repeated addition of 5:
(5)(1) + (5)(1) : shows the addition or sum of the partial products of
5 + 5 which are also partial sums of the final product
10

Or, since the result is the same if the order of the multiplication is changed:

(2) (5)
(2) (1 + 1 + 1 + 1 + 1)
(2)(1) + (2(1) + (2)(1) + (2)(1) + (2)(1)
2 + 2 + 2 + 2 + 2 : extra, for example, = (2+2)+2+2+2 = (4+2)+2+2 = (6+2)+2 = 8+2 = 10
10 A summation by grouping , showing partial or intermediate sums

Here is a verification of distribution

Since: (5)(2) = (5)(1 + 1) : a factor expressed as a sum of partial sums
And: (5)(2) = 5 + 5 : multiplication is repeated addition of that value, equating the equivalences:

(5)(1 + 1) = 5 + 5 : which would result after multiplying the first factor to each partial sum of the second factor

Consider these partial sum integers that an integer value can have, and even these partial sums can be expressed as other partial sums.

9	10
(1 + 8)	(1 + 9)
(2 + 7)	(2 + 8)
(3 + 6)	(3 + 7)
(4 + 5)	(4 + 6)
	(5 + 5)

: after this, the values start repeating, for example (6+4) was already expressed as (4+6)

Clearly, an integer (N) will have the whole portion of (N/2) integer partial sums. 9/2 = 4.5, and 10/2 = 5
Note that all the values from 1 to (N-1) will be included in these partial sums. Any value plus 0, is not considered as a partial sum. Any number or factor can be expressed as partial sum:

number x 9 = number(9) = number(sum of 9's partial sums) , for example:

number x 9 = number(1+8) = number(1) + number(8) = partial sum or product + partial sum or product = product

In short, if given number added once, and given number added eight times, the sum of those two values is equivalent to number being added nine times. Number being added nine times is the same as number times 9.

A factor can also be expressed as the sum of more than two partial sums, even up to the point where that factor is expressed entirely as the sum of values that are 1.

Ex. 7(4) = 7 (+1 + 1 +2)
28 = +7(+1) +7(+1) +7(+2) :shown with indicated (ie. expressed) distribution
Now multiplying (it is also distributing, even if just one term), and it will "clear the grouping symbols".
28 = +7 +7 +14 Now combining the terms on the right hand side:
28 = 28

Ex. +3(2+1) Using distribution (D):
+3(+2) +3(+1) : an expression with indicated distribution and the resulting indicated products
+6 +3 : after distributing to clear grouping symbols, and here its a single multiplication in each term
+9

Ex. +2 +(3-7) -(5-7) this can be rewritten for better clarity as:
+2 +1(+3 -7) -1(+5 -7) distribute (+1) and (-1) to clear grouping symbols:
+2 +1(+3) +1(-7) -1(+5) -1(-7) : expressing the distribution, now multiplying to clear grouping symbols:
+2+3-7-5+7 optionally summing up all positive terms, and negative terms separately:
+12 -12
0

Here is another generalized formula of the concept of distribution. Variables, or placeholders representing any value, are used rather than specific values:

$(A + B)(C + D)$
$A(C + D) + B(C + D)$: an intermediate step often not shown. Note that each term now includes the same factor of $(C + D)$.
$AC + AD + BC + BD$

OR : $(A + B)(C + D)$
$(A + B)(C) + (A + B)(D)$ this can be expressed as:
$(C)(A + B) + (D)(A + B)$: here, each term has a factor of $(A + B)$
$CA + CB + DA + DB$ this can be expressed as:
$AC + BC + AD + BD$ which can be expressed as:
$AC + AD + BC + BD$: the same as above

Ex. (+2+3)(+1+4)
+2(+1) +2(+4) +3(+1) +3(+4) : showing the sum of the indicated products of distribution
+2 +8 +3 +12 : after distributing again to clear grouping symbols
+25 : after combining terms

checking: (+2+3)(+1+4) = (+5)(+5) = +25 : here the expressions in the grouping symbols were simplified first

Ex. -1(+4 -5 +2) Distributing the -1:
-1(+4) -1(-5) -1(+2) Distributing or multiplying to clear grouping symbols in each term:
: Note, this is three terms being combined (summed): -1(+4) and -1(-5) and -1(+2) since it is understood that these terms are being summed or combined, and that the combine symbol (+) is hidden or understood as being there
-4 +5 -2 Combining these terms:
+5 -6
-1

checking: -1(+4 -5 +2)
-1(+6 -5)
-1(+1)
-1

Ex. Given a value of 1.5, this value can be expressed as a sum of the whole value and the fractional value:

1.5 = 1.0 + 0.5

If this value of 1.5 is multiplied by 2, then 1.5 and any possible expressed equivalence of 1.5 is effectively multiplied by 2:

2 (1.5) = 2 (1.0 + 0.5) distributing:
3 = 2 (1.0) + 2 (0.5) : expressed or indicated distribution and the sum of the products
3 = 2 + 1 : the right hand side is the sum of a multiple of 2 + a fraction (part) of 2
3 = 3

Notice that the specific fractional portion of 0.5 in 1.5 corresponds to a specific fractional portion of the value of 2, that is, the portion will be less than 2, and here, that this portion of 2 is the value of 1. 1 is numerically this portion (ie. fraction) of 2:

$\frac{1}{2}$ = 0.5 : the decimal equivalent of the fractional part (here, it's 1) of 2

Ex. 5.2 (10) = (5 + 0.2) (10) = (10)(5) + (10)(0.2) = 50 + 2 = 52

The distributive property is used widely in algebra, especially when the expression inside a grouping symbol of one factor cannot be simplified further. To continue simplifying, distribution will create a sum of terms which is a simpler expression. The ultimate goal when simplifying expressions is usually to produce a single value or term if possible.

Note: (+2 +3)(+1 +4) and +2+3(+1+4) are actually very different.

In the expression +2+3(+1+4), only the value of +3 is being multiplied to (+1+4), that is, this is a two term expression. The first term from left to right is +2, the second term is +3(+1+4) which represents a product of two factors +3 and (+1+4). Multiplication has a higher precedence of performing than combining. In the expression (+2+3)(+1+4), the total quantity or value inside the first grouping symbol is being multiplied to (+1+4). That is, (+2+3) is an expression of a factor to be multiplied to (+1+4). The whole expression (+2+3)(+1+4) actually represents only one term, and since the sign is not indicated, it is considered positive in sign, however this does not necessarily mean the result after evaluating (simplifying, solving) the expression will be positive in sign.

(+2+3)(+1+4) /=/ +2+3(+1+4)
+2(+1)+2(+4)+3(+1)+3(+4) /=/ +2+3(+1)+3(+4) , and:

+2+8+3+12 /=/ +2+3+12
+25 /=/ +17

Checking by first using simplification within the grouping symbols:

(+2+3)(+1+4) /=/ +2+3(+1+4)
(+5)(+5) /=/ +2+3(+5)
+25 /=/ +2+15
+25 /=/ +17

Here is a generalized or algebraic version or representation of this process where variables represent any valid number:

(a + b)^2 = (a + b) (a + b) = (a)(a) + (a)(b) + (b)(a) + b^2 = a^2 + 2ab + b^2

Ex. (a + 1)^2 = (a + 1)(a + 1) = a^2 + a + a + 1 = a^2 + 2a + 1 : (a + 1)^2 = (1 + a)^2

CANCELING TERMS IN FRACTIONS

Only terms that are factors of the entire numerator and denominator can be canceled. This is a very important concept such as when there are variables being used in algebraic or symbolic expressions.

Ex. $\frac{(2+10)}{2}$

Here, the 2 in the denominator cannot be directly canceled with the 2 in the numerator or the factor of 2 in the value of 10 since neither term in the numerator is a factor of the entire numerator. That is, the entire numerator, and not just a term or selected terms of it, is being divided by 2. If canceling were to occur as just described, you would get the incorrect result. The correct result is actually 6. This can be checked by using the order of operations more closely where the grouping symbols in the numerator and denominator are to be cleared first if possible. Actually, the grouping symbols in the numerator are not needed in this example. To clear grouping symbols here, distribute a (+1) factor preceding it:

$$\frac{(2+10)}{2} = \frac{+(+2+10)}{+2} = \frac{+1(+2+10)}{+2} = \frac{+2+10}{+2} = \frac{+12}{+2} = \frac{6}{1} = 6$$

As a check, consider the expression as the result of a sum of fractions with like or common denominators:

$\frac{(2+10)}{2} = \frac{2}{2} + \frac{10}{2}$: now, both fractions have a common factor of 2 in their entire numerators and denominators, and can easily be divided out or canceled:

$$\frac{(2+10)}{2} = 1 + 5 = 6$$

Another check can be made by factoring the expression or value in the grouping symbols so as a possible common factor to both the entire and numerator can be created. Factoring each of the terms in the numerator we find that each has a common factor of 2, and this is also a common factor to both the entire numerator and denominator:

$$\frac{(2+10)}{2} = \frac{((2)(1) + ((2)(5))}{2} = \frac{\cancel{2}(1+5)}{\cancel{2}} \text{ or= } \frac{2}{2}\frac{(1+5)}{(\ 1\ )} = (1)(6) = 6$$

Notice that the factorization of the terms in the numerator expression is the same as the distributed form of those factors.

Ex. $\frac{-3(2+6)}{15}$ Here, rather than performing distribution to simplify the numerator, "canceling out" can be used due to that the entire numerator and denominator have a common factor (here, its 3, or -3, and we will use 3) among them:

$$\frac{\overset{-1}{\cancel{-3}}(2+6)}{\underset{+5}{\cancel{15}}} \text{ or } = \frac{\overset{-1}{\cancel{-3}}(2+6)}{\cancel{3}(5)} = \frac{-1(2+6)}{5} = \frac{-2-6}{5} = \frac{-8}{5} = -1.6$$

: note 15 = (3)(5) or= (-3)(-5)

Checking using distribution to clear grouping symbols (ORD., step 2):

$$\frac{-3(+2+6)}{15} = \frac{-6-18}{15} = \frac{-24}{+15} = -1.6$$

Checking using an alternate version (ORD., step 4, combining, simplification within the grouping symbols first):

$$\frac{-3(+2+6)}{15} = \frac{-3(+8)}{15} = \frac{-24}{15} = -1.6$$

Dividing A Several Or Multi-Term Expression

A several or multiple ("multi", meaning several or many) term expression is called a multinomial. For the specific number of terms present or used, for example, a binomial is a 2-term multinomial, and a trinomial is a 3-term multinomial. A single (one or "mono") term expression is called a monomial. To divide a sum of terms by a monomial or single term, divide that term into each and every term of the given sum of terms. To initially express this concept in the example below, the order of operations is not strictly followed in the following example:

Ex. $\frac{(+7+14)}{7} = \frac{7}{7} + \frac{14}{7} = \frac{+21}{7} = +3$: a multinomial divided by a monomial

Expressing or explaining this in word form:

$\frac{\text{value1 + value2}}{\text{value3}} = \frac{\text{value1}}{\text{value3}} + \frac{\text{value2}}{\text{value3}}$ even these expressions can be expressed as something like:

$\frac{v1 + v2}{v3} = \frac{v1}{v3} + \frac{v2}{v3}$: expressed by essentially using a "shorthand" version, simple indication, notation or identification of the values used.

This concept above should be clearly seen since for the addition of like fractions (have the same denominator), the numerators are combined (added) and placed over that same denominator:

$+\frac{7}{7} + \frac{14}{7} = \frac{(+7+14)}{+7} = \frac{+21}{+7}$ Ex. $\frac{25}{5} = \frac{(20 + 5)}{5} = \frac{20}{5} + \frac{5}{5} = 4 + 1 = 5$

Each term of a multinomial or several term expression can be divided by a multinomial denominator expression as long as that entire multinomial denominator is considered as like one single divisor and-or value within grouping symbols, and it is then essentially representing and acting as one single value, monomial or single term.

Ex. $\frac{+3+9}{+2+1} = \frac{+3+9}{(+2+1)} = \frac{+3}{(+2+1)} \ (+) \ \frac{+9}{(+2+1)} = \frac{3}{3} + \frac{9}{3} = \frac{+12}{3}$, or = 1 + 3 = +4

checking: $\frac{+3+9}{+2+1} = \frac{+12}{+3} = +4$

Ex. $\frac{(+3+9)+6}{(+2+1)}$

Note that the numerator is indicated or expressed as having two terms: +(+3+9) and +6. Dividing each of these terms by this multinomial denominator:

$\frac{(+3+9)}{(+2+1)} + \frac{+6}{(+2+1)}$: here, the numerator of the first fraction contains two terms: +3 and +9, then we have:

$\frac{+3}{(+2+1)} + \frac{9}{(+2+1)} + \frac{6}{(+2+1)} = \frac{3}{3} + \frac{9}{3} + \frac{6}{3} = \frac{(3 + 9 + 6)}{3} = \frac{18}{3} = +6$, or = 1 + 3 + 2 = +6

Multinomials are sometimes called polynomials, but more formally and in terms of algebraic variables and advanced mathematics, polynomials are multinomials that are a sum of terms that are algebraically related such as having the same variable but which are raised to different indicated powers - often to the next whole number exponent so as to produce a sum of terms that are ascending integer powers of the variable. A polynomial may then be simply said to be a sum of terms which contain several powers of the same variable.

CDC Is Another Helpful Plan For Simplifying Expressions

C, D and C are the first letters for the words: Cancel, Distribute, and Combine. Taken together, they represent an easy method, steps or plan to remember or follow when expressions are to be simplified. This plan closely follows and can be used along with the standard order of operations plan. To help you learn this plan, you should frequently write these letters, or the words they represent, next to the problems you are solving:

C - Cancel : Try to reduce or possibly cancel (eliminate) fractions, that is, rid the denominators of fractions (make them +1 if possible). One way is to cancel out common factors to both the numerators and denominators.

D - Distribute : Use the distributive property to eliminate complex (multiple term) factors such as expressions within grouping symbols, or simply multiply to produce a product. Basically, to distribute is to clear grouping symbols.

C - Combine : Combine (add) terms using the rules for signed numbers, or variables as in algebra where the terms must be "like terms" that have the same exact variables.

Ex. Simplify $\frac{8(2+1)}{4} + \frac{5}{1}$ C , cancel common factors in the fraction (here, the +4):

$\frac{\overset{+2}{\cancel{8}}(2+1)}{\underset{+1}{\cancel{4}}} + \frac{5}{1}$ D , distribute, or multiply (here, the +2) :

$\frac{+4+2}{1} + \frac{5}{1}$ C , combine or add terms (here, in the numerator):

$\frac{+6}{1} + \frac{5}{1}$: C , combine, add terms (here, that are fractions with the same denominator of 1):

$\frac{+11}{1} = +11$

Ex. Simplify $2(6\text{^}2 + \sqrt{16}\,) +5$

Note that this is a two term expression (a binomial). The terms are: $+2(6\text{^}2 + \sqrt{16}\,)$, and +5. 2 is a factor of this first term. $(6\text{^}2 + \sqrt{16}\,)$ is also a factor of the first term, and which itself contains multiple terms, hence it is surrounded or "delimited" (set-apart, bounded, identified) by grouping symbols. It is a factor of the indicated product of 2 and $(6\text{^}2 + \sqrt{16}\,)$.

First, there is nothing to "cancel" here since there are no fractions or division in the expression.

$2(6\text{^}2 + \sqrt{16}\,) +5$ ORD. step 1 (simplify powers and roots), ORD means ORDer Of Operations:

+2(+36+4) +5 ORD. step 4 (combine values or like terms):
+2(+40) +5 D , Distribute ("distribute to clear grouping symbols"), ORD. step 3:
+80+5 C , Combine, ORD step 4:
+85

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Dividing Or Factoring A Several Term Expression

This procedure stems from, and is effectively the inverse of distribution. Given a several or multiple term expression (a multinomial) as a dividend or numerator of a fraction, we can divide by any value:

Factor or divide this expression by 2: (value1) + (value2)

$$\frac{\text{(value1) + (value2)}}{2} = \text{quotient} \qquad \text{mathematically:}$$

$$\text{(value1) + (value2)} = 2\,\text{(quotient)} \qquad \text{: a factored from of the multinomial dividend}$$

$$\frac{\text{(value1) + (value2)}}{2} = \frac{\text{value1}}{2} + \frac{\text{value2}}{2} = \text{quotient}$$

We see that the left side expression is equal to that of the right side of adding the like fractions with the same denominator, hence this division or factorization process of the numerator is essentially the reverse of adding like fractions together. It can also be said that a multinomial divided by a monomial or single term expression is equal to each term of that multinomial divided by that monomial.

Ex. Divide (8 + 6) by 2.

We see that this is a multinomial (8+6) divided by a monominal (2).

$$\frac{(8+6)}{2} = \frac{8}{2} + \frac{6}{2} = 4 + 3 = 7 \qquad \text{checking:}$$

$$\frac{(8+6)}{2} = \frac{14}{2} = 7$$:checks , multiplying the divisor and quotient together as a check of the dividend:

$(2)(7) = 14 = (8+6)$: checks

A good way, if possible, to factor a several term expression is to is to divide it all, or each term, by a factor that is common to all the terms of that expression. Usually, when greater simplification is concerned, the largest, greatest, or highest common monomial factor (HCMF) of all the terms is chosen as the divisor.

The result after canceling common factors in both the entire numerator and denominator will be another multinomial. As similar to basic concepts of division, the product of this quotient and divisor (denominator) is equal to the original multinomial numerator or dividend. The indicated or expressed product of the quotient and divisor is formally called a factored form of the multinomial.

Ex. Factor the expression: +10+20

First, checking for any common factors, we find that the highest common factor (HCF) of or within all the terms is 10. Other examples of common factors of both terms are 2 and 5. Since 10 is larger, dividing the entire expression by 10 we have:

$$\frac{+10+20}{+10} = \frac{+10}{+10} + \frac{+20}{+10}$$: dividing each term by 10, or "factoring 10 out of each term"
: here 10 is the highest common factor to all the terms in the dividend, and:

$$\frac{\overset{+1}{\cancel{+10}}}{\underset{+1}{\cancel{+10}}} + \frac{\overset{+2}{\cancel{+20}}}{\underset{+1}{\cancel{+10}}}$$: Cancel

$$\frac{+1+2}{1}$$ Combine (the fractions):

(+1+2)

Hence $\frac{+10+20}{+10} = (+1+2)$

Expressing the product of the divisor and multinomial quotient is a factored form of the multinomial dividend:

+10(+1+2) : this is also the result after multiplying both sides of the above equation by 10 and canceling. Hence: +10+20 = (+10)(+1+2) in a factored form, and the left side is the equivalent result after using distribution on the right side.

The "golden rule of factorization" is to first factor out any factors common to all the terms. Usually, the HCMF is chosen since the resulting expression will be completely factored.

As already indicated, to factor a multinomial, you don't necessarily have to divide by a factor common (ie common factor) to all the terms, but can perform a division by any value:

Given +10 + 20 we can factor by any value such as 2, 5, 7, 9, 1.2, -5.3, or any other value needed:

$$\frac{+10 \ +20}{7} = \frac{10}{7} + \frac{20}{7} = 1.4285\underline{7} + 2.8571\underline{4} = 4.2857\underline{1} \text{ hence:}$$

$+10 +20 = 7\,(4.2857\underline{1})$: = 29.99999$\underline{7}$ = almost 30.0 which is the correct result. A slight error of only (30.0 - 29.999997) = 0.000003 is due to rounding. This error value could be indicated with a negative sign since the result we calculated of 29.999997 is 0.000003 less than the true result. You can also rather use this formula to find the error value, and the sign will be automatically placed:
(calculated value) - (true value) = (error value)
The sign of the error value will be in reference to the calculated value being more (+), or less (-), than the true value.

FACTORING A BINOMIAL OF PERFECT SQUARES

To factor a binomial (a two term expression) that is composed or consists of terms that are both perfect squares (indicated with an exponent, or not), one, and only one, of the terms must be a negative term. The expression is therefore much like a difference of perfect squares. The result of factoring a binomial of perfect squares is a product of two binomial factors where one binomial factor contains a sum of the terms square roots, and the other binomial factor contains the difference of the terms square roots. This is a "special factorization" that you should try to memorize so as to recognize it and-or quickly solve some math problems when it can be applied, and including the reverse of this procedure. Symbolically, this concept can be expressed as:

(squared value1 - squared value2) =

(square root value1 + square root value2) (square root value1 - square root value2) : factored form of the binomial of perfect squares

Ex. Factor $(3^2 -9^2)$: both terms are (indicated or expressed) squares of a value, here 3, and 9
The second term (-9^2) is the negative term, or it could be said that the entire binomial expression is a difference of perfect squares.

$(3^2 -9^2) = (+3 +9)(+3 -9)$: the binomial of perfect squares factored, continuing:
$(3^2 -9^2) = (+12) \quad (-6) = -72$ checking, using distribution:

$(+3+9)(+3-9)$ expressing the distribution of the factors:
$+3(+3) +3(-9) +9(+3) +9(-9)$ distribute:
$+9-27+27-81$ combine, note that the "counter part" terms of -27, and +27 cancel out or sum to 0 :
-72 checking, using the order of operations:

$(3^2 -9^2)$ ORD. step 1, simplify powers and roots:
$(9-81)$ combine:
-72

If $(3^2 -9^2)$ was initially expressed as $(9 - 81)$ you would have the same result, and a general or algebraic representation for the above procedure would be:

$\mathbf{(a - b) = (\sqrt{a} + \sqrt{b})(\sqrt{a} - \sqrt{b})}$: **Formula for factoring a binomial of perfect squares.** You can check this by using distribution in the right side expression:

$$\sqrt{a}\ \sqrt{a} + \sqrt{a}\ (-\sqrt{b}) + \sqrt{b}\ \sqrt{a} + \sqrt{b}\ (-\sqrt{b})$$

$$a - \sqrt{a}\ \sqrt{b} + \sqrt{a}\ \sqrt{b} - b$$ combining like terms:

$$a - b$$

In short, a difference in terms could be viewed or thought of as equivalent to the product of the sum of the terms square roots and difference of the terms square roots. Actually, in more simplified words, and perhaps where the above mathematical derivation stems from, it could be said that the product of the sum and difference of two values is equal (=) to the difference of those two values when they are both squared. A general symbolic or algebraic representation of this might look like:

$\mathbf{(a^2 - b^2) = (a + b)(a - b)}$: **Factored form of a binomial (a sum of two terms) of the difference between two perfect squares**

Ex. Given (8+2)(8-2) , this can be expressed in simpler terms with the values squared as:
Expressing each value as the square of their square roots:
(8^2 - 2^2)
(64 -4) = 60 : As a check, all these expressions equal (8+2)(8-2) = (10)(6) = 60.

Ex. Factor (25-1)

First, there are no common factors here except 1 of course. Each value (term) here is (considered to be, or can be considered as) a perfect square, and therefore, there is the expressed difference (ie. has the - symbol) of perfect squares.

Note, actually the term of: -1 is not a perfect square of 1, it is though a perfect cube of (-1):

(-1)^3 = (-1)(-1)(-1) = (+1)(-1) = -1

Hence, when factoring binomials of perfect squares, the sign of the negative term is meant to be taken as meaning the difference of positive values, or adding the negative of a positive value. For the above example, "think of"
(25-1) as (25-(+1)), which are equivalent.

(25 -1) with 5 as the square root of 25, and 1 as the square root of 1:
(5^2 - 1^2) : expressing the binomial of perfect squares, simplifying:
(5+1)(5-1) : the factored form of the initial binomial. Here, we have the product of the sum and difference of the square roots of the terms of the binomial of perfect squares.
(6)(4)
24 checking:

(+25-1) combining terms:
+24

or

(+5 +1)(+5 -1) distribute:
+25 -5 +5 -1 combine:
+25-1
+24

As an extra note or thought, when two binomials (here, as a factored form of the difference of two squares) differ only in sign as shown above, they are often called **conjugate** forms of each other. Do not confuse the above discussion with the square of a binomial which is usually expressed in the form of: (a + b)^2 or (a - b)^2

COMMON FACTORS CONTAINING THE SAME TERMS WITH DIFFERENT SIGNS

Sometimes, common factors to both the entire numerator and denominator of a fraction appear that they cannot be canceled because all the signs of the terms within one factor are not the same as the corresponding terms of the other factor. That is, all the corresponding terms are opposite in sign. A method to overcome this problem will be shown below.

Ex. Simplify $\frac{(+5+3)}{(-3-5)}$ By use of the commutative law of addition, this can be expressed as:

$\frac{(+5+3)}{(-5-3)}$ After factoring (-1) out of each term in the denominator, and expressing the factored form or product:

$\frac{(+5+3)}{-1(+5+3)}$

$\frac{\overset{+1}{\cancel{(+5+3)}}}{-1\underset{+1}{\cancel{(+5+3)}}}$: now we can cancel, using the common factor of (+5+3), and then distributing in the denominator:

$\frac{+1}{-1}$:clearly, the result here should be a negative value since the signs of the numerator (or dividend) and denominator (or divisor) are different.

$\frac{\overset{-1}{\cancel{+1}}}{\underset{+1}{\cancel{-1}}}$:canceling to clear the fraction of its denominator and the ridding of the fraction. Here, both the numerator and denominator are divided by -1. This will leave a standardized or commonly expected value of +1 in the denominator. In short, any value divided or multiplied by (-1) is the "(number line) inverse" or signed number counterpart of that value. For example: -1 (-1) = +1, and -1/(-1) = +1

$\frac{-1}{+1} = -1$: since any value divided by 1 is that value, or: (+1)(-1) = -1 when checking by multiplying the divisor and quotient to produce the dividend.

Checking, here the numerator and denominator expressions will be simplified first:

$\frac{(+5+3)}{(-3-5)} = \frac{+8}{-8} = -1$: checking (-8)(-1) = (-1)(-8) = +8 , "the negative of negative eight, is positive eight"

The initial problem above could have been solved, perhaps more easily, by "factoring out" or dividing all the terms of the numerator by -1. This would be similar to the concept of dividing a multinomial by a monomial (and here it would be -1), and expressing the product of that monomial divisor and multinomial quotient:

(+5 + 3) = -1 (-5 -3) and therefore

$\frac{-1\overset{+1}{\cancel{(-5-3)}}}{\underset{+1}{\cancel{(-5-3)}}} = -1$: note that canceling does not mean to eliminate completely or to make it a value of 0, but rather make it a value of 1, which does not change anything when using it as a factor or divisor.

Ex. $\frac{(+5-3)}{(-5+3)} = \frac{\overset{+1}{\cancel{(+5-3)}}}{-1\cancel{(+5-3)}} = \frac{+1}{-1} = -1$ checking: $\frac{(+5-3)}{(-5+3)} = \frac{+2}{-2} = -1$

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AVERAGE AND STANDARD DEVIATION

Average, or average value, which has been previously discussed in this book, is a basic requirement for understanding the concepts of both average deviation and standard deviation. Average gives us a single value to represent a set of data, but it does not guarantee that the data values used are even close to that ("central") average value. Are the date values dispersed (set apart or away from, their difference) close in value to the average, or are the data values dispersed far away in value from the average? The standard deviation value was defined to answer this question so as to get a better idea of the actual values of a given data set.

Here is a simple example to observe:

Data Set 1: 48.5, 50.5, 51
Data Set 2: 2, 50, 98

The average of both data sets above is the same value of 50. However the dispersion or deviation of data values from the average value is clearly greater (in difference) in the second data set. 2 and 98 are not very close to 50, as compared to the data values in the first data set. How far are they from the average? The answer is the amount they differ (as in a difference value) or deviate away from the average:

data - average = deviation : the (signed) deviation that a data value is from the average value, The sign of this deviation indicates if the deviation was greater (ie.: +deviation_value) or less (ie.: - deviation_value) than the average value.

Though the sign of the deviation is not required for the computation of the average and standard deviation values, the sign with its corresponding deviation can be used elsewhere to indicate if that deviation value is less than (- , in sign) or greater than (+ , in sign) the average. The deviation with it's sign is the (pos. or neg.) change or difference from the reference value, and here, it is the average value of all the data. For example, if the average of a given set of data was 10, and one of the data values was 7, that data value has a deviation of:

data - average = deviation
7 - 10 = -3 : the negative deviation indicates that this data value is three less than the average value of the data set. This value can be said as being "below (the) average (value)".

If you find out the deviation from the average for each member of a set of data, you could sum up all the deviations and divide by the number of deviations (or number of data values of the set) to find the average deviation for that set of data (values, "members" or "elements"). This seems to be a correct method but it will not work (without an adjustment) due to the mathematical fact that the average of the data has created, in which the sum of deviations below (less than) the average, plus the sum of deviations above (greater than) the average is always equal to 0. To overcome this, an adjustment is used, and that is to simply use the absolute (ie. unsigned, or positive) value of each deviation. Mathematically, when a value is placed within two vertical lines (ex. | value |) it means to take the absolute or signless value, whether the value is positive or negative. The average value of a set of data (values) may be either positive or negative in value, but the average deviation is considered as a signless value, or as a generic or basic "distance" value from the average in order to represent all the deviations of the all the data values. Due to using only the absolute value of the deviation, you may optionally use this next formula that still gives the deviation a sign, but it wont be used in the formal average deviation formula shown below: average - data = deviation

$$\text{A.D.} = \text{Average Deviation} = \frac{\text{sum of } |\text{ deviations }|}{\text{number of data values}}$$: **AVERAGE DEVIATION , A.D.**

Ex. For the first data set above, the (signed) deviations are:

data - average = deviation

48.5 - 50 = -1.5, use for example: | -1.5| = 1.5 when finding the A.D.
50.5 - 50 = +0.5
51 - 50 = +1.0

The average deviation is:

$$\text{average deviation} = \frac{1.5 + 0.5 + 1}{3} = \frac{+3}{3} = 1$$

: Average, and average deviations are mathematical concepts of the entire data considered, and it does not necessarily mean that any, or all data value(s) are equal to the average or average deviation value.

Ex. For the second data set above, the deviations are:

2 - 50 = -48 use: |-48| = 48 when finding A.D.
50 - 50 = + 0
98 - 50 = +48

The average deviation is:

$$\text{average deviation} = \frac{48 + 0 + 48}{3} = \frac{96}{3} = 32$$

Standard deviation (S.D.) is close in value to the average deviation, but the standard deviation is more like an expectation or prediction, rather than an actual mathematical fact like average deviation. Standard deviation is the deviation that most data values could have from the average deviation, hence standard deviation could be called or thought of as something like: "the most likely deviation".

For example, given the data set of: -5, +5

,the average is 0 and it is easy to see that most values (hear, actually all the values) deviate from it by 5.

Given the data set of: -5, 0, +5,

,the average is 0 again, and most values (the -5 and +5) deviate from it by 5.

Given the data set of: -5, -5, -5, -5, -4, +4, +5, +5, +5, +5

,the average is 0, and most, but not all, values deviate from it by 5. You should expect the standard deviation value to be close to 5. Let's derive the formula for standard deviation:

The derivation of standard deviation is similar, at first, to average deviation:

1. For each data value, find its deviation from the average.
   average - data = deviation

2. Rather than take the absolute value of negative signed deviations for their effective positive value, square each deviation since any value squared is positive in sign. This also has the effect of magnifying a larger deviation (at least >1) much more than a smaller deviation. Small deviations (particularly those less than 1) are therefore very small in value and practically eliminated since any fractional value squared is a smaller value since a "fraction of a fraction" yields a smaller value.

3. Find the average of the squared deviations:

$$\text{average of the squared deviations} = \frac{\text{sum of squared deviations}}{\text{number of data items} - 1}$$

Notice 1 is subtracted from the total number of data items. This is due to the fact that one of the data items is predicted to be at least near or equivalent to the average, and that this data value should not be counted (included) in the set since it does not deviate from the average and therefore cannot play much of a role in any expected or predicted deviation. Remember, we are not looking for a true mathematical value as like average deviation, but we are looking for a "most-likely" value. If one is not subtracted, you will have another form of standard deviation that yields a "compromise" value that is a value between the average and standard deviation values of a set.

4. Since we squared the deviation values in the formula, the result is essentially another squared value, and we need to do the inverse (or reverse if you will) and take its' square root to effectively arrive at the square root value before it was squared:

$$\text{Standard Deviation} = \sqrt{\text{average of the squared deviations}}$$:**STANDARD DEVIATION (S.D.)**

For example, given the set of:

-5.0, 0.1, +5.0

The average of this set is +0.033333333.

By observation, you could then say that most values have a deviation, from the average, of about 5. and that the standard deviation for this set should be about 5.

The deviations are:

-5.033333333, +0.066666666, +4.966666666

The deviations squared are:

25.33444444, 0.004444444, 24.66777777

Using the formula shown above for the average of the squared deviations (using data items -1), we get:

25.00333333

Taking the square root of this we arrive at:

standard deviation = 5.000333322

Notice that this value is slightly larger than the predicted value of 5. This is due to the "greater magnification" of the larger deviation values after squaring them.

For comparison purposes:

$$\text{average (unsigned) deviation} = \frac{5.033333333 + 0.066666666 + 4.966666666}{3} = 3.355555555$$

For the data set given, most values had a deviation of 5 from the average value, and the standard deviation value yields a better deviation value for most of the data than the average deviation value shown here does.

The "compromise value", as discussed above, is calculated to be: 4.082755061

SCIENTIFIC NOTATION

Scientific Notation (S.N.) is a standard numerical notation to expresses any numbers, and in particular, very large or small numbers that have many digits where its value perhaps can't be easily realized or worked with. The notation still uses the same digits values of that number, but has only one whole number between, or including, 1 and 9, and with all the fractional (less than one) digits appended on, and a multiplying power of 10 attached to the number. This notation helps with manually processing calculations, or for expressing the results of calculations to make them appear presentable and intuitively manageable. Since numbers expressed in scientific notation contain powers of 10, the rules for multiplying and dividing like values with exponents, such as with powers of 10, will allow for greater agility and speed for calculating or estimating some results.

To convert a number greater than one to scientific notation, move the decimal point to the right of the most significant digit (MSD), and the multiplying power of 10, to this newly created lesser value, will have a positive exponent equivalent to the number of digit positions ("decimal places") it was moved. Any trailing zeros can be omitted.

Ex: 70 = 7.0 (10^1)
Ex: -3579.2 = -3.5792 (10^3)

To convert a number less than one to scientific notation, move the decimal point to the right of the most significant, non-zero digit (MSD), and the multiplying power of 10 will have a negative exponent equivalent to the number of digit positions the decimal point was moved. Any trailing zeros can be omitted. Note, that since the number was made apparently larger, there must be a value that would decrease or divide it so that it would be equivalent to its original (lesser) value from which it was created or derived from. This value is the negative power of ten which actually means division by an equivalent positive power of ten.

Ex: -0.00123 = -1.23 (10^-3)

Ex. Convert 35,000 to scientific notation (SN)

35000 = 3.5(10^+4)

checking: 3.5(10^+4) = 3.5(10,000) = 35,000

Ex. Convert 50.0250 to scientific notation

50.0250 = 5.0025(10^+1)

checking: 5.0025(10^+1) = 5.0025(10) = 50.025

Ex. Convert 0.0047 to scientific notation

0.0047 = 4.7(10^-3)

checking: 4.7(10^-3) = $\frac{4.7}{1}\left(\frac{1}{(10^{+3})}\right) = \frac{4.7}{1000} = 0.0047$

Ex. Multiply 20,000 by 3,000

(20000)(3000)
2(10^+4)3(10^+3) : each factor represented or expressed in scientific notation
(2)(3)(10^+4)(10^+3) : associative law

$6(10^{(+4+3)})$: when multiplying like values or variables, add their exponents
$6(10^{7})$: 10^{7} effectively places 7 zeros after the value and moves the decimal point seven places rightward:

60,000,000 = "sixty-million"

Another method to convert decimal values to fractional values and vice-versa, is to use some of the concepts of scientific notation.

Ex. Convert 0.247 to an equivalent (proper) fractional form

$0.247 = 2.47(10^{-1}) = \frac{2.47}{1}\left(\frac{1}{10^{+1}}\right) = \frac{2.47}{10}$: "two point forty-seven, tenths", OR:

$0.247 = 24.7(10^{-2}) = \frac{24.7}{1}\left(\frac{1}{10^{2}}\right) = \frac{24.7}{100}$: "twenty-four point seven, hundredths", OR:

$0.247 = 247(10^{-3}) = \frac{247}{1}\left(\frac{1}{10^{3}}\right) = \frac{247}{1000}$: "two-hundred and forty-seven thousandths"

Note, as seen above, the first factor in the indicated products need not be between and including 1 and 9, as in true scientific notation, as long as the power of 10 is adjusted correctly and the resulting expressed values are equivalent. This type of (exponential) notation is then a modified form of the standard scientific notation.

Ex. Convert 52.3/100 to scientific notation, and then convert this equivalent scientific notation to its equivalent decimal form.

$\frac{52.3}{100} = \frac{52.3}{10^{+2}} = 52.3(10^{-2})$: = 0.523

$52.3(10^{-2}) = 5.23(10^{+1})(10^{-2}) = 5.23(10^{(+1-2)}) = 5.23(10^{-1})$: scientific notation

$5.23(10^{-1}) = \frac{5.23}{1}\left(\frac{1}{10^{1}}\right) = \frac{5.23}{10} = 0.523$: decimal notation. Hence the same as the above, but without any multiplying factor such as the power of 10.

checking: $\frac{52.3}{100} = 0.523$

Scientific calculators and computers often represent values in a modified scientific notation format with a symbol of (E), meaning "exponential", instead of indicating or displaying the common base value of 10. The exponent (of 10) will follow the (E) symbol.

Ex. If the calculator display is: 1.234E-12
This is to be interpreted as: 1.234E-12 = $1.234(10^{-12})$ = 0.000,000,000,001,234 : "1.234 pico"

This notation is very helpful since the calculator may only have 8 or 10 digits. It should also be pointed out that these digits of the calculator display are for only the significant digits of the result of an operation, and any lesser valued. least significant digits not capable of being displayed, stored, processed, or entered by the user, are essentially lost and won't be available for further calculations. If the calculator "truncates" or discards some least significant digits of a value, this "error" or deviation from the true value can eventually "build up" (increase in value) and may need to be considered when using that value (with some slight, or insignificant error) for any possible further calculations.

MULTIPLYING AND DIVIDING BY NUMBERS TOO LARGE FOR AN ELECTRONIC CALCULATOR

Most hand-held electronic calculators are either the 8 to 10-digit display types, hence they are limited to that maximum amount of digits, and therefore, any value or calculated result with more digits than this maximum amount of digits cannot be fully entered or displayed. The value, as entered, is too large for that specific calculator to use and work with (ie., process). The least significant digits will then be lost, and the result will usually be rounded at the least significant digit that is displayed. A loss of trailing digits and fractional amounts can eventually lead to incorrect accuracy of the result(s) if this "shortened" or approximated value is used as an operand for further operations. In general, multiplying an operand having N1 (some number of) digits by another operand with N2 digits will yield a product with about (N1 + N2) digits, give or take 1 digit, and which may be more than your calculator can display. If the product is too large for your calculator to display, your calculator will either signal an "overflow error" (the value is greater than the maximum value that can possibly be displayed on and-or processed within that calculator), or it may display a "scientific" (exponential, power of 10) form with only the most significant digits and "dropping" (disregarding, truncating or loosing) the least significant, "last" digits. For most cases this wont be a problem, and is acceptable. Below are some basic examples of how to overcome this number of digits limitation if you need to. These examples assume that your calculator is an 8-digit (for internal calculations, and display) type and you want to maintain the least significant digits within a calculation using that calculator. However, as will be shown below, you may need to perform a final calculation by hand.

Ex. (7)(4321.567895) : the second factor contains ten digits, and this can be expressed as:
(7)(4321 + 0.567895) distributing :
(7)(4321) + (7)(0.567895)
30247 + 3.975265 summing by hand (ie. the calculation performed by you using a pen on paper) :
30250.975265 : an eleven digit result

Ex. (5) (1234567891) : the second factor contains ten digits, and this can be expressed as:
(5) (1234500000 + 67891)
(5) ((12345)(10^5) + (5) (67891))
(61725) (10^5) + 339455 multiplying by observation :
6172500000 + 339455 summing by hand :
6172839455 :a ten digit result

Ex. Divide 3 by 10,000,000,000 factoring 10,000,000,000 into two (or more) factors:

$$\frac{3}{10{,}000{,}000{,}000} = \frac{3}{(1{,}000{,}000)\,(10{,}000)} = \frac{0.000{,}003}{10{,}000} = 0.000{,}000{,}000{,}3 = 3(10\text{^}{-10}) = \frac{3}{10\text{^}10}$$

The example above can be solved in a variety of ways, and specifically determined by the powers of 10 used.

METRIC SYSTEM OF UNITS

The sub-units (ie., fractions of a unit), and multiples (ie., powers of a unit) of a unit of measurement within the metric system are all based on 10 of the specific "base" or fundamental units in use, such as for example, the fundamental unit of length called a meter. Why 10? The answer is that since the decimal system that we count with is based on 10, there should then be a standard system of units that are also based on 10. The word "metric" is based on the words and concepts of "to meter", which means "to measure", and "numeric". The metric system was created to be a practical, world-wide accepted standard system of measurements, and so as to avoid the many, and with some being antiquated (old and-or unused), systems of measurements in use which can cause many problems for those not familiar with them and-or the conversions between two or more different measuring systems. The metric system was created in France in 1795 as their new standards of measurement.

Type of Measurement	Base units in the U.S.A. system of measurements (Many are also now using the metric system.)	Base units in the metric system of measurements.
Distance or Length	mile or feet (mi)	meter (m)
Weight	pound (lb)	Newtons (N) for weight or force, and gram (g) for mass or matter
Volume	gallon (gal)	liter (L)

The Appendix Section of this book contains many conversions and constants.

Within the metric system of measurements, to change from larger to smaller units (technically called a reduction of units within the system, and commonly (although somewhat incorrectly) referred to as a "conversion" of units which is actually for converting between units of two different measuring systems and their units) that are some multiple or fraction of the base, fundamental or standard reference units, simply multiply or divide by a power of 10. The same process of simple multiples or division of units cannot generally be said about the English or British system of units. For example, in reference to the English length measurement units, 1 foot is 12 inches, 1 yard is 3 feet, 1 mile is 5280 feet. Here, the (conversion) factors vary widely: 12, 3 and 5280, and therefore, there is no systematic or logical approach to the relation of the units within that measurement system. One advantage of anything based on 12 units, such as a dozen or 12 inches, is that it can be evenly divided by or into 2 (a half), 3 (a third), 4 (a fourth or quarter), and 6 (a sixth) smaller fractional parts or fractional units. Half of (1/4) will give (1/8), half of the (1/8) will give (1/16), and half of the (1/6) will give (1/12). A system based on 10 units can only be evenly divided by 2 and 5, and with half of 2 being 1 step or unit.
.
In the metric system, the powers of 10 of the derived units are frequently represented as word or alphabetical abbreviation prefixes before the base unit. These prefixes are usually written as abbreviations of the corresponding name given to that power of 10. For example, all the units of measurement for distance or length values are based on, or are in reference to, the unit of measurement called the meter (m). Hence the base units for length are called meters. A centimeter (cm) is defined as one-hundredth of a meter. A centimeter is a sub-unit or fractional unit and part of a meter. The word or unit prefix "centi" (as previously discussed with the concept percentages) numerically means the multiplying factor of: (1/100), "one-hundredth", or= (1)(10^-2), and its abbreviation is (c). Thus, 1 centimeter = 1cm = (1)(10^-2)m = 0.01m, which is one-hundredth of a meter. There are 100 centimeters in one meter. In other words, 1 meter is equivalent in length as 100 centimeters, and vice-versa. A kilometer (km) is one-thousand meters. The prefix kilo (k) means (1000) = (1)(10^+3) = 10^3. 2km = (2)(10^3)m = (2)(1000)m = 2000m.

Here's an example of a conversion (although it's technically or formally a "reduction") within the metric system:

Since there are 1000 meters per kilometer, (1000m/1km), or one kilometer per 1000 meters, (1km/1000m), to convert meters to kilometers, simply divide the quantity of meters by 1000. In short, when converting from smaller units to a larger units, use division, otherwise, use multiplication when converting from larger units to smaller units since there is many more smaller things within a larger thing such as larger sized units of measurement. Here is a mathematical verification and example:

$1000m = 1km$: this is the known, given and basic equivalence between these two units of measurement. Solving for (1) meter by dividing both sides of the equation by 1000:

$$\frac{1000m}{1000} = \frac{1km}{1000}$$ after canceling common factors:

$1m = \frac{1}{1000}$ km : "one meter is one-thousandth of a kilometer". This can also be expressed as: 1m = 0.001km

Multiplying each side by some number (N) such as some number or multiple of meters:
N (1 meter) = N (1) meters = N meters, and we must do the same to both sides to keep their equivalence:

$$(N)1m = N\left(\frac{1}{1000}\right)km = Nm = \frac{N}{1000}\ km = (0.001\ N)\ km$$

or:
$1000m = 1km$
$N\ (1000m) = N\ (1km)$
$(1000N)\ m = N\ km$

Hence. when converting (some number or multiple of) meters to kilometers, divide the magnitude (value) of meters by 1000. For example, to convert 4752 meters to it's equivalent distance with kilometers units:

$$4752\ m = \frac{(4752)}{(1000)}\ km = 4.752\ km$$

You may also wish to consider this method which considers the ratio of one unit to that of the other to find the proper multiplier to use when converting between those units. With the division of like or similar values rules, the process to find the multiplier is easy, such as by subtracting the exponents. Consider this example of converting centimeters to kilometers:

$$\frac{1\ cm}{1\ km} = \frac{10^{-2}\ m}{10^{+3}\ m} = \frac{10^{(-2 - (+3))}}{1} = 10^{(-2-3)} = 10^{-5}$$

: shown with some intermediate steps.
: the units essentially cancel out or are eliminated since its a strict numeric, or unit-less value, here just a ratio.

Now, mathematically, (dividend = quotient x divisor), and from this equation above we have:

$1cm = 10^{-5}\ (1\ km)$: the multiplier is: $10^{-5} = 1/100{,}000 = 0.000{,}01$
$1cm = 0.000{,}01\ km$

Multiplying both sides by N, as for some given number of centimeters:

$N\ cm = (0.000{,}01\ N)\ km$ mathematically, we also find:

$$N\ km = \frac{N cm}{0.000{,}01} = \frac{N cm}{10^{-5}} = N\ 10^{5}\ cm$$

Ex. $$1km = \frac{1cm}{0.000{,}01} = 1(10^{5})cm = 100{,}000\ cm$$

Since the metric standards were created, many of the units have also been modernly, well defined scientifically, and in often in terms of universal constants so as to improve their precision and standard reference accuracy. An example to express why this needs to be done is: suppose the Earth gains more mass due to cosmic dust and meteorites, it will then have more gravity, and the reference kilogram of mass will weight more, but its mass is still the same.

Here are the common metric prefixes for the base-units of measurement:

tera(T) = 1(10^+12) = 1,000,000,000,000 : trillions
giga (G) = 1(10^+9) = 1,000,000,000 : billions
* mega (M) = 1(10^+6) = 1,000,000 : millions
* kilo (k) = 1(10^+3) = 1,000 : thousands Ex. 5km = 5 (1000)m = 5000m
hecto (hc) = 1(10^+2) = 100 : hundreds
deca (dc) = 1(10^1) = 10 : tens (of units)

1 = 1 : "ones" or "whole", complete or entire units,
no unit prefix used, but explicitly indicated
deci (d) = 1(10^-1) = 0.1 : tenths (of a unit)
centi (c) = 1(10^-2) = 0.01 : hundredths Ex. 100cm = 100 (0.01)m= 1m
* milli (m) = 1(10^-3) = 0.001 : thousandths
* micro (u) = 1(10^-6) = 0.000,001 : millionths
nano (n) = 1(10^-9) = 0.000,000,001 : billionths
pico (p) = 1(10^-12) = 0.000,000,000,001 : trillionths

* Note that these (*) indicated base-unit prefixes differ by a factor of 1000, and including the prefix centi, these prefixes are perhaps the most commonly used. A discussion on CHANGING UNITS given further ahead in this book shows how to make conversions between the Metric and English systems of measurement, or within a single particular system.

The base unit of mass (a measure of the amount of actual physical substance [like atoms], matter or material) in the metric system is the gram (g). The word of "gram" is rooted in words of: "grain", "measure", "measurement", and "weight", and originally meant a small weight or piece of something. A gram of water is defined as having the measurement of one cubic centimeter (1 cc, a volume measurement, ie., "space" or "three dimensional spacial area") when its temperature is just above the freezing temperature of water. 1000 grams of matter or a substance is equal to 1 kilogram of that mass: 1000g = (1k)g = 1kg = (10^3)g. 1cc volume of water is defined as equal to 1g of mass of water. 1cc of other substances will be more or less grams of mass since their density (mass/volume = amount of matter per volume) is different than that of water. 1000cc volume of water is defined as having 1000g of mass = 1kg of mass. A cube shape, 10 centimeters long, 10 centimeters wide, and 10 centimeters high is (10cm x 10cm x 10cm) volume = 1000cm^3 = 1000 cubic centimeters = 1 liter unit of volume size.

For spacial, space, or volume measurements, the liter (L) is the base unit of measure in the metric system. 1liter = 1L = 1000cc. 1 liter volume of water is defined as having 1kg of mass, and has a "proportional, corresponding or associated weight (technically a force)" indicated as 1kg on a weight scale, however, 1 liter volume of some other substance like copper, gold, zinc, sodium, lithium, plastic, foam, etc., will have a "corresponding weight" of more or less than that of water, due to that the (mass) density (= mass/volume) of that particular substance is not the same as the density of water which is 1gram/1cc.

One-thousandths of a liter = 1L/1000 = 1000cc/1000 = 1cc = 1(10^-3)L = 1 milli-liter = 1ml

For conversion purposes:

1 fluid ounce , water = **1fl. oz** , essentially a volume oz measurement, = **29.5735cc** = 29.5735 grams if water (Often in the U.S.A food industry, 1 fl. oz. is set as equal to an even 30 grams of "corresponding scale weight".)

1 pound = 1lb , a weight measurement, = 0.453592 kilograms = 453.592 grams
1 ounce = 1oz , a weight measurement, = 1lb/16 = (1/16)lb = 0.0625lb = 453.592g/16 = 28.3495 grams

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SECTION 2: BASIC ALGEBRA

ALGEBRA

Algebra is a form of mathematics where symbols or letters are used to (symbolically) represent numbers. The English word "algebra" is just a couple of hundred years old and is derived from the older Arab word al-jbar. The basic interpretation of this word is: "the combining of several things, so as to have one thing". Simple algebraic-like methods have also been used since ancient times, primarily by the Egyptians, Greeks, and Hindus. This mathematical "invention" or knowledge could be said as always being someplace within the ability of humankind right from the start of our first basic civilizations, and therefore it was surely destined to happen, grow and "sprout up" like a nurtured seed as time went by, and to this day, it is still growing and being useful.

When a value remains the same and-or cannot change in value, that number is said to have a constant value, and it is therefore called a **constant**. When a number is not a constant or a specific known value, or that number is an unknown value yet to be known and-or solved for, it is or has a value that can variate (from the words vary (changeable), and various (several)) and which means that it is non-specific, non-constant, and that it can be one of many possible values, and-or change in value. The symbolic names, letters, identifiers, representatives or any other symbolic placeholders for numbers that can vary in value and-or are yet unknown in value are therefore called **variables**. Variables are often found in mathematical expressions and equations. Constants can also be given symbolic identifiers or names. On of the most common natural constants is Pi ("pie") which equals about 3.141592654 and is the result of any circle's circumference length divided by the diameter length of that circle.

If the temperature of something can variate, fluctuate or change in value, we can use something like the word: "Temperature" itself since it makes a good identifier for the value it represents. Simply using the letter T is more compact and usually acceptable, and when it is understood as to what it represents. Temperature or T, or some other identifier, can be used as a symbolic placeholder yet to be set to a specific numeric value and-or for an updated numeric value of the temperature.

Consider this analogy: Within a box, the current temperature is written on a piece of paper. That box is essentially a placeholder for the temperature value, and this box could be identified and labeled with the letter T on it. That box and value it contains within it can also be mathematically represented and expressed with that identifier of T. Since the temperature can fluctuate or vary in its value, the identifier of T is a therefore also called or identified as "variable T". T is now understood to identify, represent, and mathematically express and be equivalent to a varying and-or unknown value of the temperature.   The value of T can be set (assigned, written), changed (updated, adjusted), or read (viewed, obtained), since it is a placeholder for a value.

T = this letter or symbol identifies and represents both the temperature and its corresponding numerical value. If the temperature can vary or change in value, T is called a variable, and is specifically known as:"variable T". If the temperature measurement was 70 degrees, T will be set or assigned with this value of 70:   temperature = 70, and since T = temperature, T = 70.

To use identifiers or variables as equal to, and as if they are, actual numbers is very helpful and useful in mathematics. For example, unknown values can therefore be expressed and mathematically worked with as if they were a (literal, actual or specific) numeric value. Another advantage is that variables don't then need to be a specific numeric value so as to be mathematically useful, and it is understood that the variable is a general representation of one of many possible and-or changing values. Note that even though a variable can have a changing value, it is still only equal to one specific value at a time or measurement, and it should be said, indicated or understood that its value is a current (as of now, or at this time) value. Even after saying all this, it is still possible for a variable to have the same value for quite a long time, and therefore it acts or appears as if it were a (unchanging) constant.

Mathematical operations involving variables are performed in a manner very similar to using actual numeric values. It is easy to work with variables when you begin to "think of" or treat them just like actual numbers. Some examples of variables are: A, B, C, X, Y and Z. Usually, if you are to create a variable's name (symbol(s)), or "identifier" (often used as a general name for any variable as for computer languages), and as previously mentioned, it is best to choose one that will be descriptive to what the value represents. Sometimes, rather than use A or L to represent length, it is better to use a descriptive name like: LENGTH since this will be helpful for your future reference and-or for anyone else who sees it used in a formula or calculations.

Two or more variables written side by side are considered as being multiplied together. For example, AB means "A times B", (A x B) , or (A)(B). Since multiplication is commutative, the order of performing or expressing it does not matter. for example: AB = BA. A value in front of a variable(s) is called it's numerical coefficient (ie. cofactor). A numerical coefficient is a multiplying factor of the indicated product. For example, 2A = (2)(A). Due to multiplication being associative, the numerical coefficient can be, or perhaps was previously, associated with any other factors of that term of which it is in. For example, 2AB = 2(AB) = (2A)(B) = A(2B) = AB2. For conformity, the numerical coefficient is usually placed first. If a variable does not have a numerical coefficient indicated, it is considered to have a numerical coefficient of one (1) which of course does not change its value. Likewise, for example, 2A can be considered as the coefficient of B in the term of: 2AB. This coefficient then has a numeric part and factor: 2, and a variable part and factor: A.

It is also of importance to understand that a variable, being just a placeholder, can represent any numeric value, positive or negative in sign, and may even represent the resulting value of some other expression. It should be generally understood that the actual numeric value of a variable can be either positive or negative in sign.

Ex. If variable A is equal either to: -10, this can be correctly expressed as: A = -10,
It is incorrect to express this situation with the equation of: -A = -10, that is: -A is not equal to -10. As a verification of this, if you divided both sides by -1, you will have this equation: A = 10 which we know is wrong.
It is possible to express the correct equation of: A= -10 as: -A = 10 because the equation is still in balance and you can check, determine or verify this if you divide both sides by -1, and the result is A = -10.

Ex. Total = reference value + C : here C represents some change (from and with respect to, or in reference to, the given reference value)

If C = +10, Total = reference value + (+10) = reference value +10
If C = -10, Total = reference value + (-10) = reference value - 10

Notice that with two possible signs for the value of the variable, here C, that only one algebraic formula is needed as originally stated in the example. If the formula was expressed or written as: Total = reference value - C, hence as a difference between the values, the resulting value of the equation would be different, and specifically for this equation, it would result in opposite signed values.

Another thing to be aware of with variables is when a variable is expressed as being raised to an indicated (with an exponent) power. For example:

XY^2

Above, only the Y variable is to be squared, not both (the product of X and Y) of the variables. The exponent (ie. indicated or expressed power) is applied only to the immediate variable and not to a product of variables or an expression unless those are explicitly indicated within grouping symbols as being raised to the indicated power. Perhaps this rule would be more clear if the commutative or associative law was considered, and the expression was re-written as:

From: XY^2 to= Y^2X to show that only Y is being raised to the second power. or, by using parentheses:

From: XY^2 to= X(Y^2) = (Y^2)X

If the indicated product of X and Y were to be squared, it would be noted with grouping symbols such as parentheses as:
(XY)^2 expressing or "extending" the indicated operation:

$(XY)(XY) = XYXY = XXYY = X^2Y^2$, hence: $(XY)^2 = X^2Y^2$: a product to a power, then distributed the exponent

Note that with: XY^2 , it is incorrect to think that the two factors here are just (X) and (Y). The factors of this product are actually: (X) and (Y^2).

In daily life, we often need to make what are actually, and can be expressed as, simple algebraic equations to solve a problem. Take the information from the problem stated and apply variables and constants, known or unknown, so as to form an equation to mathematically represent the situation. For example, Mike is three years older than Tom, and we can express this as an equation. Actually, this can be expressed three ways, and that each equation can be mathematically derived from each other:

Let variable M = Mike's age and
Let variable T = Tom's age

1. $M = 3 + T = T + 3$

2. $M - T = 3$: here the left side expression is a difference of the variables set equal to the constant difference value of 3.

3. $M - 3 = T$ and this is equivalent to after switching sides: $T = M - 3$

Mathematically (or "algebraically"), all of the equations above can be derived from any one of the other shown equations. Any initial choice in the equation you do create and use, and if correct mathematically such as one of these equations above, will still yield the correct result.

LIKE TERMS

Terms that have the same exact variables are called similar or like terms. The exponents of corresponding variables of two or more like terms must also be the same. That is, the indicated powers of a variables must also be the same so as to still represent the same numeric value. The numerical coefficients or multiplying factors of like terms do not need to be the same.

Examples of like or similar terms:

AB and BA
A^2B and BA^2
5X^2 and 2X^2 : here, the numerical coefficients are different.
4XYZ and 3YZX : Often, you will want to arrange these terms so that they appear more visibly as being like terms. Ex., 4XYZ and 3XYZ , or any other possible way.

If A=10, and B=2: AB = (10)(2) = 20 and BA^2 = (2)(10^2) = (2)(100) = 200 = A^2B

Clearly AB and BA^2 or = A^2B are not like terms since they represent different values.
Here, 20 is not equal to 200. AB and A^2B are not like terms.

ADDING LIKE TERMS

"Like" or similar terms can be combined (added, summed). For example, 2X + 3X is a two-term (algebraic) expression. The terms +2X and +3X are like terms (both have the exact same variable(s), here X, including their indicated powers, and therefore can be combined just like being the same numeric value to produce a multiple of that value.

Given +2X and +3X, how do we combine these (like) terms?

First notice that a common factor (in both terms) of X can be divided (ie. factored) out of each term:

2X + 3X = X(2 + 3) = (2 + 3)X = (5)X = 5X

If you had 3 of a certain value, and also 5 of that same value, you then have (3+5) = 8 of that value, and which also means 8 times that value.

Hence, **to combine like terms, combine their coefficients, and keep the same variable(s).**

Here is a check using the concept of multiplication to represent repeated addition:

(2X) + (3X) "extending" the indicated products and expressing their "partial sums" or terms:
(X + X) + (X + X + X) distributing (+1) to clear the grouping symbols:
X + X + X + X + X using multiplication to represent this repeated addition of X, here five times:

X5 or = 5X : The left hand side is the "common mathematical form" of expressing this result, and it has the form of a number or variable to be multiplied 5 times. The right hand side is the equivalent "algebraic form" where the multiplier is placed first and is called the numerical coefficient of the variable.

We see that summing or combining the numerical coefficients of like terms is a shorthand way to repeatedly add like terms (which may be already some existing multiple of the variable(s))..

Ex. $4X + BAX = X(4 + BA)$: here X, the common factor of each term, was factored out (divided out of each term) and is expressed as a factor to the resulting quotient so as the two expressions remain in balance.

Another, perhaps simpler, analogy or verification of the addition of like terms stems directly from the basic concept of adding things which are similar or have like units. To use units as variables is non-standard, but it serves as a good illustration. In a strict sense, variables are a symbolic (representative) numeric concept of representing a numeric magnitude or quantity (of the units) that is known or unknown with symbolic letters, text or other symbols. For example, we could let letter, or the symbol, B (or perhaps: *, or notches such as | or 1) represent an object such as a brick or "brick unit", rather than some numeric or variable value. If there are five bricks, this can be algebraically represented as 5B, meaning 5 Bricks = "five bricks". If you divided or separated the bricks into a group of three bricks, and another group having two bricks, the total sum of the bricks is still equal to five bricks:

2 bricks + 3 bricks = (2 + 3) bricks = 5 bricks
B, B + B, B, B = (B + B) + (B + B + B) = B + B + B + B + B : optional, for clarity
$2B + 3B = (2 + 3)B = 5B$

If we then let B=Bricks, variable X = 2, and Y = 3 for the quantity of that variable or unit, the above could then be expressed as:

$XB + YB = (X + Y)B$
2 Bricks + 5 Bricks = (2+5)Bricks = 7Bricks

In the standard sense of mathematical understanding, when B represents a numeric value, variable or constant, and not some object identifier or unit of measurement as temporarily used here to illustrate the summing of "like" or similar terms concept and method, the equations and operations as expressed here are still the same, except that then the numerical coefficients (here, X and Y) of variable or value B, are actually to be considered as (multiplying) factors to B. Likewise, B would also be considered a multiplying factor in each term, and since it is in both terms, it is a common factor to and of those terms, and can be factor or divided out from each term and then expressed in the "factored form" (of the original terms). The factored form of the expression is the product of B and a product that is the expressed sum of the remaining terms. Doing the reverse is called the "distributed form" of the expression.

Ex. Simplify $5X -3X$

$+5X -3X = (+5-3)X = (+2)X = +2X$: After combining the numerical coefficients of like terms and keeping the variable(s). You can check this result by first factoring variable X out from each term, and that it is indicated here in the second equivalence: $(+5 -3)X$ or= $X(+5 -3)$

The intermediate steps shown above are often considered as "understood" (known) and hence they are usually omitted in practice for the sake of some speed, efficiency and clarity, however, its always good to see what has actually taken place.

Ex. Simplify $-3X +4B +3B -X$

The terms -3X and -X are like terms, and the terms +4B and +3B are like terms.

$-3X -X = -3X -1X = -4X$, and $+4B+3B = +7B$

The sum is therefore:

-4X+7B or = +7B -4X : since combining terms is commutative, so therefore, the order of which term comes first, or is expressed first (perhaps for some clarity), has no affect on the result.

If X=3 and B=2 above, the value of the expression is found by assigning or substituting these values into their corresponding variable numeric placeholders:

-4X +7B = -4(3) +7(2) D (distribute, from the CDC method mentioned previously):
-12 +14 C (combine terms using the laws for the signs):
+2

ALGEBRAICALLY VERIFYING DISTRIBUTION

The concept of distribution has already been mentioned, discussed and used several times in this book, and now we are at the point to where it can be algebraically described and verified below:

$a + a + a + a + a + \dots$: The mathematically expressed repeated addition, of a variable or value. an endless or infinite amount of times, hence the sum is not an expressible value, other than mentioning that it is infinitely large.

If we consider a total, or limited number, say (n) of the above terms by using grouping:

$(a + a + a + a + a) = an = na$: Expressing the repeated addition as multiplication.

Now grouping terms of this set of (n) terms into two groups or sets of terms of variable (a), with one set containing a total of (b) like terms ,and the other, a total of (c) like terms; (here, (b) will actually be 2, and (c) will be 3 for this analysis example):

$(a + a) + (a + a + a)$ expressing these groups of repeated addition, as multiplication:

$ab + ac$: $ab + ac = ba + ca$ is a more common form with the coefficients of (a) first

Since $n = (b + c)$, : n = the total number of repeated additions of the like terms

$a + a + a + a + a = an = na = (b + c)a = ba + ca = a(b + c) = ab + ac$: verifies distribution

$2a + 3a = a(2 + 3) = a5 = 5a$

Also notice that $a(b + c)$ is the factored form of $(ab + ac)$, where each term has a common factor of (a).

Ex. $(a + b) (c + d)$ expressing an important intermediate step of distribution that is often omitted:

$a (c + d) + b (c + d)$: Notice, and as a verification, that each term here has a common factor of $(c + d)$, and if you were to factor this out of each term, you would then have the original expression.

$ac + ad + bc + bd$: this is the distributed form of the above expression, and you can verify that this is correct by factorization, and it is essentially a reverse process - such that during multiplication, a factor is put into each term, and during factorization or division, a factor is essentially pulled back out from each term:

$(ac + ad) + (bc + bd)$: grouping like terms, one group has (a), and the other group has (b) for each term.

$a (c + d) + b (c + d)$: factoring out the like variable(s), or values, from each grouping of like terms

$(c+d) (a + b)$: here, $(c + d)$, an expression, is treated like a single variable or value contained in the grouping symbols, and was factored from each term. This can be expressed as, using the commutative or order law:

$(a + b) (c + d)$: this is the initial expression shown above. This gives verification to how distribution was used and expressed starting from the initial expression.

Below, a few generalized examples of expressing and working with algebraic values, equations, and distribution will be shown.

Ex. If you were to add and subtract the same value, say (b), to or from another value, say (a), what is the difference (d) in the resulting two values?

Letting: $N1 = a + b$, and $N2 = a - b$: N1 and N2 represent binomials (2 terms), and N2 is a difference of terms.

Let: $d = N1 - N2$

$d = (a + b) - (a - b)$: subtracting (or adding the negative of a value) to find the difference
Distributing (+1) and then (-1) to clear grouping symbols:

$d = a +b -a +b$ combining like terms of variable (a), and variable (b):

$d = 2b$: You can imagine or see this when you take any value (a) on a numbered-line and increase it by (b), and also decrease it by (b). This change in value, or (b), can be thought of as a line segment graphically, and the difference in the values of N1 and N2, is 2b. The specific value of (a), which we can think of as the starting or reference value, which is also centered on the entire length of 2b, does not affect or play any role in the difference value, and is therefore absent from the resulting equation or formula (for d) for this situation. If (b) was called the change of or from the starting value (a), then (d) could then be expressed as: d = 2(change).

Ex. If a=0 and b=5: $d = (0+5) - (0-5) = 5 - (-5) = 5 + 5 = 10$
$d = 2(b) = 2(5) = 10$

Ex. What is the formula for the difference (d) between (a + b)^2 and (a - b)^2 ? Note that this is similar to the above example, but here the terms, or its (equivalent) value, are now squared.

$d = (a + b)^2 - (a - b)^2$: mathematically expressing the difference. Extending the squared terms:

$d = (a + b)(a + b) - [(a - b)(a - b)]$ using distribution (D) within the grouping symbols:

$d = [a^2 + ab + ab + b^2] - [a^2 -ab -ab +b^2]$

$d = +1[a^2 +2ab + b^2] - 1[a^2 -2ab +b^2]$: notice in the two grouped terms, that the term 2ab is present and either pos. or neg. Distributing +1 and -1 to clear grouping symbols:

$d = a^2 +2ab +b^2 -a^2 +2ab -b^2$ combining (C) like terms:

$d = 4ab$: When a=b, $(a-b)^2 = (a-a)^2 = (b-b)^2 = 0^2 = 0$, yet this formula is still valid.
When (a) and-or (b) is equal to 0, there is no difference possible, and d is always 0.

If (a) and (b) are always equal, a=b, this difference (d) formula here could be written as (using algebraic substitution):

$d = 4aa$ or $= 4bb$

$d = 4a^2$ or $= 4b^2$

Ex. The sum or difference between a variable squared and that variable results in an multiplication expression:

$a^2 + a = a (a + 1)$, ex. $5^2 + 5 = 5 (6) = 30$, $a^2 - a = a(a - 1)$, ex. $5^2 - 5 = 5(4) = 20$: check with dist.

Ex. What is the difference (d) between the sum of two squared values and the difference of two squared values?

$(a^2 + b^2) - (a^2 - b^2) = d$ after clearing each grouping symbol by distributing (+1), and (-1):
$a^2 + b^2 - a^2 + b^2 = d$ after combining like terms, and switching sides:
$d = 2b^2$

Ex. Algebraically show that (a/b) and (b/a) are reciprocals of each other. The easiest way to do this is by knowing the fact that a reciprocal times its own reciprocal is equal to 1:

$$(\text{reciprocal})\left(\frac{1}{(\text{reciprocal})}\right) = \frac{\text{reciprocal}}{\text{reciprocal}} = 1$$

$$\left(\frac{a}{b}\right)\left(\frac{b}{a}\right) = \frac{ab}{ba} = \frac{ab}{ab} = 1$$: their product of 1 indicates the values are reciprocals of each other

Let $(a/b) = n$, then $1/n$ is its reciprocal, $= 1/(a/b) = (b/a)$: algebraic verification that (a/b) and (b/a) are reciprocals of each other

MULTIPLYING LIKE VARIABLES

Multiplying like variables is performed just like the multiplication of like values or numbers. To multiply like variables, add their exponents, and multiply their numerical coefficients together.

Ex. (A)(A) = A^2

Checking: (A)(A) = A A = (1)A^1(1)A^1 = (1)(1)A^(1+1) = (1)A^2 = A^2

If A = 5, (A)(A) = AA = (5)(5) = A^2 = 5^2 = 25

Ex. A^2A^3 = A^5

Checking: A^2 A^3 = A^(2+3) = A^5
Checking: A^2 A^3 = (AA)(AAA) = AAAAA = A^5

Ex. This example shows that when multiplying like variables that have numerical coefficients other than 1, multiply those numerical coefficients, and add the exponents of the variable.

(2A)(2A) = 4A^2

Checking: (2A)(2A) = (2)(A)(2)(A) = (2)(2)(A)(A) = 4A^2

Ex. (10A^2)(2AB) = 20A^3B

Checking: (10A^2)(2AB) = (10)(2)A^2A^1B = 20A^3B

Ex. (2A^3)(B)(-5A^ -4) = -10BA^ -1 Or=: $\frac{-10B}{A}$ Or: _ $\frac{\_10B\_}{A}$

A POWER TO A POWER

To simplify a "power to a power", that is, an indicated power that is itself raised to an another indicated power, simply distribute (multiply) the exponents.

Ex. (A^2)^3 = A^(2x3) = A^6

Checking by "extending" the indicated operation, and then using the multiplication of like variables rule of adding the exponents:

(A^2)^3 = (A^2)(A^2)(A^2) = A^(2+2+2) = A^6

If the base of the indicated power contains multiple factors, "distribute" (multiply) the (outer) indicated power to each of the factors exponents.

Ex. (x^3y^1)^2 = x^((3)(2)) y^((1)(2)) = x^6y^2 checking by extending (using multiplication) the indicated power:

(x^3y^1)(x^3y^1) = x^3y^1x^3y^1 = x^3x^3y^1y^1 = x^(3+3)y^(1+1) = x^6y^2

MULTIPLYING SIMILAR (INDICATED) POWERS

To simplify the product of like or similar indicated powers (have the same indicated exponent), raise both the product of the numerical coefficients and then the product of the variables to that indicated power.

Ex. $(2X)^2 (3XZ)^2 = ((2)(3))^2 ((X)(X)(Z))^2 = (6)^2 (X^2 Z)^2 = 36 X^4 Z^2$ checking:

$(2^2 X^2)(3^2 X^2 Z^2) = (4X^2)(9X^2 Z^2) = 36X^4 Z^2$ or:

$(2X)(2X) (3XZ)(3XZ) = (2)(2)(3)(3)(X)(X)(X)(X)(Z)(Z) = 36X^4Z^2$

Another way to simplify this is to consider that the two (same) indicated powers have been previously factored out from a single power of:

$(6X^2Z)^2$: this is the result after multiplying the bases of the same indicated powers together and indicating that same power. This can be expressed as:

$(6XXZ)^2$
$((2)(3)(X)(X)(Z))^2$ which can be expressed as:
$((2X) (3XZ))^2$ distributing the exponent to each of the indicated factors:
$(2X)^2 (3XZ)^2$: this is the original example given

Now checking by distributing the exponent of the single (or unfactored base product) power:

$(6X^2Z)^2$ we have:
$6^2 X^{(2(2))} Z^{(1(2))}$
$36 X^4 Z^2$:checks

Hence the bases of similar (indicated) powers can first be multiplied together, and then the exponent can be distributed. Also, as indicated here, you can factor the base of an indicated power, and each factor will have that same indicated power or exponent: $(ab)^n = a^n b^n$, which is the same result after initially distributing the exponent to each factor.

DIVIDING LIKE VARIABLES

As with numerical values, just take the difference (or add the negative of) of the exponents when dividing like variables.

Ex. $\frac{A^5}{A^2} = A^{(+5-2)} = A^{+3}$ checking: $(A^2)(A^3) = A^{(2+3)} = A^5$

Ex. $\frac{A^2}{A^5} = \frac{A^{(+2-5)}}{1} = A^{-3}$: with a negative exponent

OR: $\frac{A^2}{A^5} = \frac{1}{A^{(5-2)}} = \frac{1}{A^3}$: with a positive exponent

checking: $\frac{A^2}{A^5} = \frac{\overset{+1}{\cancel{A^2}}}{(A^3)\underset{+1}{\cancel{(A^2)}}} = \frac{1}{A^3} = \frac{A^{-3}}{1}$ or: checking: $(A^5)(A^{-3}) = A^{(5+(-3))} = A^{(5-3)} = A^2$

OR: $\frac{A^2}{A^5} = \frac{A^2\,A^{-5}}{1} = A^{(2+(-5))} = A^{-3}$: using the multiplication of like values or variables rule to add or combine their exponents and keep that base value

Dividing A Several Term Expression By A Single Term Expression

To divide a several or multiple term expression (often called a multinomial) by a single term expression (often called a monomial), divide the denominator term into each term of the numerator expression. As discussed previously using only numeric values, this concept is simply the reverse of the concept of adding "like" fractions that have the same denominator. The process is also a form of simplifying a fraction.

Ex. $\frac{10X + 5}{5} = \frac{10X}{5} + \frac{5}{5} = 2X + 1$

Checking: $5(2X + 1)$: Multiplying the quotient and divisor together, and this is also a factored form of the multinomial dividend. Distributing the factor of 5 to each term in the second factor which is an expression that is a binomial since it is two terms being combined:

$5(2X) + 5(1)$

$10X + 5$: checks, since this is equal to the dividend

Ex. $\frac{30X^3 + 15X^2 + 10X}{5X}$

$\frac{30X^3}{5X} + \frac{15X^2}{5X} + \frac{10X}{5X}$

$6X^{3-1} + 3X^{2-1} + 2X^{1-1}$: $X^{1-1} = X^0 = 1$, and this instance of variable X is essentially eliminated as being a factor.

$6X^2 + 3X^1 + 2$

Checking: $5X(6X^2 + 3X^1 + 2)$ indicating the distributing of the 5X factor:

$5X(6X^2) + 5X(3X^1) + 5X(2)$ distributing:

$30X^3 + 15X^2 + 10X$: checks with the original numerator or dividend

Dividing A Several Term Expression By A Several Term Expression

This is about dividing a several or multiple term expression, often called a multinomial, by another several or multiple term expression. First, so that you can accept the concepts and process, here is a simple verification using only numeric constants, partial sums, equivalent fractions, and summing fractions, and without using any (algebraic, or symbolic placeholder) variables:

$$\frac{200}{100} = 2 = \frac{100 + 100}{100} = \frac{100}{100} + \frac{100}{100} = \frac{100}{(50 + 50)} + \frac{100}{(50 + 50)} = \frac{100 + 100}{50 + 50}$$

: expressed as a multinomial expression divided by a multinomial expression

Now an example using variables:

since $(x+2)(2) = 2x + 4$, then mathematically, dividing both sides by the $(x+2)$ factor we should have (2):

$x + 2 \overline{)\,2x + 4}$: the quotient should obviously be equal to 2, since dividing a product by one factor yields the other factor of that product.

To get the first term of the quotient, divide the first term of the divisor (or denominator if you will) into the first term of the dividend (or numerator if you will). Then continue as in normal division.

$$\frac{2x}{x} = 2$$

$$\begin{array}{r} 2 \\ x + 2 \overline{)\,2x + 4} \\ -\ \underline{(2x + 4)} \\ 0x + 0 \end{array} \quad = \quad \begin{array}{r} 2 \\ x + 2 \overline{)\,2x + 4} \\ +\ \underline{-2x - 4} \\ 0x + 0 \end{array}$$

: after using distribution (of -1) to rid the grouping symbols.

: remainder of 0, division is complete

Checking by multiplying the quotient and divisor: $2(x+2) = 2x + 4$: checks, since the product equals the dividend

Also: $$\begin{array}{r} x + 2 \\ 2 \overline{)\,2x + 4} \\ -\ \underline{2x} \\ 0 + 4 \\ -\ \underline{+4} \\ 0 \end{array}$$, and $2(x+2) = 2x + 4$

A method to make the process of the division of values that contain variables easier is to first place the terms containing the variable in an order of descending or decreasing (indicated) powers or exponents. Do this for both the dividend and divisor. If a power of a variable is absent or unused, replace it with a "0 placeholder". That is, express as a term that (unused) power of the variable, and with a 0 constant. This effective 0 value added in to the sum will obviously not change the value of the dividend (or numerator) or divisor (or denominator). For example, if the numerator or denominator is:

3 + x + 5x^3 rearrange this for the division process as:

5x^3 + 0x^2 + x^1 + 3 :the last term could also be indicated as having a factor that is a power of x: 3X^0 , since 3x^0 = 3(1) = 3

Actually, you can think that you have been dividing polynomials by polynomials all along without even realizing it. This can easily be seen by representing the strict (number only) numeric values of the numerator and denominator in an expression form where each term is the positional product of each digit of that number:

Ex. $\frac{385}{11} = 35$ Creating a positional sum by expressing each digit as the positional product of the digit value and its corresponding decimal weight in the decimal number:

$$\frac{385}{11} = \frac{3(100) + 8(10) + 5(1)}{1(10) + 1} = \frac{3(10^2) + 8(10^1) + 5}{1(10^1) + 1}$$

Here above, it is quite obvious that instead of a "polynomial in x (a variable)" we have a "polynomial in 10". If you were to replace each power of 10 above with a corresponding power of x, the results would be the same.

$$\begin{array}{r|l} & \underline{\ 3(10^1) \ + \ 5\ } \\ 1(10^1) + 1 & 3(10^2) + 8(10^1) + 5 \\ & - \underline{[\,3(10^2) + 3(10^1)\,]} \\ & 0(10^2) + 5(10^1) + 5 \\ & \qquad - \underline{[+5(10^1) + 5]} \\ & \qquad\quad 0(10^1) + 0 \end{array}$$

:start by dividing 3(10^2) by 1(10^1), these are the first terms of the dividend and divisor

: Note, the negative sign preceding the [ ] grouping symbols and it can be thought of as the factor (-1) to distribute and then do the combining.

Checking, using multiplication of the divisor and quotient so as to see or compare if it equals the dividend:

(1(10^)1 + 1) (3(10^1) + 5) distributing:
3(10^2) + 5(10^1) + 3(10^1) + 5 combining:
3(10^2) + 8(10^1) + 5 distributing:
300 + 80 + 5 combining:
385

Another check is to use the order of operations more closely, simplifying the expressions within the grouping symbols first:

(1(10^)1 + 1) (3(10^1) + 5) using distribution on the inner grouping symbols:
(10 + 1) (30 + 5) combine (add):
(11)(35) distribute (multiply)
385

As like during the process of dividing two plain numbers, when dividing multinomials, if the difference after subtracting is 0, the division is complete. If the first term (ie. monomial) or variable in the difference has a lower power than that of the first term of the divisor, then the divisor cannot divided (not even once) into that difference, and then the remaining part of the dividend becomes the remainder.

It should also be mentioned that it is mathematically possible, but maybe not practical, to divide each term of the dividend (or numerator if you will) by the entire divisor (or denominator if you will). Here, the entire multinomial divisor, perhaps set within grouping symbols for clarity, is used to represent a single value divisor.

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EQUATIONS

Two expressions that represent the same value can be expressed or set as being numerically equivalent (ie., "equal", =) in value, and this expression of equivalence is called an equation. An equation can be mathematically worked with and manipulated much like a conceived imaginary numerical weighing scale or balance. Whatever you do mathematically to one (entire) side of the equation, you must also do to the other (entire) side of the equation so as to keep both expressions or sides equal and the entire equation in balance. More specifically, whatever change in total value you apply to one side, you must also apply that same change in total value to the other side. All of this will be of great importance since it will allow us to solve for the equivalent value or expression of a variable or unknown value used in that equation.

When both sides do not represent the same value, they are said as "not equal", and the equation is "unbalanced". When the two sides or expressions are equal, there is no mathematical difference between their values, and the equation is therefore balanced. When the two sides or expressions are not equal, there is a mathematical difference between them, and therefore, those expressions or sides are unbalanced with respect to each other. One side is greater in value than the other, and therefore, one side is less in value than the other. Given an equation, the difference of a side value with respect to the other side value = side value - the other side value. A difference can also be thought of as the amount of separation or unbalance between two things such as the two sides or expressions of an equation.

Ex. Given the two expressions of: Eq.1: $(5+2)$ and Eq.2: $(3+5)$, are they equivalent in value?

First simplifying each expression to find their resulting net worth, total, sum, value or weight:

Eq..1: $5+2$ Eq.2: $3+5$
7 8

$7 - 8 = -1$:The amount that Eq.1 is different in value than Eq.2 ,
Since the result is not 0, the equations are not in balance.

$8 - 7 = +1$:The amount that Eq.2 is different in value than Eq.1

Notice, the absolute (or signless) value of each difference (or numerical separation), or numeric separation or "distance" between them is the same value of 1.

Since the net worth, sum or weight of each expression does not have the same value, there is a difference in or of value among the sides, and therefore, we can not equate those two expressions. We can however indicate or express their mathematical relationship with a special (mathematical, values) relational or relationship operators:

$7 < 8$: < , when read from left to right, it is called the "less-than" symbol. , "seven is less-than eight"
$8 > 7$: > , when read from left to right, it is called the "greater-than" symbol. , "eight is greater-than seven"

In either case, the arrow's smaller point end always points to, indicates or expresses the smaller or less value, and the open or large end of the arrow is near the greater or larger value.

7 |=| 8 : expressing that 7 is not equal to 8, and-or that 8 is not equal to 7

Ex. Given the two expressions of: Eq. 1: $(10-3)$ and Eq. 2: $-(257-249)$, are they equivalent in value?

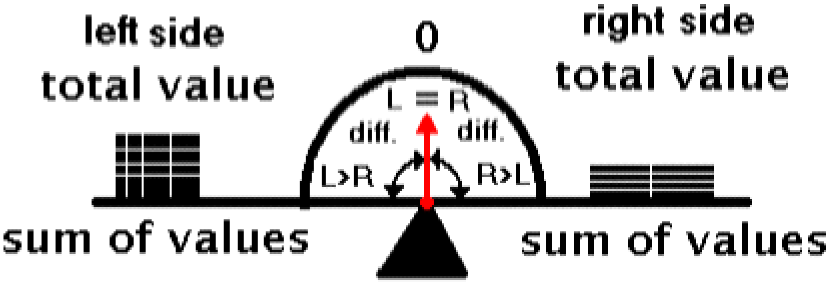
First simplifying each expression to find their resulting net worth, total, sum, value or weight:

Eq.1: $+10-3$ Eq.2: $-(-257+249)$
$+7$ $+256 -249$
$+7$

Checking for any difference: (+7) - (+7) = 0, hence there is zero or no difference of the values of each expression. Since the net worth, sum or weight of each expression has the same value (here, 7), we can equate those two expressions and create an equation with sides of those equations:

10 -3 = -(-256 +249)

Here is a graphical representation of a balanced equation where: left side = right side: [FIG 22]



A BALANCED EQUATION

When the scale is balanced, both sides are equivalent in weight, and if any weight is added to or removed (ie., subtracted) from just once side, there will be an unbalance. When two values on either side of the scale are not equivalent (ie., =|=), there is a difference (diff.) between those two values. If we subtract one value from the other, we can find if there is any difference between those two values, and the actual value of that difference.

Given: value1 and value2 — Checking to see if those two values are equal or different:

value2 - value1 = difference : difference is the amount that value2 is different than value1

Note also that:

(value1 - value2) = difference

Here, difference will be the same (signless or absolute) value as:
(value2 - value1) = difference
, but will now have a different sign indicating if value1 is greater or less than value2. Expressed this way, the sign of the difference is in reference to the first term,and that is value2.

value2 and value1 can be in placed in balance or equivalence when the specific amount of their unbalance, here the difference value, is combined with value1, and their difference will then be 0. This is much like a checking a subtraction problem when the difference is added to the subtrahend, so as to equal the minuend.

Now given:

(value1+ difference) and value2 or: value1 and (value2 + difference) if the signed difference is used

Checking to see if those two values are equal or different:

value2 - (value1 + difference) = 0 : since there is no difference, these two values are equivalent:
value2 = (value1 + difference) : both sides of this equation balance. This can be expressed as:
value2 = value1 + difference : as like all variables, these should be considered as signed values which can be either positive or negative in sign.

Ex. If value1 is 3, and value2 is 7:

We immediately can see that these two values are not equal since one value is larger or greater than, or with respect to, or in reference to the other value. Likewise, one value is therefore less than the other value:

value2 > value1 and-or: value1< value2 and value2 /=/ value1 : mathematical or numerical relationships

$7 - 3 = 4$: +4 is the difference between 3 and 7, which could also be said as the difference between 7 and 3. 7 is 4 greater than 3. Likewise it could be said that 3 is 4 less than 7. Note: $3 - 7 = -4$, here the negative sign of the difference means that the initial term and reference value of +3 is 4 less than +7.

$7 - (3 + 4) = 0$
$7 = (3 + 4)$
$7 = 3 + 4$: a balance scale having a weight of 7 units on one side will be in balance if the other side
$7 = 7$ has a weight of 3 units and another weight of 4 units.

Below are some examples of equations and mathematical manipulations on them.

Ex. $14 = (7)(2)$: each side can be simplified to a value of 14, therefore the equation is true, and not unbalanced and false.
$14+6 = (7)(2)+6$: Lets add 6 to each side
$20 = 14 + 6$: after combining on the left side, ORD step 3 (ie. mult.) on the right side
$20 = 20$: after combining the right side, ORD step 4 (ie. combining). The resulting equivalent values on both sides of the equals sign indicate that both sides were, and are still in balance after the same mathematical operations were applied to them.

Ex. $(8+2) = (5)(2)$: each side can be simplified to a value of 10
$(3)(8+2) = (5)(2)(3)$: multiplying 3 to each side
$+24+6 = +30$: after distributing
$+30 = +30$: after combining

Ex. $75 = (5)(15)$

$$\frac{\overset{+15}{\cancel{75}}}{\underset{+1}{\cancel{5}}} = \frac{\overset{+1}{\cancel{(5)}}(15)}{\underset{+1}{\cancel{5}}}$$

: dividing each side by 5, and canceling (canceling out, dividing both the num. and den. by a common factor(s))

$15 = 15$

Ex. $100 = (25)(4)$ taking and expressing the square root of both sides:

$\sqrt{100} = \sqrt{(25)(4)}$ factoring the radical:

$10 = \sqrt{25} \quad \sqrt{4}$ simplifying (here, rationalizing) each radical:
$10 = (5) \quad (2)$ distribute or multiply:
$10 = 10$

Let's say expression1 equals some value that we can set equal to and call as: x. If another different expression, such as equation2 also equals this value of x, then we can equate expression1 and expression2 since we know they are equal to the same value, here x. Showing or expressing this mathematically:

expression1 = x = expression2 therefore:
expression1 = expression2

Ex. If, or given: $(7 + 3) = 10$ and $\sqrt{100} = 10$ therefore, the expressions can be equated since they both equal the same value of 10:

$$(7 + 3) = \sqrt{100}$$
$$10 = 10$$

ALGEBRAICALLY EXPRESSING THE MORE ADVANCED MATHEMATICAL OPERATIONS

Previously in this book, examples of concepts, expressions, equations and formulas, with many using a pure or strict numerical form were given, such as for powers, roots, logarithms, etc. Some simple algebraic or introductory algebraic-like methods were also utilized where needed. Below, in a more pure algebraic sense, the mathematical and corresponding relationships of a base (b), exponent (e) or (x), and number (N) are expressed in a pure or strict algebraic mathematical form using variables, and in three fundamental mathematical operations and equations. Also take note of the various locations of the variables (N, b, x) within each equation, and that each variable also has the same value in each equation:

$N = b^x$: **Powers**

In a power expression or equation, the number (N) is often called the power value, (b) is often called the base or "base of the power value", and (x) is often called the exponent or the "indicated power".

$b = \sqrt[x]{N}$: **Roots**

In a radical expression or equation, the base (b) is often called the root, (N) is often called the radicand, and the exponent (x) is often called the index (i) or "indicated root".

$x = \log_b N$: **Logarithms**

In a logarithmic expression or equation, the exponent (x) is often called the logarithm, (b) is called the base of the logarithm, and (N) is often called the Number. Number is actually equal to a power of that indicated base: $b^x = N$

Expressing the above equations in a somewhat unconventional format or way so as to help make the three relationships more clear using just the same three names or identifiers for the parts of all the equations:

powers	roots	logarithms	: expression type
$\text{power} = \text{root}^{\text{logarithm}}$	$\text{root} = \sqrt[\text{logarithm}]{\text{power}}$	$\text{logarithm} = \log_{\text{root}} \text{power}$	

Expressing the above by using the more formal, common, standard identifiers or names for the values, along with an equivalent variable or identifier:

$$\text{power} = \text{base}^{\text{exponent}} = \text{Number}$$

$$\text{root} = \sqrt[\text{index}]{\text{radicand}} = \text{base} = \sqrt[\text{exponent}]{\text{power}} = \sqrt[\text{exponent}]{\text{number}}$$

$$\text{logarithm} = \log_{\text{base}} \text{Number} = \text{exponent}$$

We see in all three equations above that each of the three values can have a new or alternate meaning as a different part of each equation of each of the three mathematical operations, but it's actual value is the same in all three equations:

These will have the same numeric values:

base = root
exponent = logarithm = index,
power = Number = radicand

Given all the corresponding values for (N), (b), and (x) from one equation, they will satisfy (ie. keep both sides in balance and equivalence) all of the three equations given above. For example, if b=10, x=2, and N=100:

Powers	Roots	Logarithms
$N = b^x$	$b = \sqrt[x]{N}$	$x = \log_b N$
$100 = 10^2$	$10 = \sqrt[2]{100}$	$2 = \log_{10} 100$

Given any two values of: (b), (x), and (N), there is only one possible and corresponding third value. The instance of, and the specific set of three corresponding values is unique and essentially happens only once, and all the corresponding values are specifically related according to the equations shown above.

Some of the following has been mentioned about exponents, but here it is presented in algebraic terms:

$b1^x = b2$ therefore, $b2^{(1/x)} = b1$ checking that this is true using substitution for b1:

$b1^x = (b2^{(1/x)})^x = b2^{(x/x)} = b2$: as shown above. Also:

$b1^x = b2$ therefore, $b2^{(1/x)} = \sqrt[x]{b2} = b1$: the exponents of b1 and b2 are reciprocals.

$b1^x = b2$ therefore $b1 = \sqrt[x]{b2}$: or $b1^x = N$ and $\sqrt[x]{N} = b1$

Ex. $3^{2.5} = 15.588\underline{5}$ and $15.5885^{(1/2.5)} = \sqrt[2.5]{15.5885} = 15.5883^{0.4} = 3$

BASIC METHODS OF SOLVING FOR A VARIABLE

To solve for the value of a particular variable that's part of an equation is to isolate the variable on one side of the equation, and then the other side of that equation will be the equivalent value of that variable. First, if possible, simplify each side of the equation using the order of operations and-or the CDC method discussed in this book. Some other helpful steps are to place all terms that have the variable in question on one side of the equation, and place all the other variable terms and constants on the other side of the equation. This moving, shifting or arranging of terms to the other side of an equation is referred to as transposing (T). To transpose, add the negative of the term to both sides of the equation, then after combining like terms, the variable that was transposed will have a numerical coefficient of 0, and therefore, that variable is effectively eliminated from that side of the equation since anything times zero is equal to zero, and adding zero does not affect the value of that side.

equation

expression1 = expression2

term1 + term2 = term3 : To solve for term1, or a variable that is within term1, isolate term1 to one side of the equation; and here, by keeping term1 where it is and removing term2 from that side of the equation by the transposition (change position) process of adding the negative of that term to both sides of the equation:

term1 + term2 -(term2) = term3 -(term2) using distribution, here of and by: -1, to clear grouping symbols:

term1 + term2 -term2 = term3 -term2 combining like terms:
Note, +term2 -term2 = (+1)term2 (+) (-1)term2 =
(+1 + -1)term2 = 0 term2 = 0

term1 = term3 -term2

Ex. Solve for (x) given: $x+4 = 7$

$x+4 = 7$
$x+4 -4 = 7 -4$:T. +4, so we then add the negative of $(+4) = -(+4) = -4$ to each side of the equation
$x = +3$: after combining

Once the value of a variable is found, it can be checked by substituting (setting equal to, replacing) the value back into the variables place (placeholder) or position in the original equation. That is, the variable itself is to set equal to that found value. If both sides of the equation are found to be equal, then the value of the variable is correct.

Using the last example where $x=+3$:

$x+4 = 7$: original equation, substitute the solved or calculated value for x:
$+3+4 = 7$ combine:
$7 = 7$: checks

Ex. Solve for X given: X - Y = 5

$X - Y = 5$ First, let's T. (-Y) = Transpose (-Y) = add the negative of (-Y) = -(-Y) = +Y to both sides
$X - Y + Y = 5 + Y$ combining (C.) like terms:
$X + 0Y = 5 + Y$
$X = 5 + Y$: We find that X is equal to an expression containing a variable (here it's Y) whose actual value has an affect on determining the corresponding and specific value of X.

If a variable being solved for is being multiplied by some multiplying factor, other than one, that variable can be isolated by dividing both sides of the equation by its multiplying factor or numerical-coefficient.

Ex. 25X = 50, solve for X

$\frac{25X}{25} = \frac{50}{25}$: to rid X of its multiplying factor of 25, simply divide by 25. This would also be equivalent to multiplying each side by (1/25).

$\frac{\overset{+1}{\cancel{25}}X}{\underset{+1}{\cancel{25}}} = \frac{\overset{+2}{\cancel{50}}}{\underset{+1}{\cancel{25}}}$: cancel

$+1X = 2$ or simply: $X = 2$

If the variable to be isolated is being divided by some value, the variable can be isolated by multiplying both sides of the equation by this value, and then using canceling to effectively rid it from that side.

Ex. $\frac{X}{2} = \frac{50+Y}{1}$: solve for X

$\frac{(2)(X)}{(1)(2)} = \frac{(50+Y)(2)}{(\quad 1 \quad)(1)}$: to rid (x) of its "dividing factor", here 2, multiply both sides by this "dividing factor" and then cancel it out:

$\frac{(\overset{+1}{\cancel{2}})(X)}{(1)(\underset{+1}{\cancel{2}})} = \frac{(50+Y)(2)}{(\quad 1 \quad)(1)}$: cancel on the left side, and using distributing on the right side:

$X = (+2)(+50) + (+2)(+Y)$:indicated distribution

$X = +100 + 2Y$:after distributing

Ex. Solve for X given: $\frac{X}{5} - \frac{X}{7} = \frac{6}{35}$

There are several ways that X can be solved for, and for the first, we will combine the like terms of: (1/5)X and (-1/7)X :

$\frac{X}{5} - \frac{X}{7} = \frac{6}{35}$ combine like terms on the left side (here, essentially combining fractions with the LCD = 35):

$\frac{7X}{35} - \frac{5X}{35} = \frac{6}{35}$: showing the newly created and equivalent fractions on the left side

$\frac{(+7X-5X)}{35} = \frac{6}{35}$: indicating the combining of the numerators of the like or similar fractions that have the same denominators

$\frac{+2X}{35} = \frac{6}{35}$ now rid X of its' dividing coefficient factor of (35), and multiplying coefficient factor of (2): This can also be done in a single step by multiplying both sides by the reciprocal of 2/35 which is 35/2.

$\frac{\overset{+1}{\cancel{(35)}}+2X}{(1)\ \underset{+1}{\cancel{35}}} = \frac{6\ \overset{+1}{\cancel{(35)}}}{\underset{+1}{\cancel{35}}(1)}$: cancel fractions (rid denominators)

$2X = 6$ Isolate X, rid X of its multiplying factor of 2, by dividing by that value of 2:

$\frac{\overset{+1}{\cancel{2}}X}{\underset{+1}{\cancel{2}}} = \frac{\overset{+3}{\cancel{6}}}{\underset{+1}{\cancel{2}}}$: cancel

$X = +3$

The method below uses the CDC method. Here, the fractions are to be eliminated or canceled first rather than combining them. To eliminate the fractions on one side of the equation, multiply both sides of the equation by a common denominator (LCD preferred) of that side of the equation. To eliminate all fractions of the entire equation at once, then multiply both sides by the LCD of all the fractions.

$\frac{X}{5} - \frac{X}{7} = \frac{6}{35}$ multiplying both sides by 35:

$\frac{35}{1}\left(\frac{X}{5} - \frac{X}{7}\right) = \frac{6}{\underset{+1}{\cancel{35}}}\frac{\overset{+1}{\cancel{(35)}}}{(1)}$: cancel on the right side, now distribute on the left side:

$\frac{+35X}{5} - \frac{35X}{7} = \frac{6}{1}$ cancel on the left side:

$\frac{+\overset{+7}{\cancel{35}}X}{\underset{+1}{\cancel{5}}} - \frac{\overset{-5}{\cancel{35}}X}{\underset{+1}{\cancel{7}}} = \frac{6}{1}$: cancel

$+7X - 5X = 6$ combine, add the like terms numerical coefficients or factors:

$+2X = 6$ isolate X, divide both sides by 2:

$\frac{+\overset{+1}{\cancel{2}}X}{\underset{+1}{\cancel{+2}}} = \frac{\overset{+3}{\cancel{+6}}}{\underset{+1}{\cancel{+2}}}$: cancel

$X = +3$, Checking using substitution:

$\frac{X}{5} - \frac{X}{7} = \frac{3}{5} - \frac{3}{7} = 0.6 - 0.428\underline{6} = 0.1714\underline{3} = \frac{6}{35}$: checks

Ex. The formula relating Fahrenheit (F) and Celsius (C) temperature is given below. Solve for variable F. If water normally boils at 212° Fahrenheit, then at what corresponding Celsius temperature does it boil?

$C° = \frac{5}{9}(F° - 32°)$ after multiplying both sides by 9 and canceling:

$9C° = 5(F - 32°)$ dividing each side by 5 and canceling:

$\frac{9C°}{5} = F° - 32°$ after transposing (-32) by adding the negative of (-32) = -(-32) = (+32) to each side, and afterwards, switching sides:

$F° = \frac{9C°}{5} + 32°$ or $F° = 1.8C° + 32°$

Notice that for every degree change in Celsius temperature that the corresponding Fahrenheit temperature changes not by 1 degree, but by 1.8 (almost 2) degrees. As to the boiling point of water:

$C° = \frac{5}{9}(212° - 32°)$

$C° = 5\frac{(180°)}{9} = 5(20°) = 100°$: Celsius temperature scale, boiling point of water. (At 0° C = 32° F , water freezes)

Ex. Show that $\left(1 + \frac{1}{x}\right)$ = one plus the reciprocal of x, is equivalent to: $\left(\frac{x+1}{x}\right)$ or = $\frac{(1+x)}{x}$

$\frac{x+1}{x} = \frac{x}{x} + \frac{1}{x} = 1 + \frac{1}{x}$: after putting each term in the numerator over the common denominator. You can also show they are equal by combining fractions in the first expression given for this example.

Ex. If x = 2: $1 + \frac{1}{2} = 1 + 0.5 = 1.5$ and:

$\frac{2+1}{2} = \frac{3}{2} = 1.5$

Ex. If x = 100: $1 + \frac{1}{100} = 1 + 0.01 = 1.01$ and:

$\frac{100+1}{100} = \frac{101}{100} = 1.01$

Ex. If: $a = b + c$ this can be mathematically expressed as:

$a = b + c$ dividing each side by a:

$\frac{a}{a} = \frac{b + c}{a}$ which can be expressed as:

$\frac{a}{a} = \frac{b}{a} + \frac{c}{a}$

$1 = \frac{b}{a} + \frac{c}{a}$ we see that (b) and (c) essentially represent a fraction or percent of (a):

$1 = 100\% = \left(\frac{b}{a}\right)\left(\frac{100}{1}\right)\% + \left(\frac{c}{a}\right)\left(\frac{100}{1}\right)\%$

Ex. $5 = 3 + 2$

$1 = \frac{3}{5} + \frac{2}{5} = 0.6 + 0.4 = 60\%$ of 5 + 40% of 5 = 100% of 5
: 3 and 2 are expressed as relative values with respect to the sum = 100%=1

When the variable being solved for is part of a radicand, rationalize the radical by raising it, and also the other side of the equation (to keep their balance and equivalence), to an indicated power equal to the index (indicated root of the radicand), and then convert the radical expression to its equivalent exponential form for easy simplification.

Ex. $2\sqrt{+X+5} = 7$ first, squaring both sides of the equation:

$(2\sqrt{+X+5})^2 = (7)^2$ placing the radical form into exponential form:

$((+X+5)^{(1/2)})^2 = +49$ power to a power, distribute the power:

$(+X+5)^{(2/2)} = (+X+5)^1 = +49$

$(+X+5) = +X+5 = +49$ T. +5:

$+X +5 -5 = +49 -5$ combine terms:

$X = +44$

Be sure to check this result in the original equation since raising values to powers, though algebraically correct, can possibly introduce some extraneous (extra, and usually undesired) results. As a simple verification, consider this:

$10 - Z = 0$ after solving for Z we find:
$Z = +10$

If both sides are squared, the solution of Z should still be the same since the equation is still in balance:

$Z^2 = 10^2$ or:
$Z^2 = 100$ Solving for Z by taking the square root of both sides, we now find:
$Z = \pm 10$: the solution of (-10) will not satisfy the original equation, and it is therefore an extraneous root.

Below is an example where understanding a problem can be more difficult than mathematically solving it, and the problem was not even intentionally written to be difficult.

50 workers belong to a machine shop. Some of them can make nuts, and some can make bolts. Some workers are known to be able to make both. If 35 workers said they can make nuts, and 40 workers said they can make bolts, how many workers make both nuts and bolts? A **bolt** is much like a (machine, metal) screw (essentially a rod (ie., long cylinder shaped) with a spiral of thin metal "threads" along its outer length) of which a **nut** with matching threads on the inside surface of its cylinder ring shape can be twisted upon so as tighten and create a clamping force upon the surface(s) between the larger head structure of the bolt and the nut.

The total sum of nut makers (N), and bolt makers (B), where some of these people can make both:

Total = N + B
75 = 35 + 40 : total "virtual" number of makers or "jobs" possible, where only 50 are actual people and makers

However, for each given day only 50 real or actual workers out of this total of 75 (nut or bolt makers, which include dual (both) makers) work at the machine shop making either nuts or bolts, but not both. That is, regardless of the actual number of workers who are making nuts or bolts that day, this sum is always 50 workers. Subtracting this value of 50 will leave the number of people who can make both nuts and bolts.

75 = N + B + workers who can make both nuts and bolts
75 - (N + B) = workers who can make both nuts and bolts
75 - 50 = virtual workers - actual workers = both
both = 25 workers

Even though the problem was solved using a simple equation, it may need some clarification. First, consider the possibility that none of the workers can make both nuts and bolts, then the total "possibilities", "jobs" or "positions" (nut makers or bolt makers) is always 50 (instead for example of the 75 shown above), and cannot be any less since there is always the same 50 workers at the machine shop. Now, if any workers do both, the total "possibilities" will then exceed 50, and this excess above 50 can only be those who do both. The subtraction shown will find this excess or people who can make both nuts and bolts. Note also that if every one of the 50 workers can make both nuts and bolts, that the maximum "positions" or "possibilities" is: 50 people can make nuts, and 50 people can make bolts = 100 people can make both.

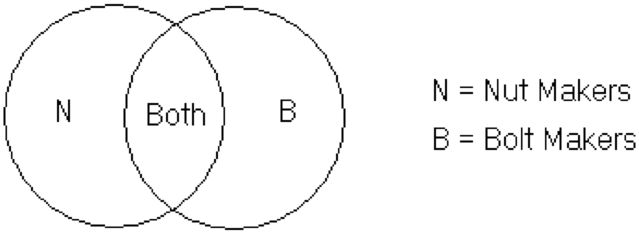
As a simple verification of the results of this problem, consider just 2 workers belonging to the machine shop, and each will either make nuts or make bolts during their work at the machine shop, that is, both workers are known to be a nut maker and a bolt maker. How many workers are both nut makers and bolt makers?

Total = N + B
4 = 2 + 2 : total possible or virtual "positions"

Since only 2 people are at the machine shop:

Both = Total - 2 : 2 workers at the machine shop
Both = 4 - 2
Both = 2 : 2 workers can do both (make nuts or bolts)

Though not discussed much further in this book, problems like the above can be represented and analyzed graphically with the aid of what are known as Euler (Leonhard Euler) or Venn (John Venn) diagrams. These diagrams represent sets (similar things) by using simple geometric figures, and their relationships and relative values are seen by observation and-or by using a special mathematical notation for sets. A simple Venn diagram of the above example might look like this: [FIG 23]


N
Both
B
N = Nut Makers
B = Bolt Makers

GRAPHING AN EQUATION TO VIEW AND SOLVE IT

Sometimes an equation cannot be solved easily using normal algebra. By graphing the equation you can get a close estimate of the solution. A graph of the relationship of variables is also very powerful and a useful visual tool for quickly understanding the numerical values and relationship between two variables over a wide range of possible values. For example, to view if one variable is increasing or decreasing and-or at what rate (ie. "how fast") as the other variable changes (increases or decreases) in value. Once an equation is graphed by making many calculations with it and plotting the results or points, it also then effectively becomes an automatic visual calculator or computer that can quickly solve for a corresponding value of either the independent or and-or dependent variable. It is even possible to make a mechanical computer with the curve drawn, and such as using the intersection of two movable or user adjustable lines, one vertical (for the x or independent value) and one horizontal (for the y or dependent value).

Solve for (x) in : $2x + 4 = -2$

Though algebra can be used to easily solve this equation for x, it will be only used to check or validate this demonstration of using a graph to solve an equation. To solve this equation, you are to find what value of (x) will make the left side equal to the right side of -2.

By transposing (-2), the equation can be algebraically expressed as:

$2x + 4 + 2 = -2 + 2$
$2x + 6 = 0$: x can be mathematically solved here, but also we want to solve this equation graphically Expressed this way, we can see that there is a certain or specific value of x that will make the expression on the left side equal to 0.

Here, we are to find the value of (x) that will make the left side equal to the right side of 0, still, the value of (x) that solves this equation will also solve the original equation since the new "derived" equation was derived (ie., created directly from) from the original equation. It is simply another mathematical expression of that original equation. This can easily be verified after solving for (x) mathematically. When the equation is expressed this way (set equal to 0), the value of (x) that satisfies or solves this equation is called the "root" of the equation, although don't confuse this concept with that of radicals.

Since the equation given is a linear (has a line-like graphing, of the proportional mathematical variable relationship) or "first-order" equation (where the exponent of the variable is only one = first = 1st), there will only be one root or solution. In general, if the power of the variable is 1, it is a linear equation. Linear equations always graphically plotted (ie. scribed, indicated , graphically expressed locations (ie., address) of corresponding variable values) as marks or points of a straight line.

To plot or graph an equation, there must be a mathematical relationship between two variables. In this discussion, the relationship is a mathematically proportional or linear ("one to one" or "balanced") type of relationship. As one "independent" variable changes in value by some factor (ie. a multiplier applied to the variable), then the other "dependent" variable will then change at the same constant rate since its value is essentially multiplied by that same constant factor value. Consider:

$y = 1x$: this is the same as: $y = x$, and this will actually graph as a horizontal line as long as the multiplying factor or coefficient to x is 1, but if it changes to another value, then it will no longer be horizontal, but will have a vertical part, either upward if the coefficient is positive, or downward if the coefficient is negative. The greater this multiplier, the "steeper" the line, because the rate of change of the y values with respect to the x values is mathematically more such as for : $y = 5x$

We can also set the entire equation, or its equivalent result, as equal to another variable such as (y) for plotting purposes. Rather than just expressing a mathematical relationship between (x) and the result or numeric "output" of the expression, and we will now have a mathematical relationship between two expressed variables such as (x) and (y). (x) can be thought of as the input or "independent" variable, and (y) can be thought of as the output or "dependent" (dependent or depending upon the value of x) variable.

$2x + 6 = 0 = y$ or by switching sides:
$y = 2x + 6$

By setting the equation equal to 0, the root or solution will actually be where the graphical line (considered as a special instance of a curve, or curve plot on a graph) of the equation crosses the (x) or horizontal axis which is where the (y) value, of that corresponding point or part of the curve, is always 0. When finding corresponding values of (y) and (x), that is, finding the coordinates (corresponding values which define a points location) of each point on the line, it is best to find points both above and below the (x) or horizontal axis so that a smooth line (or curve) can be drawn (by hand, with the aid of a straight-edge drawing guide or a ruler) through or across the axis when using the points as a drawing guide. To draw a line, at least two points on it must be known, and a third or more points gives some verification that the two other points and line is accurate. You could begin by selecting random-like values for (x), and then mathematically finding (solving for, using the equation) each corresponding (y) value:

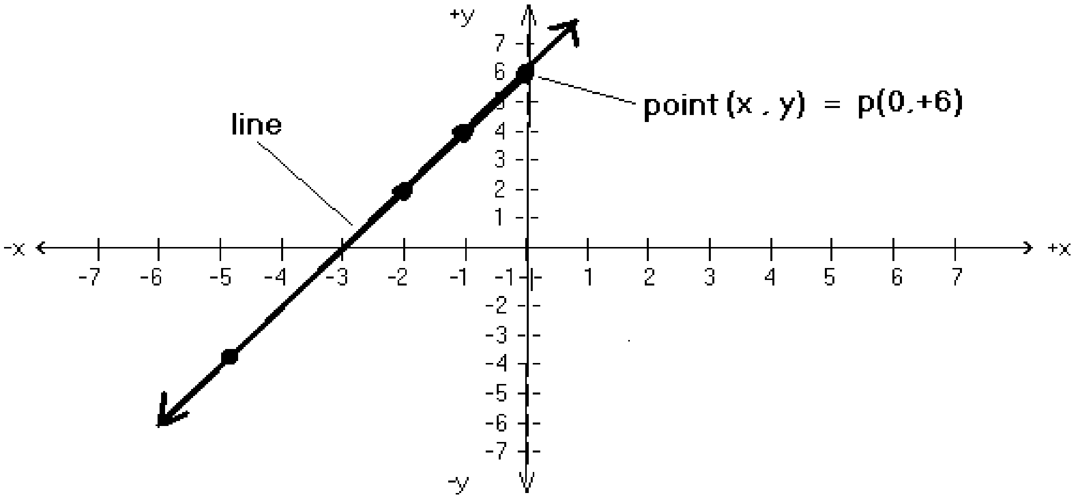
Make a list of several corresponding values of the (x) and (y) variables to be graphed or expressed visually as the (coordinate, or coordinated) location of points on a (flat) plane or rectangular (square-like) coordinate (location) system. These points are identified by that corresponding pair of values for the variables such as: (x , y). The coordinates (corresponding locations) are fundamentally in reference to an arbitrary point chosen as the starting or reference point, usually identified with coordinates (x=0,y=0), and which is sometimes called the "axis center point" of the coordinate system. The horizontal and vertical dimensions are called the axis (lines) of the system, and these are numerically scaled for values and measurements of each variable.

x	y = 2x+6
0	+6
-1	+4
- 2	+2
-5	-4

: each pair of corresponding values of the variables can be represented graphically as a point whose location, address or coordinates is identified as point(x,y) = p(x,y)

Plotting the points:

[FIG 24]



The solution (visually, and by the chart or list of the corresponding values of the variables) appears to be about: $x = -3$. Let's check by substituting this value into the original equation:

$2x + 4 = -2$
$2(-3) + 4 = -2$ distributing:
$-6 + 4 = -2$ combining;
$-2 = -2$: checks

The point where the line crosses the x-axis is called the x-axis intercept. Here, the corresponding value of y is always 0. If not already done, solve the equation for y, and then set this equation and y equal to 0. This is important since it will create a completely solvable equation with only one variable in it.

$y = 0 = 2x + 6$: setting y equal to zero, to find the corresponding value of x. Solving for x we have:

$x = \frac{-6}{2} = -3$: after transposing +6, and dividing both sides by 2
The x-intercept, or x-axis intercept, point is: P(x,y) = P(-3,0)

Likewise, given $y = 2x + 6$, we can solve for variable x and set its value equal to 0. Then we can solve for y since there will be only one variable in this equation since variable x will be essentially eliminated (by using a constant or numeric value for it, such as 0). y is then solvable, or is said as being "completely solvable" since it will have a numeric value:

$x = \frac{y - 6}{2}$ setting x=0 for the purpose of finding its corresponding y value:

$0 = \frac{y - 6}{2}$ when x=0, solving for y so as to find the corresponding y value of that point:

$y = 2(0) + 6$

$y = 0 + 6 = +6$: The y-intercept point is: P(x,y) = P(0,+6) as seen in the graph above.
The intercept points also can be used to draw the line, and-or to verify if a previously plotted and graphed (drawn) points or line is correct. Any more points should also be plotted on that same existing line.

SOLVING EQUATIONS CONTAINING MULTIPLE VARIABLES

If an equation contains multiple variables, a specific variable and its value can still be solved for if all the other variables can be placed on the other side of an equation and be expressed "in terms" of the given variable . This is known as algebraic substitution (as opposed to substituting actual numeric values for numeric substitution). The overall method is to produce a new equation which will have only one unknown or variable of which it will then be solvable.

Ex. In a certain animal sanctuary, there is a total of 60 animals consisting of cats and dogs. There are twice as many cats as dogs. How many cats, and how many dogs are there?

We see that we have a problem with one known value: 60 animals, and two unknown values: the number of cats and the number of dogs. From the given statement, creating a simple algebraic equation:

60 animals = number of cats + number of dogs

Let's let variable D represent the number of dogs, and variable C represent the number of cats.

Notice that the number of cats was stated in terms of the number of dogs. For each (1) dog, there are two (2) cats. There are twice as many cats as dogs.

First, it is incorrect to state or express this situation mathematically as: 1 dog = 2 cats or $1D = 2C$ or $1D/2C$ ("one dog per two cats") as a ratio. If used that way, dog, D, and cats, C, are being incorrectly used as units of measurement, rather than as variables or numeric values that can change in value. Saying, that there is one dog per two cats is normally acceptable, but if you actually express or divide the values: $1D/2C = 0.5(D/C)$, hence the units would be (D/C), and again, D and C are to be variable here, rather than units. It would be better to express the ratio as: 1 dog/2 cats.

The actual number of cats: (C) equals the number of dogs times two, hence the number of cats depends on the number (D) of dogs, well mathematically anyway:

$C = 2D$: C depends on D, and C can be called a dependent variable whose value depends (is mathematically related to, depends upon, and is determined by) on the value of an independent variable, here D.

$C = 2D$ can also be expressed as: $D = C/2$

$60 = C + D$ since $C = (2D)$, and using algebraic substitution for C:
$60 = 2D + D$: hence, now a solvable equation (because it has just one unknown value, here variable D)
$60 = 3D$ isolating D by dividing each side by 3 :
$D = 20$: number of dogs

$C = 2D$: original equation
$C = 2(20)$: using (numeric) substitution for the value of D
$C = 40$: number of cats

OR:

$60 = C + D$ after solving for C:
$C = 60 - D$
$C = 60 - 20$
$C = 40$: number of cats.

If you used algebraic substitution for $D = C/2$, you would also arrive at this value:

$60 = C + (C/2) = 40 + (40/2) = 40 + 20 = 60$

Ex. A box contains 75 coins consisting of pennies and nickels with a total sum of $1.75. There are twice as many pennies as nickels, how many pennies and how many nickels are there?

Let P = value of a penny = (0.01)
Let N = value of a nickel = (0.05)

Let X = number of pennies
Let Y = number of nickels
Total coins = X + Y = 75

Total monetary sum due to pennies + total monetary sum due to nickels = 1.75

$X(P) + Y(N) = 1.75$ substitution the monetary values:

$X(0.01) + Y(0.05) = 1.75$

Since there are twice as many pennies as nickels, Y must be multiplied by 2 to equal X: $X = 2Y$

Now, X can be algebraically substituted or replaced by 2Y, therefore producing a solvable equation which has only one unknown (variable):

$2Y(P) + Y(N) = 1.75$
$2Y(0.01) + Y(0.05) = 1.75$ distribute:
$0.02Y + 0.05Y = 1.75$ combine like terms (add their numerical coefficients):
$0.07Y = 1.75$ isolate Y:

$$Y = \frac{1.75}{0.07} = 25$$: there are 25 nickels

Since $X + Y = 75$
$X + 25 = 75$
$X = 75 - 25$
$X = 50$: there are 50 pennies

Or: $X = 2Y = 2(25) = 50$: as noted above

Checking, using the original equation:

$X(0.01) + Y(0.05) = 1.75$
$50(0.01) + 25(0.05) = 1.75$
$0.50 + 1.25 = 1.75$
$1.75 = 1.75$: checks

Here is another way, perhaps a simpler way of solving this problem, and which uses only the number of coins involved, and it's very similar to the previous example:

We were given:

$X + Y = 75$ and as stated in the original problem:
$X = 2Y$ by algebraic substitution:
$2Y + Y = 75$ combining like terms:
$3Y = 75$ solving for y, isolate it by dividing both sides by 3:

$Y = \frac{75}{3} = 25$: number of nickels

$X = 2Y = 2(25) = 50$: number of pennies

Check: $X + Y = 75$
$50 + 25 = 75$
$75 = 75$: checks

SIMULTANEOUS EQUATIONS

One method to solve multiple variable equations that are all related to a particular situation or instance is referred to as the "addition and subtraction" method. The basic requirement is to have N equations for N unknowns (variables). These initial equations must be independent of one another, that is, none can be algebraically derived or created from another because that would essentially be expressing the same equation as just another way.

Things that happen together or function together, and sometimes at the same exact time or "instantaneously together", are called simultaneous and their (numerical) relationship is that they behave and change simultaneously or together, and may be dependent on the values of each other. A set or group of related multiple variable equations is often called "simultaneous equations" when the solution for each variable is also simultaneous (ie. the same) for all of the given equations. After solving any equation in a set of simultaneous equations for a particular variable(s), its value will satisfy all the other equations..

Here is a system of simultaneous equations that has only one unknown (variable):

Equation 1: 5 + N = 8 : N is a placeholder (ie. a variable) for some unknown value yet to be found
Equation 2: (7.9)(N) = 23.7
Equation 3: 3N + 2 = 11
Equation 4: N^2 - 2 = 7

Find the value of N. Since the system of equations shown contains only 1 variable, solving for the value of the variable can be done using just one equation, and with or without the assistance of any of the other equations. This will also solve all the equations of the system of equations shown.

5 + N = 8 T. (5), add the negative of it to each side = -(+5) = -5
5 + N - 5 = 8 - 5 combining:
N = 3

Checking using equation 4:

N^2 - 2 = 7
3^2 - 2 = 7
9 - 2 = 7
7 = 7 : checks

Before a multi-variable example is shown, some verification or practical proof of the "addition and subtraction" process often used comes from the basic fact that adding, subtracting, multiplying, or dividing both sides of an equation by the same value still keeps the equation in balance or "true". For instance, given these two equations:

Eq. 1: (3+2) = 5 and Eq. 2: (4+6) = 10 Adding 10 to each side of the first equation:

$(3+2) + 10 = 5 + 10$ but since $(4+6) = 10$, and substituting this expression for it:

$(3+2) + (4+6) = 5 + 10$: which is effectively the same as adding both corresponding sides of the original equations (Eq. 1, and Eq. 2) horizontally. Simplifying:

$5 + 10 = 5 + 10$
$15 = 15$: checks

But now take note:

$(3+2) + (4+6) = 5 + 10$ clearing grouping symbols:
$3 + 2 + 4 + 6 = 15$ since addition is associative, we can regroup some other terms:
$(3+4) + (2+6) = 15$
$7 + 8 = 15$ showing this vertically, we can visualize adding both sides or the entire equations together:

$$\begin{array}{rcl} (3+2) & = & 5 \\ +\underline{(4+6)} & = & +\underline{10} \\ 7+8 & = & 15 \end{array}$$

here, the left sides were combined, and then the right sides were combined:

: $15 = 15$ checks

Perhaps an expressed pseudo-algebraic (algebraic-like) verification can also help explain this concept:

left side eq. 1 = right side eq. 1 : eq. = equation
left side eq. 2 = right side eq. 2

Taking the first equation, and then adding anything to both sides of it, even if its the left side of the second equation, still keeps the equation in balance:

left side eq. 1 + left side eq. 2 = right side eq. 1 + left side eq. 2

However, since: left side eq. 2 = right side eq. 2, and then substituting this into the above equation:

left side eq. 1 + left side eq. 2 = right side eq. 1 + right side eq. 2 : horizontally adding the equations
sum = sum

Showing this vertically:

$$\begin{array}{rcccl} & \text{left side eq. 1} & & & \text{right side eq. 1} \\ + & \underline{\text{left side eq. 2}} & & + & \underline{\text{right side eq. 2}} \\ & \text{sum} & = & & \text{sum} \end{array}$$

: vertically adding the two equations

Ex. You purchased some fruit at a grocery store on several occasions, but have forgotten the price for each piece of fruit. You have receipts of your purchases showing the amount and type of fruit purchased, and the total purchase price:

6 apples + 8 pears = \$ 1.10 : a receipt (of a purchase, receiving and payment record) total from one purchase
3 apples + 2 pears = \$ 0.35 : a receipt total from another purchase several days later

How much is an apple? How much is a pear? (also assuming their prices did not fluctuate (change) between the two purchases shown, and perhaps since then).

Let A = cost of each apple
Let P = cost of each pear

Optionally considering each value with units of cents, rather than dollars. Therefore:

Eq. 1: $6A + 8P = 110$ cents
Eq. 2: $3A + 2P = 35$ cents

Before the addition (or subtraction) of these two equations, you must first multiply (or divide) either or both equations by some value so that when they are either added or subtracted that one variable will essentially be eliminated from the new equation formed, hence producing a solvable equation which has only one variable or unknown value. Whenever you multiply or divide both sides of equations by the same value, this actually creates a (derived) "equivalent equation" in the sense that the solution will be equivalent or the same for both. Related equations where the solution is the same for both are also called simultaneous equations as mentioned previously.

Ex. Eq. 1: $2x + 3 = 7$ multiplying both sides by 2 :
$2(2x + 3) = (2)(7)$ we have created an equivalent equation of:

Eq. 2: $4x + 6 = 14$

For both Eq. 1 and Eq. 2 directly above, the solution is the same value of $x=2$.

Extra: If you were to graph (plot or place, and draw) these two (line or linear) equations (as discussed previously, and ahead in this book), they would intersect (cross, meet, join) at a single point. Since the second equation was created from the first, the solution here is actually a point on the "x" (horizontal or independent variable) axis, and which is also the "x" axis intercept (where $y=0$) for both lines. Algebraically re-arranging the equations for plotting:

From eq. 1, we get: $2x - 4 = 0 = y$: let $y = 0$ (and the eq.), and then plot this relationship.
From eq. 2, we get: $4x - 8 = 0 = y$: let $y = 0$ (and the eq.), and then plot this relationship.

Note however, that if an expression, not an equation, such as these is added by a value, that the new expression will yield a line that is parallel to the original line expression. The reason that it will be parallel is that this added in constant value will essentially shift the line (or curve) vertically (up and down) in reference or respect to the original line. The vertical steepness (or "slope") will still be the same for both lines. Consider:

Ex. $2x + 3$ after adding 5:
$2x + 8$

Setting both expressions equal to (y) for plotting:

Eq. 1: $y = 2x + 3$
Eq. 2: $y = 2x + 8$

Since parallel lines do not intersect at some point, there is no (simultaneous) solution to this system of equations.

Continuing the original example and equations:

Multiplying the second equation by 2 since $6A/3A = 2$, or simply $6/3 = 2$ after canceling A in the num. and den.

Eq. 2: $2(3A + 2P) = 35(2)$ distribute:
New Eq. 2: $6A + 4P = 70$

$6A + 8P = 110$ Eq. 1

$\underline{-(6A + 4P = 70)}$ New Eq. 2. Subtracting (or adding the negative of each term) the equations:
$0A + 4P = 40$: The result is one equation with one unknown (here P) which can be solved for.
Or by first distributing the (-1) to both sides to clear grouping symbols, and then combining terms (instead of actual subtracting) we would have this for the new Eq. 2:
$\underline{-6A - 4P = -70}$ Now combining (adding) instead of subtracting:
$0A + 4P = 40$ Variable A was effectively eliminated, and we now have an equation that has only one variable, and we can solve for the value of that variable.

$4P = 40$ isolating P, by dividing both sides by its multiplying factor of 4, we have:

$P = \frac{40}{4} = 10$: The price of each pear is 10 cents.

By substituting the value of P into any of the original equations, you can solve for A.

$6A + 8P = 110$
$6A + 8(10) = 110$
$6A + 80 = 110$ transpose the non-A terms:
$6A + 80 - 80 = 110 - 80$ combining:
$6A = 30$

$A = \frac{30}{6} = 5$: The price of each apple is 5 cents.

Checking these results in an original equation, by using substitution:

$3A + 2P = 35$
$3(5) + 2(10) = 35$
$15 + 20 = 35$
$35 = 35$: checks

Another possible method to solve simultaneous equations is by equating similar variables. That is, at the solution to the set of equations, each particular variable in question will have the same solution or value, and the expressions for them can therefore be equated. It can also be said that the difference between similar variables is 0 at the solution. For example:

Eq. 1: $a + 2b = c1$: here, identifier (c) is some constant numeric value, and not a variable value
Eq. 2: $3a + b = c2$

We could have also used subscripts (to identify two or more different instances and their values) for variables (a) and (b), but at the solution, both values of (a) are identical, and both values of (b) are identical. Therefore, for now, we can identify both as just (a) and (b) rather than for example (a_1) and (a_2).

From Eq. 1: $a = c1 - 2b$: variable (a) in terms of variable (b)

From Eq. 2: $a = \frac{c2 - b}{3}$: variable (a) in terms of variable (b)

At the solution: $a = a$: or $a - a = 0$

Therefore: $c1 - 2b = \frac{c2 - b}{3}$ transposing the right hand side:
(Could also begin to solve for (b) by multiplying
both sides by 3, and then canceling.)

Or: $c1 - 2b - \left(\frac{c2 - b}{3}\right) = 0$ with some distribution:

$c1 - 2b + \frac{-c2 + b}{3} = 0$ combining fractions on the left hand side:

$\frac{3c1 - 6b - c2 + b}{3} = 0$ multiplying each side by 3 and canceling:

$3c1 - 6b - c2 + b = 0$ combining like terms:

$3c1 - c2 - 5b = 0$ transposing (-5b), and then dividing each side by 5, and switching sides:

$b = \frac{3c1 - c2}{5}$:since c is a constant and there are no other variables in
the expression, the numeric value of (b) is solved for

In a similar manner to the above process, variable (a) could have been found. Since (b) is now known, (a) can be found by substituting this value of (b) into one of the original equations and then solving for (a).

There are a few other methods to solve a system of equations besides the last two methods shown. For example, there is the "graphical method" discussed previously where the solution is the point of intersection (crossing) of the graph (plotting, drawing) of the equations. However, the result is usually only a good approximation since it's a non-algebraic (ie. non-exact) method that does not produce a specific or exact numeric result. Still, don't underestimate graphs since for instance they have the power to quickly show the relationships of variables over a wide range of possible values and can be used for quick estimates.

Problems involving percentages might seem to be awkward at first, but if you can write some simple equations, the problem usually resolves to one of basic algebra.

Ex. If 5% of N is 30, what is 15% of N ?

Writing equations with the information given:

1) $N(0.05) = 30$: note that the percentage rate is expressed here in (strict numeric) decimal form for the equations
2) $N(0.15) = X$

By algebraically solving for N in the first equation, we can algebraically substitute its equivalent value into the second equation that has two unknown values, and therefore make that equation solvable with only one unknown value:

From the first equation: $N = \frac{30}{0.05} = 600$ substituting this value for N into the second equation:

$N(0.15) = X$
$600(0.15) = 90$: hence $X = 90$ and 15% of 600 is 90

An alternate approach to finding X is to use the concepts of (equal or similar) proportions, and which will be discussed shortly ahead, but that method can simply be thought of as an equivalent fractions concept where if one value is magnified (basically by a multiplying factor, greater or less than one [ie. technically, demagnified]), all the other corresponding values (between the two systems or magnifications) will have the same (ie. equal or similar) value of magnification.

A basic example of proportions could be if you were using a basic recipe where half (50%) of something (ex. weight, mass, volume, some other ingredient, cake, batch) is water. If you were to make more or less of that entire recipe, still, half (50%) of that something (whatever size it is) should still be water. It will be more or less (as for this example, less water) than used in the original recipe, but it is still half of that something. It could be said that the proportions, relative amount (here, with respect to the previous instance), fractional part value(s), or even corresponding ratio values among its other ingredients, are still the same value.

Given the original equations above, here is where the proportions type of equation shown below comes from:

$N(0.05) = 30$ and $N(0.15) = X$ Solving for N in both equations:

$N = \frac{30}{0.05}$ $N = \frac{X}{0.15}$ Equating the two expressions for N, we have a "proportions" type of equation, or "equivalent (in value) portions":

$\frac{30}{0.05} = \frac{X}{0.15}$: can be read as: "If 30 is 5%, then X is15%"
or: "30 is to 5%, as X is to 15%", or: "30 is to 0.05 as X is to 0.15",
or: "5% of some value corresponds to 30, as 15% of that value will correspond to X",
or : "5% is to 30, as is 15% is to X". (etc)

Clearly, if you increase the percentage, or fraction of something, you should have more of it, so X should be greater than 30.

You can also invert the fractions since their new (reciprocal) values will still be identical:

$\frac{0.05}{30} = \frac{0.15}{X}$ as mentioned above, for example: "5% is to 30, as is 15% is to X"

First, solving by using the equivalent fraction concepts:

since: $\frac{0.15}{0.05} = 3$, multiplying both the numerator and denominator by this value to create the equivalent fraction:

$$\frac{30(3)}{0.05(3)} = \frac{X}{0.15}$$

Note above that (3/3)=1, and that multiplying any side of an equation by 1 will not unbalance the equation.

$\frac{30}{0.05} = \frac{90}{0.15} = \frac{X}{0.15}$ hence, by simple observation:

$$X = 90$$

Now using the concepts of proportions, and of which are expressed as equations and usually solved as such:

$\frac{30}{0.05} = \frac{X}{0.15}$ after multiplying both sides by 0.15 and canceling :

$$X = \frac{30(0.15)}{0.05} = 30(3) = 90$$

N can also be found using a proportion type of equation.

Since 100% = 1.0 = 1

$$\frac{90}{0.15} = \frac{N}{1} \quad \text{or} \quad \frac{30}{0.05} = \frac{N}{1}$$

$N = 600$

Ex. In a bag, there are some marbles. Of the total number of marbles, 25% (of the total) are red, 10% are white, and exactly 13 are blue. What is the total number of marbles?

Notice that the number of red and blue marbles was not given, however their values were indirectly stated in terms of the total number of marbles, and more specifically, a portion or percentage of the total number of marbles.

Let us represent the total number of marbles with the letter or variable M, then we have:

Total Marbles = red marbles + white marbles + blue marbles or:
$M = \text{red} + \text{white} + \text{blue}$
$M = (25\% \text{ of } M) + (10\% \text{ of } M) + 13$ which can be expressed as:
$M = 25\%M + 10\%M + 13$ expressing the percentages as their decimal number equivalent:
$M = 0.25M + 0.10M + 13$ combining like terms on the right side:
$1M = 0.35M + 13$ transposing +0.35M, add the negative of this term to each side:
$1M - 0.35M = 0.35M + 13 - 0.35M$ combining like terms:
$0.65M = 13$ after dividing both sides by 0.65 we have:
$M = 20$:a total of 20 marbles in the bag

Note that the total percentage of red and white marbles in the bag is: 25% + 10% = 35%. The percentage of blue marbles in the bag is therefore the remaining (of 1.0 = 100%) percentage of the marbles left:

100% - 35% = 65% of all the marbles are blue. : or $1.0 - 0.35 = 0.65$

checking: (Total Marbles)(0.65) = (20)(0.65) = 13

$$\frac{\text{blue marbles}}{\text{Total marbles}} = \frac{13}{20} = 0.65$$ and the problem could have been solved using:

$$\frac{13}{M} = 0.65$$ solving for M:

$$M = \frac{13}{0.65} = 20$$: total marbles

The number of red marbles is therefore: 25% of 20 = (0.25)(20) = 5
The number of white marbles is therefore: 10% of 20 = (0.10)(20) = 2
red marbles + white marbles + blue marbles = 5 + 2 + 13 = 20 marbles

Ex. The sum of two numbers is 198, and their difference is 48, what are the two numbers?

equation 1: $a + b = 198$
equation 2: $a - b = 48$

From equation 2, we find: $a = 48 + b$, [or: $b = a-48$] , and using substitution of this into the first equation:

$(48 + b) + b = 198$ combining like terms:

$48 + 2b = 198$ after transposing 48:
$2b = 198 - 48$ after combining:
$2b = 150$ after dividing each side b 2:
$b = 75$

$a + b = 198 = a + 75 = 198$, after transposing 75, $a = 123$. and $a + b = 123 + 75 = 198$: checks

The solution for one variable was made by placing it in terms of the other variable, so as to have a solvable equation. As an extra note, there are many two number pairs that can sum to 198, and many two number pairs that have a difference of 48, but only one pair does both, and for this example , that pair is only 123 and 75.

ALGEBRAICALLY VERIFYING THE LOG RULES

Below are algebraic verifications of the fundamental log rules.

Given: N1 = b^x and N2 = b^y : power equations, therefore:

N1 N2 = b^x b^y = b^(x+y) by the log definition:

$\log_b (N1\ N2) = x + y$ the equivalent expressions for x and y are:

From N1 = b^x, we have by the log definition: $x = \log_b N1$.

From N2 = b^y, we have by the log definition: $y = \log_b N2$.

Using substitution in:

$\log_b (N1\ N2) = x + y$

$\log_b (N1\ N2) = \log_b N1 + \log_b N2$: LOG PRODUCT RULE

Ex. Given log (X + Y) , factor it to where the log product rule can be used:

Factoring out X from each term of the given binomial, we now have two factors:

$\log (X (1 + Y/X)) = \log X + \log (1 + Y/X)$:

Letting N = N1, and given: $\log N + \log N + \log N$

Since repeated addition of any value can be represented using multiplication of that value, we have:

$1 \log N + 1 \log N + 1 \log N = (1+1+1)\log N = 3 \log N$

By treating (log N) like a variable and adding the numerical coefficients of each of the like variables, we can let a variable (such as x shown below) represent this sum of numerical coefficients (c) of (log N). Unlike for repeated addition as shown above, the numerical coefficient(s) need not be 1. Expressing this as an algebraic equation, we have:

$x \log N = (c_1 + c_2 + c_3 + \ldots) \log N = c_1 \log N + c_2 \log N + c_3 \log N + \ldots$

Note, you can also consider this as factoring (log N) from each of the terms, leaving a sum of the numerical coefficients.

Ex. $x \log N = \log N + \log N + 0.3 \log N = (1 + 1 + 0.3) \log N = 2.3 \log N$: x = 2.3

Now, the concept of the log product rule will be used on the following equation:

$x \log N = \log N + \log N + \log N + \ldots$

Using the log product rule in reverse (ie. a "log factor [of a product] rule") on the right hand side:

x log N = log (NNN . . .) : x equals the (integer) power of N
x log N = log N^x : (form of) LOG POWER (OR EXPONENT) RULE

The equation above almost verifies the log power rule, but since all the coefficients of log N were (1), (x) is always an integer and not a real (possible integer plus a possible fractional value) value. Here is a derivation when (x) can be any (ie. real) value:

Eq. 1: N = b^a raising each side to the x power:
N^x = (b^a)x using the power to a power rule, this can be expressed as:
Eq. 2: N^x = b^(xa)

Using the log definition on the two equations above:

$\log_b N = a$ and:

$\log_b$ N^x = xa substituting the above expression for (a):

$\log_b$ N^x = x $\log_b N$: LOG POWER (OR EXPONENT) RULE

Now, the log quotient rule will be algebraically verified, and it also uses the exponent rules for dividing like variables:

$\left(\frac{N1}{N2}\right) = \frac{b\text{^}x}{b\text{^}y} = b\text{^}(x - y) = b\text{^}(\log_b N1 - \log_b N2)$: with both logs having a base of (b).

Using the log definition, or by taking the log (base b) of both sides and simplifying with the log power or exponent rule:

$\log_b \left(\frac{N1}{N2}\right) = \log_b N1 - \log_b N2$: LOG QUOTIENT RULE
Also: $\log_b$ (N1/N2) = $\log_b$ [(N1) (N2)^-1] = (log(b) N1) + (log(b) N2^-1)
= log(b) N1 -1 log(b) N2
= log(b) N1 - log(b)N2

If the base of an indicated power is the same as the base of the logarithm:

$\log_b$ b^x using the log power rule:

x ($\log_b b$) since b^1 = b, $\log_b b = 1$ therefore:

x (1) = x therefore:
x $\log_b b$ = x by the log exponential rule:

$\log_b$ b^x = x : "SPECIAL" log rule. If the log base, and the base of N are the same, then the log expression is simply equal to the indicated power of N.

The number in a log expression, can also be an expression.

Ex. ln (a + 1) = 3.218875 solve for a: Using the log definition, and N = number = (a+1):

$\ln N = \ln_e N = \log_e N$, e is a special number in mathematics, like PI is special.
e is about 2.71, and PI for a circle is about 3.14
When e is used for logarithms, it is called the
"natural logarithm (nl = ln) base"
$\log_e (a+1) = 3.218875 = \ln (a+1)\ 3.218875$

e^3.218875 = (a + 1) = a + 1 Solving for a:
a = e^3.218875 - 1 Using a calculator which can easily calculate powers of (e):
a = 25 - 1
a = 24 : A calculator can easily check this if it can calculate natural logarithms.
$\ln (a + 1) = \ln (24 + 1) = \ln_e 25 = 3.21887\underline{6}$

SOLVING FOR THE BASE OF A POWER

Ex. Solve for x, the base of an indicated or expressed power, given:

x^(2/3) = 10 raising each side to the 3/2 power, we can isolate x as: x^1 = x :

(x^2/3)^(3/2) = 10^(3/2)

x^(6/6) = 10^(3/2)

x^1 = x = $\sqrt[2]{10^3}$ = $(\sqrt[2]{10})$^3 or $\sqrt[2]{10^3}$ = $\sqrt[2]{1000}$ = 31.62

x can also be solved using logarithms:

x^(2/3) = 10 taking the log (use same base) of both sides:

log x^(2/3) = log 10 using the log exponent rule:

(2/3) log x = log 10 solving for log x by dividing by its multiplying factor:

$\log x = \frac{\log 10}{(2/3)} = \frac{\log 10}{0.67}$ from the definition of logs (here x = N, the number):

x = 10^((log 10)/0.67) = 10^(1/0.66667) = 10^1.5

x = 31.62 and 31.62^(0.67$\underline{7}$) = 10

Note, you could also begin to solve for x by first simplifying its fractional exponent as:

x^(2/3) = 10
x^0.6666$\underline{7}$ = 10 and after solving for x: x = 10^1.5 as shown above.
If you were to also take the 0.66667 root of both sides of: x^0.66667 = 10^1, the result would also be the same value for x.

SOLVING FOR AN EXPONENT VARIABLE

Solving for an exponent generally involves using a logarithm expression. Most of this topic has already been discussed previously in the topics of logarithms and solving for exponents, however, here is a more complex example in purely algebraic terms:

Solve for x given:

$\frac{a}{b} = \frac{c^x}{d}$ multiplying both sides by d and canceling:

$\frac{a d}{b} = c^x$ taking the logarithm (can use any base) of both sides:

$\log c^x = \log \left(\frac{a d}{b} \right)$ using the log of an indicated power rule:

$x \log c = \log \left(\frac{a d}{b} \right)$ dividing both sides by log c :

$$x = \frac{\log \left(\frac{a d}{b} \right)}{\log c}$$

Ex. Given $5^x 7^x = 1225$, solve for x.

$5^x 7^x = 1225$ multiplying the bases of similar indicated powers and keeping the exponent:

$((5)(7))^x = 1225$

$35^x = 1225$ taking the log (use any base) of both sides:

$\log 35^x = \log 1225$

$x \log 35 = \log 1225$ isolating x:

$$x = \frac{\log 1225}{\log 35}$$

$x = 2$

Ex. $2^{(5-x)} = 8$ Solve for x: Two methods will be shown.

Method 1:

$2^{(5-x)} = (2^5)(2^{-x}) = 8$: factoring the left hand side

$\log ((2^5)(2^{-x})) = \log 8$ using the log product rule:

$\log 2^5 + \log 2^{-x} = \log 8$ using the log exponent rule:

$5 \log 2 - x \log 2 = \log 8 = (5 - x) \log 2$ if terms factored. isolating the term that contains the x variable, use transposition (transpose):

$-x \log 2 = \log 8 - 5 \log 2$

$x = - \frac{\log 8 - 5 \log 2}{\log 2} = 2$: this is easily solved here when you use a base of 2 for the logarithms

Method 2:

$$\log 2^{(5-x)} = \log 8$$

$$(5-x) \log 2 = \log 8$$

$$5 - x = \frac{\log 8}{\log 2}$$

$$x = 5 - \frac{\log 8}{\log 2} = 5 - 3 = 2$$

Ex. Solve for x:

$$2 = 4^x$$

$$\log 2 = \log 4^x$$
$$\log 2 = x \log 4$$

$$x = \frac{\log 2}{\log 4} = \frac{\log 2^1}{\log 2^2} = \frac{1 \log 2}{2 \log 2} = \frac{1}{2} \quad : = 0.5$$

Note $4^{(1/2)} = 4^{0.5} =$ square-root(4) $= \sqrt[2]{4} = 2$

Note also that this is an example where if you can express the numbers (N1 and N2) as indicated powers having the same base (2 in this example), then the result is simply the quotient of those indicated powers.

Ex. Find x given: $5^x + 6^x = 61$

A more advanced analysis will be discussed, and you may skip over this discussion for now if you want. Since the (x) variables cannot be combined, the method below solves for one instance of (x) in terms of the other instance. Of course you cannot (immediately) solve for a variable when that same variable is also part of the solution for itself. To overcome this limitation, a guess, calculated or not, for the value of (x) is first used with the hope that by repeatedly using the formula, that a very close value of (x) will gradually and eventually be found. When this happens, it is said that the result is converging (becoming closer to, approaching (a certain final value or "limit"), "focusing" or "zeroing in on" and becoming equal) to the true value of (x), and the difference between each new value of x approaches 0, hence continue this process till this difference is as close to 0 as needed. This method of solving for a variable is called successive approximation with recursion (reusing the same formula over and over with each new closer approximation of x to be used in it). More examples of successive approximation will be shown throughout this book. Successive approximation will not always work as shown below:

First, as a very simple example to help consider the above described method:

Find x in the equation: $5 + x = 8$. Rather than mathematically solve and express the equation in terms of, and equal to x, you can take an initial guess, of say, 1 for x. $5+x = 5+1$, and since this left side equals 6 and does not equal the right side of 8, the difference with respect to, or compared to the right side is: $(8-6)= 2$. Therefore, the difference in the sides is not 0, and you can keep increasing the value of x and comparing the left side to the right side of 8. This comparing is done mathematically by taking their difference. If the difference between both sides is 0, they are therefore equal and the actual value of x has been found, and that is 3.

$5^x + 6^x = 61$

$6^x = 61 - 5^x$ taking the log of both sides:
$\log 6^x = \log(61 - 5^x)$ using the log of a power rule:
$x \log 6 = \log(61 - 5^x)$ isolating (x) by dividing by log 6:

$x = \frac{\log(61 - 5^x)}{\log 6}$: The result of (x) here is a better approximation of (x). log 6 is also a constant for this problem and therefore it needs to be calculated only once. After substituting an initial guess for (x) into this expression, a new and closer value to the true value of (x) will result. Continue to use these closer values of (x) until you are satisfied with the result. The value for(x) will converge to 2. Checking using substitution:

$5^2 + 6^2 = 61$
$25 + 36 = 61$
$61 = 61$: checks

The other equation that you can derive from the original equation above for finding the solution for (x), as shown below, will not converge to a single value when using the computer program shown ahead in this book. The solution is still 2:

$$x = \frac{\log(61 - 6^x)}{\log 5}$$

Even though the value of (x) is not initially known, we can find a rough value by noting that if x=0:

$5^0 + 6^0 = 1 + 1 = 2$: since 2 is less than 61, clearly, x must be larger than 0

If x=1:

$5^1 + 6^1 = 11$: since 11 is less than 61, clearly, x must be larger than 1

If x=3:

$5^3 + 6^3 = 125 + 216 = 341$: x must be less than 3 since 341 is greater than 61

For the maximum value of (x), consider that logs are only defined for positive numbers. Since 0 is considered as the smallest positive number:

$0 = 61 - 5^x$: 0 ensures 5^x will be at it's maximum possible value; when x is at maximum
$5^x = 61$
$\log 5^x = \log 61$
$x \log 5 = \log 61$

$x = \frac{\log 61}{\log 5} = \frac{1.785329835}{0.698970004} = 2.554229543$: maximum value possible for a guess of x
Note, this is not the solution to the original equation.

If (x) were larger, the result would be a negative number for (61 - 5^x). 5^x was used instead of 6^x since division by log 6 would result in a slightly lower value. The true solution of (x) is 2. See A METHOD TO SOLVE DIFFICULT EQUATIONS in the APPENDIX section for a computer method that can be used to solve equations such as that shown above.

SOLVING FOR AN INDEX VARIABLE

The easiest way to solve for an index (the indicated root of the radicand in a radical expression or root operation) is to first rationalize the radical by raising it to an indicated power equivalent the index, then convert the radical to its exponential form and simplify, or vice-versa. This effectively transforms the problem into one of solving for an exponent.

Ex. What "root" of 1024 is equal to 4 ?

Expressing this mathematically:

$\sqrt[x]{1024} = 4$ raise each side to the (x) power:

$(\sqrt[x]{1024})^x = (4)^x$ convert the radical to exponential form:

$(1024^{(1/x)})^x = 4^x$ using the power to a power rule:

$1024^{(x/x)} = 4^x$

$1024 = 4^x$: This could have easily been found by inspection by considering:
"What is the exponent of a base of 4 so that the power would equal 1024".
OR:

$\sqrt[x]{1024} = 4$ expressing the radical with its equivalent exponential form:

$1024^{(1/x)} = 4$ raising each side to its x power:

$(1024^{(1/x)})^x = 4^x$ using the power to a power rule, multiplying exponents:

$1024^{(x/x)} = 4^x$ simplifying:

$1024 = 4^x$: the same equation as shown above
Taking logs of both sides:

$\log 1024 = \log 4^x$ using the log exponential rule:

$\log 1024 = x \log 4$ dividing each side by log4:

$\frac{\log 1024}{\log 4} = \frac{x \overset{+1}{\cancel{\log 4}}}{\underset{+1}{\cancel{\log 4}}}$: isolating x

$x = \frac{\log 1024}{\log 4} = \frac{3.0103}{0.60206}$

$x = +5$, hence $\sqrt[5]{1024} = 4$, and $4^5 = 1024$

If you were to write a simple formula for the above process, it would look like this:

$i = \frac{\log \text{radicand}}{\log \text{root}}$: **FORMULA FOR THE INDEX , THE INDICATED ROOT OF A RADICAND**

Also note the similarity of this formula, and the actual values used, to that of the formula for solving for an exponent shown below. The only thing different is the wording or description for the values. This is due to the close relationships of the power, root, and log concepts and expressions, as shown previously in this book. Using the last example:

From: $1024 = 4^x$

$$x = \frac{\log 1024}{\log 4} = \frac{3.0103}{0.60206} = 5$$

$$x = \frac{\text{log power}}{\text{log base}}$$: **FORMULA FOR THE EXPONENT OF AN INDICATED POWER**

EQUATING RADICALS (Solving For A Radicand)

As mentioned previously, the easiest method to solve for variables involved with radicals is to first place the radicals into exponential form.

Ex. $\sqrt[2]{100} = \sqrt[3]{N}$ placing radicals into exponential form:

100^(1/2) = N^(1/3) taking the logarithms of both sides of the equation, use any base (b) and use it consistently in a problem.

log 100^(1/2) = log N^(1/3) using the "log exponent rule":

$$\left(\frac{1}{2}\right) \frac{\log 100}{1} = \left(\frac{1}{3}\right) \frac{\log N}{1}$$

$\frac{\log 100}{2} = \frac{\log N}{3}$ solving for log N, and switching sides of the equation:

$\log N = \frac{3 \log 100}{2}$ since the right side is essentially the exponent of the base to equal N, and choosing a base of 10:

N = 10^((3 log 100)/2) : after expressing the anti-logarithm or "removing or reversing the logarithm" (here, from N) using the basic definition of logarithms. Now, after simplifying the exponent:

N = 10^(6/2)

N = 10^3 = 1000

hence: $\sqrt[2]{100} = \sqrt[3]{1000}$
checking: 10 = 10

Writing a general formula for the above procedure:

$\sqrt[i1]{N1} = \sqrt[i2]{N2}$: i1 and i2 are the indexes of their corresponding or associated radicals

N2 = b^((i2 $\log_b$ N1) / i1) : EQUATING RADICALS (RADICAND FORMULA)

EQUATING RADICALS (Solving For An Index)

Ex. $\sqrt[2]{100} = \sqrt[x]{1000}$ converting both sides to their equivalent radical form:

$100^{(1/2)} = 1000^{(1/x)}$ taking logarithms of both sides, be sure to use the same base:

$\log 100^{(1/2)} = \log 1000^{(1/x)}$ after using the log exponent rule:

$\frac{\log 100}{2} = \frac{\log 1000}{x}$ after solving for x:

$x = \frac{2 \log 1000}{\log 100} = \frac{6}{2} = 3$

hence: $\sqrt[2]{100} = \sqrt[3]{1000}$

checking: $10 = 10$

Writing a general formula for the above procedure:

$i2 = \frac{i1 \log N2}{\log N1}$: EQUATING RADICALS (INDEX FORMULA)
N1 and N2 are the radicands of the radicals.

Some other observations and derivations can be made here. Considering the above formula, we then have:

$\frac{i2}{i1} = \frac{\log N2}{\log N1}$: when two radicals are equal, the ratio of the indexes is equal to the ratio of the logarithms (with the same base) of the numbers. N1 and N2 are the radicands of the radicals.

Some of the following may, or may not, appear obvious according to the logarithm and radical definitions. We know that the log of a value is an exponent (here of any base that is used consistently in a problem), hence:

$\frac{i2}{i1} = \frac{\text{exponent2}}{\text{exponent1}} = \frac{\log N2}{\log N1}$: base^exponent2 = N2 and base^exponent1 = N1

EQUATING A POWER OF A NUMBER TO THE LOG OF THAT NUMBER

Here, the base of a power of (N) is the same as the number (N) of a logarithm and we are to find the proper exponent that will make that power value of (N) equal to the logarithm of (N).

$$N^x = \log_{b1} N$$

Taking the log of both sides (can use any base (b2) as long as you use it on both sides): Note that the base for the logarithm on the right hand side can also be any value. It could be 10, 2, 7, (e) or any other value.

$$\log N^x = \log (\log_{b1} N)$$

using the log of a power rule:

$$x \log N = \log (\log_{b1} N)$$

isolating x, by dividing both sides by (log N):

$$x = \frac{(\log (\log_{b1} N))}{\log N}$$

: note, log N in the numerator is not a common factor of the entire numerator, and therefore it cannot be canceled out by the denominator. (Log N) in the numerator is the Number value of another logarithm. Note that (Log N) in the numerator might even have a different base than (Log N) in the denominator, but it depends on how it was expressed in the original problem.

Here is a solved example where x = about 0.1505

$$100^{0.1505} = \log_{10} 100 = 2$$

: Another way of solving for x is to first simplify the right hand side as shown to a value of 2, hence this example would then become: $100^x = 2$, and then of solve for exponent x in the normal manner using a logarithm of both sides.

EQUATING A ROOT OF A NUMBER TO THE LOG OF THAT NUMBER

Here, a radical is set equal to the logarithm of the same radicand value, and we are to find the proper indicated root (index) of that radicand.

$\sqrt[x]{N} = \log_{b1} N$: any base for the logarithm. Expressing the radical in exponential form:

$N^{1/x} = \log_{b1} N$: taking the log (with any consistently used base) of both sides:

$\log N^{1/x} = \log(\log_{b1} N)$ using the log "exponent" or "(indicated) power" rule:

$\frac{\log N}{x} = \log(\log_{b1} N)$ solving for x:

$$x = \frac{\log N}{(\log(\log_{b1} N))}$$

Observe that this value for (x), the index of a root, is the reciprocal of the value for (x) as just shown above as for the indicated power when equating the power of a number to the log of the number. When you compare $N^{1/x}$ to N^x, or a power of N to the same root of N such as in the above two formulas, you will see that the exponents and their values are reciprocals of each other, and this leads to the reciprocal-like results of the two formulas.

Here is a solved example for the index (x), where N=100, and x=6.644:

$\sqrt[6.644]{100} = \log 100 = 2$: again, simplifying the right hand side, as shown to a value of 2, is the easiest way to solve this equation, as: $\sqrt[x]{100} = 2$

As an extra note, when the index of a radical is less than one, it actually represents an indicated power greater than one for that radicand.

Ex. Here is the half-root (1/2 = 0.5) of N, and it's actually equivalent to a larger value than that base of N:

$\sqrt[0.5]{N} = N^{(1/0.5)} = N^2$ Ex. $\sqrt[0.5]{10} = 10^2 = 100$

Of course, if N is less than one , a (proper) fractional value, then the expression would actually represent a value less than N since repeated multiplication of a fractional value is essentially a fraction of that fraction, and which results in a lesser value. For example,
$\sqrt[0.5]{0.3} = 0.3^{(1/0.5)} = 0.3^2 = 0.3^2 = (0.3)(0.3) = 0.09$

SIMPLE EQUATIONS CONTAINING RECIPROCALS

Given: $\frac{1}{G} = c\,G^x$ Here, G and cG^x are reciprocals of each other. (c) is a constant. The expression of: (1/G) is the reciprocal of G. The expression of: $c\,G^x$ is a multiple of G^x since (c) is multiplying factor to G^x. (c) and G can have any value except 0. Solve for c, G and x:

$\frac{1}{G^1} = c\,G^x$: note here that $G^x = (1/c)(1/G^1) = (1/cG^1)$, multiplying both sides by G:

$1 = cG^{(x+1)}$ dividing both sides by $G^{(x+1)}$, and switching sides:

$c = \frac{1}{G^{(x+1)}}$ For example, if x = 1: $c = \frac{1}{G^2} = \frac{1}{G^1}\ \frac{1}{G^1}$

Ex. Given: $\frac{1}{5} = c\,5$, $x=1$, $c = \frac{1}{G^{(x+1)}} = \frac{1}{5^2} = \frac{1}{25} = 0.04$: $G=G^1=5$

Note also that $1/cG = G^x$, here,for this example: $1/((0.04)(5)) = 1/0.02 = 5$

After mathematically solving for $G^{(x+1)}$, such as by taking the reciprocal of both sides::

$G^{(x+1)} = \frac{1}{c}$ taking the (x+1) root of both sides:

$$\sqrt[(x+1)]{G} = \sqrt[(x+1)]{\frac{1}{c}} = \frac{\sqrt[(x+1)]{1}}{\sqrt[(x+1)]{c}} = \frac{1}{\sqrt[(x+1)]{c}}$$

Given:

$c\,G^x = \frac{1}{G}$ solving for x:

$C^x = \frac{1}{cG}$ taking the log of both sides:

$$\log c^x = \log\left(\frac{1}{cG}\right)$$

$$x \log c = \log\left(\frac{1}{cG}\right)$$

$$x = \frac{\log\left(\frac{1}{cG}\right)}{\log c}$$

From the initial equation this can also be had:

$1 = cG^x\,G^1 = c\,G^{(x+1)}$ switching sides:

$c\,G^{(x+1)} = 1$

$G^{(x+1)} = \frac{1}{c}$ taking the logarithm of both sides:

$\ln G^{(x+1)} = \ln(1/c) = -\ln c$

$(x+1)\ln G = \ln(1/c) = -\ln c$

$x + 1 = \frac{\ln(1/c)}{\ln G} = \frac{-\ln c}{\ln G}$

$x = \frac{\ln(1/c)}{\ln G} - 1 = \frac{-\ln c}{\ln G} - 1$

If $G^x = \frac{1}{c}$ solve for x. Taking the log of both sides:

$\ln G^x = \ln\left(\frac{1}{c}\right) = -\ln c$

$x \ln G = \ln(1/c) = -\ln c$

$x = \frac{\ln(1/c)}{\ln G} = \frac{-\ln c}{\ln G} = -\frac{\ln c}{\ln G}$

SOME EXAMPLES OF ALGEBRA WITH GEOMETRIC FORMULAS

Geometery is a word that means the measure (meter, metery, metric) and analysis of earthly (geo), physical (real) or virtual (as if real) things or ideas.

Since this book has a trigonometry section, the first example will be directed towards finding the formula for the area of a triangle.

Ex. What is the formula for the area of a triangle?

Previously in this book, some basic concepts about area have been discussed in the topic of Powers. Below are some more fundamental details about the concept of area.

Distance is a measurement of how many units, or "long" (ie. length), something is, or the length between two points or locations, hence it is a measure of the amount of separation, which is therefore a difference, between things. Distance or length is a linear or line-like concept. The length or distance between two points is found by measurement and-or the difference between those two points having a known linear value. In short, linear (having a line-like quality or relationship) means one-dimensional which basically means having only a single direction for its structure and measurement. Lines are one-dimensional geometric concepts. Lines are infinitely thin, that is, in theory or concept, they actually have no thickness, and are infinitely long, and each extends infinitely outwards in one specific direction. Being one dimensional, a line and its can graphically represent one dimension and its concept. Likewise, one dimensional concepts are therefore often associated with lines and are said as being linear. A line can also be considered as being composed of an infinite number of infinitely small points, all being side by side or next to each other.

One dimensional means that something can only be measured, so as to quantify it, in only one way, such as the length of a line segment that is a small part of a line. A line segment is a defined or (length) bounded region, section or segment of a line.

Two dimensional basically means that something can be measured, so as to quantify it, in two ways, such as a length (dimension) and width (dimension) of a defined or bounded area that is part of the infinite area of a plane. Planes are essentially infinitely "flat surfaces" that have only two dimensions: a length dimension and width dimension, also known as the first and second dimensions, but no thickness or third dimension. Two dimensional measurements will naturally build upon and use the one dimensional concept of length. Some examples of two dimensional structures are squares, circles and triangles which are a small defined or bounded (having defined boundaries) part of an infinitely large plane. A plane can be thought of as an infinite number of side by side parallel lines so as it creates a width, thickness, second dimension. Parallel lines will never intersect or cross each other. Each of the two dimension of a plane are defined and understood as being perpendicular (90° of rotation apart, right-angle, exactly "sideways" or "cross") to each other.

Three dimensional means that something can be measured in three ways such as the length, width, and height (or "thickness") of a bounded or defined volume, space or solid structure that is part of the larger and infinite volume of space. Three dimensional measurements will be done using the concepts of two dimension measurement of which are based on the linear or one dimensional concept. Therefore, it is possible to conceive a solid or volume as being a bounded or defined segment, section, or region of an infinite number of side by side parallel planes. The third dimension of a volume is defined and understood as being perpendicular to the other two dimension of a plane.

A drawn horizontal (left to right) line can be thought of as graphically representing the first dimension, and a vertical (up and down) line can be thought of graphically representing the second dimension. A line extending straight through the paper can be thought of as graphically representing the third dimension.

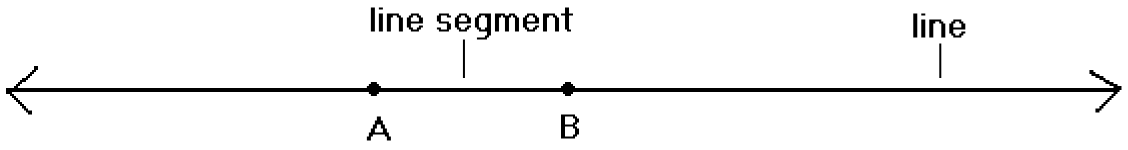
The perimeter (the bounding and defining sides, and-or its corresponding length values) of a triangle or the circumference of a circle, both two-dimensional objects, can still considered as a single (total) length value even though it is not a single (straight) line or segment but rather the sum of them, and that the objects are two-dimensional. This total distance or sum of various lengths can then be thought of and represented as one single line segment.

Again, a distance or length concept is considered as a one-dimensional concept. A distance or length value is a measurement of just one-dimension that is bounded or defined. Because the second and third dimensions are based on the concept of one-dimension, one-dimensional measurements and values of distance or length will also be used to bound, define and-or measure areas (ie., surfaces) and volumes (ie., space).

One of the most fundamental or basic concepts of geometry is the point. A point can be thought of as both an infinitely small object and-or a location for it. Hence a point is either physical (object), and-or logical (a location, address, numeric). A line can be thought of as composed of an infinite number of points. How do you then locate or identify a point on a line or line segment? It is very difficult without another known or agreed upon point that can be used as a reference of measurement or addressing point for all other points. Like numbers being in reference to some value, often 0 as for the common counting numbers, a location is always in reference to some other known (reference) location which can be even considered as 0 (or having a address of 0). A location of something can be described or represented as a numeric address or distance value (using some standard unit of measurement) from a reference address, point or location. The units of (distance or length) measurement used are a defined length unit. It takes two points to create or define a line segment. These points can be thought of as side by side or separated by some distance, with an infinite number of points between them. The number of points possible on a line segment of any size of length is infinite (unending, uncountable), and the definition of a line can then be thought of as one continuous (unending) line segment. The smallest, least, or shortest distance between two points is equal to the length of a line segment between those two points.

[FIG 25]

line , the arrows indicates it extends infinitely, and therefore it cannnot be measured to express its length. It's length is therefore said to be infinite.

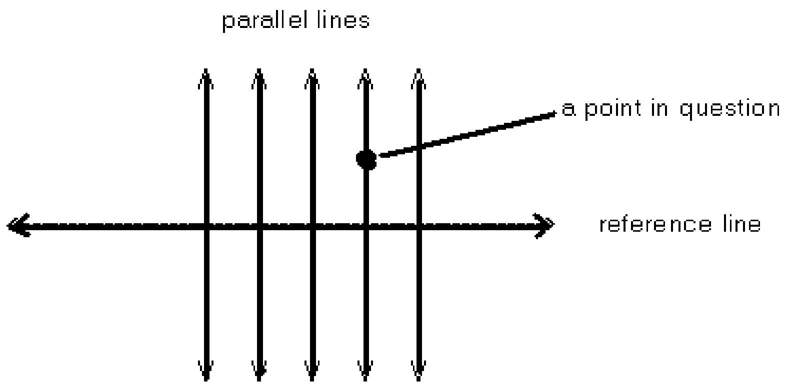


line segment, a (bounded or defined) part or portion of a line that can be measured to express its specifc or definite length

line segments can be identified with two defining end-points, here labeled or expresed as point A and point B for line AB = $\overline{AB}$

All points (or locations) on a plane can be identified (ie. its numerical distance or location) by two perpendicular distances from any point chosen as a reference point of measurement and-or addressing for the system. This distance and-or its (address) location is the point's linear (distance in each dimension of a plane or surface) coordinates in a rectangular coordinate system of measurement. Coordinates is a term used to describe a point's two corresponding (or "coordinating" or "coexisting") location values. Point (0,0) is usually used to indicate the common and starting (measurement and location) reference point for all dimensions or axis of the measurement system. Why perpendicular? First, all points on a line are unique to that line only, that is, they can only be identified as a distance from another reference point on that one specific line. To identify or locate a point on a plane or surface would require that you know which particular line (of an infinite number of parallel lines), so perhaps from a line chosen as a reference line, and the distance to that point along that line from a "base" or reference point on that line. Since there is an infinite number of parallel lines that are uniform to each other, it is best to make all the reference points of these parallel lines uniform also, hence placing them all along a common fixed "reference" line in this (location) system. Still, a problem now would be identifying a line among an infinite number of parallel lines, and that is not practical. This reference line need not be perpendicular to these parallel lines, but it is usually set perpendicular (a "right-turn" or "right-angle" = a quarter of a full 360° circular turn = ninety degrees = 360°/4 = 90°) since it is essentially the least parallel-like of all possible parallel lines. It is also done in order to create a rectangular coordinate (measuring) system. Rectangular (which means "right-angular") is a term that is used to describe something that is: "square-like", where each dimension or side is rotated 90 degrees in reference to each other.

Some initial concepts for creating a coordinate system on a plane. [FIG 26]



As mentioned, finding the location of a point on a plane using this system, shown in the above figure, is still difficult since there is an infinite number of parallel lines to identify so as to locate a specific parallel line, and then to locate the specific point in question along that specific line. To overcome this problem of identifying and-or address a line, you could assign a particular parallel line as a reference (of measurement) line and then identify each line by its practical (and not the infinite and hence undefinable line number) length or distance from this reference line using a standard (agreed upon) unit of measurement. The outcome of all this is that a points location on a two-dimensional plane or area is identified by it's two perpendicular distances (which could be represented as dotted lines) from two reference lines.

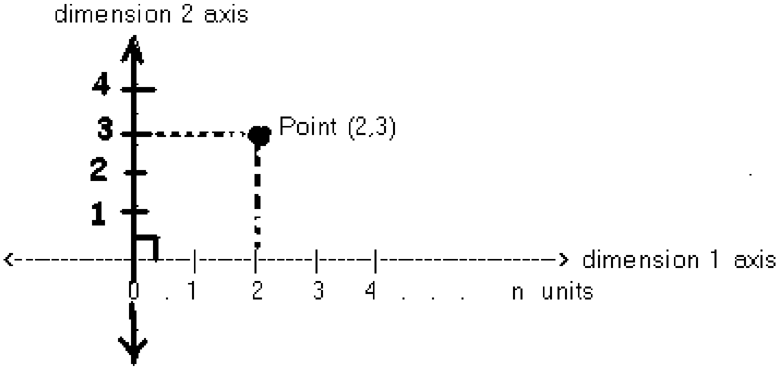
Each point on a plane or surface is at a unique (only one instance of) intersection of two specific lines in a rectangular measuring and addressing system. This perpendicular system is the simplest and most practical method to locate (give an "address" to, or identify and refer to) a point on a plane or surface. The location or point where the two reference lines (of measurement in this system) intersect or cross is called the reference point (that the location of all other points on that plane will be in reference or respect to) or **origin** ("original", source, reference, or starting) point of the system, and it is therefore has an address, "offset" or distance value of 0 with respect to the two reference lines of the system. The location of this point, and any other points in that system, is typically written as a pair of coordinates (its two corresponding location values). The location of the origin of the system is written as: (0,0) or p(0,0) which can be read as: "point zero comma zero". The two perpendicular reference lines drawn through the origin are commonly called **axis lines** of the coordinate system. Axis is a word based on "axel" and "center" location such as of a wheel. These two lines represent the reference lines of measurement for each of the two each dimensions of a plane or rectangular the coordinate system. Each of the two coordinates of a point are effectively the (perpendicular) distance to (or from) the other coordinates axis line of reference. If given an address of (x,y), the x-coordinate is the points distance from the y-axis, and the y coordiante is the points corresponding distance from the x-axis.

The coordinates of a point (p) are often expressed using the basic format of:

p**(**length or position in dimension1 **,** length or position in dimension2**)**

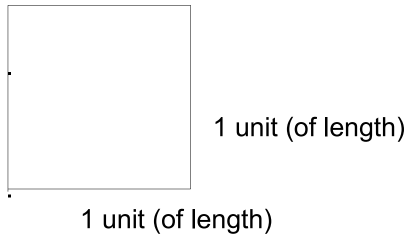
Examples: p(x , y) , p(horizontal, vertical) p(left and right, up and down)
p(variable1,variable2) , p(independent variable, dependent variable)
p(time, temperature) , p(age,weight) , p(day,sales)

[FIG 27] A Rectangular Coordinate System



For an analogy (a similarity or likeness) of this addressing scheme, consider this: If a plane represented a city, the origin would be the town-square or center, the (x) co-ordinate (distance in dimension 1) would be the street location from the town-square, and the (y) co-ordinate (distance in dimension 2) would be the location along that street.

Since plane or flat surfaces are now defined as two-dimensional, the basic unit of measurement of a plane surface (area) cannot be in length units alone - which are used to measure only one-dimensional lines. The obvious choice is to have a reference piece of surface as the basic unit of measurement of areas (bound, defined regions) of a plane, and this is similar to the method of measuring one dimensional distances by using a standard reference piece of length as the unit of measurement or reference to compare and quantify all other lengths to. Other surfaces or portions of surfaces can therefore be measured (comparing and counted) in reference or with respect to this special piece or unit of reference or measurement surface. Since a measurement for each dimension will be in reference to the (hopefully similar) units being used for each dimension, this basic unit for area or surface measurement should then also include both dimensions and the units used for both dimensions. Since dimensions are at right angles, this reference piece or unit of surface area will have the shape below which is called a square. A square is a bounded or defined piece, segment, or "area" of a plane, and can therefore be used as a common reference or measuring unit to measure and-or represent the size of other areas. [FIG 28]



: basic unit of surface measure
: 1 square-unit

Like a scaled (marked with several equivalent unit distances) ruler line is to measuring distances, these special squares are to measuring surface areas. Since the shape is a square with each side having a length of 1 unit (feet, inches, meters, centimeters, etc), it is common sense to call this unit of measurement a "square unit" or "unit square". A square-unit is the basic unit of measurement for measuring surfaces. In mathematical notation:

1 square unit = 1 unit square = 1 sq. unit = (1 unit) x (1 unit) = (1)(1)(unit)(unit) = 1^2 unit^2 = 1 unit^2 = unit^2

Consider the algebraic notation when width = length, as for a square: (length)(width) = (length)(length) = length^2
Also, if that length was just one unit long, that length unit squared is also the unit of area measurement:
length^2 = unit^2 = squared unit

The actual measured size of a portion of a plane or surface is called the surface's area, area of surface, or just area. The word "area" is sometimes loosely used as meaning surface itself, instead of a measurement of it. Since a surface is measured in reference to a square-unit, the actual number of units of a measured surface or "area" will be equal to a specific quantity or number of those square-units of measurement.

If you had 4 separate surfaces, each with an area of 1 unit^2 = 1 square-unit, their total area would be:

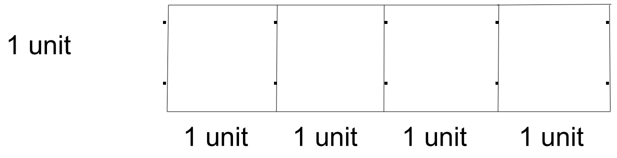
Total Area = (1 unit^2) + (1 unit^2) + (1 unit^2) + (1 unit^2) adding like units:
Total Area = (1 + 1 + 1 + 1) unit^2 = 4 unit^2 : "4 units square" = 4 units^square = "4 square units"

If the reference length units used to define a reference square-unit were feet units, the units of surface measurement must therefore be expressed with those same units:

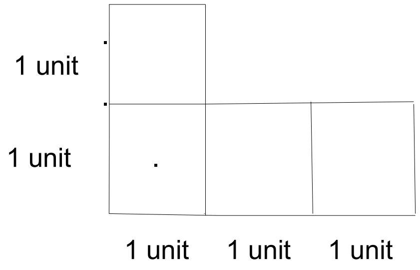
Total Area = 4 unit^2
Total Area = (4)(1 ft x 1 tt) = (4)(ft^2) : the word feet is actually the multiple or plural of foot (ft)

= 4 feet^2 = 4 ft.^2 : note, this expression: units^2 , is not a mathematical operation, and it is
= 4 square-feet just the mathematical notation used for these types of units of measurement since it was defined this way from two linear units: (1 unit)(1 unit) = 1 unit^2 = unit^2 . That is, the unit of area reference is an area that is bounded or defined as a square, 2 dimensional shape, and the unit of reference is therefore not just a line or one dimensional unit only.

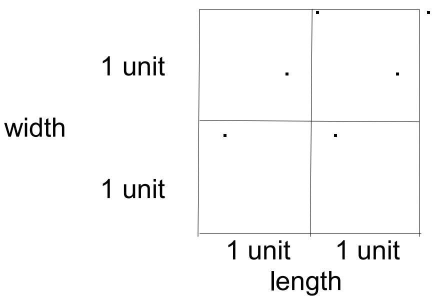
These different shaped surfaces shown below can be measured as a sum of unit areas, and the total area is equivalent to any other possible surfaces having the same total area of 4 ft.^2. This can easily be seen when these separate surfaces are joined together, particularly when the entire shape becomes a rectangle and the area can be easily calculated.
[FIG 29]



: total area = 4 square-units =
= 4 units-square = 4 units^2



: total area = 4 units^2



: total area = 4 units^2

The first and the last surface shapes shown above can help verify an important concept. When a bounded or defined surface has straight (or linear) sides that are set at a right (or "square", 90 degree) angle with respect to each other, and remain constant throughout the surface, the corresponding sides (or "lines") of that surface are said to be parallel and "opposite (across from) from each other", and the shape or structure of this surface is called a rectangle. First, the word "**angle**" is an old French word that means a "corner" where two sides, such as two walls, intersect, meet or join at. A rectangle ("**rect**" is an old Latin word that means a "right" turn, bend or direction, hence 90° like that of a square shape) is essentially a square surface that has been elongated (in one direction or dimension) by increasing the distance between one pair of parallel sides. The longest or largest side of a rectangle is commonly called it's "length side", and the shortest or smallest side is commonly called it's "width side". The word "width" is similar to the word and meaning of the word "wide". The expression for the total area of any rectangle shape will becomes one of repeated addition (of square units), which as we know, can be expressed simply or more compactly as multiplication:

To find the area of a given rectangle, divide the entire rectangular surface into rows or columns of square units, where each division or part is 1 unit long for each of the length and width dimensions or sides (a side is a portion of the entire boundary or perimeter of the surface or object) of each square unit. If the side of the rectangle is (n) units long, there will be (n) such rows of square units, and the total area (A) of that rectangle (r) can then be expressed as:

Ar = area of row 1 + area of rows 2 + area of row 3 + . . . + area of row n

Since each row of a rectangle is equivalent in area, to say the area of row 1, and there are (n) such rows, by algebraic substitution, we have:

Ar = area of row 1 + area of row 1 + area of row 1 + . . . + area of row 1

Expressing this repeated addition with a multiplication expression:

Ar = (n)(area of 1 row)

Since the area of each row is equivalent to length (L, of the row) square units, of which each unit automatically includes a unit of width (W), or a width length units, and that (n), the number of rows, is equivalent to the numeric value of the rectangle's width, that is, it is the number of width (W) units:

Ar = (W)(L units^2) By multiplication:
Ar = (LW) units^2 : **AREA OF A RECTANGLE = (length units)(width units)**

We see that the formula is equivalent to multiplying the number of length (L) units by the number of width (W) units:

Ar = (L units)(W units) = LW units^2

For a similar check, if there are L squares, or L square units per row, and there are (W) rows, the total number of squares in a rectangle is just a matter of repeated addition or multiplication:

Ar = (L squares)(W) = (LW) squares

If each square with sides 1 unit long is the unit of measurement for area, that is, each is a square-unit:

Ar = LW square-units = LW units^2

The same formula for the area of a rectangle is also naturally used if the area being measured also has or includes a fractional part of a square unit. For example if a rectangle is measured to be 5.2 centimeters long, and 2 centimeters wide, the total area of that rectangle is: Ar = LW units^2 = (5.2 cm)(2 cm) = (5.2)(2) square-centimeters = 10.4 cm^2

As a final description of (two-dimensional) area, the corresponding change or increase in the amount of area is not a linear (balanced, direct, one-to-one, matching, same, constant portion or ratio) concept when its (one-dimensional, or linear) sides change in size or value. The formula for area in relation, respect or in reference to its sides is not a linear expression or resulting value. When a line has or will be increased in length by some factor, you can find its new or resulting length value by adding repeatedly or multiplying a value to the old length and value. If you add a length to itself, you double its length and value. This is the same as multiplying that value by 2. However, if you were to double the sides of a square, it's corresponding area does not just double, but actually increase by the square (the second power) of this (increasing) factor. The area will then increase by the (multiplying) factor of: (the increasing or multiplying factor of its side length)^2. For this example, where the side length of a square increased by a factor of 2, the corresponding area will increase by a factor of = 2^2 = 4 The new area is 4 times more or bigger. It could be said that the area has been magnified or multiplied by a factor of 4. Here is an algebraic or symbolic mathematical verification:

A = (s)(s) square-units : s = side length of a square, where length=width,
A square is actually a special or unique instance and form of a rectangle

A = s^2 square-units : **FORMULA FOR THE AREA OF ANY SQUARE** = **(side)(side)**

Since the variable does not have an exponent equal to just 1, this
mathematical relationship of (s) and A, will not graph as a line.
Their mathematical relationship is said to be non-linear.

Doubling each side of a square we find::

A = (2s)(2s) units^2 = A1 =
A = (2)(2)s^2 units^2 = 2^2 s^2 units^2 = 4s^2 units^2 : when each side of an area is increased by a factor, that area increases by: (factor^2).

A2 = ((factor)(s))^2 = (factor^2)(s^2) = (factor^2)(area) = (factor^2) A1

Dividing this new or second area A_2 by the older or original area A_1 that was initially shown and expressed above as just variable A, we can find the number of times that A_2 is bigger than A_1, or the number of times that the original area A_1 has been magnified (a multiplicative or factor increase = increased by a multiplying factor value, rather than being just a simple linear increase by adding some value).

$$\frac{A_2}{A1} = \frac{4\,\cancel{s^2}}{\cancel{s^2}} = 4$$

: the new area is four times bigger or larger than the old area, such as is the result when that old area has been increased by a factor or four after doubling (2) its sides.

If the side length of an area tripled (increased by 3 times), the area will increase by: (increasing factor)^2 = 3^2 = 9 times more.

A = (3s)(3s) units^2
A = (3)(3)s^2 units^2 = 3^2 s^2 units^2 = 9 s^2 units^2 , hence A2 = 9 A1

The mathematical relationship between area and length is said to be a non-linear relationship, and specifically, it is an exponential relationship. Area grows or increases in value much more rapidly than the side lengths of that area. As the sides length changes or increases, the change in the areas increases by an exponential rate. This is due to the squared value: side^2 = s^2 in the formula for area. Consider this small chart below, and the descriptions on the right side as a more advanced analysis of the mathematical relationships of side length and area, and may be skipped over for now.

Side length of the square	Area of that square	
1	1 sq. units	
2	4 sq. units	: the side increased by 1, and the area increased by a value of 3 1 is the difference between the new and old side lengths values. 3 is the difference between the two area values At this point, as the side increases by 1, the area increases by 3. At this point, the areas is increasing by 3 for each 1 increase in the side. The rate of a change in area to a 1 unit change in a side is 3. (change in area) / (change in side) = 3 : rate of changes or (difference in areas) / (difference in sides) = 3/1 =3 : rate of changes Generally, a rate should rather be thought of as a multiplicative increase, magnification, or factor, rather than a difference. This value of 3, or the rate is not a fixed or constant value but depends upon the specific value of the side length and area.. When the mathematical relationship of: area with respect to side is graphed, the higher the rate, or mathematical relationship, the more "steeper" the curve becomes as the side increases in value. In a linear relationship, this rate, depending on the equation, would be a constant value. For example y=2x, and no matter what value x is, if x changes by 1,the corresponding change in the value of y will always be 2. (change in y) / (change in x) = 2, therefore: (change in y) = 2 (change in x). The mathematical relationship between and y and x is said to be a constant or linear mathematical or numeric relationship. This rate (of changes) value of 2 is also the mathematical expressed steepness or "slope" value of their graphed relationship or "curve" that shows all the corresponding values of those two variables, here, x and y, as points on the graph.
3	9 sq. units	: again, the side increased by 1, however, the area now increased by a value of 5 square units. Clearly area is increasing at a faster rate than the sides, and therefore their numeric relationship is not linear or proportional. 1 is the difference between the new and old side lengths values. 5 is the difference between the new and old area values. At this point, as the side increases by 1, the area increases by 5. At this point, the areas is increasing by 5 for each 1 increase in the side. The rate of a change in area to a change in side is 5: (change in area) / (change in side) = 5 or: (difference in areas) / (difference in sides) = 5 This value of 5, or the rate of changes, is not a fixed or constant

		constant value but depends upon the specific values of the variables being considered. Previously, when the side increased by 1, the area only increased by 3.
4	16 sq. units	: the side increased by 1, and the area increased by 7
5	25 sq. units	: the side increased by 1, and the area increased by 9

Notice that when the side length increase or changes by 1, a constant value, that the area increases or changes (ie. a difference) not by a constant value (which would make the relationship between area and length a linear mathematical relationship), but by a variable and growing (not constant) value.

Likewise, if the area of a square is only doubled, the corresponding sides are, or were not actually doubled either, as would be the case for some kind of linear or direct relationship, but they are only increased by a (multiplying) factor of:

$\sqrt{2}$ which is about 1.414, and here is the verification:

$A = s^2$: formula for area of any square. Solving for the side variable (s) given the area:

$s = \sqrt{A}$: **FORMULA FOR THE SIDE OF A SQUARE SHAPE OR AREA**

Given: $A = s^2$ multiplying each side of the equation by 2, so as to double the area:

$2A = 2s^2$ factoring 2 (here, for its two sq. root factors):
$2A = \sqrt{2}\,\sqrt{2}\,(s)\,(s)$ by the associative law :
$2A = (\sqrt{2}\,s)\,(\sqrt{2}\,s)$: each side or dimension of a square must have the same length
$2A = (\sqrt{2}\,s)^2$: if area of a square doubles, the sides only increase by the square root of 2 (about 1.414), and which is lesser value than 2.

In general, if some area and value is said to change by a factor of N, the formula for the sides of a representative square that has the same equivalent area is:

From: $s = \sqrt{A}$ If area increases or is magnified by some factor N:

(Note, we cannot use: $s = N\sqrt{A}$, since this is actually only a multiple of $\sqrt{A}$ or= s, and this is not equal to the area value = A)

$s = \sqrt{NA}$ which can be expressed as:

$s = \sqrt{N}\,\sqrt{A}$: If area increased by a factor of N, the corresponding sides of that (equivalent square shaped) area will only increased by the factor of: $\sqrt{N}$, and not N. Shown above, when the area doubled (2), the side length of the square increased by the (multiplying) factor $\sqrt{2}$ = about 1.414

If it is said that the side length of an area (being thought of, represented or considered as a square) changed by a factor of N, a general expression for area would be:

From: $A = s^2$ if the side length changed by a factor of N:

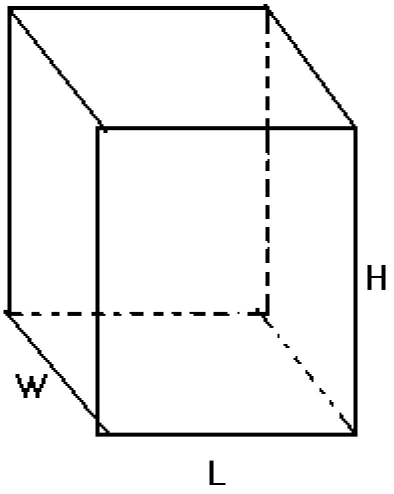
$A = (Ns)^2$ which could be expressed as:
$A = N^2 s^2$: we see that when the sides of a square change by a factor of N, that the area of that square will actually change by a factor of: $A / s^2 = N^2$

Much of the reasoning that went into the discussion about area can also be applied to the concept of volume or spatial (spaces) measurement. Volume is considered as a three-dimensional concept and measurement. The standard unit of measurement or reference of a volume is a cube shape, and since it is defined so to have sides of 1 unit of length long, it is called a cube unit = cubic unit.

A cube or "square-block" is essentially constructed from a square plane shape that is moved a unit distance that is perpendicular to the (planar) square's two (ie., length and width sides) reference lines or dimensions. For the volume (V) of a cube:

V = (Length units)(Width units)(Height units) : volume of a cube , and where all the units are identical
V = ("Base Area" units^2)(Height units^1) : Base or "Bottom" square area will be moved Height units.
V = LWH cubic-units = LWH units^3 : a cube is the reference (unit) of measuring volume [FIG 30]

A 2 dimensional visual representation of a 3 dimensional cube.



L = W = H = 1 unit long

Cube = (base) x height = (1 unit x 1unit) x 1 unit = 1 unit^3 = 1 cubic unit

Ex. An irregular shaped container is said to contain a volume of 100 cubic units of some liquid. If this liquid is put into a (uniform, equally sided) cubic shaped container, what would the length of each dimension be?

First: volume = unit^1 x unit^1 unit^1 = units^2 x units^1 = units^3 , taking the third or "cubic" root of each side:

$\sqrt[3]{\text{volume}} = \sqrt[3]{\text{units}^3} = \text{units}^1$

The side length of a cubic structure having a volume of 100 cubic units is:

$\sqrt[3]{100} = 4.64\underline{2}$ units long : the length units are usually those indicated with the original volume (ex. feet, inches, meters, centimeters, etc)

Checking: Using the above value, the total volume of the corresponding container is:

Volume = length x width x height = $4.64\underline{2} \times 4.64\underline{2} \times 4.64\underline{2} = 4.64\underline{2}^3$ cubic units = 100 cubic units = 100 units^3

If each linear dimension of a volume is increased a factor of (n) times, the volume will increase by a factor of n^3 times:

New Volume = (n) old_length x (n) old_width x (n) old_height : n = linear magnification or factor
New Volume = n^3 (old_length x old_width x old_height)
New Volume = n^3 old_Volume

For example if the 3 dimensions of a volume, space or solid object are doubled (2), such as after a object and-or size magnification, the old volume is increased by:

n^3 = 2^3 = 8 times.

Another related concept is that the mass and-or weight of a 3 dimensional object will also increase by (linear_magnification)^3 = the cube of the linear magnification of the volume ("or size") of an object.

new_mass = (linear magnification)^3 old_mass and new_weight = (linear magnification)^3 old_weight

If any one factor or physical dimension of a volume changes by (n), either multiplied or divided by that factor of (n) such as its size decreasing, the volume will likewise decrease by that same factor of (n). Here is a verification for when the height is half:

V1 = (length)(width)(height) = LWH if the height is now half, or divided by a factor of n=2:

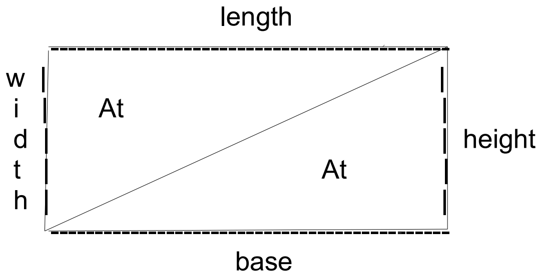
$$V2 = (\text{length})(\text{width})\frac{(\text{height})}{2} = L\,W\,\frac{H}{2} = \frac{LWH}{2} = \frac{V1}{2}$$

:the volume is divided by that same factor of n=2
If the height was rather doubled, V doubles:

V2 = (length)(width)(2 height) = L W (2H) = 2 LWH = 2 V1

Due to the previous discussions, we now have more information to help us develop the formula for the area of a triangle.

First, the area of a rectangle is: Ar = (length)(width) , the units of measurement for length and width must also be the same so that the area's units of measurement resolve to square units = units^2 of measurement. Now, if you diagonally (corner to opposite corner, that is, being not on the same side or line itself) divide any rectangle in two, and the result is two identical (same = congruent = identical or "exact same size") triangles. Hence, each one of those two triangle's is equivalent to half the area of a rectangle that it was or can be derived from. Expressing this as a mathematical expression we have the formula for the area of any right triangle:



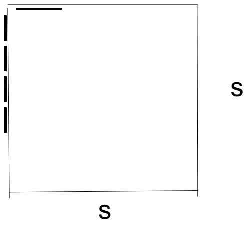
[FIG 31]

$$At = \frac{(\text{base})(\text{height})}{2}$$: **AREA OF A RIGHT TRIANGLE**

Here, base (side of the triangle) is equivalent to the length of the rectangle, and height (side of the triangle) is equivalent to the width of the rectangle. This formula is essentially that of half (divide by 2) the area of a rectangle which contains "square corners" or "right-turn" angles (90 degrees in value). All right-triangles will have a 90 degree internal angle = 90° angle.

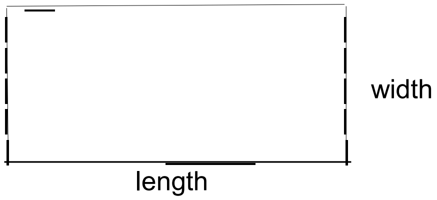
Now we will derive the formula for the area of a parallelogram in order to create a more general formula for the area of any triangle that may or may not contain a right angle so as to be a right triangle of which the above formula can only be applied to.

A square, a figure having four equal sides at right ("square", "corner", "exactly right or left", "exactly (vertical) up and down", or "ninety-degree") angles to each other, is actually a special case or instance of the geometric figures called rectangles. For a square or "square rectangle", the length of each pair of parallel sides, happens to be the same value. [FIG 32]



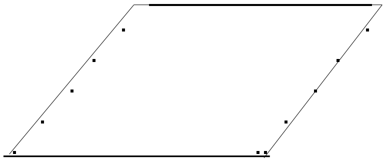
: a square

Rectangles have all sides at right angles with respect to each other. Parallel (ie. sides directly opposite each other, "side by side", "next to", "along side") sides must be the same length, but the pair of adjacent parallel sides (at a right angle to the parallel sides) can vary in length unlike that for a square where they must all be the same length. **Rectangles are actually a special case or instance of the geometric figures called parallelograms.** [FIG 33]



: a rectangle

A **parallelogram** has 2 pair of opposite sides which are parallel and equal in length, however, the non-parallel or adjoining sides can be at any angle with respect to each other rather than be perpendicular (90 degrees or right-angle) only. [FIG 34]



: a parallelogram

The formula for the area of the square is based on that of the area of rectangles:

A = length x width   For a square, the length and width sides (s) are always equal, therefore:
A = s x s
A = s^2   : area of a square

What is the general formula for the area of a parallelogram? It's very similar to that of a rectangle:

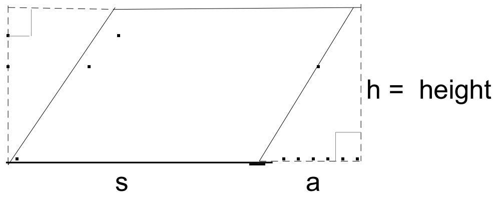
Ar = length x width : area of a rectangle

Below, the sides of the given parallelogram are extended to construct a rectangle whose length dimension is (s + a), and whose height dimension is (h):

The area of the indicated rectangle below in which the parallelogram resides is:

Ar = length x width
Ar = (s + a) x (h) : here h = height of parallelogram = width of rectangle. Distributing :
Ar = sh + ah

[FIG 35]



The area of the parallelogram (Ap) can be found if the area (At) of the 2 identical small triangles is subtracted from the total area (Ar) of the rectangle:

Ap = Ar - 2 At using substitution:

$Ap = [(s + a) \times h] - \cancel{2}\frac{(ah)}{\cancel{2}}$: canceling like terms, then using distribution:

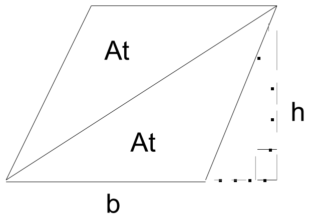
Ap = sh + ah - ah combining like terms:
Ap = sh letting s= b= base of the parallelogram :

Ap = bh : AREA OF A PARELLELOGRAM

As the parallelogram is tilted away from where the sides intersect, one pair of diagonal angles will decrease, and the other pair will increase by the same angle. The more the sides of a parallelogram are "tilted", the more its height will decrease, and therefore its area will decrease. The parallelogram becomes "flattened".

If you draw a diagonal within the parallelogram you will create two identical (exactly the same or congruent) triangles, and then it is easy to see the general formula for the area of any triangle, and not just for right-triangles, and that it is one-half of the area of the corresponding parallelogram: [FIG 36]

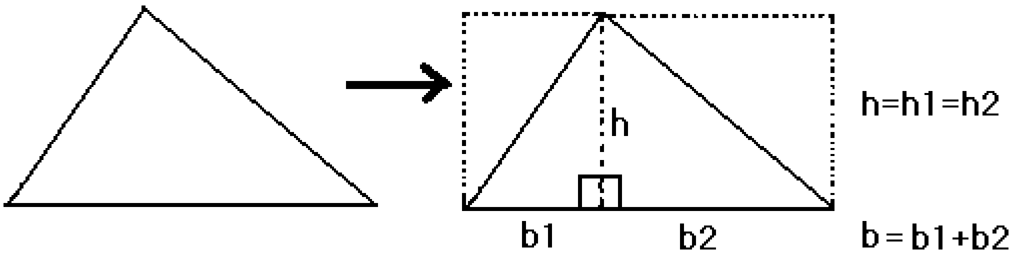
$At = \frac{Ap}{2} = \frac{bh}{2}$



: AREA OF ANY TRIANGLE

Note, the base (b), and height (h) lines and-or indicated extensions, are defined as perpendicular to each other. This is obviously the same formula for the area of a right triangle.

Consider this alternate analysis and derivation of the area of any triangle: [FIG 37]

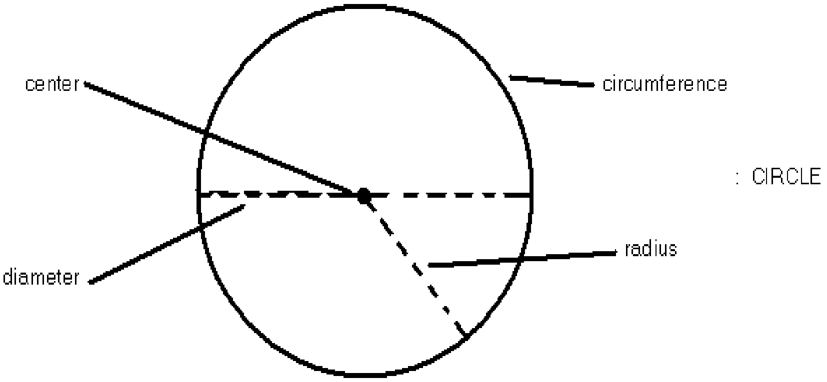


Let :   h = h1 = height of left rectangle = h2 = height of right rectangle

$$\textbf{At} = \frac{\text{area of left rectangle}}{2} + \frac{\text{area of right rectangle}}{2} = \frac{(b1\ h)}{2} + \frac{(b2\ h)}{2} = \frac{(b1\ h) + (b2\ h)}{2} = \frac{h\ (b1 + b2)}{2} = \frac{h\ b}{2} = \frac{\mathbf{b\ h}}{\mathbf{2}}$$

Here are some formulas concerning circles, or that which is circular-like:

A **circle** is a defined area of a plane. A **circle** is a set of points where each point is exactly the same distance (called the **radius** = r or R) from a common (reference) point and-or location called the center or center point of the circle. This set of points is called the **circumference** (C) of a circle and is essentially the perimeter (defined border, boundary) of a circle. The **diameter** or greatest length within or across a circle, and it is equal to twice the radius length: (D = 2R). One concept of a circles circumference is that it is like a line segment that was bent or curved so that the end points meet. A similar concept is that a circles circumference is composed of an infinite number of infinitely small line segments joined together and are set or positioned at an infinitely small angle, nearly 0°, with respect to the previous one. If that circle then "grew" or increased to a very large size, a small portion of its circumference would begin to appear as if it were a straight (non-curved), unbent or "flat" (level) line segment. A circles area can be closely approximated using a large number of these infinitely small line segments as a base of identical triangle with a height equal to the radius (r) of that circle. [FIG 38]



If the circumference (C) of a circle is C=10.75ft., what is the radius and area of that circle?

The formula (standard expression) for the circumference (C) of a circle in relation to its radius (R) is:

first, $\mathbf{C = \pi D}$: π is the pi (pronounced as "pie") symbol, and you may sometimes see this expressed using a simple text symbol as (p) or (pi), as in this book. For example: C=pD
Pi is an abbreviation for the Greek word "periphereia" (periphery, ie., perimeter), and its modern use for the circle is generally credited to mathematician Leonhard Euler in about 1750.

p , (("pi") or PI), is not a variable but is a symbol for a numeric constant that has a value of about 3.141592654. The appendix section of this book contains a derivation of pi, area, and the circumference of a circle. The value of p is equal to the circumference (C) of any circle divided by its corresponding diameter (diagonal or across measurement) (D) which is the greatest distance across the circle. **p = C / D**. Mathematically, (p or pi) is the ratio of a circles circumference to its diameter. It is how many times longer that the circumference is than the diameter. Pi is a strict numeric value that does not have any associated units such as for length, and is simply a plain, numeric factor value. Given any circle, its circumference is always larger by a factor of about 3.14 times more than its diameter. The diameter, when considered or represented as a line segment, always passes though the center of the circle from and to the circumference. The radius (R, or r) of a circle is the distance from the center of a circle to its outer perimeter, hence it is equal in length of half the circle's diameter. The word "radius" is rooted in the word "ray", such as a ray of light that extends outward from a source.

For a specific given circle that remains a constant size, the radius, diameter, and circumference are not variable in value, but are constant (specific, unchanging) values for that circle. All circles are similar, but are not identical (the same) if they are of different size and-or have different values or parameters (radius, circumference, diameter, area) in their construction.

$R = \frac{D}{2}$ hence: $D = 2R$

therefore, $\mathbf{C = pD}$: **CIRCUMFERENCE OF A CIRCLE** (in terms of its diameter)

$C = 3.1415\underline{9}\ D$: a simplified, approximate formula for the circumference of a circle

$C = p2R$: Rearranging the values to express a more standard form so that the decimal
$\mathbf{C = 2pR}$ number comes first as a typical numerical coefficient of a term.
: This is the circumference of a circle in terms of its radius.

$C = 2(3.1415\underline{9})R = 6.283\underline{2}\ R$: a simplified, approximate formula for the circumference of a circle

From $C = 2pR$, after solving for R:

$\mathbf{R = \frac{C}{2p} = \frac{D}{2}}$: **RADIUS OF A CIRCLE**

$R = \frac{C}{2(3.1415\underline{9})} = 0.15915\underline{5}\ C$: a simplified, approximate formula for the radius of a circle. This indicates that the radius is about 16% of the length of the circumference.

Substituting the values given in the example:

$R = \frac{10.75\text{ft.}}{2(3.14159)} = \frac{10.75\text{ft.}}{6.2832} =\sim 1.71091$ ft.

Also, since $C = pD$, $\mathbf{D = \frac{C}{pi}}$ **: DIAMETER OF A CIRCLE** , pi = about 3.1415$\underline{9}$

$D = \frac{C}{3.1415\underline{9}} = 0.3183\underline{1}\ C$: a simplified, approximate formula for the diameter of a circle. This indicates that the diameter is about 32% =~ (1/3 = 0.3333...) of the length of the circumference, or that the circumference of a circle is slightly more than 3 times its diameter length.

The following example may be too advanced, involved or lengthy for some people, and they can just skip it for now, and consider viewing it at another time.

Ex. The orbit of Earth around the Sun is circular-like. Since the Earth is about 93 million miles away from the Sun, how many miles does it travel in its orbital path around the Sun in the following periods of time : In 1 year, in 1 day, in 1 hour, in 1 minute, and in 1 second? Here are some useful formulas:

distance = speed x time **: DISTANCE FORMULA** (the units of distance are length units)

This will find the distance that a moving object has traveled or moved. Distance is directly related to speed and time. The faster you travel, and the longer the time you travel will increase the distance you traveled. "speed" is sometimes called the "rate" (ie. intensity) or velocity of travel. **velocity = (change in distance) / (change in time)**
If 0 is the initial values for distance and time, the equation reduces to:
speed = distance / time : this is much like an average velocity

An object that can or is moving will travel a distance or length from its starting and reference (ie. 0) position. The total distance or length it will travel depends on how fast it is going. The word "fast" used in this way, is equivalent in meaning to the word "speed". If something is moving greater or faster per, or for each, unit of time, such as a second or an hour, the greater the distance it will have moved after that amount of time.

Ex. If you traveled at 50 miles per hour (50mi/1hr) for 2 hours total time, the total distance you traveled is:

$$\text{distance} = \text{speed x time} = \frac{50\text{mi}}{\underset{+1}{\cancel{1\text{ hr}}}}\ \frac{\overset{+2}{\cancel{(2\text{hr})}}}{(\ 1\ )} = 100 \text{ miles}$$

Mathematically, from the distance formula, we can derive the formulas for its associated variables:

$$\text{speed} = \frac{\text{distance}}{\text{time}} \quad \text{and} \quad \text{time} = \frac{\text{distance}}{\text{speed}}$$

Notice that for a fixed distance value, that speed and time are inversely related. In short, as time increase, speed decreases, or vice-versa. If speed is higher than a previous speed, the time to travel a given (fixed value, constant) distance will be lower.

The distance of 93 million miles = (93)(10^6)mi. is actually Earth's radius (R) of orbit or path about the Sun. Hence:

Circumference of orbit about the Sun = 2pR = 2p(93)(10^6) miles = 584,336,233.6 miles

Since the orbit takes 1 year to complete, Earths speed is therefore: 584,336,233.6 miles/year.

Since there are close to 365 days in 1 year (1 year = 365 days), we can find the orbit speed or distance traveled per day if we divide the yearly orbit distance by 365, essentially creating an equivalent fraction.

$$\frac{584{,}336{,}233.6 \text{ miles}}{365 \text{ days}} = \frac{1{,}600{,}921.188 \text{ miles}}{1 \text{ day}} = 1{,}600{,}921.188 \text{ miles/1 day} = 1{,}600{,}921.188 \text{ miles per day}$$

: 1 year is actually 365.25 days

We would also get this result if you simply canceled out the denominator (ie. set it equal to 1) by dividing the numerator and denominator by 365 without actually creating the equivalent fraction.

Since there are 24 hours in 1 day, (24h = 1d), we can find the orbit speed or distance traveled per hour if we divide the daily orbit distance by 24:

$$\frac{1{,}600{,}921.188 \text{ miles}}{1 \text{ day}} = \frac{1{,}600{,}921.188 \text{ miles}}{24 \text{ hour}} = \frac{66{,}705.04949 \text{ miles}}{1 \text{ hour}} = 66{,}705.04949 \text{ miles/1hour}$$

Since there are 60 minutes in 1 hour, we can find the orbit speed or distance traveled per minute if we divide the hourly orbit distance by 60:

$$\frac{66{,}705.04949 \text{ miles}}{1 \text{ hour}} = \frac{66{,}705.04949}{60 \text{ minutes}} = \frac{1{,}111.750825 \text{ miles}}{1 \text{ minute}}$$

Since there are 60 seconds in 1 minute, we can find the orbit speed or distance traveled per second if we divide the minute orbit distance by 60, and to keep the fraction in balance or equivalence, it must also be done to the numerator:

$$\frac{1{,}111.750825 \text{ miles}}{60 \text{ seconds}} = 18.52918042 \text{ miles/second}$$

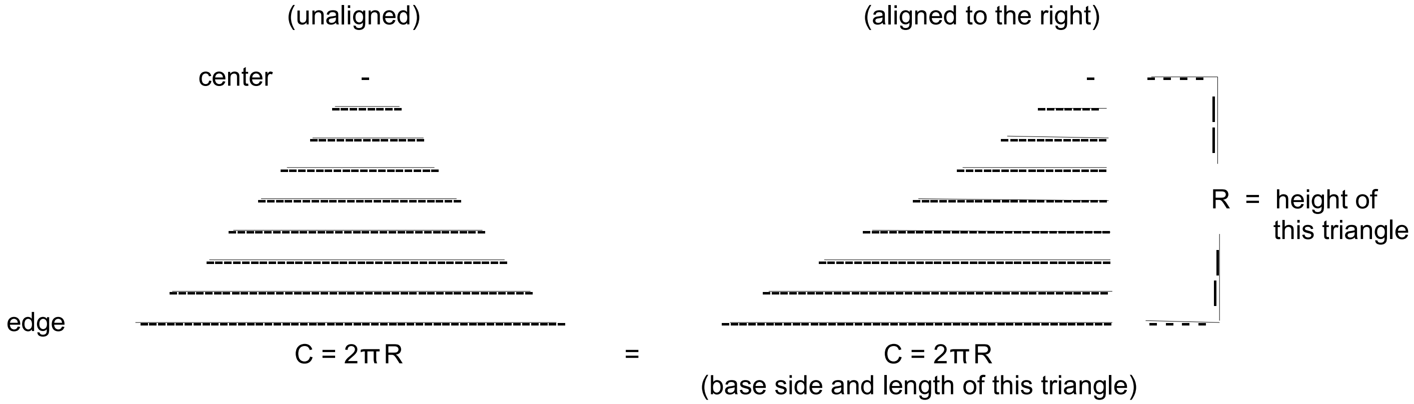
We don't notice this speed since we are relative (in relation to) to Earth's speed. That is, we are also going a distance of 18.53 miles per second around the Sun so there isn't any new accumulated distance between us and the Earth so as to even notice this speed. The difference between our speed and the Earth's speed is practically 0.

The formula or standard equation for the area (A) of a circle or disk surface in relation to its radius is derived below:

Given a circle, and creating an infinite number of concentric (all having the same center point) circles, "rings" or "shells" within it, the width of each ring will approach a thin width close to 0 (but never actually reaching 0 since there would not even be a possible ring to consider). The length (ie. circumference) of each ring will go from C=2(pi)R at the outer edge of the circle to 0 at the center point of the circle. The area of each ring will essentially be that of an infinitely thin rectangle. Here is the area of the outer ring:

-- = width : C = Circumference of the circle
C = 2πR = outer length : R = radius of the circle

Stacking all the rectangle areas and aligning them to the right, their combined width or height is obviously equal to the radius of that circle, and the structure has the shape of a triangle: [FIG 39]



The area of the circle is then equivalent to the sum of the areas of all the thin rectangles, or simply the area of the right-triangle created as seen above when all the rectangles are aligned to their right side. The triangle created is a right triangle and its area is therefore equivalent to the area of the circle from which it came from:

$$Ac = At = \frac{bh}{2} = \frac{CR}{2} = \frac{(2\pi R)(R)}{2} = \boldsymbol{\pi R^2}$$

: AREA OF A CIRCLE OR DISK
: b = base side of triangle , h = height of triangle

(The letters (c) and (t) are not variables, but are called subscripts (of the variable). Here, they are used to specifically identify what kind of area or what the area is in reference to. Here, (c) is used for circle, and (t) is used for triangle. Variables Ac and At are now unique variables, that is, they are not similar or "like" variables or terms, and they have and represent a unique thing and value each. If more than one circle was to be identified, a subscript or number would usually follow (c), for example: Ac1, Ac2, C1, C2, r1, r2, etc.).

This derivation of the area of a circle above is a good hint of what the branch of mathematics called calculus is about in general: the summing of the very small parts of the "whole" in order to find out what the "whole" actually is, or, it is finding a particular (the mathematical value, or equation of a) small part. The appendix contains another derivation of the area and circumference of a circle.

This derivation above is also a simple verification to the general formula for the area of any triangle: At = (bh)/2. Clearly a right triangle is half the area of a rectangle which also has the same dimensions, but as graphically shown above using two triangles, for as long as the base and the height of any two triangles are equivalent, their areas are always equivalent. Using this method, a formula for the area of parallelogram can be verified by using a square. The general formula for the area of a triangle will be verified in the TRIGONOMETRY section of this book.

Continuing the problem given previously:

A = πR^2

A = (3.14159)(1.71ft.)^2
A = (3.14159)(1.71ft.)(1.71ft)
A = 9.18632(1ft. x 1ft.) = 9.18632ft.^2 or=: 9.18632 square-feet

While on the topic of circular areas, here is a discussion of the size and comparison of circular areas, specifically when the diameter (or radius) is doubled or halved:

Ex: There are two circular areas, one with a diameter of 5 inches and the other with twice the diameter with 10 inches. How many times greater is the area of the larger circle? Since 10 is double or twice of 5, you might first assume the area is double (2), triple (3), or perhaps some value like 3.14. Since area is actually based on 2 linear dimensions, and if they are doubled, the result is that the area is 4 times greater, and this increase and factor will be shown algebraically below.

The area of a square (s) is: A=s^2 , with (s) being the side length of the square. Given a square with side (s), and when this length is doubled to (2s), the area of that square is:

As = (2s)^2 = 4s^2 : hence 4s^2 / s^2 = 4 times greater than that before doubling the side length

Likewise, if the side length is tripled, the area would be increased by 9. We have verified that when the sides of a square or rectangle increase by a factor, then the area increases by the square of that (linear) multiplying factor.

For the two circles mentioned in the example: r1 = d1/2 = 5/2 = 2.5 , r2 = d2/2 = 10/2 = 5 , and their areas are:

A1 = (pi)(r1)^2 A2 = (pi)(r2)^2 : the second circle, with area A2, has twice
A1 = (3.14159)(2.5^2 in^2.) A2 = (pi)(5^2 in^2) the diameter as the first circle with area A1
A1 = 19.635 in^2 A2 = 78.54 in^2

The (factor) increase (expressed as a ratio) of A2 in comparison, reference or respect to A1 is:

$$\frac{A2}{A1} = \frac{78.54\ \text{in}^2}{19.635\ \text{in}^2} = 4$$

Showing this in general terms algebraically if the radius (r) of a circle increases by a factor of N:

$$\frac{A2}{A1} = \frac{(\text{pi})Nr^2}{(\text{pi})\ r^2} = \frac{(\text{pi})N^2r^2}{(\text{pi})\ \ r^2} = N^2$$

: Area will multiply in value equal to the square or second power of the multiplying factor to the radius.

Using this information, it is now reasonably clear that if it is said that the diameter or radius of a circle is halved, then the area will be 4 times less:

$$A2 = (\text{pi})(r1/2)^2 = (\text{pi})(r1^2/4) = (\text{pi})(r1^2) / 4$$

:where r1 is the radius of the original circle. The area will be divided by 4, hence 4 times less.

Dividing by 4 is the same as multiplying by $(1/4) = 0.25$

r2 is noted at being half of r1: $r2 = r1/2 = 0.5\ r1$, Hence $r2/r1 = 1/2 = 0.5$

If r1 was changed by this factor value of 0.5, then area will change by this factor value squared:

$$\frac{A2}{A1} = N^2 = 0.5^2 = 0.25 = (1/4)$$

: A2 is a "fourth" of A1, as mentioned above

Extra: $\frac{\text{area of circle}}{\text{circumference of circle}} = \frac{Ac}{C} = \frac{(\text{pi})\ r^2}{2\ (\text{pi})\ r} = \frac{r}{2} = 0.5\ r$, mathematically: * $Ac = \frac{C\ r}{2}$, and $C = \frac{2\ Ac}{r}$

* : hence with this equation: Ac = **(**one semi-sphere length of the circle = C/2**)** **(** r **)** =
Ac = (Half the Circumference) (Half the Diameter) = (length) (width) units^2 . This can be seen by drawing a circle and "straightening" out the upper semi-sphere length, and then swinging the two radius lengths connected to that so as to be at a right angle to it. This rectangular area (which can be thought of as being constructed of many rectangles whose length is that radius value) is greater than the semi-sphere area due to that gaps (extra space) in that area were introduced by straightening the curve of the circumference. After overlaying the lower semi-sphere area, a solid rectangle is created whose area is equivalent to the area of that circle. A **sphere** is a three dimensional geometric structure and-or volume that is basically a **ball shape**.

$$\mathbf{Acircle = Arectangle = (length)(width) = (C / 2)\ (r) = (2\ (pi)\ r / 2\ )\ (r) = (pi)\ r^2}$$

$$Ac = (C / 2)\ (D / 2) = ((\ 2\ (\text{pi})\ r) / 2\ )\ (\ r\ ) = (\text{pi})\ r^2$$, also:

$$Ac = (C / 2)\ (D / 2) = CD / 4$$

Another derivation for the area of the circle is to consider it being composed of the summed area of a larger to infinite number (N) of thin triangles with their base (b) side along the circumference and whose height is the equal to the radius of that circle. $Ac = N\ (At) = N\ (b\ r / 2) = (Nb)\ r / 2 = C\ r / 2 = (2(\text{pi})r)(r) / 2 = (\text{pi})r^2/$
Note above that: $C\ r / 2$ = (area of a rectangle or parallelogram) / 2 = (area of a triangle)

Extra: From: $C = (\text{pi})\ D$ and $A = (\text{pi})\ (D / 2)^2 = (\text{pi})\ D/4 = (\text{pi})\ r^2$

$$(\text{pi}) = \frac{C}{D} = \frac{A}{r^2}$$, we have: $$r^2 = \frac{AD}{C}$$

It must be mentioned that the mathematical relationship between area (A) and perimeter (P) of any two objects is not always a corresponding or direct relationship. The relationship can be an inverse relationship, that is, It is possible for the area of an object-1 to be smaller than the area of object-2 even though the perimeter of object-1 is larger than the perimeter of object-2. Consider the simple example below where the units of measurement are considered the same for both rectangles: [FIG 40]

Ex.

From: Perimeter = sum of each side of an object:

P1 = 10 + 10 + 1 + 1 = 22　　　　P2 = 7 + 7 + 2 + 2 = 18

From: Area of a rectangle = (length)(width):

A1 = (10)(1) = 10　　　　A2 = (7)(2) = 14

The mathematical conclusion of this example can be expressed as:

$P1 > P2$: > means "greater than". Arrow points to the smaller value. "Perimeter 1 is greater than perimeter 2."

$A1 < A2$: < means "less than", still, the arrow points to the smaller value. The larger or "open" end of the arrow indicates the larger value. These symbols and their use are discussed next. "Area 1 is less than area 2."

When the length and width of a rectangle are the same value, the ratio of the length and width of the perimeter of a rectangle are 1, and the rectangle will have the largest possible area, and this shape is that of a square which is a special instance or case of all the possible rectangles. When the length or width are not equal, one side, usually the width, is a fractional of the length, the longest side.

More about expressing mathematical differences or relationships will be discussed next in the topic of inequalities.

INEQUALITIES

An inequality is when two or more things are not the same or mathematically equal. An inequality can be considered as an equation that is out of balance and-where the left and right sides are not equivalent (=). The way to determine if things are equal is to simply subtract their values, and if the result is 0, there is no difference among them, and they are therefore equal. If they are not equal in value, there is a difference between those two values, and one value will be greater than the other value, and one value will be less than the other value. < and > are symbols which are examples of what are known as relational or conditional syntax (grammar, notation). This syntax is used to show or express a mathematical or numerical relationship of the actual values, or a supposed (conditional) relationship as for a some kind of question ("if such and such") or test (performed mathematically, usually by subtraction, such as for computers) or determination of a value or relationship so as to simply express, or to do, or not do something.

The mathematical expression of a mathematical relationship is called a (mathematical) statement. A statement is sometimes noted as being a (mathematical) "sentence". Here is a symbolic, general format of a mathematical statement:

(expression) (mathematical relationship) (expression) : statement

Statements where the mathematical relationship is not one of equivalence (ie. equals) are often called inequalities. Here are the common mathematical relationship operators and what they mean:

relation or condition	meaning, or read as:
>	"greater than", ex: a>b is read from the left to the right as: "(a) is greater than (b)"
<	"less than"
= or: ==	"equivalence" : used in statements called equations, there is no inequality here
>=	"greater than or equal"
<=	"less than or equal"
≠ or: !=	"not equal" or: /=/ series of standard text characters as sometimes used in this book.
~ , ~= or =~	"approximately equal to", "about equal to", "equals about" ex. 5.1 ~= 5

Ex. Express all values of (x) being greater than five.

$x > 5$: This can be read as "x is greater than 5"

Here, (x) can be a specific value greater than 5, or it is a general representation for all values greater than 5. This can also be expressed as:

$5 < x$: This can be read as "5 is less than x",
Expressed this way, it gives more of a significance to 5, such that
it makes it like the focus of the statement, or reference value.
The statement also indicates: "x is greater than 5":

Hence: $x > 5$ or= $5 < x$: again, the tip of the arrow points to the smaller value or side

Ex. a /=/ c : "a is not equal to c"

Ex. Express the quantity of (x+1) being greater than or equal to 5.

$x + 1 >= 5$

To solve for values of (x) that satisfy this relationship, solve the inequality just like solving an equation:

$x + 1 >= 5$ T. +1, so add or combine -1 to each side :
$x + 1 - 1 >= 5 - 1$ C. and we find that:
$x >= 4$

If you multiply or divide each side of an inequality by a negative signed value, the direction (or "sense") of the inequality sign must be changed. The example below will illustrate and verify this:

Ex. We already know by common sense, or according to a number line, that both of the following statements are true:

$5 > 4$ and $-5 < -4$: both statements are right, correct, true

```
      -8 <--------|--------|--------|--------|--------|--------|--------|--------|--------|--------|--------|---------> +8          : [FIG 41]
 (lesser)  . . . - 5     -4      -3      -2      -1       0      +1      +2      +3      +4      +5 . . .   (greater)
```

If this number line was a road, this is what some direction signs located at any point on it would look like:

(greater -->) : values that are greater, are in the rightward direction, (regardless of your current position)
(<-- less) : values less are in the leftward direction, (regardless of your current position)

Multiplying each side of both of the inequalities expressed above by (-1), but not changing the "sense" or relationship:

$5 > 4$ and $-5 < -4$
$(-1)(5) > (4)(-1)$ $(-1)(-5) < (-4)(-1)$
$-5 > -4$ $5 < 4$: both relationships are wrong, incorrect, false

According to the number line, both of these statements are now false. To true these statements, simply change the direction of the relationship indicated between the members (expressions) or sides of each statement. The results will then be equivalent to the true statements shown at the beginning of the example.

HOW A COMPUTER MAKES DECISIONS

After considering the previous discussion, the same mathematical grammar, notation or syntax is often used with modern electronic, digital computer programming languages to make decisions on what to do next, such as displaying a value on the screen or adding some values, or perhaps what not to do next - such as even jumping-over (skipping-over, not-processing) some computer code (program, steps) statements. You can skip over this section if you are not currently interested in this topic and-or computer programming. Since the syntax and it's corresponding meaning was already defined before computers, their is no need to replace it with something else for computer use. To determine if the computer should do something, some condition (essentially a test, or determination) must be met (be satisfied or true) by determining that a certain mathematical relationship is true. This is accomplished with some appropriate computer code (a piece or part of the entire computer program that instructs the computer to do things). This particular type of code is formally called a conditional statement. When the computer is processing or "running" a program, it will process the mathematical (relational) statement part in the conditional statement. The computer (or even you can) determines if a conditional statement is true by subtracting the two values (operands) on either side of the conditional symbol. For example, to determine if two variables, or perhaps a variable and a constant, are equal, their difference must be 0. As another example, to determine if one variable or value is less than or greater than another, their difference will be checked to be either positive or negative in sign. In short, a computer makes decisions by the mathematical comparison (comparing) of two or more values by essentially using subtraction to find their difference.

Ex. Using a basic, generic or "pseudo" computer programming instruction, language, or code:

if (a=b), then do (this) : "if a equals b then do this" , "this" could be for example, incrementing the value of a variable, going to another part of the program, make a sound, display some text, a print command, or to do something else. "do this" means process the following command(s) and-or other statement(s) if the stated relationship (here (a=b) is true.

When a = b, they both have the same value and therefore, there is no change or difference (separation, change) from one to the other. The computer will subtract b from a, and if the result is 0, they are equal, and the "do this" part of the computer program will be processed since the expressed relationship is true, otherwise, the "do this" will not be processed since the conditional (or test) statement is false, and will be skipped over to the next line of the computer instructions (ie., computer program). For C-language computer programs, since (=) is used to set a value of a variable, the notation or "syntax" of: (==) is used to check or determine an equivalence, and the above pseudo conditional statement would be: if(a==b){ then do this; };

Ex. if (5 > 10) :Check and determine if 5 is greater than 10 , or= is 10 is less than 5 , since: (5 > 10) = (10 < 5)

To find out if this condition stated is true, subtract that which is considered, or assumed, as being truly less from that which is assumed more or greater, as indicated by the conditional symbol. Since 5 was the first value expressed, it is the reference value for this statement which asks if 5 is greater than 10. Performing a subtraction operation to find out:

$$\begin{array}{r} 5 \\ -\underline{\ 10} \\ -5 \end{array}$$

or: $(+5) - (+10) = 5\ (+)\ (-1)10 = +5\ (+)\ -10 = -5$

: the negative sign of the difference indicates that 5 is not greater than 10, and the condition, test or question of: is (5>10)? , = is false = 0 : Computer systems often use a binary number that is physically or electronically represented using a voltage value of 0 for false and 1 for true. true (1) and false (0) are logic (truthiness, "truth" and-or deciding "decision") values.

Here, for the question (if) and (logic, truthfulness = truth) condition and mathematical relationship (5>10)? , that the actual value of the result, other than 0, does not matter and has no further use here, but what does matter is the sign of the result since we are looking to see if the relationship is true or false only. The negative sign indicates that the mathematical relationship is not possible, hence the relationship cannot be true, and therefore the conditional statement evaluates to false = untrue = not-true. As a simple illustration, it is false that you can physically give away 10 apples if you (only) have 5 apples. It is true that you can give away 3 apples if you have 5 apples since you will have +2 apples leftover or remaining. A computer has a special memory (data or information storage implemented via electronic circuitry) location that indicates ("flags") the mathematical sign (pos. + , = 1 , or neg. -, = 0) of the result of an arithmetic operation after performing it. Another memory location indicates if the result's value is 0 which means no difference. This will allow for conditionals such as: "equivalence", and the "greater-than-or-equal" dual-conditional where both the "sign" and "zero" flags are then both checked after the subtraction, and both need to be set to 1 for that conditional statement to evaluate (or result, "return") as true = 1 to the computer program being used to control or direct the computer system to do things. If a certain bit at a memory address or location, such as a "flag register" and "flag bit" is set to 1, it is sometimes said as that bit or location is "set", otherwise if is 0, it is said as "not set". The word "**bit**" has its origin in the word of "bite", and that a bit is the piece of the entire object or food that was bitten. Later, a grouping of 8 bits was called the word "byte" in digital and-or computer terminology. Interestingly, the word "bit" could also be considered as based on the ancient word of "cubit" which is unit and-or measure of an amount of length.

As an example consider this pseudo-code (generic code expressed with literal statements, generally adaptable for any computer language) of a conditional statement, and the possible syntax for it:

```
 if (condition is true) then do something     or:   ?(condition) then (do something)  ,  Ex:  if ( P2  <  P1) then do something
```

The "do something" in the above statement is computer program code (coding, instructions, statements, "steps", commands, often written in plain text) to perhaps change the value of a variable, or code to branch (go to) to another portion of the program to do something else, or to evaluate the logic (truth or falseness) of another conditional statement. If the condition that the relationship stated is true, it will be "flagged" (remembered or indicated) as 1=true, then the "do something" code will be processed by the computer. If the relationship stated is false, it will be "flagged" as 0=false, then "do something" will not be performed and that portion of computer code will simply be ignored by "ignored", "jumped over", or "skipped over", and the program will then continue processing ("running") at the next computer code.

With modern-day "high-level" (plain text) computer programming languages, the eventual and actual computer (strict numeric) code (steps) to find the difference and branch to the "do something", or not, is then converted and written as an equivalent machine language program. Machine language and programs are a "low-level", fundamental computer language which is strictly a numeric (digital, binary) code language so as to be compatible with the electronics of the computer, and specifically the CPU which is the "Central Processing Unit" to execute (ie., "run") programs, usually built as an integrated circuit ("chip") inside the computer system. Higher level languages such as BASIC convert the much more mentally easier to understand, more readable (as ASCII text and understandable words) programs into machine code.

A plain text, readable "high-level" computer language that the computer programmer types in will later be processed with the aid of another general computer program called a compiler program that gathers and compiles (ie., combines to a single piles) together the "readable" program parts (variables, values address, etc.) and transform it into its equivalent, low-level, machine program that an electronic machine, such as a computer, can effectively understand or work with, and which is very much like (logically to physical electronic voltages) controlling, setting and resetting, many on-off switches of that machine. Note that it is still possible for a programmer to initially write (faster) actual machine code or language programs, and a helpful program to do this is with a program called an "assembler", of which itself often has its own (minimal) syntax called "assembly language", and these programs are then called assembly (language) programs. In pure (digital, numeric input, control) machine language, "strict (ie., only)" numbers that are input and stored as digital, binary numbers in its memory would be used for commands and-or data to process (ie., use, work with). For example, 00000100b = 4d might be the documented instruction (ie., command, or "opcode" = operation code = machine code) to add the values (ie., the operands of the addition operation) in two memory locations and-or (operand) "registers" and place the output or result in another memory location, and the equivalent for the assembly language program would use a mnemonic (pronounced as "nuh-maa-nik", and is associated with the word memory, and is another word, abbreviation, or symbol(s) so as to represent and be easier to use) for that instruction or command such as ADD, and this helps the programmer understand what is being done in the program and makes it easier to write and-or edit later. In a high level language such as C, we can more easily read and write (for example, using more common English language) expressions and-or program statements (much like a complete set of instructions or a sentence) for what we want done, and the compiler program will quickly create all the corresponding machine code from that for us. Here, the machine is the computer's central processing unit (CPU, digital circuit, on chip(s)) and the rest of the computer system that will run (ie.

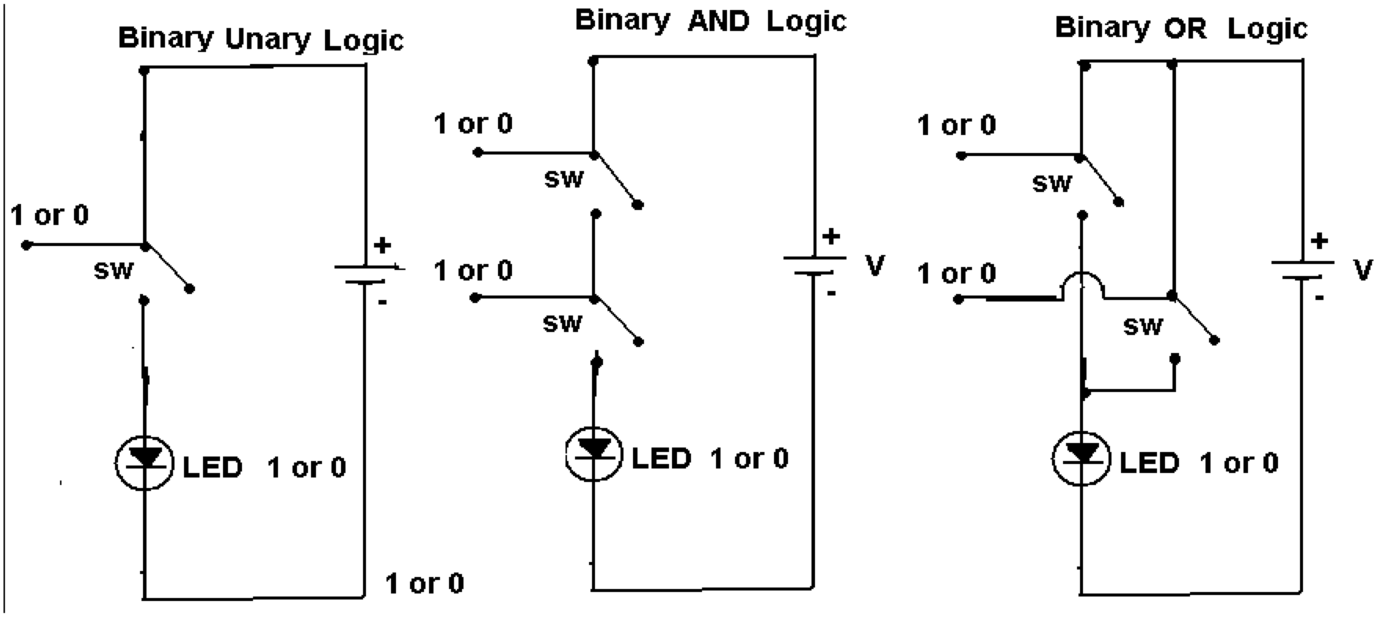
process, execute, utilize) that computer program. For example, the C statement: c = a + b; will add the operands (a) and (b) that are each stored at a memory location that is identified and accessed by the program as (a) and (b) respectively, and the result is assigned to variable (c) that it and-or its value is assigned to another memory location to hold (numeric, digital) data.

Finally, if something or some process is to be performed depending upon the truth (ie. the logic or logic state) of more than one conditional statement, a special syntax is often used. The symbols of this syntax are called logical operators. The operands of a logical operation where truth (numerically represented as 1), or falseness (numerically represented as 0) is to be determined are conditional statements used to determine the truth of a mathematical relationship. The two most common logical operators are called the AND and OR operators. The AND symbol is usually something like: &&, and the OR symbol is something like: ||, however for some programming languages (such as BASIC), or a generalized pseudo-code language, they may be simply noted as: AND and OR. AND is used when all the conditional statements must evaluate (result) to true for a logically true result. OR is used when either or both conditional statement must evaluate to true for a logically true result.

if (condition1) AND (condition2) then do something : This logical operation uses the logical operator: AND Logical operations operate on the results of conditional statements whose results are either true (which can be mathematically represented as a value of 1), or false (which can be mathematically represented as a value of 0). Electronically, a condition could be considered as an input of a 1 or 0 value or result.

Ex. if ((P1 < P2) && (A1 < A2)) then do something
The English translation is typically something as the following:
"If P1 is less than P2, AND if A1 is less than A2, then do something."

Because the logical AND operator is used, for something to be performed, both of the two (logical, true or false) conditions shown must first evaluate to be logically true (1). For OR logic or an OR Gate, if either input is 1 or set, the output is 1. [FIG 41B]



SW = switch, here shown as a mechanical on-off (ie., on or off) switch, but it could also be an electronic switch or "current valve" such as a transistor. Connected to each switch is a logic or data, input control line or pin which can electronically open or close the switch, and that if it set at a high enough voltage (often 5V standardized), which can be described as a binary 1 value or set, then the switch will also be set or closed so as to allow current to flow and the LED will be lit, on or set which can also be described logically as binary 1, logical 1 or digital 1 when lit, or 0 when off or not set or lit. In binary, unary logic, only one binary (either 0 or 1) and-or logic value (true=set=1, or false=notset=0) at the input will determine the binary and-or logical output of either 0 or 1. Simple binary logic or binary **unary** or single logic can be thought of as a **single switch** that can be ON=1 or OFF=0 and have the same corresponding logical output of 1 or 0. Negative or inverse logic is when a binary value is converted to its opposite value where 1 becomes a 0, and a 0 becomes a 1. A single transistor can be used to create either a switch (ie., a gate) to control current flow, much like a relay, or to be an "inverter" which is also called a "not gate" or "(binary) complement" of which will invert the logic or state (ie., logic status) of a binary value. The above circuits are also called "logic gates" for it can be said that it is the input logic (truth, binary on, or binary off, open or closed, set or not-set, or logical 1 or 0) and (logic) gate (ie., like a "logic or binary filter", here implemented as circuits) that determines output (ie., 0 or 1) which is based on the input values. Note for example that it is possible to construct an AND gate with more than two or more inputs, and that all would have to be set to 1 so as to produce a 1 output. A NAND gate is a Negative-AND gate of which inverts the logic of the output of the initial AND gate. If the logic result of the AND section is 1, it will be inverted by an inverter and sent to the output as a 0.

Before the invention of the modern-day electronic, digital or binary computers, there were mechanical, analog computers. These often contained gears and rods. One of the oldest found mechanical, analog computers is known as the **Antikythera mechanism** which is a (hand cranked powered, turned) mechanical computer having thin, fine bronze gears and which could calculate and-or predict many astronomical events in the past or future. It was made by the ancient Greeks and has been scientifically estimated to have been made around 75BC to 100BC, due to the carbon dating of wood pieces, and other items having a known antiquity date. The device was found in a submerged shipwreck in 1901 near the Greek island Antikythera in the Mediterranean Sea, and it is considered as an incredible discovery. The delicate device was encrusted by debris and had to be x-rayed so as to find much further details about it. An obvious speculation is what other mechanical computers, metallurgy, machine and-or **gear** (a rotational disk [cog or cogwheel] that is a part of a mechanical [ie. force and-or power] transfer method, and having teeth or protrusions about its circumference so as to interconnect [without slipping in their motion] with another gear) knowledge and technology was available at its time and that which came before this, and was the production of such devices very limited, and were there many less complex, and less costly mechanical devices available to the general public. Later, similar fine metal gear work would be found in the clocks made in the 1500's. Surely, hundreds of years of astronomical observations, calculations and knowledge had to be already known so as to produce a **automatic computing** device such as the **Antikythera mechanism or calculator**. By 1000AD, the Arab civilization of the middle-east made further progress in the sciences, including mechanical devices, most likely assisted by the decimal, positional numeric system first invented in India and then refined and used by the Arabs so as to express formulas and make calculations. Numeric counters and-or displays before the electronic digital age used gears and-or wheels. The Antikythera mechanism is a "hard coded" computer, that is, it is a (mechanically) pre-programmed (by using specific gear sizes, etc) device to do only specific (astronomical) calculations that it was designed to do. Many computer processes are repetitive and-or cyclic, and a wheel or gear is a mechanical device that can be used to do and-or control such processes. As for electronic computers, at the core of their repetitive and or cyclic processes is a (high speed) electronic oscillator (ie., creates repetitive vibrations or pulses having a uniform clock-like timing, that which causes oscillations, "back and forth" motion) which for its specific use here in a computer, it effectively replaces the older, slower and larger sized technology, mechanical rotating wheels or gears, springs, cams, etc. of a mechanical oscillator and-or clock which needs to be regularly adjusted and-or reset. An electronic computer is not analog (analogue, analogy, similarity or representation) in nature, but rather digital (ie., discrete steps, approximation) and capable of being very fast, especially in terms of numerical calculations and data processing.

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RATIOS

A ratio is a value and-or numerical expression for the numerical relationship of two values. This (ratio) numerical relationship and value is created when the two values are expressed in fractional form and-or divided form, and then calculated to find the resulting value or ratio. In "simple terms" (ie., a simple explanation), it is how many times (like a multiplying factor) larger that the dividend or numerator value is with respect to the divisor or denominator value. Likewise, the denominator value could be said as being that many times smaller. Hence a ratio value it is much like a quotient or result value of a division problem. It must also be noted that even though quantities with unlike units are often (mathematically, numerically) compared like a ratio, true or formally defined ratios will have quantities with the same units so that the resulting ratio value is unitless and only a simple numeric value that is how many times that the numerator value is greater than (more, magnified) or less than the denominator value. Ratios and their concepts are very important and useful, for example, for magnification values, and for an equivalent portion ("ration") or fraction of something that is bigger or less than another similar thing of reference.

In terms of true fractions representing a value less than 1, a (fractional) ratio can be thought of as a simple numerical representation of an expressed fraction, and that ratio value will always be between 0 and 1 after dividing the numerator by the denominator. Here, two fractions are shown to have and represent the same ratio or portion of something, and therefore both have the same ratio value and are therefore equivalent fractions: 3 / 4 = 0.75 and 4.875 / 6.5 = 0.75
The numerical value of a fraction can be thought of as a relative value or percentage that the numerator is part of the denominator. A value between 0.0 and 1.0 in percentage form is a value between 0% and 100%

First, here is a short and practical discussion of why ratios are used.

We can compare two values, and can do it first with words that simply say or express that one value is less than, equal to, or greater than another value. We can express these states or conditions mathematically with the relational operators of: (<, = and >). If the values are different, we can mathematically quantify and express how different they by finding their difference with a subtraction operation. This is one way to compare two values. This is not practical if the values are not in relation or respect to the same thing, for example: If there is 5 objects in one box, and 8 objects in another box, the number of objects is different in each box, and their difference is (8 objects - 5 objects) = 3 objects. However, if there is 5 houses and 8 total people in all the houses, we cant compare people to houses and express it as a difference of either houses or people because it just doesn't make much practical sense: 8 people - 5 houses? , or 5 houses - 8 people? What we can do is create, express and quantify their numerical relationship as a ratio:

8 people/5 houses or 5 houses/8 people : "8 people per 5 houses" , or "5 houses per 8 people"

Note that with ratios, the parts of a numerical relationship don't need to have the same units of measurement. Here, the amount of people, and amount of houses are not even fractional parts of the same thing as would be for example: 5 objects out of 10 objects expressed as: 5 objects/10 objects, and even then, the result is strictly a numeric value, here 0.5 = 50%. Since the units were the same, it can be said that 5 objects is 50% of 10 or all the objects being considered. With the ratios of people and houses, the ratio can be expressed as a (non strict) numeric value or ratio, or perhaps even expressed with a new named unit that is a composition of the specific units used:

The ratio of people to houses is: 8/5 = 1.6 : This result is expressed as a strict or pure numerical quantification, and the result is to be considered more like a numerical average or possibility only , rather than the actual number of people in each and every house.

8 people / 5 houses = 1.6 (people/house) = "one point six people per one house"

The ratio of houses to people is: 8/5 = 0.625
5 houses / 8 people = 0.625 (houses/people)

Ex. If you have two apples and five oranges, the ratio of apples to oranges (meaning the quantity of apples compared with, or with respect or in reference to, the quantity of oranges) is said to be "two to five", which is mathematically expressed as: 2/5 = 0.4/1 ("zero point four, to one", or simply 0.4) after reducing the fraction to lower terms. It is incorrect to convert this decimal value to its equivalent percentage and say that 40% of all the fruit is apples. NOTE: This value of 0.4 really means that the quantity of apples only is 0.4 times larger than the quantity of oranges only. The value of "all the fruit" is actually the number of both apples and oranges, and which is (2+5) = 7 fruits.

The ratio of apples to oranges is: $\frac{\text{apples}}{\text{oranges}} = \frac{2}{5} = 0.4$: apples and oranges, being different fruits, cannot be physically or mathematically compared or combined to produce more of either. The expressions here just represent the quantities of the apples or oranges. The words of: "apples" and "oranges" are being used here like mathematical variables that are only numeric in value.

Mathematically the number of: apples = 0.4 oranges

If the number of apples equaled the number of oranges, their ratio would be 1. For the result value of 0.4, it can be said that, for every 1 orange, there are 0.4 apples. That's almost half an apple, or 0.5 apples for ("per") each orange

As mentioned, tt is incorrect to convert this decimal value to its equivalent percentage and say that 40% of all the fruit (both apples and oranges) is apples. This value of 0.4 really means that the quantity of (only) apples is (only) 0.4 times larger than the quantity of (only) oranges. Since 0.4 is less than one, it is a fraction, hence the number of apples is only a fraction and will be less than of the number of oranges. The can be clearly seen since 2 is less than 5, and therefore, 2 is only a fraction or part of 5.

The ratio of oranges to apples is: $\frac{\text{oranges}}{\text{apples}} = \frac{5}{2}$: "five to two", or by canceling:

$\frac{\text{oranges}}{\text{apples}} = \frac{2.5}{1}$: "two-point-five to one", or simply 2.5 times more oranges
This form is an expression for a "fundamental or unit" ratio.
Mathematically: oranges = (2.5) apples

Clearly with the numbers shown that for every 1 apple, it can be said that there are 2.5 oranges. Likewise, it can be said that there are 2.5 oranges per 1 apple. Note that 0.4 and 2.5 are reciprocals of each other and that these two corresponding ratio's may sometimes be noted as being "reverse" or "inverse" ratios. If you converted both of these ratios to their corresponding percentages, their combined percentage is: 40% + 250% = 290%. Clearly, this is incorrect since the total quantity of fruit cannot exceed its whole or entire self = 1 = 100%. Since apples is part of the total quantity of fruit, to find the percentage of (all) the fruit that is apples only, the quantity of apples must be mathematically compared, by a division by the total quantity of fruit, hence by taking the ratio of apples with respect to the total quantity of fruit (and not just oranges):

$\frac{\text{apples}}{\text{fruit}} = \frac{\text{apples}}{\text{apples + oranges}} = \frac{2}{2+5} = \frac{2}{7} = 0.2857...$: The ratio of apples to, or of, all the fruit is 0.2857... "~ 28.6% of all the fruit is apples", and the ratio of apples to oranges is 0.40 = 40%

The remaining fruits' (oranges in this example) percentage can be found by:

Whole = Sum of parts and for this example:
Whole = (part apples) + (part oranges) algebraically:
part oranges = Whole - part apples
part oranges = 1.0 - 0.285714285 : can optionally use percentages: 100% - 28.57%
part oranges = 0.714285714 : "71.43% of the fruit is oranges"
As a check:

$\frac{\text{oranges}}{\text{fruit}} = \frac{\text{oranges}}{\text{apples + oranges}} = \frac{5}{2 + 5} = \frac{5}{7} = 0.714\underline{3} = 71.43\%$: 5 parts of 7 total parts

PROPORTIONS

The concept of proportions (same, "similar", proper or equal portions) is a concept of having the same portion or fraction of the whole or entirety as that of a similar reference has. When two things (items, parts, portions) are mathematically compared to each other as a fraction, portion or rate, and then if two other things are compared and have that same exact value, then the mathematical relationship between those to fractions, ratios or portion values is said to be proportional (ie. having the same or portion, "within proper proportions"). The word "portion" which means: "a part or fraction of". The word proportion, as a noun, is a combination of the prefix "pro", which means: "for", "in favor of", "consistent with", or "similar". **Proportional** or "**proportional in value**" means having and keeping a consistent or constant mathematical portion and-or relationship (among the parts of the whole). Since a portion of something is a fraction of that something, proportions also means equal portions or equivalent fractional parts, either physically and-or numerically. The concept of proportions has been briefly mentioned previously in this book.

Ex. ratio1 = ratio2

$\frac{2}{4} = \frac{5}{10}$ Since ratio1 = ratio2 , the numerator represent the same part or portion of denominator, and that ratio and equivalent portion of the whole is 0.5. These two equivalent fractions are proportional.

If you were to magnify (make larger, or possibly smaller - a demagnification) a construction such as a square or triangle to be smaller or larger, that construction is said to be a similar construction because it looks (and is mathematically, as in a certain measurements, such as the ratios of any two corresponding parts) the same as that of the source or initial reference construction. Given a source construction, and you magnify it by a value to produce a larger, new and similar shaped construction, all the individual parts (or portions) of the new construction will also be magnified by that same value. To magnify values mathematically, we can "grow" (increase, or possibly decrease) or change all those values by multiplying them by the same (factor) value. Note that adding the same value to a group of different values will not equally magnify all of them by the same value, and this was mentioned when creating equivalent fractions where multiplication by the same value must be used.

Lets look at some of the basic mathematics of proportions:

Given (a) and (b), perhaps two sides of one construction, we can "mathematically magnify" all of them so as to create a similar looking new construction, either bigger or smaller, by multiplying each by the same value of (n):

$a(n) = (an)$ and $b(n) = (bn)$: parenthesis are used around both (an) and (bn) to help show, or give clarity, that each values represents a different value than (a) or (b).

(n) is also the multiplying or "magnification" factor of the entire new construction in relation to the source or reference construction. It could be said that the second construction is "n times bigger" than the first construction. If (n) is not known, it can be found by dividing the two values of the corresponding (similar) parts of the two similar constructions, so as to find their ratio:

$\frac{(an)}{a} = n$ and $\frac{(bn)}{b} = n$

Ex. Construction2 is similar to, and larger than construction1. If a certain part, say part A1 of construction1 has a value of 5, and the corresponding part, say part A2 of construction2 has a value of 35, how many times bigger is that part and-or construction since all of its parts were also magnified by this same value so as to have a similar and larger construction?

$\frac{\text{part of construction2}}{\text{corresponding part of construction1}} = \frac{A2}{A1} = \frac{35}{5} = 7$: "seven times" bigger. All the parts, and and the construction, are n=7 times bigger.

This ratio value indicates that part A2 is seven times bigger than the similar (same) or corresponding part A1. Since the constructions are similar, each part has also been magnified or increased by the same (factor) value and the ratios of all corresponding parts will be this same value of 7. All the parts of construction2 can be said as being proportional or "of or with the same portion" to all the corresponding parts of construction1. Likewise, construction2 is therefore also said as being proportional to construction1. When constructions are proportional like this, even the ratio of any two parts of construction2 is equal to the ratio of the two corresponding parts of construction1, and these parts and-or their values are also said to be proportional or having the same portion (fraction of) or ratio value. In general, the ratio of any two parts within a single (unmagnified) construction does not necessarily have or need this same ratio value, however, this ratio of parts within one construction will be equal to the corresponding ratio of the two similar parts in a magnified or similar construction.

The reciprocal of (n) could be thought of as the "demagnification" factor. Given (an) and (n), to solve for (a), we divide (an) by (n), which is the same as multiplying by the reciprocal of (n), which is (1/n):

$a(n) = (an)$ dividing both sides by (n):

$\frac{a(n)}{n} = \frac{(a\cancel{n})}{\cancel{n}} = a$ or by indicating the reciprocal of (n) more clearly:

$$\frac{(1)\,a(n)}{(n)} = \frac{(1)\,(a\cancel{n})}{(\cancel{n})} = a$$

Ex. For the last example, the magnification factor was 7. If we multiply A2 by the reciprocal (here, 1/7) of this magnification (specifically here, by a shrinking, or reducing) factor, we will find A1.

Since $\frac{A2}{A1} = 7$, $A2 = 7\,A1$, $A1 = A2/7 = A2(1/7) = (A2)(0.1428\underline{6}) = 35/7 = 5$

We can take the ratio of the values of these two parts (a and b) of one construction before and after a magnification. One value will become the numerator, and the other value will become the denominator. This will essentially create two fractions that are equal since both the numerator and denominator have been multiplied by the same value. Expressing this mathematically:

$\frac{a}{b} = \frac{an}{bn} = r$ or perhaps a this subscript notation that indicates which construction that instance of the part, side or variable (value of that side) is in reference to:

$$\frac{a1}{b1} = \frac{n\,(a1)}{n\,(b1)} = \frac{a2}{b2} = r$$

(r) is the common or constant ratio value of all these and any other equivalent fractions, and similar (but magnified) constructions. Note, as shown above, (r) is not the magnification value that is to be (or was) applied or to all the parts of any one construction (so as to make a larger construction) since (n) is that value. (r) is sometimes called the "constant of proportionality" which indicates a constant (same value) numerical relationship among (with respect to) any two, or more, specific parts of any one construction, or of the same specific parts of a magnified (similar) construction.

Given a two parts of one construction, their mathematical relationship (ie. ratio), (r), may not be the same as that from another two of that same construction. Though the magnification (n) values between various sized similar constructions can be different, (r), the mathematical relationship of a specific set of parts of one construction, is always constant for all similar (magnified) constructions. Each part of a similar construction is numerically the same portion, fraction or percentage of that entire construction or other parts of that construction. Each part(s) is said to be of, or as having similar proportions as that of a similar reference or source construction.

Observe these equivalent fractions:

$\frac{2}{4} = \frac{4}{8} = \frac{6}{12} = 0.5$: r = 0.5, = ratio value of numerator to denominator for all similar fractions, and-or the ratio value of two parts of each similar construction

$\frac{2(2)}{4(2)} = \frac{4}{8}$: n = 2 = "magnification value" applied to an entire set of parts of one (reference) construction, or the "magnified value" within or of all parts of a similar construction. r=0.5

$\frac{4(1.5)}{8(1.5)} = \frac{6}{12}$: n = 1.5 , r = 0.5

$\frac{2(3)}{4(3)} = \frac{6}{12}$: n = 3. , r = 0.5 , with same values of (r), the portions are the same

We see that (n), the magnification factor of similar constructions can vary, and yet the ratio of any two parts of one construction is equal to the ratio of the two corresponding parts of all similar (magnified) construction.

When both numerator and both denominator values of two ratios differ in value by the same factor, those values and each expressed ratio are said to be proportional or in proportion to each other. In simple wording, when two ratios are equal in value, they are said to be (equally) proportional to each other. These ratios are equivalent fractions that represent, and are numerically equal to, the same portions or parts. Hence, two fractions with the same value can also be set equal to each other just like an equation (of two equivalent fractions) and solved just like one. This type of equation of two equal portions or fractional values is therefore often called "equivalent portions", "proportions" or a "proportion problem", and they are of great use when changing units or when something is grown (magnified, expanded) bigger or even smaller such as for the amounts to be used in, for example, a construction or a recipe or other mixture.

Proportional means having and keeping a consistent or constant mathematical portion and-or relationship (among the parts of the whole). Since a portion of something is a fraction of that something, proportions also means the same or equal portions or equivalent fractional parts, either physically and-or numerically. The concepts of equivalent fractions has been mentioned previously in this book.

In short, proportional means equivalent fraction parts or portions of the whole part. Two parts or entire constructions consisting of several parts, can only be said as being similar if all the corresponding parts of both constructions have the same portions with respect to each other, hence being proportional (same portions or fraction of). To magnify a given construction by a certain (magnification) value, so as to create a similar construction, you must also magnify each and every part by that same value or factor. The entire two constructions are then said to be similar in (visual) shape and function, and (mathematically) proportional since the ratio value of any two parts of one construction will have the same ratio value of the two corresponding parts of any similar construction.

The example below will verify the common mathematical constructions and methods of solving a proportion type of problem.

Ex. A certain mixture of metals is 15% copper and 85% tin. If 5 pounds are tin, how many pounds of copper are in the mixture?

First, since the percentage of tin is more than that of copper in the mixture, the weight of the tin in that mixture will be likewise be more than the weight of the copper in that mixture.

.

From the information given in the question, we can write some equations:

85% of total weight = 5 lb : can think of 85% from: part weight/total weight = 5lbs/total weight = 0.85,
15% of total weight = x lbs therefore mathematically: (0.85)(total weight lbs) = 5lbs

(total weight) (0.85) = 5lb : tin , 0.85 is the decimal equivalent of 85% , to be used for the calculations
(total weight) (0.15) = xlb : copper

From these, we algebraically have:

$$\text{total weight} = \frac{5\text{lb}}{0.85} \qquad \text{and:}$$

$$\text{total weight} = \frac{x\text{lb}}{0.15}$$

Equating the two (equivalent) expressions for total weight, we have a common expression of an equation for a proportion type of problem. With practice, this could be written just from the information given in such problems:

$$\frac{5\text{lb}}{0.85} = \frac{x\text{lb}}{0.15}$$

: This can be read as: "5lbs is to 85%, as xlbs is (equivalent or equal) to 15%".

Solving for xlb in the equation of (equivalent) proportions, or "proportion equation", we have:

$$x\text{lb} = \frac{(0.15)(5\text{lb})}{0.85} = 0.882\text{lb}$$

: pounds of copper in the mixture

The reciprocal of each side of the equation for the above proportion will also yield the correct result for xlb, if you happened to write the fractions and equation that way:

$$\frac{0.85}{5\text{lb}} = \frac{0.15}{x\text{lb}}$$

: This can be read as: "85% is to 5lbs, as 15% is to xlbs".
xlb = 0.882lb

Though the total weight could have been found from the original equations written, we can also find the total weight from the left hand side of the proportion equation by performing the indicated division:

total weight = 5lb/0.85 = 5.882lb : since (amount or weight of copper) = (total amount or weight)(0.85)

However, finding the total weight this way (with the specific right side expression known to be equal to the total weight) from the proportion equation may not of been intuitive (natural to you) that this was the only equation needed to solve the problem. As a check to this result, we can also create an expression for the total weight from:

total weight = (weight of tin) + (weight of copper)
total weight = 5lb + xlb = (5lb + xlb) = (x + 5)lb

Substituting this value into the original expressions:

(x + 5)lb (0.85) = 5lb : tin
(x + 5)lb (0.15) = xlb : copper

Let's solve for x, the weight of the copper, using the second equation:

(x + 5)lb (0.15) = xlb distributing:
0.15x + 0.75 = x transpose 0.15x:
0.75 = 1x - 0.15x combining like terms:

$0.75 = 0.85x$ dividing both sides by 0.85, and switching sides:

$x = \frac{0.75}{0.85} = 0.882\underline{4}$ lbs :checks

Given any other sized mixture or volume of the two metals, as long as 15% is copper and 85% is tin (as specified in the formula or recipe), the mixtures are said to be proportional ("within or having proper portions") to each other since they both contain the same portion, ratio or fractional part value of each metal used. Also, the "internal ratios" of each single construction or mixture, such as he ratio of copper to tin, or tin to copper, will also be a constant value. Those parts are also said to be proportional to each other in each instance of the mixture or construction..

From the previous equation:

$\frac{5\text{lb}}{0.85} = \frac{x\text{lb}}{0.15}$ From this we can derive and can mathematically express the ratio of the two ratios or parts percentages, and the ratio of their corresponding weight values:

$\frac{0.15}{0.85} = \frac{x\text{lb}}{5\text{lb}}$: ratio of portions = ratio of weights

$\frac{\text{copper}}{\text{tin}} = \frac{15\%}{85\%} = \frac{0.15}{0.85} = \frac{0.8824\text{lb}}{5\text{lb}} = 0.17648$: percentages and weights ratios of copper to tin, are equivalent

Note that the weight of copper is 15% of the entire weight of the mixture of the metals:

$\frac{\text{copper}}{\text{tin}} \;/\!=\!/\; \frac{\text{copper}}{\text{copper + tin}}$: these two expressions cannot be equal (have the same quotient or ratio) since the denominators are different. (copper + tin) = (total weight)

$0.17648 \;/\!=\!/\; 0.15$

Total Weight = weight of tin + weight of copper
Total Weight = 5 lbs + 0.8824 lbs
Total Weight = 5.8824 lbs

$\frac{\text{copper}}{\text{total weight}} = \frac{\text{copper}}{\text{copper + tin}} = \frac{0.8824}{5.8824} = 0.15 = 15\%$

Ex. In a mixture from a recipe, there are 2 cups of sugar, 3 cups of flour, and 1 cup of another ingredient. The ratio of sugar to flour can be mathematically expressed as:

$\frac{2 \text{ cups}}{3 \text{ cups}} = 0.667$: sugar/flour

However, it is incorrect to say that 66.7% of the entire mixture is sugar since the 1 cup of another ingredient was not even included in the calculation. Since there are more ingredients, the percentage of sugar in the mixture will then be less than 66.7%. To find the actual percent that is only sugar in the entire mixture is to have the ratio of the amount of sugar with respect to the total amount used for all the ingredients in the mixture. The units of each value must all be the same and consistent. For this example, cups are the units of measurement:

$$\frac{\text{amt. sugar}}{\text{amt. of all ingredients}} = \frac{2\text{ cups}}{2\text{ cups} + 3\text{ cups} + 1\text{ cup}} = \frac{2\text{ cups}}{6\text{ cups}} = 0.333 = 33.3\%\text{ sugar}$$

$$\frac{\text{amt. flour}}{\text{amt. of all ingredients}} = \frac{3\text{ cups}}{2\text{ cups} + 3\text{ cups} + 1\text{ cup}} = \frac{3\text{ cups}}{6\text{ cups}} = 0.50 = 50\%\text{ flour}$$

$$\frac{\text{other ingredient}}{\text{amt. of all ingredients}} = \frac{1\text{ cups}}{2\text{ cups} + 3\text{ cups} + 1\text{ cups}} = \frac{1\text{ cups}}{6\text{ cups}} = 0.16\underline{7} = 16.7\%\text{ other ingredients}$$

Summed together, we see that the sugar and flour comprise 33.3% + 50% = 83.3% of the mixture.
The sum of all the ingredients or parts portions is: 0.333 + 0.50 + 0.167 = 1.0 = 100% : the whole or entire mixture

If twice the amount of each ingredient is used to double (2) the amount or size of the mixture, there would be 2x2=4 cups of sugar, 3x2=6 cups of flour, and 1x2=2 cups of another ingredient. The percent of sugar, and flour in this mixture is then:

$$\frac{\text{amt. sugar}}{\text{amt. of all ingredients}} = \frac{4\text{ cups}}{4\text{ cups} + 6\text{ cups} + 2\text{ cups}} = \frac{4\text{ cups}}{12\text{ cups}} = 0.333 = 33.3\%\text{ sugar}$$

$$\frac{\text{amt. flour}}{\text{amt. of all ingredients}} = \frac{6\text{ cups}}{4\text{ cups} + 6\text{ cups} + 2\text{ cups}} = \frac{6\text{ cups}}{12\text{ cups}} = 0.50 = 50\%\text{ flour}$$

We see that the portion, or relative amount, of sugar in each mixture is always 33.3%, and the portion of flour in each mixture is always 50%. Each of the two mixtures is said to consist of the same fractional value or portion of a given ingredient. This is the basic essence of the concept of proportion - when the portions of something are always consistent regardless of the actual sizes or specific amounts (of units) being considered. Also, the ratio of the portions of sugar and flour in any one and all similar mixtures or "constructions" of the recipe (plan, steps, method, ingredients)) will always be the same value of:

$$\frac{\text{sugar}}{\text{flour}} = \frac{0.333}{0.50} = 0.667 \quad \text{: for any similar (magnified=increased or demagnified=decreased) batch or construction using that recipe}$$

Because of the portions (ie., percentages) of sugar and flour are to be always the same value, the actual amounts of sugar and flour used for any similar mixture of this recipe are also said as being proportional to one another. Since the ratio of the portions of sugar to flour is always 0.667, a constant, all similar ratios of sugar to flour in various sized mixtures or instances of this recipe will be equivalent, and these equivalent ratios are called (equal, same) proportions.

Using the actual amounts of each ingredient in a mixture, the amount of sugar will always be 33.3% of the mixture, and the amount of flour will always be 50% of the mixture, and the ratio of the amount of sugar to the amount of flour will always be 0.667

Checking: $\frac{2\text{ cups}}{3\text{ cups}} = \frac{2(2)}{3(2)} = \frac{4}{6} = 0.66\underline{7}$:66.7% sugar

Considering just the mixture of the sugar and flour, what percent of this mixture is sugar?

$$\frac{\text{sugar}}{\text{sugar + flour}} = \frac{2}{2 + 3} = \frac{2}{5} \quad 0.40 = 40\%\text{ sugar}$$

You will often read that a construction such as a model or "blueprint" (a type of photographic copy of a drawing, and where the background is a blue color due to a chemical reaction to light, and the subject or drawing is a white color) is "geometrically proportional" to the actual (reference) construction of something. What does this mean? The easiest way to answer this question is that by observation. They are proportional since they look exactly alike, except that one is larger or smaller than the other. This size, its "largeness" or magnification effect can even be achieved simply by viewing any one construction at different distances that are closer or farther from you. Now, the mathematics of constructions that are said to be proportional will be considered:

Ex. Now observe these two simple and similar constructions that have a triangle shape: [FIG 42]

construction 1 construction 2

In these similar constructions, sides (a) and (c) are corresponding sides, and (b) and (d) correspond to each other. Even though there are only two constructions, they are said to be (mathematically) proportional and are truly similar constructions if all the corresponding ratios of sides are the same.

Construction 1 and 2 are proportional if:

$\frac{a}{c} = \frac{b}{d} = r$: r = ratio value of each equivalent fraction

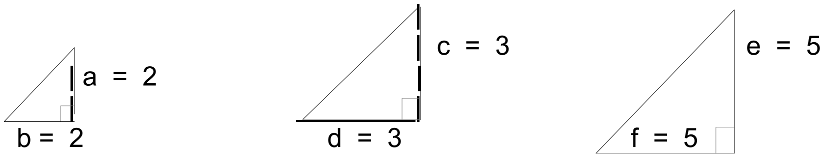
(n), the "magnification value" of each corresponding part is not indicated, but it is the value used to create any equivalent fraction and (magnified or indemnified) construction by multiplying all its parts by this same value of (n). In general (n) is a different value and concept than (r). Expressing this:

$\frac{a}{c} = \frac{a(n)}{c(n)} = \frac{b}{d} = r$: r = ratio value of each equivalent fraction or portion of all similar constructions
: n = magnification (factor) value used to created a unique similar construction

Given two corresponding sides of two similar constructions, you can find the magnification value. For example, since (c) is the magnified value and corresponding side to (a):

From: $a(n) = c$, therefore: $\frac{c}{a} = n$

Any two corresponding sides, or lengths, of any two geometrically proportional constructions will have a specific ratio. Note also that this ratio is not necessarily the same constant as shown for the two constructions in the example above. The specific value of the ratio of similar constructions, depends on the specific parts being considered. To help you understand this, consider all the infinite and different "magnifications" possible. As long as all the ratios of any sides or parts of one construction are equal to those of another construction, the constructions are still geometrically proportional or similar. For example, here are three constructions that are geometrically proportional. Basically, they are magnifications of one another: [FIG 43]



construction 1 construction 2 construction 3

$\frac{c}{a} = \frac{d}{b}$: ratios of corresponding parts of two similar constructions

$\frac{3}{2} = \frac{3}{2} = 1.5$

$$\frac{e}{c} = \frac{f}{d}$$

$$\frac{5}{3} = \frac{5}{3} = 1.6\underline{7}$$

Likewise, for example:

$$\frac{e}{a} = \frac{f}{b}$$

$$\frac{5}{2} = \frac{5}{2} = 2.25$$

Clearly, the ratios are different (due to the magnification value (n) applied to the values of all the parts of one construction, so as to create a similar construction), but still, the ratio (r) among specific parts of any one (similar) construction will always be the same as that of any another similar construction. Here, for example, one specific ratio of sides happens to be:

$$\frac{a}{b} = \frac{c}{d} = \frac{e}{f} = 1.0$$

: note for example that a and b are sides of the same construction, c and d are the corresponding (similar, or in reference to a and b) sides of another, such as the first construction.

It is important to note that we can mathematically derive ratios among corresponding sides of two similar constructions (here, in the example they happen to be triangles):

From: $\frac{a}{b} = \frac{c}{d}$

These are equivalent fractions and corresponding ratios of two sides of each construction. Here, each is an equal ratio value, this (before magnification is even considered) is the fundamental meaning of things or parts of a construction being proportional, or kept in proportion, among themselves. If a construction is magnified, all of its parts are automatically kept in proper proportion having the same fractional value of the whole. From this, we can mathematically derive other ratios between two constructions. From the above equation, if we multiply each side by (b), and divide each side by (c) we get:

$$\frac{a}{c} = \frac{b}{d}$$

or : $\frac{c}{a} = \frac{d}{b}$

: Ratios of corresponding sides of two similar constructions. Due to the various possible magnifications (n) possible for similar constructions, equations of their sides, as shown here, may not have the same values, as shown previously. The only thing that can be said here is that given any two specific similar constructions, the ratio of corresponding sides will be the same. This value is also the "magnification factor" (n, m, or f) for only these two specific similar constructions.

One method often taught for solving for a variable in an equivalent fraction or proportion type problem is to "cross multiply" the numerators to the denominators, effectively ridding the expressed fractions on both sides of the equation, but you should realize that this is essentially the result after proper multiplication and canceling has occurred.

Ex. Solve for (x) given: 2 is to 4, as 5 is to (x).

Mathematically, this can be expressed as:

$\frac{2}{4} = \frac{5}{x}$ this may also be expressed as: 2:4 as 5:x , 2:4 = 5:x , 2 to 4 = 5 to x
2 per 4 = 5 per x , 2 parts of 5 = 5 parts of x , and:

2 is to 4 as is 5 is to x

$$\frac{\overset{+1}{\cancel{(4x)}}(2)}{(1)\underset{+1}{\cancel{(4)}}} = \frac{(5)\overset{+1}{\cancel{(4x)}}}{\underset{+1}{\cancel{(x)}}(1)}$$

: C. clearing or canceling fractions by multiplying both sides by the LCD of the entire equation, and here it is 4x.

$2x = 20$: after D. distributing. This would be the same result after "cross multiplying" numerators and denominators together on opposite sides of the original equation

$$\frac{\overset{+1}{\cancel{2}}x}{\underset{+1}{\cancel{2}}} = \frac{\overset{+10}{\cancel{20}}}{\underset{+1}{\cancel{2}}}$$

: isolate x, so divide by its multiplying factor of 2

$x = 10$ Checking using the decimal equivalents of the ratios on both sides of the equation:

$$\frac{2}{4} = \frac{5}{10}$$

$$\begin{array}{r} 0.5 \\ 4\overline{)\,2.0} \\ \underline{2.0} \\ 0 \end{array} = \begin{array}{r} 0.5 \\ 10\overline{)\,5.0} \\ \underline{50} \\ 0 \end{array}$$

$$0.5 = 0.5$$

Ex. At an apple orchard, the workers picked 15 baskets of apples. They counted a total of 450 apples in all the baskets. The next day, they picked 24 baskets of apples. How many apples are expected or estimated to be in the 24 baskets without going through the long process of actually or physically counting each one?

This problem can be solved using an equivalent proportion or equivalent fraction equation such as:

450 apples is to 15 baskets, as (x) apples is to 24 baskets.

$$\frac{450 \text{ apples}}{15 \text{ baskets}} = \frac{x \text{ apples}}{24 \text{ baskets}}$$

Obviously, whenever the number of baskets of apples is larger, the total number of apples should likewise be larger. That is, the corresponding (such as the numerators, here apples, or the denominators, here baskets) numerical values that compose each equivalent fraction and ratio are said to be "proportional" or more correctly as: "directly proportional" or "directly related". If you increase (ie. magnify) or decrease the number of baskets by some factor, the total number of apples will correspondingly increase or decrease by this same factor. If the number of apples picked increases or decreases by some factor, the number of baskets needed to hold those apples will correspondingly increase or decrease by the same factor. The number or quantity of apples or baskets is said to be directly proportional. This keeps each physical portion or part of each new similar construction (here, apples or baskets) to have the same mathematical portion or fractional value with respect to the other parts of that construction. Mathematically, a proportion is essentially an equivalent fraction concept, and you can solve for the variable as if you were creating an equivalent fraction, otherwise, just solve it just like an equation.

First, in a true ratio form. the numerators and denominators are the quantity of similar things or quantities with the same or like units. The result (the ratio value) is a unitless or pure mathematical value:

$\dfrac{\text{new number of apples}}{\text{old number of apples}} = \text{a ratio} = \dfrac{\text{new number of baskets}}{\text{old number of baskets}}$: this ratio here is actually the magnification (n) value of similar (proportional) constructions

Mathematically, we can get another ratio from the above proportion:

$\dfrac{\text{old number of apples}}{\text{old number of baskets}} = \dfrac{\text{new number of apples}}{\text{new number of baskets}}$ = this a different ratio (r) and value than (n) is

Continuing the example:

Equation Method:

$\dfrac{450\text{ apples}}{15\text{ baskets}} = \dfrac{x\text{ apples}}{24\text{ baskets}}$ isolate x, here, we will multiply each side by (24 baskets):

$\dfrac{\overset{+1.6}{\cancel{(24\text{ baskets})}}(450\text{ apples})}{(\quad 1 \quad)\underset{+1}{\cancel{(15\text{ baskets})}}} = \dfrac{(x\text{ apples})\overset{+1}{\cancel{(24\text{ baskets})}}}{\underset{+1}{\cancel{(24\text{ baskets})}}(\quad 1 \quad)}$: c. cancel

$(+1.6)(450\text{ apples}) = x\text{ apples}$ d. distribute:

$720\text{ apples} = x$ or by switching sides:

$x = 720\text{ apples}$: 15 baskets is to 450 apples, as is, 24 baskets is to 720 apples , or
450 apples is to 15 baskets, as is, 720 apples is to 24 baskets

Equivalent Fraction Method:

$\dfrac{450\text{ apples}}{15\text{ baskets}} = \dfrac{x\text{ apples}}{24\text{ baskets}}$

The denominator on the right side of the equation is larger than the denominator on the left side by a factor of:

$\dfrac{\overset{+1.6}{\cancel{24\text{ baskets}}}}{\underset{+1}{\cancel{15\text{ baskets}}}} = +1.6$

To make a proper equivalent fraction, the numerator must also be changed (here multiplied) by the same factor.

$(450\text{ apples})(+1.6) = 720\text{ apples}$

$\dfrac{450\text{ apples}}{15\text{ baskets}} = \dfrac{(450\text{ apples})(+1.6)}{(15\text{ baskets})(+1.6)} = \dfrac{720\text{ apples}}{24\text{ baskets}}$

Note: Performing the indicated division on either side of the equation will give us the number of apples per basket (apples/basket). It also creates an equivalent fraction.

$\dfrac{450\text{ apples}}{15\text{ baskets}} = \dfrac{30\text{ apples}}{1\text{ basket}}$: or 30 apples/basket = "Thirty apples per basket" (a "unit ratio")

$\dfrac{720\text{ apples}}{24\text{ baskets}} = \dfrac{30\text{ apples}}{1\text{ basket}}$: or 30 apples/basket = "Thirty apples per basket" (same ratio)

This value of (30 apples/basket) which is common to both of the ratios shown, or any more that can be created for this proportional (equivalent portions or fractions) system, is known as a constant of proportionality of, or in each and every similar construction or instance. Hence, the total number of apples in the baskets can be simply found or predicted by multiplying the number of baskets by the number of apples known per basket.

$$(24 \text{ baskets})(30 \text{ apples/basket}) = \frac{(\overset{+24}{\cancel{24 \text{ baskets}}})(30 \text{ apples})}{(1)(\underset{+1}{\cancel{1 \text{ baskets}}})} = 720 \text{ apples}$$

If you like, you can substitute the word(s) of: "for", "for each", "for every", "to every", "in", for the "per" (/) symbol when reading the literary statements that contain it. Ex. "30 apples/1 basket", for example can be thought of as: "30 apples for every basket", or "30 apples to one basket". This may sometimes be simply spoken as "thirty to one", or mathematically expressed as : 30:1

Ex. 15 people is to 3 houses: 15 people / 3 house , We can create two equivalent fractions by dividing both the numerator and denominator by either the numerator or denominator so as to find the fundamental results for each unit of either people or houses:

1 people / 0.2 house or 0.2 house / 1 people, and 5 people / 1 house or 1 house / 5 people

Ex. A basic formula for some product might state that for each (that is, 1 mathematically) part or unit of substance A, there should be 3 parts of substance B. Note that here, A and B are not variables in an algebraic sense, but represent some physical materials. The word "part" as used in the context above means a general representation for any units of measurement used, for example, the parts could be ounces, pounds, kilograms, etc. The two parts might even be expressed as having two different units of measurement, for example: "Mix one gram of oil into every 3 liters of gas":

The ratio of A to B is 1/3 = 0.333333... (perhaps with units of "grams per liter" as: 0.333333 grams/liter)

$$\frac{\text{number of A parts}}{\text{number of B parts}} = \frac{1}{3} = 0.333333...$$

If you only used 0.75 parts of A, how many parts of B should you then use in order to ensure the parts have the same portions of the whole, or simply, so as to keep them proportional?

We know that the ratio of A to B is always a constant value of (1/3) = 0.333333 , regardless of the quantity of A and the corresponding value of B used for any instance or similar construction:

$$\frac{A}{B} = \frac{0.75}{B} = \frac{1}{3} = 0.333333 \qquad \text{solving for B:} \qquad B = 3\,(0.75) = 2.25$$

You could also solve for any quantity of A and B from the simple formulas of:

From: $\frac{\text{number of A parts}}{\text{number of B parts}} = 0.333333...$ we can mathematically or algebraically get:

$A = 0.333333\ B$ and:

$B = \frac{A}{0.333333} = 3A$: when given just two parts in a proportional system, the numerical coefficients of those values are reciprocals of each other

Here is a note about proportions:

Ex. You have the equivalent fractions or proportion of:

$\frac{X}{10} = \frac{1}{2}$, you can say that:

"X is one-half of 10" or = "one-half of 10 is X" : remember, "is" means "equals"

As a verification to these statements, consider this:

$\frac{X}{10} = (1/2) = 0.5$

Solving for X, we get: $X = (0.5)(10) = (1/2)(10) = \frac{10}{2}$: "X is one-half, or 0. 5, of 10". Any value divided by 2 is half of that value.

Ex. You have the equivalent fractions or proportion:

$\frac{5}{x} = \frac{1}{2}$: You can say that: "five is one-half of x" or = "one-half of x is 5"

Letting 1/2 = 0.5 be the quotient of 5/x,:

$\frac{5}{x} = 0.5$

Multiplying the divisor and the quotient to yield that dividend of 5, we see that the word "of" in "one-half of x is (equal to) 5" is mathematically expressed as a multiplication symbol, and the word "is" is expressed as the equals symbol:

$(0.5)(x) = 5$ after dividing both sides by 0.5:
$x = 10$

Ex. A cup of a liquid is defined as 8 fluid-ounces (oz.), which is actually a measure of volume. Weight measurements can also have units of ounces and is a measure of force. This similarity is due to that 1 fluid ounce volume of water is defined as having a weight 1 ounce. However, if the substance is not water, then 1 fluid ounce volume of that substance will no longer weigh 1 ounce of weight. How many (fluid) ounces are in 1/6 of a cup?

First, a more general or fractional approach:

Writing what we know about the relationship between cups and fluid-ounces of measurement:

1cup = 8oz. : 1 cup equals 8 ounces , this is an equation where each equivalent side is expressed as a quantity of different units.
1 is not equal to 8, but 1cup is equal to 8oz

1/6 ("one-sixth") of a cup implies a fraction of a 1 cup. (1/6) cup is one part of a cup divided into 6 equal parts = 1 part of 6 parts. (1/6) of 1 cup = (1/6) x 1cup. Which can be expressed as the following where a division (here of 1cup into smaller and equal parts) by 6 on both sides of the equation is used to express this fractional part:

$\frac{1\text{cup}}{6} = \frac{8\text{oz}}{6}$ simplifying the right side:

$\frac{1}{6}$ cup = 1.333...oz. : "one-sixth of a cup is (equivalent to) one point three, three ounces"

As an extra note, given that 1cup = 8oz., mathematically dividing both sides of this equation by either side, 1cup or 8oz., we can get:

$"1" = \frac{1\text{cup}}{8\text{oz}} = \frac{8\text{oz}}{1\text{cup}}$: since the numerators and denominators are equal (represent the same actual amount of substance, but with different units), the physical (not mathematical) volume ratio is 1. This can be read as "1 cup per 8 oz", and "8 oz per 1 cup".

Fractions like this are called "unity" (1) fractions or ratios. Here, the numerator and denominator represent the exact same amount or value of something, even though they are described differently using different values and units. Unity fractions will be discussed some more in the topic of changing units.

Now with the concept of proportions:

$\frac{8\text{oz}}{1\text{cup}} = \frac{X\text{oz}}{(1/6)\text{cup}}$ or=: $\frac{8\text{oz}}{1\text{cup}} = \frac{X\text{ oz}}{0.166...\text{cup}}$ solving for X oz.:

$X\text{ oz} = 1.33\text{oz.}$

Note also that: 1 cup / 8 oz , and dividing the num. and den. by 8: (1/8) cup / 1 oz or 1 oz / (1/8) cup = 1 oz / 0.125 cup

Ex. An objects true or actual height (H) is known to be 3.8 feet, and is measured to have a height (h) of 0.25 inches in a photograph. If the objects width (w) is measured to be 1.125 inches in the same photograph, what is a good estimate of the objects true width (W) ?

We can consider this a system with two similar constructions. One construction or instance is the real object, and the other similar and proportional construction is represented by an image on a photograph. These similar constructions are essentially magnified versions of each other, and the parts of each similar construction have the same mathematical portion of the whole part. One construction (here the photograph or image) is magnified or multiplied in value to create the larger construction. In a reverse type of manner, the larger construction is essentially demagnified" or multiplied by the reciprocal of the same value so as to create the smaller construction. It can be said that the larger construction is essentially divided by that same value (the magnifying or multiplication factor, n) to create the "demagnified" similar construction. We can solve this problem using proportions or equivalent fractions.

ratio in photograph = actual ratio of the reference construction or object

$$\frac{w}{h} = \frac{W}{H}$$

$H = 3.8\text{ feet} = (3.8\text{ feet})(12\text{ inches}/1\text{ feet}) = 3.8\,(12\text{ inches}) = 45.6\text{ inches}$: 1feet = 12 inches And used substitution, or by mult. both sides by 3.8

$$\frac{1.125\text{ in.}}{0.25\text{ in.}} = \frac{W\text{ in.}}{45.6\text{ in.}}$$

$$W\text{ in.} = \frac{(1.125\text{ in.})\,(45.6\text{ in.})}{0.25\text{ in.}} = \frac{51.3\text{ in.}\hat{}2}{0.25\text{ in.}\hat{}1} = 205.2\text{ in.}$$

$$W = 205.2 \text{ in.} = \frac{205.2 \text{ in. } (1 \text{ ft.})}{(12 \text{ in})} = 17.1 \,(1 \text{ ft.}) = 17.1 \text{ ft.}$$

: (1ft/12.in) is a "unity ("1") fraction"
Each inch is, (=), a fraction of a foot.
From 1ft = 12 in, equivalency or "unity":
Dividing both sides by 12 to solve for 1in:
1in = 1ft/12 = 0.083333ft, and to
have multiple inches, such as 205.2 here,
we multiply both sides of this by 205.2 to
have the equivalent number of feet.
From the expression above, we also get
these equivalent fractions:
(1ft/12in) = (0.08333ft/1in)
Here, the "magnification factor" is:
0.083333. Mathematically. from these
equivalent fractions. we can get:
(1ft/0.08333ft) = (12in/1in) = 12 which is
the reciprocal of the "magnification factor":
(1/0.083333) = 12

Checking: $$\frac{1.125 \text{ in.}}{0.25 \text{ in.}} = \frac{205.2 \text{ in.}}{45.6 \text{ in.}} = 4.5$$: the ratio of the width to the height in each similar construction

The similar construction magnification ratio (n) or factor is:

$$\frac{H}{h} = \frac{45.6}{0.25} = \frac{W}{w} = \frac{205.2}{1.125} = 182.4 \quad \text{or} \quad (1/182.4) = \text{about } 0.005482456 \text{ for demagnified constructions}$$

Up to now, the examples of proportions have been of the direct proportion type. Another type, perhaps not as common, is the inverse proportion type. With inverse proportions, a quantity is mathematically inversely related to another quantity. One way to think of an inverse relationship is to think of a quantity where getting "more when there is less" , or when getting "less when there is more". Algebraically, we know that a quotient (Q) is directly related to a numerator (N) value , and inversely related to the denominator (D) value:

$$r = Q = \frac{N}{D} \qquad : r = \text{ratio}$$

If the numerator increases, the ratio increases. If the numerator decreases, the ratio decreases. This is a direct mathematical relationship. If the denominator increases, the ratio decreases. If the denominator decreases, the ratio increases. This is an inverse type of mathematical relationship.

If an equation expresses the relationship between only two variables, and they are known to be inversely proportional or related to each other, then the numerator variable is not needed to express their relationship and it is replaced by (1), or changed to (1) using division if necessary:

$$B = \frac{1}{A}$$: essentially then, each variable is the reciprocal or "inverse" of the other. These variables are said to be (mathematically) inversely related to each other.

$$A = \frac{1}{B}$$ mathematically: $1 = AB$: the product of two corresponding reciprocals is 1.

Ex. It takes a (1) person to do a certain (1) job (such as make or repair something) in 1 hour. 2 people can work together and do that job in half the time or (1 hour / 2 =) 0.5 hours. Clearly, the variables of the number of people doing the work, and time to do that work, are inversely related. The more people working, the faster (less time) the job can get done. As we now know, this relationship cannot be expressed in the form of:

$\frac{N}{D} = r$ for example:

$\frac{1 \text{ people}}{2 \text{ people}} = \frac{1 \text{ hour}}{0.5 \text{ hour}}$: this is wrong since after canceling:

0.5 /=/ 2

Nor could this be expressed algebraically as something like:

$\frac{1 \text{ people}}{1 \text{ hour}} = \frac{2 \text{ people}}{0.5 \text{ hour}}$ since mathematically:

1 people/hour /=/ 4 people/hour : after canceling, or:

1 (people/hour) /=/ 4 (people/hour) : expressing or showing that (people/hour) is like a new unit of work

The proper form to use is that first discussed:

$r = \frac{1}{D}$ or $A = \frac{1}{B}$ using the variables stated in the example:

$\text{people} = \frac{1}{\text{time}}$ or mathematically:

$\text{time} = \frac{1}{\text{people}}$

Note, algebraically from each equation:

(people)(time) = 1 = 1(people)(time) = 1 (job completed) = 1 job = 100% job completed

(people)(hour) = 1

Checking that which was stated in the example:

$1 \text{ hour} = \frac{1}{1 \text{ people}}$

$0.5 \text{ hour} = \frac{1}{2 \text{ people}}$

If 4 people were all helping each other getting the job finished, it would take one-fourth the time: time/4 = time(0.25)

$0.25 \text{ hour} = \frac{1}{4 \text{ people}}$, or perhaps by using this equation:

1 job = 1 job : since 1 job = (people)(time), and using substitution of the problems values:

1 ((people)(hour)) = ((4 people)(x hours)) solving for x hours :

$$x \text{ hours} = \frac{1((\text{people})(\text{hour}))}{4 \text{ people}} = 0.25 \text{ hours} \quad \text{, or } = 15 \text{ minutes}$$

Note that for both instances:

(people) (time) = 1 : = 100% , one job completed

(people A) (time A) = 1 = (people B) (time B) = 1 job completed : here, A=of job A, B=of job B
(4 people)(0.25 hours) = (2 people)(0.5 hours) = (1 people)(1 hour) = 1 = 1 job finished or completed

We can now make ratios which will then express the problem as an inverse proportion or inverse equivalent fractions:

From: (people A) (time A) = 1 = (people B) (time B) we mathematically have:

$$\frac{\text{people A}}{\text{people B}} = \frac{\text{time B}}{\text{time A}}$$: A BASIC FORMAT EXAMPLE FOR AN INVERSE PROPORTION

Notice that the ratios are inverted, instead of having corresponding units in their numerators and denominators. In more general terms, if A is inversely related to B, as C is inversely related to D, the proportion is of this form:

From: $A = \frac{1}{B}$, and $C = \frac{1}{D}$ mathematically:

$AB = 1 = CD$
$AB = CD$ mathematically we can then get this proportion (but it's an inverse one):

$\frac{A}{C} = \frac{D}{B}$ Using the values from the example:

$$\frac{\text{people A}}{\text{people B}} = \frac{\text{time B}}{\text{time A}}$$

$$\frac{1}{4} = \frac{0.25}{1}$$

$0.25 = 0.25$: since the ratios are equal, the values are correct

DEFINING THE MATHEMATICAL RELATIONSHIPS OF VARIABLES

The formal name of how variables and values are mathematically related is called variation, specifically, it is how a variable varies (ie. changes) when others vary (change). Most of the concepts of variation have already been mentioned in this book.

When a value is mathematically directly related to another value, this is sometimes noted as direct variation. For example:

> The formula for the circumference (C) of a circle is: circumference = (pi)(diameter) = C = pD, and this has a more generic mathematical expression form that is technically called a "linear equation" (that graphs as a line since the ratio of the variables is constant) of:

$y = cx$: Where (c) is an arbitrary constant which is sometimes noted as the constant of variation. This is easily seen since c = y / x always (ie a constant). It is also the multiplying factor or numerical coefficient to x. c is often found from experimentation and/or calculation. A common example of a constant is in the formula for the circumference (C) of a circle where C=pD. Here, p="pi" is a constant of about 3.14159, and D is the diameter of the circle. For any circle with it's own circumference and diameter, p=C/D always. When c, the constant is not indicated, it is understood as being 1. The constant of variation can also be thought of as the constant in the relationship of the variables that can change or variate.

Here, (y) is directly related to (x). An indication that this is so is to recognize that (x) is essentially placed in the numerator of the right hand expression whose denominator is 1, or understood as 1 when not indicated. If (x) increases, (y) increases. If (x) decreases, (y) decreases.

If (y) is mathematically inversely related to (x), the expression would look something like this:

$y = \frac{c}{x}$: or can be expressed as:

$y = c\left(\frac{1}{x}\right)$: to explicitly express the reciprocal or inverse relationship of y and x.

This is known as inverse variation. If (x) increases, then (y) decreases in an opposite or inverse manner. Likewise, if (x) decreases, then (y) increases. The indication that (y) is inversely related to (x) is that (x) is in the denominator of the right hand expression. (y) is mathematically related to the reciprocal of (x) which has the value of (1/x), and (x) is mathematically related to the reciprocal of (y) which is the value of (1/y):

$$x = c\left(\frac{1}{y}\right)$$

It is also easily possible for a variable to be directly related to one or more variables, and at the same time be inversely related to one or more variables. For example:

$y = \frac{cx}{z}$: again, here (c) represents a constant that does not change or variate like a variable can. Here (y) is directly related to (x) and inversely related to (z).

Knowing the concepts of variation can help you write equations and formulas. If given all the values of the variables in an expression, you can then solve for the constant of variation when needed, and write a formula showing the mathematical relationship among those variables.

Ex. The cost of a certain rectangular piece of material was $24 and this cost was directly related to its' size (LW = length x width), that is, the cost was directly related to the area of the material. The length of the material was 4ft. and the width was 2ft. Calculate the cost per square unit of area of this material. The basic equation or formula for this is:

$\text{total-cost} = c\,(LW)$: where c is the price per square foot.
$\$24 = c\,(4ft)(2ft)$
$\$24 = c\ 8ft^2$ solving for c:

$c = \frac{\$24}{8\text{ ft.}^2} = \frac{\$3}{1\text{ ft.}^2}$: $3 per square foot is the cost per square foot, and the basic formula becomes:

total-cost = (cost per square unit)(total number of square units)

$\text{total-cost} = (\$3/\text{ft.}^2)(LW)$ or = $(\$3/\text{ft.}^2)$ Area

If another section of that same kind of material was 5ft. long , and the width was 7ft., what would be the cost?

$$\text{total-cost} = \left(\frac{\$3}{1\text{ ft.}^2}\right)(5\text{ft.})(7\text{ft.})$$

$$\text{total-cost} = (\$3)\left(\frac{35\text{ft.}^2}{1\text{ ft.}^2}\right) = (\$3)(35)$$

$$\text{total-cost} = \$105$$

As a check, solving this problem using proportions:

$\frac{\$24}{8\text{ft.}^2} = \frac{x}{35\text{ft.}^2}$ solving for x:

$$x = \frac{(\$24)(35ft^2)}{8ft^2} = (\$24)(4.375)$$

$$x = \$105$$

In terms of the proportions, notice that the constant of variation, ($3 per square foot or= $3/1ft^2) above, is the constant of proportion or proportionality for this specific system, and it is the cost per square foot of a certain material.

If a variable is directly related in value to another, when one increases, the other will increase. When one decreases, the other will decrease.

If a variable is directly proportional to another, the variables are directly related, and when one increases by a factor or multiple value, the other will increase by that same factor or multiple value.

Ex. $a = b$, If (b) increases by a factor of (n), (a) will increase by (n), and so as to keep the equation in balance:
$na = nb$

Ex $a = b$, If (b) decreases by a factor of (n), (a) will increase by (n), and so as to keep the equation in balance:

$$\frac{a}{n} = \frac{b}{n}$$

If a variable is inversely proportional to another, when one increases by a factor or multiple value, the other will decrease by that same factor or multiple value and which could be stated that the other will decrease by the reciprocal of that factor value.

Here, (a) and (b) are inversely related. The numerator (1) in the fraction of (1/b) could also be any general value, but this expression of (1/b) explicitly shows the inverse relationship, and here it is also a reciprocal, mathematical relationship since the numerator is 1. For a reciprocal relationship, the two mathematically, inversely related values are factors of 1. (a)(b) = 1 , and a = (1/b) and b = (1/a)

Ex. $a = \frac{1}{b}$, If (b) decreases by a factor of (n), (a) will increase by that same factor of (n):

$$\frac{1}{\frac{(b)}{(n)}} = \frac{\frac{1}{1}\frac{(n)}{(b)}}{\cancel{\frac{b}{n}\frac{(n)}{(b)}}} = \frac{n}{b} = n\left(\frac{1}{(b)}\right) = n\,a$$

This book often uses the words "a proportional problem", and this is meant as how to solve for an unknown value if the relationships of the values and-or variables have a proportional, mathematical relationship. The process could then be called one of solving a proportional problem or "proportion equation":

Ex. $\frac{a}{b} = x$ and if both the num. and den. are multiplied by the same value: $\frac{a}{b} = \frac{a(n)}{b(n)} = \frac{a\,(1)}{b\,(1)} = \frac{a}{b} = x$

We see the multiplying both the numerator and denominator by the same value still produces the same ratio value as the result.

$\frac{a}{b} = \frac{c}{d} = x$ Since both fractions or ratios have the same value, here (s) it means that each corresponding set of values in the num. and den. were changed by the same factor value, say (n) or (1/n). Showing this intermediate step:

$\frac{a}{b} = \frac{an}{bn} = \frac{c}{d} = x$: the process created an equivalent fraction, and each faction is said as having the same portions and-or ratio value as the other, and this may be said as the numerator and denominator values are within proportion or have the same portions as that of another set of values. Equivalent portions are verified by having the same quotient or ratio, and here it is shown as (x).

In this example it could be said that the ratio of (c) to (d) has the same as the ratio or fractional value as that of (a) to (b), and here that value is shown as(x). For an explicit numerical example, rather than the algebraic example just given:

$\frac{5}{10} = \frac{2}{4} = 0.5$: Each numerator value is half of the denominator value. The numerators have the same portion, part or fraction value of the denominator. These fractions are equivalent, and have the same ratio and-or quotient value.

It could be said that one set (ie. the numerator and denominator of a fraction) is a magnified version of the other set, and-or one set is a demagnified version of the other set, and yet the parts of each set have the same multiple or fractional value with respect to each other. Consider this:

$\frac{10}{5} = \frac{50}{25}$: here the ratio of the numerator to the denominator parts of each fraction is 2, and that the parts of the fraction on the right are 5 times bigger since (num2/num1) = (50/10) = 5, and (den2/den1) = (25/5) = 5, hence a magnified set, instance or version of the portions.

$\frac{10\ (5)}{5\ (5)} = \frac{50}{25} = 2$, also consider the reverse, where the multiplier is the reciprocal of 5: (1/5) = 0.2 , hence a demagnified set, instance or version of the portions. $\frac{50\ (1/5)}{25\ (1/5)} = \frac{10}{5} = 2$

CHANGING UNITS

It is very possible to change units of a given measurement or value if the units being changed to are similar, such as being a length unit, such as converting a quantity of feet to a quantity of meters or vice-versa, or a volume unit, such as converting a quantity of gallons to liters or vice-versa.

Changing or converting units from one to another is easily accomplished by setting up an equivalent fractions or proportion type of problem. First, a basic relationship among the units given and the units you are solving for must be known. This relationship is commonly called a "unity", "units or "equivalency" ratio. This will be on one side of the equation. The other side will be the ratio of the value with its units being converted, and a placeholder variable for the converted value and its units. It is a usually easier to solve for the variable when it's in the numerator, and then be sure to keep corresponding units in both numerators, and both denominators on both sides of the equation.

Ex. Change 5000 feet to its equivalent value in miles.

The known relationship between miles and feet is that there is 1 mile unit per 5280 feet units, or 5280 feet per (1) mile.

From: 1 mile = 5280 feet After dividing one side by the other we have:

$\frac{1 \text{ mile}}{5280 \text{ feet}} = 1$ and $\frac{5280 \text{ feet}}{1 \text{ mile}} = 1$: 1 here means equivalence or unity since both 5280 feet and and 1 mile represent the same length or distance. If you were to subtract rather than divide, the result would be 0 since when you subtract equal values, their difference is 0.

$\frac{1\text{mi.}}{5280\text{ft.}}$ or $\frac{5280\text{ft.}}{1\text{mi.}}$: unity fractions for this problem

$\frac{1 \text{ mi.}}{5280\text{ft.}} = \frac{x \text{ mi.}}{5000\text{ft.}}$: setting up the proportion
Can be read as: "If 1 mile is to 5280 feet , then x miles is to 5000 feet".

After solving for x using the equation (here, multiply both sides by 5000ft) or (creating an) equivalent fraction method:

$$x = \frac{1\text{mi}(5000\text{ft})}{(5280\text{ft})}$$

$$x = 0.947 \text{ miles}$$

Hence, 5000 feet = 0.947 miles.

After a general observation of the above, you can convert feet to miles simply by dividing the number of feet by 5280. Division, rather than multiplication, is the essence of converting from smaller (sized) units to larger (sized) units since there is a lesser quantity of larger units.

From: x mi. = 5000 ft.

,we know that we can divide both sides of an equation by the same value and they will still be in balance. Letting that value be (use either equivalent side of) :

1 mi. = 5280 ft. : that is, you can divide one side by 1mi, and the other by 5280ft since these lengths are equivalent, even though they are mathematically represented differently with different quantities and units.

We can get (which can algebraically be found from the initial proportion shown) :

$\frac{5000 \text{ ft.}}{5280 \text{ ft.}} = \frac{x \text{ mi.}}{1 \text{ mi.}} = x$: x is the number of units which are known to be miles.
We see that x is basically the ratio of the left hand side.

On the other hand, multiplication is the basic essence of converting from a quantity having larger (sized) units to a quantity having smaller (sized) units since there are many more of the smaller units per each larger units of measurement. For example:

Given: 1 mi. = 5280 ft.

To convert 5 miles to its equivalent value of feet, you can set up a proportion type of problem, or simply multiply each side of the equation by 5 since 5 miles is a factor of 5 times larger than 1 mile; (5 = 5mi./1mi.) :

(5)1 mi. = 5(5280) ft.
5 mi. = 26400 ft. putting this in a general formula, for a "miles to feet" conversion formula:

x mi. = x (5280) ft. : which is the same as multiplying both sides by (x), and the equation will still be in balance

Here's how to convert between miles and inches if you do not know the direct "conversion factors" between miles and inches, but do know the conversion factor between feet and inches:

From: 1 mi. = 5280 ft. : eq. 1
And: 1 ft. = 12 in. : eq. 2 , multiplying each side of this equation by 5280:

5280 ft. = 63360 in. substituting this fact into eq. 1:

1 mi. = 63,360 in. multiplying both sides by x:

x (1 mi) = x 63,360 in

x mi = x 63,360 in : a general formula for converting miles to inches

Or, since as shown previously:

x mi. = x (5280)ft. : converting miles to feet

And likewise:

x (1) ft. = x (12) in. after mult. both sides by x, and substituting this into the above:

x mi. = x (5280) ft. = (5280)(x ft.) = (5280)(x 12 in.) = x (5280)(12) in. =

x mi. = x 63360 in. : a formula to convert miles to inches

Perhaps a similar and simpler derivation is from: 1 mi = 5280ft, then x miles would be equivalent to:

x mi. = x (5280)ft. = x 5280 (1 ft) = x 5280 (12 in) = x 63360 in.

Here, a proportion is used to solve for the number (N) of inches in a given amount (x) of miles.

$\frac{1 \text{ mi.}}{63360 \text{ in.}} = \frac{x \text{ mi.}}{N \text{ in.}}$ after solving for N inches:

Nin. = x 63360in. : an expression, or formula, for the relationship between a quantity of inches and (x) miles. Here, 63,360 is a constant in the relationship between inches and miles.

Ex. Here is a more advanced example. Given there are 231 cubic inches (231cu. in. = 231 in.^3 = 3.785$\underline{4}$ liters) per 1 American (U.S.) gallon of water, if it rains 1 inch deep over on an area of 1 square mile (1 sq. mi. = 1 mi.^2), how many total gallons did it rain? Extra note: 1 American (U.S.) gallon has 128 fluid ounces. The British gallon is 277.274 cubic inches = 4.546 liters and has 160 fluid ounces. The gallon unit is slowly becoming obsolete in favor of the metric system unit of liters, however the gallon unit is still needed for any past references and conversions.

From : 1 mi. = 5280 ft. squaring each side, we get:
(1 mi.)^2 = (5280 ft.)^2 : 1 square-mile in terms of square-feet, or vice-versa
1 mi.^2 = 27,878,400 ft.^2 : about 28 million square feet

From: 1 ft. = 12 in. squaring each side, we get:
(1 ft.)^2 = (12in.)^2
1 ft.^2 = 144 in.^2

Setting up a proportion problem:

$$\frac{1\ \text{ft.}^2}{144\ \text{in.}^2} = \frac{27{,}878{,}400\ \text{ft.}^2}{x\ \text{in.}^2}$$

$$x\ \text{in.}^2 = \frac{27{,}878{,}400\ \text{ft.}^2\ (144\ \text{in.}^2)}{1\ \text{ft.}^2}$$

x in.^2 = 27,878,400 (144 in.^2) = 4,014,489,600 in.^2 : a square mile in terms of square inches. Approximately 4 billion square inches.

Multiplying this by 1in. since the third dimension of the volume is 1 inch deep (of rain) in this problem:

(4,014,489,600 in.^2) (1 in. ^1) = 4,014,489,600 in.^(2+1) = 4,014,489,600 in.^3 : cu. in. of rain

$$\frac{1\ \text{gallon}}{231\ \text{in.}^3} = \frac{x\ \text{gallons}}{4{,}014{,}489{,}600\ \text{in.}^3}$$

$$x = \frac{4{,}014{,}489{,}600\ \text{in.}^3\ (1\ \text{gallon})}{231\ \text{in.}^3}$$

x = 17,378,743 gallons : about 17 million gallons of rain water when it rains 1 inch deep over a square mile

As an extra note, 1 gallon of water weighs about 8.3456 pounds (lbs.). Also, the equivalent gallons in 1 cubic foot (1 cu. ft. = 1 ft.^3) of water can found using the above proportion by expressing 1 cubic foot of water in terms of cubic inches of water: Since 12 inches equals 1 foot:

1 ft.^3 = (1 ft.)^3 = (1 ft.)(1 ft.)(1 ft.) = (12 in.)(12 in.)(12 in.) = (12)(1in)(12)(1in)(12)(1in) = (12^3)(1in)^3
1 ft^3 = (12^3)(1^3 in^3) = 12^3 (1 in^3)
1 ft^3 = 1728 in.^3

After solving the proportion for x gallons:

$$\frac{1 \text{ gallon}}{231 \text{ in.}^3} = \frac{x \text{ gallons}}{1728 \text{ in.}^3}$$

Therefore, 1 ft.^3 of water = 1728 in.^3 = 7.4805$\underline{2}$ gallons of water , and:

One ton (2000lbs) of water is equal to : 2000lb / 8.3456 lb / gallon = 239.6$\underline{5}$ gallons =~ 240 gallons
One cubic foot of water, or 7.4805 gallons of water weighs: (7.4805gal)(8.3456 lbs/gal) = 62.43lbs
One ton (2000lbs) of water is equal to : 2000lb / 62.43 lbs per cubic ft. = 32.03$\underline{6}$ =~ 32 cubic feet of water
Taking the cube root of a cube with 32.036 cubic units, here: ft.^3, is 3.17$\underline{6}$ ft per side: (l, w, or h).
This is just slightly more than a yard (3ft) per side, and also just slightly less than 1 meter (3.28 ft.) per side.

Ex. A basement is to be resurfaced with concrete. The dimensions of the basement are as follows: Length=30 ft., Width=12 ft., and the thickness or Height of the concrete is to be 1 inch deep. How many cubic yards of concrete will be needed?

First, we will convert all the measurements to their equivalent measurements with units of yards.

Since 3 ft. = 1 yd. we can write a proportion problem as:

$$\frac{1 \text{ yd.}}{3 \text{ ft.}} = \frac{x \text{ yd.}}{n \text{ ft.}} \qquad \text{solving for } x \text{ yd:}$$

$$\frac{(n \text{ ft.})(1 \text{ yd})}{3 \text{ ft}} = x \text{ yd.}$$

: Hence to find x yd. of a given length of feet, simply divide the number (n) of feet by 3. .

Here are the converted values: L = 30 ft. = 10 yd.
W = 12 ft. = 4 yd.

For the Height, we will first convert its value with units of inches to an equivalent value with units of feet, and then proceed as usual:

$$\frac{1 \text{ ft.}}{12 \text{ in.}} = \frac{x \text{ ft.}}{1 \text{ in.}}$$

$$x \text{ ft.} = \frac{(1 \text{ in.})(1 \text{ ft.})}{12 \text{ in.}} = (1/12) \text{ ft.} = 0.083333333 \text{ ft.}$$

: 1in =~ 0.0833ft

Dividing by 3, or multiplying by (1yd/3ft), to convert feet to yards, we have:

H = 0.083333333 ft. = 0.027777777 yd.

Since (L yd.)(W yd.)(H yd.) = x yd.^3 , we have:

(10 yd.)(4 yd.)(0.027777777 yd) = 1.111111111 yd.^3 : the minimum needed
Slightly more than 1.11 cubic yards.

Below is a small table that can be used for making typical conversions between the metric and American systems of units of measurement for length or distance:

Metric English-American

1cm = 0.393700787 in. : cm=centimeter, in.=inch. 1cm = (1/100)m = 0.01m , Note that 1 in = 2.54 cm, exact
1m = 3.280839895 ft. : m=meter, ft.=foot

1m = 1.093613298 yd. : m=meter, yd.=yard
1km = 0.621371192 mi. : km=kilometer, mi.=miles , 1km = 1000m

How about expressing these relationships the other way around? For example, here is 1 mile found in terms of kilometers without using the standard methods of conversion:

1km = 0.621371192 (1) mi. : units equivalence of distance

Dividing both sides by 0.621371192 will rid (1) miles of it multiplying constant:

$$\frac{1\text{km}}{0.621371192} = \frac{0.621371192\ (1)\ \text{mi.}}{0.621371192}$$ after dividing and switching sides:

1 mi. = 1.609344001 km : **When given a unit equivalence of two different units, the unit equivalence of the other unit is equal to the reciprocal of the given factor value of that other unit.**

Here is a similar example showing the number of centimeters per inch:

1cm = 0.393700787 (1) in. after dividing both sides by 0.393700787 and switching sides:
1 in. = 2.54 cm.

Conversions of weight: units:

A gram (g) or sometimes a kilogram, is the unit of mass (ie. substance or matter) in the metric system. A gram is physically defined as being equal to the mass of 1 cubic centimeter of water (which must have a temperature of only a few degrees above freezing, since as water is heated, it expands, and hence it's dimensions increase). The (corresponding) weight of a gram is also commonly used as the (adopted) unit of weight in the metric system, although the Newton unit is the actual unit of weight or force. Most modern scales will automatically convert a weight value to its equivalent gram value.

1g = 0.035273961 oz. : g=gram, oz.=ounce
1kg = 2.204622622 lb. : kg=kilogram, lb.=pound

Conversions of liquid volume:

A cubic centimeter (abbreviated as cu.c., or cc, or cm^3) is a unit of volume or three dimensional spacial area in the metric system. (10cm x 10cm x 10cm) or 1,000 cubic centimeters is defined as 1 liter = 1l.

1L = 0.264172052 gal. : l=liter, gal.=gallon

1 liter of water weighs 1 kg or 1000g since 1l = 1000 cu. c. , and each cu. c . weighs 1g.

Here is an example that is a conversion (called a reduction) within the metric system:

How many millimeters (mm) are in a centimeter (cm)?

Considering the basic facts: $1\text{ cm} = \frac{1}{100}\text{ m} = 0.01\text{m}$ and:

$$1\text{mm} = \frac{1}{1000}\text{ m} = 0.001\text{m}$$

Dividing the equivalent meter value for centimeters by the equivalent meter value for millimeters, we can find out how many times larger that a centimeter is than a millimeter:

$$\frac{1\text{ cm}}{1\text{ mm}} = \frac{0.01\text{m}}{0.001\text{m}} = 10$$

Therefore, a centimeter is 10 times larger than a millimeter. Considering the division this reciprocal way:

$$\frac{1\text{ mm}}{1\text{ cm}} = \frac{0.001\text{m}}{0.01\text{m}} = 0.1$$

We find that a millimeter is only a tenth of a centimeter. This can be mathematically verified from the above equation by multiplying each side by 1 cm:

$$1\text{ mm} = 0.1\ (1\text{ cm})$$
$$1\text{ mm} = 0.1\text{ cm} = \frac{1}{10}\text{ cm}$$

It can also be said that there are 10 millimeters in a (1) centimeter. This can be mathematically verified from the above equation by multiplying each side by 10:

$$10\text{ mm} = 1\text{ cm} \qquad \text{or} = \qquad 1\text{ cm} = 10\text{ mm}$$

This could have also been found from the original division shown above after multiplying each side by 1mm.

Before leaving this topic of changing units, consider these example equations below.

1ft. = 12in. Having some multiple of feet to convert to inches, its a matter of repeated addition or simply a multiplication. Multiplying each side by (x):

$$x\ (1\text{ft}) = x\ (12\text{in})$$

$x\text{ft} = x12\text{ in}$: to convert a number of feet to its corresponding number of inches

Given 1ft = 12in , and dividing both sides by 12:

$$1\text{in} = \frac{1\text{ft.}}{12} = 0.0833333\text{ ft}$$ when having some multiple of inches to convert to feet:

$$x\text{in} = \frac{x\ (1\text{ft})}{(12)} = \frac{x\text{ft}}{12} \quad \text{or} = \quad 0.0833333x\text{ ft}$$

You can also use equivalent fractions or "proportions" to solve conversion problems.

AN UNCOMMON METHOD OF CREATING AN EQUIVALENT FRACTION

There is a somewhat non-standard, perhaps novel, way worth mentioning of making an equivalent fraction. Given two equivalent fractions, you can add (or subtract) their numerators together, and add their denominators together to create a third fraction that is equivalent:

$$r = \frac{a1}{b1} = \frac{a2}{b2} = \frac{a1 + a2}{b1 + b2} \qquad : r = \text{ratio of the fraction(s)}$$

At first sight, this appears to go against everything learned so far, that is, you cannot add a (same) value to both the numerator and denominator to create an equivalent fraction. This is true, but here, the values added to the numerator and denominator are not the same (constant) value. Here is the verification that the sum of equivalent fractions is another equivalent fraction:

We know that to create an equivalent fraction, we multiply (or divide) the numerator and denominator by the same value, say (n) as shown below:

$$r = \frac{a1}{b1} = \frac{na1}{nb1} = \frac{a2}{b2}$$

Hence, we must verify that:

$$r = \frac{a1 + a2}{b1 + b2} = \frac{a1 + na1}{b1 + nb1} \qquad \text{factoring out common factors to each term on the right hand side :}$$

$$r = \frac{a1 (n + 1)}{b1 (n + 1)} \qquad \text{after canceling out the common factor of } (n + 1) :$$

$$r = \frac{a1}{b1} \qquad : \text{This checks since the ratio is still the same value}$$

Summarizing the results:

$$r = \frac{a1}{b1} = \frac{na1}{nb1} = \frac{a2}{b2} = \frac{a1 + a2}{b1 + b2} = \frac{a1 + na1}{b1 + nb1} = \frac{a1 (n + 1)}{b1 (n + 1)} = \frac{a1}{b1}$$

A small advantage of using this method to create an equivalent fraction is that addition is usually faster than multiplication.

Ex. $\frac{2}{4} = \frac{15}{30} = \frac{(2 + 15)}{(4 + 30)} = \frac{17}{34} \qquad : r = 0.5$

In short, to create a third equivalent fraction, you can add a fraction's numerator to the numerator of another equivalent fraction, and add the fraction's denominator to the denominator of that same equivalent fraction. The process, as shown above, appears to be that of "adding equivalent fractions" to create a third equivalent fraction, though it is really adding corresponding numerators and denominators of equivalent fractions to create a third equivalent fraction. As an extra note on this topic, since the ratios (r) of equivalent fractions are the same value, adding equivalent fractions will result in a value that is a multiple of (r) and which is therefore a different value than (r) which is needed so as to have an equivalent fraction.

AN OBSERVATION ON RECIPROCALS

When two values are reciprocals of each other, say (a) and (b), we know that their product equals 1.

$(a)(b) = 1$ and $a=1/b$ and $b = 1/a$

But what do we get when we divide these reciprocals by each other?

$$\frac{a}{b} = c$$

$$\frac{b}{a} = d$$

It may or may not be obvious or intuitive, but the quotients of dividing two reciprocals, here (c) and (d), are also reciprocals. For example:

Given: $a=0.5$ and $b=2$:

$(0.5)(2) = 1$ and $0.5 = 1/2$ and $2 = 1/0.5$: Verifying that 0.5 and 2 are indeed reciprocals since their product equals 1.

$$\frac{a}{b} = \frac{0.5}{2} = 0.25 = c$$

$$\frac{b}{a} = \frac{2}{0.5} = 4 = d$$

Observe that 0.25 and 4 are also reciprocals:

$(c)(d) = (0.25)(4) = 1$, and:

$1/c = 1/0.25 = 4 = d$, and:

$1/d = 1/4 = 0.25 = c$

So in a way, we are right back where we started with two reciprocals, and the process can be continued by dividing these new reciprocal values, and here they are 4 and 0.25.

Some other observations:

From: $ab = 1$, $a = 1/b$, $b = 1/a$: (a) and (b) are the reciprocals of each other

And: $\frac{a}{b} = c$: c = the quotient of the reciprocals (a) and (b), mathematically: It will be shown below that the square root of this quotient (here c) of two reciprocals is equal to the numerator, here a solving for c:

$a = bc = \frac{1}{b}$ solving for c:

$c = \frac{1}{b^2}$ after taking the square root of both sides of this equation:

$\sqrt{c} = \frac{1}{b} = a$, also, squaring both sides we have: $c^2 = a$

Ex. Given these two reciprocals: a=0.25 and b=4:

$\frac{a}{b} = c$: hence, a division of two reciprocals

$\frac{0.25}{4} = 0.0625$

$a = \sqrt{c} = \sqrt{0.0625} = 0.25$,checking: $a = bc = (4)(0.0625) = 0.25$

Likewise:

$\frac{b}{a} = d$

$\frac{4}{0.25} = 16$: 16 and 0.0625 are also reciprocals

$b = \sqrt{d} = \sqrt{16} = 4$

If you took two values that were not reciprocals and divided them, their quotients are still reciprocals, but the square root of the results generally would not be equal to these values.

Ex. $\frac{5}{10} = 0.5$: 5 and 10 are not reciprocals since their product does not equal 1. (5)(10)=50

$\frac{10}{5} = 2$ These quotients are still reciprocals of each other: (2)(0.5)=1

Since the these values, 5 an 10, were not reciprocals to start with, the square roots of their quotients are not equal to those values as shown here:

$\sqrt{0.5} = 0.707106781$ and $\sqrt{2} = 1.414213562$

The roots of the two quotients (0.5 and 2) are reciprocals. Checking:

$\sqrt{0.5}\sqrt{2} = \sqrt{0.5(2)} = \sqrt{1} = (0.707106781)(1.414213562) = 1$

Analyzing the results:

$ab = 1$: (a) and (b) are reciprocals

$\frac{a}{b} = c$ and $\frac{b}{a} = d$

$cd = 1$: (c) and (d), the two quotients of two reciprocals (a) and (b), are also reciprocals

$\sqrt{c} = a$ and $\sqrt{d} = b$: This is true only when (a) and (b) were reciprocals to begin with.
Here is a verification of this:

From: $ab = 1$

$$\frac{a}{b} = \frac{\frac{a}{1}}{\frac{1}{a}} = a^2 = c$$: a reciprocal divided by its reciprocal is equal to: numerator^2
taking the square root of both sides we have:

$$\sqrt{c} = a$$

An interesting pair of reciprocals is the square root of 10, and the reciprocal of this square root of 10. Both values have the same digits, but they are different by a factor of 10.

$$\frac{a}{b} = \frac{\sqrt{10}}{1/\sqrt{10}} = \frac{3.16227766}{1 / 3.16227766} = \frac{3.16227766}{0.316227766} = c = 10$$ checking:

$$\frac{a}{b} = \frac{\sqrt{10}}{1/\sqrt{10}} = \frac{10\text{^}0.5}{1/ 10\text{^}0.5} = (10\text{^}0.5)(10\text{^}0.5) = 10\text{^}(0.5+0.5) = 10\text{^}1.0 = 10 = c$$

Clearly, when two values differ by a factor of 10, as just shown, their digits will be identical.

$$a = \sqrt{c} = \sqrt{10} = 3.16227766$$

You can also use other values besides 10 for the radicand (N), and the two values ($\sqrt{N}$, and $1/\sqrt{N} = \sqrt{1/N}$) will differ by a factor of that specific radicand., but their digits will not be (visually) identical as when the radicand is 10. Also expressed is that the reciprocal of a root of a number is equal to the root of the reciprocal of that number:

$$\sqrt{1 / N} = \sqrt{1} \;/\; \sqrt{N} = 1 / \sqrt{N}$$: The root of the reciprocal of a number is equal to the reciprocal of the root of that number.

Note that::

$$\sqrt{1/10} = \sqrt{0.1} = \sqrt{1 / 10} = \sqrt{1} \;/\; \sqrt{10} = 1 / \sqrt{10} = 1 / 3.16227766 = 0.316227766$$ and

$$\frac{\sqrt{10}}{\sqrt{0.1}} = \frac{3.16227766}{0.316227766} = 10$$ checking by multiplying the divisor and quotient together:

$$10 \sqrt{0.1} = \sqrt{100}\,\sqrt{0.1} = \sqrt{(100)(0.1)} = \sqrt{10}$$

As an extra note or concept involving reciprocals, of which much has already been discussed:

If you have say 10 items, each (1) item can be said or indicated as: 1 in 10, 1 of 10, 1 out of 10 , and these can be numerically expressed as:

$\frac{1}{10}$ in standard fractional form as an expression or= 0.1 in standard decimal fractional form as a single numeric value

Hence given any number of items, say N, the numeric value of each item is simply the reciprocal of that number N = (1/N)
Ex. If given 5 items total, each item can be numerically expressed as: (item) / (total items) = 1 / 5 = "1 of 5" = 0.2

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QUADRATIC EQUATIONS

In this book, we have already discussed and solved equations of the first degree or order. That is, the exponent of the variable being solved for has only been the first (1) power or degree. Below is a discussion on solving equations of the second order or degree, more formally known as quadratic equations. Here, the exponent of the variable being solved for is 2. The word quadratic comes from the word prefix of quad which means 4. Since these equations have a "squared variable" often associated with square shapes with 4 equal sides; (s) and area is (s^2), it makes some sense to call them quadratic. Another reason that they are called quadratic equations is in reference to their graph. When the graph of the quadratic equation is plotted on a perpendicular or rectangular-coordinate system of measurement and location on a plane, the curve will eventually occupy two or more quadrants (a 1/4, or a quarter section) of the rectangular-coordinate system. When plotted, the geometric shape of a quadratic equation or curve is called a **parabola**. "Parabolic" is a word that basically means "two-sided", and this curve is two-sided as being symmetrical (ie., symmetry, likeness, identical, mirrored) about the axis of the parabola curve. A common use for the parabola shape is with the shape of the large main (light collector) mirror of a reflector telescope where all the dim or faint incoming light is reflected to be gathered together and concentrated into a much smaller area or point of brighter or more intense light which can then be magnified and viewed by the eyepiece lens. This point is formally called the focus, or focus point of the curve or system, and which basically means the central or center of locus (of a path of related points) or location point of the parabola curve.

The general format of the basic linear (line) expression and equation is:

$ax^1 + b = ax + b$: (a) is a multiplying factor, or numerical coefficient to the variable (x).
(a) cannot equal 0, since the variable would essentially be eliminated
since anything times 0 is 0. (a) can be 1, and the expression would
simply be: $(1)x^1 + b = x + b$
(b) is some arbitrary constant added in and it could be 0, and the
expression would simply be: $ax + 0 = ax$
If (a)=1, and (b)=0, the expression would simply be: x

If the value of this expression was set equal to y, such as for graphing or plotting the relationship of, or between, y and x, and where y can be thought of as the output of the expression, and x as the input of the expression. The mathematical relationship between y and x is a linear (ie. proportional) type of relationship.

If the basic linear expression is multiplied by a constant (c), the result is another linear expression "in (variable) x":

$$c(ax^1 + b) = acx^1 + bc$$

That is, the expression still consists of a term which has (x) to the first power, and which is multiplied by a numerical coefficient, and has a (constant) value term added to the first or "variable term".

However, if it is multiplied by the variable (x) that is also within and defines that specific linear expression, the result is a quadratic expression:

$$x(ax^1 + b) = ax^2 + bx$$

As an extra note to consider: This is equivalent to repeatedly adding the expression or its resulting value a total of (x) times. For example:

$x(ax^1 + b) = (ax^1 + b) + (ax^1 + b) + \ldots + (ax^1 + b)$: here, the number of terms depends on the value of x
If x=3:

$$3(a3^1 + b) = (a3^1 + b) + (a3^1 + b) + (a3^1 + b) = a3^2 + 3b = 9a + 3b$$

We can add in an arbitrary constant (c) term, and set the expression equal to another variable such as (y) to hold the variable results of the expression. We can also set the expression specifically equal to 0 (ie., setting y=0) so as to have a more general format of a quadratic equation, and here with (x) representing any independent or "determining (of y)" variable that is used:

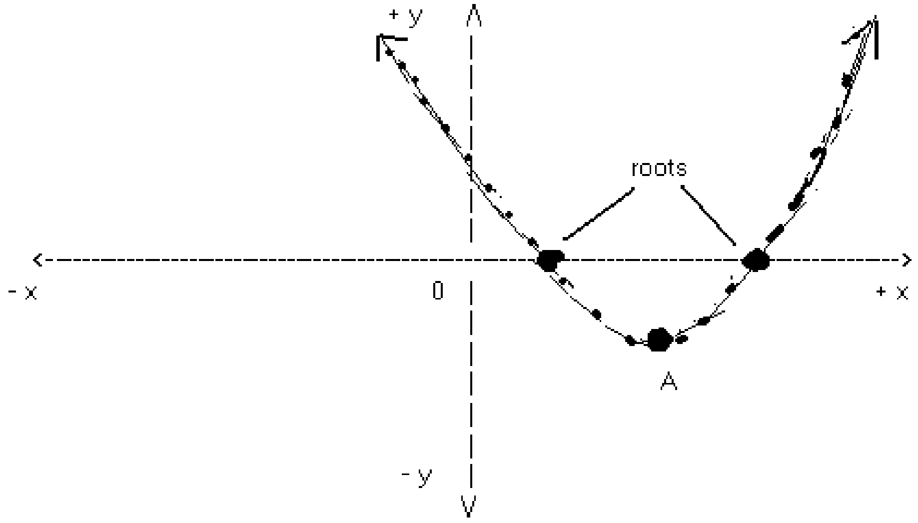
ax^2 + bx^1 + c = 0 : **BASIC QUADRATIC EQUATION** , (a /=/ 0, since the x^2 term must exist to be a quadratic equation)

Notice the descending powers of the variable in each term: (x^2, x^1, and x^0 = 1), this is the basic essence of formally defining equations of a higher degree. Variables a and b are numerical coefficients of the terms containing the variable x, and c is an arbitrary constant. Variables b and-or c can even have a 0 value, and hence those terms would not be present as expressed here:

ax^2 + c = 0 and ax^2 = 0 : these are sometimes called "partial quadratic equations"

By choosing values for variable x, and plotting (showing pairs of corresponding values of (x) and (y) as points or locations) a graph of the equation, it will show the overall relationship between the (x) or (independent or "input") variable and another corresponding variable (that is affected or determined by x), such as (y) that is set equal to the "output" or result of that equation. The graph or "curve" of the equation will visibly show the relationship of the variables over a large range of values.

If variable (y) is set equal to this basic or general form of the quadratic equation that was solved for and-or set equal to 0, it will essentially be setting (y) also equal to 0, and the corresponding value(s) of x are the coordinates or location of a point called the x-axis intersect (intersection or crossing) of the curve, and where all the corresponding values of y are equal to 0. Here is the general shape of the curve of a quadratic equation when plotted (placement of its points) as a graph with a natural-like smooth flowing curve between its points: [FIG 44]



Notice the point indicated as A on the graph. This is called the minimum (here, as in the minimum y value) point of the curve, or the minimum (least, smallest possible) value, output or result of the equation. If the curve opened downwards, instead of upwards as seen here in the graph, the point would be called a maximum point of the curve, or the maximum value, output or result of the result of the equation. There is a mathematical method, discussed briefly in this book, that can be used to find the exact location of this point. This method uses a special equation called a derivative (ie. an equation derived (ie. created) from or out of a given equation), for the slope (a measure of the steepness or incline) of the curve at any given point and-or value of x. At point A, the curve (as in the (y) values) is neither decreasing or increasing as the (x) variable increases, hence the "steepness of the curve" or slope at that point on the curve is said to be 0. Setting (y) of the derivative equation (which is basically an equation for the slope of the curve at a given point or value of x), or "slope [rate of change of the variables] equation" equal to 0, you can then easily solve for its corresponding value of (x), and then use this value of (x) in the actual quadratic equation to find its corresponding value of (y). For example, if the (x) variable was actually time (t), and the (y) variable was actually temperature (T), you can mathematically find or predict the exact point in time where the temperature was either at minimum or maximum value, and the corresponding value of that temperature.

The two axis of a rectangular measuring and location system will divide that plane into 4 areas called quadrants. (Quadrants (Q) are numbered counter-clockwise from the upper right quadrant which is considered as the first quadrant (Q1). The other quadrants are identified as Q2, Q3 and Q4.

Many other things can be observed from the curve of an equation. For example, in the above drawing, the co-ordinates of the minimum point, or any point, within the fourth quadrant will have the form of:

> Numerical signs of the co-ordinates of a point in the 4th quadrant are:
> p(x is positive in value, and y is negative in value)

Another observation is that the two roots (or "zeros") or x values where the curve crosses the x-axis and where y=0 will both be positive in sign since they are on the positive side of the x axis.

The word root (or "root of an equation") is often used to describe the solution of a quadratic equation expressed in the general format where y is set equal to 0. The description is used since the solution of any squared value, such as the variable (here x^2), is to take the square-root of it so as to have the "plain", "unsquared" and root value.

Another thing that can be observed is that the curve leftward of the minimum point slopes downward (as x increases). Here, the overall mathematical relationship between (y) and (x) is an inverse relationship. As (x) increases, the corresponding (y) co-ordinate of each point on the curve decreases. The curve to the right of the minimum point slopes upward. Here, the overall mathematical relationship between (y) and (x) is now a direct type of relationship. As (x) increases, the corresponding (y) co-ordinates of each point on the curve is also increasing. Due to this behavior of the slope of the curve not being constant, the mathematical relationship between variables (y) and (x) is not constant or linear in value, and is said to be a non-linear relationship.

The graph also indicates the value of (y) when the corresponding value of (x) equals zero. This point, p(0, y), on the curve is called the y-axis intercept.

When b=0 in a quadratic type of equation, the x^1 term is eliminated, and the resulting equation is reduced to a partial quadratic equation which can be easily solved:

$ax^2 + c = 0$ transposing c :

$ax^2 = -c$ dividing both sides by a:

$x^2 = \frac{-c}{a}$ taking the square root of both sides:

$x = \pm\sqrt{\frac{-c}{a}}$: note that since (c) and (a) are constants, the x-axis intercepts, p(x,0) where y=0, the value of x will be a constant

And for a real root (ie. not having an unknown or imaginary value), the value of the radicand must be positive in sign since the square root of negative numbers are not defined as being a real number and are considered as imaginary numbers or values that are to be imagined or conceived as being mathematically possible.

Ex. If (c) has a value of -16 and (a) has a value of +4:

$$x = \sqrt{\frac{-(-16)}{4}} = \sqrt{\frac{+16}{4}} = \sqrt{4} = +2 \text{ and } -2$$

If the given equation does not include the added in constant (the "constant term" (c)), the quadratic equation can easily be solved by factoring:

$ax^2 = bx$ placing this given equation into a form similar to the standard quadratic form:
$ax^2 - bx = 0$ factoring (x):

$x(ax - b) = 0$ since the product is 0, either or both factors must be, or can be considered as being, equal to 0:

$x = 0$: After dividing both sides by (ax-b). this is one root of the two possible roots.
root = p(x,y) = p(0,0)

After dividing both sides by x:

$ax - b = 0$: after solving for x:

$x = \frac{b}{a}$: this is another (or second) root of the two possible roots
root = p(x,y) = p((b/a) , 0)

If both the x^1 term and the constant term are present, the solution of (x) may not be so easy.

Ex. $x^2 + 2x = 15$ placing this equation into the std. quadratic equation form, so as to help solve it:

$x^2 + 2x - 15 = 0$

We see that a=1, b=2 and c = -15. Lets begin solving for (x) by transposing -15 and factoring (x) out of the left side:

$x(x+2) = 15$ dividing both sides by x:

$x + 2 = \frac{15}{x}$ transposing +2:

$x = -2 + \frac{15}{x}$ also note above that $x = \frac{15}{(x + 2)} = \frac{15}{2 + x}$

Now, how do we solve for a variable when that variable is part of the solution? In general, it cannot be done, that is, it can't provide an exact or algebraic solution as is. However, if used repeatedly, it can provide a solution which is as close to the actual or true solution as we need it to be. Methods of this type are called repeated or successive approximation (of the solution). First consider this simple analogy example. You are to guess a number someone is thinking. The assistance you will receive is an indication to go higher or lower of each next guess. You can do this by using an adjustment or change value (positive or negative in sign) to apply (combine) by adding it to that initial and any further guesses. Expressing this in a formula:

closer guess = guess + adjustment : of course, initially there is no adjustment, as if the adjustment is 0, and then the expression is essentially: guess = guess , and if it is incorrect (ie. the difference between it and the actual or true value is not close to 0), the adjustment process will need to be used.

This equation can also be expressed as:

new guess = guess + adjustment or:
closer guess = guess + adjustment

new guess will then be substituted into the guess position, and this "refining" or "zeroing in (getting closer, a low difference)" process can be continued.

After each step you are to take the value of the left side of the equation and substitute it into the right side of the equation and so as to automatically make adjustments and create a new closer value. When the new updated or modified guess or the numeric result of the equation is reasonably close to the true number that is being guessed or solved for, the process can stop. As you get close to the number, the adjustments should be "smaller and smaller" in absolute value. The concepts of "higher or lower" and "smaller and smaller" adjustments are automatically built into the mathematical method.

Continuing the example:

By replacing (x) in the denominator with its corresponding value (equation), we will have an equation where the computed value of (x) is nearer to the actual or true value of (x). Here, this method will create a "continued fraction":

$$x = -2 + \cfrac{15}{-2 + \cfrac{15}{x}}$$

: and so on. You will still have to guess at x at some point. It could be 0 if you want, or some value that you know is close to the true value of x. As you can imagine, if the initial guess or value chosen for x as a possible solution is close to the actual solution, the less computation needed to find the true result.

If you want to know some more about continued fractions, see Methods For Solving Square Roots in the appendix.

Let's call or assign the (x) in the denominator xn and the new, and more accurate solved for, or computed value of (x), as xn+1. In general, with the example given, if you already know or previously calculated a value for (x) such as from the continued fraction method just shown, then xn+1 can be found (without solving a lengthy continued fraction) by substituting the value of (x) for Xn and using the original equation again:

$x_{(n+1)} = -2 + \frac{15}{x_n}$: (where $x_n = x$) , or as noted above, $x_{n+1} = \frac{15}{(2 + x_n)}$: here, n and n+1 are not variables but are subscripts of the variable x that indicate a specific instance and-or value of that variable x. Hence, you cannot combine these variables: x_n and $x_{(n+1)}$ cannot be combined.

Note, to use this second formula noted, you must have calculated all previous values of (x) from this formula only. Likewise, to use the first formula noted, you must have calculated all previous values of (x) from that formula only.

Since an initial value of (x) might not be given or known, a guess close to -5.0 would be great for this example, but in general, if you don't like to guess, being in the "true" or exact realm of algebra, simply set (x) equal to any value up to infinity, of course using a value near infinity will usually take you longer to get sufficient accuracy in the result. A computer can do this type of mathematical repetition for you in usually less than a second of time. Note that a computer will usually signal an error and may stop processing the program if it (ie. the electronics) tries to divide by 0, hence, you may have to use an initial value of something other than 0 if you use a computer and-or a specific program being used by the computer. When doing the computation by hand, setting xn equal to infinity in the example shown will set the value of that term essentially equal to zero. This is due to the inverse relationship of the resulting quotient (ie. ratio value) and the denominator (ie. divisor). As the denominator increases the quotient decreases. A quotient does have a direct relationship to the value of a numerator.

$\frac{N}{\infty} \rightarrow 0$: --> means "approaches", or "the value will get nearer to this value". Here, as the denominator value approaches an infinitely high value, the quotient value will approach a value of 0.

For the example given, after repeatedly performing the indicated computation, the value of (x) will approach (and actually is) -5.0.

Checking:

$$\begin{aligned} x^2 + 2x &= 15 \\ (-5)^2 + 2(-5) &= 15 \\ +25 - 10 &= 15 \\ 15 &= 15 \quad \text{: checks} \end{aligned}$$

It should be of no surprise that there are two solutions (or "roots") to quadratic equations that are expressed using the basic quadratic equation form. These root values set the expression equal to 0. The two square roots of a positive value are the same in absolute value, and one root will be positive and the other root will be negative, however, with quadratic equations, the two roots' values are usually not the same in magnitude (absolute value), and might even have the same sign. Don't confuse the meaning of "roots" of equations with square-roots only because they are actually different concepts. If the quadratic equation was just $y=x^2$, then it is easy to understand where the term or word "root" comes from in relation to these types of equations since the solution of (x) is simply the root (here, the square root) of x^2 and-or its equivalent value of (y).

Here's how to get the other root of our example, and it uses a "continued radical" method:

$x^2 + 2x = 15$ solving for x^2 by transposing 2x:
$x^2 = 15 - 2x$ taking the square root of both sides:

$x = \sqrt{15 - 2x}$: and in general, we know that a variable cannot be part of it's own (strict algebraic) solution

As done in the continued fraction method just shown, replacing each (x), here in the radicand, with its equivalent value:

$x = \sqrt{15 - 2\sqrt{15 - 2\sqrt{15 - 2x\ldots}}}$ (and so on)

The initial value of (x) can be 0. This continued radical method will actually "zero-in" on a root more quickly than that of the continued fraction method. Here, (x) will approach (as is actually) +3.0.

Checking:

$$\begin{aligned} x^2 + 2x &= 15 \\ 3^2 + 2(3) &= 15 \\ 9 + 6 &= 15 \\ 15 &= 15 \quad \text{checks} \end{aligned}$$

This continued radical notation can also be placed into a more expressive notation for successive approximation:

$x_2 = \sqrt{15 - 2x_1}$ or : $x_{n+1} = \sqrt{15 - 2x_n}$ or: $x_{new} = \sqrt{15 - 2x_{old}}$

Here, x_2 is a closer approximation to the actual or true value of x. Repeat the process by setting x_1 equal to x_2 and evaluate the expression again until the desired accuracy (overall correctness) and precision **(**the smallest or "fineness, minute or fractional part of the value" considered, hence the least significant position [as in what decimal position] or weight**)** is achieved. When the difference between the input (here, x_1) and output (here, x_2) is 0 or as close as needed to it, the result has essentially been calculated or found:

When: $(x2 - x1) = 0$, the difference between the new and old values is 0, or is as small as needed, and the result has been found.

Factoring is another method to solve for the variable in a quadratic equation, however, its only sometimes feasible (especially when the coefficient of the x^2 term is not equal to 1) and therefore, it will not be discussed in great depth. It goes something like this: You can factor an (n) order equation (n=2 for a quadratic equation, a "second order equation") into a product of (n) linear equations (where the highest power of x=1).

First, here is the extended product of two binomials "in x":

$(x + a)(x + b) = x^2 + bx + ax + ab = x^2 + (a + b)x + ab$: clearly, this side is a quadratic expression

If you are given the quadratic expression or trinomial "in (powers of) x" (or whatever variable you happen to be working with) such as on the right hand side above, we now know that it can be factored to a product of two (linear, to the first power) binomials (sum of two terms) "in x" as shown on the left hand side of the equation. The first term of each binomial factor will be the square root of x^2, which is (x). Notice that when the factors (here, a and b) of the third term are summed, that this sum equals the coefficient of (x) in the second term. If the product (ab) is positive in sign, the second terms in the two (linear, first order) binomial factors are either both positive or both negative in sign (remember, a negative times a negative is a positive). When (ab) is positive and the sum of (a) and (b) is negative, that is, if the coefficient of (x) in the second term is negative, then both of the second terms in the binomial factors are negative in sign:

$x^2 - (a + b)x + ab = (x - a)(x - b)$

Ex. $x^2 -9x + 14 = (x -7)(x -2)$: $ab = (-7)(-2) = + 14$ and $(a+b) = (-7) + (-2) = -7-2 = -9$
Note, you will need to figure out which factors of +14 will actually sum to -9; algebraically, which factors of (ab) will sum to (a+b). Here is a pseudo algebraic form of this factorization method:

x^2 + (sum of factors of third term)x + (product of factors of the third term) = $(x+a)(x-b)$

, and where (sum) is the sum of the coefficients of (x^1), and (product) is the product of the coefficients of (x^1).

When (ab) is negative, and the sum of (a) and (b) is positive, the highest factor of (ab) is positive and the lower factor is negative. In general, the second term in only one of the two binomial factors will be negative in sign:

$x^2 +(a+b)x - ab$: here, expressing that when the product of (a) and (b) is negative, hence the term is negative. That is, think of (a) and (b) here as numeric values rather than variables which can generally have any (positive or negative) sign. This is the best we can do to mathematically to represent or express values that are signed constants, for this analysis.

$(x + a)(x - b)$: when (a) is positive and greater than (b) which is negative

Ex. $x^2 +5x -14 = (x +7)(x -2)$: $(+7)(-2) = -14$ and $(+7) + (-2) = 7-2 = +5$

$(x - a)(x + b)$: when (b) is positive and greater than (a) which is negative

When the sum of (a) and (b) is negative, the highest factor of (ab) is negative and the lower factor is positive. In general, when the third term is negative, the second term in only one of the two binomials factors will be negative in sign:

$x^2 -(a+b)x - ab$
$(x - a)(x + b)$: when (a) is negative and greater in absolute value than (b)

Ex. $x^2 -5x -14 = (x -7)(x +2)$: $(-7)(+2) = -14$ and $(-7) + (+2) = -7+2 = -5$

$(x + a)(x - b)$: when (b) is negative and greater in absolute value than (a), that therm is negative in sign

Ex. $x^2 -5x -14 = (x + 2)(x -7)$

Given: $(x + a) (x + b)$, the general factored form of a quadratic equation, it will now be pointed out that $-(a) = -a$, the negative or "negative counterpart" of (a), regardless if the actual value of (a) is either positive or negative. A variable can always be assumed to represent either a positive or negative numeric value unless told otherwise of the value(s) and its sign). $-(b) = -b$ are the actual roots of the corresponding quadratic equation. Note that (a) and (b) in these binomial factors are not to be confused with the variables used in the standard quadratic equation form of: $ax^2 + bx + c = 0$. With this in mind,now consider:

Setting $(x + a)(x + b) = 0$, we know that any value times 0 is equal to 0, so either factor or both can be set equal to 0, and we can solve for x:

$x + a = 0$ and, or $x + b = 0$
$x = -a$ $x = -b$: the two solutions, and roots of the quadratic equation of:
$(x+a)(x+b) = x^2 + (a+b)x + (ab)x = 0$

Letting: $r1 = - a$: r1 for "root one", and:
$r2 = - b$: r2 for "root two"

, and creating binomial factors where the constants are r1 and r2, and setting them equal to the standard quadratic equation, we get the generalized factored form of the generalized quadratic equation:

$(x + r1) (x + r2) = x^2 + bx + c = 0$ extending, or distributing on the left hand side:

$x^2 + xr2 + xr1 + r1r2 = x^2 + bx + c = 0$ factoring x out of the like terms:
$x^2 + (r1 + r2) x + r1r2 = x^2 + bx + c = 0$

By observation, we see that in the standard quadratic equation: $ax^2 + bx + c$:

$b = r1 + r2$: b = the sum or the roots and :

$c = r1 r2$: c = the product of the roots

Ex. Factor the following trinomial:

$x^2 + 5x + 6$: factors of 6 are: 6, 1, 3, 2. The factors of +3 and +2 sum to +5:
$(x+3)(x+2)$: you can check this with distribution
: $a=+3$, $b=+2$

Setting this equal to 0 (as when solving for the "roots" (where y=0) for the standard quadratic equation format) and solving for x:

$(x+3)(x+2) = 0$ dividing both sides of the equation by (x+2):

$x + 3 = \frac{0}{x+2} = 0$

$x = -3$: root , on the graph, this corresponds to point(-3 , 0)

Doing the same for the other binomial factor, we find:

$x = -2$: root , on the graph, this corresponds to point(-2 , 0).

In short, since the product shown above is 0, the method essentially assumes one or both factors are equal to 0.

Checking: $x^2 + 5x + 6 = (-3)^2 + 5(-3) + 6 = 9 - 15 + 6 = 0$ and:
$x^2 + 5x + 6 = (-2)^2 + 5(-2) + 6 = 4 - 10 + 6 = 0$

A quadratic equation or curve will only cross or intercept the y-axis once. Setting x=0, and solving for the corresponding value of y in either equation shown above, we find the y-axis intercept to be point(0 , 6).

There is a direct and preferred method to solve for the specific value(s) of the variable in a quadratic equation, and it is the same method so as to produce the two roots. This method uses the quadratic formula:

$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$: **QUADRATIC FORMULA** , given: $ax^2 + bx^1 + c = 0$,
a /=/ 0 , because the x^2 term must be present to be "quadratic" equation.
(b) and (c) are usually constants for a given "quadratic equation in (x)".

A derivation of this is as follows:

$ax^2 + bx + c = 0$: basic quadratic expression, transposing c:

$ax^2 + bx = -c$ dividing both sides by a:

$\frac{ax^2 + bx}{a} = \frac{-c}{a}$ the left side can be expressed as a sum of fractions:

$\frac{ax^2}{a} + \frac{bx}{a} = \frac{-c}{a}$ after canceling out the common factor of (a) in the first term:

$x^2 + \frac{bx}{a} = -\frac{c}{a}$

(*) The perfect square of a binomial is expressible in the general algebraic form of:

$(x+b)^2 = (x+b)(x+b) = x^2 + xb + xb + b^2 = x^2 + 2bx + b^2$, or=: $(1)x^2 + (2b)x^1 + (b^2)x^0$

The last expression, a trinomial, is equivalent to a perfect square of a binomial value, and specifically for this example, it is (x+b). Likewise, the square root of a trinomial that happens to be a perfect square of a binomial, is equal to that binomial:

Since: $x^2 + 2bx + b^2 = (x+b)^2$ taking the square root of both sides:

$\sqrt{x^2 + 2bx + b^2} = (x+b)$: the square root of a trinomial that is a perfect square

Now given only $x^2 + 2bx = 0$, what is a formula for the third term that can be added in (to both sides of course) so that $x^2 + 2bx + ?$ equals a perfect square of a binomial. Notice in the second term that the coefficient of (x) is represented as (2b). If we divide this value by 2 and square it, we will have the third formal term (of a perfect square of a binomial) which is represented as b^2. This process is known as "completing the square" and can be used as a way to check or determine if a trinomial is a perfect square of a binomial.

Given a general trinomial: $x^2 + (a + b)x + ab$, that is not a perfect square, but you can still create one out of it by using the above method. Take half of the coefficient of the x^1 variable and then square it:

$$\left(\frac{(a+b)}{2}\right)^2$$

You can then add this to the trinomial if you first transpose the (ab) term, a product and constant value, to the other side of the equation. If you do not transpose (ab), you must find the difference between (ab) and this value to be added in, and then add this difference to both sides so as to complete the trinomial that is a perfect square of a binomial. For example:

$(x + 2)(x + 4)$ these two different binomial factors will produce this trinomial:
$x^2 + 6x + 8$

By the above two different binomial factors, we know that this trinomial is therefore not a perfect square of a binomial. You can also find if it is a perfect square of a binomial by just considering the first two terms, completing the square and comparing this to the third term (here its 8):

$x^2 + 6x + ?$ completing the square:

$$\left(\frac{6}{2}\right)^2 = 3^2 = 9$$

Since the third term is not 9, it is also a confirmation that this trinomial is not a perfect square of a binomial. The third term or the "constant value term", the 8, is not correct for this to be a perfect square of a binomial, so an adjustment must be made. The difference between 9 and 8 is 1, therefore, if it is added to one side, it must be added to the other side of the equation so as to keep the equation in balance. Here is the resulting trinomial:

$$x^2 + 6x + (8 + 1) = 0 + 1$$
$$x^2 + 6x + 9 = 1$$

Taking the square roots of the first and last terms, we can find the terms of the binomial that this trinomial is a perfect square of.

$\sqrt{x^2} = x$ and $\sqrt{9} = 3$ therefore:

$x^2 + 6x + 9 = (x + 3)(x + 3) = (x + 3)^2 = 1$ taking the square root of both sides:

$\sqrt{x^2 + 6x + 9} = \sqrt{(x+3)^2} = (x + 3) = \sqrt{1} = 1$: this is one method to solve for (x) or a quadratic equation:

$x + 3 = 1$ transposing +3 :
$x + 3 - 3 = 1 - 3$ combining:
$x = -2$ If you put this in the original quadratic equation set equal to 0 to find the roots, you will see that this value of $x = (-2)$ solves it:

$0 = x^2 + 6x + 8$
$0 = (-2)^2 + 6(-2) + 8$
$0 = 4 - 12 + 8$
$0 = +12 - 12$: $0 = 0$: checks

Also: $(x+3) = 1$, $(-2) +3 = -2 + 3 = 1$: checks

For completeness, when the second term of a squared binomial is negative, the expansion becomes:

$(x -b)^2 = (x -b)(x -b) = x^2 -1xb -1xb +b^2 = x^2 - 2bx + b^2$

, so given any trinomial, if the second terms value (algebraically and formally expressed as: $+2bx$ or $-2bx$) is twice the square roots of the first and third terms, it is a perfect square of a binomial. The sign of the second term of this binomial will correspond to the sign of the second term of the trinomial.

In the above discussions, it is assumed that the coefficient of x^2 (usually indicated as (a) in ax^2) is 1. If it is not, then "divide through" each side by it, which is essentially dividing all terms on both sides of the equation by this coefficient (usually noted as (a)) if possible. At the above location in this book indicated as (*) in this derivation of the quadratic formula, the left side has an x^2 term and an x^1 term (which we know are the necessary terms of a formal quadratic trinomial or equation), hence we can solve for a value that when added in (to both sides of course) to that side, will make that side an expression equivalent to a perfect square of a binomial. The net result after some simplification is that we will have only one (solvable) unknown which is the variable we are solving for and its' equivalent value on the other side of the equation.

Even after dividing (b) by (a), in the trinomial, (b) which is the coefficient of the x^1 term is still twice (2) of what it needs to be for the binomial to be squared that represents the given trinomial. Dividing the co-efficient of this x^1 term by 2 so that we can find the value of (b) for the second term of the binomial squared that will be equal to the given trinomial. Algebraically expressing this process:

$\dfrac{\frac{b}{a}}{\frac{2}{1}} = \dfrac{b}{2a}$ squaring this, so as to help during some following rationalization of roots, we get:

$\left(\dfrac{b}{2a}\right)\left(\dfrac{b}{2a}\right) = \dfrac{b^2}{4a^2}$ adding this to both sides of the equation, so as to keep the balance on both sides:

$$\frac{x^2}{1} + \frac{bx^1}{a} + \frac{b^2}{4a^2} = \frac{b^2}{4a^2} - \frac{c}{a}$$

Now, the left side is equivalent to a perfect square of a certain binomial that represents or is equivalent to a given trinomial ("in x"). By taking the square roots of the first and third terms, (which are the (perfect) "squared" terms) we will have the terms of that binomial.

$\sqrt{x^2} = x$ and:

$\sqrt{\dfrac{b^2}{4a^2}} = \dfrac{\sqrt{b^2}}{\sqrt{4a^2}} = \dfrac{\sqrt{b^2}}{\sqrt{4}\ \sqrt{a^2}} = \dfrac{b}{2a}$: or= 0.5 (b/a). , hence:

$\left(\dfrac{x}{1} + \dfrac{b}{2a}\right)^2 = +\dfrac{b^2}{4a^2} - \dfrac{c}{a}$: the left side is a binomial squared that is equivalent to the trinomial ("in x") of $ax^2 + bx + c$ less the constant term c] . Algebraically, and-or mathematically, the right side is also equal to this trinomial.

Adding the fractions on the right side, (LCD = $4a^2$):

$$\left(\frac{x}{1}+\frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$$

Let's isolate (x) by first taking or expressing the square root of both sides

$$\frac{x}{1}+\frac{b}{2a} = \pm\sqrt{\frac{b^2 - 4ac}{4a^2}}$$ now solving for x by transposing non (x) or non-like terms:

$$x = -\frac{b}{2a} \pm \sqrt{\frac{b^2 - 4ac}{4a^2}}$$ the root of a fraction can be expressed as:

$$x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{\sqrt{4a^2}}$$: taking the square root of the radicand factors in the denominator:

$$x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}$$ combining like fractions that have the same denominator, here (2a):

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$: **QUADRATIC FORMULA**
For finding (x) in a trinomial of the form: $ax^2 + bx + c$ $(a \neq 0)$

If (c) was equal to 0, the general form of the quadratic equation is reduced to a quadratic equation having the form of:

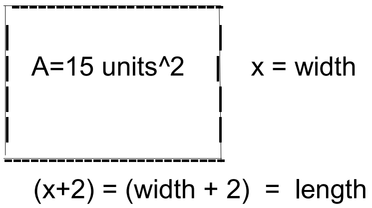
$$ax^2 + bx = 0$$

(x) can be solved for by "dividing through" (dividing each entire side of the equation) by (a), and then completing the square (of a binomial). (x) can also be solved for by using the quadratic formula, and the results are:

$x = -\frac{b}{a}$ and $x = 0$

Ex. A company manufactures a certain rectangular shaped object where its length must be 2 units longer than its width. If the total area of this rectangle is specified to be 15 square units, how long is the length and width?

Drawing a diagram of the situation: [FIG 45]



We know that: Area = (length)(width), and for our specific example, letting width = x, and length = (width + 2) = (x+2), we get:

$$\begin{aligned}(\text{length})(\text{width}) &= \text{Area}\\ (x+2)(x) &= 15\end{aligned}$$

Notice that this is the same equation that we had at the beginning of the discussion.

$(x+2)(x) = 15$ placing this into the std. quadratic eq. form, here by using distribution:
$x^2 + 2x = 15$
$x^2 + 2x - 15 = 0$ in relation to the standard quadratic equation form: $a=1$, $b=2$, $c = -15$

Substituting these values into the quadratic formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-(+2) \pm \sqrt{(+2)^2 - 4(1)(-15)}}{2(1)}$$

$$x = \frac{-2 \pm \sqrt{64}}{2} = \frac{-2 \pm 8}{2} \quad \text{hence:}$$

$$x = \frac{-2 + 8}{2} \quad \text{or} \quad x = \frac{-2 - 8}{2}$$

$$x = \frac{+6}{2} \quad \text{or} \quad x = \frac{-10}{2}$$

$$x = +3 \quad \text{or} \quad x = -5$$

As is the case for the solution of many problems with quadratic equations, the useful solution is usually the positive root only. For the last example, there is simply no physical length that can ever be negative in value, and for here, (-5) satisfies the quadratic equation mathematically, but it does not satisfy the given real or physical situation.

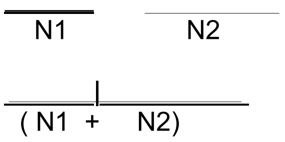
Checking:

$x^2 + 2x = 15$
$(3)^2 + 2(3) = 15$
$9 + 6 = 15$
$15 = 15$ checks

width = x = 3
length = (width + 2) = (x + 2) = 3 + 2 = 5

Area = (length)(width) = (5)(3) = 15 : checks

Ex. This example will discuss and derive the constant called the **Golden Ratio** = (GR). This value is about 0.618033988749894... and is used in many types of constructions (both man-made and natural) due to the supposed beauty it brings about. The Golden Ratio constant is an irrational value (like "pi" = 3.14159265... is) since it cannot be expressed as a ratio (ie. as a fraction) of two values since it does not in-fact have an exact value that could be easily representable mathematically since its digits never end, hence an infinite amount of digits. The derivation of the value of the GR is based on the magical-like concept when the ratio of N1 to N2, is equal to the ratio of N2 to the sum of N1 and N2. Graphically representing these values as line segment lengths:



: Let N3 = (N1 + N2)

And mathematically expressing a relationship:

$\frac{N1}{N2} = \frac{N2}{N3} = \frac{N2}{(N1 + N2)} = GR$: note here also, $(N2)^2 = (N1)(N3)$ and $N2 = \sqrt{(N1)(N3)}$

Now, we need some actual numeric values to find out just exactly what the value of this Golden Ratio constant is. Considering the value of N3 = (N1 + N2) as the whole (100% = 1) length, we will therefore assign it a value of 1, and then the problem can be analyzed using percentage or relative (to the whole or entirety) values.

$N1 + N2 = 1$ mathematically:
$N1 = 1 - N2$
$N2 = 1 - N1$

Placing these (algebraically) "derived" values back into the equivalent fractions or proportion equation, so as to use only the N2 variable and-or expressions that are "in terms" (ie., expressed with or as) of N2, it will effectively create a solvable equation which has only one unknown value.

$\frac{N1}{N2} = \frac{(1 - N2)}{N2} = \frac{N2}{1}$ solving for N2 :

$(1 - N2) = N2^2$
$N2^2 - (1 - N2) = 0$ distribute to clear grouping symbols:
$N2^2 + N2 - 1 = 0$: this has the basic form of: $1x^2 + 1x - 1 = 0$, and therefore: $x = \sqrt{1 - x}$
$x=0.618...$ Note alto that: $0.618^2 + 0.618 = 1$ and that $1.618^2 - 1.618 = 1$

We see that this is a quadratic equation with a = (+1), b = (+1), c = (-1). Using the quadratic formula:

$$N2 = \frac{-(+1) \pm \sqrt{(+1)^2 - 4(+1)(-1)}}{2(+1)}$$

$$N2 = \frac{-1 \pm \sqrt{1 + 4}}{2}$$

$$N2 = \frac{-1 \pm \sqrt{5}}{2}$$

: golden ratio mathematical formula, and the decimal equivalent of this is:
The square root of 5 is 2**.**236067978

N2 = 0**.**618033988749894... and N2 = -1**.**618033988749894... : these two values are negative reciprocals

Note that the second (root) value is actually the same value (has the same digits) as the first value plus 1, and is negative in sign, and it may not be apparent that they are also negative reciprocals of each other. The difference between these two signed values is the square root of 5 = 2.236067978**...** Due to the derivation, since the second value for N2 is negative in sign, and it will not be chosen as the ratio value we were seeking since there is no negative physical length, and hence, no ratio of real lengths that can actually be negative in value.

0**.**618033988749894... (about 61**.**8%) : **GOLDEN RATIO (GR, or "phi")**
The "reciprocal ratio" of this is:
$1/GR = 1.618033988749895... = 1 + GR$
$GR^2 = 0.381966011... = 1 - GR$

$1/GR = 1 + GR$ solving for GR by multiplying both sides by GR:

$GR^2 + GR = 1$ placing this into a quadratic equation form for the quadratic formula:

$GR^2 + GR - 1 = 0$: here, $a=1$, $b=1$, $c=(-1)$, after placing these into the quadratic formula and simplifying:

$$GR = \frac{-1 \pm \sqrt{5}}{2}$$

As an extra note: The sum of all the (integer powers of GR) = $(GR)^1 + (GR)^2 + (GR)^3 + \dots = 1/GR = 1.618\dots$

$N2 / N1 = 1.618$ and $N3 / N2 = 1.618$ or the reciprocal ratios: $N1 / N2 = 0.618$ and $N2 / N3 = 0.618$

If we let: $a=N1$, $b=N2$, and c=the total length of $(N1 + N2)$, we obtain the proportion or equivalent fractions of:

$$\frac{a}{b} = \frac{b}{c}$$ from this we mathematically obtain:

$$b^2 = ac$$ taking the square root of both sides:

$$b = \sqrt{ac}$$

: hence (b), (ie. N2), is the "**geometric mean**" (= square root of the products) of $(c) = (N1 + N2) = N3$, and $(a) = (N1)$.
Extra note ex.: the geometric mean of 10 and 20 is $\sqrt{(10)(20)} = \sqrt{200} = 14.142\dots$
$= \sqrt{10}\,\sqrt{20}$ or= $\sqrt{100}\,\sqrt{2} = 10\sqrt{2}$
$= 14.142$

The geometric mean of two vales tends to be closer in value to the least value, and this is due to the nature of squared values that increase rapidly in value.

In a simple way, you can think of the geometric mean of (10) and (20) as this example: If you had squares with areas of: 10 square units and 20 square units , their sides are about 3.162 and 4.47 units long respectively. The average of these two sides is about $3.81\underline{7}$ which corresponds to the side of a square having an area of about $14.5\underline{7}$

Another way to think of the geometric mean is this: Given an area with sides (a) and (c), and multiplying (a) and (c) together to find that area, we have: (a)(c), and to find the side length of an equivalent square shaped area which will have the same value for all its sides, we take the square root of this area value: $b = \sqrt{(a)(c)}$ b is therefor the side of a geometric equivalent area of another area having sides of (a) and (c).

A geometric mean could be considered like a geometric average, however, this average is of another form and calculated a different way because of the way the squared values grow or increase from one value to the next, and it is not an "arithmetic" or "linear" average that is located exactly between the two given values, For example, if we have the integer or "arithmetic" values of 2, 3 and 4, the average of 2 and 4 is 3. Now given the square of those values: 4, 9, and 16, the average is not 9. The linear average of 4 and 16 is actually: $(4 + 16)/2 = 20/2 = 5$. The geometric average is: $\sqrt{(4)(16)} = \sqrt{64} = 8$. This geometric series would be: 4, 8, 16, and we see that the common ratio= r=2. If we take the square root of 8, the corresponding series of these roots would be: 2, 2.828427... (not 3), and 4.

$$N2 = \sqrt{(N1)(N1 + N2)} = \sqrt{(N1)(N3)}$$ and also:

$$N2 = \sqrt{N1^2 + N1N2}$$

also:

$$c = \frac{b^2}{a} \qquad \text{and} \qquad a = \frac{b^2}{c}$$

$$N1 = \frac{N2^2}{N1 + N2} \qquad \text{and} \qquad N1 + N2 = \frac{N2^2}{N1} = N3$$

From the formal definition of GR, we can also obtain this fact:

$$\frac{N1}{N2} = \frac{N2}{(N1 + N2)}$$ simplifying to rid fractions:

$$N1(N1 + N2) = N2^2$$

$$N1^2 + N1N2 = N2^2 = N1(N1 + N2) = N1\,N3$$ The left side can be expressed as:

$$N2 = \sqrt{N1\,N3}$$

$N1N2 = N2^2 - N1^2$: this has the basic format of:
(product of values) = (difference in the squared values)
Due to the derivation of all this, it is only applicable to values that have
a golden ratio factor between them, such as $N2 / N1$ and $N3 / N2$.

The golden ratio (GR = 0.618033988749894. and is sometimes noted as "phi") is typically found or used in geometry and constructions. If you take a pentagon (5 equal sided plane figure) and draw a diagonal within, the ratio of this diagonal to a side is that of the golden ratio. A certain and simple series of integer numbers or terms that get larger is called the Fibonacci numbers and they are closely related to the golden ratio, and it is discussed further ahead in this book.

Also consider for example, by using 1.619 for (phi):

$(\mathrm{phi}) - (1/\mathrm{phi}) = 1$, and $\mathrm{phi}^2 + (1/\mathrm{phi})^2 = 3$, which is a rational value from the irrational golden ratio (phi).
$(\mathrm{phi})^3 - (1/\mathrm{phi})^3 = 4$, A pattern here is to increase the exponent by 1, alternate the sign, and the result is the sum of the previous two results. The result is a Fibonacci-like series: 1 , 3 , 4 , 7 ,11, 18 , 29 , 47 . . . : the ratio of successive terms approaches (phi).

If you were to multiply a quadratic equation by (x), the result is a cubic equation:

$x\,(ax^2 + bx + c)$, a simple example of this is: $x\,(x^1) = x^2$ and $x\,(x^2) = x^3$
$ax^3 + bx^2 + cx^1$ adding in an arbitrary constant, here (d):
$ax^3 + bx^2 + cx + d$: general or formal cubic expression or (algebraic) polynomial, here, "in x"

When there is no known formula for direct solution of an equation, the methods of numerical computation can be employed, as shown previously. Successive approximation is typically used. The appendix contains a computer program to solve an equation, particularly the square root of a value, using successive approximation. There are also many other algebraic and-or basic calculus methods to solve for the roots of equations, and these methods are often found in the mathematical topic that is called such as : Theory Of Equations, or Equation Theory.

Here is another successive approximation example.

Ex. $x^3 + x^2 = 12$ solving for (x):
$x^3 = 12 - x^2$ isolating (x) by taking the cube root of both sides:
$x = \sqrt[3]{12 - x^2}$

Expressing this in successive approximation form:

$x2 = \sqrt[3]{12 - x1^2}$:where x2 is a better approximation than x.

Continue till the difference between x2 and x1 is as small as desired. For this example, the true value of (x) is 2. Another recursion formula that can be derived from the original example is:

$x2 = \sqrt[2]{12 - x1^3}$: Note that the value of the second term cannot exceed 12 (ie. > 12) because it would cause the creation of a radicand that is negative in value and we can't solve for the square root of a negative value.

For equations other than linear (where the variable is only to the first power), it is possible, as shown previously, that there is more than one solution to an equation. Though not discussed in this book, there are some formally defined ways to determine the number of roots of an equation. By graphing an equation that is placed in "root form" (ie. algebraically set equal to 0) you can determine the number of roots of an equation, and you can also find the approximate values of those roots. You can then use each of these approximate values in a successive approximation equation. Using the last example, the "root form" of the equation for plotting (and for finding the roots or solutions) is:

$x^3 + x^2 - 12 = 0$: the result of the expression is set to 0, hence y=0, and we are then solving for a specific value(s) of x such that its corresponding value of y is equal to 0.

By also setting (y) equal to this expression, we create an equation and numerical relationship between two (or more) variables. This relationship can now be graphed as:

$y = x^3 + x^2 - 12$

The examples above were mainly quadratic or second degree equations since they are encountered more often in mathematics than equations of any higher degree. Another use for being familiar with different types of equations and their graphs is in design. The graphs or curves of many equations, or some combinations of equations, have a beauty in their own right. A part of the equation's curve can be used as the model for some form of construction, and can always be described, or duplicated to any size if given the (same) source equation and "boundary points" (ie. the two "end" or "extreme points" bounding or defining what specific part) of the curve is in question. By using the concepts of calculus, the (bounded) curve's length, and other useful facts (such as the area beneath a curve, or area between two curves) of that portion of the curve can be calculated.

Given two equations, each with their own graphing or curve shape, they can be expressed as being mathematically summed together and the result is a point per point, or instance per instance, amplitude (ie., the y values) summation of each, and it will produce a third or resulting curve or waveform. As a very simple example to this consider two waves colliding, combining or summing such that the peak (say +1 in value) of one wave is combined with the trough (say -1 in value) of another wave, and the result is: (+1) + (-1) = 0, no signal and-or energy. This is the case when the waves have the same frequency and amplitude, and are 180° (ie., opposite) out of timing or phase at the intercept or combining location. This concept is sometimes called destructive interference, as opposed to when the signals would combine to produce a larger signal (ie., amplitude, strength) and which is called constructive interference. For example: (+1) + (+1) = +2. In electronics, the electric signals (waveform, frequency, amplitude or voltage strength) might be combined in or across a resistor or transistor. The third or resulting wave is sometimes said as being composed of two or more other waves. As a simple and practical verification to all this, consider how two colors of light can be combined to produce a third color.

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ALGEBRAIC EQUATIONS WITHOUT A SPECIFIC SOLUTION

For equations that do not have a specific solution because of having unknown values that are part of the solution, simply express the relationship of the variables,

Ex. $x + y = 20$ solving for x and y:

$x = 20 - y$ and $y = 20 - x$: variables expressed "in terms" of the other variables, this is a linear or line equation, and it can even be (point per point) plotted and graphed (ie., connecting the points) as a curve. Technically, a line is considered as a unique instance of all the possible curves.

The corresponding solution depends upon the specific value of the other variable, hence if there are an infinite number of values for the value of this variable, their is an infinite number of corresponding results.

Here, we see that the values for either variable, x or y, depends on the other variable. We could say that each variable is a function of the other variable. If you pick a value for y, there is a certain or corresponding value for x such that their sum is equal to 20. If you pick another value for y, then there is a different value for x such that their sum is still equal to 20. In other words there is an infinite number of solutions rather than just a single specific solution. In short, when a variable is expressed in terms of another variable(s), there is no specific numerical solution, and only a certain expressed relationship of the variables. A formal equation of a line is one such example:

$y = cx$^1 $+ b$: the relationship of (y) to, or in terms of: (x), after solving for (x):

$x = \frac{y - b}{c}$: the relationship of (x) to, or in terms of: (y)

Ex. $y = 3x + 2$

$3x - y = -2$

$x = \frac{y - 2}{3}$

A relationship between variable(s) is typically called a "dependence" or "function". For example, it could be said for the above equations that "y is a function of x" or that "the value of y depends upon the value of x". If the equation is rearranged as shown above, we find that "x is a function of y", and a function expressed this way is sometimes called an "inverse function", such as to solve for the corresponding value of the independent variable when initially given a value of the dependent variable.

SOME ADVANCED EXAMPLES OF USING ALGEBRA

These examples use the mathematical operations previously covered in this book, and they also show some new concepts related to that particular type of example, and are good mathematical tools to have available. They will be helpful to your understanding of further topics in this book and elsewhere. A main point of these examples is to give more insight into algebra and how mathematical formulas can be made and utilized.

WHEN TWO OR MORE THINGS ARE WORKING TOGETHER

When two or people work together on a project or job, it is called cooperation, and that job can then get done or completed in a shorter amount of time and less effort and energy required from just one person. An easy to realize descriptive example is this: If it takes 1 person working for a year (365 days) to complete a project or job, then 365 people can complete the project in 1 day. In both cases, it takes 365 "man-days" (ie., a unit of measurement) or= 365 "people-days" to complete the project.

365 people-days = (number of people)(number of days) : basic formula to complete this specific job

365 people-days = 1 person x 365 days or:
365 people-days = 365 persons x 1 days If there are only 10 people on the team cooperating:

365 people-days = (10 persons) x (X days) After dividing both sides by 10 people: X days=36.5days
365 people-days = 10 persons x 36.5 days : 10 people can complete the project in 36.5 days

In terms of the typical 8 working hours per day for a person, you can multiply the number of days needed to complete the project by 8 so as to find the number of "man-hours" or "people-hours" needed to complete the project:

1 people-days = 1 people (day x 8hours/day) = 1 people (8 hours) = 8 people-hours
Also: people-hours / 8 = people-days
365 people-days = 365 people (day x 8 hours/day) = 2920 people-hours

Ex. A tank of water takes 10 minutes to be filled by pipe 1 (P1). The flow rate of water through and out of the pipe is 5 gallons (a volume unit) a minute (5gal / 1min). If pipe 2 (P2) is added to the system and its flow rate is 2 gallons per minute, how long will it take both to fill the tank?

Clearly since another pipe, P2, was added to the system, the tank will fill "faster", "quicker" or in less time than 10 minutes as it would with just P1.

Total flow per minute = Flow of P1 per minute + Flow of P2 per minute : gal/min
Total flow per minute = 5 gal/min + 2 gal/min
Total flow per minute = 7 gal/min : flow or flow rate

The total amount of water the tank holds can be found from what we know about with just information or data given for P1:

From : distance = (rate) (time) : rate = rate of change, here it's the change in distance per unit time, and it's usually called the speed or velocity of motion or moving. For a constant distance value, note that rate and time are inversely related.

volume = Total water = (rate) (time) = (amt. of flow per unit time)(time of flow) : here, rate is the change in volume per unit time
rate = R = flow rate

$$\text{Total water} = \frac{5\text{gal}}{\underset{1}{\cancel{1\text{min}}}} \times \frac{\overset{10}{\cancel{10\text{ min}}}}{1} = 50\text{ gal} = \text{Total Volume} = V_t = R\ T$$

Setting up a proportion or equivalent fractions equation:

$$\frac{50gal}{xmin} = \frac{7gal}{1min}$$ solving for x:

$$x = \frac{50gal\,(1min)}{7gal} = 7.14\underline{3} \text{ minutes}$$: mathematically: Total Volume = (Flow Rate)(Time) = R T , and here:

From this example, we will now solve it another way.

For each pipe filling the tank alone. We have 50gal to be filled by P1 that has a flow rate of 5gal/min.

From Total = (rate) (time)

$$time = \frac{Total}{rate}$$

$$time = \frac{\frac{50gal}{1}}{\frac{5gal}{1min}} = \frac{\frac{\overset{10}{\cancel{50gal}}}{1}\;\frac{(1min)}{\cancel{(5gal)}}}{\frac{\cancel{5gal}}{\cancel{1min}}\;\frac{\cancel{(1min)}}{\cancel{(5gal)}}} = 10\ (1\ min) = 10\ min$$

Doing the same for P2, we find that it will take 25 min.

Setting up a proportion from the data for P1:

100% is to 10min, as x% is to 1min. : note that full or whole = 100% = 1, and that relative values will be used here:

For P1:

$$\frac{1}{10min} = \frac{x}{1min}$$

$$x = \frac{1min}{10\ min} = \frac{1}{10} = 0.1$$: P1 will fill 1/10 = 0.1 = 10% of the tank per one minute

Now doing the same for P2, we find that it fills 1/25 = 0.04 = 4% of the tank per minute.

Full = 100% = 1 = total amt. due to P1 + total amt. due to P2
1 = (flow rate P1)(time) + (flow rate P2)(time)
1 = (R1)(T1) + (R2)(T2) When full, T = T1 = T2, the same value, therefore:
1 = (R1)(T) + (R2)(T) : an equation with only one unknown value to solve
1 = 0.1 T + 0.04 T Or perhaps expressed as:
1 = 1/10 T + 1/25 T = T (1/10 + 1/25)
1 = 0.14T solving for T: Notice the sum of rates, or= "total rate".

$$T = \frac{1}{0.14} = 7.143\ \ min$$

Checking:

Total gallons = total amt. due to P1 + total amt. due to P2
Total gallons = (5gal/min)(7.143min) + (2gal/min)(7.143min)

Total gallons = 35.715 gal + 14.286 gal
Total gallons = 50 gal : checks

Another note to make clear here is that time (T) and rate (R) are inversely related. At completion = 100% = 1.0 we have this:

$1 = (R)(T)$:R=total rate, T=total time, mathematically then:

$T = \frac{1}{R}$ and $R = \frac{1}{T}$: this is very similar to the relationship between cycles or frequency and time in the study of mechanical motion-or electronics: $T = 1/F$ and $F = 1/T$

At completion of a system with several things or parts of it working together, and with each having their own rate of doing or performing it, the time of completion of doing something is the same for all the parts of the system working together. The total rate is simply the sum of all the rates of the parts of that system as shown in the example above:

$Rt = R1 + R2 + R3 + \ldots$ and we have:

$1 = (T)(R1 + R2 + R3 + \ldots) = (T\,R1) + (T\,R2) + T\,(R3) + \ldots = (T)(Rt)$ and:

$$T = \frac{1}{R1 + R2 + R3 \ldots} = \frac{1}{Rt}$$

If each pipe took a certain amount of time to fill the tank, with having several pipes working together for the system, the total time would be inversely related, or less than just one pipe in the system.

From $Tt = \frac{1}{R1 + R2 + R3 + \ldots}$ this can be expressed as:

$$Tt = \frac{1}{\frac{1}{T1} + \frac{1}{T2} + \frac{1}{T3} + \ldots}$$

If only two pipes, machines or perhaps people are working in the system then we have:

$$Tt = \frac{1}{\frac{1}{T1} + \frac{1}{T2}} = \frac{1}{\frac{T1 + T2}{T1\,T2}} \quad \text{therefore:}$$

$Tt = \frac{T1\,T2}{T1 + T2}$: product divided by the sum

Checking using the last example:

$Tt = \frac{(10m)(25m)}{10m + 25m} = \frac{250m}{35m} = 7.14\underline{3}m$: checks with the previous results

LINEAR EXAMPLE

A linear mathematical relationship is when the rate of change between one variable with respect to another is constant. A graph of their relationship will also look like a line.

Ex. Two objects, say A and B are traveling, or are to travel, along the same path. A is 1 mile ahead of B, and is traveling at a rate of 25 milers/hour. B is traveling at 50 miles/hour. How long (in time) will it take B to catch up to and reach A, where at this point, the distance (ie., the difference) between them will be 0 miles.

First, it must be noted that B will never catch up to A if B is going slower than A. B's speed must be at least be slightly faster than A's speed if it is to ever reach A's position. Of course, if B is only slightly faster, it may take a considerable time and distance (since distance traveled is directly related to time of travel) to catch up to A.

Solution: Write the distance equations relating to each object. The general distance equation (or formula) is:

distance = speed x time **:** this is an example of an equation having the linear equation format

This formula comes from the fact that distance is directly related to the speed of travel. We also see that distance, a length measure of the change in location or position due to movement, is directly related to the time needed to move, travel or "go" that amount of distance. If either or both speed or time of the movement increases by some factor, the corresponding distance associated with that change will likewise increase by that same factor value. If either or both speed or time decreases, the corresponding distance associated with that change will decrease. Consider:

If you traveled at a speed of 7 miles/hour for 2 hours, and then traveled a speed of 5 miles/hour for 3 hours, the total distance you traveled is:

total_distance = sum of distances
total_distance = D1 + D2 : or= Dt
total_distance = (speed1)(time1) + (speed2)(time2)
total_distance = (7mi/h)(2h) + (5mi/h)(3h)
total_distance = 14mi 15mi
total_distance = 29mi :"29 miles"

total time = sum of times = time1 + time2 = 2h + 3h = 5h :"29 miles in 5 hours"

average speed = total_distance / total time = 29mi / 5h = 5.8 mi / hour :"an average speed of 5.8 miles per 1 hour"

Mathematically rearranging the basic distance equation, we have speed and time::

$$\text{speed} = \frac{\text{distance}}{\text{time}}$$: the units for speed are of the units for distance divided by the units for time. Mathematically:

$$\text{time} = \frac{\text{distance}}{\text{speed}}$$: we see that time and speed are inversely related. The faster you complete the distance, the shorter (less) the time taken.

For both equations, we will use miles for the distance units, hours as the units of time, and therefore use (miles/hour) for the speed units. These units were chosen since these were the units stated in the problem. Actually, any units could have been used as long all their quantities are properly converted to be in reference to that new unit.

Let's use variable D to represent distance, S for speed, and T for time. Hence:

D = S x T Writing the distance equations for each object:

$DA = SA\ TA$ + distance A already traveled or "lead" distance at the start, where $T=0$: distance for object A
$DB = SB\ TB$ + distance B already traveled or "lead" distance at the start, where $T=0$: distance for object B

The added term to the distance value is included as a possible adjustment, and it is a constant added to the equation.

At the start of the system in question, where $T=0$, we know that:

$DA = 25m/h\ Th + 1m$: when $T=0h$, $DA = 1m$ as mentioned in the statement of this example
$DB = 50m/h\ Th + 0m$

At a point in time, which is what we are solving for, the difference between the distances of the two objects is 0m. That is: $DA - DB = 0$ (or: $DB - DA = 0$). Likewise, it could be said or easily verified algebraically, that at a certain point in time, the distances of the objects are equal: $DA = DB$, and the distance or difference between them is 0m.

$DA - DB = 0$ substituting values:
$(25m/h\ Th + 1m) - (50m/h\ Th) = 0m$ clearing some grouping symbols:
$25m/h\ Th + 1m - 50m/h\ Th = 0m$ combining like terms and transposing the +1m:
$-25m/h\ Th = -1m$ isolating Th:

$$Th = \frac{-1m}{-25m/h} = 0.04h$$

Let's convert this fraction of an hour value to a more practical value of minutes:

$$\frac{60min}{1h} = \frac{xmin}{0.04h}$$

$x = 2.4min$

Let's check using substitution:

$DA = 25m/h\ (0.04h) + 1m = 2m$
$DB = 50m/h\ (0.04h) = 2m$: checks, and:

$DA - DB = 2m - 2m = 0m$:checks, at the point of intersection or meeting, their difference in distance, and-or distance between them, is 0m.

This 2m value also tells us the total distance that B will travel in order to catch up to A. A will only travel only 1m (check: 25m/h x 0.04h = 1m) further since it already had a lead or initial distance of 1 mile from the start of this system where Th is considered as 0h for both objects. A's "meet-up" to B distance, or A's distance to go, in this system (where T starts at 0) - A's lead distance = 2m - 1m = 1m.

Note that in the stated problem that object A's distance lead could mean several things. For example, it could mean that the objects started from the same initial physical point or place as B, or it could also mean that A's starting position is in fact 1m further away or distant along the route from that of B's starting position.
If it was stated that A also had a time lead before the start of the current system or situation of both the objects, its' equation would look something like this:

$DA = Sm/h\ (Th + \text{time lead}) + \text{distance lead}$: extra, this could be thought as: DA = lead distance, when $T=0$, and then substitute this in:
$DA = (Sm/h)\ (Th + \text{time lead}) + \text{lead distance}$

Likewise, the system such as that just mentioned can just as well be analyzed where one object lags behind in time (ie. a "time lag"). For example if Th represents the time object A has traveled, that is, Th is the system time for both object A and object B, and object B lags behind in (the systems) time by 1 hour (ie. it effectively sat idle and not changing in distance after the start (where T=0) of the system, or perhaps was late to arrive at the start of the system). The equations for this situation are:

DA = speed x time = speed x Th : regular equation, no lags
DB = speed x time = speed x (Th - 1) : if there was a lag. A time lag effectively causes a distance lag

Note, that as long as speed (or velocity)) remains constant, then these distance equations will all have the same format as that of the basic linear (line) equation or formula of:

y = mx + b : for linear or line studies, the formal variable (a) that is the coefficient of (x) is often replaced and identified as: (m) as shown. Note that (a) and-or (m) is a constant value, and it and its value is associated with the specific linear mathematical relationship between (y) and (x).

Some forms of this equation will also note (b) as (c), and this is simply a constant (often used as an adjustment needed) added in. Using letter (c) helps indicate it as a constant.

In this equation, (y) is the dependent variable whose value is mathematically related to and depends upon the value of the independent variable (x). In perhaps simpler words, (y) is said to be a result, output, or "a function of" the given expression and here it's (mx + b). Since this expression is equal to (y), the expression is said to be the (processing, working, source of) function (in a noun sense). Since (m) and (b) remain constant for linear equations, (y) is realistically, a function of (x), and would then be often mathematically expressed or indicated with the function (f) syntax (grammar) of: y = f(x), and this notation is read or spoken as: "y is (or equals) a function of x". In this example, y = f(x) = mx + b. The group or set of independent values (here, all the possible values of x) of a function is often called the "domain" of the function, and the corresponding group of dependent values (here all the y values) to those domain values is often called the "range" of the function. If this line equation is graphed or plotted using the common rectangular (planar or two-dimensional) coordinate system, variable (m) (ie,.the speed for this example) is the slope (or "steepness") of the line, or more technically, the rate (a measure, or ratio, of change) of the corresponding change of (y) with respect to (x). Variable (m), or whatever happens to be the coefficient of (x), is always a constant value for linear or line equations. Slopes are also discussed ahead in the TRIGONOMETRY section. (b) is a constant that is the initial or "lead" value of (y) when (x) = 0. In relation to the example, D corresponds to (y), S corresponds to (m), T corresponds to (x), and the (distance) lead corresponds to (b):

y = mx + b : the basic form of a linear equation
D = ST + distance lead : the example's specific and-or linear equation

Linear equations, when plotted on a graph where one axis, usually the x or horizontal axis, of reference or dimension represents increasing values of the independent variable, and the other axis, usually the y or vertical axis, of reference or dimension, represents the values of dependent variable, always produce a straight line. Lines indicate that the rate (expressed as a ratio) of change (ie., the slope = m value) of the dependent variable (such as y) with respect to the independent variable (such as x) is a constant value. This type of mathematical relationship is then called a linear or a (mathematically) constant relationship. If b=0, this can be clearly seen:

From: y = mx : here, b=0. If m=1, the equation reduces to: y = 1x = x , clearly a direct relationship. From the equation we algebraically get:

$m = \frac{y}{x}$: **m = slope** of the line, it's a constant value for a certain equation and line and is not a variable, and it is the numerical coefficient of x. This value of (m) is the rate of change of the corresponding values of (y) in relation, or with respect to changes in (x). Basically, it is a measure of how the value of (y) changes when (x) changes by 1.

For example, if $m=3$, when (x) changes by 1, the change of, or in (y) with respect to (x) is (the factor of) 3.

From the equation: $y = 3x$, we have: $3 = m = \frac{y}{x}$: when variables are said to have a linear mathematical relationship, the rate of corresponding changes in their values is constant.

The above quotient, ratio, or "magnification factor" indicates that the corresponding value of y will always be three times greater than the value of x. This value also indicates that for each (1) unit change in or of the value of (x), then the corresponding value of (y) will change in value by 3 times that change. If (x) changes by 1, (y) will change by: (3)(change in x) = (3)(1) = 3. If the value of (x) changes by 2, the corresponding value of (y) will change by: 3(2) = 6. More specifically for any change in (x), the corresponding change in (y) will be 3 times that change. This is the mathematical relationship between (y) and (x), and it is a linear or constant mathematical relationship as indicated by the constant factor value (m), and here for this example or equation, it is 3.

change in y = (3) change in x since $m=3$, mathematically then:

$m = \frac{\text{change in y}}{\text{change in x}}$: regardless of what b, the arbitrary constant, is. Let's give some more verification:

Consider one instance of the line eq.: $y1 = mx1 + b$: eq. 1
Consider another instance of the same line eq.: $y2 = mx2 + b$: eq. 2

This second equation can be expressed (using substitution) in relation (or in "terms of") to the first equation as:

First, below, y2 will be expressed in terms of y1, and x2 will be expressed in terms of x1. Change is essentially, and calculated as, the difference between y2 and y1. It could also be said as how y1 was changed, or the change applied to y1, so as to have the new value of y2:

From: $(y2 - y1) = (\text{change in y1})$: or= (change in y values) = (change in y) , solving for y2:
$y2 = y1 + (\text{change in y1})$: likewise, $(x2 - x1) = (\text{change in x1})$ or= (change in x)

$$\begin{aligned} y2 &= m\,x2 + b \quad \text{, and using substitution for the values:} \\ y1 + (\text{change in y1}) &= m\,(x1 + \text{change in x1}) + b \\ y1 + m\,(\text{change in x1}) &= m\,x1 + m\,(\text{change in x1}) + b \end{aligned}$$

$$mx1 + b + m\,(\text{change in x1}) = m\,x1 + m\,(\text{change in x1}) + b$$

Transposing +(m)(change in x1), we find this must be true (especially, since we used: m(change in x1) for (change in y1)) because we arrive at the original equation (eq 1) that we know is true:

$y1 = m\,x1 + b$: b is also known as the "y-axis intercept or crossing" where $x=0$, that is, the coordinates of this point are $(x,y) = (0,b)$
When $b=0$, we can still see that $y1/x1 = m$

Also from the above equation, and substituting the values of: (mx1+b) for y1, and (change in y1) for m(change in x1) :

$$\begin{aligned} y1 + m\,(\text{change in x1}) &= m\,x1 + m\,(\text{change in x1}) + b \\ mx1 + b + \text{change in y1} &= m\,x1 + m\,(\text{change in x1}) + b \end{aligned}$$ after Transposing. mx1, and b:

$(\text{change in y1}) = m\,(\text{change in x1})$

Hence:

$$m = \frac{\text{change in } y1}{\text{change in } x1} = \frac{y2 - y1}{x2 - x1}$$

: showing that (m) can be calculated from (any) two points on a line, or two pairs of corresponding values of variables (x) and (y). This mathematically verifies that variable (m) is also equal to the rate of change in the values of the variables, here (y) and (x). If the change in (x) is 1: m = (change in y1) / 1 = (change in y1). In other words, when (x) changes by 1, (y) will change by (m), and this is a simple way to think of what (m) does.

Since this will also be the case for any two points on the line, the formula for (m) can be generalized and noted as simply:

$$m = \frac{\text{change in } y}{\text{change in } x}$$

Also to consider:

$$y1 = m\,x1 \text{ , and: } \quad y2 = m\,x2 \quad = \quad y1 + \text{change in } y1 \quad = \quad m\,(x1 + \text{change in } x1)$$

$$m = \frac{y1 + \text{change in } y1}{x1 + \text{change in } x1} = \frac{y2}{x2} = \frac{y1}{x1}$$

The coordinates ("x,y pairs") of two or more points on a line are said to be proportional to each other,and whichis basically an equivalent fraction concept. Actually, for this to be true, (b) must equal 0. For example considering two points on a line:

$$\text{From: } \quad \frac{y1}{y2} = \frac{x1}{x2}$$

$$r = \frac{y1}{x1} = \frac{y2}{x2}$$

let's verify this, also note that here, r = m (the slope of the line):

$$\frac{y1}{x1} = \frac{y2}{x2} = \frac{y1 + (\text{change in } y1)}{x1 + (\text{change in } x1)} = \frac{y1 + m(\text{change in } x1)}{x1 + (\text{change in } x1)}$$

Observing the first and last fractions:

$$\frac{y1}{x1} = \frac{y1 + m(\text{change in } x1)}{x1 + \text{change in } x1}$$

We note that this is very odd since the right hand (equivalent) fraction is created by adding to the numerator and denominator, and not the typical way where both the numerator and denominator are multiplied (or divided) by the same value. Adding is allowed if it is done right, as mentioned previously, as a unique way for creating an equivalent fraction, and the value to be added cannot be the same in the numerator and denominator, nor just any values. The right hand side has the form of:

$$r = \frac{y1 + a}{x1 + b}$$

: where (a) and (b) are any proper pair of values to make an an equivalent fraction. Setting n(y1) = (a), and n(x1) = (b):

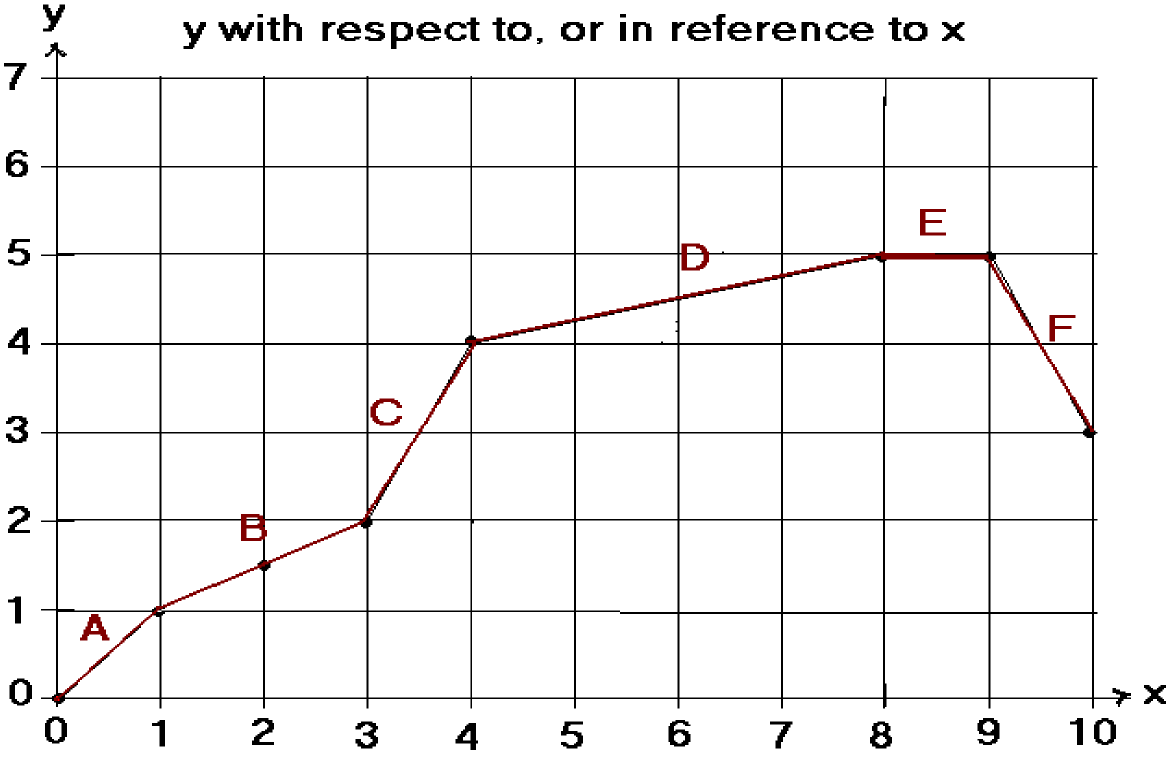
$$r = \frac{y1 + n(y1)}{x1 + n(x1)} \qquad \text{extra: } = \frac{y1\,(1 + n)}{x1\,(1 + n)} = \frac{y1}{x1}$$

: shows they are equivalent fractions. Here, the value that was essentially added to the numerator and denominator had the same multiple (here n) of itself.

Letting n(y1) = change in y1, and n(x1) = change in x1 :

$$r = \frac{y1 + \text{change in } y1}{x1 + \text{change in } x1} = \frac{y2}{x2}$$

Here is graph for a basic example of seeing, calculating and understanding slope values on a graph: [FIG 46]



This graph would be typical of a object that occasionally changed its speed while moving. The slope of the indicated line segments (indicated as A through F) is as follows:

The main changes in the slope (and its mathematical value) of the line are indicated where the curve or line segment on the graph changes to be more or less vertical which indicates an increased or decreased slope value. The slope value visually and mathematically expresses the relationship between the changes in (y) and (x). The slope value is calculated by using the (ratio of) corresponding changes in the (y) and (x) values. When there is no corresponding change in the relationship of the (y) values when the (x) or independent variable changes, the line segment or curve will be horizontal. The slope of the line is 0 or "flat" meaning the object is not moving and not increasing its distance.

When there is no noticeable change in the current slope or "vertical tilt" of the curve or line segment, then the slope is a constant value, and the mathematical relationship is then said to be "linear" (a linear mathematical relationship) which means a constant mathematical relationship. Between any two points on any line segment, such as seen in the above graph, the slope is constant. At any point where the slope changes, the (rate of changes) or "speed" has changed. If the speed is greater, the slope on the graph is greater, and the distance traveled will be greater per unit change in x or time.

A: From x=0 to x=1, the slope is: (change in y) / (change in x) = (y2 - y1) / (x2 - x1) = (1-0) / ((1-0) = 1/1 = 1
This value can be thought of as: As (x) changes by 1, (y) changes by 1.

B: The next part of the curve, or relationship between (y) and (x), is not as steep or vertical.
From x=1 to x=3, the slope is: (change in y) / (change in x) = (y2 - y1) / (x2 - x1) = (2-1) / ((3-1) = 1/2 = 0.5
This value can be thought of as: As (x) changes by 1, (y) changes by 0.5

C: The next part of the curve, or relationship between (y) and (x), is steeper and more vertical.
From x=3 to x=4, the slope is: (change in y) / (change in x) = (y2 - y1) / (x2 - x1) = (4-2) / ((4-3) = 2/1 = 2
This value can be thought of as: As (x) changes by 1, (y) changes by 2.

D: The next part of the curve, or relationship between (y) and (x), is not that steep, and is somewhat horizontal.
From x=4 to x=8, the slope is: (change in y) / (change in x) = (y2 - y1) / (x2 - x1) = (5-4) / ((8-4) = 1/4 = 0.25
This value can be thought of as: As (x) changes by 1, (y) changes by 0.25 , or "not that much"

E: The next part of the curve, or relationship between (y) and (x), is that the curve or line segment is horizontal.
From x=8 to x=9, the slope is: (change in y) / (change in x) = (y2 - y1) / (x2 - x1) = (5-5) / ((9-8) = 0/1 = 0
This value can be thought of as: As (x) changes, (y) is not changing and is a constant value of 0.

F: The next part of the curve, or relationship between (y) and (x), is that the curve or line segment tilts in a vertically downward direction as (x) increases. From x=9 to x=10, the slope is: (change in y) / (change in x) = (y2 - y1) / (x2 - x1) = (3-5) / ((10-9) = -2/1 = -2. This value can be thought of as: As (x) changes by 1, (y) changes by -2, hence (y) decreases by (-2) for each change in (x).

GRAPHING AN EQUATION

Below is an example of graphing an equation that is for a line.

Given an algebraic equation, it can be represented geometrically, and most commonly as a graph on a (flat) plane. If the equation is plotted and produces a line, the equation is a linear equation. For example:

$$x + y = 10$$

This is a two variable linear equation. There isn't a single specific solution to it since an infinite number of solutions is possible. Each corresponding pair of variables is unique to one solution only when the variables are considered ordered, that is, as an "ordered pair" (ie., coordinates of a point) they will only occur in that unique (corresponding) order or way only once. This pair of variables is therefore often called an "ordered pair". For example, the following pairs of variables will solve the equation shown above:

x	y
11	-1
10	0
9	1
8.5	1.5
5	5
1	9
0	10
-10	20

Mathematically: $y = 10 - x$ and $x = 10 - y$

$y = -x + 10$ $x = -y + 10$

Notice that the variables (y) and (x) are inversely related. That is, as one variable increases, the other will decrease, and vice-versa, to satisfy (make it true) the equation. Clearly with the data we have, as variable x decreases, variable y increases. It could also be said that as variable x increases, variable y decreases.

For plotting purposes, it is best to arrange the equation in terms of just one variable since the graph of the equation plotted on a rectangular coordinates, location and-or measuring system will show or represent the relationship of two variables, and in particular, one variable "against" or "versus" (ie., with reference or respect to) the other. Usually (y), is expressed in terms of (x), and the actual variable names, symbols or identifiers you can use depends on the information expressed in the given problem and-or equation.

From: $y + x = 10$ we algebraically have:

$$y = -x + 10$$

The connection between algebra and geometry is that every set or pair of variables that satisfies this equation will produce a point which is a geometric concept. Likewise, any point on the geometric graph of the equation will satisfy (be a solution to) the algebraic equation.

In theory, any two points can be used to define a line on a graph. In practice, the two points used are separated by a reasonable distance (not so close together, and the farther, the better results displaying, plotting or verifying all other points on the line), and a third point, also on the line, will be used as a check to ensure that the two "end" or "defining" points (of that line segment,or line) are not two completely random points on a plane, but are also defined by the equation and plotted (placed, indicated) at the right location on the graph.

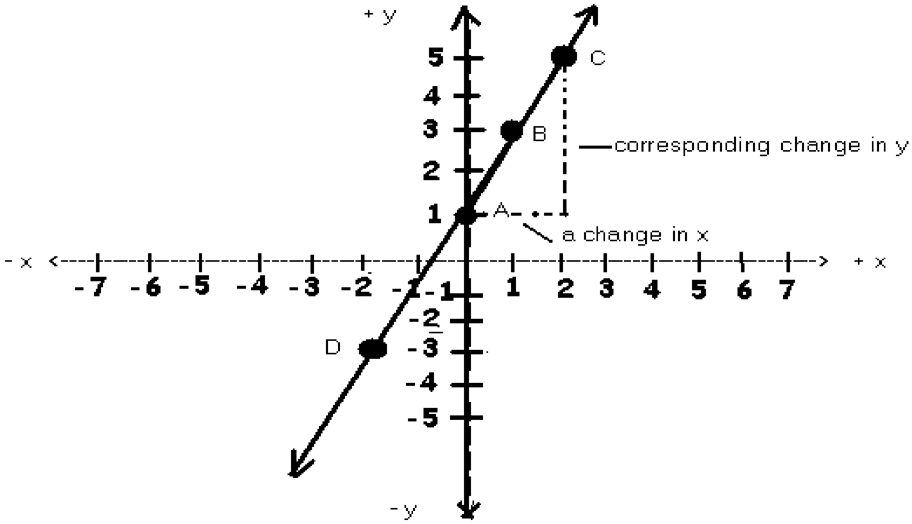
Any two points can be used to define and draw a line for plotting on a graph. In practice, the two points used are separated by a reasonable distance, and a third point will be used as a check to ensure that the other two "end" or "defining" points (of that line and-or line segment) are correct.

Ex. Graph the line: $y = 2x + 1$

Substitute three values for (x) and solve for the corresponding values of (y). Each corresponding value of (x) and (y) will be the coordinates ((coordinating, coordinated, or corresponding) "address" or location) of a point on a line within the rectangular co-ordinate system that is usually used for identifying points and places (locations) on a plane. A rectangular co-ordinate system, as described previously, is simply two number lines that cross perpendicularly at both their reference or "0" positions. Each point in this system is located at the horizontal and vertical intersection of its defining co-ordinates.

Point	x	y
A	0	1
B	1	3
C	2	5
D	- 2	- 3

Once you have the co-ordinates of some sample points of and on the line, plot (display, draw or indicate) them, and then use a straight surface, such as a ruler's edge, as a helpful guide to draw and connect these points on the line. [FIG 47]



The point where the line crosses the y-axis is called the (y) or vertical (axis) intercept. Here, the corresponding (x) co-ordinate of that point is always 0. To find this specific (y) co-ordinate, first set (x) to a value of 0, and then solve for (y):

$y = 2x + 1$
$y = 2(0) + 1$
$y = 1$: this value is always equal to the (b) variable in the basic expression of a linear equation: $y = mx + b$

Algebraically, and graphically, the coordinates of this point are: P(0,b) : "y (axis) intercept point" of a linear equation, or "dependent axis intercept". This could be thought of as a "lead value" for (y) when (x) has no value yet and-or is equal to 0.

The point where the line crosses the (x) axis is called the (x) or horizontal intercept, or "root" of the equation. Here, the (y) co-ordinate of that point is always 0. To find the corresponding (x) co-ordinate, set y=0 and then solve for x:

$y = 2x + 1$ Let's rid non x termed variables to one side, or place x termed variables on one side to isolate x.
$0 = 2x + 1$ T. (+1), transpose +1 by adding the negative of that term to each side of the equation :
$-1 = 2x$ isolating x by dividing by its multiplying factor or coefficient, and then switching sides:

$$x = \frac{-1}{2} = -0.5$$

In general, the algebraic formula for this x coordinate can be created using the basic (slope-intercept form, of a) linear or line equation:

$y = mx + b$ setting y=0 and solving for x, we find:

$x = \frac{-b}{m}$ or $= -\frac{b}{m}$: x coordinate of the x-axis intercept of a line

Algebraically, the coordinates of this point is: P(-b/m , 0) : "x intercept point"

Checking the slope (m) between points A and C, as indicated on the graph:

$\text{slope} = m = \frac{\text{change in Y}}{\text{change in X}} = \frac{YC - YA}{XC - XA} = \frac{3 - 1}{1 - 0} = \frac{2}{1} = 2$: checks

What does this value of 2 mean? In short, and as discussed previously, the answer is that when (x) changes by 1, (y) changes by 2. For each and every one unit change in (x), (y) will change by 2 units. It's basically an equivalent fraction concept where the ratio (here m, the slope value) of the corresponding changes in the values of the variable is a constant value of 2. The rate of change in a linear equation is a constant value. If the rate of change between the two variables was not a constant value, then there would be a variable involved in the calculation for that (non-constant, or varying) rate of change of one variable with respect to, or in reference to, another variable. In a linear, first order or degree equation, the rate of change of Y with respect to X is a constant value of (m) which is the coefficient of the (X) or independent variable. In the **Extras And Late Entries** in this book, there is **a graph to help under linear equations and slopes**.

$m = 2 = \frac{\text{change in Y}}{\text{change in X}} = \frac{n\,(\text{change in Y})}{n\,(\text{change in X})}$: if the change in X was increased by a factor of (n), the corresponding change in Y will be increased by that same factor.

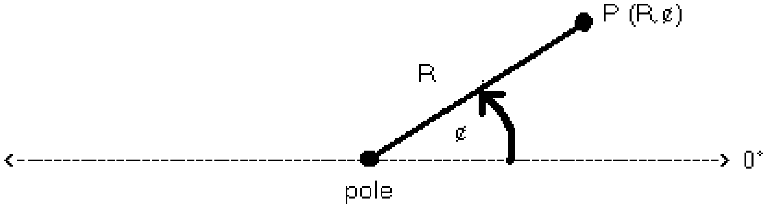
From this, we can find out how much Y will change when X changes by some value.

(change in Y) = m (change in X)

The higher (m), the slope, is in absolute value, the more vertical the line. The lower (m) is in absolute value, the more horizontal the line. When slopes are negative in value, this indicates that the line or line segment will slope downward (ie., y coordinates are decreasing) as the x coordinates increase in value. A negative slope indicates an inverse relationship between the (x) and (y) values at that location on the graph or curve of an equation.

The rectangular co-ordinate system is simply an extension of the geometric concept of a plane. In simple terms, a plane is an infinitely flat surface in all directions. Hence a plane can be thought of as having only a horizontal and vertical dimension, and lacking any third or "depth" dimension used to define space or volumes. All points on a plane can therefore be identified by their corresponding two location or dimension values from any point chosen as a reference (of measurements) or "origin" location, position or point, usually p(0,0), on that plane. Each location, or coordinate, is actually the measured or scaled (dimensional, and perpendicular) distance in units from the origin. Each coordinate is therefore also equal to the perpendicular distance from the other coordinate's axis (reference (of measurement) line). The point is located at the intersection of these (imaginary) perpendicular "distance or coordinate lines". The coordinates of a point is essentially the location or address of that point with respect or in reference to the origin of that (addressing and measuring system) system.

The graph (geometric representation) of an inequality is a segment or portion of a plane. As a simple example, the inequality ($y>5$), "y if greater than 5", represents the plane portion or segment that is composed of all the points (x, y) where the corresponding (y) coordinate of any point within that plane segment is greater than 5. This could be expressed as: $p(x, y>5)$. Note, the number of points within a segment or portion of a plane is just as infinite as the plane itself. There is another standard, though perhaps less used, co-ordinate or location addressing system for points and it is called the polar co-ordinate system. Here, a point's (P) location is defined as its straight-line distance, known as the radius vector (R), from the origin (or "pole") to that point, and to distinguish it from all other possible points at this same distance (like the points on the circumference of a circle, all being the same distance since they all have the same (radius) distance from the center of a circle), its location includes the radius vector's angle from a reference line (called the polar (system) axis). The figure below shows polar co-ordinate system within a two dimensional rectangular system. It is very possible to have three-dimensional co-ordinate system for either or both methods. [FIG 48]



Briefly, you can convert between polar and rectangular coordinates of a point when you consider the radius-vector line as the hypotenuse side or line (segment) of a right triangle and whose two other sides or lines are equal to the rectangular coordinates of that point. These two other sides can be used to determine the angle of the radius-vector line, and vice-versa. A minimal amount of the knowledge of trigonometry, as discussed ahead in the TRIGONOMETRY section of this book, will be of assistance for these calculations. Polar coordinates will not be discussed further in this book, but further ahead, this book does include a basic discussion about vectors.

Given two lines, there are several methods to solve for the point of intersection of those lines. The lines cannot be parallel lines since parallel lines are defined as never intersecting. Here is a simple algebraic method.

$y1 = m1x1 + b1$: a line equation general format
$y2 = m2x2 + b2$: another line equation general format.

Note, if $m1=m2$, and the value of (b) is different for each unique line, the lines are parallel. Parallel lines have the same slope both graphically, and numerically, which is the same rate of change of the dependent variable (y) with respect or in reference to the independent variable (x).

At the **point of intersection of two lines**, the difference between both the (y), and-or both the (x) co-ordinates of both points located at the point of intersection is 0 since they will have the same value for both points. This concept can also be used for many types of equations and curves.

$y1 - y2 = 0$ substituting their equivalent expressions:
$m1x1 + b1 - (+ m2x2 + b2) = 0$ distributing:
$m1x1 + b1 - m2x2 - b2 = 0$ at the intersection, $x1 = x2$, so let's express this value as just (x):
$m1x + b1 - m2x - b2 = 0$ factoring (x) out of the left side:
$x (m1 - m2) + b1 - b2 = 0$ transposing b terms:
$x (m1 - m2) = b2 - b1$ isolating x:

$$x = \frac{b_2 - b_1}{m_1 - m_2}$$: **X CO-ORDINATE OF POINT OF INTERSECTION OF TWO LINES**

The corresponding (y) value of the point of intersection can be found by direct substitution of this value of (x) into one of the original linear equations.

Ex . Eq 1: $y_1 = 2x_1 + 1$ Eq 2: $y_2 = 3x_2 + 2$

$m_1 = +2$ $m_2 = +3$

$b_1 = +1$ $b_2 = +2$

At the point of intersection: $x = \frac{b_2 - b_1}{m_1 - m_2} = \frac{2 - 1}{2 - 3} = \frac{1}{-1} = -1$

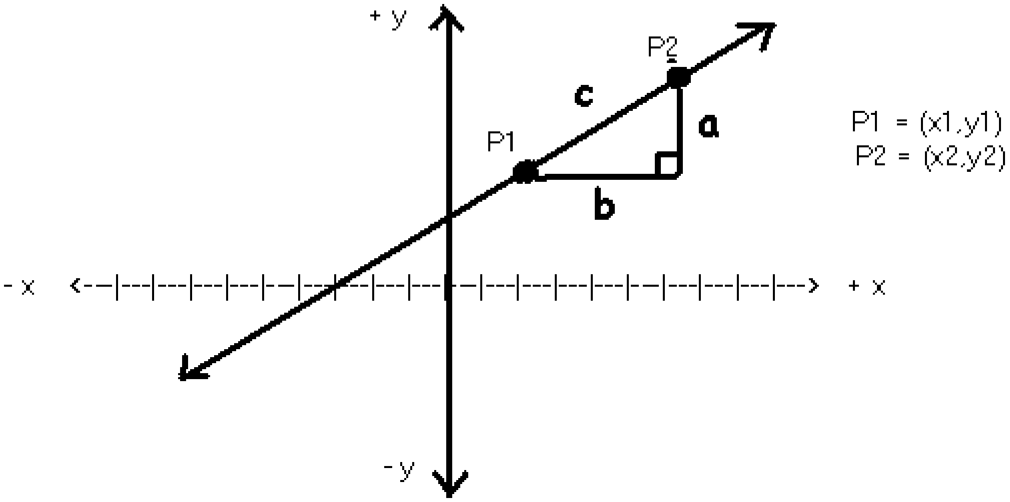
By substituting this value of (x) into either of the original equations, we will have the corresponding (y) value of the point of intersection (Pi) of those two equations or lines:

$$\begin{aligned} y_1 &= 2(-1) + 1 &= \quad y_2 &= 3(-1) + 2 \\ y_1 &= -2 + 1 &= \quad y_2 &= -3 + 2 \\ y_1 &= -1 &= \quad y_2 &= -1 \end{aligned}$$

Pi = (-1, -1) : point of intersection for the given and specific system of two equations or lines

Writing An Equation Of A Line:

For plotting points on a line, or for predicting data (information, values or points) that is line-like (having a linear-like relationship), you must know or create (write, express) the linear equation for the mathematical relationship that corresponds to these points. The minimum number of points needed to plot or draw any specific line is two, and it is also the minimum number of points needed to create the corresponding linear equation. If it is stated or understood that a line crosses at the origin (0,0), then this point can also be used as one of those two points necessary to define a line. Observe the drawing below: [FIG 49]



First, lets find the slope (m) of the line with the given co-ordinates of two points on it:

$$\text{slope} = m = \frac{\text{vertical change}}{\text{horizontal change}} = \frac{(y2 - y1)}{(x2 - x1)} = \frac{\text{side (a) of a right triangle}}{\text{side (b) of a right triangle}} = \frac{\text{"rise"}}{\text{"run"}}$$

Now, find the Y-axis intercept value. This is the corresponding y value where x=0. It is often assigned the variable identifier of: (b) since it is the next letter following the letter (a) in the alphabet, and of which is the formally expressed coefficient of the variable in a general linear equation of: $y = ax^1 + bx^0 = ax + b$

$y = mx + b$
$y = m(0) + b$
$y = b$

Now that (m), the slope of the line, has been solved for, take the coordinates of a known point on the line and substitute them into the basic form of a linear equation and solve for the actual value of (b) for the specific line:

$y = mx + b$ solving for b:
$b = y - mx$

Finally, write the equation of the line, substituting the found values for (m) and (b):

$y = mx + b$

Ex. $y = 2x + 3$: where m=2, and b=3

Given a set of points or data that appear (perhaps after plotting some points) to be linear or line-like, you can do the above process to find the equation of the corresponding line. Perhaps this equation can be used for predictions or estimates of data values, etc., however, given real-world data points, it is possible that several lines (although probably close together) could be created. There are advance methods of writing line equations that takes this into account and to create a single, "average" or "best fitting" line and equation to represent the given data. The appendix contains a simple or practical approach to finding a "reasonable equation" of a line given a set of points.

The following is a method of writing an equation of a line, however, it is one of the few discussions in this book that has a forward reference, particularly of the Pythagorean Theorem that is found in the TRIGONOMETRY section.

Notice in the drawing that if you let (c) represent the length of the line segment between the two points, that this is the (hypotenuse, longest side) side of a right triangle. The other two sides (a and b) of this triangle are the distances (here, calculated by difference) between the corresponding coordinate values of the points that define that line segment. By using the Pythagorean Theorem, we can write a general formula for the distance between any two points on a line, or anywhere within a coordinate system where both axis (reference of measurement lines) use the same units of measurement and scaled (indicated, successive increments of the units) distances.

$c^2 = a^2 + b^2$: **Pythagorean Theorem** for right triangles where (c) is the longest side

$c^2 = (\text{vertical change})^2 + (\text{horizontal change})^2$ taking the square-root of both sides :

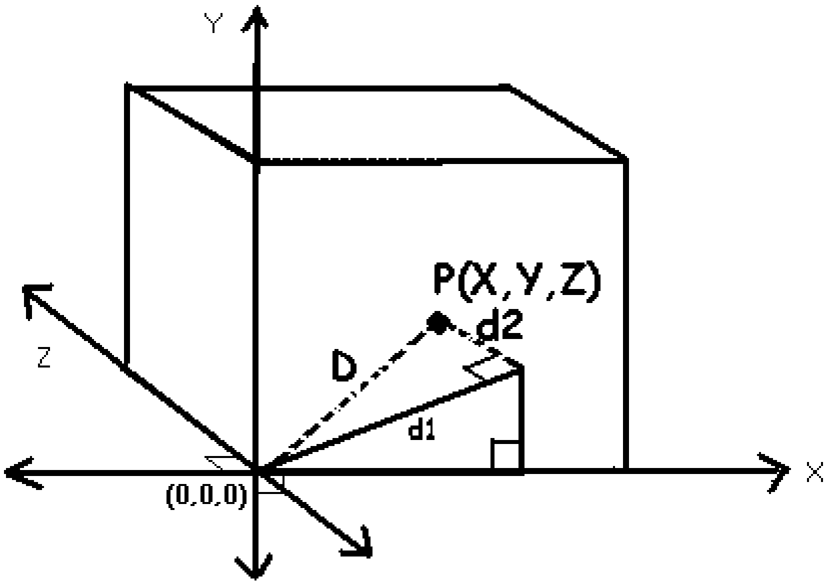
$c = \sqrt{(\text{vertical change})^2 + (\text{horizontal change})^2}$ letting c = distance = d , this is calculated as:

$d = \sqrt{(y2 - y1)^2 + (x2 - x1)^2}$: **DISTANCE FORMULA BETWEEN TWO POINTS (2-D)**
(on a plane or two-dimensional "flat" surface).

For this formula, it makes no difference which point (with its coordinates) is chosen as "point 1" and which point is chosen as "point 2" since the square of the difference of the points corresponding coordinates, whether this difference (or length if you will) is positive or negative in sign (both differences would have the same absolute value), is always a positive value. In particular, a negative times a negative is still a positive value, and here, a positive value as required for any length to be physically real.

When a plane (2 dimensions) is moved perpendicularly, along another axis that is perpendicular to its' own two (reference or start of measurement) axis, a three-dimensional (3-D) space, or volume is defined. For example, if you take a part (ie. bounded and defined region such as a square) of a common two-dimensional (2-D), X and Y plane and move it perpendicularly above or below itself along a different axis (typically called the Z axis) or dimension, a three-dimensional space is defined. A common geometric example of three-dimensional space is a cube, and a cubic unit is naturally and commonly used as the reference unit of measurement for any volume or space. If you take a square (a region of a plane) which has 1 unit for its two (perpendicular) side dimensions and move it 1 unit perpendicularly to both of those side dimensions, a cube is created and the cubic unit (of measurement for space or volumes) is defined. Another common example of a three dimensional space is when a circle, or disk is moved a height, and this creates a cylinder or solid rod shape, "space" or volume.

The formula for the distance (D) between any two points within a three-dimensional space is very similar to that used for a two-dimensional system. The formula for the two-dimensional system is also used in the analysis for creating the formula to use for a three-dimensional (measuring and location) system. As shown in the drawing below of a three-dimensional system, (d1) is the distance between the two points when considering only the (X) and (Y), two-dimensional, coordinates of the point. This value of (d1) is also a side of a right-triangle within the three-dimension system, and the other side of this right-triangle is (d2) and which is perpendicular (ie. at a right-angle) to (d1) and extends into the (Z) plane or third-dimension, and its length is the difference between the (Z) coordinates of the two points in question that we are to find the distance or length between them: [FIG 50]



Dotted or broken lines in drawings typically indicate a construction that cannot be seen normally in a 2-D diagram or graph that tries to represent a 3-D diagram or graph. In the above drawing, the broken or "dotted" lines are for parts or constructions that extend into the Z plane. This is as if the broken or dotted lines extend behind, through or into this page that you are viewing.

change in Z = d2 = (Z2 - Z1) : in particular, considering the third dimension for d2

d1^2 = (X2 - X1)^2 + (Y2 - Y1)^2 : considering just the first and second dimension for d1

Substituting this value for $d1^2$ into the Pythagorean Theorem to solve for D:

$$D^2 = (d1)^2 + (d2)^2$$

$$D^2 = (X2 - X1)^2 + (Y2 - Y1)^2 + (Z2 - Z1)^2$$ taking the square root of both sides:

$$D = \sqrt{(X2 - X1)^2 + (Y2 - Y1)^2 + (Z2 - Z1)^2}$$:**DISTANCE BETWEEN TWO POINTS (3-D)**
(in a three-dimensional system)

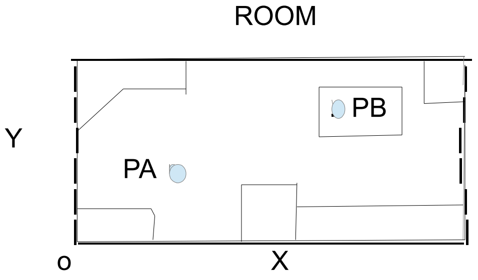
OR simply:

$$D = \sqrt{(\text{change in } X)^2 + (\text{change in } Y)^2 + (\text{change in } Z)^2}$$

If one of the "end points (at the farthest or extreme ends of the line segment, hence it could also be an initial, defining or starting point)" of this line segment was located at the origin (0,0,0), then P1(X1,Y1,Z1) = P1(0,0,0), and the equation reduces to simply:

$$D = \sqrt{X2^2 + Y2^2 + Z2^2}$$

Ex. Below is a simple "blueprint" (a drawing, often with a blue colored background, and which is a demagnified and proportional (often spoken as: "drawn to scale") image or drawing of the construction it represents. Here, the image is of a room as viewed from above it. In the room, you are to find the distance (D) between point A and point B. These points are located at different heights above the floor of the room. The length of the room can be considered as the X dimension, the width as the Y dimension, and the (vertical) height or depth as the Z dimension. The lower left corner on the floor of the room is arbitrarily chosen as the origin (o) or point of reference for this three-dimensional analysis and co-ordinate system. [FIG 51]



Let PA = P(X1 , Y1 , Z1)
Let PB = P(X2 , Y2 , Z2)

SCALE: 1/8 inch (on blueprint) = 1 foot (real)

$$D = \sqrt{(X2 - X1)^2 + (Y2 - Y1)^2 + (Z2 - Z1)^2}$$

Ex. Given a cube which is a 3-dimensional, volume shape with square sides (s) all of equal length, what is the length of the diagonal within it? We will be using the full side lengths where a side starts at 0 length, hence, for example: Letting: X = (x2 - X1) = (x2 - 0) = X2 = s . If each dimension (length, width, and height) side has a length of 1: side X = side Y = side Z = s

$$D = \sqrt{s^2 + s^2 + s^2} = \sqrt{1^2 + 1^2 + 1^2} = \sqrt{1 + 1 + 1} = \sqrt{3} = \mathbf{1.73205808}$$:**Cube Diagonal Constant**

$$\mathbf{D} = \sqrt{3s^2} = \sqrt{3}\sqrt{s^2} = \mathbf{1.7320808\ s}$$ **: Diagonal formula of a cube with side (s)**

If the point of reference (ie., p(0,0,0) is considered as: p(X1, Y1, Z1) , the equation reduces to : $D = \sqrt{X2 + Y2 + Z2}$

Given the side of a cube, the diagonal length of that cube will be greater by the factor of 1.73205808

Ex. If s = 5 cm, D = 1.7320808 (5 cm) **=** 8.660254038 cm

If the side length (s) increases by a factor of (n), the diagonal length of that cube will also increase by that same factor of n. For a square, the diagonal constant is 1.414213562 which is the square root of 2. The sum of the square root of 2 and the square root of 3 is 3.14626437... and it is very close to the value of (pi) = 3.141592654... , and with a difference of just 0.004671716...

$$D = \sqrt{(ns)^2 + (ns)^2 + (ns)^2} = \sqrt{3(ns)^2} = \sqrt{3}\ \sqrt{n^2}\ \sqrt{s^2} = \sqrt{3}\ n\ s = 1.73205808\ ns$$

A square also has a diagonal constant. $D = \sqrt{s^2 + s^2} = \sqrt{2 s^2} = \sqrt{2}\ s =$ **1.414213562 s :Diagonal in a square**

Besides the two endpoints of a line segment, or anything analyzed like a line, the point exactly between them is called the **midpoint of a line** (segment) and is often of importance for various constructions. Since midpoint means the "half-way" (of full length), half-length (length/2 = distance/2), "middle", "midway", or center point, simply divide the given length or distance by 2 so as to find this half-way length. half-way length + half-way length = 2 (half-way length) = full length

If the three dimensional equation for points is set equal to a constant such as r=radius, all the points that satisfy or complete this equation will be part of a **sphere** (= a disk or circle and-or its circumference rotated 360° [= the standardized angle of 1 full rotation] about its center point), and with the center point being at p(0,0,0). If the result of the equation is less than (r) then the point is withing the sphere. If the result of the equation is greater than (r), then the point is outside the sphere.

Continuing the main discussion:

Ex. If the length of some material is 20 inches long, the half-way point is at:

$\frac{20 \text{ inches}}{2} = 10$ inches : half-way point of 20 inches

This length of 10 inches can be thought of as starting at the 0 point of a number line since from the concept of length, all lengths start from nothing and are in reference to this nothing or "0" point of length. For this example, 10 means 10 from or with respect to 0, and not from any other reference value: [FIG 52]

```
|-------------------|-------------------|--------------->
0                  10                  20             n units
```

A length can also be thought of as a difference or change between any two points on this number-line or one-dimensional measurement system. For example, here we are to find the midway point of the length (L) of the line segment bounded by the point identified as 100 and 120: [FIG 53]

```
|------------------------------------------------|------------|-------------->
0                                              100          120            n units
                                                 |--    L   --|
```

The difference or change between these two points which defines the "length" or distance between those two points is:

(max. value - min. value) = change = difference = length
120 - 100 = 20

Half of this length is:

$$\frac{\text{length}}{2} = \frac{20}{2} = 10$$

This value of 10 is half of the length of the line segment between the numbers, points or locations of 110 and 120, and is only in reference to the start (ie., 110) of that line segment only. Though 10 is the center point of half the distance value between the two defining points on the line, what is that center point's corresponding value, location or point on the line of which the defining, delimiting or bounding locations or points of that line segment were given and which are in reference to 0? We must now adjust this center point or midway value of 10 to be in reference to this entire system or line, hence in reference to 0. Notice that this line segment (from point 100 to point 120), and all points on and associated with it, are effectively shifted or increased from the "0" position (of reference for the entire system) by a value equivalent to the minimum point or location of the two points bounding that line segment in question. This minimum point is effectively the "0" or reference point of just the line segment or length in question, and not the entire number-line system. Since this "new 0" or reference point of length is shifted (ie. added to) by a value equal to the lengths starting or min. value within the entire system:

First,here are some abbreviations for the points or locations on the reference (of measurement) line:
min. = minimum = smallest, least , mid. = midway = center , max. = maximum = largest

For the example given: the min. and max. points or locations are in reference to the 0 reference point of the entire system. min = 100, and max = 120. These points also define a line segment, and they are its boundary or defining points.

$$\text{mid.} = \frac{\text{length}}{2} + \text{min.}$$

: length = length of the line segment, here its 10, using substitution: This mid. point or location is the value that corresponds to the line segments center point's location on and in reference to the entire line.

Since the length of the line segment is the difference between its boundary points or locations: (max. value - min value):

$$\text{mid.} = \frac{\text{max.} - \text{min}}{2} + \text{min.}$$

: for this example, min=100, and max=120

This formula algebraically leads to a "simple average" formula:

$$\text{mid.} = \frac{\text{max.} - \text{min}}{2} + \text{min.}$$

combining fractions:

$$\text{mid.} = \frac{\text{max.} - \text{min}}{2} + \frac{2\ \text{min}}{2} = \frac{\text{max.} - \text{min.} + 2\ \text{min}}{2}$$

Combining like terms in the numerator:

$$\text{mid.} = \frac{\text{min} + \text{max.}}{2}$$

: "middle or center" value between two other values.
: This is similar to an average of two values formula

Or, as indicated above, and will be shown further below:

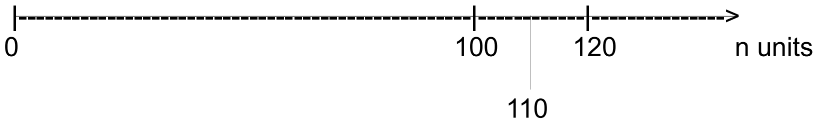
$$\text{mid.} = \text{min.} + \frac{\text{change from min.}}{2}$$

, or = min. + 0.50 (change from min.)
: = "min. value plus half the total change"

For the last example:

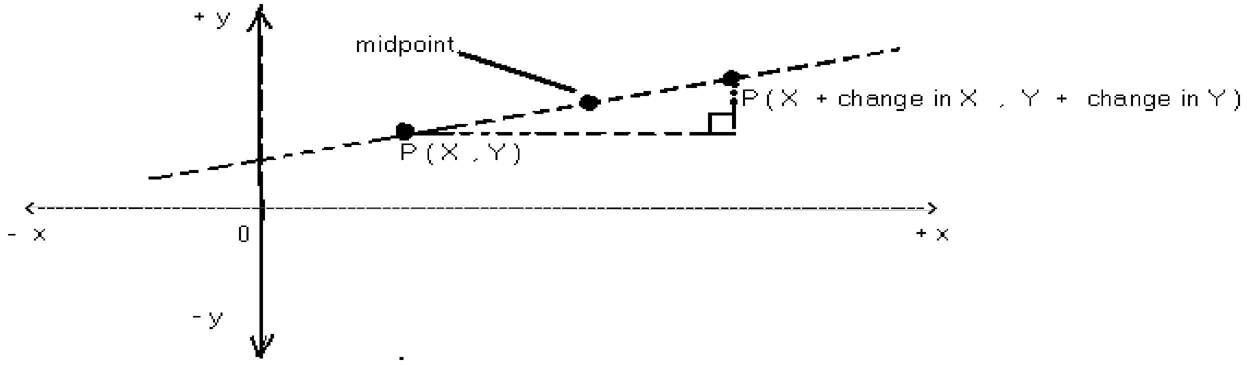
$$\frac{100 + 120}{2} = \frac{220}{2} = 110$$

: the center point of 10 of the line segment corresponds to the point or location of 110 on the main reference line or system



: [FIG 54]

The co-ordinates of the midway point between two points in a two-dimensional co-ordinate system is analyzed as follows: [FIG 55]



Here, X and Y are effectively the min. values for each dimension and are "offsets" or distances from the origin (0) in the system.

From: $\text{mid.} = \dfrac{\text{min.} + \text{max.}}{2}$ and observing the diagram:

$$\text{midx.} = \frac{\text{min. X} + \text{max. X}}{2}$$: X coordinate of midpoint

$$\text{midy.} = \frac{\text{min. Y} + \text{max. Y}}{2}$$: Y coordinate of midpoint

$$\text{Pmid.}\left(\frac{\text{min. X} + \text{max. X}}{2} , \frac{\text{min. Y} + \text{max. Y}}{2} \right)$$ **: COORDINATES OF MIDPOINT BETWEEN TWO POINTS ON A PLANE**

Or by using a substitution for max X:

$$\text{midx.} = \frac{X + (X + \text{change in X})}{2} = \frac{2X + \text{change in X}}{2}$$: X = min. X
(Change in X) = (X2 - X1)

$$\text{midx.} = X + \frac{\text{change in X}}{2}$$: X coordinate of midpoint

$$\text{midy.} = Y + \frac{\text{change in Y}}{2}$$: Y=min. Y, midy = Y coordinate of midpoint, hence :

$$\text{Pmid.}\left(X + \frac{\text{change in X}}{2} , Y + \frac{\text{change in Y}}{2} \right)$$: COORDINATES OF MIDPOINT BETWEEN TWO POINTS ON A PLANE

Hence: Pmid = (X + half the change in X , Y + half the change in Y)

As was previously noted about a variable being a place holder for any value, being either positive or negative in sign, and therefore, the actual value of (b) in a linear equation can either be negative or positive. Given a basic linear equation of:

$y = mx + b$

, the mathematical relationship between the (y) and (x) variables is a direct relationship when (m), the slope, is positive in sign. If (x) increases, (y) increases and vice-versa. However, if you know of a linear like relationship where as one variable increases and then the other variable decreases, or vice-versa, the mathematical relationship between those two variables is said to be an **inverse relationship**. This would happen in a linear equation when (m), the slope, is negative in sign.

We know that the inverse or counterpart of a value is expressed by taking the negative, or "counterpart" of that value:

The inverse of $+y$ is $-(+y) = -y$
The inverse of $+x$ is $-(+x) = -x$

If (y) was equal to (x) as in the most basic of linear relationships:

$y = x$: a linear equation, where m=1, b=0

and if the relationship between (y) and (x) is an inverse relationship, that is, if the inverse of (y) is (x), or vice-versa:

$-(y) = x$
$-y = x$ by multiplying or dividing each side by (-1), this can be expressed as:

$y = -x$: y equals the inverse or negative of x. Note that x, a variable, can still be either positive or negative in value, and this expression essentially means the inverse of any value that variable may have. Here, as the value of x increase, the value of y decreases, and this is an inverse relationship.

If the value of (y) is related to (x) (either directly or indirectly) by some constant factor (called m) other than 1 as seen above in: $y=(-1)x = -x$, this must be noted in the equation. That is, if the ratio of (y) to (x) is always the same constant:

$\frac{y}{-x} = m$ therefore: $y = m(-x)$: the variable name assigned for this constant is (m), and mathematically, we then have:

$y = -mx$: this indicates that y is always the same multiple of x. Adding in the arbitrary constant of (b):

$y = -mx + b$: an "inverse form" of the basic linear equation of: $y = mx + b$

This form now implies the same linear equation but which now has the negative of a given slope. Linear equations where the (x) term is negative, such as when the coefficient (the slope, m) of (x) is negative, the corresponding (y) values for each (x) value get smaller as the (x) values increase. The effect on the line is that it will slope vertically (upward, in the (+y) direction) downward, rather than vertically upward, as (x) increases, as should be expected for an inverse relationship where as one value increases, the other value decreases.

Even though the line of a linear equation and the line of the inverse of that linear equation will cross, don't assume they cross perpendicularly. Lines that cross perpendicularly will and must have negative reciprocal slope values:

$m1 = -\frac{1}{m2}$ or = $m2 = -\frac{1}{m1}$: **SLOPES OF PERPENDICULAR LINES**
This is verified in the TRIGONOMETRY section.

Note also that (m1)(m2) = -1 always. For any two perpendicular lines (other than the axis or reference of measurement lines), one line will have a positive slope and the other line will have a negative slope, hence their product will always be a negative value (such as -1). As indicated in the equations above, these slopes are also inversely related. If one line has a very low absolute value for its slope, the other line will have a high absolute value for its slope.

Note, if the (b) term is present in the equation of a line, the fundamental relationship between (x) and (y) is still linear, but the numerical relationship, as in the form of a constant ratio, between each corresponding pair of (y) and (x) is no longer a constant, and only the slope remains constant, defining what is known as a linear relationship. Consider this example where the ratios of corresponding values is not constant:

Ex. $y = 2x + 3$: m = slope of the line = 2

$$y(1) = y = 2(1) + 3$$
$$y = 2 + 3$$
$$y = 5$$

: y(1) is a simple notation which means solve for the value of (y) when x=1, a more general or formal expression for this notation is: y(x) = "y at x" which means to evaluate the value of y when x is this value". When no value of x is specified, y(x) means: "y is a function of x".

$$\frac{y}{x} = \frac{5}{1} = 5$$

$$y(3) = y = 2(3) + 3$$
$$y = 6 + 3$$
$$y = 9$$

$$\frac{y}{x} = \frac{9}{3} = 3$$

Though the ratio of (y) to (x) is no longer a constant due to the (b) term, the slope (m) of any line, as in the above example, is constant, and here it is a value of 2. Besides being obvious in the equation that the slope is 2, using the two points found using these two instances of the equation, we can calculate the slope:

$$m = \frac{\text{change in y}}{\text{change in x}} = \frac{9-5}{3-1} = \frac{4}{2} = \frac{2}{1} = 2$$: when x changes by +1, y will change by +2

The higher (x) is, the value of (y/x) will approach that of (m) since the value of (b) will have less and less effect on the value of (y) as compared to the larger effect a high value of (x) would. For example, using the last examples equation, and letting x = 1000:

$$y = 2(1000) + 3 = 2003 \quad \text{and} \quad \frac{y}{x} = \frac{2003}{1000} = 2.003$$: nearly 2

We see that although there can be a constant numeric relationship, such as the rate of changes between the values of variables such as (y) and (x) as for a linear equation, the relationship between (y) and the entire expression or function value that (x) is in, may not be always be constant as shown above for a linear equation. Clearly given an equation of:
$y = mx + b$, y is obviously not equal to x, but y is equal to the whole expression or function that x happens to be in. It is said that y is a function of x, rather than y is equal to x. y = f(x) is the mathematical notation or expression for "y is a function of x", and even this can be stated as meaning: "y is equal to the entire expression or function containing the variable x". The word "function" is used and meant to imply or stress the concept of the dependent relationship of y and-or its value due to x and-or its value.

Given a certain construction, when lengths of corresponding pairs of line segments are said to be proportional (equivalent in ratio), the ratios of their lengths is constant. When the ratios of corresponding sides (or side lines if you will) of two triangles (discussed in the TRIGONOMETRY section) are equivalent, the sides are often spoken of as being proportional since a proportion is two (or more) equivalent ratios. This is the case when the triangles are said to be similar (as in similar looking or similar shaped constructions) triangles, and which are essentially magnified versions of each other.

An extra note, before this example is finished, is about using subscripts to indicate a particular instance of the same equation, and therefore, a particular instance of the variable(s). In this manner, the subscripts effectively identify (with mathematical notation) the corresponding results and values used in that equation. For example, given:

$y = 4x + 32$: $f(x) = 4x + 32$, hence $y = f(x) = y(x)$ = "y is a function of x"
Again this is just notation, f(x) and y(x) does not mean multiplication.
Given (4x + 32) is a stated function in or of x, and if we set y equal to this, it gives
that function a specific name or identification. The function can now be called or
indicate as y, and instead of using f(x), we can use y(x) to be more specific as to
what specific function is a function of x. There could be other functions of x such as:
g(x), h(x), etc., which generally would have different expressions containing the
independent variable x.

$y1$ would correspond to the value $x1$ used in the equation.
$y1$ is a specific instance of y and $x1$ is a corresponding and specific instance of x.
$x1$ and $y1$ would be the coordinates of a point: $p(x1 , y1)$ or $p1(x1 , y1)$

$y2$ would correspond to the value $x2$ used in the equation, and so on.
$x2$ and $y2$ would be the coordinates of a point: $p(x2 , y2)$ or $p2(x2 , y2)$

$y = f(x) = y(x) = 4x + 32$: main function or relationship

$y1 = f(x1) = y(x1) = 4(x1) + 32$: instances of the function
$y2 = f(x2) = y(x2) = 4(x2) + 32$

LINE AND CIRCLE EXAMPLE

In this example, the circumference of a circle will be intercepted and passed through by a line. There will be two points of interception associated with this. Write a line (linear) equation of this intercepting line if the radius (r) of the circle is 3, and the (x) coordinate of the point of interception is 2. This line passes through the center of the circle which happens to be located at the center or origin (0,0) of a rectangular co-ordinate system.

The basic equations (including subscripts for this example) of a circle and line are:

$y_1 = mx_1 + b$: basic line equation
$r^2 = x_2^2 + y_2^2$: basic circle equation , and this also resembles the Pythagorean Theorem

(Note above, the subscript identified as 1 refers to the line, and the subscript identified as 2 refers to the circle).

Solving for and expressing the equations in terms of y:

$y_2^2 = r^2 - x_2^2$
$y_1 = mx_1 + b$

When $x_2 = 2$, and evaluating the equation or function for y_2 at this value; this can be noted with functional notation as f(2), or $y_2(2)$, which means to evaluate the expression or function when (x)=2: :

$y_2^2 = r^2 - x_2^2$
$y_2^2 = 3^2 - 2^2$
$y_2^2 = 9 - 4$
$y_2^2 = 5$ taking the square root of both sides:
$y_2 = 2.236067977$

At the point of interception, the two points (one on the line, and one on the circle) have the same location, and therefore have the same value: $x_1 = x_2$, and $y_1 = y_2$, hence:

$x_1 = x_2 = 2$
$y_1 = y_2 = 2.236067977$ substituting these values into the linear equation:

$y_1 = mx_1 + b$
$2.236067977 = m(2) + 0$: letting (b), the vertical or (y) intercept, equal 0 since this line passes through the y-axis at (0.0). The line also passes through the center of the circle which is located at the center or origin of the rectangular co-ordinate system. If the center of the circle is not at the origin, all points associated with the circle are effectively offset or "adjusted" by some (same) value from the origin, and therefore, the basic equation of the circle must be "adjusted" if the circle is to be in reference to this specific, "new" or local rectangular system located within the larger rectangular system. If the center of the circle is now at p(a,b), rather than (a=0, b=0) as in the larger or standard rectangular coordinate system, the (x) coordinate of the center and all other points associated with the circle's system must be incremented by (a) to so as to be in reference to the larger rectangular coordinate system. Likewise, the circles (y) coordinates must all be incremented by (b) so as to be in reference or with respect to the larger rectangular coordinate system. The equation of the circle therefore becomes this so as to be in reference to the main system:

$r^2 = (x + a)^2 + (y + b)^2$: **GENERAL CIRCLE EQUATION**

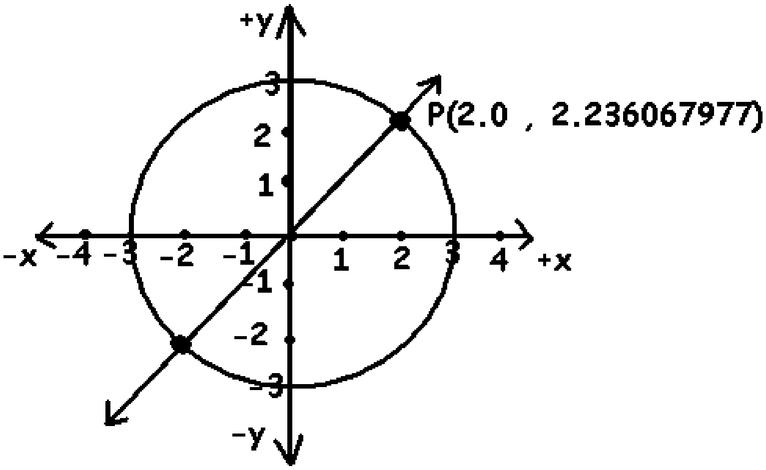
(When the center is not at the origin at (0,0), and the coordinates of the circle's and other (x,y) points are in reference to another location other than (0,0) of the larger rectangular system. Still, (a) and (b), and point(a,b), the center point or origin of the specific, local, "sub, or" micro" rectangular system. are in reference to the larger macro rectangular system. The value of (r) does not need to be modified since it is a fixed length value, and not a (variable) location.)

After solving for m:

$m = \frac{2.236067977}{2} = 1.118033989$ putting this value of m into the line function or equation:

y1 = 1.118033989 x1 : equation of the line in question , and where b = the y-axis intercept = 0 for this line. The y-axis intercept, where x=0, is: p(b,0);

[FIG 56]



Given the equation of the circle and the equation of the intercepting line, the point of intersection of the two curves can still be found. We know that at the point of intersection, that the difference of the (x) and-or (y) values of both equations is 0 since the equations at that point will have the same value.

In theory or consideration, a line is a special case or instance of a curve, and you can think of this as a portion, or (line-like) segment of the circumference of a circle that has a very large or infinitely large radius or diameter.

y2 - y1 = 0 : at the point of interception of two curves, therefore, mathematically:
y2 = y1

And the known equations are:

y2^2 = 9 - x2^2 : circle

y1 = 1.118033989 x1 : line

There are several ways to solve for (x) when subtracting the equations, for example, you can square both sides of the linear equation and subtract it from the other equation, and with the difference in y values defined as being 0. Here is a method that uses algebraic substitution:

from $y2^2 = 9 - x2^2$ solving for y:

$$y2 = \sqrt{9 - x2^2}$$

Removing the subscripts since (x) and (y) are the same value at the intersection point, and substituting this into the linear equation:

$y2 = y1$: at the point of intersection of the two curves or equations

$\sqrt{9 - x^2} = 1.118033989\,x$ squaring both sides of the equation :

$9 - x^2 = 1.25\,x^2$ transposing ($-1x^2$) to solve for (x) :

$9 = 2.25\,x^2$ isolating the (x) variable, and then taking the square roots of both sides:

$x = \sqrt{\dfrac{9}{2.25}} = \sqrt{4} = 2.0$ solving for (y) using algebraic substitution:

$y = 1.118033989\,x$
$y = 1.118033989\,(2.0)$
$y = 2.236067977$: checks

By observing the above procedure carefully, a formula for the (x) coordinate of the point of interception can be derived:

$x = \pm\sqrt{\dfrac{r^2}{m^2 + 1}}$: X COORDINATE OF CIRCLE AND LINE INTERCEPTION (center at origin (0,0), and b=0 for the line)

This can also be expressed as:

$$x = \frac{\sqrt{r^2}}{\sqrt{m^2 + 1}} = \pm\frac{r}{\sqrt{m^2 + 1}}$$

(y) can be found by substituting the value for (x) into one of the original equations, or from:

$m = \pm\sqrt{\dfrac{r^2}{x^2} - 1}$: this was derived from the above expression for x

and since $m = \text{TAN}\ ¢ = \dfrac{y}{x}$: TAN ¢ = tangent of the angle. More exactly, it is the ("tangent") ratio of two specific side lengths of a triangle. tan = opposite side/adjacent side. For this example and line, The angle is between the line and the x-axis.

When (change in y) = y, and (change in x) = x,(ie,. one point being at the origin, p(0,0)) we can algebraically get:

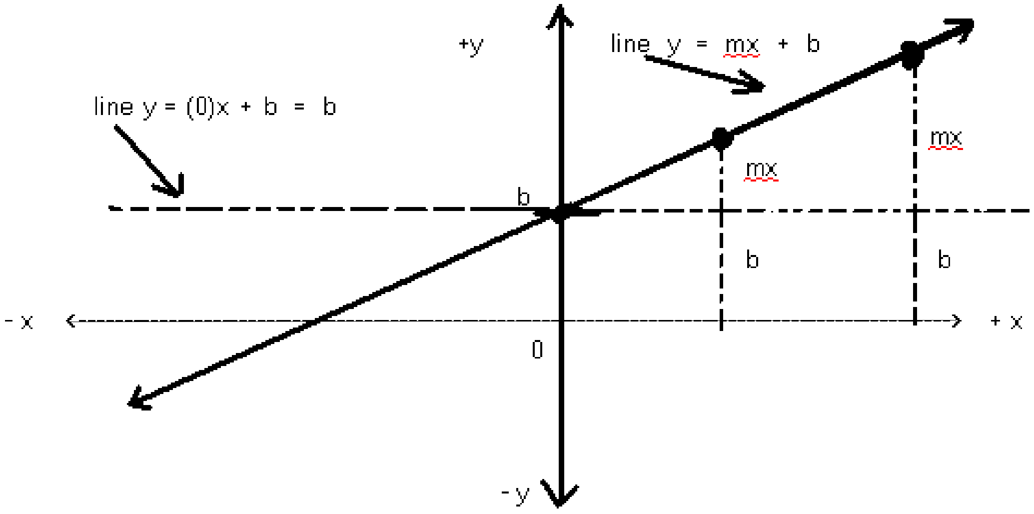
$y = mx$: the basic linear equation (when b=0). Since (m) is a initially an algebraic variable, it can have any value, including (1). If m = 1, the result is the simplest linear equation of: $y = x$.

Given: y = mx, mathematically, we have: (y / x) is always a constant value of (m) = the slope of that line. Given: y =mx, (y) is a (constant=m) multiple of (x).

When the line does not cross at the origin point of the system, the vertical intercept (b) is not at y=0 but some other value of y we will note as (b). When x=0, the corresponding y value or coordinate on the line is this value of (b). This point is therefore p(x,y) = p(0,b). It could be said that (y) already has an offset, initial or "lead value", and this value is y=b. The basic equation of y = mx is still used, but now with an adjustment term added or combined in to it. For all values of (x), each corresponding value of (y) is increased, changed, or adjusted to automatically account for and contain this constant value of (b). Reflecting this as a linear equation:

y = mx + b : the basic linear equation.

In the graph of a general line equation below, note that when (b) is positive in value, causes a "lead" in the (y) values which corresponding to the (x) values, and is seen as an "upward vertical shift or offset" (from the origin and-or x-axis) and-or a "leftward horizontal shift or offset" (from the origin and-or or y-axis) for the entire line. [FIG 57]



The general linear equation is also verified ahead in the discussions of similar triangles.

LINEAR TEMPERATURE EXAMPLE

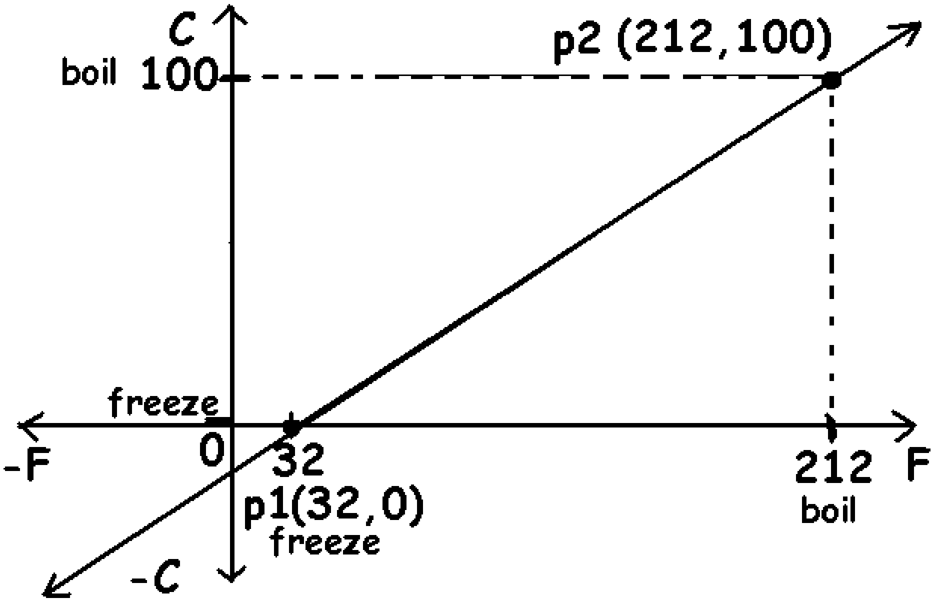
Here, we will derive the mathematical relationship between the Fahrenheit and Celsius temperature scales. Temperature is basically defined as the amount of energy something has. The more energy, the hotter (higher in temperature) it becomes. This is a direct relationship. The temperature measured is linear for both the Fahrenheit and Celsius temperature scales. If the objects (including the air) temperature doubled, the measured temperature indicated on a thermometer (a temperature metering or measuring device) will double in value. For both systems, the temperature with respect to the energy of an object, can be plotted as straight lines. If the energy of an object doubles (2), its temperature likewise increase by that same factor value and double. The Celsius temperature scale is named after **Anders Celsius**, from Sweden, who originally conceived of it in 1742. "Centigrade" (ie., 100 gradients) is an older word for the Celsius scale.

What is generally or commonly known about the Fahrenheit and-or Celsius scales is the temperature of the freezing point and boiling point of water:

	Freezing	Boiling
Fahrenheit:	32°	212°
Celsius:	0°	100°

: the Celsius scale appears to be more "metric-like". Some have even proposed that time and clocks should also be based on a "metric-like" system with 10 hours before noon, and 10 hours after noon. Each hour would have 100 minutes, and each minute would have 100 seconds.

Observe the drawing below: [FIG 58]



For finding the slope of this line shown, we will use these two known points on the line that have the format of: p(x , y) = p(Fahrenheit , Celsius). The values of the two shown points are: p1(32,0) and p2(212,100):

$$\text{slope} = \frac{\text{vertical change between points}}{\text{horizontal change between points}} = \frac{\text{change in Celsius}}{\text{change in Fahrenheit}} = \frac{(100 - 0)}{(212 - 32)} = \frac{100}{180} = \frac{0.5555\ldots}{1} = 0.5555\ldots$$

Hence for each increment (1), or degree change in Fahrenheit temperature, the corresponding Celsius temperature will change by 0.5555... degrees which is a little over a half (1/2 = 0.5) a degree Celsius.

By taking the reciprocal of this value, we can find that for each degree change in Celsius, Fahrenheit will change by (1/0.5555..) ~= 1.8 degrees, or derived as:

$$\frac{\text{change in Fahrenheit}}{\text{change in Celsius}} = \frac{212 - 32}{100 - 0} = \frac{180}{100} = 1.8$$: almost 2 degrees (Fahrenheit temperature units)

If we use P(32,0) consistently as the second point in the above formula we would still have the same rate or slope value:

$$1.8 = \frac{F - 32}{C - 0} = \frac{F - 32}{C}$$: using p(F,C) as general variable notation for the first point. Solving for F:

$F = 1.8C + 32$: relationship between corresponding Fahrenheit and Celsius temperatures
This is a linear equation with m=1.8, and b=32.
We can see clearly that when C=0, F=+32
Solving for C:

$$C = \frac{F - 32}{1.8}$$ which can be expressed as the sum of two fractions:

$$C = \frac{1F}{1.8} - \frac{32}{1.8}$$ by canceling fractions, this can be expressed as:

$C = 0.55\underline{6}F - 17.7\underline{8}$: relationship between corresponding Celsius and Fahrenheit temperatures
This is a linear equation with $m=0.55\underline{6}$, and $b = -17.7\underline{8}$
We can see clearly that when F=0, $C=-17.7\underline{8}$
This slope for the Celsius equation is the reciprocal of the slope for Fahrenheit equation: $1/m2 = 1/0.55555 = 1.8 = m1$

Extra: -40°F = -40°C : same numeric value for these different temperature units, and this is only a special instance

Although the Celsius and Fahrenheit temperatures scales are each linear, that is, if the energy doubles, the temperature or thermal energy measurement will double on each scales, however the Celsius and Fahrenheit values are not proportional to each other due to that each linear or line-like scale has a different rate of change, hence a different slope. For example, trying to make a conversion using proportions will not work, for example, consider this for when trying to convert 30° Celsius temperature to its corresponding Fahrenheit temperature:

$$\frac{212°F}{100°C} = \frac{X°}{30°C}$$, after solving for X°: $$X° = \frac{30°C\ (212°F)}{100°C} = 63.6°F$$:**the is wrong**
The correct result is X° = 86°F

It should be noted that the outside **air temperature** is, in most cases, standardized as the temperature of the air only, and the thermometer and-or temperature sensor is placed the shade such as in a shaded, and air-vented box with no sunlight energy directly or indirectly (by reflections of light) upon the thermometer, and which would otherwise significantly (and incorrectly) increase the measured, true air temperature to be displayed as say 130°F instead of correctly as 85°F. To prevent an incorrect measurement by possible sunlight and-or reflections of sunlight and its thermal energy heating of the box and air in it, the box should be in the shade and-or double or triple insulated (via air gaps or spaces) and ventilated to prevent any heat build up and a corresponding increase in the temperature reading or measurement.

When wind contacts the surface of your bare unprotected skin, the wind essentially gets heated up by the warm skin (typically about 98.6 degrees Fahrenheit) and removes some thermal energy from it. This will cause you to feel cooler in those areas initially, and if it continues, it will effectively be like having an air-conditioner to cool you down in that uncovered or uninsulated area, and your entire body temperature can drop dangerously low because the healthy thermal energy in your body tissues and blood will be wasted to the outside world. This initial effect on the surface of your skin (or any other object such as the ground, water, buildings) is called **windchill** (ie., cooling due to the wind), and it is similar to the concept of evaporative cooling where water humidity (water vapor ="water-gas", cool steam) or moisture can effectively absorb and remove heat from an object, and then that moisture rises or is blown away by a wind and the object then has a reduced temperature. This is essentially what happens when a person sweats so as to help the body cool down. If the air temperature is at or below the freezing temperature of water, the wind can hasten frostbite (frozen, damaged body tissue) in those areas since it is removing what little amount of heat was still in those areas so as to keep them warm and unfrozen. The temperature that the bare unprotected skin will feel when wind is blowing upon it is called the windchill temperature. The windchill temperature is a calculated value using a formula that depends mainly on both the air temperature and the wind speed. Obviously, for a constant air temperature value, the "windchill temperature" depends mainly upon the wind speed. The windchill temperature will be lower than the actual air temperature. When the air temperature is of a lower value, and for a constant wind speed, the difference between the actual air temperament and the windchill temperature increases. In wintertime with cool air temperatures, the wind has a significant impact on how cold you will feel, especially if not wearing several layers of typical clothing (like insulation), and usually shirts, so as to retain much body heat so as to feel warm. For a basic idea of the (subjected, human perceived) mathematical relationships, given a constant wind speed, for every 10 degrees colder in actual temperature, the windchill temperature feels about 12.6 degrees colder. The windchill temperature formula usually reflects only the data for temperatures 50° or lower where the effects are most noticeable. The windchill formula is not overly complex, but its derivation is beyond the space and scope of this book. Due to the average person only having a temperature reading or estimate, and maybe a windspeed estimate and no other data, here is a very simplified formula to obtain a rough value for the windchill or "feels like" temperature:

A windchill estimate temperature = F° - windspeed mph : A simplified and practical windchill estimate formula

The author of this book has found that the wind is one of the worst parts of winter, and so therefore, you need to first check the outside temperature and windspeed, and then dress accordingly such as with several layers of shirts and possibly pants before venturing outside. The lower the air temperature, the more the wind will have a cooling effect. Denim pants are not ideal for cool climates because they do not seem to keep the heat in and keep you warm. "Sweat pants" are much better than denim (ie. tightly woven cotton material) "blue-jeans" in the winter if you will be outside doing things, and without much of the worry of constantly trying staying warm. Wear a winter hat and gloves, and keep them immediately available and close by (ie., in a pocket or bag) for when you will need them. Wearing two coats is sometimes necessary, depending on the temperature and-or windspeed. If in a difficult situation with no warm shelter, brisk or fast walking can help raise your body temperature. Shelters can be made from sticks, rocks, mud, leaves, snow, etc.

The concept of **heat-index** is essentially the opposite of the windchill concept. The heat-index is the temperature of how hot the air temperature feels like depending on both the actual current air temperature and the humidity level. Surely, whether hot or cold, the sunlight will always make the air temperature feel warmer, perhaps an additional 10°F warmer , and so this measurement, like the windchill temperature, also uses the measured air temperature in the shade. **Humidity** is moisture in the air which can carry much heat with it and make the air temperature feel hotter. It is essentially a "steam (heat) bath". **Danger:** If a persons core body temperature is 104°F = 40°C or greater, they are in severe health danger, and will most likely have symptoms of various mental confusion, especially if they are dehydrated and in need of plain drinking water. Overworking and in the sunlight and-or stress can quickly cause dehydration in the body. A person having these conditions needs to be cooled down fairly quickly. Place that person in the shade and remove any restrictive and warm clothing. Let them drink some water and sit in front of a fan and-or cool off in water with constant supervision by the persons(s) aiding them. A person can also be cooled off by immersion of their feet or hands into a container (such as a dishpan) of water that feels cool to them. The higher the humidity, the higher the heat-index temperature because evaporation to cool the body has slowed down. The transfer of heat energy from the body to the air is less or slower in higher humidity or moisture climate conditions. Occasionally, cool moist air can actually make a person feel cooler, and the heat-index temperature will be slightly lower than the actual air temperature.

Heat Index estimate temperature = °F + (%humidity in decimal form)(10) °F :Generally when the humidity is >= 50% and Temp. >= 50 °F :

:**A simplified and practical, heat-index estimate formula. This is the "feels like temperature" when the humidity is considered. Ex. If the humidity is 70% , add (0.70)(10) = 7 °F .**

If the wind-chill effect or skin cooling due to of the wind essentially absorbing heat from your body is to be considered in the above heat index formula, the wind will reduce the "feels-like" temperature of the heat index temperature. A simple value to use is to subtract 1°F for each mile / hour of wind speed. With sunlight upon a dark colored, non-reflective surface, it will heat or warm up, but it will generally release that heat back into the air like a heat sink does. You will feel about 10°F warmer in the direct sunlight than in the shade without direct sunlight.

The average or normal, internal body temperature is: 98.6°F = 37°C =~ 310.15°K :**Normal Human Body Temperature**

When the human body can not create enough thermal energy to maintain its core or internal temperature (normally 98.6°F) due to the external cool air and-or cool water, or possibly an internal health problem, it will begin to reduce in temperature. This can lead to a dangerous health condition called **hypothermia** when the temperature of the body is just 95°F = 35°C and obviously less. In this condition, a persons normal mental and physical activity will slow down. The person may have uncontrollable shivering at first so as to create some heat, and then could become unconscious (ie., "pass out", faint), and will die if they are not heated up externally. It is also common for people in the cold weather to forget things, particularly when they are physically exhausted such as climbing a mountain. Swimmers in apparently warm comfortable water that is about 82°F =~ 27.78°C can eventually, over time, loose enough body heat by the contact (ie., thermal conduction) with that water, and can get hypothermia. In fact, being in water that is 95°F = 35°C for a very long time, perhaps a few days, will eventually cause a person to get hypothermia if they cannot maintain a higher internal temperature than this. Try to keep yourself and others warm so as to prevent hypothermia, and to warm a person and seek help from others if someone has the conditions associated with hypothermia. Going into cool water for just a quick swim can also be a health risk for some people with existing medical problems because their body will try to get warm by increasing their heart rate and blood pressure, and they are then at an increased risk for a heart attack, stroke or other medical problems. If a person is to go swimming, it is best to mention to them about hypothermia, and to gradually acclimate their body response to cool water, and that is to initially place their feet into the water for a few minutes, and then their legs for a few minutes before attempting to "go swimming". A person can drown if they cool down too much and get sluggish (slow, low-energy) at swimming. People also need to be reminded to not panic if they fall into cool water, but rather go to and-or or swim to safety, even if its just by grabbing onto something that can keep them a float until they are rescued. To help prevent such dangerous issues, people on boats need to be reminded of having a flotation device(s) such as a "life vest", "life ring" or a container filled with air, and-or wearing a hollow container(s) capable of supporting and-or floating their body in the water. It would also be helpful for people on boats to have a signaling device(s) such as a flashlight (a.k.a. (electric) torch), strobe-light, flare, signaling flag, signaling mirror, communication radio, etc., so as rescuers can locate them more easily if they fall into the water and-or the boat may sink.

TEMPERATURE AND ALTITUDE

As the elevation increases, the air is less compressed (ie., squashed by force) because the effective column of weight of air above that location is less. Since the air is less compressed, the air has less pressure and density (mass/volume), and therefore fewer atoms to have heat energy (ie., thermal capacity) and the (average) temperature will be less.

When the air temperature is measured with a thermometer, it is taken in the shade because the Sun's light will cause the thermometer to get heated, and this will cause an incorrect air temperature measurement.

The global average temperatures is 57°F at sea level, and the value for a specific location on Earth depends on that locations specific weather conditions (ie., altitude, rain, snow, season, current temperature, sunlight energy, etc) and the Sun angle (location [latitude or "lateral side" of the Earth's location from the equator line] on Earth, time of day, day of year). The example temperatures shown below correspond to heights above sea level at 57°F, and the formula shown below is generally a average global estimate for altitudes up to 30000 ft. At 60000 ft., the temperature begins to increase slightly above the calculated value, and at about +5 °F per 10000 ft higher in altitude, and to up to about 100000 ft. in altitude at where it begins to decrease again.

Altitude: 0ft = 0m = Sea Level
Temperature: 57°F ~= 13.889° = 14°C

Altitude: 7200ft
Temperature: 32°F = 0°C = water freezes

Altitude: 16300ft
Temperature: 0°F =~ -18°C

Altitude: 60000 ft =~ 11.4 miles ~= 18.3 km
Temperature: -68°F =~ -56°C

Altitude: 30000ft   : about the height of Mt. Everest
Temperature: -47F ~= -44°C

The temperature will decline at a fairly constant or steady rate (change in temp/change in altitude) from sea level to about 30000ft high. Lets derive a basic formula for the typical and average change in temperature per change in altitude:

$$\frac{\text{(change in temperature)}}{\text{(change in altitude)}} = \frac{(32°\text{F} - 57°\text{F})}{(7200\text{ft} - 0\text{ft})} = \frac{-25°\text{F}}{7200\text{ft}} = \frac{-0.0034722°\text{F}}{1\text{ft}} \sim= -0.0035°\text{F per foot higher in altitude}$$

Since a meter is 3.28 times longer or higher than a foot, the above rate corresponds to: -0.01148°F / meter

For every 1000ft higher the predicted, average change in temperature would be; according to the above equation:

(change in temperature) = (change in altitude)(-0.0035°F / 1ft)

change in temperature = (1000ft)(-0.0035°F /1ft) = -3.5°F

A general formula for the global average temperature at an altitude above sea level and up to 30000 ft is:

temperature at altitude = 57 °F - (altitude)(change in temperature / change in altitude)

temperature at altitude = 57 °F - (altitude in feet)(-0.0035°F / 1ft)   : *

*** : In place of the value of 57°, you can substitute your local air temperature.**

For conversions, and as shown in the book, a change of 1°F corresponds to a change of 0.55555°C, therefore, using proportions or equivalent fractions, a change of -0.0035°F corresponds to a change of -0.0019444°C =~ -0.002°C, and the above formula can be expressed as:

temperature at altitude = 57° - (altitude in feet)(-0.002°C / 1ft)

The change in temperature per 1000ft ~= 305m =~ 0.305km is about -3.5°F ~=   -1.9444°C =~   - 2 °C

The change in temperature per 1km = 1000m =~ 3280ft is about -11.48°F ~=   -6.38 °C .

For some other conversions (approximate):   1m = 3.28ft   , 1 ft = 0.3048 m ,   1km = 3280ft = 0.6214 miles ,
1mile = 5280ft = 1.60934 km = 1609.34m

A change or increase of 1 °C corresponds to a change or increase of 1.8 °F. From this, we can solve for 1°F by dividing each side by 1.8, and we have: A change or increase of 1°F corresponds to a change or increase of 0.555...°C.   =~   0.56 °C = (roughly) 0.5 °C

0 °C = 32 °F

Due to the non proportional scale relationship, and even though each scale is linear in temperature, here we **cannot** then simply solve for 1°F by dividing each side by 32. A change of 1°C does not equal a change of 1°F,and vice versa, and as noted above, the change is another (constant) value other than 1.

0 °F = 17.7<u>8</u> °C

Extra: More about humidity and cloud cover of the Earth's entire surface and during a year of time.

Surely there are places on Earth's surface where there is not much rain, and yet other places with much rain, sometimes seasonal throughout the year. In the cooler regions of Earth, such as at the higher latitudes, the climate is cooler in temperature, and having less humidity on average - even if the ground is covered in snow, and this is due to that the humidity in the atmosphere there has became snow and is no longer in the air.

* About 70% of the Earths surface is ocean water which composes about 95% of all the water on Earth. About 30% of the surface of Earth it land. It is also of note that only a fraction of that land is usable for farming and reasonable living conditions, hence only about 10% of the surface of Earth s moderate in climate and enough for humans to grow food and exist without too much problems. Since the oceans cover most of Earth's surface, most of the water vapor or moisture in the air is due to the ocean surface water evaporating into the air, molecule by molecule, and without any salt attached to it. It is this water vapor that can later condense into clouds and rain which brings fresh drinking water for all forms of life.

If there are signs of drought (lack of moisture, rain and-or water) coming, then it is a good idea to begin to save vast quantities of it for high-priority use later, and this can be various cisterns, ponds, lakes, ditches (even in and-or along a stream) and various water containers. Drinkable water is often called "potable water", and which had any possible debris filtered out of it, and-or sanitized (sterilized of harmful germs, etc) by various methods (a few drops of bleach, vitamin C, UV light, etc, and you will have to research these). The low tech. version of water sanitation would be to filter and then boil the water for several minutes, and-or let it sit in the bright sunlight for several house.

* The amount of cloud area or "cloud cover" over the Earth is about 67% on average and this is due to humidity (warm water vapor molecules) rising upward in the more dense, cooler air and then condensing enough to form clouds.

* The amount of cloud cover over the oceans is about 90% on average.
Due to this there is a great amount of desalinated water vapor in the air there, but warmer air and water temperatures, and sunlight (ie., without clouds), greatly helps the process of evaporation.

* The amount of cloud cover over land is about 70% on average.

* The average humidity over the oceans is about 80% near the surface.

* The average humidity over the surface of the entire Earth is about 50%, and with nearly all of it being less than three miles high.

* The humidity level is lower at higher altitudes since there is less air and cooler, and therefore, less water vapor which is actually denser than the air gasses and tends to be at lower elevations.

* Higher latitudes are cooler in temperature due to the tilt of the horizontal surface of the Earth in those locations, and the reduced solar thermal radiation due to having a longer path through the atmosphere (ie., air gas).

* The warmer or higher in temperature ocean water is, the more thermal energy it will have, and the easier it can then evaporate upward into the as water vapor molecules, and the more humid the air area above the ocean level will be. Warmer air can hold more moisture or water vapor. As the air gets cooler, the water vapor (water molecules having less mass than both oxygen and nitrogen) starts to condense and fall out of the air due to gravity.

* When it is said that the air is saturated , it means it cannot hold any more moisture.

* The average amount of water vapor or moisture in the air near Earth's surface is 0.25% of the mass of the air. Water vapor molecules in the air also affects the local air pressure. The STP **d**ensity of air is about 1.24g / L depending upon the temperature, pressure, and moisture (humidity). density = **p** = m/V, and m = pV , (1.224g air / 1L) (1L) = 1.24g of mass of air in 1L of air. (mass of air in 1L) / (mass of water vapor in 1L) =~ 1.24g / 0.02g = 0.62g / 0.01 g = 62. Therefore, in theory, 62 L of air will contain about 1 g of water vapor or moisture. Note that on a cool day, it can still be foggy or misty with visible, condensed humidity, and therefore, this air can be denser than that without that amount of humidity.

* The ratio of actual amount of water vapor in the air to the maximum amount it can possibly be at a given

temperature is called the **relative humidity**. When the relative humidity is at 100%, the air is said to be saturated with water vapor or humidity, and it cannot hold any more and be higher, and the temperature when this happens is called the dew point (temperature) and this actually happens at a cooler temperature where the water moisture or vapor in the air has lost much of its kinetic energy and is condensing together into (larger, visible) dew or rain drops.

SQUARED VALUES EXAMPLE

Find an algebraic expression for the value of the next squared integer from any given squared integer, and the expression for their difference.

If we can let (x) algebraically represent any integer, then (x + 1) represents the next integer in terms of that (previous) integer (x). If we can find an algebraic formula for the difference between their squares, we can add this difference to any squared integer value so as to find the value of the next squared integer.

$(x + 1)^2 - x^2 = D$: D = difference , Extending the first term (into a trinomial):

$(x + 1)(x + 1) - x^2 = D$ distributing:
$x^2 + x + x + 1 - x^2 = D$ combining like terms:
$x^2 + 2x + 1 - x^2 = D$ combining like terms and switching sides:

$D = 2x + 1$: difference between consecutive square integers
where x is the first integer to be squared. Then:

$(x + 1)^2 = x^2 + D$: from the first equation above, and with substitution for D:
$(x + 1)^2 = x^2 + 2x + 1$: the formula we seeked; the value of the next squared integer

Notice that this difference is not a constant since it depends upon the value of (x) used. Since this difference directly related to the specific value of (x), and the higher (x) is, the greater the difference (D) between consecutive squares.

For the sake of algebraic notation, we can also let Xn be any integer and Xn+1 be the next consecutive integer in the "chain of values or terms" for this specific system, then:

$(Xn+1)^2 - (Xn)^2 = D$ therefore, algebraically:

$(Xn+1)^2 = (Xn)^2 + D$ or:

$(Xn+1)^2 = (Xn)^2 + 2Xn + 1$: formula for the value of the next consecutive squared integer

Ex. Find the value of the next squared integer after 4.

$(4+1)^2 = 4^2 + 2(4) + 1$: here, x=4, and (x+1)=5 , 5 is the next integer after 4
$5^2 = 4^2 + 2(4) + 1$
$5^2 = 16 + 8 + 1$
$25 = 25$: checks

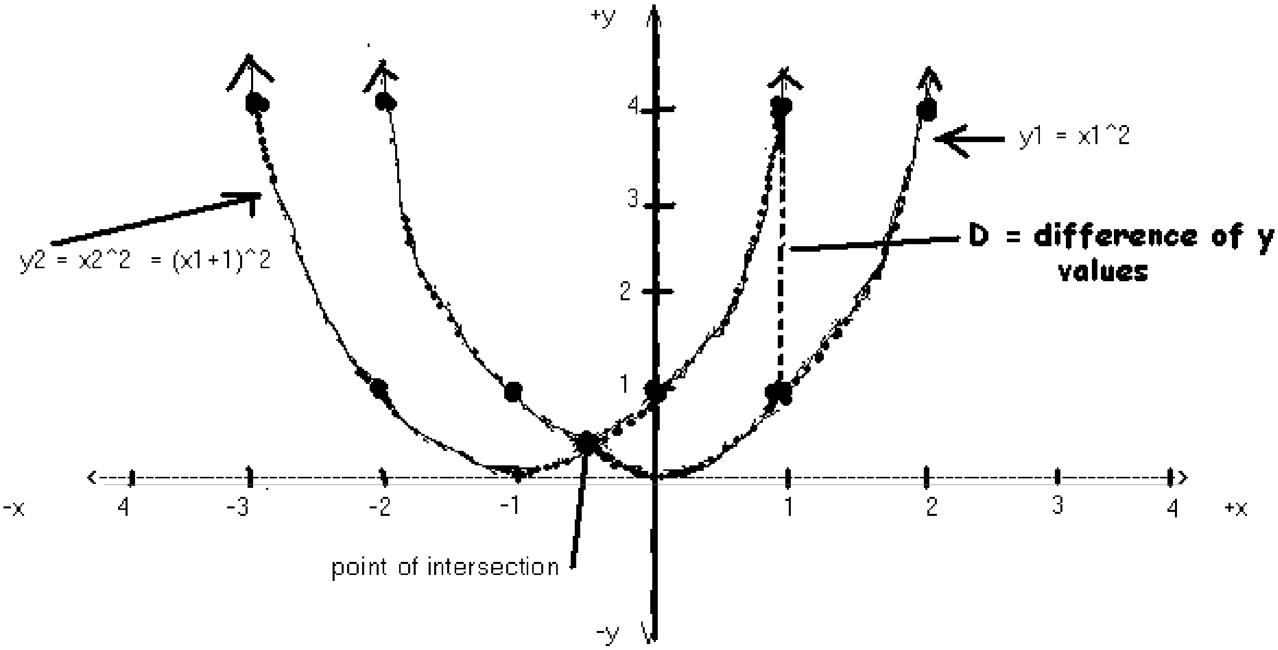
The curves of $y = x^2$, and $y = (x + 1)^2 = x^2 + 2x + 1$ are identical in shape except that the $y = (x+1)^2$ curve is effectively shifted or offset leftward ("has a left shift") by 1 along the x-axis. Consider this example data:

x = 0 1 2 3 4 . . . (and so on)

y = x^2 0 1 4 9 16 . . .

y = (x + 1)^2 1 4 9 16 25 . . . : notice the same values as the previous set of values

Here is a graph of the two curves: [FIG 59]



As an extra note; clearly, the curves cross or intersect at some point. The difference between their x values (and y values) is 0 since the x values are identical at this point. You can find this (x) value by setting the y values equal, that is, set the two equations as equal:

y = y1 = y2 : at the point of intersection, the values are equal. Using substitution:
x1^2 = (x1 + 1)^2 extending the right side:
x1^2 = x1^2 + 2x1 + 1 transposing the left side:
0 = 2x1 + 1 : essentially setting the difference D = (y2 - y1) = 0.
After solving for x1 (or the general x variable at the point of intersection):

x = x1 = x2 = $\frac{-1}{2}$ = - 0.5 : the x coordinate of the point of intersection

Even though -0.5 is not an integer number, it goes to show that it is possible that the expressions of x^2 and (x+1)^2 can and do have the same values at one point. The y value of this intersection point can be found by the substitution of this value of x into the any or both of the original equations:

y = y1 = x1^2 = x^2 = -0.5^2 = 0.25 = y2 = (x1 + 1)^2 = (-0.5 + 1)^2 = (0.5)^2 = 0.5^2 = 0.25

The point of intersection is therefore: p(x, y) = p(-0.5, 0.25)

Now, let's discuss the value of the difference (D) a bit more.

The difference between $(x + 1)^2$ and x^2 is:

$(x+1)^2 - x^2 = 2x + 1$ substituting a value for x:

Ex. let x = 2: $(2+1)^2 - 2^2 = 2(2) + 1 = 4 + 1 = 5$

The difference between $(x + 2)^2$ and $(x + 1)^2$ is:

$$(x + 2)^2 - (x + 1)^2 = 2x + 3$$

Ex. let x=2: $(2 + 2)^2 - (2 + 1)^2 = 2(2) + 3 = 7$

The difference between $(x + 3)^2$ and $(x + 2)^2$ is:

$$(x + 3)^2 - (x + 2)^2 = 2x + 5$$

Ex. let x=2 $(2 + 3)^2 - (2 + 2)^2 = 2(2) + 5 = 9$

Now note that the change in the differences between two consecutive squared values:

$(2x + 3) - (2x + 1) = 2x + 3 -2x -1 = 2$
$(2x + 5) - (2x + 3) = 2x + 5 -2x -3 = 2$

We see that when (x) changes by 1, that the difference (from the previous difference) changes or increases by 2, that is, 2 is a constant in the relationship between any two consecutive differences. You could say that it is the difference or change between any two consecutive differences, or simply as: "the difference in the differences". Also, even though the difference of (2x + 1) between two squared successive integers is a fixed or constant expression that doesn't change, the actual numeric value of this difference or expression is not a constant since it depends upon the value of (x) used, as was already previously noted. The higher (x) is, the greater the differences will be, but it will only be a value of 2 greater than the previous difference.

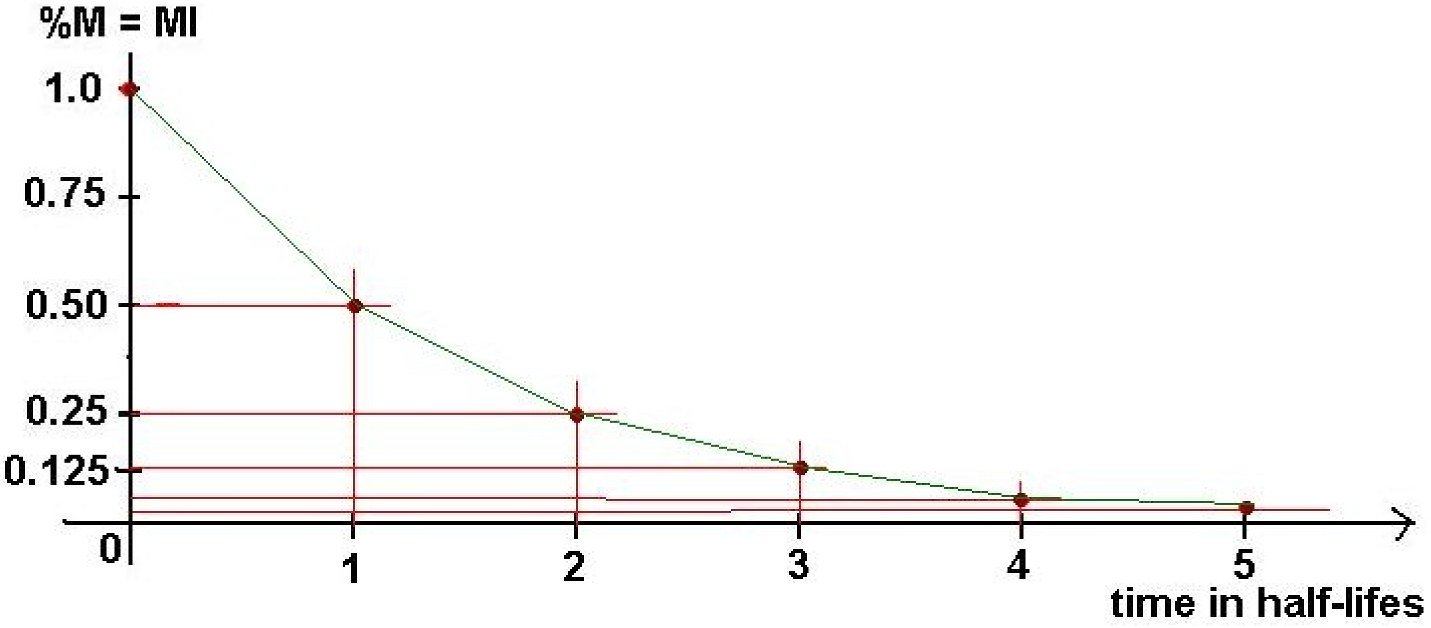
In more advanced mathematics, the expression for the value of the difference (D) as (x) increases or changes by 1 is defined as the rate of change of-or in (y) with respect to a change in (x). It (here D) is an expression of how (y) will change when (x) changes by (1). As with lines, this rate is commonly known as the slope of the curve at that point. This concept of the "instantaneous slope" (rate of change [of the dependent variable with respect to the independent variable] at a specific point only) or "derivative" is directly related to the concept of slope. The expression for D above is often formally called the (first) derivative. The change in this difference (D) as (x) increases or changes by 1 is formally defined as the slope of the slope, that is, it is the rate of change of the slope value. This value is therefore a derivative of a derivative, and it is therefore often called the second derivative. For the last example, the second derivative was a constant value of 2. In perhaps more simple wording: The slope of an equation is the rate of change of one variable (say y) with respect to the other (say x). Given different equations to start with, this slope, or derivative, value may not always be a constant or specific numerical value, and hence it changes or varies since it could be an equation of which includes a variable (such as x). The rate of change of this slope or (first) derivative with respect to the other variable (ie here, the independent variable = x) is called the second derivative. It is even possible to have further derivatives of derivatives, but once a derivative is a constant numerical value, instead of an equation, there will be no further derivatives possible since the rate of change of a constant is nonexistent or 0 since a constant value does not change in value. For the example above, it could be said that the rate of change of D with respect to (x) is 2.

$\frac{\text{change of y}}{\text{change in x}} = D = 2x + 1$: a rate of changes, and this value is how the dependent variable (y) will change with respect or in reference to the independent variable (x) for the given equation. This equation is the first derivative.

$\frac{\text{change of D}}{\text{change in x}} = 2$: this is due to and signifies that the difference, D=2x+1, is not a constant and changes by 2 for each increment of (x). That is, the difference is also changing in value, and for this example, is by a constant, steady value and rate of 2. This is a "derivative of the derivative", hence the second derivative.

HALF-LIFE EXAMPLE

Half-life is the period (time to complete a repeating cycle or event) it takes a decaying radioactive ("giving off", releasing, radiating energy) mass to effectively equal one-half (1/2) of it's original or "starting" radioactive "mass" and-or measured value. Mass is essentially the quantity or amount of the substance (ie. it is number of atoms which occupies a specific volume [a three-dimensional measure of space]). Remember, mass is not the same as the weight (a force) of something, but a mass under the influence of-and being constantly accelerated by gravity (a force) will produce a weight (force). The same mass or amount of matter on one planet will weigh differently on another planet where gravity is stronger or weaker than that of Earths gravity. A good example of the use of the concepts of half-life is in the dating old objects such as by using a process called "carbon-14 dating" which is also called "radiocarbon dating". This process is credited to **Willard Libby** in 1949. For example, a reference mass of 1 gram of a certain type of living tree contains a certain mass of carbon-14 while it is living and absorbed. Carbon-14 is produced high in the thin atmosphere after **cosmic rays** (usually highh speed protons and nuclei (nucleus, and-or particles) of atoms, from stars) have struck the air gas atoms and caused some free neutrons which might collide into a nitrogen-14 atom and creating a (unstable, slightly radioactive) carbon-14 atom of which then combines with oxygen to form (slightly radioactive) carbon dioxide which plants absorb or "breath" for food. After a tree or animal dies, the carbon-14 in it will start to decay (decrease in amount and radioactivity) back to nitrogen-14. To determine how long ago a similar tree died (which is roughly how old the artifact is, give or take 20 years for a large animal, or 200 years for a typical tree), the carbon-14 mass remaining in a size equivalent to the reference size (one gram for this example) is determined from the artifact (a tree or piece of wood in this example), and that value is then used in an algebraic formula to determine how old it is. Carbon-14 dating is practical for objects as old as about 50,000 years. [FIG 59A]



Let: Mo = Original or starting radioactive mass of the decaying substance. %M is shown in decimal value in the figure.
MI = The remaining radioactive mass of the decaying substance left after a period of time (such as the half- life time). "Mass left". The higher MI is, the higher the emitted radiation and decay.
P = Half-life time, or time the period (length of time) of the half-life of a certain radioactive, decaying material
N = (Number of the) Period in question.

T = Time elapsed.

Clearly, the total time elapse (T) is equal to the time of each period (P) times the number (N) of periods gone by. That is:

T = 1 period of time + 1 period of time + 1 period of time . . .
Since all terms are the same, and there are N terms, this repeated addition can be expressed with multiplication:

T = N x period of time = NP , and:

$T = NP$: 0P ------------ 1P ------------ 2P ------------- 3P ------------ 4P ---- . . . (and so on)

$$Ml = Mo \qquad \frac{Mo}{2} \qquad \frac{(Mo)}{\frac{(2)}{\frac{2}{1}}} = \frac{Mo}{2^2} \qquad \frac{Mo}{2^3} \qquad \frac{Mo}{2^4} = Mo\ 2^{(-4)}$$

: an inverse or reciprocal exponential decay

Hence, to find the value of Ml after the next period of time, the value of Ml is divided by 2, and we find that the denominator to Mo is equal to an indicated power with a base of 2 which has an exponent that corresponds to, and is equal to the period in question, that is, the denominator after the end of each period in question (N) and-or start of the next period of time is: (2^N).

from: $T = NP$, $N = \frac{T}{P}$

hence: $2^N = 2^{(T/P)} = 2^{(\text{total Time elapsed / time of Period})}$: for this example, time of period = half-life time

By using this expression for N, we can account for the mass left at all times, even when the time is in-between periods (ie. portions or fractions of a period gone by). Expressing all this data into an algebraic formula for the mass left after any period or amount of elapsed time in question:

$$Ml = \frac{Mo}{2^N} = \frac{Mo}{2^{(T/P)}} \quad \text{or} = \quad Mo\ 2^{-(T/P)}$$

: **MASS LEFT FORMULA**, , and here, of the Carbon-14
(for half-life, or 50% decay rates per period)
T=total time elapsed, P=period time of the half-life
Mass lost and-or decayed = Md = (Mo - Ml)

Solving for T which is the time elapsed or age of the artifact:

$Ml = \frac{Mo}{2^{(T/P)}}$ clearing fractions and ridding non-variable (T) terms to one side:

$2^{(T/P)} = \frac{Mo}{Ml}$ taking the log of each side to isolate (solve) for the exponent:

$\log 2^{(T/P)} = \log \frac{(Mo)}{(Ml)}$ using the log exponential rule:

$\frac{T}{P} \log 2 = \log (Mo/Ml)$ dividing each side by log 2, and multiplying each side by P:

$T = P \frac{\log (Mo/Ml)}{\log 2}$: TIME ELAPESED OR "AGE" FORMULA
(Use a consistent (ex. e, or 10) log base for both the dividend and divisor)

Also, solving for P by multiplying each side by $\underline{\ \ \log 2\ \ }$ and canceling:

log (Mo/Ml)

$$P = T \frac{\log 2}{\log (Mo/Ml)}$$: PERIOD OF HALF-LIFE FORMULA

Ex. You started with 1 gram of some substance that decays naturally, and after 70 days the amount of the substance remaining was 0.825 grams. How long (total) will it take till 0.750 grams (ie. three-quarters of a gram = 3/4 grams) remains?

First calculate the half-life period. "Natural-Logarithms" (have base of (e) = a constant value of about 2.71828)) will optionally be used, however you can use any base, perhaps 10, as long as you use it consistently.

$$P = T \frac{\ln 2}{\ln \left(\frac{Mo}{Ml} \right)} = \frac{70 \text{ days } (0.69314718)}{\ln \left(\frac{1 \text{ gram}}{0.835 \text{grams}} \right)} = \frac{48.52030264 \text{ days}}{0.180323554} = 269.07 \text{ days}$$: period of "half-life"

You could have initially converted 70 days to its' equivalent number of years so as to have the period expressed with time units that is also years. You can also convert 269.07 days to its equivalent number of years. Since P is less than 1 year, it is perhaps better to express it, and other calculations using it, in terms of days rather than awkward fractions of a year.

Now using this period value to find the time till 0.750 grams remains:

$$T = P \frac{\ln (Mo/Ml)}{\ln 2} = \frac{(269.07 \text{ days}) \ln (1.0g/0.750g)}{0.69314718} = \frac{(269.07 \text{ days}) \ln (1.333333333)}{0.69314718}$$

$$T = \frac{269.07 \text{ days } (0.287682072)}{0.69314718}$$

$$T = 111.67 \text{ days}$$

Given 1 gram of this particular substance, it will take 111.67 days till 0.750 grams of it remains. Likewise, it will take this long for (1g - 0.750g) = 0.250 grams of it to disappear or change to another state or substance.

Frequently, the period of half-life of a substance, when not being calculated, is found from a "look-up" or reference table. For example, the **half-life of carbon-14 is noted as being about 5,730 years**.

If you like, you can use "relative" fractional or percentage values for Mo and Ml. To do so, place these percent values in numerical (decimal) form. Usually, the original mass (Mo), or any mass considered as the original or "starting" mass of the system (where T is considered as 0), is considered as 100% since it has not decayed yet, and therefore
Mo = (100/100) = 1.0 will be used in the formulas. For example, if the mass left over (Ml) is 50%, this will be numerically represented as Ml = 50% = fifty parts per hindered parts = (50/100) = 0.50 in the formulas. The mathematical reason that these percent or fractional values can be used is that the ratio of the masses is also equivalent to the ratio of their corresponding percentages.

Below, another method of calculating half-life is derived, and some more generalized formulas will be developed.

After one period (T) of half-life time, the ratio of the mass or substance left (Ml) to that of the original mass (Mo) is always found to be 50% (or 0.50) in half-life situations. Since mass left or remaining is in question, this value could be called the percent remaining after each period of time.

$\frac{Ml}{Mo}$ = PR = percent remaining (PR) after any one period. Mo = mass at start of the period in question, Ml=mass at end of the period in question.

Algebraically: $M_l = M_o(PR)$

Regardless of the period in question chosen to find PR, PR will always be a constant as long as the physical decay rate is always a constant value.

	Period Number =	0	1	2	3	. . .
	Ml =	Mo	Mo(PR)	(Mo(PR))(PR)	((Mo(PR))(PR))(PR)	. . . these can be simplified to:
OR:	Ml =	Mo	Mo(PR)^1	Mo(PR)^2	Mo(PR)^3	. . .

Notice that Mo and PR are common factors of the remaining mass left (Ml) at the end of each period. Also notice that the exponent of PR is equal to the number of the period (N) in question. Expressing this data into a general algebraic formula:

Ml = Mo(PR)^N or= $Mo\left[\frac{Ml}{Mo}\right]$^N :Mass Left Formula
(PR corresponds to the specific decay rate used, and what remains)

Notice that the physical decayed or lost value (a change or difference = mass at start of period - mass at end of period) becomes smaller and smaller as the periods increase since the same (constant) percentage (PR), when used a decimal value, is multiplied to a smaller value (the mass starting each new period = Ml). Overall, in decay situations, PR is less than 1, and any value less than 1, a proper fraction, raised to a power greater than 1, or multiplied by itself, is smaller than itself. A fraction of a fraction results in a smaller value. Therefore, in the first few periods, a relatively large decay (loss of the substance in question) will be produced, and as the periods increase, the actual physical decay values will be less and less and it will eventually appear as a very slow, prolonged or endless decay where there is always some minimal or minuscule value still left (Ml).

The actual value of the decay between the starting mass and the mass at the end of a period in question can be calculated as:

Total Decay = Mo - Ml : where Mo is mass at the start of the system, and Ml becomes the mass at the end of the period in question.

The actual value of the decay between consecutive periods can be calculated as:

Period Decay = Mo - Ml : where Mo or Mn is mass at the start of the period in question, and Ml becomes the mass at the end of the period in question (or= at the start of the next period). Perhaps, a better notation could be:
Period Decay = $(M_n) - (M_{n+1})$: n=N , the period in question

Ex. For half-life study, the ratio of Ml to Mo for each period is: Ml/Mo = PR = 0.50, and if Mo= 100g (g = grams).

Ml = Mo(PR)^N
Ml = 100g (0.5)^N

Period Number =	0	1	2	3	4	. . .
Ml =	100g	50g	25g	12.5g	6.25g	. . .
Decay (g) =	0g	50g	25g	12.5g	6.25g	. . .

(Note, only when PR = 0.50, the decay of the (last) period of time equals the mass left = Ml.)

Observe that the value of the decay is decreasing as time increases, but still, the decay rate is the same constant value of 0.50

Solving for the period (N) in question:

$Ml = Mo(PR)^N$ dividing both sides by Mo:

$\frac{Ml}{Mo} = (PR)^N$ taking the log of both sides:

$\log\left(\frac{Ml}{Mo}\right) = \log (PR)^N$ using the log (of a) power rule:

$\log\left(\frac{Ml}{Mo}\right) = N \log (PR)$ dividing each side by log (PR) and switching sides:

$N = \frac{\log (Ml/Mo)}{\log (PR)}$: PERIOD NUMBER
: this denominator is a constant for a given problem, for half-life problems, PR=0.50

Total time or age can be found using the expression: $T=NP$

Again, like the formula shown previously but now using different variable names or identifiers, the total time elapsed (t) is the product of the period time (now using T, instead of P) and the number (N) of the period in question.

$t = TN$: t = Total Time Elapsed Formula
(using either upper and lower case letter (t) for the time values)

Therefore, mathematically:

$N = \frac{t}{T} = \frac{\text{time elapsed}}{\text{time of period}}$: Determining The Period In Question Formula

The (time) units of measurement of (t) and (T) must be the same, such as seconds, minutes, hours. days, or years.

since: $Ml = Mo(PR)^N$,we can derive a more generalized formula:

$Ml = Mo(PR)^{(t/T)}$: Mass Left Formula
PR = percent remaining = 50% = 0.50 for half-life problems.
t=total time, T=half-life period time, Mo=starting mass

Ex. If you began with 270g, (Mo), of substance, and after a period (T) of 3 years, 50% of it or 135g (Ml) remains. How much will you have after a total of 12 years of elapsed time (t).

In this example the percent remaining (PR) is given as 50%, it can be calculated from:

$PR = \frac{Ml}{Mo} = \frac{135g}{270g} = 0.50$: PR after a 3 year time period, and in its usable decimal form for calculations.
Since half of the substance is remaining after this length of time, this length of time, 3 years, is also the half-life time or period.

$Ml = Mo(PR)^{(t/T)}$

$Ml = 270g(0.50)^{(12yrs./3yrs.)}$

$Ml = 270g(0.50)^4$: the half life period is 3 years, and 4 of these (3 year) periods is 3 x 4 = 12 years

Ml = 270g(0.0625)

Ml = 16.875g or = 16g + 875(10^-3)g = 16g + 875mg

Note, the ratio (PR) of Ml to Mo per period can be greater than 1. This happens when there is a growth or gain such as an account that has gained interest, growth of living things, growth of crystals, etc.

INTEREST EXAMPLE

You may have a savings account which receives periodic interest (usually as a small percentage value) that is added to the account balance or amount. The amount of interest is based on, or in reference to, the current balance in the account. The current balance or amount in the account is the initial deposit plus all the interest already received. This concept is sometimes called "compound interest", but remember, the interest rate is a still a constant percent value, and that the interest money received per period is rather increasing. Compound interest means that you will receive interest based on both the current balance and any previous interest already added to the account. If you initially deposit $200.00 and the interest rate is 5.35% (0.0535 decimal) on the net or total amount in the account after each 3 month period, what is your accounts total worth after 1 year (12 months) has elapsed?

First, you may wish to review the previous example in this book called: Half-Life Example

We were given: Mo = $200.00 : we can consider variable M as a variable for money instead of mass.
Mo = original, initial or starting value of money

$N = \frac{t}{T} = \frac{12 \text{ months}}{3 \text{ months}} = 4$ N = number of intervals or period (T) durations during the complete time duration duration (t).

After the first period:

Ml = Mo + Interest : here, Ml = Mass (of money) left = Money left or current value
Ml = 200 + 200(0.0535) = $210.70

$PR = \frac{Ml}{Mo} = \frac{\$210.70}{\$200.00} = 1.0535$: which is essentially; PR = (1.0 + interest rate) in terms of relative values
PR = "Percent Remaining" of the original starting mass, quantity or value.

Ml = Mo(PR)^(t /T)

Ml = $200(1.0535)^(12months/3months)

Ml = $200(1.0535)^4

Ml = $246.36 : after 1 year

Checking by "long-hand" or manually:

N Elapsed	Interest (at 5.35%)	Net Account	
0		$200.00	$200 + (0.0535)($200) = $200 (1 + 0.535) = $200 (1.535) = $210.70 :
1	$10.70	$210.70	
2	$11.27	$221.97	
3	$11.88	$233.85	
4	$12.51	$246.36	:checks

Below is another way to mathematically express Ml:

$Ml = Mo + (\text{change of } M)$ or for this example above: $Ml = Mo + Mgained$

If the change is a decay or a loss, as with half-life problems, this can be algebraically expressed as negative in sign:

$$Ml = Mo + (-Mlossed)$$
$$Ml = Mo - Mlossed$$

Since PR = percent remaining, assigning a variable name of PC (Percent Changed) to the percent that was decayed or lost from the 100% value:

$$PC = \frac{Change}{\text{Initial value}} = \frac{Mo - Ml}{Mo} = \frac{Mlossed}{Mo} \quad \text{: (per period), therefore:}$$

(Note, for purposes other than this immediate discussion, if you want to keep PC with a sign indicating if the change was greater or less in value, use (Ml - Mo) for the numerator above.)

$$Ml = Mo - Mlossed$$
$$Ml = Mo - Mo(PC)$$

The sum of PR and PC is equal to 100% (or 1.0). This will now be verified:

$$PR + PC = 1.0$$

$$\frac{Ml}{Mo} + \frac{(Mo - Ml)}{Mo} = 1 \qquad \text{combining fractions:}$$

$$\frac{Ml + Mo - Ml}{Mo} = 1 \qquad \text{combining terms in the numerator:}$$

$$\frac{Mo}{Mo} = 1 \qquad \text{: checks}$$

From: $1 = PR + PC$: Here, 1 is the decimal equivalent of 100%, therefore:
$PC = 1 - PR$ and:
$PR = 1 - PC$

Ex. When $PR = 90\%$:
$PC = 1 - PR$
$PC = 100\% - 90\%$
$PC = 10\% = 0.10$

$Ml = Mo - Mo(PC)$ factoring out the HCF of Mo:

$Ml = Mo(1 - PC) = Mo(PR)$: Formulas For Money or Mass Left For A Single Period

Therefore, the formulas for the money (such as in an interest account) or mass left (such as of a decaying mass), can be expressed in terms of either the percent remaining or the percent changed.

From: $Ml = Mo(PR)^{(t/T)}$
$Ml = Mo(1-PC)^{(t/T)}$: Money or Mass Left Formulas (decays)

If the change was an increase, such as from a growth or interest account, this value is algebraically noted with a positive sign.

$Ml = Mo + (\text{change of } M)$
$Ml = Mo + (+\,Mgained)$
$Ml = Mo + Mgained$

When the ratio of Ml to Mo is greater than one, PR = percent remaining will be greater than one. PC will be a growth rather than a loss, and then PR = 1 + PC. This will now be verified since it may not obvious:

$PR = 1 + PC$

$PR = 1 + \frac{Mchanged}{Mo}$: when there is a growth, $Ml > Mo$, $PC > 0$, $PR > 1$

$PR = 1 + \frac{Ml - Mo}{Mo}$ combining fractions (LCD = Mo):

$PR = \frac{Mo}{Mo} + \frac{Ml - Mo}{Mo}$

$PR = \frac{Mo + Ml - Mo}{Mo}$ combining terms in the numerator:

$PR = \frac{Ml}{Mo}$: checks

$Ml = Mo + Mgained$ Since $Mgained = Mo(PC)$:

$Ml = Mo + Mo(PC)$ factoring out the HCF of Mo:

$Ml = Mo(1 + PC) = Mo(PR)$: Formulas for Mass Left After A Single Period

$Ml = Mo(PR)^{(t/T)} = Mo(1+PC)^{(t/T)}$: Mass Left Formulas (growths)

A more generalized formula which accounts for decaying (-) or growing (+) substances in terms of percent remaining (PR) or percent changed (PC) per period is:

$Ml = Mo(PR)^{(t/T)} = Mo(1 \pm PC)^{(t/T)}$: Mass Left Formulas (decays or growths)

However, we know algebraically, that we can use a more generalized formula as long as the proper sign for the value of PC is always accounted for:

$Ml = Mo(PR)^{(t/T)} = Mo(1 + PC)^{(t/T)}$: Mass Left Formulas (decays or growths)
(use proper sign for PC)

Ex. If a substance decays by 5% each period:

$PC = -5\% = -0.05$: assigned the negative sign since its a decay or loss
$PR = (1 + PC) = (1 + (-0.05)) = (1 - 0.05) = 0.95$:95%

Many times the "natural" constant of (e), about 2.718, is used in mathematical formulas for (natural and continuous) decays and growths. Here is a derivation where (e) can be used for half-life study:

At the end of the first (1) half-life period (T):

$$M1 = \frac{Mo}{2}$$

From the concepts of logarithms:

If e^x = 2, then: x = loge 2 = ln 2, therefore:

e^x = e^(ln 2) = 2 and:

$$M1 = \frac{Mo}{2} = \frac{Mo}{e^{(\ln 2)}} = Mo\, e^{(-\ln 2)}$$

: Often, a negative exponent indicates a decaying function or value rather than a growth. Here, e^(-ln 2) is actually a numerical coefficient (a multiplying factor to) of Mo. It will cause the value of Mo to decrease or decay.

By the end of the second half-life period:

$$M2 = \frac{M1}{2} = \frac{\frac{Mo}{2}}{2} = \frac{Mo}{2^2} =$$

$$\frac{Mo}{(2)(2)} = \frac{Mo}{(e^{\ln 2})(e^{\ln 2})} = \frac{Mo}{(e^{\ln 2})^2} = \frac{Mo}{e^{(2 \ln 2)}} = Mo\, e^{(-2 \ln 2)}$$

Notice that at the end of each half-life period in question (n), that the numerical coefficient of the exponent of (e) is the same as the period. Expressing this in a formula:

$$Mn = Moe^{(-n \ln 2)}$$

: Formula For Mass Left After A Decay , (expressed with e), where the decay is 50% for each period. (n) is the period number.

Here's how to express the formula for times in between periods, and hence, at any time in question:

Since total time (t) is the number of periods (n) gone by times the time-length or duration of each period (T), as shown previously:

$$t = nT \quad \text{therefore,} \quad n = \frac{t}{T}$$

Substituting this for (n) in the formula we get:

$$Ml = Mo\, e^{(-t/T \ \ln 2)} = Mo\, e^{(-0.69314718\ t/T)}$$

: Mass Left After A Decay Formula (for 50%, "half-life" decays)

Notice that the formula can be written using the associative law as:

$$Ml = Mo\, e^{(-t \ \ln 2/T)}$$

The value (ln 2/T) is a constant for the given problem, and therefore it only needs to be calculated only once. For example, if the half-life period is 10,000 years, perhaps for a half-life analysis using carbon-14 dating:

$$\frac{\ln 2}{T} = \frac{0.69314718}{10000} = 0.000069314$$

and the resultant formula is simply:

$$Ml = Mo\, e^{(-0.000069314\ t)}$$

Ex. In this example, a person gives you the following equation that looks similar to those just discussed, and you are asked if you can algebraically solve for T, hence create a formula for the mathematical relationship between T and all the other values.

$Vl = Vo\ e^{(-\,0.69314718\ t/T)}$ The equation can also be expressed as:

$$Vl = \frac{Vo}{e^{(0.69314718\ t/T)}}$$

solving for the time of each period: T
In this example T will be the time of the half-life or value.

$$e^{(0.69314718\ t/T)} = \frac{Vo}{Vl}$$

taking the natural log. of both sides:

$$\ln e^{(0.69314718\ t/T)} = \ln (Vo/Vl)$$

after using the log exponent rule, and some simplifying:

$$0.69314718\ t/T = \ln (Vo/Vl)$$

after multiplying each side by T, and then isolating T:

$$T = \frac{0.69314718\ t}{\ln (Vo/Vl)}$$

: T = the "half-life" period of time. This is the time taken or needed for Vl to be equal to half of Vo. T is a constant for the specific system.
t = the time taken for some change to take place, such as the time needed for Vo to change (here decay) to Vl.
Vo = the starting or initial value that will be decreasing or decaying naturally
Vl = the current value left after the natural decrease or decay of Vo

Ex. One of the most common formulas for a savings account where compound or compounding interest is added to an account is shown below. Compound interest is simply a method where the interest is calculated on the current total or net (accumulated, gathered) savings balance, and this balance includes any interest that was previously calculated and added to it. The interest rate value is usually a constant or fixed value, and is not "compounded" in value.

$$T = M (1 + I)^n$$

M = Starting deposit
T = Total = M + Total Interest
I = Interest rate per period
n = nth period, or nth interval of interest being added into the savings account balance,
This is the specific interest period or interval number in question.

Of course, for this formula to hold true, it is assumed that there are no withdrawals from the savings account. A derivation of this formula will now be presented:

$T0 = M$: the total at the start or at interval or period 0

$T1 = T0 + \text{Interest of } T0 = M + MI = M (1 + I)^1$: the total after the end of the first interval

$T2 = T1 + \text{Interest of } T1 = M (1 + I) + M (1 + I) I = M (1 + I) (1 + I) = M (1 + I)^2$

$T3 = T2 + \text{Interest of } T2 = M (1 + I)^2 + M (1 + I)^2 I = M (1 + I)^2 (1 + I) = M (1 + I)^3$

Clearly, the exponent of the (1 + I) factor corresponds to the interval of the savings. This pattern confirms the formula. Since total time, (t) = (Time of a period)(n periods), you can use this formula for (n):

$n = t / T$: t = total time, T = Period of the interest rate **:**

TOTAL = M (1 + **I**)^(t / T) : total savings in an account with compound interest of **I**
M=starting balance, t=total time elapsed, T=time between interest payments or payment interval time

Ex. If the initial deposit into a savings account was \$25, and with 5% compounded interest every 4 months, how much will the account grow to after 12 months of time?

Here, t = total time elapsed = 1 year = 12 months
T = period of calculating the interest rate = 4 months
TOTAL = M (1 + **I**)^(t/T)

TOTAL = \$25(1+ 0.05)^(12months/4months) = \$25(1.05)^3 = 25(1.15$\underline{8}$) = \$28.95

A gain of: (\$28.95 - \$25) = \$3.95

Next, a general discussion about paying interest on a loan will be included in the example given. Again, observe how algebraic formulas can be created.

LOAN EXAMPLE

A bank gave a person a $15,000 loan (Mo) to be paid in full by 1 year after 12 interval (N) base payments (B). With each base payment, the bank also charges you a 2% interest (I) on the remainder (Ml = money left to pay) of the loan not yet paid back. This is similar, in a reverse type of manner, to the bank paying you interest on the total amount you have in the bank.

Defining variables to be used in the algebraic formulas:

Mo = Original amount of the loan, and in this example, it is $15,000.
Ml = Amount of the loan left or remainder of the loan yet to be paid back. Obviously, after each base payment, the value of Ml is less, and therefore, the interest on that amount will be less.
I = Interest rate, use the decimal equivalent for the formulas, here, it is 0.02
B = Base payment which is the interval payment due on the loan. This is a constant value. This does not include the interest to be paid on the remainder (Ml) left to be paid.
P = Interval payment which is the sum of the base payment (b) and the interest payment (I). This value will be smaller for any next interval(s) since the interest payments will be less since the remainder of the loan (Ml) to be paid back is less.
N = Interval in question, here, the total intervals or number of payments is Nt = 12

The total you will eventually pay back to the bank is:

Total = Loan + Total Interest   expressing this algebraically:
$Mt = Mo + It$: Total Formula

The base payment of the loan is simply the total value of the loan divided by the total number of interval or base payments to be made.

$B = \frac{Mo}{Nt}$: Base Payment Formula, also, algebraically:

$Mo = NtB$: Loan Value Formula, substituting this into the Total Formula:

$Mt = NtB + It$: alternate Total Formula

The payment (P) due for each interval (letting n=N) in general terms is:

Payment = Base payment + Interest payment   and algebraically:
$P = B + In$

Each interest (I) payment at interval n=N is the remainder of the loan yet to be paid times the interest rate. That is,

In = (remainder of loan)(interest rate)
$In = (Ml)(I)$

Algebraically, here you may also note: interest rate = $\frac{In}{Ml\text{=remainder of loan at N intervals}}$

Interval Payment = Base payment + (remainder)(interest rate)   algebraically:

$P = B + MI\ I$: basic Payment Formula

The remainder or amount of the loan left to be paid (without regard to any interest) can be equated or expressed as the total amount of the loan less the amount already paid back:

From: $Mo = \text{loan paid back} + MI$
$MI = Mo - \text{loan paid back}$

Since the amount of loan paid back can be calculated in terms of the base payment (B) and the number of intervals (N) or times it was paid, payment (P) can be calculated in terms of an interval in question:

from: $\text{loan paid back} = (\text{base payment})(\text{intervals})$: $= Mo - MI$
$\text{loan paid back} = BN$: (without adjustment, see below)

When calculating the loan paid back at the start of a new interval, N must be reduced by 1. For example, starting when the loan was first taken out, this is considered interval 0 since no (0) payment or interest is yet due. When the first payment is due, interval 1 is entered and the next payment will be due at the end of this interval or start of interval 2. In other words, the intervals and-or time actually start at 0, and not 1. Therefore, when the first (1) payment or "interval 1's" payment is due, no amount of the loan has yet been paid back. To correct the loan paid back formula, an adjustment must be made, and that is to subtract 1 from N:

$\text{loan paid back} = B(N-1)$:loan paid back at the interval (N) in question

$MI = Mo - B(N-1)$:this will be substituted into the Payment Formula, but first:

Here is another derivation of the remainder (MI) of the loan to be paid:

Payment N	MI in which interest for a payment is to be calculated with.					
0						
1	Mo					: payment due at the start of the interval
2	$Mo - B$	=	$Mo - B$	=	$Mo-1B$	
3	$(Mo - B) - B$	=	$Mo - B - B$	=	$Mo-2B$	
4	$((Mo-B)-B) - B$	=	$Mo-B-B-B$	=	$Mo-3B$	
(and so on)						

Observe that for each interval or payment, B, the base payment, is to be subtracted [algebraically adding (-B) since the subtraction of a value really represents an addition, or combining with, of a reduction or loss, (ie. a negative value)] from the remaining part of the loan (MI). We also observe that this is essentially adding (-B) to Mo, (N-1) times. In short, the coefficient of the B term is 1 less than the interval in question. Expressing these facts algebraically for MI, we see that this checks with the equation previously shown.

$MI = Mo - (N-1)B$ or $MI = Mo - B(N-1)$: by using the commutative law

$P = B + \text{Interest}$
$P = B + MI\ I$
$Pn = B + (Mo - B(n-1))\ I$: n=Nth (interval) Payment Formula

Since $Mo = NtB$, we can get an alternate payment formula:

$Pn = B + (NtB - B(N-1))I$ factoring out the common factor of B:

$Pn = B + I(B(Nt - (N-1)))$
$Pn = B + BI(Nt - (N-1))$: alternate Nth Payment Formula

: Here, the value of the loan, Mo, need not even be known or used

Notice the factor BI in the last derivation is a constant value,therefore only needs to be calculated once for a given problem. This value of BI can be known as the "Base Interest" for or per each interval. To calculate the interest payment due at a certain interval payment, simply use that part (here, the second term) of the above expression that represents it:

$In = (Mo - B(N-1))I = BI(Nt - (N-1))$: n=N , Interest Payment N Formulas (*)

To find out the total amount that will be paid on the loan, the total interest must be calculated:

Total = Loan + Total Interest or in a more algebraic sense:
$Mt = Mo + It$: Total Formula, also, algebraically:

It = sum of interest payments for all the intervals

$$It = I(N=1) + I2 + I3 + I4 + \ldots + I(N=Nt)$$

$$It = MoI + (Mo-1B)I + (Mo-2B)I + (Mo-3B)I + \ldots + (Mo-(Nt-1)B)I$$

Distributing (I):

$$It = MoI + Mo\,I - BI + Mo\,I - 2BI + MoI - 3BI + \ldots + MoI - (Nt-1)BI$$ OR:

$$It = MoI + MoI - BI + MoI - 2BI + MoI - 3BI + \ldots + MoI-(Nt-1)BI$$ Factoring (I) from each term:

$$It = I(Mo+Mo-B+Mo-2B+Mo-3B + \ldots + Mo -(Nt-1)B)$$

Combining like terms in the multiplying factor of (I):

First, notice that there are Nt=12 like terms of Mo and that each numerical coefficient of Mo is one (1). Summing up the numerical coefficients of Mo is essentially repeated addition of one (1), twelve (12) times. The numerical coefficient of Mo in an algebraic sense is therefore:

$(1)(12) = (1)(Nt)$ or simply $Nt = 12$ Expressing this in the general algebraic formula:

$$It = I(NtMo -B -2B -3B - \ldots (Nt-1)B)$$

To make things easier, we will "divide through" the B terms by (-1) so that the coefficients of B will all be positive. A general format example will be shown first:

Given -B, dividing by -1:

$$\frac{-B}{-1} = +B$$ and therefore:

$-1(+B) = -B$:checks with the given value of: $-B$

This is basically the same as factoring (-1) from each variable B term:

$$It = I(NtMo -1(B +2B +3B + \ldots +(Nt-1)B)$$ or simply:

$$It = I(NtMo - (B +2B +3B + \ldots +(Nt-1)B)$$

We can combine the terms that contain B by summing up B's numerical coefficients by hand or we can use a general formula for the sum of integers from 1 to (Nt-1):

The general formula for the sum of the first N integers from 1 to N is:

$$\text{Sum} = \frac{N^2 + N}{2} \quad \text{or} \quad = 0.5N^2 + 0.5N = 0.5(N^2 + N)$$: **sum of the first N integers from 1 to N**

Here is a simple verification/derivation of the above formula:

Ex. We are to sum (S) up (from an initial sum of 0) the first 4 integers (1, 2, 3 and 4), hence letting N=4 in the above expression.

$$\begin{array}{rrl} & S & = 1 + 2 + 3 + 4 \\ + & \underline{S} & \underline{= 4 + 3 + 2 + 1} \\ & 2S & = 5 + 5 + 5 + 5 \end{array}$$

: this is N=4 terms, adding the sum to itself, we will have S + S = 2S:
: remember, addition is commutative, so the sum (here S=10) should still be the same regardless of order, and here, it's a descending order of values.

The right hand side can be represented with multiplication since multiplication is basically a repeated addition of the same value. Notice that each term is algebraically (N + 1) = (4 +1) = 5, and that there are N terms, and expressing this with multiplication:

$$2S = (N+1)(N)$$ solving for S:

$$S = \frac{(N+1)(N)}{2}$$ clearing grouping symbols by distributing N:

$$S = \frac{N^2 + N}{2}$$ substituting the given value of N:

$$S = \frac{4^2 + 4}{2} = \frac{16 + 4}{2} = \frac{20}{2} = 10 = 1 + 2 + 3 + 4$$

Checking:

$$\begin{aligned} 2S &= 5 + 5 + 5 + 5 \\ 2(10) &= 20 \\ 20 &= 20 \end{aligned}$$: checks

Extra: Here's the sum of the first 100 integers from 1 through 100:

$$S = \frac{100^2 + 100}{2} = \frac{10000 + 100}{2} = \frac{10100}{2} = 5050$$

For terms which are not necessarily integers, and-or don't have a difference between them of 1, a similar derivation and more generalized formula can be found in the study of mathematical progressions and series.

In the interest example, the highest coefficient of B is (Nt -1) = (12 - 1) = 11. The sum of the first eleven integers is:

$$\frac{11^2 + 11}{2} = \frac{121 + 11}{2} = \frac{132}{2} = 66$$

checking: 1+2+3+4+5+6+7+8+9+10+11 = 66

Substituting the algebraic expression for the coefficient of B into the total interest formula:

$It = I(NtMo - (B + 2B + 3B + . . . + (Nt-1)B)$

$It = I\left(NtMo - \left(\frac{(Nt-1)^2 + (Nt-1)}{2} \right) B \right)$: Total Interest Formula or:

$It = I\,(NtMo - ((Nt-1)^2 + (Nt-1))\, 0.5B)$: Total Interest Formula or:

$It = NtMoI - ((Nt-1)^2 + (Nt-1))\, 0.5BI$: Total Interest Formula

Substituting the arguments or actual values used in the initial problem given at the start of this discussion, and also the derived or calculated arguments, we can find the total interest that will be paid to the bank.

$It = (12)(\$15{,}000)(0.02) - ((12-1)^2 + (12-1))\,(0.5)(1{,}250)(0.02)$ we find:

$It = 1{,}950$ dollars $= \$1{,}950$

Manually checking this result will also give it a check, and give an algebraic verification to the total interest formula:

Payment Number	MI	I = (MI)(0.02)
1	$15,000	$300
2	$13,750	$275
3	$12,500	$250
4	$11,250	$225
5	$10,000	$200
6	$ 8,750	$175
7	$ 7,500	$150
8	$ 6,250	$125
9	$ 5,000	$100
10	$ 3,750	$ 75
11	$ 2,500	$ 50
12	$ 1,500	$ 25

$It = \$1{,}950$:checks

Notice that the interest payment for each interval is reduced by a constant equivalent to (BI). This value is seen in the (In) formula shown previously and is marked with an asterisk symbol (*). Therefore, each payment (P = B + I) will be reduced by (BI). In the example shown, the reduction in interest will be ($1,250)(0.02) = $25 for each payment. A certain bank might allow the average of the total interest charge to be added to each base payment so that all of your payments are the same value or constant.

$Iav = \frac{It}{Nt}$: Average Interest Payment Formula

For the example shown:

$Iav = \frac{\$1{,}950}{12} = \$162.50$

Therefore, the average payment will be:

$Pav = B + Iav$: A "Constant Payment", or "Average Payment" Formula

For the example shown:

Pav = $1,250 + 162.50 = $1,412.50

Since there are Nt average payments, the total you will have to pay back to the bank can be found by:

Total = Mt = Loan + Total Interest = NtPav : Total Payment Formulas

Mt = $15,000 + $1,950 = (12)($1,412.50) = $16,950

REFLECTION EXAMPLE

A certain type of mirror reflects 90% (or 0.90) of the incident (incoming) light. If a beam of light is reflected to and off of a series of five (5) of these mirrors, what percent of the original beam will be reflected off of the last (5th) mirror?

Letting Mo = Starting light beam value, quantity or intensity
Mn = Value, quantity, or intensity of the beam after reflecting off of the nth mirror.
I = Percentage of incident beam reflected off each mirror and available for the next mirror.

Here, $I = PR = \frac{Mn+1}{Mn} = 0.90$: 10% of the light is lossed due to absorption and scattering at the mirror (reflective surface).
PR = percent remaining, as used in a previous example

After reflecting off of the first mirror (n=1), the remaining light beam is:

$M1 = (Mo)I$

After reflecting off of the second mirror, the remaining light beam is only 90% of the first reflection:

$M2 = (M1)I = (MoI)I = MoI^2$

By observing the correlation (relationship, likeness, similarity) between the value of the stage (n) or interval (physically each mirror reflection) of the reflection series, and the exponent of (I), we can generalize this and express it into an equation:

$Mn = Mo\ I^n$

Since the numerical value of Mo was not given we can consider it the whole (numerically as 1) starting value, or 100% (= 100/100 = 1) of the original beam. This is sometimes referred to as doing calculations in percentage, portion, or relative terms. For this example, the ratio of Mn to Mo will always be the same for a given value of (n) since the loss is a "simple loss" and not compounded like interest.

$Mn = MoI^n$ for the 5th mirror:
$M5 = (1)(0.90^5)$
$M5 = 0.59049 = 59.049\%$

Checking: If Mo has a value of 34 units, the value of M5 is:

$M5 = (34 \text{ units})(0.90^5)$
$M5 = (34 \text{ units})(0.59049)$
$M5 = 20.07666 \text{ units}$ and:

$$\frac{M5}{Mo} = \frac{20.07666}{34.0} = 0.59049 = 59.049\%$$

Checking: If Mo only had a value of 15 units, the value of M5 is:

$M5 = (15 \text{ units})(0.90^5)$
$M5 = (15 \text{ units})(0.59049)$
$M5 = 8.85735 \text{ units}$ and:

$\underline{M5} = \underline{8.85735} = 0.59049 = 59.049\%$: the same ratio value, checks

Mo 15.0

From Mn = MoI^n algebraically:

$$I^n = \frac{Mn}{Mo}$$: here, I^n is the percentage of the initial or incoming light reflected off mirror n.

The problem could have been stated or interpreted that there is a 10% loss after each step or period, and an alternate method to solving problems, such as the above one, is given below, and it is similar to those already presented and will yield the same formula as above.

M1 = M0 - loss : Or in general, Mn+1 = Mn - loss
For problems with gains, use: M0 + gain , or M0 + interest

M1 = M0 - (0.10)M0 Factoring out the H.C.M.F = M0 from the right side terms:
M1 = M0 (1 - 0.10)
M1 = M0 (0.90)

M2 = M1 - loss
M2 = M1 - M1(0.10) using substitution:
M2 = M0(0.90) - M0(0.90)(0.10) : H.C.M.F. = M0(0.90)
M2 = M0(0.90) (1 - 0.10) : note, as shown previously, PR = (1 + PC), and here, PC = -0.10
M2 = M0(0.90)(0.90)
M2 = M0(0.90)^2

Hence, this leads to the general formula:

Mn = M0(0.90)^n : M0 = Mo , and n = mirror and-or reflection number, which here is a counting number

Extra: **This is the technical description of what a reflection is, and how a mirror functions.** A shiny, rust free, metal surface has many free electrons (unbound electrons, no longer orbiting the nucleus of an atom) on it since it is where the repulsion force by other electrons inside that metal is weakest, and until the surface cannot place any more there due to repulsion by the accumulated negative charge of the electrons there on that that surface. A metallic surface will appear to **reflect** an incident or incoming ray(s) of light, and at an angle equivalent to that it received. The reflection is actually due to the electrons absorbing the electromagnetic energy from the light and gaining kinetic energy, causing oscillations (ie., vibrate and move a short distance of movement) and releasing or re-radiating that energy as an electromagnet wave of the same frequency, here, particularly as visible light. The more electrons in motion in a particular direction, the more the electromagnetic (EM, electric and magnetic field, essentially photons of EM ("light") energy). Any energy not transmitted was due to some of the energy being converted to thermal energy and-or radiation and some of it being absorbed by the surface atoms of the metal. As you can imagine, light striking a non-metallic, non-crystalline surface is rather re-radiated in in a wider angle than that of the angle of incidence due to the greater probability of internal reflections of that material and-or its atom structure. Non-metals tend to absorb many wavelengths of light and apparently only re-radiate one specific color or frequency depending on the specific surface elements or coating (such as paint) of that light and at a reduced intensity in the viewers direction.

GROWTH OF ORGANISMS EXAMPLE

If a certain type of organism divides into two organisms after about every 20 minutes, how many organisms will be present after 4 hours if given just 1 organism to start with?

Let's work out the basic data first and then try to derive a formula from that data. First let each growth interval (n) have a period (T) of time of 20 minutes long. Also, for this analysis, we do not know the average life span of the organism, so therefore, we will not consider any possible organisms that have died into the equations. The analysis is therefore more correctly for how many organisms have been born.

The basic formula between successive intervals or periods is:

organisms at n+1 = (organisms at n) (2) : after each interval, the number of organisms is twice the previous interval

n	0	1	2	3	. . .	
organisms	1	1 x 2	(1 x 2) x 2	((1 x 2) x 2) x 2	. . .	which could be expressed as:
organisms	2^0	2^1	2^2	2^3	. . .	

Mathematically, we see that the number of organisms is actually a power of 2 whose exponent is equal to the interval or period number. Expressing this we have:

organisms n = 2^n : basic growth formula in terms of the interval elapsed

Since each interval is a 20 minute long time period (T), the total time (t) for this growth process can be expressed as the number of intervals times the time of each interval:

$t = n T$ therefore, algebraically:

$n = \frac{t}{T}$ expressing this into the basic growth formula we have:

organisms n = 2^(t/T) : basic growth formula in terms of time

Substituting the values given we have:

organisms = 2^(4 hr./20 min.)

Since the units of time must be the same here, we must convert one to the other. We will convert 4 hours to an equivalent time of minutes:

$\frac{60 \text{ min.}}{1 \text{ hr.}}$ as= $\frac{x \text{ min.}}{4 \text{ hr.}}$ after solving for x min. :

$$x \text{ min.} = \frac{(4 \text{ hr.})(60 \text{ min.})}{1 \text{ hr.}} = 4\ (\ 60 \text{ min. }) = 240 \text{ minutes}$$

organisms = 2^(240 min. / 20 min.) = 2^12 = 4096

If you want to express the formula for the number of organisms using (e), you can convert the value of 2 to its' equivalent

power value of (e):
Let 2 = e^x solving for x by taking the log of both sides:

log 2 = log e^x : arbitrarily using the common log base = 10, though any base such as (e) could also be used

log 2 = x log e

$$x = \frac{\log 2}{\log e} = \frac{0.30102995}{0.434294481} = 0.69314718$$

Therefore, e^0.69314718 = 2 , and substituting this into the formula:

organisms = (e^0.69314718)^(t/T) = e^(0.69314718 t/T)

MUSIC EXAMPLE (FREQUENCIES AND FRET SIZES)

This is an advanced example, and the reader may skip over this topic if uninterested, or till another time. The author has previously experimented in the field of lutherie (lute or string instruments) and has some helpful concepts to contribute and share in terms of math. These concepts presented are also useful for anyone interested in sound and music study. In modern times, the fundamental music scale is normally divided up into 12 acceptable sounds or tones or "notes", and the 13th note, or the first note of the next higher pitch (sounding or frequency) scale, is called an octave note since it is similar in sound to the first note of the scale but twice its frequency or pitch. The higher the frequency, the higher it sounds. This fundamental music scale is often called the chromatic music scale. In ancient times the (chromatic or "stepped") musical scale only had 8 successive notes, and today, the range from a frequency to double that frequency is still called an octave, and for example, there is 8 tones or notes in any "major scale". The mathematical relationship between successive notes in the chromatic scale is a geometric relationship. With this type of relationship, there is a constant ratio between each successive frequency value of the notes. Therefore, this ratio value can be used as a constant multiplier to find the frequency of any next note in the chromatic or 12-note, divided "octave" scale. Let's give an algebraic identifier name or variable of (R) to this ratio value. In this example we are now going to find many things such as the actual value of the successive note or frequency ratio of R.

Some other important facts:

Any chromatic scale of 12 notes is usually said to contain 7 pleasurable sounding notes in reference to the first, fundamental or "root" note of that chromatic scale. The first note, for example "C", chosen also defines the name of the entire scale that is based on that note. Each successive note or sound in the chromatic scale increases in "pitch", that is, it is higher in frequency. Frequency is a measure of how frequently or often the pulses or waves of sound are. The units for frequency are cycles or vibrations (pulses) per second. This composite unit is sometimes called hertz (hz = cps = cycles/second), and which is the name of the scientist who first described these things.

If each note position was represented as a point on a line (or actually on a vibrating, sound producing string of a "stringed instrument" such as a lute [origin of the word "luther" for lute maker], violin, or guitar), each successive next note or point will be a less distance apart. This is due to the physical relationship of vibrating structures, where the relationship of physical size (such as lengths, and volumes) and resulting natural frequency is an inverse relationship. A small bell or tuning-fork produces a high pitch or frequency sound. A big bell or tuning-fork produces a low pitch or frequency sound. In a pseudo equation, this mathematical relationship could be noted as: (frequency or pitch) = 1 / (size or length)

$R = \frac{\text{next note}}{\text{note}}$: using frequencies of the notes, therefore:

next note = note (R) or=: note1 (R) = note2

Given an initial note or frequency (F1), the frequency of higher pitched notes of the scale can be calculated as:

F1 : 1st note
F2 = F1(R) = F1R^1 : 2nd note
F3 = F2(R) = (F1R^1)R = F1R^2 :3rd note
F4 = F3(R) = (F2R)(R) = (F1R^2)(R) = F1R^3 :4th note

Notice the correlation of the indicated power of R to the n_{th} note in question, and the indicated power of R is one less. Writing an algebraic formula to express this:

frequency of note n = (previous notes frequency)(R) = (first notes frequency)(R^(n-1))

Fn = F(n-1)R = F1R^(n-1) : Frequency of the nth note of a common (chromatic, or 12 steps or notes) musical scale

At the 13th note, or octave note, the frequency is always twice the first note:

$$2F1 = F1R^{(13-1)} = F1R^{12}$$

Now, R can be completely solved for since the F variable can be eliminated. After dividing both sides of the equation by F1, and switching sides:

$$R^{12} = 2$$

Here, we can use the concepts of logarithms to solve for the base (here R) of the (indicated) power. Taking the logarithm of both sides of the equation using any base you choose:

$\text{Log } R^{12} = \text{Log } 2$ Using the "log power rule":

$12 \text{ Log } R = \text{Log } 2$

$\text{Log } R = \frac{\text{Log } 2}{12}$ Hence, by the definition of logarithms:

$R = 10^{((\text{Log } 2)/12)}$ again, using a log base of 10:

$R = 10^{(0.301029995/12)}$

$R = 10^{0.025085832}$

$R = 1.059463094$: (R), Ratio Of The Frequencies Of Successive Notes (of the common 12 note, "chromatic" musical scale)

Checking: If F1=1000hz, the 13th note, or "octave note", should be twice this frequency or 2000hz:

$F13 = F1R^{12} = 1000\text{hz}(1.059463094^{12}) = 1000\text{hz}(2) = 2000\text{hz}$: checks

Given a frequency of 500hz, the frequency of the next (or second "step" in the chromatic scale) note is:

Since: $R = \frac{\text{next note}}{\text{note}} = \frac{Fn+1}{Fn}$ algebraically:

$$Fn+1 = FnR^{(n-1)} = FnR^{(2-1)} = FnR^{1}$$

$F = Fn+1 = Fn(1.059463094)$

$F = 500\text{hz}(1.059463094)$

$F = 529.7\text{hz}$: (roughly 530 hz)

Given a scale which spans one complete octave, rather than speak in terms of frequencies for the notes of a scale, musicians speak in terms of the symbols and steps such as that given for each successive step or note of the chromatic scale. The successive notes are usually given a text character symbol with names (from A to G) of:

1 , 2 , 3 , 4 , 5 , 6 , 7 , 8 , 9, 10 ,11 , 12
A, A#, B, C, C#, D, D#, E, F, F#, G, G#, A (A = the 13th note in the chromatic scale, or the octave note of an 8 note modern scale)

(# = "sharp" , this is a note raised or increased to the next note (typically called a half-tone or half-step) of a whole entire step such as from A to B.)

Note that the notes are identified with the first letters of the alphabet and that there is no note identified as "H" or by any other letter. Further notes are identified by the same repeating or periodic series of letters since these higher pitched notes (in a next higher octave range or "pitch") sound similar due to that their corresponding frequencies are a multiple (often 2, twice, etc.) of the frequencies corresponding to those notes.

The first note of a certain defined and pleasing scale composed of selected notes from the chromatic scale can be any one of the above notes, for example. If the initial or "root" note of the scale is G, the 12 notes of the G (chromatic) scale in order are:

G, G#, A, A#, B, C, C#, D, D#, E, F, F#, G (the 13th note a the scale = first note of the next higher pitched scale)

As mentioned previously, these twelve notes, semi-tones or half-steps are formally called the notes of a chromatic scale. It is possible to create a scale of either less or an infinite number of notes. The chromatic scale selects the more humanly discernible tones, and is therefore more manageable and reproducible by other musicians. Usually, other formally defined and more pleasing to hear scales contain only some of these notes of the chromatic scale. These notes can be played and heard one after the other in any desired timing and order, and including at the same time with any other of the pleasing notes in that scale. Playing two or more notes at once is called a chord of notes, or simply a chord. For example, the Major scale that is used in a modern or "Western" sounding, or type of music, consists of these 7 selected notes or steps from the 12-note chromatic scale:

1, 3, 5, 6, 8, 10, 12, 13 : with the 8th note (being the 13th note of the chromatic scale) of this scale being the octave, these 8 notes formally comprise an octave (oct is a prefix meaning "eight") range of notes or tones

For example, The C-Major scale is:

C, D, E, F, G, A, B, C : This is typically called the "key" (octave, or scale) of C-Major.
1, 2, 3, 4, 5, 6, 7, 8 : Corresponding tone or note numbers of the 8 note or octave scale, and these numbers help someone to easily reference and "transpose" (move or change position, convert) a scale to a similar higher or lower pitched (ie., in frequency) scale or key.

A peculiarity of the C-Major scale is that it was defined so that it does not formally contain any sharps (#) or flats (b) and is therefore often chosen when simplicity and-or a clearer understanding is needed. A "sharp" or "sharped note" is where a note is changed to a semi-tone or half-step higher in frequency which is essentially the next note in the chromatic scale. A flat (b, ex. Eb = "E flat") is where a note is change to a semi-tone or half-step lower in frequency. Sharps and flats can result in the same note, for example, the A# = A-sharp note is equivalent to the Bb = B-flat note. Both have the same frequency. Note for example that this scale starts at (C) and the next note is (D) which is an increase of two semi-tones or notes of the chromatic scale: C, C#, D. Such an increase of two semi-tones (or half-steps) is often called a whole-tone or whole-step. The notes of any scale, or selected notes of a scale are often generalized by noting or speaking in terms of whole-tones and half-tones rather than always use specific notes (such as C then D), numbers or frequencies. For example, this will allow a song to be easily transposed to another key. For a major scale:

1, 3 , 5 , 6, 8 , 10, 12, 13 : tone numbers of the chromatic scale
1, 2 , 3 , 4, 5, 6, 7, 8 : tone numbers of "major scale".
Fundamental or "Root" Tone, W, W, H, W, W, W, H : W = whole-tone or step
: H = half-tone or step

Knowing how to calculate the frequencies of the notes of a scale can be very handy for people such as computer programmers and some musical instrument designers and makers. Stringed instrument makers are often called luthiers, and the etymology (word source) of this word is derived from the stringed instrument called the lute.

From the study of physical science (physics), the "fundamental" or "natural" frequency of a vibrating string or wire can be calculated from the formula:

$$\text{Fhz} = \frac{1}{2\text{Lm}} \sqrt{\frac{\text{Tn}}{\text{M kg/m}}}$$

: Formula by **Marin Mersenne** (1588-1648) from France, a part of 'Mersene's Laws'. Similar types of equations were later developed by Christian Huygens for pendulum motion, and Isaac Newton for the laws of gravitation. He was also an associate of Galileo Galilei. Ts = 1 / Fhz = 1 / cps

L = Length of the string with meters (m) as the units of measurement

T = Tension (Force) applied to the string, and here it is essentially a "stretching" force.

Force = (mass of an object)(acceleration of an object) = ma. In simple terms, a force is the application of energy. The units of force are commonly newtons (N), 1N = (1 kg)(1 m/s^2) = (mass)(acceleration) = force

A force of 1N constantly applied to a 1K mass will accelerate (an increase in speed) it by 1 meter faster per second. Note that when a force being applied, there must be an acceleration or change, otherwise the force is not being applied. This force applied to a musical instrument string is usually adjusted by turning a tuning knob, and therefore adjusting the tension and natural (ie., "resonant", inherent) frequency of that string. Strings are subjected to some increased tension when pressing down on it to play a note, and the frequency heard could be slightly higher than expected. This is more noticeable on the thicker strings, and the design of the instrument may increase those string lengths so as to effectively reduce the frequency to the correct frequency. This concept is called something such as "bridge compensation" and "intonation". This may then create a problem that the "open" (unfretted) or full length string will have a note that is then slightly in error, and some instruments do consider this.

M = Mass of the string in kilograms per unit (meter) length. This is also called the linear or length density of the string. "Deep", "low" or "base" sounding stringed instruments usually have thicker and heavier (ie. more mass/length) strings so as to produce lower frequency waves and-or sound more efficiently. F is inversely related to L and M and directly related to T. The units of mass = M used here are (kg / meter of length = kg/m)

As a basic verification to the above formula, here is a derivation of a similar formula: Work = **Energy** = (force)(distance) = (f)(v)(t) = (Mass)(acceleration)(distance) = (M)(a)(d) = (M)(d/t^2)(vt) = (M)(d/t)(v) = (m)(v)(v) = mv^2 = m (d^2/t^2) = md^2 / t^2, and after dividing both sides by (d), we have: Force = md / t^2 , if we use M=m, and Tension for force we have: T = Md / t^2 , solving for t, we have: t = sqrt (Md / T) , and taking the reciprocal of time, we have frequency = = f = sqrt (T / Md) = sqrt (T / M Length)

Right now, we will be mainly interested in the relationship between the length of the vibrating string and frequency produced. These variables in the radical, for all practical purposes, will remain constant for a given situation, such as of a particular string. The radical factor (which resolves to a constant for our particular study here) can be completely eliminated since it is theoretically possible to have a tension of 1 newton applied to a string which has a mass of 1 kilogram per meter, and the square root of 1 is always 1. The formula is then reduced to just the fundamental (inverse) relationship of the frequency (f) and length (L) variables:

$$\text{Fhz} = \frac{1}{2\text{Lm}} = \frac{1}{t} \quad \text{solving for L:} \quad \text{Lm} = \frac{1}{2\text{Fhz}} = \frac{1}{2\,\frac{(1)}{(t)}} = \frac{t}{2} \quad \text{and} \quad t = \frac{1}{\text{Fhz}} \;:= 2\text{L} \text{ for this string where } \text{T=1N and M=1kg/m}$$

Clearly, frequency and length of the string are mathematically inversely related. In general, each is half (1/2 = 0.5) the reciprocal of the other. If Lm increases by a factor of (n), F will decrease by that same factor (here a divider) of (n) and which would be the same as multiplying by the reciprocal of (n) or (1/n).

A stringed musical instrument maker or player can alter the note (ie. frequency) heard by effectively shortening the length of the vibrating portion of the string. As indicated in the formulas above, this will increase the frequency of vibration and the tone heard will be "higher". This is usually accomplished by the stringed instrument players by using their finger(s), or a combination of a fret and finger as for fretted guitars and bases. A fret is an aid so that the note (ie. string length and frequency) is always the same regardless of slight position or finger misplacement, and therefore making the instrument easier to learn and-or use (ie., "play"). In the sketch below, the nut and bridge hold, fix, set, bound and define the ("open", longest, "unfretted", "natural" or "initial") specific length of the string allowed to vibrate.

Below, the numbers indicate the position of the frets and notes in a chromatic scale beginning at 0 as the note produced by the entire string. A violinist will try to position and hold their finger in these locations, but for fretted instruments such as guitars, the player will position and hold their fingers anyplace between the previous fret and the desired notes corresponding fret so as to allow the fret to physically set the proper string length for the note. [FIG 60]

```
                                                                                       string
Fret  0        1         2        3        4       5       6      7      8     9    10  11 12       .   .   .            |
     o-|----------|-----------|----------|---------|--------|-------|-------|------|-----|-----|----|---|----  - - - - - - - - - - - - - -  ---------------|-o
       |                                                                                  |                                                  /
     Nut                                                                             Half length                                         Bridge
```

Using the formula for R given above, we can calculate the frequencies of the notes in a scale and see where each fret (or finger as for unfretted instruments such as for violin-like instruments) should be placed using the length formula.

If the fundamental or "natural" frequency of the entire string is set to 262hz (which happens to be the usually defined frequency associated with the "standard" C note):

Note	Frequency (hz)	String Length (m) = 1 / (2Fhz)	
C	262	0.001908396	The higher the note frequency, the shorter the string length:
C#	277.5793306	0.001801286	: C# = (C)(R) = CR^1 = (262hz)(1.059463094) ~= 277 hz
D	294.0850565	0.001700188	: D = (C#)(R) = (CR^1)(R) = CR^2 ~= 294 hz

Taking the ratio between the string lengths of successive notes, we find a constant ("geometric") ratio and relationship: successive notes and string lengths:

$$\frac{\text{C\# length}}{\text{C length}} = \frac{0.001801286 \text{ m}}{0.001908396 \text{ m}} = \mathbf{0.943874313...}$$

: **Fret Size Ratio = FSR** (of the distance length, size or space between successive frets, and is also the **ratio of string lengths of successive notes** of the standard 12-note chromatic scale)

$$\frac{\text{D length}}{\text{C\# length}} = \frac{0.001700188 \text{ m}}{0.001801286 \text{ m}} = 0.943874313...$$

Hence, given a string of length L, at the first fret where the next successive note of the chromatic scale is located, the ratio of this string length to the previous notes string length should still be 0.943874313.

If the (open or unfretted) string or entire possible "scale length" is 25 inches, the length of the string where the first fret should be placed is:

$$\frac{\text{Length of string at Fret 1}}{25\text{inches}} = 0.943874313 \qquad \text{algebraically:}$$

Length of string at Fret 1 = 0.943874313(25in) = 23.59685783in

Hence the difference in string length, or the fret size (length or space) for the next note on that 25in long string is:
25.0 in - 23.59685783 in = 1.403142175 in :first fret size

Due to the constant ratio of successive string lengths of successive notes, their is a constant ratio between successive note or fret positions. The length of string between those two successive notes is also known as the "fret size" to musical instrument makers. Given just a fret size, the distance to the next successive fret, and so on, can be calculated without knowing or calculating the frequencies of notes and their string lengths. For example, once the first fret size has been calculated for a length of string, the next fret size and others can be calculated from that.

$$\frac{\text{second fret size}}{\text{first fret size}} = 0.943874313$$

second fret size = 0.943874313 (first fret size) Using the above value for the first fret size for a 25.0 in string:
second fret size = 0.943874313 (1.403142175in) = 1.324389856 in

Likewise, the size of the third fret can be calculated using the size of the second fret:

third fret size = 0.943874313 (second fret size)
third fret size = 0.943874313 (0.943874313 (first fret size))
third fret size = 0.943874313^2 (1.324389856 in.) = 1.250057565 in.

Finding all these fret sizes can be done quickly with either a computer with this equation programmed into it, and-or with a calculator. Since many calculators retain the last input "function" or operation (here, multiplication), it is simply a matter of pressing the "equals" or "enter" key so as to find the next successive result (here, fret size) of using the last displayed result or "output value" as the operand of that same function.

Notice that in order to calculate the (n)th fret size, that the exponent of the FRET SIZE RATIO (FSR) is one less than the (n)th fret size in question, and that the first fret size for a given string length or "(complete, possible) scale length") can be used as a constant factor for successive fret sizes. Expressing this in a formula:

Fret Size n = (1st fret size) FSR^(n-1) = (1st fret Size) 0.943874313^(n-1) : **Fret Length (size, space) Formula.**
1st fret size = (FSR)(string length)

Incidentally, and very conveniently, note that FRET SIZE RATIO = 1/(Note Ratio) = 1/NR , and this is verified below where NR is the ratio of the frequencies of successive notes in the standard chromatic 12-note (per octave, or frequency doubling) scale:

Letting L2 be the string length of the next higher note in the chromatic scale, and produced on a string of length L1:

$$\text{FRET SIZE RATIO} = \frac{L2}{L1} = \frac{\frac{1}{2(F2)}}{\frac{1}{2(F1)}} = \frac{2F1}{2F2} = \frac{F1}{F2} = \frac{F1}{F1(R)} = \frac{1}{(NR)}$$

$$\text{FRET SIZE RATIO} = \text{FSR} = 0.943874313 = \frac{1}{1.059463094} = \frac{1}{NR}$$

: **FSR and NR are reciprocals of each other**.
: NR = ratio of frequencies of successive notes

NR = Note Ratio = 1.059463094 = the ratio of frequencies of successive ascending notes in the chromatic scale. This value is slightly greater than 1, and the frequencies and tones heard of the next successive note is higher. NR is essentially the frequency ratio (FR) of successive notes.
NR = (next note frequency in the chromatic scale) / (note frequency) . NR =~ 1.06 =~ 1 / 0.94

FSR = Fret Size Ratio = 0.943874313 = Ratio of successive fret spacing sizes or string lengths of successive notes in the chromatic scale. This value is slightly less than 1, and as the fret size spacing and string length get shorter, the frequencies and tones heard of the next note are higher due to NR being greater than 1. FSR can also be called a "Fret Scale Ratio". Wind or "pipe" instruments also have corresponding note and pipe lengths based on FSR.

Since NR > 1, each note or frequency will increase in value, and this is an ascending or increasing series or steps.
Since FSR < 1, each fret length will decrease in value, and this is a descending series or steps.

Ex.	Chromatic Scale #	Note Name	Major Scale #	Ex. Frequency in hertz = cps	F ratio F/F1	Fractional or Relative Value Of The Total String Length = L , From Bridge	Fret #
	1	C	1 (root)	F1 = 262	1.0	1.0 L = 100% of L = full string length	0 (open string)
	2	C#		278	1.06	0.943874313 L ~= 0.95L	1
	3	D	2	294	1.12	0.890898718 L ~= 0.90L	2
	4	D#		312	1.19	0.840896416 L ~= 0.85L	3
	5	E	3	330	1.26	0.793700527 L ~= 0.80L	4
	6	F	4	350	1.34	0.749153530 L ~= 0.75L	5
	7	F#		370	1.41	0.707106782 L ~= 1 / $\sqrt{2}$ ~= 0.70L	6
	8	G	5	392	1.5	0.667419928 L ~= 0.65L	7
	9	G#		416	1.58	0.629960526 L ~= 0.63L	8
	10	A	6	440	1.68	0.594603559 L ~= 0.6L	9
	11	A#		466	1.78	0.561231026 L ~= 0.55L	10
	12	B	7	496	1.89	0.529731549 L ~= 0.53L	11
	13	C	8 (octave root)	524	2.0	0.5 L : half the string length = 50% L	12

Extra: Also note that the 13th note of the chromatic scale is twice the frequency of the 1st note, and the 7th note is $\sqrt{2}$ times the frequency of the 1st note. At the 13th note of the chromatic scale, the string length is half of the length of the string, and that 0.5L is equal to (0.707106782)^2 L as of the 7th note in the chromatic scale The frequency of the 8th note of the chromatic scale is equal to the average of the 1st and 13th note which has twice the frequency as the 1st note. Note also that 0.707106782 = Sin and Cos of 45 degrees and is located in the center of the octave or scale in terms of the number of half-steps.

The 6th note frequency of F6 is: (F1) + (F1 / 3)
The 7th note frequency or F7 is: (F1) 1.414213562 = F1 / 0.707106782
The 7th note frequency or F7 is: (F13) (0.707106782)
The 8th note frequency or F8 is: (F1) + (F1 / 2) = (F1 + F13) / 2

As seen in the above chart, many of the frets are spaced at a difference of 0.05L., except that between 0.5 and 0.55 or 0.56, there is also 0.53, and between 0.6 and 0.65 or 0.66, there is also 0.63. This could be 0.6 called a "5-hundredths rule", and may be used for quick calculations with reasonable results.

For reference, one-sixteenths of an inch = 1in / 16 = 0.0625 in , and this is slightly greater than 0.05.

Given a frequency number, Fn, then: F(n+6) = Fn (1.4142135562) , then mathematically:
F(n-6) = Fn (0.707106782)

When marking where the frets should be placed onto the fretboard, measure the length(s) in the **middle of the fretboard**., and this itself may be a topic for a disscussion amongst luthiers since the outer strings will have slightly longer fret spacings, and also knowing that the nut and the frets are set to be parallel to each other.

The most common scale and-or string lengths are: 24.75", such as for many guitars by the Gibson guitar co., and the 25.5", such as by the Fender guitar co. and Martin guitar co. A short scale guitar has a string length of 20" to 24". A short scale bass guitar, such as the Hofner co., violin shaped electric bass guitar is said to have a 30" scale length, and the neck feels somewhat like a standard electric guitar neck. A Fender Co. style Precision Bass and-or Jazz bass has a 34" scale length. A recent and helpful discussion about playing guitar and music notation was placed near the end of this book, and for luthiers, it includes the fret spacings of some common scale sizes.

There are various inexpensive kits available for many instruments, particularly for the electric guitar and electric bass guitar, and of various scale lengths. Some relatively inexpensive "guitar packages" often include a small amplifier and cord (plug) to it, and some may include a pick(s) and some type of guitar tuner (audio, sound) device. If a guitar tuner is not included, some audio players can generate tones at a desire frequency, and-or some computer and-or phone apps, and videos have some reference notes and-or tuning available. A metal tuning fork is low tech, but is reliable. A note played on another tuned instrument can be utilized as a reference note so as to tune other instruments.

EXAMPLES OF CHANGING THE PERCENTAGE OF A SUBSTANCE IN A MIXTURE

Ex. If a bleach solution for household use is noted as being 6% chlorine bleach per volume (say 1 cup of volume = 8 fluid ounces (fl. oz) of that solution is being used for this example) and it is diluted (ie., combined with, and will reduce a value) with 1 cup of water, what percent of that total mixture or solution will be chlorine bleach?

Since plain water substance is being added into this volume of solution, it will reduce or dilute the percentage of any other substance (here chlorine bleach) in that total volume.

6% chlorine bleach of 1 cup or 8 ounce is: 8oz (0.06) = 0.48 oz of each cup is chlorine bleach

Total volume of the mixture or solution = 1 cup of bleach solution + 1 cup of water solution = 2 cups =
= 8oz + 8oz = 16oz

This 16 oz of solution still contains only 0.48 oz of chlorine bleach, and the percentage of bleach in this total solution or mixture is:

$$\frac{\text{0.48 oz chlorine bleach}}{\text{Total mixture or solution}} = \frac{\text{0.48 oz}}{\text{2 cups}} = \frac{\text{0.48 oz}}{\text{16 oz}} = 0.03 =$$

3% of this mixture is chlorine bleach, regardless of the volume used. "a 3% solution of chlorine bleach". (Total mixture) must be > 1 unit volume.

Note, the total mixture contains: 2 cups = 1 cup of bleach solution + 1 added cup of water .

This formula will also work if the reference volume, here cups, is some other reference volume unit such as a for example 1 to several fluid ounce measuring cup, and where the bleach solution will still be 6% per that volume.

We can create an equation for the number of total number of (generic, non specific in size) volume units of solution needed so as to create a solution with a desired percentage (or dilution) of the chlorine bleach:

$$\frac{\text{6\% chlorine bleach per unit volume}}{\text{Total mixture solution in volume units}} = \text{Desired percentage in total volume},$$

from this, we mathematically have:

$$\text{Total mixture solution in volume units} = \frac{\text{6\% chlorine bleach per unit volume}}{\text{Desired percentage in total volume}}$$

Here, 1 volume unit will from the original 6% chlorine bleach mixture, and the amount of water to add into this solution will be = (Total mixture solution in volume units) - (mixture solution of 1 unit volume). The desired percentage in the total volume must also be 6% or less, since all the solutions will be a diluted ("thinned", less of a percentage of chlorine bleach) solution.

Ex. 6 units of total volume = 6% solution / 1% desired solution , : hence add (6-1)units = 5 units of water so as to create a 1% chlorine bleach solution

Rather than always use 6%, the above formula can even be made more generic or useful by using a variable such as P for the % of the substance in the original or given undiluted volume. Pharmacists and chemists use similar formulas. For the formula use entire and generic volume units, rather than a specific volume such as 8 fluid ounces. Later, you can then convert each volume unit to its actual associated units.

Extra: Volume is a different concept than weight. Fluid ounces is a measure of volume (ie., space having length, width, and height, hence a 3 dimensional structure or shape such as a cube) with units called a fluid ounces (fl. oz. = volume ounce), because 1 ounce unit of weight of water was defined as having a volume of 1 fluid ounce. 1 fluid ounce of other substances or elements will weigh more or less per unit volume. For example, liquid lead or liquid mercury weighs much more than 1 fluid ounce of liquid or fluid water, yet their volumes are still the same at 1 fluid ounce. In general, fluid ounces should not be considered as the corresponding weight of the substance unless it is water.

Ex. This example is more advanced and may be skipped over by new readers. A certain mixture or solution consists of two substances, let's say substance A and a substance B. Given a mixture where 5% of it is substance A, make the mixture to where only 3% of it is substance A. Given this data, the other substance(s) (here, substance B) in the mixture must then comprise (100% - 5%) = 95% of the solution. Substance B might be water, and if we dilute (ie. add in another substance or ingredient so as to reduce the percentage of some other substance(s) in the mixture) the mixture with some more water, the net percentage of the mixture that is substance A can effectively be reduced. In a way it could be said that substance A will be diluted, lessened or weakened, in the mixture. This method was chosen to be done here since any amount of substance A cannot be easily or cheaply removed from this specific mixture, and it is faster to just add more water to effectively decrease the percentage of A in it. Note that this is only a reduction of A's percentage of the entire mixture, and not a reduction of the specific volume, weight or amount of A, since this same volume or amount of substance A will still be in both (starting and final) mixtures since no amount of A will actually be removed or taken out of the mixture. If A was one gram or ounce in the starting mixture, there will still be one gram or ounce of A in the final mixture. The amount (ie., mass) of substance A per unit (of) volume and-or weight will decrease. Once the percentage, or "**concentration** (in the mixture)" of A is effectively reduced to 3% of the mixture, the amount of substance B will then comprise (100% - Substance A %) = (100% - 3%) = 97% of the solution, hence more of the mixture will then be substance B.

Let's look at several mathematical concepts and solutions:

Though we can solve the problem above with just the data given, it can help if we give a value to the starting quantity or volume of the mixture. Lets say the starting volume was 1 liter = 1L = 1000cc , (1cc = 1cubic centimeter and weighs 1 gram if the substance is water). 5% of this 1000cc mixture is substance A:

$$\frac{\text{volume A}}{\text{total volume}} = \frac{\text{volume A}}{1000\text{cc}} = 0.05 \qquad \text{therefore:}$$

volume A = 0.05 (1000cc) = 50cc : 50cc = amount of substance A (in both the starting and final mixtures)

We know that this 50cc value will become only 3% of the final diluted solution or mixture of:
(starting amount + added in dilution amount) = (1000cc + Xcc) Xcc is the (unknown for now) amt. of water to add in to the mixture so as to dilute, lessen and-or weaken the amount of substance A in it. Expressing this in an equation:

$$\frac{50\text{ cc}}{(1000\text{cc} + X\text{cc})} = 0.03 = 3\% \qquad \text{therefore, mathematically:}$$

$50\text{cc} = 0.03\,(1000\text{cc} + X\text{cc})$ using distribution:
$50\text{cc} = 30\text{cc} + 0.03\,X\text{cc}$ transposing 30cc:
$20\text{cc} = 0.03\,X\text{cc}$ after Dividing both sides by 0.03 to isolate Xcc:

$X\text{cc} = 666.6\underline{7}$: the amt. of water to add into the entire mixture
Note that this is a relatively (with respect to 1000cc) large amount of water needed to effectively reduce A's percentage or "concentration" by just a few (here 2) percent from 5% to just 3% of the entire mixture.

1000cc + 666.67cc = 1666.67cc : total volume of final mixture

5% of 1000cc = 50 cc : amount of substance A in the original or starting mixture
3% of 1666.67cc = 50cc : A's percentage or portion of the entire new mixture is now reduced, yet the physical amount of A in the new mixture remained the same.

95% of 1000cc = 0.95 (1000cc) = 950cc : amount of substance B (water) in the starting mixture
97% of 1666.67cc = 0.97 (1666.67cc) = 1616.67cc : amount of substance B (water) in the final mixture

Also: starting B + added B = ending B

950cc + 666.67cc = 1616.67cc : total volume amt. of water in final mixture

New amt. of substance A + New amt. substance B = New amt. of total mixture
50cc + 1616.67cc = 1666.67cc

Now we will look at some various other ideas and solutions, in particular, using fractions for the percentages:

A's volume in starting mixture = A' volume in ending mixture

100%A = 100%A : A's physical volume or amount is the same in both the starting and ending mixtures.

$(0.05)(v1) = (0.03)(v1 + x)$: v1 is total starting volume, x is amt. to add to total volume. Notice that in order to keep this equation in balance, as one factor decreases (here 0.05 decreased to 0.03), the other factor to that total decreased value must increase (here, v1 increases to (v1 + x)).

We could assign or let v2 = (v1 + x) = ending volume:

$(0.05)(v1) = (0.03)(v2)$ Solving for v2 we would get: v2 = 1.667v1.
Note here the "reverse ratios": (v1 / v2) = (0.03 / 0.05)
If we let v1=1=100% of the whole or initial starting volume, then x essentially becomes the percent increase in volume:

$(0.05)(1) = (0.03)(v2)$ solving for v2:

$$v2 = \frac{(0.05)(1)}{0.03} = 1.66\underline{7}$$

: hence the starting volume must be increased by this factor, where: $v2/v1 = 1.66\underline{7}/1 = 1.667 = 166.7\%$, also:

$(0.05)(1) = (0.03)(1 + x)$ solving for x:

$0.05 = 0.03 + 0.03x$
$0.05 - 0.03 = 0.03x$

$0.02 = 0.03x$

$$x = \frac{0.02}{0.03} = 0.666\underline{7}$$

v2 = v1 + x   Using relative or percentage values:
v2 = 1 + 0.6667   : x = 66.67% increase from 1 = 100%
v2 = 1.666$\underline{7}$: v2 = 166.67% the volume of v1

If we knew the volume of A as 1 (fluid) ounce = 1oz:

A = 1oz = (0.05)(v1)   : v1 = volume 1, the initial or starting volume before adding any water. Mathematically:

v1 = A / 0.05 = 1oz / 0.05 = 20oz   In the final mixture, A is also:

A = 1oz = (0.03)(v2)   :v2 = volume 2, the initial or starting volume plus the added water volume
v2 = (v1 + x),   v2 = the ending volume

v2 = A / 0.03 = 1oz / 0.03 = 33.3333   hence:

1 oz = A = (0.05)(20oz) = 0.03(33.3333oz)   : A is 1 oz in both the starting and ending mixtures. Mathematically:

$\frac{33.3333}{20.0}$ = 1.6667   : ratio of ending mixture to starting mixture   hence:

33.3333 = 20 (1.6667) = 20 (1.0 + 0.6667) = 20 + 20(0.6667) = 20 + (a change of 66.67% of 20)
= (starting volume + (66.67% of starting volume))
= 1.667 (starting volume)

From: starting volume + added volume = ending volume

ending volume - starting volume = added volume

33.3333oz - 20oz = 13.3333oz   : extra water to add into the mixture so as to dilute it. For a more general (algebraic) formula:

From: v2 = v1 + water added into the mixture

v2 - v1 = water added into the mixture
v1(1.667) - v1 = water added into the mixture   : from v2/v1 = 1.66$\underline{7}$, hence v2 = 1.667 v1
Factoring v1 from each term:
v1(+1.667 - 1)
v1(0.667)   Formula for the amount of water to add into the mixture for similar problems with various starting volumes of the same starting mixture with 5% substance A, and 95% substance B (water), so as to have a 3% substance A in the mixture:

v2 = v1 + v1(0.6667)   : ending volume for the 3% solution or mixture only , or:

v2 = v1 + v1(1 - (starting percent A / new percent A))   : Formula only for lowering percent A by adding in water. Percent A cannot be increased by adding in water, but only by adding in more A and-or removing water, but removing only water with A already dissolved in it may not be practical or easy. To increase A from 3% to 5%, simply add in (5-3)% = 2% of the total amount of water volume or weight in the mixture.

The concept of proportions will not work for the problem or example above, basically since we are not making a similar mixture or construction with the same proportions or fractional parts of substances. Substance A will be 5% in one mixture or construction, and only 3% in the ending mixture or construction being made. The percentage of water in both mixtures is also a different value. Whereas one percentage (here, of water) will increase, another percentage (here, substance A) will effectively decrease and this is not a direct mathematical relationship but an inverse mathematical relationship. Also, whereas the physical amount (here, volume) of water increases, the total physical amount (volume) of substance A remains the same in both (non-mathematically similar, or non-proportional) mixtures or constructions.

This below was briefly mentioned above, and will be mathematically explained further here:

Consider for example that: 2 is 2% of 100 and 2 is 1% of x

The solution to this considers that to keep the same value (here 2) but only reduce the percentage that is of the whole or total, then the whole or total must be increased. This is an inverse mathematical relationship.

$2 = 2\%$ of $100 = 1\%$ of x : here 100 is the starting quantity or "whole entire value". This can be expressed as:

$(0.02)(100) = (0.01)\,x$ this can be expressed as two equivalent and inverse ratios: solving for x:

$\frac{0.01}{0.02} = \frac{100}{x}$ in pseudo: $\frac{\text{(new percent)}}{\text{(old percent)}} = \frac{\text{(old total)}}{\text{(new total)}}$: notice the "inverse" ratios

$x = \frac{(0.02)}{(0.01)}100 = (2)100 = 200$: the percent ratio decreased by a factor of (0.02/0.01) = 2, and the whole value increased by that same factor of 2. Or in other words:
(0.01/0.02) = 0.5 and the reciprocal of 0.5 is: (1/0.5) = 2
The percent was multiplied or "increased" by a factor of 0.5, and the whole value was multiplied or "increased" by the reciprocal of that factor = 1/0.5 = 2

Here are some common cooking examples involving percentages:

Ex. A recipe for a certain type of bread requires 1/4 = 0.25 = 25% of the total flour mixture to be whole wheat. This can be easily done by using 3 measurements (scoops, cups, bags, etc) of white flour, and 1 of the same measurement of whole wheat. This will create a flour mixture with a total of 4 measurements, and with 1/4 being whole wheat flour. 1/4 could be thought of as 1 part being whole wheat of 4 total parts. If the recipe was for 1/3 = 0.33 = 33% of the flour mixture being whole wheat, for each 2 measurements of white flour, use 1 measurement of whole wheat flour. 1/3 could be thought of as 1 part being whole wheat of 3 total parts. For a 1/5 = 0.20 = 20% whole wheat mixture, use 4 measurements of white flour, and 1 measurement of whole wheat flour which makes 1 part of the 5 total parts. For a 1/10 = 0.10 = 10% of the whole mixture, use 9 measurements of the main ingredient(s) and 1 measurement of the ingredient that will compose a 10% of that entire mixture.

Ex. A recipe states to use 2 and 1/4 cup flour = 2.25 cups of flour. If you decide to use 25% = 1/4 of the total flour mixture with whole wheat flour, how much whole wheat flour will you need, and how much regular white flour will you need?

whole wheat flour = 25% of the total flour mixture
whole wheat flour = (0.25)(2.25) cups
whole wheat flour = 0.5625 cups

total flour = white flour + whole wheat flour
2.25 cups = white flour + 0.5625 cups
white flour = 2.25 cups - 0.5625 cups
white flour = 1.6875 cups

Since 25% is whole wheat flour, the remaining portion: (100% - 25%) = 75% is white wheat flour:

(amount of white wheat four) / (total amount of flour) = 1.6875 cups / 2.25 cups = 0.75 = 75% : checks

(0.5625 cups) x (8oz/1cup) = 4.5 fl. oz (dry, equivalent, fluid volume) white wheat flour
(1.6875 cups) x (8oz/1cup) = 13.5 fl. oz (dry, equivalent, fluid volume) whole wheat flour

Ex. If you have 15 cups of regular white wheat flour, how much whole wheat flour would you need to add or combine to that amount of white wheat flour so as to create a flour mixture that is 1/3 = 0.3333... = 33.3% whole wheat flour. Two methods will be shown to solve this problem.

First note that it is a common, but incorrect, thought to simply calculate the amount of whole wheat needed as just 33% of 15 cups:

(0.3333)(15 cups) = 5 cups : nice try, but it is the incorrect amount of whole wheat

The correct amount requires a different calculation:

$$\frac{\text{Z cups whole wheat}}{\text{(X + Z) total cups}} = 0.3333 = \frac{\text{Z cups whole wheat}}{\text{X cups white wheat + Z cups whole wheat}} = \frac{\text{whole wheat cups}}{\text{15 cups white wheat + Z cups whole wheat}}$$

0.333 (15 cups white wheat + Z cups whole wheat) = Z cups whole wheat
5 cups white wheat + 0.333 (Z cups whole wheat) = 1 (Z cups whole wheat) : Consider: 1 = 100%
5 cups white wheat = 1 (Z cups whole wheat) - 0.333 (Z cups whole wheat)
5 cups white wheat = 0.667 (Z cups whole wheat)

5 cups of white wheat / Z cups of whole wheat = 0.667 , mathematically using just the numbers in the ratio:

$$\frac{\text{5 cups}}{\text{Z cups}} = \frac{5}{Z} = 0.667$$ mathematically:

$$\text{Z cups} = \frac{\text{5 cups}}{0.667} = \text{7.75 cups}$$: use 7.5 cups of whole wheat in the mixture

(X + Z) total cups = X cups white wheat + Z cups whole wheat , (X + Z) = (15 + 7.5) = 22.5 :
22.5 cups = 15 cups + 7.5 cups

whole wheat / total cups = 7.5 cups / 22.5 cups = 0.333 = 33.3%

white wheat / total cups = 15 cups / 22.5 cups = 0.667 = 66.7% , The relative sum of portions of the whole (1) are:

% white wheat + % whole wheat = 0.667 + 0.333 = 1.0 = 100%

This problem can also be solved using: If 15 cups is to 66.7% then x cups is to 33.3%:

66.7% = 100% - 33.3% = 1.0 = 0.333 : Total % = % white wheat + % whole wheat , therefore:
% white wheat = Total% - % whole wheat = 66.7% = 0.667

$$\frac{\text{15 cups}}{0.667} = \frac{\text{x cups}}{0.333}$$, after solving for x: , x = 7.5 cups

Ex. A certain product for sale is a mixture of 60% of substance A, and 40% of substance B.
total mixture = substance A + substance B
total mixture = 100% (total mixture) = 60% (total mixture) + 40% (total mixture)

In decimal equivalent of these relative or percentage values, and regardless of the actual volumes, weight and-or mass used:

total mixture = 1 (total mixture) = 0.60 (total mixture) + 0.40 (total mixture)

If this mixture is then modified so that substance B is only 30% of the total mixture, how much water must be added into this total mixture to dilute it for this to happen. Let us call the water as substance C, and it is also possible either substance A or B is also water, but they might be other substances also.

Since an amount of water will be added into the total mixture the volume, mass and-or weight of the resulting total mixture will be more. The resulting volume will be larger, but still, in the relative sense, it can be considered as 100%= 1. Since substance C will now occupy some of that 100%, the percentages of both substance A and substance B in the total mixture will decrease.

Let us reduce the amount of substance A and substance B, by the same percentage in the mixture so that they then account for (100% - 10% due to substance C) = 90% being due to substance A and substance B in the mixture.

In the original mixture: B / A = 40% / 60^ = 0.4 / 0.6 = $0.66\underline{7}$, and we will try to preserve this ratio of B to A in the final mixture.

We will then have this formula and-or calculation for the mixture:

total mixture = 1 = 100% = A + B + C
100% = (< 60%) + (30% < 40%) + C

For keeping the same ratio (r) of: $\frac{B}{A} = r = \frac{0.30}{A} = 0.66\underline{7}$, $A = \frac{0.30}{r} = \frac{0.30}{0.66\underline{7}} = 0.45 = 45\%$

100% = 45% + 30% + C
100% = 75% + C , after solving for C, we have:
C = 100% - (A + B) = (100% - 75%) = 25% of the modified, final and-or new total mixture

New total mixture = 1 = 100% = A + B + C = 45% (total mixture) + 30% (total mixture) + 25% (total mixture).

Ex. If the original total mixture was 1L before adding in substance C, then A = (1000 mL)(0.60) = 600 mL and B = (1000 mL - 600 mL) = 400 mL. , The final volume = FV = (1L + C liters) = (1L + 0.25 FV) , then FV - 0.25 FV = 1L , then 0.75 FV = 1L , then **FV** = 1L / 0.75 = **1.333L** , **C** = (FV - Original Volume) = (1.3333L - 1L) = 0.333L = **333 mL** , checking: 0.333 / 1.333 = 0.25 = 25% , As an example for the volume of A, using a proportion or equivalent fraction: 0.25 / 0.333 L as = 0.45 / X liters , X = 0.6 L = 600 mL = A
A / FV = 600 mL / 1300 mL = 0.45 = 45% FV , also (45% < 60%)

Extra:. If the new total mixture is 1L = 1000 mL , then A = 450 mL , B = 300 mL , C = 250 mL : 1mL = 1 cc
A = (1000 mL) (0.45) = 450 mL

Extra: If the percent of substance A was to also remain the same at 60% in the new mixture then consider this for C:

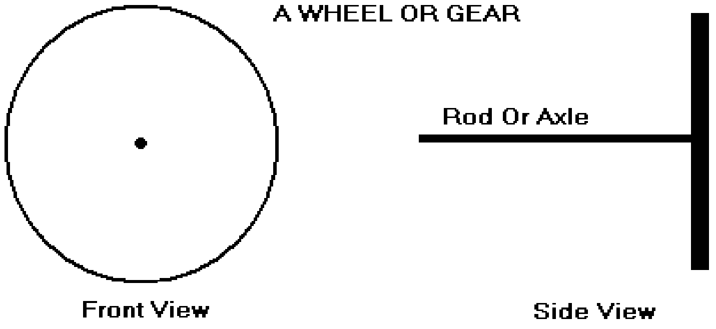
New total mixture = 1 = 100% = % being A + %B + %C = 60% + 30% + C = 90% + C =
C = (100% - 90%) = 10% , 1 = %A + %B + %C = 0.60 + 0.30 + 0.10

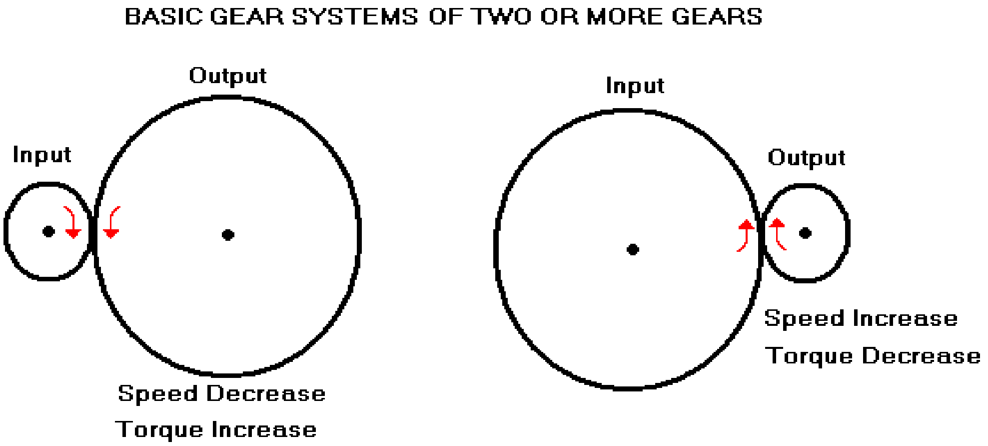
The ratio of B to A is then: 0.30 / 0.60 = 0.50 = 50% , and this is now less than 66.67%

Extra: If the percent of substance C was to be 15% , then : 1 = 100% = %A + %B + %C = A + 30% + 15% = A + 45%, and A is therefore: A = (100% - 45%) = 55% = 0.55 of (total mixture) , and B / A = 0.30 / 0.55 =~ $0.54\underline{5}$

GEARS EXAMPLE

This example and explanation will show how a simple gear system of two or more connected rotating gears or wheels of different sizes can be used to change the resulting rotational speed or the torque (a force that cause a turning or rotation about a point, hence torque is a rotating force or force of rotation) of one of those wheels or gears. The appendix also contains a few defined terms relevant to this example that you may optionally review if needed. [FIG 61]





The rod ("drive shaft" or connector) or axle is placed at the center of the wheel and can be used to hold the wheel in position for it to rotate with stability and efficiently, and-or to apply or transfer power (from elsewhere, perhaps a hand-crank or motor) to the wheel or gear so as it can rotate about it's center ("hub"). For a given rotational speed per unit of time, the larger the wheel and its circumference (ie., distance, length), the greater the distance a vehicle would travel per unit of time. It's also possible that the wheel can be used to apply power to the axle, such as in a water powered wheat grinding or saw mill. A large water wheel does not rotate very fast but it collects the high input force of the water to turn it, and this speed or revolutions per minute (rpm) of this can be changed later by using gears if needed.

Many vehicles have gearing to change the speed or torque of its wheel(s) so as to improve the vehicles movement in certain driving situations or needs. Many other machines also use some type of gearing to alter (change) the speed or torque of some moving part so as to improve it's usefulness and efficiency in some way.

A gear is much like a wheel, however the rim or edge of it has what is called the teeth of the gear, and these evenly-spaced rim protrusions are generally somewhat square and-or triangular in shape and allow efficient contact with another gear with the same size (but generally not necessarily the same number) and shape of teeth, and so as to transfer (via., contact and force) mechanical power from one to the other without slipping past each other like two wheels might. In general, the number of teeth on a gear is proportional to its size (ie., circumference, diameter, radius). The smaller each gear tooth is, the greater then number of teeth on the rim or edge of the gear and-or on the wheel. If the teeth are too small, there is a risk of gear slippage (non contact, lack of power transfer) and-or teeth wear due to friction (mechanical resistance due to the two parts [masses] and the forces in the opposite direction) and-or collisions. In general, the ratio of the gear teeth of two interconnecting or corresponding gears is the same as the ratio of their physical dimensions such as circumference, diameter and-or radius. The faster a gear spins or rotates also increases the amount of wear per amount of time and-or revolutions of it and it will then need to be replaced.

Gears or wheels of the same or different sizes (ie., due to their radius and-or circumference values) can be used to mechanically (ie., a force due to the energy of a mass and motion or movement (ie., a velocity), and then contact) transfer an input power from one to the other, and this could be described as the gears or wheels being like mechanical energy transferers (ie., like a mechanical circuit, linkage, conveyors) and-or "transformers" in terms of any torque and speed changes. Torque (rotating or twisting force) can be increased at the cost (loss, trade-off, "price") of reduced speed of the output gear. Rotating speed can be increased at the cost of reduced force (torque) at the output gear. Either way, nearly all the energy or power put into this system is "conserved" or maintained, that is, the energy just doesn't disappear, but nearly all of it is transferred from one gear to the other. No system is perfect and some small fraction of the available energy or power will be lost due to friction (surfaces in contact with each other while in motion, a resistance to motion), vibration, heat, sound, etc. This will reduce kinetic energy to the output, and therefore motion. For our basic analysis, here of a simple wheel or gear system, we will not consider these small losses. If the two gears are of the same diameter, radius and-or circumference, then there will be no increase or decrease in either speed or torque, and the gears are used to just transfer power from one location to another.

A wheel and **axle** (ie., the rod a wheel or gear can spin or rotate about, much like a perpendicular axis to the plane or disk of the wheel) is a simple machine, and of which can be used to modify (increase or decrease) the input force so as to provide some "mechanical advantage". In general, a wheel by itself provides load bearing support, and if several are used such as for a cart or car, the load or weight being moved is then distributed to each wheel. Wider wheels will have more surface area and will make less pressure (= F / A) on a surface, and this helps prevent the wheels and-or load from being stuck into soft surfaces such as soft dirt and-or gravel, and where much energy will then be wasted due to this type of friction and-or impedance to then overcome by additional force. Wider wheels will distribute the weight of the load to a wider area of which the wheel is moving upon.

Since steel metal is very hard, a drill-bit will need a high torque or force to be able to cut and drill through the metal. A "power drill" or "power driver" will likewise need a high torque or force so as to be able to place (ie., "twist") in or remove a screw from wood. In both cases, gears will be used to convert a high rotational speed from an electric motor into a slow rotational speed, but with a high output torque necessary for the situation.

Energy Out = Energy In or Power Out = Power In
100% = 100%
1 = 1

: Energy is the ability or potential to do work.
A common unit of energy is a joule (J).
Power is a measure of work done over a time, hence it is a measure of the energy used up or transferred over time so as to get the work done. A common unit of work is therefore:
joules (J) per second = J / s = Watts, hence 1J = 1Ws

If the (mechanical, power transfer) system of wheels or gears has a torque magnification factor = x, then the corresponding speed magnification factor is the reciprocal of this value = 1/x, and vice-versa. This will be verified below.

Using up or transforming energy is called work, and both energy and work have the same units of joules since energy = work, and-or energy used = work done   Power is the total amount of energy or work used during an amount of time. Power is a (average) rate of using energy and-or doing work (using energy). The units of power are **joules per second** which is also called watts (w). In terms of torque, power (Pw) = torque x rotation speed. The S.I., System International or International System (ie. metric) units of force are usually called **Newtons**, and the units of power or energy used, applied or needed is: **Newton-meters (Nm)**, hence a measure of **force** applied through a distance. Below, we see that in the equation of input and output power, or power transfer, that torque and speed are mathematically inversely related for a given (fixed, constant) amount of Power. Note that even though they are inversely related, they are generally not mathematically reciprocal in value to each other. Only the "magnification factor" of the ratio of the two torque values, and the "magnification factor" of the ratio of the two speed values, are mathematical reciprocals. For each input and output wheel or gear, the available power (P) is the same value:

P input gear = P output gear : with conservation of energy, P input / P output = 1

(input torque)(input speed) = (output torque)(output speed) :speed =angular velocity =w= radians per second
1 rev / s = (2)(pi) rads / s ,
revs = revolutions or rotations of the wheel
Speed can be thought of as a faster application
of a force or torque per time, hence more
energy being applied per unit of time,
and power = energy / time

For a constant or given amount of power, torque and speed are reciprocals.

Clearly, and strictly mathematically, to keep the equation in balance, if the value of one variable (ex. torque) is to be increased (by a multiplying factor) on one side of the equation, the other variable (here a multiplying factor, ex. the speed) on the same side of the equation must then be decreased (ie. divided by that same factor value) so as to keep both sides of the equation in balance and also have conservation (ie., same net sum of input and output values, no loss, maintain) of energy. Mathematically, from the equation we have these ratios:

$$\frac{\text{input speed}}{\text{output speed}} = \frac{\text{output torque}}{\text{input torque}}$$

: = same "magnification or gain factor" or
ratio value of these inverse ratios or
proportions

Given two fixed (having a constant circumference) sized wheels or gears, the speed or torque can be altered (changed, increased or decreased) in the output gear when it's size is not the same size as the input gear. If the size of the output gear is less or is decreased, it's speed (revolutions per second, rotation or angular speed, or velocity) will increase.

Without knowing the specific value of the applied forces (torque) or speeds of any gear, we can calculate the corresponding magnification or "gain factor" in either the speed or the torque by knowing the size of the two gears and calculating their ratio (ie., the mathematical relative, fractional, or portion of one with respect to the other). For gears and wheels, and considering speed and torque, the size value is either the circumference, diameter or radius of the gears. Given two gears or wheels, the ratio (r) of the corresponding values chosen will be the same as any other corresponding values chosen:

$$\frac{\text{circumferenceA}}{\text{circumferenceB}} = \frac{\text{(2)(pi)(radius A)}}{\text{(2)(pi)(radius B)}} = \frac{\text{radius A}}{\text{radius B}} = \frac{\text{(pi)(diameterA)}}{\text{(pi)(diameterB)}} = \frac{\text{diameterA}}{\text{diameterB}} = \text{size ratio of two circles, wheels and-or gears}$$

When a larger sized input gear is applying power (ie. force, torque) to a smaller sized output gear, the smaller gear will then rotate more (per unit time), and to help understand this, consider that a point on its circumference must and will travel and equal the same circumference or distance length as that of a point that travels on the larger input gear when it rotates just once. Expressed another way, due to the "junction", "connection" or contact point at both wheels, both wheels will actually rotate the same amount of distance, point per point, along their circumference per unit of time, however with the smaller wheel or gear, that distance corresponds to a much larger central angle and its amount of degrees. In short, when a larger input gear spins (ie., rotates) once, a smaller output gear will spin more than once. This also means that the smaller gear will rotate "faster" or more times per amount of a given time, hence its rotational or "angular" speed or velocity is more. [For a potential future reference, you may consider this equation: ¢2 = ¢1 (factor), and that this factor is equal to (C1/C2) or= (r1/r2) or= (d1/d2) , and mathematically: C1 ¢1 = C2 ¢2 , and: (¢1 / ¢2) = (C2 / C1) , which are "reverse ratios".].

Circumference = (pi) Diameter = (pi) 2 Radius = 2 (pi) Radius , and when given two wheels or gears:

larger wheel size / smaller wheel size = (x) = wheel size magnification factor of the larger wheel or gear. This value will be >1 when the numerator is larger.

therefore: (smaller wheel size) (x) = larger wheel size
or: (smaller circumference distance)(x) = larger circumference distance

Consider that 1 rotation is equal to the circumference length or distance, therefore:

(1 smaller distance)(x) = 1 larger distance

These distance values correspond to (whole or fractional) rotations of each wheel or gear:
(1 rotation of small wheel)(x) = many rotations of small wheel = 1 rotation of large wheel

Therefore, a smaller output wheel or gear will correspondingly rotate (x) times more or faster so that a point on it's circumference will travel a distance equal to a point traveled on the circumferences of the input gear or wheel. It will then be said that the speed (number of rotations or revolutions per second), also known as "angular speed or velocity" of the smaller gear (in this two wheel or gear system) is more (greater), and is sometimes, somewhat incorrectly noted as being "faster" (even though it, and-or a point on its circumference would actually travel the same (linear) distance as the larger gear or wheel would in the same amt. of time).

output speed / input speed = gain in speed : gain means a magnification, amplification or increase factor (from the input or start, to the output or end) and is expressed as a multiplier to the input to solve or determine the output value. It is also very possible that it is less than 1 (or < 1) such as when there is a reduced speed and-or torque gain at the output.

output torque / input torque = gain in torque Generally: gain in speed /=/ gain in torque , and these values are rather inversely related, and are generally not reciprocal in value to each other.

If the output gear is smaller, that is, it has a smaller radius (ie., lever arm) and circumference than the input gear, it's speed (of rotation) will be faster, and it's available torque (on its shaft, or perhaps to be applied to even another gear) will be less since the effective lever-arm (ie. its radius value) of that (smaller) gear or wheel is smaller. In short then, in a wheel or gear system, a smaller output gear will spin faster but have less torque.

A larger output gear will spin slower, but will have more torque because the effective lever-arm distance (ie. the radius distance) to the center of that wheel or gear is longer, and a force applied to a longer lever-arm will increase or magnify that force. Torque = Force x Lever-arm = (F)(L), and therefore has units of Nm. We see in this equation that the lever-arm distance is essentially a force multiplier. For example, at twice Lever-arm distance, the Torque will be twice as high. The magnification (of force, distance, and speed) is also mentioned ahead in this book in the analysis of vertical angles and the sides of such constructions in the topic of: A Simple Device To Measure Small Distances

As indicated previously, corresponding input and output speed and torque (T) values are generally not reciprocal in value for a gear or wheel system, however, the specific gain or magnification (factor value) in speed and torque are mathematical reciprocals of each other and they will always have a product of 1:

input power = output power : For torque situations: Power = P = (T)(v) = (F)(L)(v)
T = torque=rotational , amplified force at or about the axis of rotation, and F = linear force, L = lever arm or radius from the center or axis of rotation. v=velocity

(input torque) (input speed) = (output torque)(output speed) , mathematically:
(input torque) / (output torque) = (output speed) / (input speed)

(input torque) / (output torque) = gain in speed taking the reciprocal of both sides:
(output torque) / (input torque) = 1 / gain in speed or:

gain in torque = 1 / gain in speed : the gains in speed and torque are reciprocals

gain in torque x gain in speed = 1 **:** Torque and Speed Gain Relationship , and mathematically:
gain in torque = 1 / gain in speed = 1 / (input gear size / output gear size) , hence:

gain in torque = 1 / gain in speed = **(output gear size / input gear size)** : **gain in torque, by gear sizes**
gain in speed = 1 / gain in torque = **(input gear size / output gear size)** :**gain in speed, by gear sizes**

Ex. Given a rotating gear that has a certain amount of power (due to speed and torque) available, what should be the size of a connected gear so as the available torque in it is twice (double = 2) that of the input or source gears torque value?

Since specific values were not given, we will be working with generalized or relative values.

From: gain in torque = output gear size / input gear size

output gear size = (input gear size)(gain in torque) and for this specific example:

output gear size = (input gear size) (2) : the output gear size should be twice (2) the size of the input gear

Note that since torque in this specific system will be increased, that the associated speed will be decreased by the reciprocal of this gain (increase, magnification factor) in torque:

gain in speed = (1 / gain in torque) and for this specific example:

gain in speed = 1 / 2 = 0.5 : the rotation speed of the output gear will be half (1/2) of that of the input gear

If for example the speed of the input gear was 100rpm = 100 revolutions per minute of time, the output gears speed would be:

From: output speed / input speed = gain in speed mathematically:
output speed = (input speed) (gain in speed) or:
output speed = (input speed) (1 / gain in torque) and for this specific example:

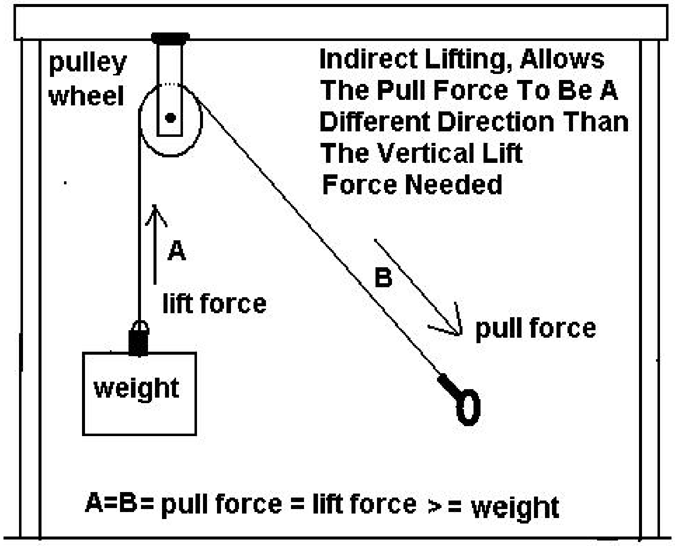
50rpm = (100rpm)(0.5)

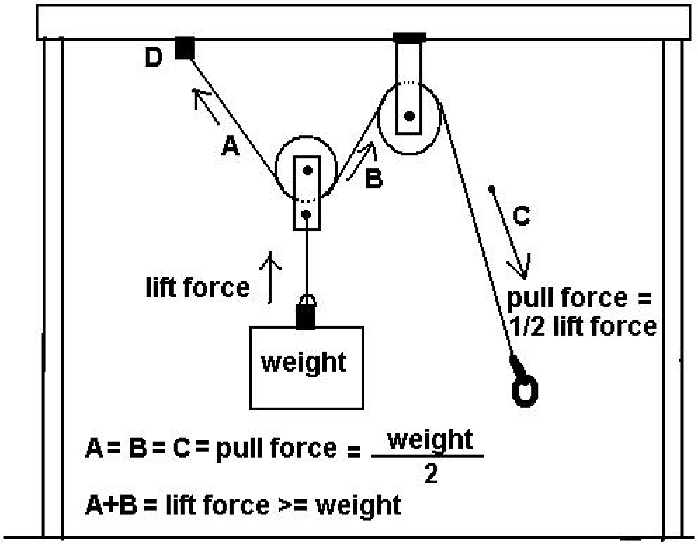
It is possible that two gears or wheels of different size can be on the same rotating shaft or axle. They will both rotate at the same speed or angular velocity (ie., rotatonal velocity, ex. degrees per second) but a point on the circumference of the larger gear will have a greater linear velocity. This type of simple machine is much like a **pulley** described below, and is essentially the reverse of a wheel and axle system and where here, the input force or power is applied to the larger wheel rather than the smaller diameter shaft or axle. This system is sometimes often seen in some simple hand powered (ie. cranked, rotated, spun, turned [torqued]) machines that get water from a deep well more easily with less input force needed, but may also be seen in pulleys for raising objects up to a height, such as into a barn. This machine may be sometimes called a compound wheel system. The gain in input force is equal to the ratio of the radius of each wheel and is essentially like the ratio of two lever arms. The larger "input wheel" applies a larger torque upon the axle and-or upon any other smaller wheel on that axle. This system is often called a "wheel and axle" system.

A common example where torque (force) will be increased mechanically is by pulleys (a rope or chain, and wheel system), and electric winches which contain an internal gear system so as to pull (apply force) heavy loads. A **pulley**, like a lever, is considered as a simple mechanical (force driven) machine that provides a "mechanical" or force advantage (ie., a greater output force than the input force) . A basic ("fixed") pulley consisting of just a wheel affixed to an above height and rope to apply ("pulled, tension) forces does not offer any mechanical advantage, but essentially changes the directions of the input and output forces, and the object is still essentially pulled, but in an upward direction instead of along the ground. See the figure below. Once an object is raised to a height, it can be set fixed at that height by "anchoring" the rope to the ground or to a heavy object. If the pulley wheel is not affixed to a height, and the object is affixed to the pulley on the rope, this pulley can still be used to lift the object to a height. One end of the rope is affixed to a height, and the other end is where the user can apply force. The advantage of this system is that anchored or affixed end of the pulley will support half the weight of the object being raised to a height. With a "compound pulley" or "movable pulley system" [also contains another pulley(s) non-fixed or movable along the rope or chain] with many pulleys working together, a larger portion of the object weight will be held by the pulley system and its fixed support, and therefor creating an improved mechanical advantage and making it even easier to lift an object, but requiring more distance and-and time by the user. The total work required to lift an object a height will still be the same as that without using a pulley, but it could be said that the work is easier and-or less stress on the body with the pulley:

Work In = (Force) x (Distance) = Work Out = (Force) x (Height) , since Distance = (Speed)(Time)
Work In = (Force) x (Distance) = (Force) x (Speed) x (Time)

Car mechanics can slowly lift a very heavy metal engine into and-or out of a car by using a compound pulley ("block and tackle" or "hoist") system. The input force applied and the speed of the object being raised is less, therefore the height will be achieved slower as the amount of time needed will increase. Essentially, if the input force is divided by (n = the "mechanical advantage" or force gain, multiplier or amplification), the length of time needed to raise an object to a height is increased by that same factor (n). [FIG 61A]


A Simple Pulley System Without Any Mechanical Advantage


A Pully System With A Moveable Pulley Where Half The Weight Is Supported At D

An **inclined plane** or "**ramp**" is another example of a **simple machine** that is said to provide "mechanical advantage", ease or gain. With a ramp a heavy object can be more easily (ie., less input force needed since the incline is now supporting some of the weight of the object, perhaps half of it) raised or lowered to an amount of height, but the path and its distance taken is longer than the height distance it is raised or lowered to and therefore the time needed is longer. A lever is another example of a simple machine.

Is the required or input energy less with a simple machine?

No, it will take the same total amount of energy or Joules to move an object to the same height. The input energy (J) and-or power (W=J/s) needed will be less per second and this helps the worker and-or machine which may have a "low power" and-or force ability, but when it is applied for a longer time, there is no actual saving of the total energy required and-or used. For example, 1 watt (ie., 1w = 1J/1s) applied for 10 seconds is a total of 10 watt-seconds, and this value is equivalent to 10 watts of power applied for 1 second. In both cases a total of 10 w-s of energy or= 10 Joules (1J = 1w-s) of energy were required and-or used to complete this work or task. A basic formula for how much of the weight of the object is supported by the incline is:

(amount of weight supported by the incline) = weight (cos ¢ of the incline)

,therefore, the effective weight that you need to move will be: **effective weight** = weight - (amount of weight supported by the incline) = weight - weight (cos ¢ of the incline) = **weight (1 - cos ¢ of the incline)**. To move the object horizontally, assuming no friction to overcome with extra force, you must apply a force, and to move or lift the object vertically, you must apply a force greater than or equal to the effective weight (ie., the objects effective amount gravitational force to overcome, much like friction to overcome) of the object. Placing the object to be moved into a carrier with wheels will greatly reduce any surface friction to overcome, and so as to reduce wasted force and energy. The larger the force applied, f=ma, the greater the acceleration (a = f/m = change in velocity per unit of time) and resulting velocity, and the less time needed to move the object a certain distance (d=vt). To say that something is accelerating, is to say it is both speeding up and going faster at the same time. For an object to o accelerate, a force needs to be applied, and once the force stops, the acceleration will stop, and it will remain at its current speed or velocity when the acceleration stopped.

The goal, assistance and practicality of using the inclined plane is to use less input force to move an object vertically.

For a measure of the work and-or energy involved, the basic equation for work is: work = energy = (force)(distance), and for this inclined plane system, the total (ie., sum) or final amount of energy required to lift an object a height will be the same as without using it, since even though the effective force is less than the weight of the object, the distance is longer: work = energy = (a lower force applied)(through, over, or during a longer distance) = (effective force)(total distance). This also ensures a "conservation of work and-or energy" where the input energy = output energy. For a given amount of energy, if one of the variables, or factors of a mathematical product increases by a factor of (n), the other variable or factor must decrease by that same factor of (n). If the effective weight decreases by a factor of 2, the total distance needed will increase by a factor of 2. Note that when an object is raised a height with kinetic energy (KE), that input energy is transferred to the object and stored as potential energy (PE) available.

Although in a gear (transfers power) system, the applying force can be greatly amplified for the output force, the number of revolutions (per same amount of time) or speed of the output gear is greatly reduced by the same factor. Therefore, a point on the circumference of a larger output gear will travel or move much slower than that of the input gear or wheel, and therefore, more time will be required to move the same distance (such as a height) as that of a point on the input gear's circumference would travel in one revolution of that input gear. Winches are slow (longer, require more time) to move something a certain distance, however, they are still able to move objects of great weight when needed.

Two gears that are used to transfer power from one to the other will rotate in the opposite (rotational) directions with respect to each other: clockwise=CW, or counter-clockwise=CCW.

As indicated above, a pulley is an example of a "simple (mechanical, with motion and force) machine" or mechanism to assist (called a "mechanical advantage") in doing something, but as we can see by the numbers, that there is also a "trade off" or "cost" of having an increase in one variable, and that there is a decrease in another variable(s). For an electric winch or pulley, the advantage is a gain in force, but the disadvantage (ie. a mechanical disadvantage) is that it will take a corresponding gain in time to go (ie. lift up to the same unaided height, or pull the same unaided distance) the same distance because speed has decreased by the same factor:

From: distance = speed x time

time = distance / speed

We see that time and speed are inversely related. For a fixed distance, if speed is decreased by some factor, then the corresponding time value necessary to go that same distance will increase by that same factor.

This note is mentioned further ahead in this book when discussing power and torque, and it is also included here due to its importance:

In terms of torque: power = (Pw) = (torque) x (rotation speed) = (force)(Lever arm)(rotation speed)

The units of force are usually called Newtons, and the units of power are Newton-meters (Nm). We see in this equation of power, that torque and speed are mathematically inversely related for a given (fixed, constant) amount of power as such for the input and output power of gears or wheels. Note that even though they are inversely related, they are generally not mathematical reciprocal in value to each other, however it is important to note that in a gear system, that the specific corresponding gains (increases, multiplying factors) or losses in torque and speed are mathematical reciprocals of each other. Please also see the article: **An Extra Note About Torque And Power** in the Extras And Late Entries section of this book.

Closely related to gear analysis is lever analysis of which an article is presented further ahead in this book. Consider a smaller output wheel, it has a smaller radius, and therefore, it essentially has a smaller leverarm. Since **Torque = (force)(leverarm)**, if leverarm decreases, the torque (T) decreases, but its speed (ie., angular velocity = degrees per time) will increase such as for a smaller gear being powered by a larger gear. Consider this mathematical relationship:
Power In = (Tin) (Angular Velocity in) = Power Out = (Tout) (Angular Velocity out)

When a wheels or gears increases its rotational or angular speed, the effective linear velocity of a point on its circumference (C) also increases by the same factor. Given a wheel with a circumference of C1 powering a wheel with a circumference equal to (n C1) = C2, it will take (n) rotations of C1 to rotate C2 once. If the time of each rotation of C1 is (t1), the time needed to rotate C2 is (n t1), hence a reduction in angular rotational speed. The linear speed of a point on C2 will be (n) times greater than that of C1.

For an example of where gearing is used to increase torque is that when a vehicle is going up a hill and it will need more force to do that than as compared to a flat or level road. This can be accomplished by using a larger power output gear so as to increase the torque (ie. [rotating] force) of the wheels, but of course it will be at a lower speed, but still, is will make it much more possible for that vehicle to reach the top of the hill. A vehicle at rest will initially need more torque to get it moving to overcome friction and to get it to its desired velocity. It takes a force (ie., torque) to alter the speed of an object. Once the vehicle is at the desired traveling speed, it will slow a little due to the wheel friction (ie., resistance, drag) with the roadway, and it will take only a relatively small applied input power and force to maintain that desired speed.

With any two or more wheel or gear system, given a system or motor that is turning just one output wheel such as for a vehicle, if that wheel size is increased, the speed of the vehicle will increase. If that wheel size is decreased, the torque of that wheel will increase.

The maximum amount of torque can be measured by using weights suspended on a string around the rotating shaft, and the weight can be lifted vertically upward. It is also possible to use a common weight or mass scale to measure torque in a mechanical type of linkage and-or manner. According to **Power = (torque)(rotational speed)** , mathematically: torque = Power / (rotational speed). Theoretically, if the input power to a very efficient motor is:

Pw = (V)(A) = Pmotor, The max. torque of an electric motor can be calculated by: **torque = (V)(A) / (rotational speed)** , and by this equation, the slower the rotational speed, the greater the torque created. Note that a motor may not be 100% efficient, but perhaps 90% efficient, and due to that some power will be lost (ie., not available as output torque) as heat energy due to the resistance of the internal coil wire and the friction of the bearings, etc. The faster a motor rotates, the greater the back-emf produced in that certain motor design, and this will reduce its power output - mainly that the rotational speed will have a maximum value. With a back-emf, the net or resulting voltage to the motor is reduced.

Gears transfer energy and-or power mechanically. Though gears are circular and provide a rotational and-or periodic (ie., repetitive) motion and timing, there are mechanical "linkages" or "cams" which can either turn a gear, or the gear moving these such as to create a linear (ie., straight) back and forth motion. Before the invention of high power motors which can be used wherever electricity is available, flowing water and wind power were used to power some machinery such as saws and grinding mills. One special gear is called a worm gear which has a screw shape, and which is sometimes used to convert between rotational and linear motion. Bevel gears can be used to change the direction of rotation such as from horizontal to vertical. If gears are separated by some distance, power can be transferred from one to the other using a shaft or rod, a belt or a chain. There are many gears and linkages already known about, and you can purchase the various books about them. New discoveries are always being thought of and made into reality. Some machinists specialize in gear creation from solid wood, plastic or metal blocks, and typically of standardized dimensions, however custom sizes can still be considered for prototypes of things being built and tested. The welding of metal is now a possibility so as to fix a broken metal gear. Gears can also be made by using hard, low friction plastics such as nylon. A common gear in most automobiles ("cars") is the **differential gear** system used to transfer the engine power and-or then a certain fraction of it to each (left and right) wheel of the car. This gear is a now considered a necessity for the car for it allows each wheel to rotate at a different rate (ie.,angular velocity, hence also the linear speed or speed on the roadway surface) as the car is turning left or right, and where one wheel is then acting as nearly a pivot or point of rotation for the other outer or more distant wheel wheel. For example, when the car is made to turn right, less power is applied to that right wheel, and the left wheel and car will then attempt to rotate about the center point of the right wheel.

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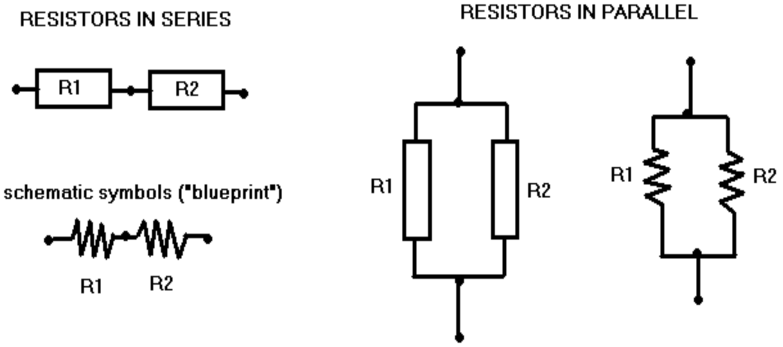
ELECTRICAL RESISTANCE EXAMPLE

This example is of the electronics or electrical realm, and shows some mathematical examples for reciprocals, and (relative) generalizations using percentages rather than specific numerical values. Near the end of this example is some further mention of the rate of changes, or the slope, between variables. If the reader is not interested, they may skip over this material and-or possibly review it at some other time.

Electrical, electric resistance or resistivity is anything which will reduce the flow of electrical current which is usually charged (with energy, and therefore has ability to help do things) electrons. Even a plain wire has some resistance, and it is therefore said as not being a "perfect conductor" (which in theory would have none, or 0 resistance) of electricity. The greater the resistance, the greater a given current will be reduced. Hence, current and resistance are inversely mathematically related and this is reflected in the (Georg) **Ohm's Law formula for electricity: current = voltage / resistance**, or in symbolic form as: I = V/R. There are electronic parts or "components" created specifically to have some measure or value of resistance, and they are naturally called resistors. Resistors have many uses in electronic circuits, for example, as current limiters (ie. to reduce current or set its maximum value permissible for that circuit), and to set voltage levels such as in a (resistive) "voltage divider" circuit. The common letter symbol or variable for resistance or a resistor is: R, and is often used mathematically as a variable for the measure of the specific resistance value of a resistor. [FIG 62]



If you were to take a wire and cut its length in half, each half of that wire will then have half the total resistance of that original wire since each is half as long. This is because resistance is directly related to the length of the (uniform sized) material. If the length is twice as long, the resistance will be twice as much. If the length is half as long, the resistance is half as much. This leads directly to the formula for the sum of two or more resistances:

Rt = R1 + R2 + R3 . . . : formula for the total resistance of resistors in series (one after the other, like two pieces of wire connected together)

Even though various resistors of various values can be in series, the same amount of current will flow through all resistors that are in series since there is no other conductive path for the current (ie., "electricity") go to travel through. This is so since the voltage (v or V) or electrical pressure or force is applied to the starting and ending points of the resistors that are in series as if it was one single resistance "seen", affecting or applied to by that voltage. The specific amount of current (It) will be determined by the total (effective) series resistance (Rt): It = V / Rt Though the current is the same through each series resistor, it does take more electrical force or pressure to force that same current through a higher resistance, and this is seen as a higher voltage across resistances of higher value. This is what causes the voltage divider effect for resistors in series, where the largest resistance will correspondingly have the largest voltage "drop" or "loss" developed across it.

By mathematically rearranging the Ohm's Law equation, we find that the voltage across a resistor (r) is equal to the current through that resistor, times the resistance of that resistor: Vr = Ir Rr A voltage across a resistor is often said to be a "voltage drop", "voltage loss" and-or a "power loss" since energy or power is "used up", lost or wasted as current is forced through a resistor, and that power is then unavailable to be used by the rest of the circuit. Though unavailable to the rest of an electric circuit, the energy or power (a measure of available or energy used. Power = voltage x current = P=VI, and units of power are called watts) wasted in and through a resistor is not actually lost but was rather converted to other forms of energy such as: heat from a resistor, motion from a motor, sound from a speaker, and-or light from a light bulb. From this concept, it is said that energy is "conserved", "not lost", or "is maintained" and that no amount of energy actually disappears. The sum or total output energy is equal to the sum of total input energy, and this is called "the conservation of energy".

So that a formula can be developed for the total (effective) resistance of two or more resistors in parallel to each other, the topic of conductance needs to be considered. Conductance (G) is a measure of the ability to pass or allow current for a given voltage or emf, and is equal to: $G = I / V$, and it is therefore, the exact opposite or inverse to the concept of resistance: $(R = V / I)$, and conductance is therefore mathematically expressed as the inverse or reciprocal of resistance: $G = \text{current} / \text{voltage} = I / V = 1 / R$

$G = \frac{1}{R}$: the units of conductance are often called **siemens,** or **mhos** which is derived from the word "ohm" spelled backwards, and that resistance and conductance are electrical opposites or inverses of each other.

If you consider a wire as like a pipe to carry or allow the conduction or flow (ie. current) of water, and if you had two pipes in parallel (along the side length of each other), you will have a greater ability or more "room", "space" or area to allow more flow, perhaps for filling up some container with water. In relation to a wire, you would have more current ability, and hence this is described as having more conductance. Total conductance is simply expressed mathematically as the sum of each (parallel) resistors conductance ability. When conductance increases, resistance decreases. When resistors are placed in parallel, the effective or total conductance of that system will increase, and the resistance will decrease. When resistors are placed in series, the total resistance increases and the conductance will decrease.

Note, it may now seem that the total resistance of parallel conductors or resistors is: $Rt = 1/G1 + 1/G2 + 1/G3 + \ldots$, but this is actually just another expression for the total resistance of series resistances or resistors. To find the right expression for parallel resistors, first consider this expression for total conductance of parallel resistors:

$$Gt = G1 + G2 + G3 + \ldots = \frac{1}{R1} + \frac{1}{R2} + \frac{1}{R3} \ldots = \frac{1}{Rt}$$

: conductance of a system of parallel conductors or resistors. Mathematically:

$$Rt = \frac{1}{Gt} = \frac{1}{G1 + G2 + G3 + \ldots} = \frac{1}{\frac{1}{R1} + \frac{1}{R2} + \frac{1}{R3} \ldots}$$

For only two parallel resistors, the formula reduces to a simple expression to memorize:

$$Rt = \frac{1}{\frac{1}{R1} + \frac{1}{R2}} = \frac{1}{\frac{R1 + R2}{R1 \times R2}} = \frac{R1 \times R2}{R1 + R2}$$

: Formula for two parallel resistors. Their product, divided by their sum.

A common example is using two of the same valued resistors in parallel, and the total effective resistance is simply half of that common value. Note that parallel resistors always create a lesser total or effective resistance because the total or effective conductance increases when two resistors are in parallel.

If R1 = R2:

$$Rt = \frac{R1 \times R1}{R1 + R1} = \frac{R1^2}{2R1} = \frac{R1}{2} = 0.5\,R1$$

: Rt is half of the common resistance value

Some considerations on the value of the effective or total resistance of two parallel resistors:

As a parallel resistor R2 increases from 0 ohms to infinity ohms, the effective parallel resistance will increase or rise from 0 ohms and approach (become nearly equal to) the value of R1, but never greater than R1. In short, a very high or "infinite resistance" (such as even the resistance of air) in parallel to R1, t than the effective resistance is practically equal to R1.

Given a parallel resistor R2 being either smaller or larger than the first resistance R1, the effective parallel resistance will always be less R1, and never greater than R1.

If a parallel resistor is the same value (or 100% of that first value), the total resistance Rt is 1/2 or 50% of that resistor.

If a parallel resistor R2 is only half (1/2) or 50% of that of the first resistor R1, the parallel effective resistance is 33.3..% , or 1/3 of R1.

If a parallel resistor R2 is only 1/3 ("one-third", or 33 percent = 33%) of R1, the total effective resistance is 25% or 1/4 of R1.

If a parallel resistor R2 is only 1/10 ("a tenth") or R1, the total effective resistance is approximately 90% or R2.

Ex. R1=10ohms in parallel to R2=1ohm, and Rt=is 0.90909... = 0.91....ohms, which is almost 1 ohm and about 90% of R2. Part of the reason this is not exactly a 10% effective resistance reduction is that values and their corresponding reciprocals don't have a linear relationship, hence their relationship is said to be non-linear, especially for small values. Also, as R2 (the parallel resistor) gets larger in value, it will cause less and less change in Rt.

If a parallel resistor R2 is double or twice that of R1, Rt = 0.667R1 or= 2/3 R1 = 66.7% of R1
If a parallel resistor R2 is triple or three times that of R1, Rt = 0.75R1 or= 3/4 R1 = 75% of R1
If a parallel resistor (R2) is nine times that of R1, Rt = 0.90 or= 90% of R1
For verification of the above results (Rt) of two parallel resistors:

From: $Rt = (R1 \times R2) / (R1 + R2)$ we can derive some "parallel resistance percent formulas".

If R2 is some known or desired fraction or percentage=P of R1 : $P = R2 / R1$, therefore $R2 = (P\,R1)$, and:

$Rt = ((R1) \times (P\,R1)) / ((R1) + (P\,R1))$: product divided by the sum. After some simplification:

$Rt = (P\,R1^2) / (R1(P + 1))$ simplifying further:
$Rt = (P\,R1) / (P+1)$ also note, mathematically, if you divided each side by R1:

$(Rt / R1) = P / (P+1)$:also remember that Rt is always less than any and all parallel resistors

If R2 is P percent of R1, the resulting parallel resistance is:

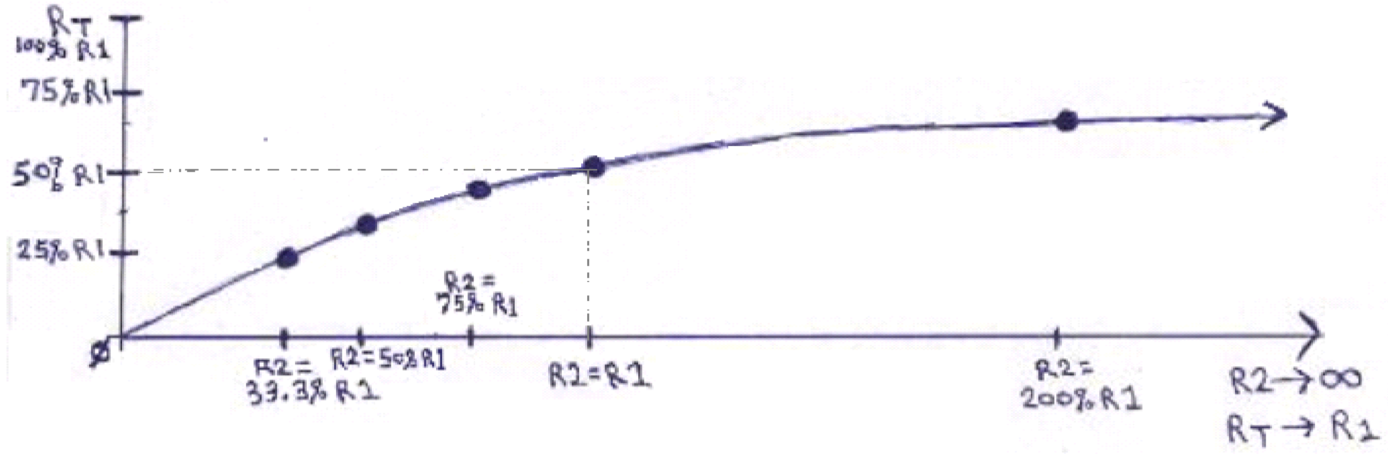
$Rt = [P / (P+1)]\%$ of R1 :Two parallel resistors, total or effective resistance percent formula.
Use the decimal value form for P, ex., for 300%, use 3.00

Ex. If R2 is 50% of R1, the parallel effective or total resistance is:

$Rt = (0.50 / (1 + 0.50))$ of R1 = $(0.50 / 1.50)$ of R1= 33.33% R1 , as mentioned above

Ex. If R2 is 20% of R1, the parallel effective or total resistance is:
$Rt = (0.20 / (1 + 0.20))\,R1 = (0.20 / 1.20)\,R1$ = 16.67% of R1

The following is a graph of the effective or total resistance (Rt) of two parallel resistors. R1 is fixed (a constant, unchanging) in value, and the value of R2 can change in value from 0 to infinity. The values of R2 are indicated on the horizontal (or "x") axis. The value of R1 is indicated on the horizontal axis. The values of Rt are indicated on the vertical (or "y") axis. [FIG 63]



As R2 increases the value of Rt (the effective parallel resistance of two or more resistances or resistors) does not change much. The slope of the curve is becoming near horizontal or level. This is due to that the rate of the changes in the corresponding values of each variable is getting less and less, and is approaching a value of 0, hence Rt is not changing by much in value with respect to the changes in R2.

Rate or ratio of changes of the dependent variable with respect to the independent variable:

$$\frac{\text{change in dependent variable}}{\text{change in independent variable}} = \text{ratio or rate of changes} = \text{slope}$$

This slope value is how the dependent variable will change when the independent variable changes.

$$\frac{\text{change in Rt}}{\text{change in R2}} = \frac{(\text{Rt.2} - \text{Rt.1})}{(\text{R2.2} - \text{R2.1})} = \text{slope}$$

Since a curve does not have a constant slope value like any line would, we can consider the curve as composed of just many small line segments, and therefore, we will make the change in the independent variable to be as small (or "close") as possible so that points chosen for this equation are as if they are on the same line or line segment, and that point or section of the curve can be analyzed like a small line segment for its slope value.

$$\frac{\text{the corresponding small change in Rt as R2 changes}}{\text{a very small change in R2}} = \text{slope of the curve at one point} = p(x,y) = p(R2,Rt)$$

The parallel resistance effect creates the condition that as Rt approaches a certain limiting value that is determined by and equal to R1, that Rt will approach (become "nearer and nearer to", or "closer and closer to") the value of R1, and that the changes in Rt, as R2 changes, get smaller and smaller until there is practically 0 change in Rt. The curve becomes near horizontal, like a horizontal line, as R2 increases.

Two, or possibly more, resistors may sometimes be placed in series or parallel so as to effectively create a third valued resistance that is otherwise unavailable.

When two resistances are in parallel, the effective or combined resistance is less than or equal to half of the largest resistance, and is slightly less in value than the smaller resistance if the difference in the resistances is large ,otherwise, it will be nearly equal to about half of any one resistor. A signal will see or be affected by the combined resistance and-or "the load". If the load resistance is low, more current can flow. If the load resistance is high, less current will flow. With impedance matching, the most electric power (= V I) is transferred.

A special resistor called a **varistor** is usually made from a semiconductor metal(s) and it can change its resistance in relationship to the applied rated max. voltage for that varistor. It is therefore a voltage sensitive, operated and-or a voltage dependent resistor (VDR). It can be placed in parallel to some electrical devices so as to protect them from excessive current due to an excessive applied voltage. It does this by sensing the applied voltage and reducing its own high internal resistance so as to conduct more and pass excessive current (ie., voltage and or current [temporary] "spikes", "surges") through it. A varistor is much like two diodes connected in reverse-parallel, and it has a non linear type of conduction conduction - essentially "on" (conducting) or "off" (not-conducting) like a diode. Due to this reasoning, a varistor is not a variable resistor. A **thermistor** is a special resistor that will change its resistance (depending on the specific type) due to the applied temperature. It is therefore very useful as a temperature sensor.

A List Of Common Metals And Their Relative Electricity Conductance

The lower the resistance, the greater the conductance and less power loss for the same size (diameter, mm^2) and length (m or cm) of wire, and at the same temperature. Room temperature is used during the measurement for the standard values. Silver has a (element specific) **resistivity constant (p)** of about 1.6 (10^-8) ohms-m = 1.6 (10^-6) ohms-cm. The values below are listed as relative values of conductance to silver which is the best know conductor with the lowest resistance. For a value of the relative conductance of a metal, use: (relative conductance) = (1 / relative resistance)

(element specific) resistivity = p = (Resistance measured in ohms) (Area in m^2) / (Length in m) ; R = p (Length / Area)

Material	**Relative conductance = (1 / Relative Resistance)**	: based on conductance and resistance are reciprocals
Silver	1.0	: an ecellent conductor , high cost , 1.6 / 1.6 = 1 = the relative resistance of silver
Copper	0.94	: an excellent conductor, relatively low cost , 1.6 / 1.7 = 0.94
Gold	0.73	: does not oxidize (ie., rust) and-or tarnish in the presence of oygen or water, very high cost if wire
Aluminum	0.60	: low cost, moderate conduction, resistance can be reduced by using thicker wire
Zinc	0.27	: this and the remaining metals listed are poor conductors having a much high resistance than silver
Nickel	0.21	
Iron	0.16	: relative reesistance of iron = 1 / 0.16 = 6.25 , 6.25 (1.6 uohms-cm) = 10 uohms-cm
Tin	0.13	: often usd in varius solder alloys, and sometmes with a small percentage of silver
Lead	0.07	: a poor conductor , but is still used mainly for fuses that melt, and electrical solder

The conductivity of silicon and germanium metal is much lower than that of lead, and they are considered as semiconductors and-or practically a high resistance. A small amount of other metals are added into these smiconductior metals so as to greatly improve their conduction such as for transistor (solid state amplifiers) devices and solarcells (solid state electricity generators).

A Note About Electromagnetic Fields

Below is a note about electromagnetic fields and current (ie., electron, matter) flow. This note ranges from basic to complex, and the reader may skip over it and review it later after viewing the more specific discussions about electricity and energy.

This book will usually describe the conventional thought about of the flow of electricity, and it is most practical and useful, such as when considering the water fluid analogy for the flow of electricity (ie., electrons). In advanced technological thought, the electrons only move a short distance and transfer their kinetic energy, usually via electrostatic (force, field) repulsion to the electron next to it in its path. This process is a chain-reaction, "one after the other" or "domino effect", and this surely gives the appearance that each electron is flowing or moving completely from one point in a closed circuit to the other. The electric force is repelling and giving the electrons kinetic energy, and these electrons will then have a motion and create a magnetic field and can be sensed by a nearby directional compass pointer ("needle") made of iron.

Stationary (static, non moving) charges such as electrons do not have a magnetic field. In a vacuum tube amplifier, electrons will travel much greater distance from one electrostatic (field, force) plate to the other before they collide with it and transfer their kinetic energy.

A voltage measurement is a measure of the electrostatic field strength between two points that are charged and are usually at different voltages, and hence the measurement is usually called a voltage (or potential energy) difference. If you were to take a resistor or long wire, the voltage difference measured between two points along it will go from say 0 to the maximum voltage value in that circuit. Lesser values of voltage can in general, always found between any two points along that wire, and that have same distance between those two points. In this case, it could be said that the electrical energy is evenly distributed.

For a traveling radio wave or light in outer-space that consists of no electrons, but both a magnetic field and an electric field, these expanding and traveling fields contain, store and convey the radio wave energy. These fields can be thought of as pure forms of, and the storage of electromagnetic energy without any moving charge carriers (ie., physical matter "storage containers" or "vessels") such as electrons which also have their kinetic (motional, mass) energy.

Radio waves and other electromagnetic energy, including light, travel at the speed of light in a vacuum, and slower in other mediums such as glass and metal wires. When electrons collide or repel with each other due to their electric charge field, they loose energy and transfer it to another electron of which it will take time to accelerate towards the next electron. It is incorrect to think that light travels instantaneously to everywhere, but it rather travels at the known speed of light, about 186000 miles / s =~ 300000 km / s per second in a vacuum (ie., no air or any other interfering, resisting and limiting matter) such as outer-space. It is currently (2021) thought in science that radio wave energy is a form of and identical to light or photon energy, and that photons are a "bit", "packet" or "particle" of light and having an electric and magnetic field, and no mass (ie., matter). These fields (or "pure electromagnetic" energy) travel at the speed of light in a vacuum. Electrons that have gained kinetic energy can also release that energy as a form of heat and-or light which is electromagnetic energy, and this is what happens in devices such as LED lights.

Even though the energy source to produce a radio-wave and-or light may be a relatively low frequency such as 50 hertz or 60 hertz AC from the utility company, or even DC from a battery or power supply which have a frequency of 0 hertz, the electromagnetic radiation such as the radio-wave or light produced has a very high frequency. This is due to the interactions of a huge amount of energized electrons. The electrons which have gained kinetic energy and escaped the orbit about an atom will eventually recombine with an atom needing an electron and release that gained energy as a photon (a particle, pulse, wave or bit of electro-magnetic radiation such as light) of energy. This photon is like it has released a small electro-magnetic field particle. Since the number of electrons involved is huge, the number of photons involved is huge, and the frequency of the electro-magnetic radiation depends on the amount of energy applied to the electrons. Consider a wire that is warm (ie., heat, thermal energy and radiation) due to its resistance and with some electricity going through it. If the amount of electric energy applied gets higher, that wire could radiate more heat and reddish light such as is the case for an electric heater. If the amount of energy applied gets even higher, heat and white light will be radiated, such as is the case for an incandescent light bulb. White light has also been found to be a mix (ie., a "rainbow") of colors such as red, green and blue, and with each color due to the specific frequency (ie., of a wave or pulses) of that electro-magnetic radiation, with each individual pulse or packet of it being called a photon particle. You may wish to view the topic **Common Colors Chart** further ahead in this book, and which has a figure that shows these colors. The electric and magnetic fields of each photon are at right angles to each other and are sinusoidal in amplitude (ie., intensity). The intensity of the light perceived depends on the amount of photons received per unit of area. In theory, once cycle of the electromagnetic wave corresponds to one photon of electromagnetic energy. It is often said that a photon does not carry any color information, but a wave of many of then, and which has a frequency, can then carry or convey the color seen. Obviously, the number of photons in a wave of them determines the amplitude or intensity of that light seen, and-or energy transferred. A wave will a higher frequency will carry and-or transmit more photons per unit of time, and therefore, more energy per unit of time.

More About Photons

All photons of light or electromagnetic radiation of any frequency will travel at the same speed through a given medium such as a vacuum or glass, and a single photon does not have a frequency. A series, number or wave (ie. of many, a wavefront) of transmitted photons can be transmitted at a certain frequency (ex. cycles per second, or photon or photon packets composing its physical wave per second, etc) and therefore having a certain wavelength or distance of travel per cycle of the wave. It may also help to know that a single photon normally travels in a straight line. A photon can be described as a invisible electromagnetic particle having no mass and inertia, and consisting of both an electric and magnetic field of potential energy. A photon is never at rest until it is absorbed as energy into another mass. In advanced particle physics it is considered that a photon having a high energy can collide with the nucleus of an atom and produce particles such as an electron and positron (ie., "antielectron", a positively charged electron, or "antimatter") pair. An electron and positron do have mass and of the same amount, but when they collide, they will annihilate (destroy, "disappear") each other and their energy is converted back to photon energy such as gamma rays. In each cycle of electromagnetic energy transmission, there is at least one photon, and if there are more photons in that cycle and-or wave, the intensity (ie., total energy) of that wave is greater. This book has many more articles about electricity, light and photons. albeit in a simple manner so as to not make things more complicated than they need to be.

WHEN A VARIABLE CHANGES IN VALUE

This example will show by how much one variable will change when another changes. It will also show how to consider and write some equations when there is a change by 1.

Let N=numerator or dividend , D = denominator or divisor , r = rate or quotient

$\frac{N}{D} = r$: If either N or D is a variable that can change, then (r) should not be considered as a constant. Mathematically:

$N = r D$: If (r) was a constant, this equation then has the form of a basic linear or "line" equation of: $y = mx$

In the following discussions (r) is a constant:

If D is to change in value, we can let (c) represent the amount that it will change. If D changes by a value, how much will N change by? Expressing this in an equation:

$N = r D$

$N + x = r (D + c)$ solving for x:

$x = r (D + c) - N$

$x = rD + rc - N$ since $N = rD$, using substitution:

$x = rD + rc - rD$ combining like terms:

$x = rc$: if D changes by c, N will change by rc

: If D changes by 1, then c=1, $x = rc = r(1) = r$

The above can also be expressed as:

$\frac{N + x}{D + c} = \frac{N + rc}{D + c} = r$: this is also a unique way to create an equivalent fraction, here of: (N / D)

If D increases or changes by a value (c), by what factor (f) does N increase? Expressing this in an equation:

$N = r D$

$N f = r (D + c)$

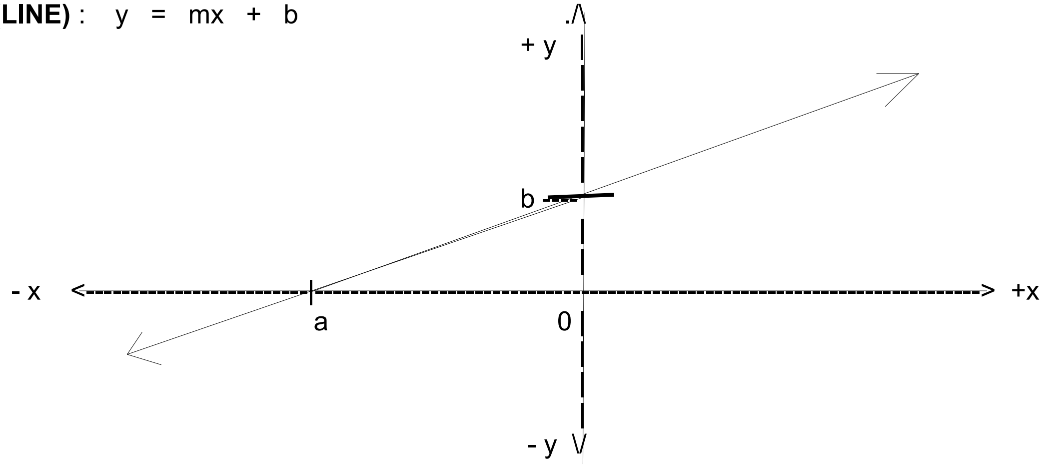
$$f = \frac{r(D + c)}{N} = \frac{r(D + c)}{rD} = \frac{D + c}{D}$$

$f = 1 + (c / D)$: if D changes by 1, then c=1, $f = 1 + (c / D) = 1 + 1/D$, also, since $D = N/r$:
$f = 1 + (cr) / N$

EQUATIONS AND SHAPES OF COMMON CURVES

For basic familiarity, below are some basic equations and a representative shape of the resulting curves when those equations are plotted on a graph. Some, such as the line, have already been discussed and others will be presented. [FIG 68]

LINEAR (LINE) : $y = mx + b$



A line is technically considered a special instance of all possible curves. As a simple verification, consider the curve or circumference of a circle being so huge that a relatively small portion of it, perhaps just a portion or arc of the circumference that corresponds to of just 1° (one degree) of rotation or movement of a point on it as it moves and rotates about the center of the circle, and is practically a line with the "bend" or curve in it being ever so slight that it is imperceptible (barely perceived or seen, very slight, small) when you are close to it. In more advanced reasoning, a very short or small segment of any curve will also resemble a line segment (a part of a line) and could be (locally, at that specific location, section, or point on the curve) analyzed as such, as for the slope of the curve at that point or location. A common example that considers this is with a concept called interpolation where an unknown value that is between two known values of an equation and-or a curve can be calculated like a point on a line or line segment.

The (y) or vertical intercept point, where x=0, is P(0, b). This can be found by substituting 0 for (x) in the linear equation and then solving for (y). To find the (x) or horizontal intercept point, where y=0, set (y) equal to 0 and solve for its corresponding (x) value. This (x) value is generally noted as variable (a) for a given line, and like (b), it is a constant for a given or certain line. The result is P(x,y) = P(xi, 0) = P(a, 0), where:

From: $y = mx + b$: setting y=0 so as to find the x-axis intercept point:

$0 = mx + b$ solving for the corresponding value of x for this point:

$mx = -b$

$$x = \frac{-b}{m}$$

$xi = a = - \underline{\ b\ }$: x-coordinate of the **x-axis intercept (i) point**, or simply the "x-intercept".

m

Algebraically, we also can find: **m = - b/a** , b= - ma , and
am + b = 0 which resembles the "slope (and, y) intercept" form of a linear equation, but this here is not actually a functional relationship of variables since all these values are constants (don't vary) for the line.
m = (change in y) / (change in x) = (0 - b) / (a - 0) = -b / a = - (b/a)

Here is a derivation of a linear equation in terms of the two (axis) intercepts or points.

From: $y = mx + b$

$mx - y = -b$ "dividing through" (each side, and hence all terms on each side) by -b:

$\frac{mx}{-b} \; (+) \; \frac{-y}{-b} = 1$ dividing the numerator and denominator of the first term by m:
(this also creates an equivalent [same value] fraction)

$\frac{\frac{x}{1}}{\frac{-b}{m}} + \frac{y}{b} = 1$ since -b/m = a , by substitution we have:

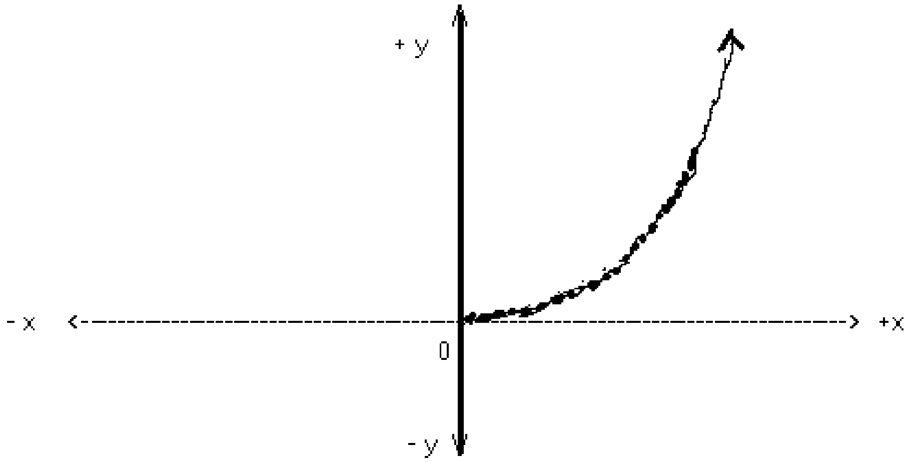
$\frac{x}{a} + \frac{y}{b} = 1$: "**dual or both intercept" form of a line**.
: Here, the slope (m = - b/a) is not explicitly indicated.
(1/a) is the numerical coefficient of (x), and
(1/b) is the numerical coefficient of (y).
(1/a)x + (1/b)y = 1 or=:
(1/a)x + (1/b)y - 1 = 0
a = x value of the x axis intercept , p(a,0)
b = y value of the y axis intercept , p(0,b)

After solving for y, we also have:

$y = b\left(1 - \frac{x}{a}\right) = b - \frac{bx}{a} = \frac{(-b)}{(a)}x + b$: this has the form of: y = mx + b
m=(-b/a) , b = (-am) , a = (-m/b)
x intercept point is: p(a,0) = p(-m/b , 0)
y intercept point is: p(0,b) = p(0,-am)

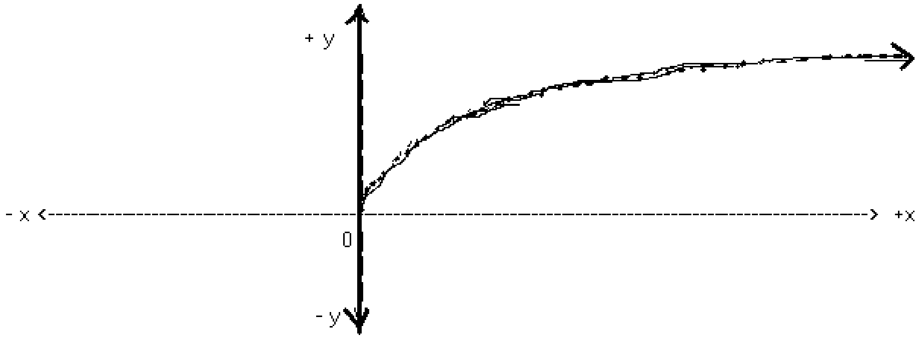
POWERS : y = x^n : a power equation is when the variable is the base value, and the exponent is constant

Ex. y = x^2 . Ex. y = x^3 . x is a variable, and x >= 0, n is a constant, and if n >= 1 , that is, the expression is not actually a root. If 0<n<1 (ie., the expression actually represents a fraction (ie., < x) amount or root of x, and with this mathematical condition, the value of y will get smaller as x increases.) [FIG 69]



For powers, is greater than 1 and gets larger, even small increases in it will cause a huge increase or change in the value of (y) which is the dependent variable. When x=1, the value of the equation y=x^n is 1. If the equation of a curve of this type (y=x^n) is unknown, it can be found such as by for example noting on the curve that when x=2, y = 8. Therefore, 8 = 2^n. Taking the log of both sides: log 8 = log 2^n = n log 2 , therefore n = log 8 / log 2 = 3 . The base of the log can be any value such as 10 or e, as long as it is used consistently. The equation of the curve is therefore: y = x^3

ROOTS : y = $\sqrt[n]{x}$ = x^(1/n) : n is the index or indicated root of the radicand x [FIG 70]



x is a variable, and x >= 0 (ie., x is not a negative radicand), n is a constant. If 0<n<1 , the expression is actually a power of x. As x grows larger in value, the curve resembles and will approaches that of a horizontal line which has a slope of 0. This means that the relationship of y to x, or the root with respect to the radicand, is not growing as much or quickly (rate) as when the radicands are lower in value. Consider x=radicands with the low values of 16 and 100.Their difference is 84, and their y=square roots differ by $\sqrt{100}$ - $\sqrt{16}$ = 10-4.0 = 6. Now consider higher value radicands of 10000 and 20000. Their difference is 10000, and yet their square roots differ only by $\sqrt{11000}$ - $\sqrt{10000}$ = 104.88 - 100 = 4.88

For indicated (ie., the index) roots greater than 1, when the independent variable is greater than 1 and gets larger and changes, then the corresponding changes or increases in the dependent variable get smaller.

Ex. y = x^0.5 : an example of a power expression, that actually results in a root of the given value, here of x.

y = x^0.5 = x^(1/2) = (1/0.5)$\sqrt{x}$ = 2$\sqrt{x}$

Obviously, variable (y) is directly related to (x), as can be seen by the graph of the curve above, but not in a linear manner.

The curve of this equation (considering both positive and negative roots) is actually that of a parabola "on it's side" with it's axis being the x-axis. To help verify this, consider $\sqrt{x}$ as equal to the perfect square of a value, say (n), then we have the quadratic equation:

y = n^2 = $\sqrt{x}$

Consider this example data:

n	y = +,- n^2	= +,- $\sqrt{x}$
1	1	$\sqrt{1}$
2	4	$\sqrt{16}$
3	9	$\sqrt{81}$
4	16	$\sqrt{256}$

: when the index is even (such as: 2, 4, 6, etc), roots, such as square (2) roots, are both positive and negative in sign, the curve will be "symmetrical" (like a mirror) with respect to a line (here, the x-axis line).

For additional verification, you can exchange the corresponding data values of (x) and (y) and find their relationship to be:

y = x^2 which is clearly a quadratic or parabolic equation.

For example, in this root equation, we will find a power equation:

y = $\sqrt{x}$ y = root = 2=index$\sqrt{\ x = \text{radicand}}$: root could be thought of as a base, and the index could be thought of as an exponent, as shown below:

3 = 2$\sqrt{9}$ therefore:

9 = 3^2 expressing this in a general algebraic sense:

x = y^2 in more of a relation to the initial equation, this could be expressed as:

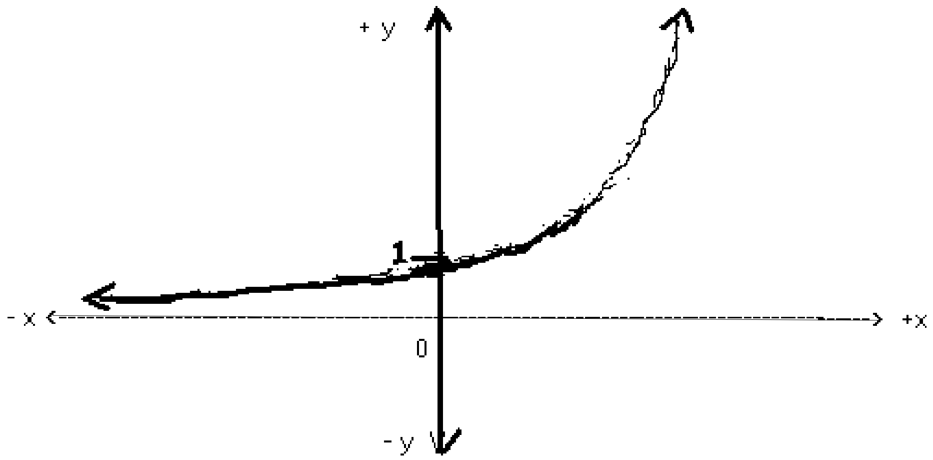
radicand = root^index this could be expressed using the more general and common relational form where y is the dependent variable, and x is the independent variable:

y = x^exponent

EXPONENTIAL : y = n^x : an exponential equation is when a variable is in the exponent, and the base is constant

Here, x, the independent variable, is an exponent, and this is why the equation is called an exponential equation. If you were to solve for (x), it is equal to: $x = \text{logn } y = \log_{n} y = \log y / \log n$

For an exponential equation, (n) is a constant. When $n >= 1$ this is the typical curve shape: [FIG 71]



When $x = 0$, $y = n^x = n^0 = 1$, and this is where the curve crosses the y-axis. Note that for a power equation as shown above, that when $x=1$, $y=1$.

Like other equations, an exponential equation may contain a multiplying constant (c):

$y = cn^x$: $c \neq 0$ so as the indicated power actually exists

The (y) or vertical intercept can be easily be found by setting (x) equal to 0:

$y(0) = cn^0$ since $n^0 = 1$:
$y(0) = c(1)$
$y(0) = c$: hence (c), the multiplying constant, is the vertical axis intercept; where $x=0$.

If the exponential equation is not known, (c) can be found by reading a graph of the corresponding data (points) of the unknown equation. If the equation is known, (c) can be easily found by:

$$c = \frac{y}{n^x}$$

Given a (y) value, the corresponding (x) value can be found by observing the graph of the equation, or mathematically by using logarithms:

$y = cn^x$ taking the logarithm of both sides:
$\log y = \log cn^x$
$\log y = \log c + \log n^x$
$\log y = \log c + x \log n$

$$x = \frac{\log y - \log c}{\log n} = \frac{\log(y/c)}{\log n}$$

If (y), (c), and (x) are known, perhaps from a graph, (n) can be solved for:

$y = cn^x$ dividing both sides by c:

$\frac{y}{c} = n^x$ taking the x root of both sides:

$\sqrt[x]{\frac{y}{c}} = \sqrt[x]{n^x} = n$ switching sides:

$n = \sqrt[x]{\frac{y}{c}}$: formula for solving for the base of an expressed power

If two or more pair of corresponding values (ie. the coordinates of two or more points) of an exponential equation are known (given, calculated or read from a graph), an exponential equation with the form of: ($y = cn^x$) can be constructed from that data:

Given two points (x,y): P(3.2 , 5.8) and P(4.7 , 8.2) that appear to be, and-or are on an exponential-like curve:

This can be expressed with functional notation as:

$y(x1) = y(3.2) = 5.8$: let = y1
$y(x2) = y(4.7) = 8.2$: let = y2, Solving for n, a base of the power, by substituting the known data into two instances of the basic exponential equation of: $y = cn^x$:

$$\frac{y2}{y1} = \frac{cn^{4.7}}{cn^{3.2}} = \frac{8.2}{5.8} = 1.413793 = \frac{n^{4.7}}{n^{3.2}} = n^{(4.7 - 3.2)} = n^{1.5}$$ therefore:

$n^{1.5} = 1.413793$ after taking the 1.5 root of each side:

$n = \sqrt[1.5]{1.413793}$

$n = 1.259671$ for the (y) values above to be shown as correct, (c) must now be solved for:

Writing equations or formulas for the above steps or procedures, we have:

$$n = \sqrt[(x2 - x1)]{\frac{y2}{y1}}$$

$$c = \frac{y}{n^x} = \frac{5.8}{1.259671^{3.2}} = \frac{5.8}{2.09325829} = 2.770799967$$

The exponential equation is therefore:

$y = (2.770799967)\ 1.259671^x$: this equation has the exponential format of: $y = cn^x$

Notice in the original stated points, that the values of (y) and (x) are directly related. As (x) increased, (y) increased. Also, if you were to divide the lesser value by the greater value, you would still arrive at the correct result:

$$\frac{cn^{3.2}}{cn^{4.7}} = \frac{5.8}{8.2} = 0.707317073 = \frac{n^{3.2}}{n^{4.7}} = n^{(3.2 - 4.7)} = n^{-1.5} = \frac{1}{n^{1.5}}$$

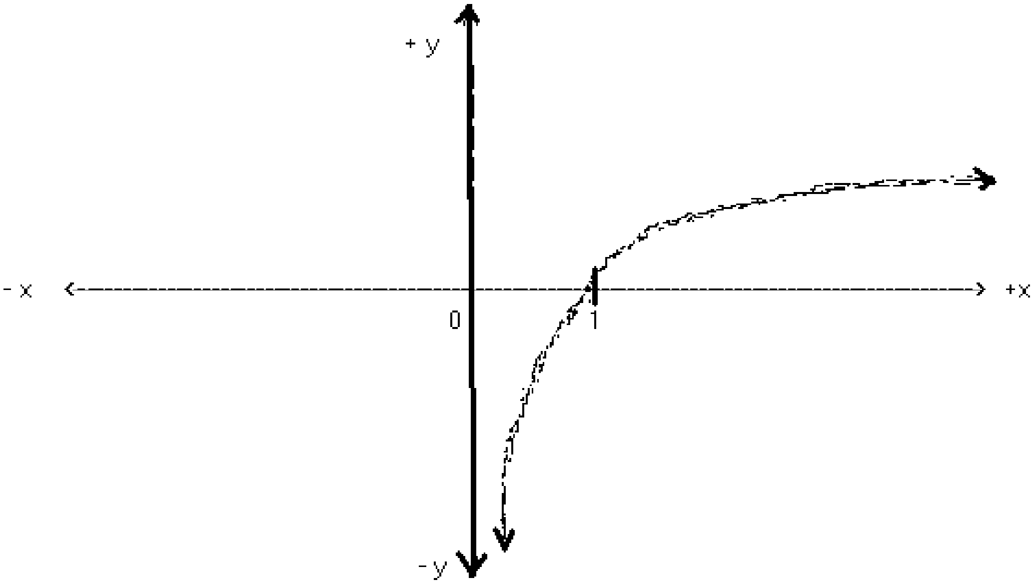
$0.707317073 = \dfrac{1}{n^{1.5}}$ taking the 1.5 root of both sides:

$\sqrt[1.5]{0.707317073} = 0.707317073^{(1/1.5)} = 0.707317073^{0.66666666} = 0.793857881 = \dfrac{1}{n}$

After solving for (n), we have: $n = 1.259671314$: checks, as shown above

LOGARITHMIC : $y = \log_b x$ [FIG 72]

When $x > 0$, and $b > 0$:



Note as (x) approaches infinity, the curve approaches the shape of a line. If you were to exchange the (x) and (y) axis of either the logarithm curve or the exponential curve, you would have each other curve. This is so since exponential and logarithms are inverse mathematical operations.

Here is a discussion and graph of the logarithm of a power of N with respect to the indicated (x, the exponent) power of N.

Given: $y = \log N^x$ using the log rules, this can be expressed as:

$y = x \log N$ or:

$y = \log N \; x$ both forms are of a basic linear equation of:
(In short, (x) the exponent, causes a multiple of log N.)

$y = m \; x$: here, $m = \log N$, and is a constant and the slope of the line

Hence the relationship between the exponent of N, which is x, and $y = \log N^x$ is a linear relationship. If (x) changes by some factor (F), (y) will change by that same factor (F). This is equivalent to multiplying both sides of the equation by F.

y = log N^x multiplying both sides by some factor (F):

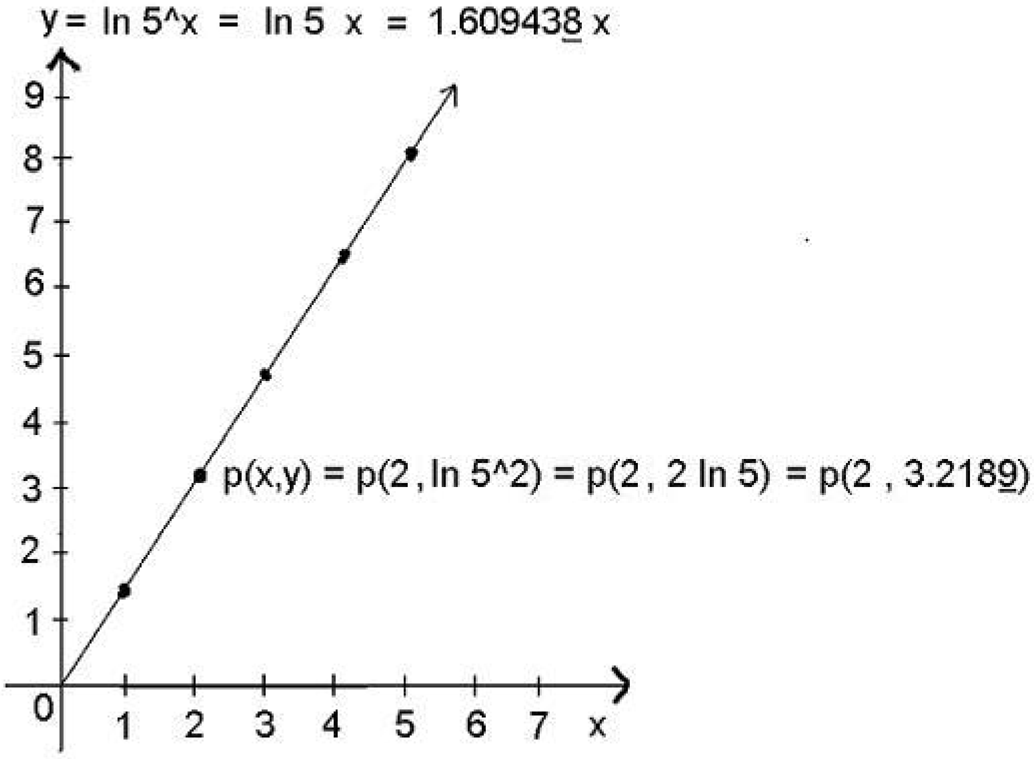
Fy = F log N^x
Fy = Fx log N
Fy = log N^(Fx)

Or, using the linear equation:

y = m x if (x) changes by some factor (F), and multiplying both sides by F to keep the equation in balance:

Fy = m Fx

Here is a graph of y = ln 5^x, the relationship between (y) and (x) is a shown to be a linear relationship and not an exponential relationship. [FIG 73]



The following curves are formally called conical curves, and are curves of various **conic sections** (cross sectional areas, "flat" planes or slices of). The word "conical" and "conic" comes from the word **cone**. A (right) cone is created when a right-triangle is rotated about one of its two "leg" sides which then becomes the central axis of that cone. These curves are the result of a planar (plane, flat), or two-dimensional, cross-sectional "slice" of a three-dimensional (right) cone. For example, when the slice or plane is at a right angle (ie., perpendicular, 90°) to the central axis of the cone, hence the slice is also being parallel to the base of that cone, the intersection of that plane and cone surface will create a **circle** or disk shape or curve. Each point on the perimeter (ie., edge, circumference) of the circular section of the cone is equidistant from a (center) point on the axis of the cone of which is also the center point of that circular shape or plane. When the plane is parallel to the side of the cone, a **parabola** shape and curve is formed at the intersection. When the plane is not parallel to the side of the cone or base, the shape and curve is an **ellipse**. If you were to take the circle section and raise one end or side up and the opposite side down, it would create an elliptical section of the cone. When the plane is parallel to the axis of that cone, it will create a **hyperbola**. The circle and ellipse are closed curves, and the parabola and hyperbola are not, but keep extending. For completeness, a point and a line can also be considered as (special, unique instances) conic sections. All of the (conic) equations that mathematically define one of the conic sections are actually a part of the general quadratic equation which will be discussed first.

First, here is the general or formally defined linear equation:

$ax + by + c = 0$: (a) and (b) are numerical coefficients (possibly 1, but not 0) of the (x) and (y) variables. (c) is an arbitrary constant (possibly 0).

Ex. After computing the resulting value (here 2) from two variables, the following equation was created:

$6x - 2y = 2$ expressing this in the general form:
$6x - 2y - 2 = 0$: by observation, and with respect to the general linear equation: a=6, b=-2, c=-2

Solving for (y), we will have (y) expressed in terms of (x), and hence an equation showing the relationship or function between the two variables:

$$2y = 6x - 2$$

$$y = \frac{6x - 2}{2} = \frac{6x}{2} - \frac{2}{2}$$

$y = 3x - 1$: this equation has the basic linear equation form of: $y = mx + b$: here b = -1

Given the general linear equation shown previously, and for the sake of simplicity, we will assume that the numerical coefficients ((a) and (b)) are 1, and letting f = c, a placeholder for an arbitrary constant, we now have:

$$(x + y + f)^1 = 0$$

By squaring the general first degree or linear equation, we will have the general second degree or quadratic equation:

$$(x + y + f)^1 (x + y + f)^1 = (x + y + f)^2 = 0$$

By using distribution and combining like terms on the left side, we have:

$$1x^2 + 2xy + y^2 + 2fx + 2fy + f^2 = 0$$

Dividing each term by (f) and assigning variables to these newly created numerical coefficients of the variables, we have the general or formally defined second degree or quadratic equation of:

$ax^2 + bxy + cy^2 + dx^1 + ey^1 + f = 0$: General Or Formal, Second Degree or Quadratic Equation

[This space for potential edits.]

CIRCULAR (CIRCLE) : x^2 + y^2 = r^2

Notice that for this simple equation of a circle, being (part) of a conic section, that it contains only some terms of the general form of a second degree or quadratic equation, that is, it need not have all the terms of that form, especially if those terms have a value of 0. This circle equation here is a simplified or unique form of the general or full formal circle equation which would have (x^1) and (y^1) terms. More specifically, this "reduced" or "partial" equation here is for when the center of the circle is at the origin (0.0) of the rectangular coordinate (rectangular, planer, two-dimensional measuring and addressing) system when it is graphed. The equation of the circle is actually a form of the Pythagorean Theorem for right triangles, where (r), the radius of the circle is essentially the hypotenuse side, and the corresponding "legs" or leg sides of this right triangle construction are the (x) and (y) values from the origin of this measurement system and center of this circle.

Algebraically arranging the equation into a form for plotting (graphing the relationship of corresponding values as points) a graphical or visible relationship between the variables:

$y^2 = r^2 - x^2$ taking the square root of both sides of this equation:

$y = \pm \sqrt{r^2 - x^2}$

Note, for a given or unique circle, r (radius) is actually a constant value.

Ex. Place the equation of: $3x^2 + 3y^2 = 9$ into standard form.

First, with some basic mathematical processing, this is of the general form of: $ax^2 + by^2 + f$
All the terms of the general quadratic equation are not present, but this partial form of the general quadratic equation that defines a circle.

Dividing through (on both sides) by 3, we find:

$x^2 + y^2 = 3$

Note that the coefficients (here 1) of both variables are equal.

Note also that $r^2 = 3$ (not 9), therefore, $r = \sqrt{r^2} = \sqrt{3} = 1.7320...$

After dividing both sides by 3 and canceling:

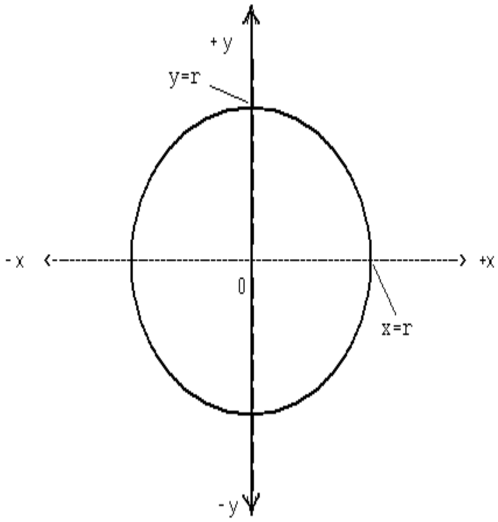
$\frac{x^2}{3} + \frac{y^2}{3} = 1$: this could optionally be expressed as: $0.333X^2 + 0.333Y^2 = 1$

Shown in this form, the square roots of the divisors (or denominators if you will) of the variables are the intercepts to the corresponding axis in the numerators. This is also the case for an ellipse curve that will be shown ahead. In general, for a circle with its center at p(0,0), the x-axis intercepts, where y=0, are clearly: $P(+- \sqrt{r^2}, 0) = P(+ - r, 0)$.

Algebraically arranging the equation for plotting by solving for (y):

$y = \pm \sqrt{3 - x^2}$

When x = 0, $y = \sqrt{\ r\ } = \sqrt{3}$.
When $x = r = \sqrt{3}$, $x^2 = 3$ and $y = 0$ [FIG 74]



All circles are similar circles to each other since the parts (circumference, diameter, radius) of each circle all have the same portion (fractional value) of that circle and with respect to each other. For example, C/D = (pi) for every circle.

Translation Of Axis And Coordinates

What if a point(s) is to be in reference to a another location and-or axis?

If a point(s), p(x,y) is to be centered about another location, say p(h, k), you must translate (ie., shift, adjust) the coordinates of each point (x, y) so as to be centered about p(h, k). The x location of each point must be increased by h, and the y location of each point must be increased by k so as to be still in reference to p(0,0).

The x coordinate value of each point in reference to this new system centered around p(h, k) will be called x**' =** "x-prime", and the y coordinate value will be called y**' =** "y-prime". p(x' , y') can also be thought of as a different reference or origin.

In relationship to the standard reference system with p(x=h=0 , y=k=0) as the origin, a translated point will have the coordinates of:

p(x , y) = p(h + x**'** , k + y**'**) or= p(x**'** + h , y**'** + k) , from this we mathematically have:

p(x**'** , y**'**) = p(x - h, y - k);

If the coordinates of a point are x = 3, and y = 2 and the center or point of reference is the standard p(0,0), and if this point is now to be in reference to, or centered about point p(10, 5), we will let h=10, and k=5:
p(x' , y') = (3, 2) and p(x , y) = p(x' + h, y' + k) = p(3+10 , 2+5) = p(13,7);

Let's write the equation of a circle that has a radius of 10, and its center point (0,0) translated to be located a p(4,6).

x^2 + y^2 = r^2 is re-expressed or translated to:

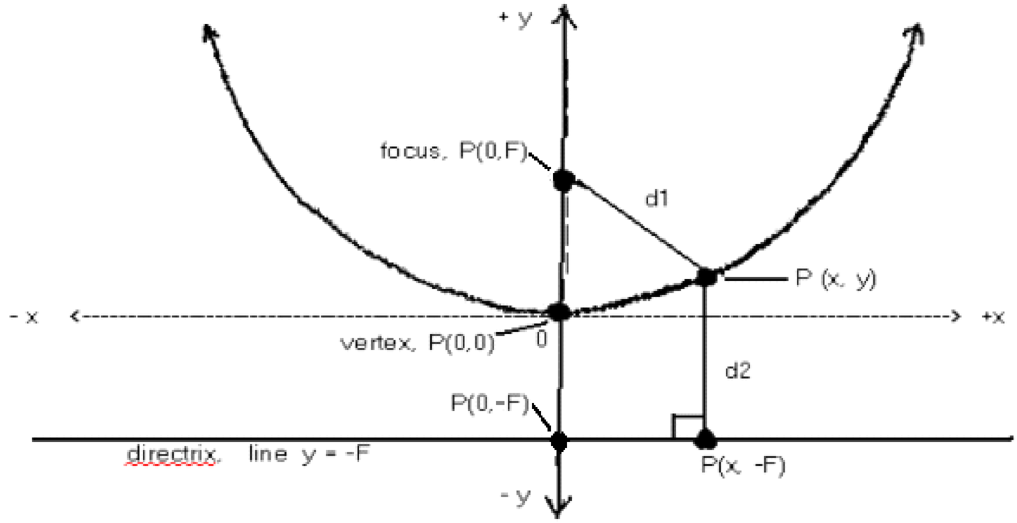
x**'**^2 + y**'**^2 = r^2 =

(x - h)^2 + (y - k)^2 = r^2 substituting the known values , if r = 10, then r^2 = 10^2 = 100 :

$(x - 4)^2 + (y - 6)^2 = 100$ and: $(y - 6)^2 = 100 - (x - 4)^2$ and $y = \sqrt{100 - (x - 4)^2} + 6$

PARABOLIC (PARABOLA) :

Observing the drawing below, a parabola curve is formed when the distance (d1) from any point on the curve to the (fixed or constant) focus point is equal to the (perpendicular) distance (d2) from that same point to an imaginary line (called the directrix) that is perpendicular to the axis of the parabola. Note, that this distance (d1 and d2) of either or both of these lines (to those points mentioned) is not constant since it changes from point to point. The focus of a parabola is located on the axis of the parabola. The parabola curve is symmetrical (ie, similar or the same in shape) about this axis and extends endlessly and appears to get wider and wider as the distance to its axis increases. The directrix line is located at the same relative distance that the focus point is from the vertex (ie. the "narrow point" and where the curve and slope changes direction and value) of the parabola, but on the opposite side of the vertex and perpendicular to the axis of the parabola. Often, the equation and shape of a parabola are constructed around an initially specified focus point. [FIG 75]



$y = ax^2$ is the equation of the above parabola curve.

The equation of a parabola when the vertex is at the origin (h=0,k=0), and the axis of the parabola is along the y-axis is:

$x^2 = 4Fy$: Where F (a constant) is the focus point (or focal length) along the (y) axis, point (0,F). If F is negative (ie., when y's numerical coefficient is negative), the parabola curve opens downward. Solving for (y) for plotting points and creating the smooth curve on a graph:

$y = \frac{x^2}{4F}$ Since 1/ (4F) is a constant (c), this also has a more general form of : $y = c x^2$, or $y = a x^2$ if we replace (c) with the typical or formal equation variable (a).

This simple equation of a parabola will now be verified. As stated above, for all points on the parabola curve:

$d1 = d2$ using the (Pythagorean or right triangle) "distance formula":

$\sqrt{(y - F)^2 + (x - 0)^2} = y + F$ after squaring both sides, to clear the radical:

$(y - F)^2 + (x)^2 = (y + F)^2$ extending the squared binomials:

$y^2 - 2yF + F^2 + x^2 = y^2 + 2yF + F^2$ transposing terms (solving for x^2):

$x^2 = 4Fy$ and solving for y:

$$y = \frac{x^2}{4F}$$

However, this equation can be expressed in another, perhaps simpler, form. Since 4F (or: 1/4F) is a constant (c) for any specific parabola:

$x^2 = (4F)y = cy$: after letting $c = 4F$, $F = \frac{c}{4}$ solving for y:

$$y = \frac{x^2}{c} = \frac{x^2}{4F}$$

$y = \frac{x^2}{c} = \frac{(1)(x^2)}{(c)(1)}$ expressing this reciprocal value of (c) as a new constant (a) :

$y = ax^2$: This is the most basic parabola equation, which is actually a (simple) quadratic equation, and also a "power (here it's 2) of x" equation. Note that if F is negative, then (a) and the (y) values will be negative and the curve will expand or open downwards.

Note: $a = (1/c) = 1/(4F) = 0.25/F$ and therefore, $F = 0.25/a = 1/4a$:the focus point for this equation and parabola would be: $F(0, 1/4a)$

Ex. Given : $y = 0.025x^2$, find the focus point of this parabola.

This has the expressed form of: $y = ax^2$, where $0.025 = a$, and $b=0$ which is essentially the vertical offset of the curve.

Since: $a = \frac{1}{c} = \frac{1}{4F}$, (c) is the reciprocal of (a):
Also of note: $F = (1/4a) = (0.25/a) = (c/4)$

$$c = \frac{1}{a}$$

Taking the reciprocal of the numerical coefficient (a) of the squared variable (x):

$$c = \frac{1}{a} = \frac{1}{0.025} = 40$$

Hence $y = ax^2 = \frac{(1)}{(c)}\frac{x^2}{1} = \frac{(1)}{(4F)}x^2 = \frac{x^2}{40}$: here, expressed in the general parabola form. Note also that: $a = (1/40) = 0.025$

Here, $c = 4F = 40$, solving for F:

$F = \frac{c}{4} = \frac{40}{4} = 10$: note also that $F = 0.25/a = 0.25/0.025 = 10$

$F = +10$: the focus of this parabola is at point: $P(0,F) = P(0,+10)$

Given the original equation of a parabola: $x^2 = 4Fy$, and taking the square root of both sides:

$$x = \pm \sqrt{4Fy}$$

This shows that for each (y) or vertical value, there are two (x) or horizontal values, one positive and one negative in sign, but both are the same in absolute value. These two points with the same (y) value or y-coordinate are on opposite sides of the axis of the parabola. Clearly then, the axis of this parabola is the vertical or (y) axis, and the parabola will be symmetrical about it.

The equation of a parabola when the vertex is at the origin, and the axis of the parabola is along the x-axis is of an identical form that has the variables switched:

$$y^2 = 4Fx \quad \text{and} \quad y = \sqrt{4Fx}$$

To graph a parabola point, you will obviously need more points than you do with the graphing of a simple straight line. Some other key (significant, helpful, guiding) plotting points of the curve are the x-axis intercepts (where y=0) which are the roots of the quadratic or parabolic equation, the y-axis intercept (where x=0) which is easily found to be P(0,c). It is possible that the curve does not cross the x-axis, and for such a case, the roots are "imaginary". Another key point is the vertex point of the parabola. Other names for this point are either the minimum or maximum point which both go by the general name of the "extreme point" of the curve. As derived ahead in the discussion about derivatives, the x-coordinate of the extreme point of a parabola is:

$$x = -\frac{b}{2a}$$

: x coordinate of extreme point of a parabola.
These variables correspond to those of the
standard quadratic equation: $y = ax^2 + bx + c$

Variable (a) cannot be 0 since anything multiplied by 0 is 0, and this would then eliminate the x^2 term and the equation would no longer be a quadratic equation. Also notice that the bx^1 term can only be 0 when (b) is 0, and then the x^1 term is not present. To find the corresponding (y) value of the min. or max. point, you can substitute this points (x) coordinate into the original equation given. Algebraically, it is found by substituting this (x) value into the general quadratic equation:

$$y = ax^2 + bx + c \qquad \text{by substitution:}$$

$$y = \frac{a}{1}\left(\frac{-b}{2a}\right)^2 + \frac{b}{1}\left(\frac{-b}{2a}\right) + c \qquad \text{after distribution:}$$

$$y = \frac{b^2}{4a} - \frac{b^2}{2a} + c$$

The first two terms have a factor of b^2 in the numerator and (2a) in the denominator. We can factor out those values:

$$y = \frac{b^2}{2a}\left(\frac{1}{2} - \frac{1}{1}\right) + c \qquad \text{combining}$$

$$y = \frac{b^2}{2a}\left(-\frac{1}{2}\right) + c \qquad \text{distributing:}$$

$$y = -\frac{b^2}{4a} + c$$

: y coordinate of extreme point of a parabola

The coordinates of the extreme point Pe are therefore:

Pe (- b/2a , - b^2/4a + c) : "extreme point" of a parabola is the max.("highest"), or min. ("lowest") point on the curve of a common parabola or quadratic equation.

Notice that variable (x) plays no role in the expression for the coordinates of this point since this extreme point is fixed, that is, it is a constant for the given equation and parabola, and does not depend upon any (x) variable value.

What is the effective "rim to rim", or "curve to curve" distance of a line, parallel to the directrix line, of the parabola at the height of the Focus point? If we set y=F, we can solve for its corresponding x value, and then double this x value:

From: y = x^2 / 4F , if y=F : F = x^2/4F , x^2 = 4F^2 , $x = \sqrt{4F^2}$ = $x = \sqrt{4}\ \sqrt{F^2}$, x = 2F and 2x =4F

All parabola curves are essentially similar when considered as being just magnified versions of each other, while still having the same exact parabolic curve shape. For a mathematical verification to this, consider that the distance from the focus (F) point to the curve at the height of the focus is always 2F. This point is always of the form: p(x,y) = p(2F,F). At great distances from the vertex, the two "arms" or sides of the parabola will be practically parallel to each other as they get nearly vertical when the slope of the curve becomes a very high value.

If a plane parabola shape or curve is rotated about its (symmetry or symmetrical) axis line, the three dimensional solid or volume created is called a paraboloid. Cross sections or planar "slices" parallel to the directrix will be circular.

Here is a short discussion about the different basic forms of a quadratic or parabola equations and their curves.

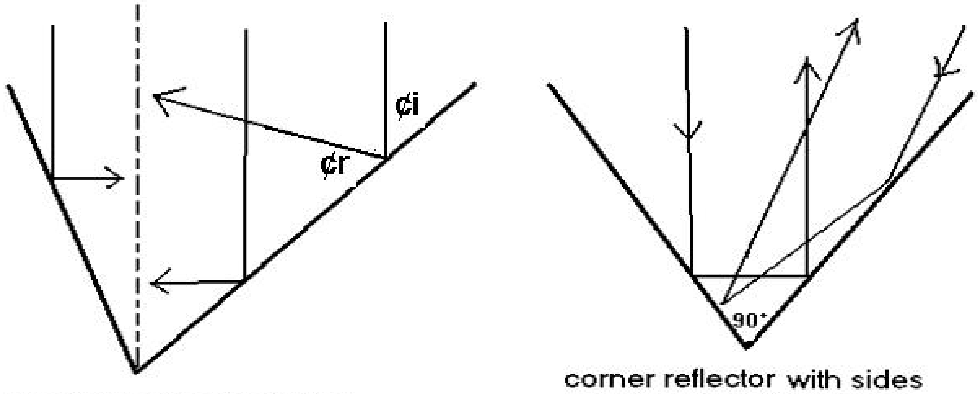
y = x^2 : This parabola curve will have a point, here the extreme or minimum point, at (0,0). When x is less than one, it's square will always be a smaller value since this is essentially a fraction of a fraction. The curve is "slow growing" for these values. When x is greater than one, the "exponential effect" of squaring of a value, is easily seen as the change in y values grows larger and larger for each change in x. The curve becomes "fast growing". Since the (rate of) change is not a constant as for linear or line equations, the equation and the relationship of the variables of a parabola equation is said to be a non-linear. When the x^2 term is negative: y= - (x^2) = - x^2, the curve will open downwards and have a maximum point. The value of x^2 without the -1 multiplier is always positive in value for all values (both pos. and neg.) of x. y=-(x^2) can also be expressed as y=(-1)(x^2), and we see its the same as the equation of: y=(x^2) but each value of y is essentially converted to its corresponding negative value by the (-1) factor. As the values of x increase more and more , the corresponding values of y will decrease more and more, and produce the inverted or "upside down" parabola curve.

y = x^2 + c : Here, a constant is added into the most basic quadratic equation. The effect. like as in a line equation, will shift the curve vertically. The vertex or minimum point of the parabola will no longer be at (0,0), and any part of that curve will not cross at that point either since for this equation, when x=0, y=c. This constant (c) is much like a "lead value".

y= x^2 + x^1 : Here, with the addition of the variable to the first power will shift the axis of the parabola horizontally (left and right). This will change the origin and vertex of the parabola and it will no longer be: (h=x=0, k=y=0). Still, the curve will cross through this point since when x=0, y=0.

y = x^2 + x^1 + c : This will be similar to the equation above, however the curve will be shifted vertically along the y-axis, and it will not cross through point (0,0) since for this equation, when x=0, y=c. (c) will not change the shape (and therefore, it also wont change the slope) of the curve, and it will have the same shape as that of: y = x^2 + x^1

Extra: If the sides of the curve were not a parabola shape, but flat or straight shape, there would not be a common single focus point of the incoming or collected rays of light as an example. A common line or plane placed between the two reflective sides could be called the focus line or focus plane instead of a focus point. The incoming rays need not even be parallel to strike this line. In a "corner reflector" (ie., right angle or 90° reflector) often used for indicating objects in dim or dark conditions, the two reflective surfaces or sides have a 90 degree angle of separation. Each reflected ray will be reflected back in the same direction or parallel to the incoming ray such as from a light source. A similar reflector is called a retro-reflector and which basically means a "return reflector". [FIG 76]



basic two sided reflector

corner reflector with sides separated by 90 degrees

. **For any mirror, the angle of reflection with respect to the surface of the mirror will equal the angle of incidence of the light or energy.**

¢i = ¢r

Several retro-reflectors have been placed on the surface of the Moon, and have been used for various purposes such as for measuring the distance to the Moon (~ 240000 mi =~ 384000 km.)

From: distance = (speed) (time) , using substitution for these values we have:

Distance to Moon = (Speed of light) ((Total time taken for light to travel to the Moon and then back to Earth) / 2) =
Distance to Moon = (Speed of light) (Time to travel to the Moon)

For a parabola shape: Incoming or receiving parallel light rays or other energy such as sound and radio waves that happen to also be parallel to the axis of the parabola will be reflected to and concentrated at the focus point. In a reverse manner, rays emitted from the focus point in every direction and those emitted towards the parabola's curved surface will be reflected parallel to the axis of the parabola.

The depth (or usually, vertical) distance from the edge of a parabola curve (or some other arcs) to the center of the parabola is called the **sagitta** distance. Sagitta is a word which is commonly called the "sag" of the curve or object. This distance is essentially equivalent to the distance from the arc or curve to the central point of a chord line which intercepts that arc or curve at two points. When rough grinding a parabolic mirror for a telescope, this distance can be checked so as to know when the grinding should stop and polishing should begin. The longer the focal length of the mirror, the less grinding needed, and the sagitta distance is perhaps only a few millimeters. This distance can be found using the standard parabola equation: y = a x^2 or as: height = vertical = y = a x^2 = a (horizontal distance)^2. See the full discussion above to calculate the value of (a = 1 / (4F)) which includes the focal (F) point distance. y = ax^2 = x^2 / 4F

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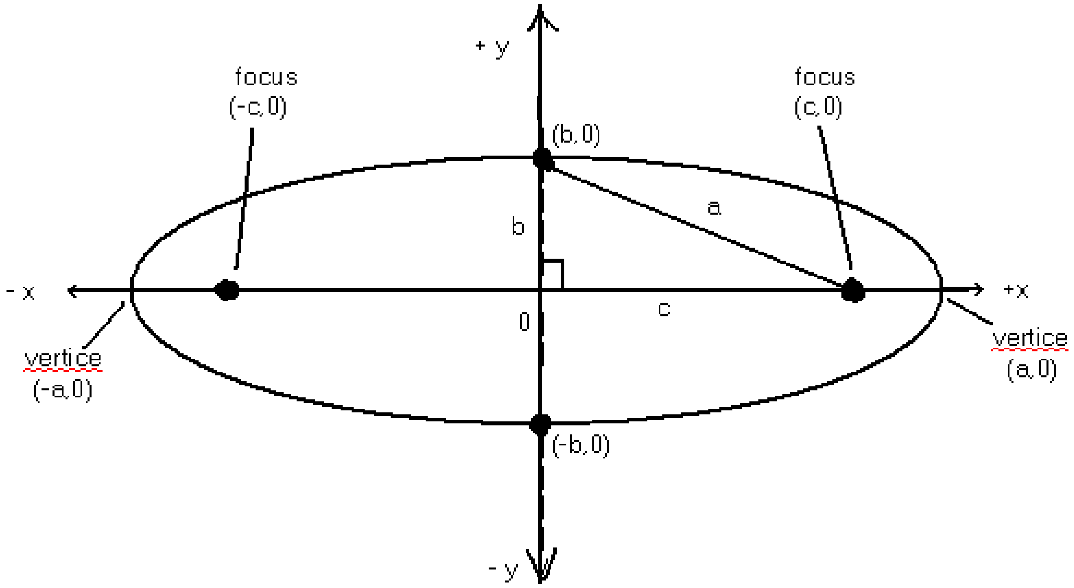
ELLIPTICAL (ELLIPSE) :

The shape of an ellipse is like that of a stretched or elongated circle, hence its shape is much like an oval or an egg. If you look at a circular ring shape and tilt it away from you, it will appear very similar to an elliptical shape. Numerically, a measure of this elongation, non-circular, or "unbalance" is called the eccentricity (e) of the ellipse. Don't confuse the variable (e) used here with the ("natural") base value used in the context of natural logarithms. The longer the elongation, the greater the eccentricity (e) of the ellipse. The eccentricity of a circle is obviously 0 since the length of the horizontal and vertical axis within a circle are the same and therefore produce no elongation in any one direction. The horizontal and vertical axis lengths within a circle are obviously equal to the diameter or (half-diameter) radius value of that circle. A circle is actually a special instance of an infinite number of ellipses. The eccentricity of a circle is 0. When the eccentricity of an ellipse is 1, the ellipse is completely elongated into a line. Most planets have a near circular orbit, hence they have a low value for orbit eccentricity. Comets usually have an orbit having a high eccentricity value. Both the circle and ellipse are examples of what are known as "closed curves" since they completely define and enclose (contain, surround) a bounded region of a plane. All circles are similar, but may vary in sizes, hence they are not identical. Ellipses that have the same eccentricity value are similar, and they may even vary in size, and are therefore not identical ellipses. [FIG 77]



Orbits of outer-space objects, such as planets and comets going around the sun are actually elliptical in shape. Here, the sun is effectively at one of the two focus points of the ellipse or elliptical orbit. This orbit shape of a planet is caused by the constant pull of the force of gravity from the Sun, and will cause the object to slowly change direction and speed.

If energy (ex, sound, light) was emitted directly outward from any focus point of a (solid, physical) ellipse shape, the energy would be reflected from the inner curve edge of the ellipse to the other focus point. An egg is somewhat of an ellipsoid (three dimensional elliptical) shape where an ellipse is fully rotated about its major or minor axis.

An ellipse ("closed") curve is defined or formed when the total distance from any point on the ellipse to both foci (plural of focus) is always the same, and that distance has a value of (2a). This distance is also equal to the length of the major (ie. longest) axis of the ellipse which passes through both foci and the center or origin of the ellipse. For a circle, it's center point is effectively both focus points being in the same place. Here (r=radius or 2r) is the common distance. The more eccentric the ellipse, the more elongated it is and the farther apart the foci are, and vice-versa, however the maximum eccentricity (not a length value, but practically a ratio value) is defined as a value of 1. The focus points on an ellipse are equidistant from it's center point along the major axis. Before the standard equation of an ellipse is shown, the ellipse will first be discussed using the common circle and its equation since a circle is a special instance of an infinite number of ellipses.

Starting from the common equation of a "representative" or unit (1) circle where the radius is understood as being 1 (or 100%), the curve of an ellipse can be created and perhaps more easily understood.

$r^2 = x^2 + y^2$: basic circle equation. After solving for y^2, and taking the square root of both sides:

$y = +- \sqrt{r^2 - x^2}$: When x=0: y = +- r, and y is at its maximum or minimum value:

$y = +- \sqrt{r^2 - 0^2} = \sqrt{r^2}$

$y = +- r$: the max. and min. y value, p(0, +- r)

When r=1 (or 100% if you will):

$y = +- \sqrt{1^2 - x^2}$

$y = +- \sqrt{1 - x^2}$

Obviously, x^2 must be less than or equal to 1 for the square root of a positive radicand value to be taken. As x grows larger and approaches the value of the radius=1, x^2 becomes closer to 1, and y becomes smaller and closer in value to:

$y = +- \sqrt{1 - 1}$

$y = +- \sqrt{0}$

$y = +- 0$: the min. y value, p(x=+- r , 0)

The values of x can be negative, and the square of a negative value is a positive value. After all, the circle is symmetrical about its origin point. Also, for each value of x there are two corresponding y values as indicated in the equations above, and is due to the fact that the two square roots of a positive value or radicand are both positive or both negative in sign.

As shown above, when x is close in value to the radius, the square root of x^2 will be close to 1 and the difference of ($r^2 - x^2$), and the corresponding (y) values, approach 0 in a non-linear (non-line-like) manner due to the squared value and the square root values that always result in non-linear values. The result is the curve of the circle, or a curve of an ellipse when the maximum (y) value is no longer equal to the maximum (x) value. In particular for the ellipse, each and every (y) value (or even each (x) value) of the normal circle equation is divided by the same (constant) value (other than, or not equal to 1), or it can also be said that each and every (y) value now has a numerical coefficient (other than 1). If each (y) value is divided by a value >1, then an ellipse with a larger (x) axis (or "major axis") will be produced.

Is there a new focus point? First of all, the center of the ellipse, like the circle, will still be at the intersection of both the (x) and (y) axes (unless the curve is "translated" or moved to a new reference or origin position within the standard rectangular coordinate system). Any straight ray emitted from the center of a circle will be reflected back (from the circumference) to the center (ie. focus point) since the angle of reflection with respect to that reflective surface will equal the angle of incidence with respect to that surface. For a circle, the angle of incidence and reflection are both actually 0° with respect to a perpendicular line at that surface or circumference point. Remember, a tangent line through a point on the circumference of a circle is always perpendicular to the radius line drawn to that point. The distance the ray or beam will travel is obviously equal to a distance of (r) to the circumference, and (r) back to the focus, hence a total distance of:

$r + r = 2r$: total distance for a ray from the focus of a circle and back to the other (coinciding) focus .
This is similar to the concept and value of 2a, as for an ellipse mentioned previously.

It is fair to guess that even though for an ellipse, which is like a circle that is essentially "squashed" and-or elongated, there is still a focus along the longest axis of it (the curve shape) due to the symmetry of the curve on both sides of that axis. It is also fair to guess that there are two focus or "foci" points due to the symmetry of the curve about the other axis. This leads to a conclusion that there are two foci points which are the same distance from the center of the curve along the major axis. If a circle was only slightly "squashed" or elongated (ie., stretched) to an ellipse shape, it is fair to guess that the foci points are still near the center of that circle as before it was "squashed" or elongated very slightly. As the circle or ellipse gets more elongated, the farther the foci points will be from the center.

A ray of energy (perhaps light, radio waves, or sound) emitted from one focus point will be received by the other focus point after reflecting from the ellipse curve shape to be collected or concentrated at that receiving focus. This is obvious for a circle. For an ellipse, this can be easily verified when the point of incidence and reflection is at the y-axis interception point of the curve. By drawing a tangent line to the curve at that point, and by using the symmetry of the angles of incidence and reflection with the y-axis, and even the angle of transmission and reception of the rays at the foci, then this fact (and the constant distance of 2a) is verified. It must be noted that all the rays emitted (in various directions) from the center (not the foci points) of an ellipse will generally not pass through both foci or even back through the center.

When the (y) values are divided by a high value (this is equivalent to multiplying each (y) value by a very small constant value = [1/high-value]), they approach a value of 0, and the square of those (y) values (being less than 1) will approach a value even closer to 0:

$r^2 = x^2 + y^2$ if $y \to 0$ (read as: "if, or as, y approaches a value of 0"), then:
$r^2 \to x^2$ r^2 will approach a value of x^2, and r will approach a value of x.
After taking the square root of both sides:
$r \to x$ With substitution:

$$y = \sqrt{r^2 - x^2}$$

$$y = \sqrt{r^2 - r^2}$$

$$y = \sqrt{0} = 0$$

So regardless of the value of (x), all the (y) values will be 0 and the curve of the ellipse approaches the shape of a (horizontal) line, hence it's the line: y=0. We then also have some verification that a line (segment) is a special instance of a conic section, and in particular, an ellipse.

Given: $y = \pm\sqrt{1 - x^2}$,

If each (y) value is multiplied by a numerical coefficient, say variable (b), as discussed above for making an ellipse, the equation for each (y) value is now:

new $y = \pm b\sqrt{1 - x^2}$ squaring both sides, we have:

$y^2 = b^2 (1 - x^2)$ after dividing both sides by b^2 :

$\frac{y^2}{b^2} = 1 - x^2$ and:

$x^2 + \frac{y^2}{b^2} = 1$ after solving for x :

$$x = \pm\sqrt{1 - \frac{y^2}{b^2}}$$

If each (x) value is multiplied by a numerical coefficient, say variable (a), and after squaring both sides of this equation to rid the radical, we will have:

$$x^2 = a^2\left(1 - \frac{y^2}{b^2}\right)$$ after dividing both sides by a^2 :

$$\frac{x^2}{a^2} = 1 - \frac{y^2}{b^2}$$ algebraically we have:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$: **BASIC EQUATION FORM OF AN ELLIPSE**
Center is at the origin, p(0,0), of both axis.

When the center of the ellipse is at the origin (x=h=0, y=k=0) and the (major) axis of the ellipse is on the (x) axis, the equation of the ellipse is of the basic form shown above. When (a>b) the (major) axis of the ellipse is on the (x) axis. If (b>a) the (major) axis is along the (y) axis. When (a = b), the equation is actually for a circle, for example, when (a) and (b) are both 1. This form above is essentially a basic "circle equation form" where the circle is a unit circle with a radius of 1 as discussed previously. This form will also lead us to find the lengths of the major and minor axis**:** When x=0, we find the maximum (y) value of $y=b=\sqrt{b^2}$, and when y=0, we find the maximum (x) value of $x=a=\sqrt{a^2}$.

Given the following equation of an ellipse, place it in a standard ellipse equation form:

$$3x^2 + 4y^2 = 24$$ after dividing through both sides by 24 and canceling:

$$\frac{x^2}{8} + \frac{y^2}{6} = 1$$: or = $0.125x^2 + 0.166667y^2 = 1$
: note that here, $a^2 = 8$ and $b^2 = 6$

Since a=8 is larger than b=6, the major axis is along the x-axis. The coefficient of x^2 is the constant (1/a), and (a) is the length along the major axis from the center of the ellipse to the edge or vertice of the ellipse. Hence, (a) is equal to half of the major axis. This (half) portion of the major axis is often called the semi-major axis. On the other hand, the coefficient of y^2 is (1/b), and (b) is equal to the length of the semi-minor axis which is perpendicular to the major axis. Again, when both (a) and (b), the coefficients of the (x) and (y) terms, are equal in value, that the equation reduces to one of a circle.

For the above example:

$a = \sqrt{a^2} = \sqrt{8} = 2.828427125$: length of the semi-major, or half of the major axis

$b = \sqrt{b^2} = \sqrt{6} = 2.449489743$: length of the semi-minor, or half of the minor axis

Note, if given this form indicated above: $0.125x^2 + 0.166667y^2 = 1$

$0.125 = \frac{1}{a^2}$ mathematically:

$a^2 = 1/0.125 = 8$:as seen above

$a = \sqrt{a^2} = \sqrt{8} = 2.828427125$ and:

$0.166667 = 1/b^2$

$b^2 = 1/0.166667 = 6$:as seen above

$b = \sqrt{b^2} = \sqrt{6} = 2.449489743$

Using the general equation and expressing (y) in terms of (x) for plotting purposes:

$$y = \pm b \sqrt{1 - \frac{x^2}{a^2}}$$: Note that when x=0, y= +-b as should be expected, and this point (0,b) is the y-axis intercept.

You can find the focus point: F(c, 0) from a Pythagorean, right triangle, type of relationship of the distances:

$$a^2 = b^2 + c^2$$: where (c) equals the ± x coordinate of a focus point

To verify that the hypotenuse is equal to (a) = half the major axis, consider the point (b,0). According to the definition of an ellipse, the total distance to both foci points is 2a after a ray is emitted from one focus and reflected off the ellipse to the other focus. Half of this distance is equal to (a).

$c^2 = a^2 - b^2$, the focus point is: F(c , 0)

Due to the elongation (of an axis) of an ellipse, you might suggest that the formula for eccentricity (e) should be the ratio of the minor and major axis, and have resulting ratios less than or equal to 1, and then and adjustment made by subtracting this ratio from one (1) since longer elongations, where the major axis is larger, should have higher (rather than lower) eccentricity values rather than lower eccentricity values. Consider the following:

$$\frac{\text{minor axis}}{\text{major axis}} = \frac{2\ (\text{semi-minor axis})}{2\ (\text{semi-major axis})} = \frac{b}{a}$$: this is sometimes called the "compression factor" or "aspect ratio"

A "proposed formula" for (e) = $1 - \frac{b}{a}$: This specific value is associated with the compression factor is sometimes called the "flattening factor", but it is not the formal formula for eccentricity.

Combining fractions, this can also be expressed as:

$$1 - \frac{b}{a} = \frac{a}{a} - \frac{b}{a} = \frac{a-b}{a}$$ = ratio of: (difference in length values) to the longest length value

As a check on the "proposed formula" for (e), a circle is defined as having no eccentricity (e=0), the length of the major and minor axis are the same, therefore (a=b):

$$1 - \frac{a}{a} = 1 - 1 = 0$$: (e) of a circle

And for a line being defined with the highest eccentricity possible, the ratio of the minor (2b) axis to the major axis (2a) is very small (since b approaches 0), and is practically 0:

$$1 - \frac{b}{a} = 1 - 0 = 1$$: (e) of a line

This "proposed formula" seems to be a very logical mathematical definition of eccentricity (e), however, it is not the one used. (e) is actually defined in such a way as to be applicable to all conic sections. A mathematical definition of the eccentricity of a conic section is:

$$\frac{\text{distance from any point on the curve to a fixed point known as the focus}}{\text{distance from the same point on the curve to a fixed line known as the directrix}} = \text{eccentricity (e) of a conic section}$$

This ratio (e) is a constant for a particular conic section. In relation to the ellipse, the definition resolves to:

$$e = \frac{\text{distance between foci points}}{\text{major axis}} = \frac{2c}{2a} = \frac{c}{a}$$

: eccentricity of an ellipse. (a) and (c) are values on the major axis. This value is always less than 1 for an ellipse.

In the above equation we find that (c) is equal to the distance from the center to one of the foci of the ellipse. (a) is the length of the half of the major axis and-or is the distance from the center to the farthest edge of the ellipse along the major axis.

As a simple verification, when the ellipse is circular-like in shape, the distance between the foci points is 0, and (e) will be a very small value, practically and equal to 0 as defined for circles. The more eccentric the ellipse is, the distance between the foci points is greater and this distance approaches that of the length of the major axis. (c) will approach the value of (a) and therefore, their ratio is near 1 as expected for the eccentricity of a line or ellipses with high eccentricity.

Since $c^2 = a^2 - b^2$ in an ellipse, we can algebraically have:

$$e = \frac{c}{a} = \frac{\sqrt{a^2 - b^2}}{a}$$

: eccentricity of an ellipse,
Also, from this: $c = e\,a$ and $a = c / e$

The eccentricity of any parabola is 1, and the eccentricity of a hyperbola is greater than 1. The eccentricity of a line is "infinity" or a "very, very high value". In short, the lower the eccentricity, the more circular-like the curve is.

A three dimensional shape and-or volume created by rotating an ellipse along its major or longest axis is called an ellipsoid. The North American ball used for the game of football has a similar shape to that of an ellipsoid. The specific elongated shape of the football is technically called a prolate spheroid (ie., elongated or "stretched" spheroid or sphere). Imagine a construction with a shape that is someplace between that of a hemisphere of a circle, a cylinder and a cone shape and you will then have an football shape. Some seeds of plants also have a similar type of this shape. The shape of the cross sections of an ellipsoid are circles or disks. A shape that looks somewhat similar to that of an ellipsoid is that of a sphere that has been "squashed" on opposite ends and it would be the result of rotating an ellipse along its minor axis. This is technically called an oblate spheroid.

An ellipse can be drawn on a piece of paper placed on a spare wooden surface by using two tacks at the foci locatons, and a loop of string of constant length equal to: 2a + 2c = 2(a + c) is needed, and of which these lengths determine the eccentricity and vice-versa. This string will be held firmly by the pen or pencil during the drawing of the ellipse curve. The tension of the string loop will guide the position of the pen so as to draw the ellipse.

The orbit of planets is given a basic discussion further ahead in this book in the topic of sun, moon and stars. It is shown that (e), eccentricity ban be calculated in another way using named variables of the orbit that correspond to the general equation of an ellipse. Some astronomers have found the specific elliptical orbit of the planets and other celestial objects by using many observations of their positions, and then predicting and checking to see if their calculation is acceptable and-or needs to be adjusted. Due to some, relatively slight gravitational interference upon a planet from other distant planets, a planets predicted elliptical orbit will change slightly from what was expected. In reference to the above drawing of an ellipse and the forward discussion about orbits, c = a planets average distance from the Sun = major axis / 2 = 2a / 2 = a = (aphelion distance + perihelion distance) / 2. This position is actually the "center" point of the ellipse, halfway between both the major and minor axes. Half the minor axis is equal to b = the square root of (a^2 - c^2) due to the right angle and Pythagorean Theorem. e = c / a = the ratio of: (distance from center to focus) / (distance from center to farthest point). Distance from the center to focus = average distance = (aphelion + perihelion) / 2. The eccentricity = e of Earth's orbit about the Sun is very low at about e = 0.017, and is practically circular. A circle has an eccentricity = e = 0.

HYPERBOLIC (HYPERBOLA) :

The word "hyperbolic" and "hyperbole" old words containing the word prefix "hyper", and are sometimes used to describe things that are excessive or extreme, and is rooted in the meanings such as "an excessive bole [throw]". The opposite in meaning to the word prefix "hyper" is the word prefix "hypo" which means low. The equation and formal definition of a hyperbola are somewhat similar to that of an ellipse, however with a hyperbola, the difference between the distances from any point on the hyperbola curve to the two foci points is always the same. Unlike the ellipse and circle, the hyperbola is not a closed curve. It has two symmetrical (similar or mirrored) branches that extend endlessly. The two branches are due to a planar section of two identical cones connected at their vertex or "top points". Each branch is considered hyperbolic. An asteroid or comet from outside our solar system may get attracted and then have a trajectory towards near to the Sun due its gravity, and yet not orbit the Sun like objects in our solar system do with an elliptical orbit, but it will rather just pass through and out of our solar system due to its high kinetic energy. That asteroid is then said to have a hyperbolic trajectory or path.

When the center of the hyperbola is at the origin, and its axis (passes between the two vertices of the hyperbola, and is similar to the major axis of an ellipse, and where the foci of the hyperbola are located upon) is along the x-axis, the equation of a hyperbola has, or can be reduced to the basic form of:

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

: (a > b) when the x-axis is the axis of the hyperbola.
This equation is similar to an ellipse equation, but the coefficients of the x and y terms have opposite signs.

There is another and simpler equation of a hyperbola (though with different axis) that is often encountered, and this is:

$$\frac{x\,y}{c} = 1$$

: For this equation to be true, (c) must equal (x)(y). (c) is a constant.
Mathematically:

$$x\,y = c$$

: Both (x) and (y) are inversely related to each other. After solving for (y) for plotting, this resembles a common "division equation", where the divisor, and therefore the quotient change in an inverse manner.

$$y = \frac{c}{x}$$

: (y) is a some multiple (c) of the reciprocal of x = (1/x). y=(c)(1/x).
The equation's curve shape is shown below. Since the graph shows the relationship of how (y) changes with respect to (x), (c) must remain constant.
In general, due to the form of this equation, it could be called a "division equation" and the graph of it could be called a "division curve". If we let c=N=numerator, or dividend, and x=D=denominator, or divisor, and y=Q=quotient, we have:

$$Q = \frac{N}{D}$$

: when N=D, Q=1
: when N>D, Q>1
: when N<D, Q<1 , a result less than one is a fraction of 1

In dividing values such as successive integers: N / (N+1) is less than 1, and approaches closer to 1 as N approaches infinity:

Ex. 1/2 = 0.5 , 2/3 = $0.666\underline{7}$, 3/4 = 0.75 , 9/10 = 0.9 , 542/543 = $0.998\underline{2}$

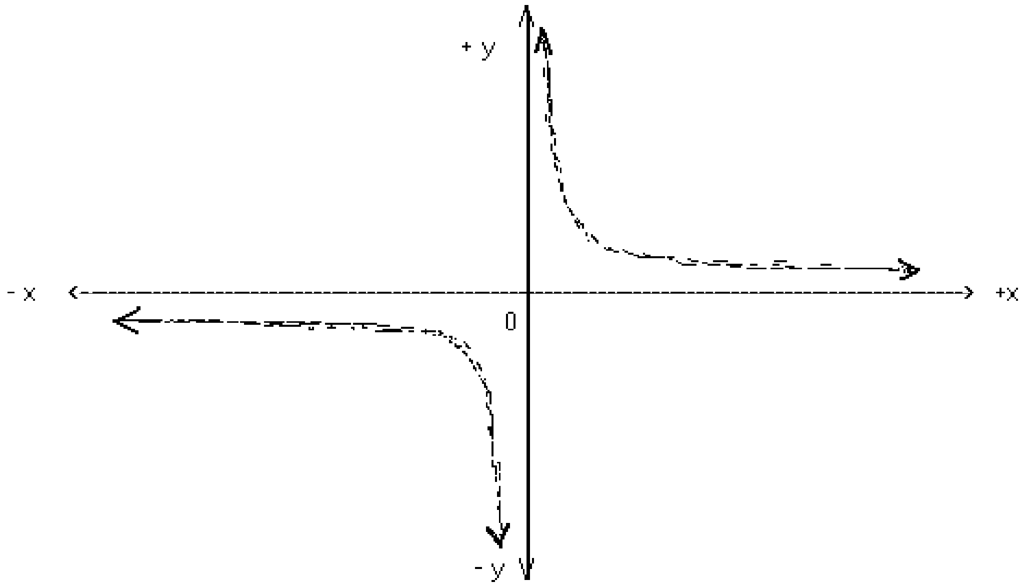
In an opposite way, (N +1) / N is greater than 1, and approaches closer to 1 as N approaches infinity.

Ex. $2/1 = 2$, $3/2 = 1.5$, $4/3 = 1.3333...$, $10/9 = 1.111...$, $543 / 542 = 1.0018...$

These quotients are reciprocal in value to the above results, since the values used were reciprocals. $(N+1) / N$ and $N / (N+1)$ are reciprocals of each other.

Ex. $3/2 = 1.5$, $4/3 = 1.333...$, $10/9 = 1.111..$, $502/501 = 1.002...$, $23510 / 23509 = 1.00004...$

When c=1, the equation $y = (c/x) = (1/x)$ could be called the "pure reciprocal equation". When the numerator is not equal to 1, you can put it into this "reciprocal form" by dividing both the numerator and denominator by the numerator (ie., canceling the numerator). [FIG 78]



The curve(s) of this equation are symmetrical to or about the origin (0,0), and to a line which passes through the origin which has a slope of 1. Note that the equation $c = xy$ may appear to have a form of a linear equation because the powers of the variables (x and y) are 1. When plotted, the curve does not resemble a line at all, but resembles, more or less, a quadratic or second degree equation. That is, this is formally a second degree equation because when variables are multiplied, the degree of the equation is the sum of the highest power of each different variable. As another example, here is a third degree equation:

$$\frac{x^2y^1}{c} = 1$$: sum of exponents = 2 + 1 = 3, hence a third degree equation

$$y = c / x^2 = cx^{-2} \quad \text{and} \quad x = \pm \sqrt{c / y}$$

A note on mathematically drawing a conic section or curve without a conic or second order equation:

You can mathematically draw a conic section on a piece of paper. Draw a line on the paper and choose a point nearby, perhaps an inch away. This line will become the directrix of the conic section or curve, and the point will be the focus (F) of the that curve being drawn.

For each point (p) on the curve being created, it will be a certain perpendicular distance from the directrix line, and a certain distance to the focus point, and the mathematical relationship (here, as a ratio) that the distance each point is from the directrix line and focus point is a constant of (e):

$$\frac{\text{(distance to focus)}}{\text{(distance to directrix)}} = \mathbf{e}$$

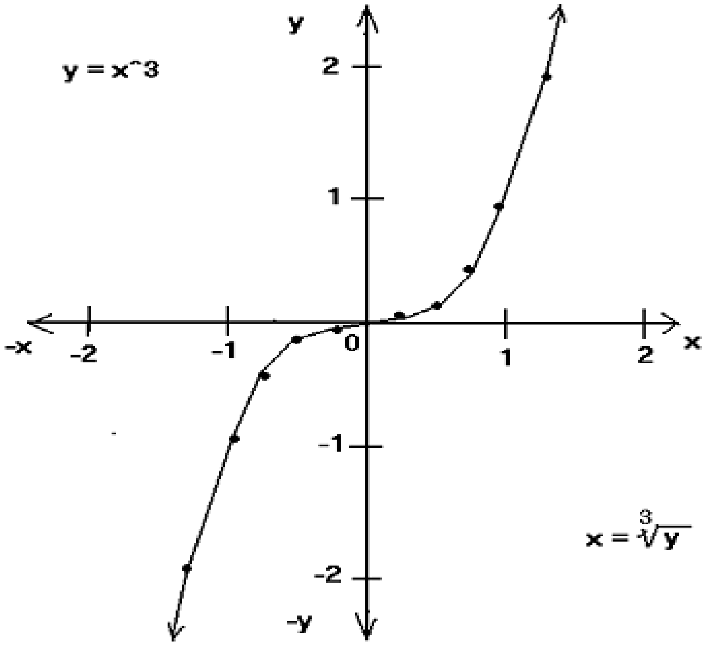
: e is a constant and is called the **eccentricity** of the conic section or curve
When (distance to focus) = (distance to directrix) , a constant, the curve is a parabola
When (distance to focus) < (distance to directrix), e<1, and the curve is an ellipse.
When (distance to focus) > (distance to directrix), e>1, and the curve is a hyperbola.

It could be said that circles have no eccentricity (e=0) or stretching, ellipses are like a circle stretched a little and have a little eccentricity (0<e<1), parabolas have moderate eccentricity (e=1), and hyperbolas have high eccentricity (e>1).

CUBE (CUBIC) , A non-conical (conic, cone) curve. This curve contains the third power of the variable.

The general cubic expression is: ax^3 + bx^2 + cx + d : constant (a) must not be 0 so as the x^3 term exists.
For a basic example: 1x^3 + 1x^2 + 1x^1 = 1 Here, the solution is about x=0.5435

There is a (somewhat complicated) formula to solve cubic equations, which can be found elsewhere if needed. Showing a graph of an example cubic function and its corresponding inverse function: [FIG 79]



If we multiply the simplest "quadratic" expression of x^2 , by x, we have the simplest "cubic" expression of:

(x^2)(x^1) = x^(2+1) = x^3

Expressing the mathematical relationship between (x) and the value or result of this expression, we have this equation:

$y = x^3$: essentially then, in this equation each (x), and the resulting (y) value of the parabola equation of: $y=x^2$ is multiplied by the factor (x).

This is similar to the graph of: $y = x^2$ except that the curve of $y=x^3$ is steeper (due to a steeper slope value) and is no longer symmetrical about the y-axis, which is essentially due to that after solving for x, we find: $x = \sqrt[3]{y}$, and here when (y) is negative in value, it produces and has corresponding negative (x) values.

A point on a graph can actually be identified by the two corresponding inverse functions, and here it would be:

$p(x, y) = p(\sqrt[3]{y}, x^3)$

Given this graph of the basic (cubic) equation of $y=x^3$, there is a specific and corresponding equation for the slope or "steepness" (from left to right, or increasing values of x) at any point of this curve, and it is:

slope at (x) = $3x^2$: Equations of this type will be discussed further ahead in the slope and derivative discussions. Technically, slope is the rate of change of (y) with respect to the changes in (x) at a given point. Because this expression has a variable, the slope value is not a constant value, but depends on the specific value of x.

Where on this curve is the steepness equal to +1? Setting this specific slope equation to 1, and solving for (x), we find:

$1 = 3x^2$, therefore: $x^2 = 1/3$, $x = \sqrt{0.3333...}$, $x = \pm 0.577...$

The corresponding (y) value is: $y = x^3 = (0.577)^3 = 0.192...$, and this point or location is: **(** ± 0.577 , ± 0.192 **)**
As (x) gets larger, the slope or rate of change increases greatly as should be expected for (increasing) values that are "cubed" (ie. number^3) or raised to any value greater than 1.

The word cube is rooted in the basic words for "elbow" (ie. a bend or angle), and possibly the words for "square", "surface", "box", and "cubit" which is a unit of measure or reference (ie, as a one-dimensional ruler for length) of the ancient world. Since a cube and other solid shapes, volumes or spaces have 3 dimensions (a length, width, and height) and therefore 3 ways to be measured, and with each having the same units of measurement, the number 3 is often associated with the words "cube" , "cubed" , and "cubic". Ex. 2^3 can be mentioned as "two cubed".

As a refresher discussion about cubed units (rather than just plain cubed numbers, ex: 5^3, without having any units):

5^3 ft is not the same as: 5 ft^3

5^3 ft = 125 ft having units of feet = ft^1, is not the same as 5 cubic-feet having units of: ft^3 = (cubic feet)

A unit of measurement: unit = unit^1 : for example, the unit of measurement or reference could be a length unit such as feet (ft), miles (mi), or meters (m), etc.
Ex: 5.23 feet

A squared unit, or a square unit: (unit) (unit) = unit^2 : Ex. 5.23 ft^2 = (5.23ft)(5.23ft) = "5.23 feet, squared"
= "5.23 square-feet"
= "5.23 feet=squared"

Ex. 2ft long, by 2ft high = (2)(2)(ft)(ft) = 4 ft^2 = "4 square-feet"
Ex. 2ft long, by 3 ft high = (2)(3)(ft)(ft) = 6 ft^2

A cubed unit, or cubic unit: (unit) (unit) (unit) or = (unit^2) (unit^1) = unit^3 : Ex. 5.23 ft^3 = "5.23 cubic-feet"
Ex. 2ft long, by 3ft wide, by 4ft high =
(2ft)(3ft)(4ft) = 24 ft^3 = "24 cubic-feet"

Ex. If considered in the shape of a cube, a million cubic feet of material would have a side or dimension length of:

$$\sqrt[3]{1000000\ ft^3} = \sqrt[3]{1000000}\ \sqrt[3]{1\ ft^3} = (100)(1\ ft) \quad \text{or} = 1\ (10^{(6/3)}\ ft) = 10^2\ ft = 100\ ft$$

That is, this cube or cubic shape of material would have a length of 100 ft, a width of 100 ft, and a height of 100 ft.

GOLDEN RATIO (In a geometric series, with a basic curve example.)

Though not often seen and-or encountered as much as the other common curves, the golden ratio (GR) is still important. For simplicity, we will use **GR =~ 0.618**

If GR was the first term of a series, and if you took 0.618 of it and of each successive term, the sum would equal:
1+GR = 1 + 0.618 = 1.618 = 1/GR = 1/0.618 :

Term :	1	2	3	4	5	6	7	. . .	
Value :	0.618	0.382	0.236	0.146	0.0902	0.056	0.034	. . .	Algebraically:
Expression:	GR	GR(GR)=GR^2	GR^3	GR^4	GR^5	GR^6	GR^7	. . .	

The value of each term (n) is: GR^n : n=term number

The sum of all the terms approaches: 1.618**...** = (1+GR) = 1/GR : as (n), the number of terms, approaches infinity

A peculiarity is that if you were to subtract each term (GR^n) from its preceding term, you would have the next term. This is similar to a (natural, basic) Fibonacci Series, if those terms above were reversed into an ascending series, and the next term was the sum of its two previous terms. Here, the next term of this series is the difference between the last two terms.

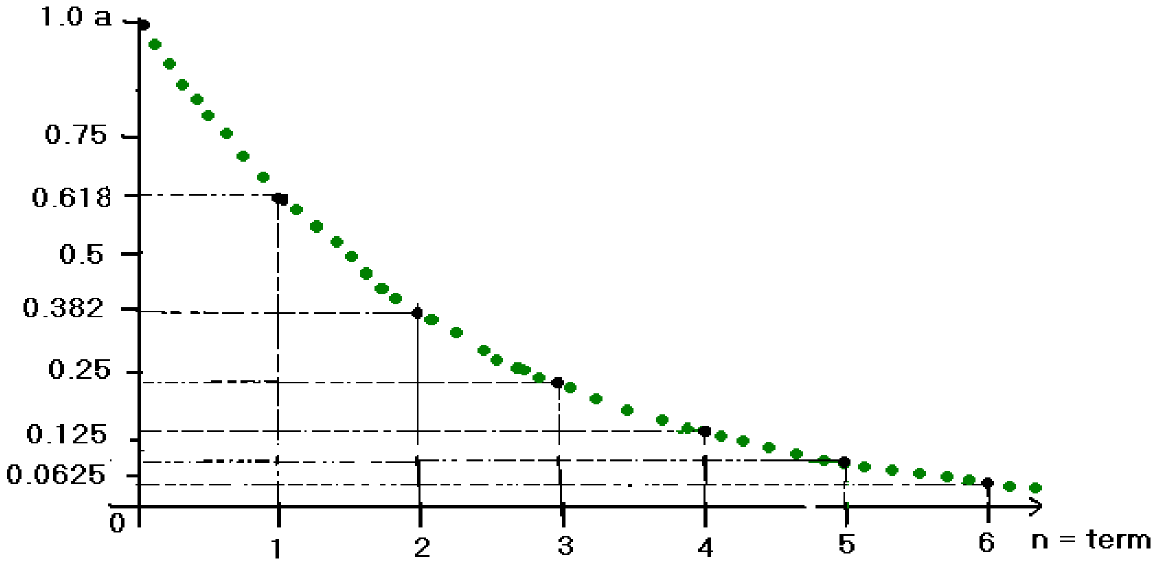
(GR)^(n+1) = (GR)^(n-1) - (GR)^n :ex. 0.236 = 0.618 - 0.382 = 0.236 , Mathematically:
(GR)^(n-1) = (GR)^n + (GR)^(n+1) Also considering just the first two terms:
1.0 = (GR)^1 + (GR)^2

If there is an initial starting term value (a), the geometric series (with multiplier r = GR = 0.618) could be expressed as:

Term:	1	2	3	. . .	
Value:	a	a(GR)	(a(GR))(GR)	. . .	with simplified expression of:
	a	a(GR)^1	a(GR)^2	. . .	**:** (a) becomes a multiplier to the value of each term above

The sum of these terms is: a + a(GR)^1 + a(GR)^2 + **...** = a (1 + GR^1 + GR^2 + **...**) = a(1+1.618**...**) = a 2.618**...**
= a(2 + 0.618...) = 2a + (GR)a

[FIG 80]



The value of each term (n) is: (a)GR^(n-1) : the term number (n) can also be a mixed number, ex. n=1.5

Ex. Given y = a GR^n , if value (a) starts out with a value of 1 (or 100%) and is processed by the formula and undergoes change in its value, and letting (y) be set to each new resulting value as (n) changes, at what term (n) will (y) be equal to half of (a's) starting value, hence where y = a/2 = 1/2 = 0.5?

Expressing the above data into the equation or formula:

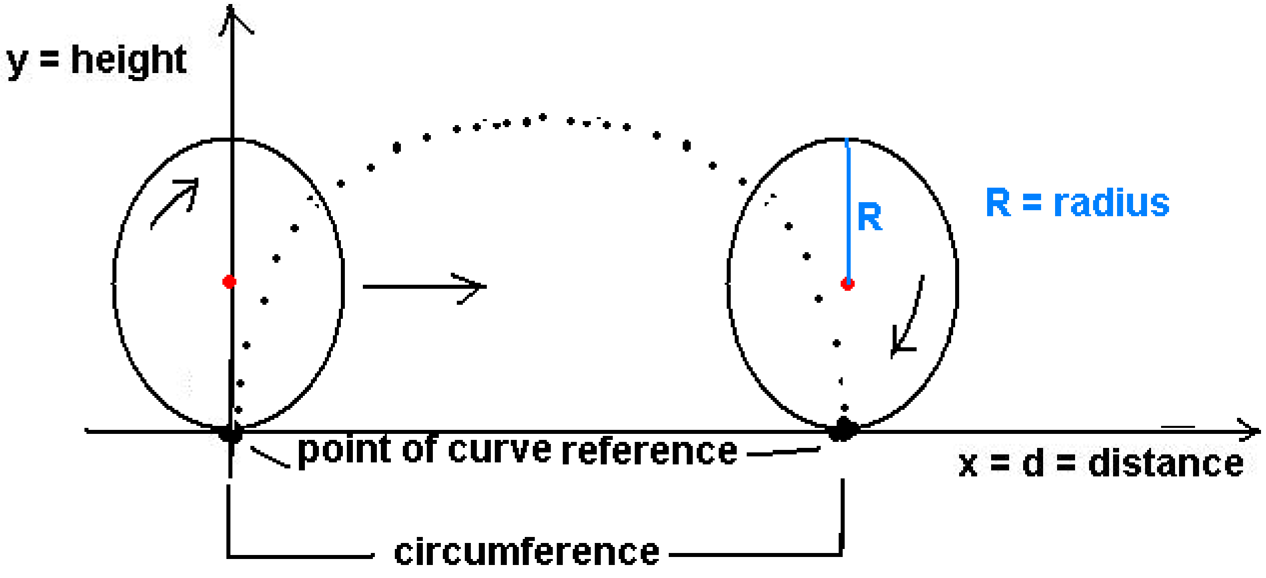
0.5 = (1) GR^n or simply:
0.5 = GR^n solving for (n) by taking the logarithm of both sides of the equation:

log 0.5 = log GR^n by using the law exponent rule:
log 0.5 = n log GR after solving for (n):

$$n = \frac{\log 0.5}{\log GR} = \frac{\log 0.5}{\log 0.618} = \frac{-0.30103}{-0.20901} = 1.4403$$

CYCLOID

The cycloid curve is a beautiful curve created when a point on a rotating wheel moving on a flat and straight surface is recorded or "traced out" for one revolution. This path or locus of this point will create a curve resembles that of a elongated half-circle and-or semi-sphere. A cycloid shape also resembles a bowl shape. [FIG 80A]



The maximum height and-or (y) value of this curve is equal to twice the radius of the circle, hence 2R, and the corresponding (x) value is equal to half the circumference of this circle.

If the distance the wheel is rotated and moved horizontally is C = 2 (pi) R, the coordinates of the max. height point are:
P(x , y) = P(circumference / 2 , 2R) = P(2 (pi) R / 2 , 2R) = P((pi) R , 2R)

Though the cycloid curve probably was discovered in ancient times, it took until Galileo Galilee to make a reasonable mathematical analysis of it. By experiments, he discovered that the area of a cycloid is three times that of the circle that was used to create it. It was later found that the arc length of a cycloid is equal to 8 times the radius (R) of the circle or wheel that created it, hence: S = 8R = 8 (D/2) where D is the diameter of the circle, hence S = 4D

A cycloid curve is one of a general class of curves called **roulettes** of which are the path traced out when one curve moves about another curve, and the resulting or third curve is the roulette of those two curves. Ways and-or patterns templates to make roulette curves have been in use for several hundred years, and possibly some since in ancient times. A modern, affordable and and very popular educational toy for mechanically drawing roulettes uses the aid of various (plastic) gears with several holes in them to place a pencil or (ink) pen for drawing (ie., "tracing out" the path or locus) the resulting curve is called the **Spirograph.** It was created by **Denys Fisher** (1918 - 2002), from England, in 1965. In this modern (2024) so called "digital world" or "virtual (simulated, as like) world", there is still much mechanical things (machines and mechanisms) still needed - both in use and-or design, and with some new ones being designed and analyzed with the aid of digital computers.

Another type of special curve is called an **involute curve**, and this is created by for example unwinding a tensioned string from a shape such as a circle, and the resulting curve is a spiral. Modern and efficient (at transferring maximum power via constant gear teeth contact any "slippage") gear teeth or protrusions are designed to have an involute of a circle. segments.

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SECTION 3: BASIC TRIGONOMETRY
THE STUDY AND MEASUSREMENT OF TRIANGLES

Trigonometry (triangle measurement) is the study, application and measurement of and with triangles, triangle-like structures, constructions and angles in general. A triangle is bounded (ie., defined) part or portion of a plane, and having three straight sides. A plane is like an infinitely "flat" surface extending in all directions, but without a thickness. A plane is a flat, 2 dimensional concept that can be described and measured in two ways: a length, and a width. A plane is formally defined as having no thickness measurement (other than being 1 infinitely small point thick). A line is an infinite amount or set of adjacent, infinitely small points on a plane, and all being in the same direction with respect to each other. For a triangle construction, this bounded or enclosed portion, shape or figure of a plane has three sides that are line segments or "sides", and of which each side will intersect the other two sides.

An angle is a separation created by the rotation of a line or point about another point (such as the center of a circle) or line that is used as a reference position, and the measurement of an angle is the amount of (circular-like) the rotation or separation. Each side of the triangle will intersect (ie. meet or join at) the two other sides, and this will create three interior angles of the triangle. The "tri" in the word "triangle" means triple or three, therefore, the word "triangle" literally means "three angles". The word "angle" as used here, is actually from an old word of "angle", which means a bend, turn, or a corner (such as the location where two walls joined, met or connected together). See [Fig 81 for an image of an angle, and Fig 83 for and image of a triangle].

Angles are usually measured with angle or rotation units called degrees which is the amount rotation about a point. The word "degree" means the intensity, quantity, amount or size of the rotation and-or angle. The word "degree" is rooted in an the old words of "step" as for stairs, steepness, and "grade" for the intensity, level or marking (as in the successive "graduations" or indicated steps of a measuring device). The word degree is also often used when stating a temperature value such as: "it is 72 degrees Fahrenheit" (= 72° F). The symbol of: (°) is often used to indicate that a value is an angle with degrees as it units measurement. For example, 35° is "thirty-five degrees" or "a thirty-five degree angle" = 35 degrees.

To give you an idea of what various sized angles and their corresponding degrees of measurement are, a complete rotation of a point around another point, regardless of the distance between those two points, is defined as a 360° ("360 degrees") angle or rotation from its initial, starting or reference (0°) position. This full rotation will create a circle when the distance (the radial, or radius distance) between those two points was always constant. This arbitrary and manageable (not too big and not too small) value of 360 degrees for 1 full rotation was chosen long ago when noting that it took planet Earth roughly, or about, 360 days to cycle through all the seasons and start again for a new year, This is roughly how long it takes Earth to orbit the Sun, and that is about 365 days, and technically it is 365.25 days. 360 is also number which can be easily divided. Any angle less than 360° is, and can be thought of as a fraction of 1 full rotation.

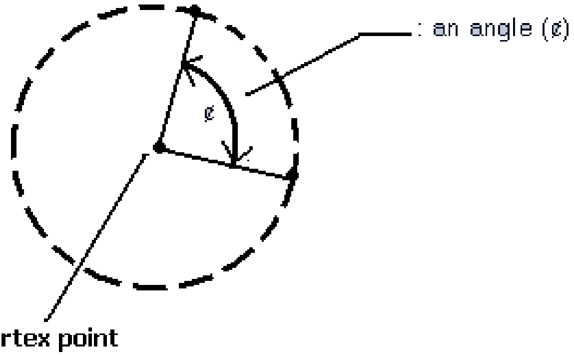
An angle can be described as the following:

Two lines (or line segments) that cross or are connected together at one ("pivot", rotation) point, such as an endpoint on each line, and can then be made to rotate about each other, creating an angle or separation between those two lines.

Given a circle, you can imagine or draw an initial (starting) line between a point rotating about or around a (fixed, or stationary) center point, and then draw another (ending) line between where the rotation of that initial line or point has stopped and the center point. The intersection (vertex point, which is similar to a pivot point) of these two lines, or any other two lines such as two sides of a triangle, will create an angle (a separation) between those two lines.

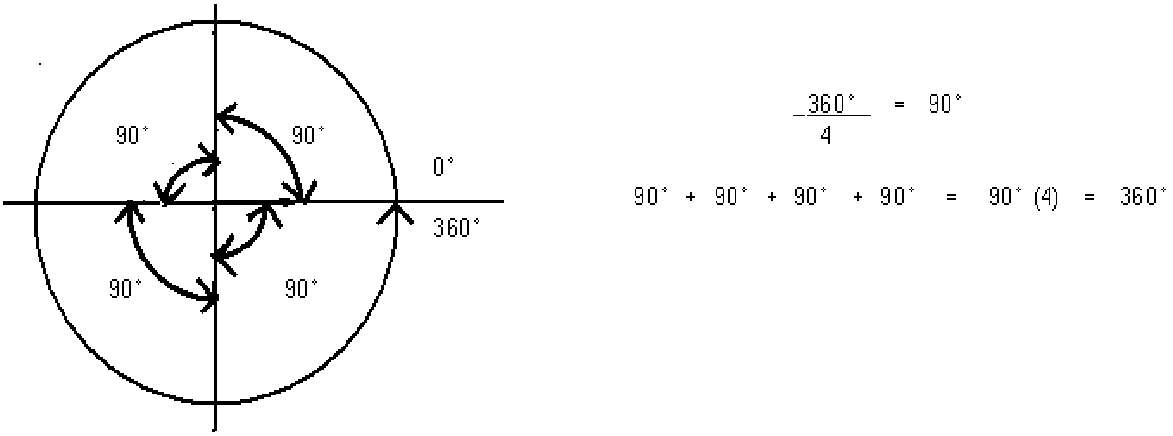
It could be said that the degrees or the measure of an angle is the amount of separation of or between two lines that intersect at a point (the vertex), and which have, or could even be considered to have rotated about that common point (the vertex, reference, or pivot point, as the "origin or center of rotation").

The amount of degree units of an angle is its (rotational, separation, or angular) measure, and this is not a linear, one-dimensional length or distance measurement. For example, circles can be very small or big, or a point or object can rotate near to, or very far from the center (of rotation), and yet each angle would still be the same value regardless of the distances or size from the (reference) center. Any circle has a 360 degree angle for a full rotation about it center, and a (360/2) = 180 degree angle for half of a full rotation, and a (360/4) = (180/2) = 90 degree angle for a quarter or one-fourth of a full rotation. A common symbol for an angle is: ¢ , and may be sometimes noted with the degree units symbol: ¢° [FIG 81]



The point where two sides (considered as lines) of a angle or triangle meet or intersect is formally called the vertex ("point", "location" , "meeting", "joining", or "intersection") of the angle. The word of "vertex" is rooted in the words such as "vertical peak", "apex" (vertex is a short and combined form for the words: "vertical apex"), or "summit" where the sides of a hill or mountain meet (intersect). The word "vertical" basically means in an upward or downward direction, as opposed to the word "horizontal" (ie. "horizon") direction which means "level" (no upward direction or angle) or in the left or right directions. The curved arrows are often used to indicate that some movement, separation or rotation has happened, hence an angle and-or its shape was created or is present.

A right angle is defined as one-quarter (1/4) of an angle that would give a full rotation of 360°, hence a right angle is (360°/4 =) 90° ("ninety degrees"). A triangle that has two sides that intersect at a right angle (perpendicularly, like two sides of a square or rectangle) is called a right triangle. The sum of any triangles interior angles is always 180°. [FIG 82]



$$\frac{360^\circ}{4} = 90^\circ$$

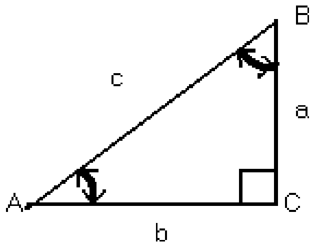
$$90^\circ + 90^\circ + 90^\circ + 90^\circ = 90^\circ(4) = 360^\circ$$

The above illustration also shows that when a line is completely rotated (a 0° to 360° angle) once (1) about the fixed center (point of rotation), that an angle of 360 degrees is created. Clearly a rectangular coordinate system or a circle can be divided up into 4 equal sections and each is called a quadrant (meaning one-fourth or a quarter of something). Each quadrant corresponds to, or has a maximum angle of (360 degrees / 4) = 90 degrees.

The amount of a rotation and its corresponding angle of measurement is often measured or can be considered to be in reference to the x-axis in the upper right quadrant. This would be considered the starting or 0 degree position, and the starting side of a rotation or angle. The x-axis is therefore often called the "reference axis" or "x-reference axis" for all "standard measured" angles. The angles measured this way will then have its second or ending side within one of these four quadrants and these angles are sometimes called as a specific (1 to 4) quadrant angle. A standard measured angle will have its "ending" side within one of these quadrants. The angle between that ending side and the x-axis (when considered as a side of an angle) is created and this angle will be less than or equal to 90 degrees and it therefore has an equivalent "mirror image" or corresponding angle of the same value in the first quadrant. [See Fig. 152] Therefore, the specific given quadrant angle can often be analyzed or considered as if it were a first quadrant angle that is less than or equal to 90 degrees. This practical property will be used later in this book when considering the very useful topic of trigonometric values. The word "trigonometric", which could actually mean any measurement pertaining to a triangles, is a usually associated with the ratio values of two sides of an angle or triangle. For basic discussions of triangles, the concepts of quadrant angles are generally not used since for a triangle, the maximum angle is 180°, and often the maximum angle encountered is just 90° such as for a common right-triangle.

The longest side of a right triangle is formally called the **hypotenuse** (side). Sometimes the longest side of any triangle is called this so as to also reverence it as the longest side of that triangle. In a right triangle, the hypotenuse side is directly opposite the right angle, and vice-versa. The somewhat (modernly) odd word of "hypotenuse" is a Latin-Greek word-form meaning "stretching [teinein] - a word similar to the modern English word of tension - a pull force, such as to straighten a cord] under [hypo], under, beneath or across from. This longest side of a triangle could probably be called the major side of a triangle.

Since the shortest distance between two points (such as the end points of the hypotenuse side) is the straight-line distance between those two points, the hypotenuse side is therefore always shorter than the other two sides summed together. Right triangles are perhaps the most studied triangles since their concepts have many practical applications. Right triangles are also used for models or theory of many concepts. Below is an example and description of a right triangle. [FIG 83]



: a right triangle

Here, capital lettered variables are used to represent the triangle's angles. As mentioned previously, angles can be explicitly (directly, surely) expressed or indicated between the two sides (lines) of an angle with a small arrow or the angle symbol: (¢) with a possible subscript indicating which specific angle it is. For conformity and understanding, the sides directly opposite of those angles in a triangle are usually indicated with a corresponding small letter variable. This is the method done so as to avoid a problem of knowing which one of the two actual sides forming, or of that angle should be indicated as corresponding to that angle(s), and then how to then possibly indicate the side opposite that angle(s). For example, side (a), is directly "opposite" or across from the corresponding angle and vertex labeled as A. The corresponding side that is directly across from an angle will also intersect the two sides of that defined angle, and so as to create or complete the triangle shape.

A small square box shape is frequently used to indicate the "right", "square", or perpendicular (at the side of), 90° angle (here, angle C) in triangles or in any other construction that has a perpendicular intersection of (perpendicular) sides or lines in drawings. This small box, square, or "turn symbol" indicates that the corresponding sides there are perpendicular (at a right angle = 90°) or "square" to each other. In the right triangle shown, the side opposite of this 90° angle is the hypotenuse side and side (here, side c).

Even though the hypotenuse side (here, side c) is the longest side of a triangle, and like any line, it is actually the shortest distance between its' two endpoints since the "straight-line" distance from one point to any other point is always the shortest distance, and any other path or directions taken between them will only result in a longer distance. The sum of any two sides is therefore always longer than the third side in a triangle. Consider a right triangle with side (c) being the hypotenuse side:

$c > a$ and $c > b$: c, the hypotenuse, is always the longest of the three sides of a right triangle

$(a + b) > c$: this can also be expressed as: $c < (a + b)$

To determine if an angle is indeed a right, 90° angle, you can first measure the same distance from the angles vertex and along on each of the sides or "legs". The distance from one point to the other will be about $\sqrt{2}$, (= about 1.414) times that of the same distance along each leg. This can be derived and checked from the Pythagorean Theorem.

SOLVING FOR AN ANGLE

If given any triangle, and one angle is unknown, this angle can be easily solved for by the fact that the interior angle sum of angles in a triangle is always 180°, and that a right angle is 90°. Placing this information into equation form:

$180° = A + B + C$ Letting angle C=90° for a right triangle:
$180° = A + B + 90°$ transposing 90°:
$180°-90° = A + B + 90°-90°$ combining:

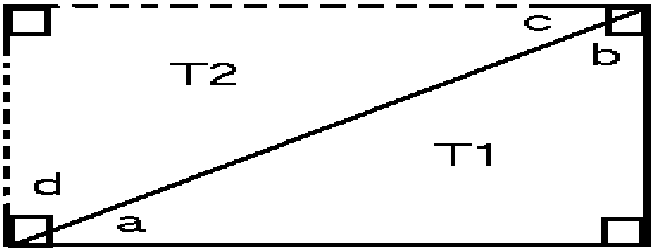
$90° = A + B$: In a right triangle, the sum of the other two angles that are less than 90°, such as A and B here, is always equal to 90°.

Ex. Given a right triangle, if A=30°, what is B ?

$90° = A + B$
$90° = 30° + B$ T., transposing 30°:
$90°-30° = 30° +B -30°$ C., combining terms:
$60° = B$: switching sides:
$B = 60°$

Given a triangle, since the sum of internal (or interior) angles is always 180°, if any one angle changes in value, one or both of the two other angles must change in value. If one angle is to remain fixed or constant in value such as when a triangle is to remain a right triangle with its 90° angle, then if one angle increases the other (the complementary angle) must decrease, and vice-versa. A and B, the non-right angles, are said to be complementary angles to each other and of the of a right triangle. Those angles complement each other by always completing a sum of 90°

Here is a quick verification that the interior angle sum of any right triangle is 180° [FIG 84]



Clearly, the interior of any rectangle has four 90° right angles, turns or corners: 4(90°)= 360° = interior angle sum of any rectangle. Since the rectangle is divided exactly in half, creating two identical right triangles, the interior angle sum of each triangle is half of 360° or 360°/2 = 180°. If you were to subtract the 90°, right angle of this triangle from the triangles interior sum of 180°, there will remain 90° left over. In the drawing or figure, clearly, (a+b) must then sum to this remaining 90°, and (c+d) will also. The two non-right angles in a right triangle will always sum to 90°.

$(a+b) = (c+d) = 90°$ since by construction, T1 and T2 are identical (the same, but mirror images), therefore (a) and (c) are identical: (a=c), and (d) and (b) are identical: (d=b).

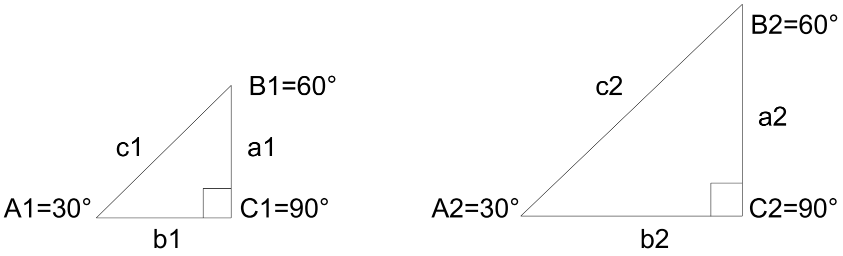
$(a+b) = (c+b) = 90°$: clearly also: (a+d)=90 , and (c+b) = 90° due to a right angle being 90°
$a+b = c+b$ after subtracting b from each side of this equation:
$a = c$: showing that the **alternate interior angles are equal**, and likewise:
$b = d$ In this instance of using a rectangle,,both the corners, and alternate interior angle sum to 90. This topic here is also a future reference in this book: Notice that the diagonal line goes from, transverses or crosses, the two parallel lines (here the upper and lower sides of the rectangle). All the angles created within the two bounding parallel lines are called interior angles. The angles created on opposite or alternate sides of that ("transversal") line are called alternate

angles, hence they are also called alternate interior angles since they are between the two lines. Here, c and a, and b and d are alternate interior angles of equal value.

Due to the construction of the rectangle having 90° internal (corner) angles, of which are called "right angles", angles (a) and (d) are complementary angles that sum up to 90°, and angles (b) and (c) are complementary angles that sum up to 90°.

SIMILAR TRIANGLES

Triangles that look very alike, but are either "bigger" or "smaller" in size, are called similar triangles. It could be said that one triangle is simply a magnified (bigger), or demagnified (smaller) view of the other. Besides from the visual point of view, certain (mathematical) conditions must also be met so that the triangles are indeed similar. They actually look alike, and are similar triangle constructions because the corresponding angles of the two are the same. The sides of similar triangle constructions usually differ in length, but they are not just any random lengths within that similar triangle. A similar triangle has sides that will have the same mathematical proportions (equivalent portions, ratios, fractions) among them as any other similar triangle. That is the result when each side of a similar triangle is a constant multiple (ie. a magnification factor) of the corresponding side of another similar triangle. Below are two right triangles that are similar, and in fact, there is an infinite number of similar triangles. [FIG 85]



TRIANGLE 1 TRIANGLE 2

The small numbers next to the variables are called subscripts. Subscripts were mentioned previously where they were used to represent a particular instance of the same variable or equation. They are also used to show that the variables are not actually the same, but are unique even though the same symbol or identifier (identifying name) for the variable is chosen for a specific set, class, group, or type of data. Here, identical variable names with the lower case letters represent corresponding sides of the similar triangles, and the subscripts represent which triangle it is in reference to. Also, capital or "upper case" letter variables represent the inner or interior angles (of the triangle) created due to its intersecting sides. Sometimes, a subscript is written or expressed in standard or normal notation, for example, A1 might be written as A_1., however this is technically a new variable name or identifier which may even be sub-scripted; ex. $A1_5$ For simplicity and for use with most text readers, this book often uses a common or standard typeset instead of standard subscripts which is a smaller print.

Though the sides of similar triangles usually differ in length, there is still a mathematical relationship among those similar triangles. The relationship, as stated previously, is that they are proportional. A part of any one triangle of all similar triangles is mathematically a certain portion or fraction value in reference to any other part in that triangle. Therefore, the ratio of any two sides of one triangle is equal to the ratio of the same two corresponding sides of another similar triangle. This is a proportions (equivalent, of or having the same portion of the whole or entirety) and-or an equivalent fractions concept when the ratio of corresponding sides are considered.

A similar triangle is essentially a magnified construction of another triangle. Each and every part of a construction must be magnified (increased, or possibly decreased) in value by the same (factor) value so that the result is truly a complete similar construction.

If the sides of a triangle1 are (a1) , (b1), and (c1), we can create a similar triangle, triangle2, by multiplying each side of triangle1 by the same factor or magnification value (n):

a1(n) = (a1)n = a2
b1(n) = (b1)n = b2
c(1n) = (c1)n = c2

Besides having all angle values the same value as another triangle, when a triangle or other construction is similar to that of another, the ratio of all corresponding parts of the two triangles must have the same (magnification) value:

$\frac{a2}{a1} = \frac{b2}{b1} = \frac{c2}{c1} = n$: the reciprocal of each ratio, or "inverse reciprocal", will also have the same value, and that is of the reciprocal of n or= (1/n).

Given that triangle 1 and triangle 2, as shown in the last figure, are similar triangles, here are some examples of all the possible equivalent fractions or proportions of their parts:

ratio of parts of a triangle = corresponding ratio of parts of any similar triangle

Ex. $\frac{a1}{c1} = \frac{a2}{c2} = r1$: the ratio of sides of similar triangle 1 = the ratio of corresponding sides of similar triangle 2. If the angles, and-or ratios are not the same, the triangles are not similar.

Ex. $\frac{a1}{b1} = \frac{a2}{b2} = r2$: these ratios (r1, r2, r3, etc.) of the sides of similar triangles are generally not the same value

Ex. $\frac{b1}{c1} = \frac{b2}{c2} = r3$

When dealing with similar triangles, there are two different types of ratios to consider. One type of side ratio is a ratio of the parts of just one triangle. For another type of side ratio, it is a ratio of corresponding parts of two similar triangles.

As indicated previously, for similar triangles, the relationship among the sides is that the ratio of any side of triangle1 to the corresponding side of triangle2, is equal to the ratio of another side of triangle1 to its corresponding side of triangle2:

Ex. $\frac{a1}{a2} = \frac{b1}{b2} = m$: the ratio of a corresponding sides of two similar constructions, such as similar triangles, is the common magnification ((m), ie., multiplication) factor for all corresponding sides of the similar triangle

For the verification of this, notice that for the previous examples, that these two different types of side ratios for similar triangles can be mathematically derived from each other:

Given: $\frac{a1}{a2} = \frac{b1}{b2} = m$: the ratios of corresponding sides of two similar triangles are equivalent after multiplying each side by (a2/b1), and canceling common factors in the num. and den.:

$\frac{a1}{b1} = \frac{a2}{b2} = r$: the corresponding ratio of sides within each similar triangle are equivalent. In general, (m) and (r) are not equal in value.

You might ask: "What is it that makes these ratio values equal to each other?" "Why is the ratio of a1 to a2 equal to the ratio that b1 is to b2?" There's nothing intuitive that the ratio values should be the same, other than the sides are now known to be mathematically proportional, a simple answer is that you are actually viewing the same triangle either closer (where its apparently bigger and measured as such) or farther away (where its apparently smaller and measured as such), hence the triangles are "visually proportional". For similar triangles, corresponding sides will be different, but the corresponding angles always remain the same. Hence in a fundamental sense, similar triangles are initially due to identical angles. These identical angles define the similar shape of all similar triangles. Although the concept of magnification and its value to create a similar triangle or construction were mentioned, a verification to this equivalent magnification ratio (m) value of all corresponding parts or sides of a similar triangles or other planar constructions is presented further ahead after the discussion about trigonometric ratios.

With the information presented above, when given two similar triangles, the length of a (unknown) side can now be solved for if there is enough information to place into a proportion (ie., equivalent fraction) problem as shown above:

Ex. Using the triangles shown previously in [FIG 85], solve for c2 given: a1=3, b1=4, c1=5, a2=5.25, and b2=7.

Since the triangles are similar, writing a known mathematical relationship of the sides of similar triangles:

$\frac{c1}{b1} = \frac{c2}{b2}$ substituting the known values:

$\frac{5}{4} = \frac{c2}{7}$ after solving for c2:

$c2 = 8.75$

Checking:

$\frac{5}{4} = \frac{8.75}{7} = 1.25$: the corresponding ratios or fractions of each similar triangle are equivalent in value

Extra: The trigonometric ratio and topics in the following not will be discussed throughout the rest of this book here and there. Some of this relates to what has just been discussed.

Given a right triangle, the ratio of the side opposite of an angle to the longest or hypotenuse side is called the sine of that angle. The ratio of the side adjacent to an angle to the longest or hypotenuse side is called the cosine of of that angle. The ratio of the side opposite to an angle to the side adjacent to an angle is called the tangent value of that angle. For this example we will use just the cos ¢ value, but sin and tan analysis also will be similar.

cos ¢ = adjacent the angle / hypotenuse = adj. / hyp. = b / c , mathematically:

b = (cos ¢) c

Though trigonometric values are not linear with respect to the angle, the above equation has the form of a linear (ie., graphs as a line) equation of: y = m x . Here, (m) is a constant value. For a constant angle, such as for a similar triangle, the (cos ¢) value is then also a constant value.

For creating a similar triangle, If say side (c) increases or changes by a factor of (n), being greater or less than 1, then all the other sides of that similar triangle, here (a) and (b) will likewise change by that same factor value. If the heights of a triangle are considered, then they will also change by that same factor of (n).

Mathematically, to keep both sides of this equation in balance, each side will need to be multiplied by that same factor of (n):

n b = n (cos ¢) c
n b = (cos ¢) n c mathematically:

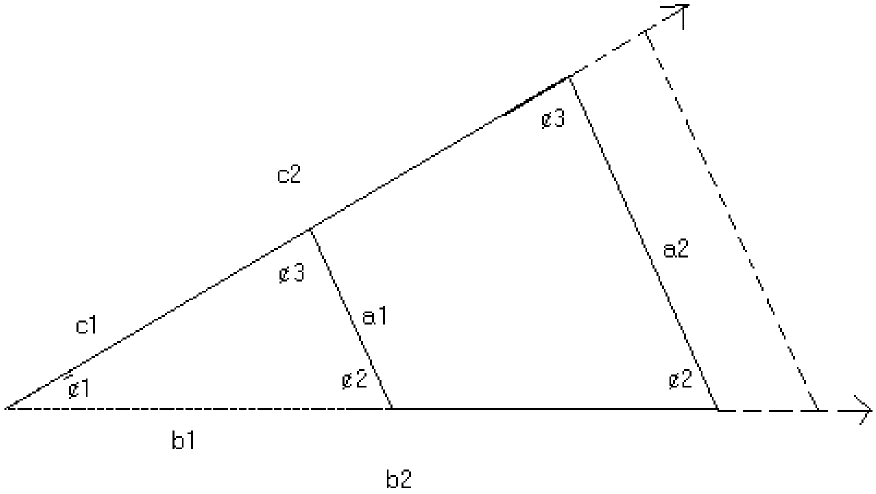
cos ¢ = $\frac{n\,b}{n\,c} = \frac{b}{c}$: showing the equivalent ratios, fractions or portions of similar triangles

CONSTRUCTING SIMILAR TRIANGLES

Constructing similar triangles, whether actual constructions, or drawings, is quite easy. Most of the facts and details to do this were previously indicated or mentioned, and a few others will be mentioned.

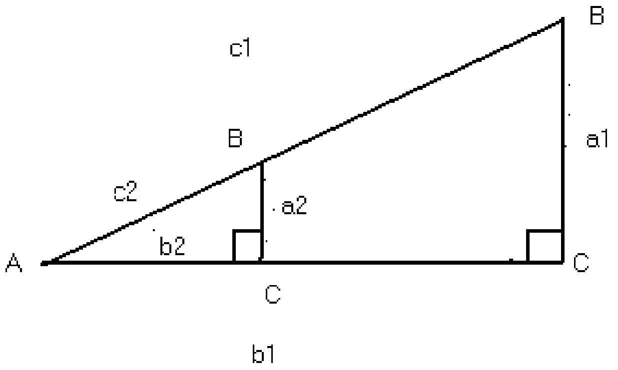
The simplest method to create a similar triangle is to use a line(s) that is parallel (and possibly even on the same line or an "extended line") to each corresponding side. Of course, to construct a similar triangle, the side lengths will be changed in value so that all its sides can intersect at some point, creating the triangle. In general, to make a parallel line to any side of a triangle, extend or reduce both (adjoining) nearby sides by the same factor or percentage value, and then connect these new end points with a (parallel) line(s). The length of each parallel lines will correspondingly be the same portion, factor or percent longer or shorter than the corresponding side of the original similar triangle.

Ex. Below, triangle1 with sides: a1, b1, and c1, is a general representation of any triangle. Sides b1 and c1 were then extended (here, partially indicated with the dotted lines, and arrow) to create a larger similar triangle called triangle2 with sides a2, b2, and c2. Sides b2 and c2 were then also extended (lengthened) by the same percentage or multiplier. Side a2 of triangle2 will be parallel to a1 and extended (ie., mathematically multiplied by) by the same factor used for b2 and c2.. After the construction below, triangle1 will becomes a smaller and internal similar triangle to triangle2. [FIG 86]



Obviously, ¢1 for each of the similar triangle's above is the same. The verification that ¢2, and ¢3 are the same for both triangles, as expected for similar triangles, is discussed ahead in the topic of: Corresponding Angles Created By A Transversal Line.

In the figure below, a line parallel to side a1 of a given triangle construction (a1, b1, c1) will be created and it will be called a2, and will be used to construct a small internal similar (right) triangle: [FIG 87]



It is quite easy to prove that they are similar triangles. For both triangles, angle A is the same value since it is bounded by the same sides or lines that intersect at a common vertex. Angle C is also the same, here it's 90° since they are right triangles. Angle B must also be the same value for both triangles due to the fact that the complementary angles (A and B), a concept that will be discussed further in this book, of a right triangle always total to 90° Hence if angle A is the same for both, angle B must be the same for both. Since all angles are the same, the triangles can only be similar, regardless of the side lengths. The result is that one triangle is simply a magnification of the other. Note that side a2 is parallel to side a1, and this "parallel side method" is actually a quick method to create a similar triangle.

By using the triangle above and the mathematical concepts of similar triangles and equivalent fractions, a very useful observation can be made. For example, at a point that is located at half of side b1, what is the (perpendicular to b1) height (to the other side) of the triangle at that specific point? The answer is the height is also halved, and this is verified below:

Since the angle (A in the above diagram) is constant, the tangent (TAN, a specific ratio of sides of a given (constant) angle which will be discussed ahead) value of that angle will also be constant:

$$\text{TAN } ¢ = \frac{a1}{b1} = \frac{a2}{b2}$$: the ratio of sides within a triangle are the same for any similar triangle

In more general terms, multiplying or dividing both the numerator and denominator by the same (factor, multiplier (m), or magnification) value will create an equivalent fraction whose ratio value is the same, and the ratio is constant, such as in similar triangles.

$$\text{TAN } ¢ = \frac{a1}{b1} = \frac{ma1}{mb1} = \frac{a2}{b2}$$: m is some multiplying factor

For the above example, half of b1 is (b1/2) = 0.5b1. Multiplying a1 by 0.5 to make an equivalent fraction (and similar triangle):

$$\text{TAN } ¢ = \frac{a1}{b1} = \frac{0.50a1}{0.50b1}$$: 0.50 = 1/2 = 50% = "half"

At half of b1, then a1, the (perpendicular) height at b1, must also be halved so as to be a similar triangle with the same portions.

Ex. At a tenth (or 10%) of b1, the height (a2) is also a tenth of a1.

$$\frac{b1}{10} = b1(0.10) = b2\,, \quad \text{and} \quad \frac{a1}{10} = a1(0.10) = a2$$

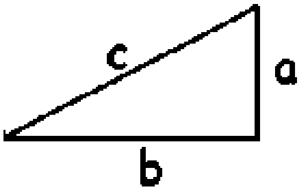
In short, to construct a similar triangle, you can reduce or increase each side by the same factor or percentage value.

If given just a single triangle, solving for the value of a side using proportions would not be possible since there is no other similar triangle to relate or be in reference to. There is however, a mathematical relationship among the sides of a single right triangle which will allow us to solve for a side if enough parts of that triangle are known. This method uses the Pythagorean Theorem, and it is discussed next. There are also some generalized methods that are discussed later that can solve for a part of any triangle if enough of its other parts are known.

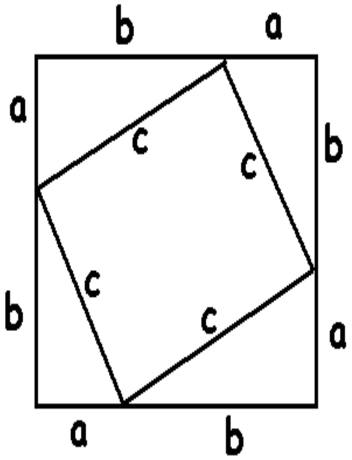
DERIVING THE PYTHAGOREAN THEOREM

Here is a derivation of the Pythagorean Theorem, and more specifically a simple mathematical **verification** of it among the several possible ways to do so. The mathematical relationship among the sides of a right triangle (ie., contains a 90° degree "right angle") is expressed by the very popular **Pythagorean Theorem** which is named after an ancient Greek mathematician named Pythagoras (570BC-495BC) who was the first to mathematically proved it in about 530BC, however the concepts of the theorem were known for perhaps a thousands years before him. A **theorem** (related to the word: "theory") is an idea or proposition of something that appears to be true by being verified by many supporting ideas and examples, and-or can be proven as true as a fact. Perhaps a more general name for the Pythagorean Theorem would be the Right Triangle Theorem. Expressing this mathematical relationship among the sides [FIG 88]:

$c^2 = a^2 + b^2$: here, c = hypotenuse or longest side of a right triangle
and (a) and (b) are the sides or "legs" of the right angle



There are many mathematical derivations of the Pythagorean Theorem. This formula for the longest or hypotenuse side of a right triangle construction should be memorized since it is one of the most useful formulas found in mathematics. Observe the drawing and derivation below for a verification of the Pythagorean Theorem: [FIG 89]



Like the area of a puzzle, the total area of the drawing above is the sum of all the areas of each piece: the inner square and the four identical (right) triangles. The total area is also equal to the area defined by the outer perimeter which is a square whose side length is (a + b), hence the total area is $(a + b)^2$

Total Area = Area of outer square = Area of inner square + Area of the 4 similar triangles
Total Area - Area of the inner square = Area of the 4 similar triangles

$$(a + b)^2 - c^2 = 4\left(\frac{ab}{2}\right) = \frac{4ab}{2} = 2ab$$

after some expanding and simplification:

$a^2 + 2ab + b^2 - c^2 = 2ab$ after solving for c^2, the hypotenuse of the triangles:

$c^2 = a^2 + b^2$: **PYTHAGOREAN THEOREM** (verified for right-angled triangles with (c) as the longest side)

$c = \sqrt{a^2 + b^2}$

Ex. A right triangle has sides: a=3 and b=4. What is side c, the hypotenuse side ?

$c^2 = \underline{a^2 + b^2}$ isolating c by taking square root of both sides:
$c = \sqrt{a^2 + b^2}$ substituting the given values for the variables:

$c = \sqrt{3^2 + 4^2}$ simplifying the radicand:
$c = \sqrt{9 + 16}$: ORD 1, order of operations step 1; powers and roots, here, in the radicand of the radical
$c = \sqrt{25}$: ORD 4, order of operations step 4; combine
$c = 5$: ORD 1, order of operations step 1; powers and roots

When the two sides or legs of a triangle are equal in value, the hypotenuse side is equal to the square root of 2 times that side length. This would also equal the diagonal length within a square shape:

$c = \sqrt{side^2 + side^2} = \sqrt{2\ side^2} = \sqrt{2}\ \sqrt{side^2} = \sqrt{2}\ side$: note, $side = c / \sqrt{2} = \sim (0.707)\ c$
$c = 1.414213562373096\ side = \sim$ about $1.414\ side$

Ex. If each side is 3, the hypotenuse side is equal to the 3rd multiple of the square root of 2, which is: 4.242640687

When one side or leg of a triangle is equal to twice (2 times) that of the other side or leg, the hypotenuse side is equal to the square root of 5 times that side length:

$c = \sqrt{side^2 + (2\ side)^2} = \sqrt{1\ side^2 + 4\ side^2} = \sqrt{5\ side^2} = \sqrt{5}\ \sqrt{side^2} = \sqrt{5}\ side$
$c = 2.23606797749979\ side$

When one side or leg of a triangle is x (a variable factor which can even be a fractional value less than 1) times longer than that of the other side or leg, the hypotenuse side is equal to:

$c = \sqrt{side^2 + (x\ side)^2} = \sqrt{1\ side^2 + x^2\ side^2} = \sqrt{(x^2 + 1)\ side^2} = \sqrt{(x^2 + 1)}\ \sqrt{side^2}$

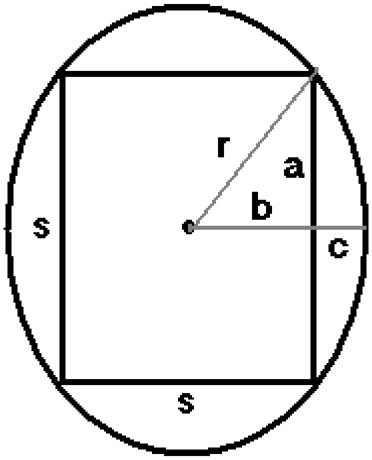
$c = \sqrt{(x^2+1)}\ side$

As the value of x gets larger and larger, the added 1 term has less of an effect to both x^2 and the result value, which is (c), and the expression approaches or becomes very close to the value of just:

$c = \sqrt{x^2}\ side$ which simplifies to:

$c = x\ side$

Ex. **THE LARGEST POSSIBLE SQUARE WITHIN A CIRCLE** [FIG 90]



In the above figure, s = the side of the internal or inscribed square, and $a = b = s/2$. The radius (r) of this circle is $r=(b+c)$. It could also be said the circle is circumscribing the square. The diagonal of the largest square in a circle is equal to the diameter (D) of that circle; $2r = 2R = D$

Due to the Pythagorean Theorem, and that $a=b$:

(radius of the circle)^2 = $r^2 = a^2 + b^2 = a^2 + a^2 = 2a^2$ therefore, by taking the square root of both sides:

$r = \sqrt{2}\ \sqrt{a^2} = 1.41421\underline{4}\ a$ solving for variable $a = b$ = half the side of the square:

$a = r / 1.41421\underline{4} = 0.70710\underline{7}\ r$

$2a = 2b = 2\,(0.70710\underline{7})\ r = 1.41421\underline{4}\ r = \mathbf{\sqrt{2}\ r = s}$: A formula for the side (s) of the largest square in a circle.
r = radius of the circle

$$\mathbf{r = \frac{s}{\sqrt{2}} = 0.7071\underline{1}\ s = \frac{d}{2}} \quad \text{and} \quad \mathbf{diameter = d = 2r = 2\,(0.7071\underline{1})\ s = 1.41421\underline{4}\ s}$$

Extra: From: $r = c + b = c + a$

$c = (r - b) = (r - a) = 1r - 0.70710\underline{7}\ r = r\,(1 - 0.707107) = 0.2928\underline{9}\ r$: = about or approximately $0.3\ r$

Note also that the diameter length of this circle is also equal to the diagonal line length of the square:

diameter $= 2r = 2\,(1.414214)\ a = 2.828427\ a$ or since $s = 2a = 2b$:

diameter$^2 = s^2 + s^2 = (2a)^2 + (2b)^2 = (2a)^2 + (2a)^2 = 2\,(2a)^2 = 2(4a^2) = 8a^2$

diameter $= \sqrt{8\,a^2} = \sqrt{8}\ \sqrt{a^2} = 2.828427\ a$

diameter$^2 = s^2 + s^2 = 2s^2$ after taking the square root of both sides:

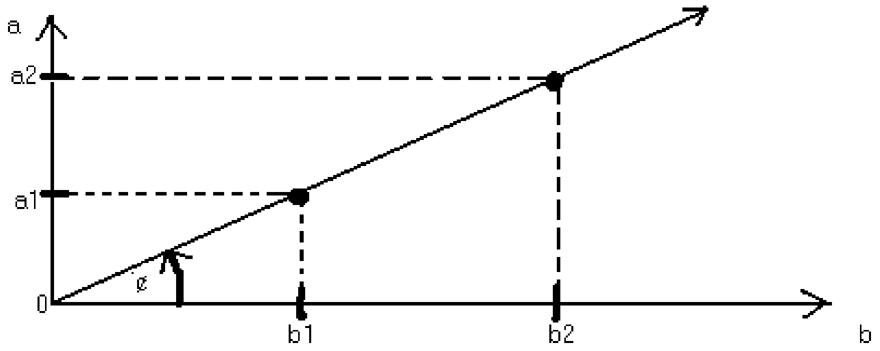
diameter = 1.414214 s = 2r

What about other triangles that are non-right triangles, does the Pythagorean Theorem hold true for them? Lets consider that if the right angle is changed to just slightly over or less than 90°. The triangle is then a non-right triangle but still appears to be as one. It is not to difficult to imagine that all the sides of this non-right triangle are nearly equivalent in length to that of a right triangle, and therefore, the Pythagorean Theorem is still practical for estimating and-or checking the results, however we will eventually need to use general (ie., for any) triangle analysis methods to find the sides and-or angles of a general triangle.

TRIGONOMETRIC RATIOS OF A TRIANGLE

Trigonometric basically means the study, measurements and mathematical relationships of the parts (sides and angles) of a triangle. Trigonometric ratios are (formal, defined, standardized) ratios among two specific sides of a right-triangle with respect, or in reference, to an angle of that triangle. The three most commonly used trigonometric ratios are given the names of: sine (SIN), cosine (COS) and tangent (TAN). They are of great use when a side or angle of the triangle cannot be found by other means. They are also used in more advanced mathematics. You may sometimes see the word "trigonomic" used for the word "trigonometric".

Below, "opposite" means the side directly opposite ("subtending") or across from the angle in question, and "adjacent" means the side directly near or adjacent to the angle in question, but it is not the hypotenuse side. ¢ is a symbol (or variable if you will) for any particular angle in question and which the ratio of sides corresponds or is in reference to. It is recommended that you memorize the three common trigonometric formulas below: [FIG 91]



$$\text{sine ¢} = \text{sin ¢} = \frac{\text{opposite}}{\text{hypotenuse}}$$

: This does not mean: (sin)(¢). It means: (sin ¢)=a ratio of the triangle's sides value.
: The word "sine" is rooted in the word "sinus" for bend, curve and cavity, and this type of shape is seen when it is graphed and it looks like a wave shaped curve.

$$\text{cosine ¢} = \text{cos ¢} = \frac{\text{adjacent}}{\text{hypotenuse}}$$

: the prefix "co" in cosine indicates the complement or complementary (of) sine

$$\text{tangent ¢} = \text{tan ¢} = \frac{\text{opposite}}{\text{adjacent}}$$

: the word "tangent" is rooted in the words for "touch" and "surface".
: A tangent value is often used to express the steepness or tilt of a line or object.

Notice that tan ¢ is mathematically equal to: (sin ¢ / cos ¢), and some of these equivalent fractions will be discussed and verified later:

$$\text{tan ¢} = \frac{\text{opp}}{\text{adj}} = \frac{\text{"vertical rise"}}{\text{"horizontal run"}} = \frac{y}{x} = \frac{\text{(opp/hyp)}}{\text{(adj/hyp)}} = \frac{\text{r sin ¢}}{\text{r cos ¢}} = \frac{\text{(hyp)(sin ¢)}}{\text{(hyp)(cos ¢)}} = \frac{\text{sin ¢}}{\text{cos ¢}}$$

: hyp = hypotenuse side
: r = radius of circle, or "radius vector" which is essentially the hypotenuse side.

Here is a quick verification:

$$\frac{\text{sin}}{\text{cos}} = \frac{\text{(opposite / hypotenuse)}}{\text{(adjacent / hypotenuse)}} = \frac{\text{opposite}}{\text{adjacent}} = \frac{y}{x} = \text{tan ¢}$$

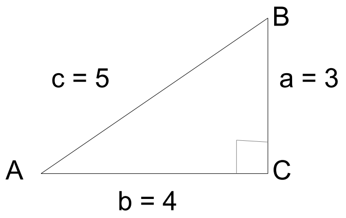
Three other, but less used trigonometric ratios are effectively the reciprocal of the three ratios just given:

cosecant ¢ = csc ¢ = $\frac{\text{hypotenuse}}{\text{opposite}} = \frac{1}{\sin \text{¢}}$

secant ¢ = sec ¢ = $\frac{\text{hypotenuse}}{\text{adjacent}} = \frac{1}{\cos \text{¢}}$

cotangent ¢ = cot ¢ = $\frac{\text{adjacent}}{\text{opposite}} = \frac{1}{\tan \text{¢}}$

Ex. For the triangle below, evaluate the sin, cos, and tan of angles A and B. [FIG 92]



SIN ¢A = SIN A = $\frac{\text{opposite A}}{\text{hypotenuse}} = \frac{a}{c} = \frac{3}{5} = 0.6$

COS ¢A = COS A = $\frac{\text{adjacent A}}{\text{hypotenuse}} = \frac{b}{c} = \frac{4}{5} = 0.8$

TAN ¢A = TAN A = $\frac{\text{opposite A}}{\text{adjacent A}} = \frac{a}{b} = \frac{3}{4} = 0.75$

SIN ¢B = SIN B = $\frac{\text{opposite B}}{\text{hypotenuse}} = \frac{b}{c} = \frac{4}{5} = 0.8$

COS ¢B = COS B = $\frac{\text{adjacent B}}{\text{hypotenuse}} = \frac{a}{c} = \frac{3}{5} = 0.6$

TAN ¢B = TAN B = $\frac{\text{opposite B}}{\text{adjacent B}} = \frac{b}{a} = \frac{4}{3} = 1.333333333...$

Usually, some of the intermediate steps would be eliminated:

Ex. SIN A = $\frac{a}{c} = \frac{3}{5} = 0.6$

Notice that the values of: sin A = cos B,
cos A = sin B,
tan A and tan B are reciprocals of each other: tan A = 1/tan B and tan B = 1/tan A

If given a trigonometric (ratio of two sides) value of an angle, you can mathematically solve for one of the corresponding sides of that trigonometric value if the other corresponding side is known:

Ex. If SIN ¢ = 0.6, and the hypotenuse is 5, what is the opposite side?

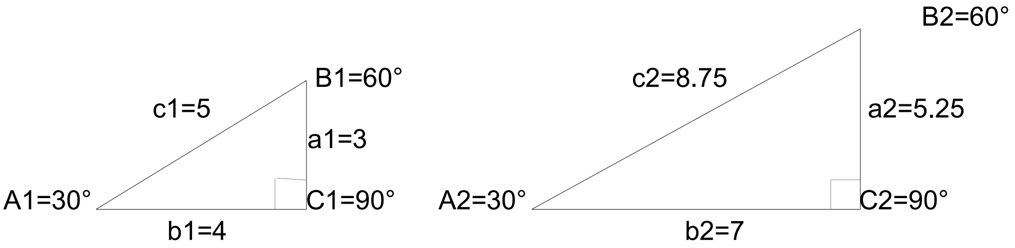
Letting ¢ = A, sin A = 0.6, a = side opposite angle A, c = hypotenuse,

From: $\text{SIN } A = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{a}{c}$ we mathematically have:

$a = c \text{ SIN } A$
$a = 5\ (0.6)$
$a = 3$

Since the ratio of any two sides of a triangle is equal to the ratio of the two corresponding sides of a similar triangle, all the trigonometric ratio values of two similar triangles will also be the same. [FIG 93]

Ex.



TRIANGLE 1 TRIANGLE 2

$\text{SIN } A1 = \text{SIN } A2$

$\frac{a1}{c1} = \frac{a2}{c2}$

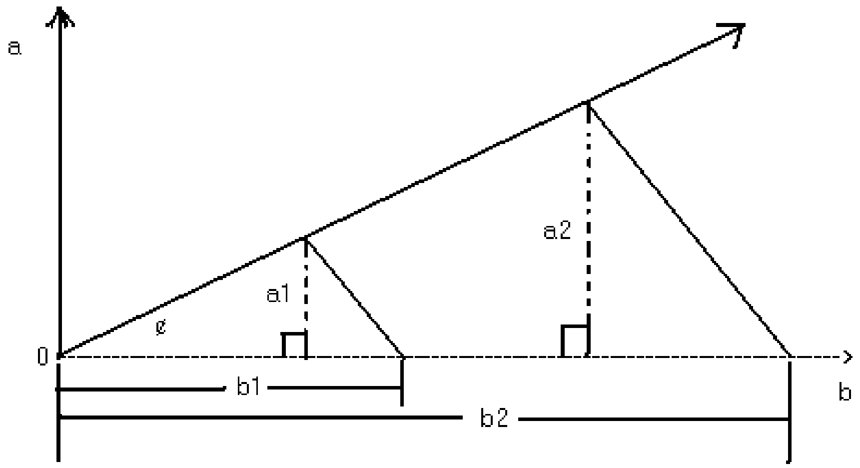
$\frac{3}{5} = \frac{5.25}{8.75}$

$0.6 = 0.6$: similar triangles have the same the trigonometric (ratio) values

Since the trigonometric ratios were in reference to an angle with the same value (here, A=30°), we could say that any other or all similar triangles would have the same ratio value. For this example, a sine value of 0.6. That is, every angle has its own particular and constant trigonometric ratio value associated with it. The sin of any 30° angle is always 0.6. The cosine of any 30° angle is always 0.866, and the tangent of any 30° angle is always 0.577. This fact that a particular angle value corresponds to a particular trigonometric ratio value, and vice-versa, allows us to solve for either.

VERIFICATION OF THE SIDE LENGTHS OF SIMILAR TRIANGLES

Here is a verification that the ratio of corresponding sides of similar triangles is equal. Observer the drawing below of a line that effectively creates a general angle (¢) -*for this analysis, and for simplicity, side (c) of each constructed (right) triangle is not indicated: [FIG 94]



If we divide the value of a1 by its' corresponding value of b1 on this line, we obtain a ratio (r) value:

$$r = \frac{a1}{b1}$$

This particular ratio value of the particular points (or sides of a triangle) is also equal to the slope (m=rise/run) of the line, and it is also equal to the tangent of the angle (TAN ¢) = opp./adj.

Since for similar triangles which have different side lengths, to retain their similar shape, the angle(s) is constant, therefore the trigonometric ratio (of sides) values (such as TAN ¢, and therefore the ratio=r) remain constant. In simple terms, the ratio of corresponding sides remains constant.

Taking another set of corresponding values of (a) and (b) on the line we get:

TAN ¢ = $r = \frac{a2}{b2}$ hence:

TAN ¢ = $\frac{a1}{b1} = \frac{a2}{b2}$ and algebraically:

$\frac{a1}{a2} = \frac{b1}{b2}$ or: $\frac{a2}{a1} = \frac{b2}{b1}$: verification

If you want to find other corresponding points (or sides of the similar triangle) on this line you can solve for a relationship between the (a) and (b) coordinates, and this relationship is the ratio value (r):

From: $r = \frac{an}{bn}$: n is some multiplying factor such as used when creating an equivalent fraction.

$an = r\,bn$: relationship between (a) and (b), and this is also the equation of the line which is composed of all the (corresponding) points on it. This has a basic line equation form of : $y = mx + b$

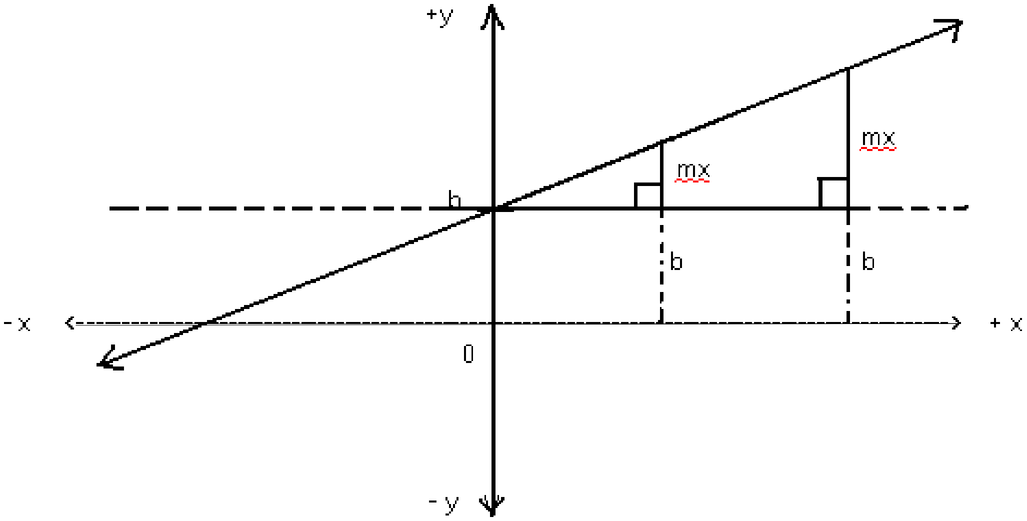
Dividing both sides by (n), this can also be expressed more simply as:

$a = r\,b$

Equating: (a) to (y), (r) to (m), and (b) to (x) , we arrive at:

$y = mx$: the basic linear (line) equation. (m) is a constant and the numerical coefficient of (x), and due to (m), (y) is a multiple (including <=1) of (x), and $(y / x) = (m)$.

Now, observe the two similar triangles below: [FIG 95]

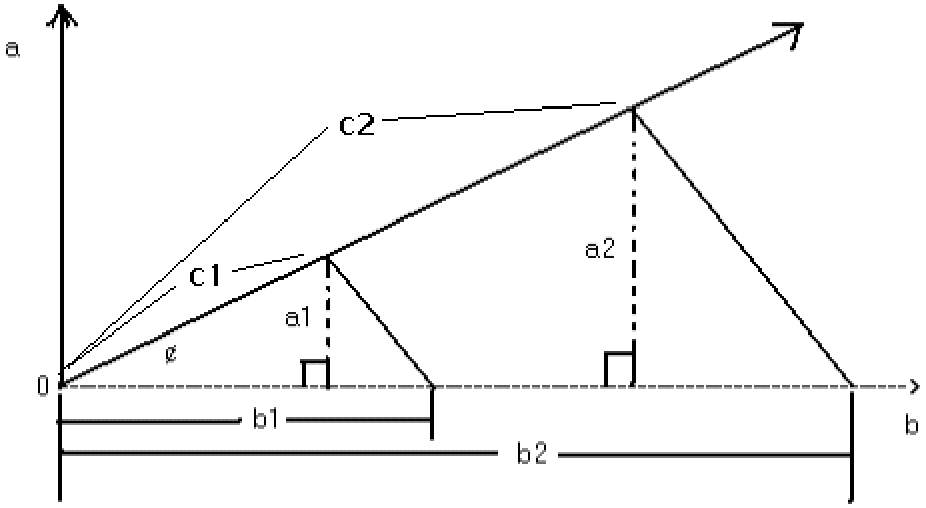


Notice that both of the right triangles have a side equal to (mx) for their height (which is related to the (y) value at those points; y=mx). This value of (mx) is not a constant, since (x) can vary, as expected for various sized similar triangles. The other (perpendicular to (y)) side, or base side of each right triangle is equal to (x). Considering just the graphing of the line, the corresponding (y) value for each (x) value is equal to the height (mx) of the similar triangle created at that point plus the value of the constant (b). Hence the (y) value on the line that corresponds to each (x) value is:

$y = mx + b$: basic linear equation for any line in a 2-dimensional, rectangular coordinate and measuring system.
: (b) essentially shifts each corresponding (y) value, and hence the entire line, vertically.
(m) can be thought of as the "steepness factor" (or slope) of the line.

Extra: Observing the above figure: $y1 = mx1$ and $y2 = mx2 = m(x1 + \text{ch. in } x1) = mx1 + m(\text{ch. in } x1) =$:ch.= change
$y2 = y1 + m(\text{ch. in } x1)$
$y2 = y1 + (\text{ch. in } y1)$
By equating the last two expressions and terms on the right sides:
$m (\text{ch. in } x1) = (\text{ch. in } y1)$, and mathematically:
$m = (\text{ch. in } y1) / (\text{ch. in } x1)$

Other useful facts can be derived here. Observe the drawing below of two general (non-right) similar triangles where the (a) value indicated within each triangle is equal to the (perpendicular) height (h, or "altitude") with respect to the base side (b) that it intersects with. These (a) values are also the (proportional) sides of similar right triangles. We can conclude from this that the ratio of (corresponding) height values of all similar triangles, and not just similar right-triangles, is equal to the ratio of any corresponding sides of the similar triangles. Remember that for a construction to be similar, each and every part, including the height or altitude sides, are magnified by the same value. For example: [FIG 96]



$\frac{h2}{h1} = \frac{a2}{a1} = \frac{b2}{b1} = \frac{c2}{c1} = r$: h is used to indicate height or altitude
: r is essentially the "magnification factor" of the similar construction that is a triangle.

The above graph shows that all sides, including the "height sides" of similar triangles are proportional (same portions or fractional values). The heights of similar right triangles are obviously proportional since any side, other than the hypotenuse, is actually the height with respect to the other side.

If corresponding sides of similar triangles are related by a ("magnification") ratio, is the perimeter (the outside edge length) of the corresponding similar triangle related by the same ratio? The answer is yes, and this is algebraically verified below where P1 and P2 are the perimeters of similar triangles:

$P1 = a1 + b1 + c1$ and $P2 = a2 + b2 + c2$

Since for example $a2 = r\,a1$, by algebraic substitution:

$P2 = r\,a1 + r\,b1 + r\,c1$ factoring (r) from each term:
$P2 = r\,(a1 + b1 + c1)$ dividing P2 by P1:

$\frac{P2}{P1} = \frac{r\,(a1 + b1 + c1)}{(a1 + b1 + c1)}$ canceling common factors of the numerator and denominator:

$\frac{P2}{P1} = r$

$\frac{P2}{P1} = \frac{h2}{h1} = \frac{a2}{a1} = \frac{b2}{b1} = \frac{c2}{c1} = r$

A logical question then arises: Is the ratio of the areas of similar triangles also equal to this ratio? The answer is: "close", but since area is a two dimensional or "square concept", the ratio (r) is the same, but squared (r^2).. This is verified below starting from the basic equations for the area of each triangle. A simple verification to this is when you double (ie., ratio=r=2) the sides of a square or rectangle, the area increases by 4 which is 2^2. If you tripled (r=3) the sides of a square or rectangle, the area increases by 9 = 3^2, hence we conclude that area increases by the square of the ratio (r) of similar (planar) constructions.

$$A1 = \frac{b1h1}{2} \qquad A2 = \frac{b2h2}{2}$$

From : $\frac{h2}{h1} = r$, we get: $h2 = r\,h1$

From : $\frac{b2}{b1} = r$, we get: $b2 = r\,b1$ by algebraic substitution:

$$\frac{A2}{A1} = \frac{\frac{r\,b1\ r\,h1}{2}}{\frac{b1\,h1}{2}} = \frac{2\,r\,b1\,r\,h1}{2\,b1\,h1} = r^2$$

: ratio of areas of similar constructions
such as triangles and rectangles.
r=the "magnification ratio" of the Area

While on this topic of similar constructions of which one is a magnification of the other and all its parts, here is a similar type of understanding for rectangles:

Given a rectangle with sides (a) and (b), its area is: A1 = (a)(b). If we want to make a similar and magnified (or demagnified, smaller) construction with respect or in reference to that first construction, we can multiply each and every part of it by the same value or magnification factor (n):

$$A1 = ab$$

$$A2 = (an)(bn) = an\ bn = (ab)n^2$$

$$\frac{A2}{A1} = \frac{(ab)n^2}{(ab)} = n^2$$

: this verifies that if all the sides of a construction are changed by the same factor value, that its area will increase by the square of that factor (here, n).

If you specifically wanted A2, rather than only the corresponding sides, to be a certain factor (n), a multiple bigger or smaller than A1:

A2 = A1 (number of times bigger) = A1 (n)

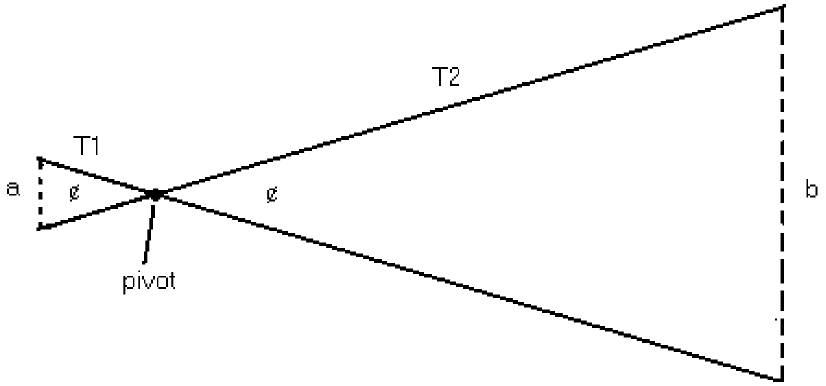
$A2 = (bh)\,n$ which can be expressed as:

$$A2 = b\sqrt{n}\ \ h\sqrt{n}$$

: to make a similar construction that is a certain factor or multiple bigger, change all of the parts by the square-root of that factor (here, n).

A SIMPLE DEVICE TO MEASURE SMALL DISTANCES

Using the concepts of similar triangles and vertical angles that are identical angles on both sides of the intersection of two lines, a device can be created to measure small distances and-or to obtain a distance measurement which will have greater precision (precise, the fineness, more least significant values, smallest capability) possible than without using it. The device presented below is only a very simplified form of such a device which can eventually be improved upon. Typically, a finely scaled ruler can only measure (precisely) to within as small as one sixty-fourth = 1/64 = 0.015625 of an inch. The device shown here is designed to measure to within about one thousandth = 1/1000 = 0.001 of an inch which is a more precise measurement. [FIG 97]



The device above is designed to measure a max. distance (or thickness) of 1 inch at side (a). That is, side (a), the "input location", of T1 (triangle 1) opens to 1 inch. Side (b) of T2, the "output or result (location or measurement)", is 20 inches when the distance being measured at side (a) is 1 inch. Hence, for any given distance measurement at side (a), side (b) will be 20 times longer or larger. Therefore, given this longer or magnified representative distance, to find out what it does actually represent (the true or actual measurement) we will divide (to reduce or demagnify) this output value by the known magnification factor or 20, hence essentially making the value of (1/magnification factor) = (1/20) as a demagnification factor.

What is the verification that side (b) is always 20 times longer? To begin with, both angles indicated are the same due to the concept of vertical angles. A vertical angle is the (equal) angle created on the opposite side of an angles vertex after extending the sides of the angle. You can recall the concepts of creating similar triangles to help verify this. The sides of T2 are extensions of the sides of T1, and side (b) is parallel to side (a), therefore, T2 is a similar triangle to T1. All the sides (and heights) of T2 are constructed to be 20 times longer than the corresponding sides of T1, hence the ("magnification") ratio (r) value is 20:

magnification ratio $= \frac{\text{output}}{\text{input}} = \frac{b}{a} = r$ mathematically:

output = input (r) and input $= \frac{\text{output}}{r}$ = output (1/r) : input = (output)(demagnification factor)

$b = a(r)$ if $(r) = 20$:
$b = a(20)$ or:

$b = 20a$: the "magnified value of (a)"
: this is of the form of a linear equation of: $y = mx$
$a = \frac{b}{20}$: the actual, true or "unmagnified" value of (a) . or:

$a = \frac{b}{20} = \frac{(1)b}{(20)} = 0.05\ b$: (1/20) = 0.05 = demagnification factor for, or applied to the value of (b)

Note that whenever the value of side (a), and therefore the angle, of T1 changes (such as when taking new measurements with this measuring instrument), the new triangle T1 will obviously not be similar to the old triangle T1, however, T1 and T2 will always similar triangles.

If side (b) is expressed or scaled in 1/64 of an inch unit increments, the decimal equivalent of this fraction is:

$\frac{1\text{in.}}{64} = 0.015625$ in. : slightly over 15 thousandths of an inch
: This is the precision of the (output or result) side, or (magnified) scale, of b.

, and if (b) indicated or was read to be (n) such increments during a measurement at side (a), the total length indicated at side (b) is a matter of repeated addition, or simply multiplication:

$b = 0.015625\text{ in. }(n) = n\ 0.015625\text{ in.}$ therefore:

$a = \frac{b}{20} = \frac{n\ 0.015625\ \text{in.}}{20}$

If only one (n=1) increment (1/64 of an inch) was indicated at side (b), we can find the measuring precision of this device:

$a = \frac{0.015625\text{ inch }(1)}{20} = 0.000{,}781{,}25$ inch :slightly less than 1/1000 of an inch. This is the (smallest) precision or measurement possible with this measuring instrument.

For the readout scale only, the smallest unit of measurement is (1/64) in. (ie. the precision = how precise, fine or minute a measurement on the this readout can be). The actual reading on this scale will be indicated, on average, as somewhere between one of these smallest units or graduations on the scale. Hence, if performed correctly, it is possible that a measurement may be inaccurate by as much as this precision of (1/64 in.) on this scale. Since the smallest unit of measurement is (1/64) in., we must either round up or round down the observed measurement to the next closest unit in order to express the measurement with this precision. The form of rounding used will depend upon the person reading the scale and-or the standard used. This (+,- 1/64) in. maximum possible "output" or reading error corresponds to an "input" error (e) of (1/64)in./20 = +,- 0.000,781,25 in. for the actual object being measured. A general formula which includes this possible error (e) can be expressed as:

$a = b/20 \quad +,- \quad e$

$a = \frac{n\ 0.015625\text{ in.}}{20} \quad +,- \quad 0.000{,}781{,}25$ in : actual measurement which includes the possible error

As mentioned above, the drawing of the device is only a basic one. Perhaps the "read-out" scale at the side (b) end would be of the form of an arc that is concentric (surrounding like a circle or part of a circle) about the pivot point. Perhaps one "arm" of the device would remain "fixed" (not movable) at the "0" or reference position end of the "read-out" scale. Note also that if the intersecting (non-parallel) line segments are equal in length, the center, halfway or mid-point of either line is located at their intersection point, and the distance between the endpoints is the same on both ends.

The device is also a very simple mechanical calculator. The operations capable with this device are multiplication and division. Here, due to its construction, the multiplier and divisor are 20. If you can adjust or set the length of the "arms" or "legs" of this device, and-or have some movable pointers or indicators, and you can then effectively make a variable (operand) multiplier and divisor calculator. The above construction of a distance multiplier is also the basis of a speed (= distance covered per time unit) or force multiplier (including demagnification) of a "simple machine", however, in the opposite direction, that is, the end at the smaller arm section will be where the (output) force is multiplied or increased, and likewise, the larger opening is where the (applied) force is less or decreased. The force gain or multiplier is determined by the ratio of the length of the sides or arms (or "lever arms") on opposite sides of the vertex or (rotational, "axis") pivot point of the construction. For premade mechanical devices that can measure small distances, such as the thickness of a piece of pipe (ex: inside diameter, thickness and outside diameter), metal or some material, research the measuring tools called (screw thread) "**micrometer**" and "**vernier calipers**". The initial concept and device of a micrometer was invented and used for astronomical measurements in about 1638 by **William Gascoigne** (1612-1644), from England. These devices have a reasonably high precision of about a tenth, hundredth or thousandth of a length unit, and for most practical measurements. Many of these devices are relatively inexpensive (perhaps under $20 USD and even less if made of plastic) as of the year 2022. Some of these devices have mechanical or printed scales, and some may have a rotating dial reading (ie., a disk with a indicator or pointer on it, much like a mechanical clock) or a digital-electronic display of the measurement taken. Vernier calipers were invented in about 1631 by **Pierre Vernier** (1580-1637), from France. This is a type of ruler with a common scale for the largest measurement, and another scale so as to calculate a fraction of the smallest unit on the common scale. This fractional length is then added to the measurement from the common scale. It is possible to make measuring device using gears, and with each next connected gear and allowing more precision and accuracy in a measurement, much like a mechanical counter. Before these fine devices there have been some other methods of obtaining more precision and accuracy such as the **Transversal Ruler** and-or concept which is thought to have been invented in the early 1300's.

With the device shown in the previous figure, it is even possible to plan and construct a similar device to find a fractional portion of any length positioned at the "output end". The input end would be scaled in fractions of its length, and it is even possible to use similar shaped and length (here, diameter) objects for this such as similar coins. For example, using 3 coins, each would then indicated (1/3) of the total input length. Using 5 coins, each would then indicate (1/5) of the input length. The output length needs to be perpendicular to the input length, and the (opposite) endpoints of each length need to be connected by an imaginary line, hence showing a complete magnification of the full input length. .Using a string(s) or rod(s) with a fine pointer at the center of its end, say (1/3) or (1/5) of the input length can then be effectively projected and-or magnified onto a corresponding (1/3) or (1/5) portion of the entire output length.

Extra: The above device can also be considered as like a lever, and with one side fixed in position or not, and various useful and similar concepts can be shown. The reader may skip over this section and consider it at another time.

When one end (a or b) of the device is brought together, the other end of the device will also be brought together at the same instant. Since (b = d2) is a greater distance than (a = d1), it endpoint(s) must be traveling faster to go that greater distance in the same amount of time. If v2 corresponds to the points at (b), and v1 corresponds to the points at (a) then:

$t2 = t1 = t$ From $d = vt$, we have: $t = \frac{d}{v}$, $\frac{d1}{v1} = t = \frac{d2}{v2}$, and mathematically: $\frac{d1}{d2} = \frac{v1}{v2}$,
$d2 > d1$
$v2 > v1$

or: $\frac{d2}{d1} = \frac{v2}{v1}$: equivalent fractions, and showing that distance and velocity are proportional in values

If d2 is (n) times greater than d1, then v2 will be (n) times greater than v1.

It can then be said that the velocity on one side is a magnification of the velocity of the other side.

$$\text{Energy} = E = \text{Work} = W = (\text{force})(\text{distance}) = Fd = (\text{mass})(\text{acceleration})(\text{distance}) = mad = mavt = m(v/2t)(v)t = mv^2/2 \text{ Joules}$$

Energy in = Energy out : for a system with no losses (ie., transfers) of energy
E in = E out A common example is plier(s) tools, and of which are essentially a lever.

From $E = Fd$, $F = \frac{E}{d}$, We see that force and distance are inversely related when given an amount of energy to apply as a force or vice-versa. If the distance decreases, the force will increase by the same factor, and this will happen at side (a) in the drawing since side (a) is the shorter and-or the side closer to the pivot, rotation or lever point. The velocity at side (a) is less than that at side (b) by the factor of n=(b/a), hence va = vb / n , and the resulting output force will be (n) times greater than the input force.

$E = \frac{n}{n} Fd = nF \frac{d}{n} = \frac{F}{n} nd$, $F = \frac{E}{d} = \frac{E}{v\,t}$, $nF = \frac{nE}{d}$, Also: If a given amount of energy is applied for a low or short distance, the force will be high. If the energy is high, the force will be high.

Levers, gear speed and torque have a similar analysis to what was shown above. When a small gear is used to transfer mechanical power (P **=** E / t **=** Fd / t **=** Fv) to a larger gear, the arc distance moved per time of the small gear is low, and the force applied to the larger gear is high. Again this maintains that: Pout = Pin and-or Eout = Ein less any friction losses, etc:

For an example of the above, a device called a **wench** that can apply a force to a cable and move (ie., pull) a heavy load attached to the end of it closer to that (anchored, set to not move) wench, and by essentially using a relatively low power motor and small gear connected to a larger gear(s). A wench mechanism often contains a ratchet mechanism so as to prevent it from becoming unwound, slipping and causing a dangerous situation such as when pulling a vehicle or tree out of a ditch. a wench can also be used as a type of electric pulley system so as to lift things vertically.

By using successive gears, say each ten times larger, it is possible to put numbers on each gear so as to create a decimal number system counter, and-or a measuring device to measure small distances to a high precision.

CONVERTING BETWEEN AN ANGLE AND ITS CORRESPONDING TRIGONOMETRIC RATIO VALUE

The easiest way to convert between an angle and one of its corresponding trigonometric ratio values, or vice-versa (in reversal), is to use a pre-written reference table or a ("scientific") calculator that has these ((automatic) calculating or conversion) functions or abilities. Section 4, the ADVANCED TOPICS section of this book, will show you some formulas for how to make these calculations, especially when a scientific calculator or reference table is not available. These calculations can be greatly assisted by using a simple ("5 function") calculator, and if that is not available, performing the calculations by with pen and paper can always be used.

Ex. What is the sin of 60° ?

Using a scientific calculator, first enter 60 and then press the SIN function key or button. The result displayed should be about 0.866 when the calculator is in standard degrees angle input and output (processing, conversion) mode.

SIN 60° = 0.866

Ex. If the sin value of an angle is 0.866, what is the angle ?

Using a scientific calculator, first enter 0.866 and then press the inverse sine key. Sometimes this key is labeled INV SIN (INVERSE SINE), SIN^-1 (this is just a certain notation or indication, so do not use the mathematical reciprocal since the notation here rather means the inverse function of, rather than an inverse or reciprocal value), ARC SIN or ARCSIN (ARCSINE, as in the arc or angle of a circle). This specific calculator key needed could be located on the calculator where some other common key (such as the SIN key itself) can be reused as another or second function, such as INV SIN if you first press the calculator's "2nd function key" first, so as to activate all the "2nd function keys". The specific "dual or second function key" here is often for the SIN key itself. The result displayed should be 60 which is to be interpreted as 60 degrees = 60°. Expressing this mathematically:

ARCSIN 0.866 = 60°

The formal or standard mathematical expression for the corresponding trigonometric function of a given angle,
and the formal or standard mathematical expression for the inverse trigonometric function to find the corresponding angle given its trigonometric value have this form:

To find the corresponding trigonometric value of a given angle:

trigonometric_value = trigonometric_function(¢) or: trigonometric_value = trigonometric_function ¢

And

To find the corresponding angle given its trigonometric value:

¢ = arc_trigonometric_function(trigonometric_value) or: ¢ = arc_trigonometric_function trigonometric_value

The specific arc_trigonometric_function to use must correspond to the given trigonometric_value:

Ex. ¢ = arcsin(sin ¢) : arc-sine , If x = sin ¢: ¢ = arcsin(x) or = arcsin^-1(sin ¢) or = arcsin^-1(x)
¢ = arccos(cos ¢) : arc-cosine
¢ = arctan(tan ¢) : arc-tangent

Trigonometric Function and Inverse Trigonometric Function

Ex. 0.866 = sin(60°) and 60° = arcsin(0.866)

0.5 = cos(60°) and 60° = arccos(0.5)
$1.732\underline{1}$ = tan(60°) and 60° = arctan($1.732\underline{1}$)

A SHORT TRIGONOMETRIC TABLE

In order for you to have some trigonometric values when a calculator is not available, a small table of perhaps the most common values is presented below. Further ahead, this book devotes much attention to showing you how to calculate such values yourself, with or without the aid of a calculator. The appendix section contains a larger table.

¢	**SIN ¢**	**COS ¢**	**TAN ¢**
0°	0.0	1.0	0.0
1°	0.01745240643728	0.9998476951564	0.01745506492822
2°	0.03489949670250	0.9993908270191	0.03492076949175
3°	0.05233595624294	0.9986295347546	0.05240777928304
4°	0.06975647374413	0.9975640502598	0.06992681194351
5°	0.08715574274766	0.9961946980917	0.08748866352592
10°	0.1736481776669	0.9848077530122	0.1763269807085
15°	0.2588190451025	0.9659258262891	0.2679491924311
20°	0.3420201433257	0.9396926207859	0.3639702342662
25°	0.4226182617407	0.9063077870367	0.466307658155
30°	0.5	0.8660254037844	0.5773502691896
35°	0.573576436351	0.819152044289	0.7002075382097
40°	0.6427876096865	0.766044443119	0.8390996311773
45°	0.7071067811865	0.7071067811865	1.0
50°	0.766044443119	0.6427876096865	1.191753592594
55°	0.819152044289	0.573576436351	1.428148006742
60°	0.8660254037844	0.5	1.732050807569
65°	0.9063077870367	0.4226182617407	2.14450692051
70°	0.9396926207859	0.3420201433257	2.747477419455
75°	0.9659258262891	0.2588190451025	3.732050807569
80°	0.9848077530122	0.1736481776669	5.671281819618
85°	0.9961946980917	0.08715574274766	11.43005230276
89°	0.9998476951564	0.01745240643728	57.28996163076
90°	1.0	0.0	(undefined, approaching infinity near 90°)

Actually, this table can be reduced to a single trigonometric value (such as SIN ¢) per angle since the other two values, COS and TAN, associated or corresponding with that specific angle, can actually be calculated from that just the SIN value as will be shown further in this book. Many formulas for the conversion of trigonometric values will be shown ahead. With the table above, you can find the SIN, COS, or TAN trigonometric value of an angle. On the other hand, if given the trigonometric value, you can work in a reverse manner so as to find the corresponding angle.

Notice that TAN 45° = 1, this happens when the side opposite the angle, and the side adjacent the angle are equal in value and their ratio is 1. Ex. TAN ¢ = (side opp ¢/side adj ¢) = 5/5 = 1 . Some other helpful values to consider memorizing are:

SIN 0° = COS 90° = 0
SIN 30° = COS 60° = 0.5
SIN 45° = COS 45° =~ 0.707
SIN 60° = COS 30° =~ 0.866
SIN 90° = COS 0° = 1 : the maximum SIN or COS trigonometric value

You can plot the above values to show the relationships graphically. For example, letting x = ¢, and y = sin x = sin ¢, so as to view the sin ¢ trigonometric curve or waveform.

SOLVING A RIGHT TRIANGLE

To "solve" a right triangle is to find the values of its 6 main parts or components, that is, its 3 interior angles and 3 sides. To solve a triangle, at least 3 of its parts need to be known, and at least one of them must be a side so as to make it a certain, specific or unique triangle among all the other similar triangles (which all have the same 3 identical angles).

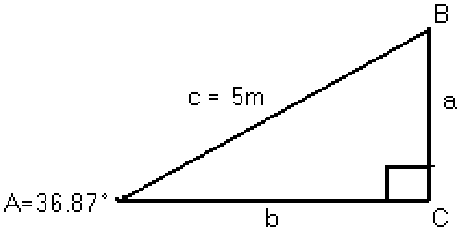
If given a trigonometric ratio value of an angle and one of the sides used to determine or calculate that trigonometric value, the other side which would be necessary to calculate that trigonometric value can be mathematically solved for. This is another method to solve for a side other than using equivalent fractions or portions of similar triangles, or the Pythagorean Theorem.

Ex. From SIN ¢ = $\frac{\text{opposite}}{\text{hypotenuse}}$: this assumes a right triangle construction or analysis, and we mathematically have:

opposite = (SIN ¢)(hypotenuse) : Since SIN ¢ is always less than or equal to 1, the opposite side length will always be less than or equal to the hypotenuse and is therefore only a fraction (ie., <= 1) of the length of the hypotenuse, and this fractional value is expressed as the SIN ¢. Mathematically, we also have:

hypotenuse = $\frac{\text{opposite}}{\text{SIN ¢}}$

Ex. Solve the given triangle: [FIG 98]



We are given: A=36.87°, c=5m, and C=90° since the triangle is indicated as a right triangle.

Note that here, 5m means: 5 units = 5 meters, and not (5)x(m) = (5)(m). Here, (m) is not a variable but it is the units of the measurement or reference.

Solving for angle B:

90° = A + B transposing A and switching sides:
B = 90° - 36.87° combining:
B = 53.13°

We can solve for side (a) since it is used in angle A's trigonometric sine ratio.

SIN A = $\frac{a}{c}$

Mathematically solving for (a) we get:

$a = (c)\ \mathrm{SIN}\ A = c\ \mathrm{SIN}\ A$ or= $(\mathrm{Sin}\ A)(c)$
$a = 5\mathrm{m}\ \mathrm{SIN}\ (36.87°)$
$a = 5\mathrm{m}(0.6)$
$a = 3\mathrm{m}$

We can solve for side (b) since it is used in angles A's trigonometric cosine ratio:

$\mathrm{COS}\ A = b/c$:

$b = c\ (\mathrm{COS}\ A)$
$b = 5\mathrm{m}\ \mathrm{COS}\ (36.87°)$
$b = 5\mathrm{m}(0.8)$
$b = 4\mathrm{m}$

Checking, using the Pythagorean Theorem:

$c^2 = a^2 + b^2$
$5^2 = 3^2 + 4^2$
$25 = 9 + 16$
$25 = 25$: checks

It is very helpful to remember that the side opposite an angle (formally) corresponds to that angle. For example, side a is opposite from angle A. Side b is opposite from angle B. Side c is opposite from angle C. It is also very helpful to remember that the two sides of an angle correspond to the other two sides. For example, angle A is composed of sides b and c. Considering the above example:

Side (a) can also be found and-or checked using angle B's trigonometric COS ratio:
$\mathrm{COS}\ B = \text{adjacent} / \text{hypotenuse} = a / c$
$a = (\mathrm{COS}\ B)\ c$

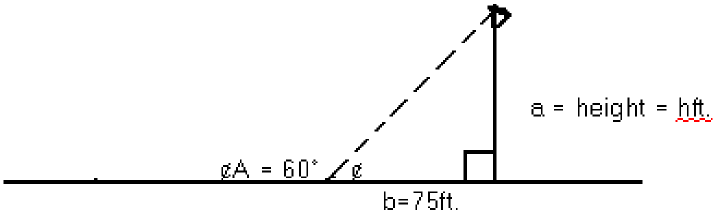
Side (b) can also be found and-or checked using angle B's trigonometric SIN ratio:
$\mathrm{SIN}\ B = \text{opposite} / \text{hypotenuse} = b / c$
$b = (\mathrm{SIN}\ B)\ c$

SOME PRACTICAL USE OF A RIGHT TRIANGLE

Many practical applications of the concepts of right triangles require only solving for a part of the right triangle construction. A helpful way to begin solving for a part is to make a drawing of the situation, information and values.

Ex. If a person is 75 feet from the base of a flagpole and measures an angle of 60° between the ground and the (imaginary) line of (eye) sight to the top of the flagpole, how high (h) is that flagpole above the horizon or eye level at the height of the observer?

Drawing a sketch of this situation, we note that we are to effectively find side (a) of a right triangle. [FIG 99]



a = h can be solved for by using the TAN A trigonometric ratio.

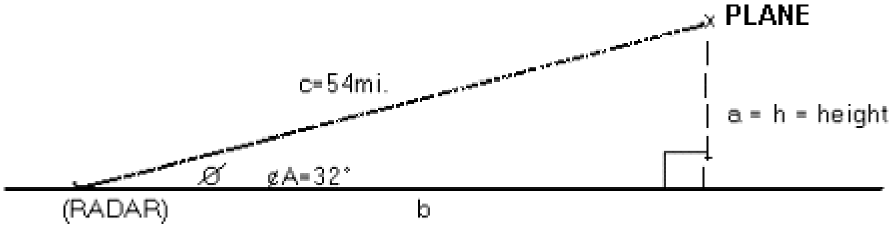
$$\text{TAN A} = \frac{\text{opposite}}{\text{adjacent}} = \frac{a}{b} = \frac{h}{b}$$

solving for h:

h = b (TAN A)
h = 75ft.(TAN 60°) = 75ft.(1.7321)
h = 129.9ft. , and if the height of the observer is to be considered in this calculation:
h = 129.9ft + height of the observer at eye level

Ex. An airplane (ie., a "plane") is detected by radar (radio waves used to sense metallic flying objects such as airplanes) to be 54 miles away (the distance of line of sight and-or radar) and at an angle of 32° above the horizon. How high is the object with respect to the ground (horizon, near horizontal level, surface of the earth), and how far away is it with respect to the radar or observer on the ground?

Drawing a sketch of the situation, we note we are to find the sides opposite and adjacent to angle A of a right triangle. [FIG 100]



h = a = c SIN A
h = 54mi. SIN 32° = 54mi. (0.53)
h = 28.62 mi. = height

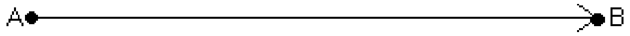
b = c COS A
b = 54mi. COS 32°
b = 54mi. (0.848) = 45.8mi. = horizontal distance. : This is the distance to the location where the plane is located directly overhead.

If the airplane is farther away, and yet at the same altitude or height, the angle it appears to be above or with respect to the horizon would be less and perhaps giving the appearance that it is at a lesser height.

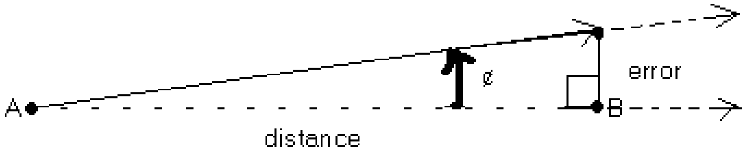
A SMALL ANGLE ERROR CAN MEAN A LOT OF ERROR

The following describes the type of situation when a small, seeming insignificant angle error can grow and become a large and significant (meaningful, large) error. First, a common or practical example is when a rod or something like a tree branch moves (ie., rotates, like on a pivot) just a small angle and distance one end, and it will correspondingly move a much larger distance at its other, more distant end.

Observe the drawing below, the shortest distance between two fixed points is the straight-line distance between those points. [FIG 101]



If something is to go from point A to point B, whether it is to move from point A to point B, or be some type of construction that connects the (A and B) points, and is "off" ("the mark", exactness, an error) by only a small angle, say a 1° error with respect to the straight-line direct course from A to B, how much will it actually be or result in the amount of (distance) error from point B? This type of question can be analyzed without unnecessary complexity by using the concepts of a right-triangle: [FIG 102]



Considering the angle remains constant, the amount "off" or error depends upon the distance between those two points (A and B). The smaller the distance, the smaller the error, and vice-versa. In many non-critical applications, small errors are typically accepted without being a problem. Notice in the drawing above that the error value (the distance between the expected location and the error location (due to the angle error) can be considered as a side of a right triangle, and the distance between the (start and ending) points can be considered and analyzed as another side of that right triangle, particularly, the hypotenuse side.

From: $\text{TAN ¢} = \dfrac{\text{opposite}}{\text{adjacent}} = \dfrac{\text{error}}{\text{distance}}$: ¢ = "error angle", solving for error:

error = (TAN ¢)(distance) : the farther (greater) the distance from the vertex ("start or origin") of the angle, the greater the error. This is expressed in the equation as a direct relationship. Also, since TAN ¢ is directly related to the angle, the larger the amount of (error) angle from the (reference) angle, the larger the "error" distance.

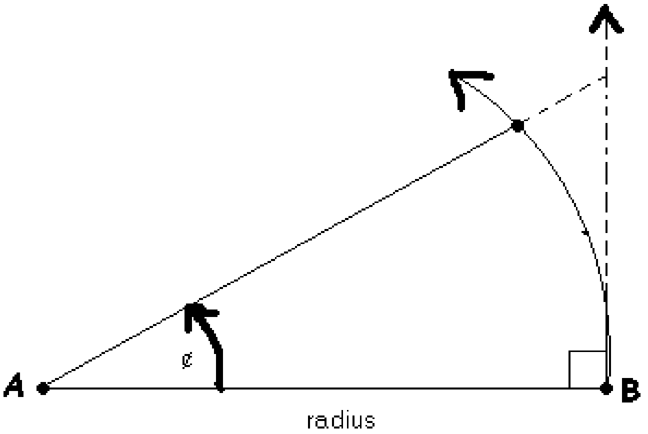
If the distance between the points is 1,000 feet, and the "error angle" is only 1°, the (perpendicular) distance error is:

error = (TAN 1°)(1,000 feet) = (0.017455064)(1,000 ft.) = 17.455 ft.

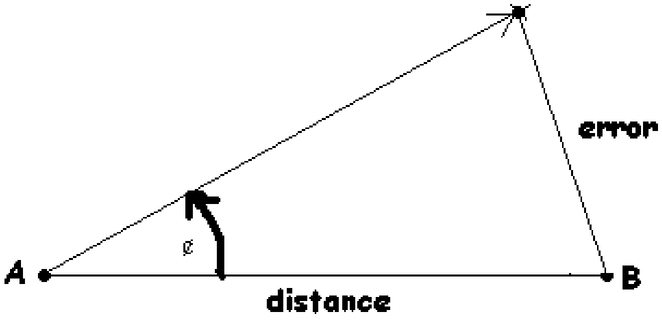
If the distance between the points was much larger, say 1,000 light-years (a light-year = light-year-distance = the distance light will travel in 1 year of time; light travels at about 186,000 miles/second), perhaps the distance to a particular star in the universe, the error distance would be an enormous value of 17.455 light-years due to a just a relatively small 1° error.

If you wish to consider a more accurate analysis:

If there was an error angle, and if the distance actually traveled was equal to the distance between points A and B, the hypotenuse side of the right triangle would now be reduced in distance. This is easily verified by rotating this distance line (from A to B) about point A, as if this distance was the radius of a circle:
[FIG 103]



The hypotenuse, being the longest side in that right-triangle, will always be smaller when it is set to a value equal to the smaller distance (here A to B) of that right-triangles other sides. When the error angle is small, the error side is practically a side of a right triangle and can be analyzed as initially shown above. As the error angle gets larger, the triangle becomes less of a right-triangle and the error distance must now be analyzed using non-right, or general triangle analysis:
[FIG 104]



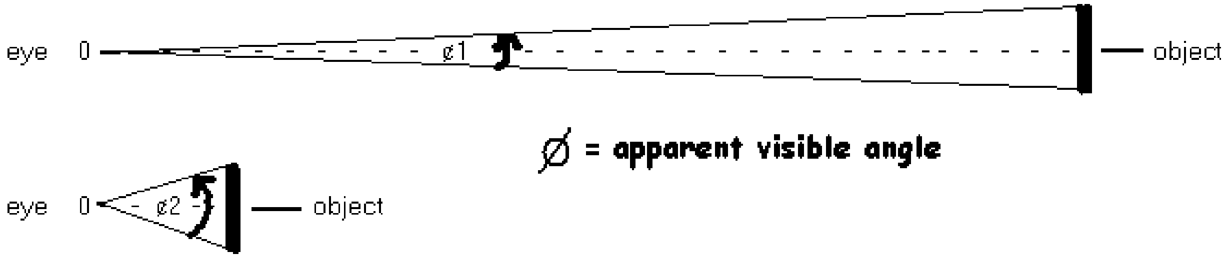
For the specific given situation, two sides of this triangle are equal. This type of triangle is called an isosceles triangle. Those two sides will intersect the third side and create two equivalent angles at that side.

THE APPARENT SIZE OF AN OBJECT

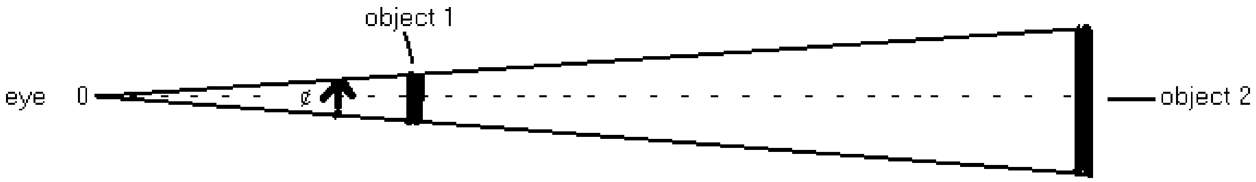
You or someone else might want to know the actual size of a distant object that you have seen. The discussion below initially assumes that there are no other nearby objects to it with their known (as a reference for measurement and comparison) size.

A object having a certain constant actual size will appear bigger when closer, and smaller when more distant. But why? The reason is that the apparent (visible with your eye) angle of the object is changing, and that the (observed, observation) angle size or value is related to the objects distance from you. The specific mathematical relationship between the observed angle and the objects distance from you is an inverse relationship. In more technical words or terms, it is said that when the distance between you and the object changes the angle that the object will subtend (be directly across from, and corresponding to) will change. When the object is closer to you, the apparent or observed angle of its size or width will be greater, yet the true or actual size of the object is still the same. As the object gets farther away from you, its apparent or observed angle and therefore, its apparent observed size will be less.

Consider these illustrations of the same object being farther, and then closer: [FIG 105]



Notice that when the same object is farther away from you, that the ("height" or "width") angle is smaller. Clearly, the distance to the object and the angle are inversely related. However, you still cannot determine the actual size of an object just by this angle alone. Two or more objects of various sizes may subtend the same observed angle and visually look the same size when the distance to each particular object is such that this same angle condition exists. For example: [FIG 106]



As seen above, even though the angle is the same for both objects, object 2 is clearly farther away and bigger. These object will appear as the same size even though they are not. This is an example of an "optical illusion". In this situation both the actual size of the objects and distances to them are not known and can only be estimated if there is not other data or a reference size known. For objects having a small viewing angle, at approximately half way to you, it will appear twice as large in width and-or viewing angle.

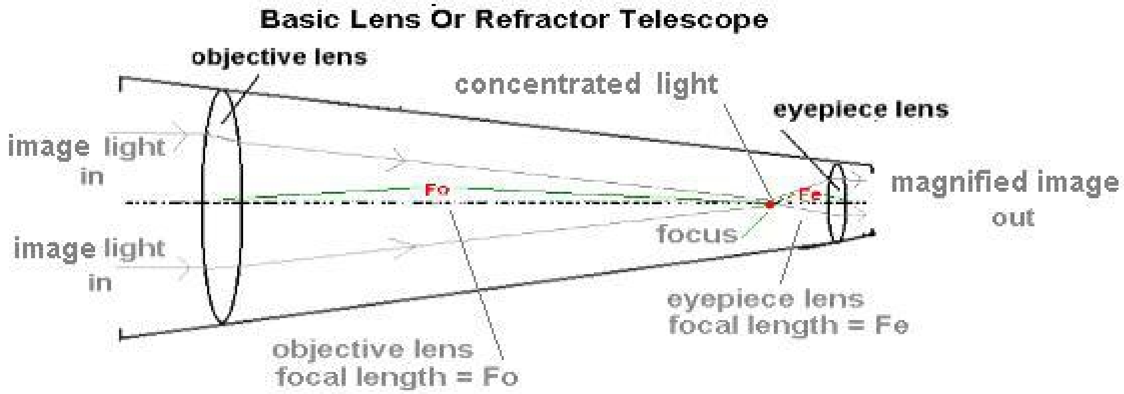
Along with the visual angle and the distance to the object, you can use right angle trigonometry (using ¢/2; "half angle") to find out the actual (length or width) size of the object, otherwise, you will need the reference of a known distance to a nearby object. If you know the size of a nearby object, you can use it as a reference (ie. for comparison, measurement and estimation) for the size of the object in question. One common example where reference sizes are used is in photographs or drawings of objects, and where it may be unclear or uncertain of the objects actual size, especially if it is a highly magnified image or view of a small object. Here, you will often see a ruler or a well known object that has a constant specific known size such as a coin being used as a reference size within the photograph itself. Most maps (which are a demagnified image of a true or actual sized image) showing vast areas and distances will also have a (greatly demagnified) distance or size reference line, rule or scale of measurement that corresponds to, or represents an actual distance between locations or points on the map or photograph. Basically, its a magnification, equivalent fractions or proportions concept. For example: Indicated on a certain map is that 1 inch is to 5 miles, and this means that 1 inch here on this demagnified image or map, corresponds to, or represents 5 miles of the true, real or actual image. If you measured a distance of 2 inches between two points on this map, how far in reality are those points apart or distant from each other?

1 inch is to 5 miles , as 2 inches is to what (x) miles?

$\frac{1\text{in}}{5\text{ miles}} = \frac{2\text{in}}{x\text{ miles}}$ after solving for x miles: x miles = (5 miles)(2) = 10 miles

A telescope and-or microscope increases the apparent viewing angle of an object or a part of it, therefore making the object or subject appear larger and-or closer, hence a magnified image of the object. The initial or objective lens gathers, refracts (ie., bends, redirects due to the slowing of the velocity of the light - much like for example: slowing the tires of the left side of a vehicle will cause it to turn leftward) and concentrates the incoming light available to the eyepiece, hence the objective lens will effectively increase the apparent visible brightness of the image (and not the actual object or subject) seen. The eyepiece lens is then used to focus (for the best clear and sharp image) and magnify the image (or part of it) received from the objective lens. As discussed in this book, if the light and-or image is magnified to a larger area, the available light energy and-or intensity is "spread out" and becomes dimmer, and that is why a large objective lens is used. A glass substance, usually silicon-dioxide, aka "quartz", can **refract** (ie., refraction,refactor, bend at an angle) the direction of incoming light, and it could be thought of as much like a surface and-or internal reflection caused by the new medium (here, glass) the light is passing through. A lens will have a certain light diffraction angle due to its material. **Diffraction** is generally associated with the scattering or spreading of light rays in various directions. When the light exits the lens there will be a second refraction due to the medium has changed from glass to the less dense air. A lens, usually one with a convex (ie., a bulging, thicker in the middle, semi-spherical) shape, can then effectively refract all the incoming light so as to eventually concentrate it all to be at one point called the **focus point** of that lens. The size (diameter) and-or curvature of a lens will determine how far the focus will be from the center of that lens, and therefore it will also help determine or set the total magnification in a particular designed telescope. A very common lens that many people have is called a **magnifier lens**, and it is a convex lens. The magnifier lens is most likely the first lens created in antiquity. They were rare and expensive, and a few have been found. From about the year 1700, the word "lens" was used since it is shaped like a lentil (a type of edible bean with a convex shape on both sides - double convex). Several magnifier lenses stacked together will increase the total magnification available. If each lens can magnify 2 times, the total magnification of this lens system is the product of each individual magnification, hence, here it would be (2) times (2) = 4. A magnifier lens can sometimes be used as a makeshift close-up and focusing adapter lens for some cameras that cannot focus on objects that are very close, say less than 1 foot or about 30cm away. An eyepiece lens from binoculars also makes an excellent high power magnifier or "jewelers loupe" to view small objects. Generally for a "hand magnifier (lens"), the object to be viewed is about 1 inch to 3 inches from the lens, and the lens curve shape is designed for this, but for a telescope, the object to be viewed is generally considered at a long to infinite distance, and the telescope lens shape is designed for this, and resulting that the parallel rays of light from the object will be at the focus point. For a microscope with its highly spherical or curved lens, the object to be viewed is very close to the lens and therefore the area of the object seen, and in focus, will also be very small. Due to the high magnification of microscopes, the light viewed from the object will be dim and will therefore need the aid of a light source (sun and mirror, or electric lamp) shined onto the object to brighten up its image. Some microscope systems use an oil drop between the subject and (sealed) lens so as have an increased magnification and-or clarity.

The basic design of a common telescope is shown below, and this one is called a refractor telescope because the glass lens will cause and-or direct the incoming light (or: light rays or lines) to refractor or bend so as to converge and be focused at one point. One of the largest (ie., objective lens) refracting telescopes is the Yerkes Observatory telescope open in 1897 near Chicago, USA, The "doublet" (2 lenses used together in series so as to help correct some image focus and chromatic and aberration distortion) objective lens of this telescope is 40 inches = 3.33 ft = 101.6 cm in diameter. Its focal length is about 63 feet. This main telescope of the Yerkes Observatory was also specifically designed to make many scientific and-or astronomical measurements and high quality photography. A "triplet" uses three lenses. [FIG 106A]



Today, the initial magnification of an image by using a telescope is still the most important factor, but with modern image sensors (ie., a digital camera) with many pixels (picture elements or parts) being able to produce a higher **resolution** (smallest discernible distance or portion of the object and-or its resulting image seen) than that possible with the human eye, it increases the effective telescope system magnification and resolution, perhaps by two or three times more than without it. Rather than state the optical magnification of a telescope, it is perhaps better to state the final image resolution of the entire telescope system including the image sensor and-or final resulting image and-or photograph. For example, 1 pixel of the image or photograph is equal to a certain value of degrees of arc (ie., viewing angle) in the sky, and-or some minimal (resolvable, resolution) distance on the subject itself. A smaller arc and-or image resolution of the entire telescope system is good for comparison the abilities of telescope systems.

Many modern telescopes, such as those having the "(Isaac) Newtonian" design, use a thin silver or aluminum surface coating for the thick glass mirror, and in such as way and combination so as to reduce the overall size of the telescope and yet still increase the focal length of the telescope to have a high amount of magnification. Again, it is rather better to state the minimal angle that a telescope can resolve (make clear, distinguish), because for example, if the (linear) resolution or minimal part of an object is say 1 meter of length of a distant object and-or in the (photographic, or screen) image of it, if the distance to that object changes then that value of linear resolution is not constant and will change, but still, the (minimal) resolution angle of the telescope system will remain the same, A (maximum) image or resolution angle is sometimes called the (total, focal, "viewing angle") **field of view** or ("viewing field" of the object width seen at its distance) of the telescope, binoculars or camera system, and may be stated as a ratio of distances such as 100 meters wide (across. horizontally or "left to right" distance seen through the telescope) at 1000 meters distance, or a specific total viewing angle extending outward from the observer. This angle can be found from: Tan ¢ = (side opposite to the angle) / (side adjacent to the angle) = 100m/1000m = 0.1 , and ¢ = arc-tan (tan ¢) = arc-tan 0.1 = 5.71° This field of view angle is usually not the (smallest) resolution angle, and for example, the resolution angle may be just a fraction of this value, say (1/100th) of 5.71° and that value is only: 5.71°/100 = 0.0571° Note that the resolution angle by a human eye vision may be not as fine or small as that of a image sensor, such as a digital camera with an enormous number of pixels in a small area, and where each is an image or light sensor.

Another concept with telescopes, microscopes, and other lens systems is called **depth of field**, and this is the range of distance to and behind the object that will still be in focus to a high degree of viewing and-or image clarity. Generally, reducing the opening to a center area of a lens will increase the depth of field, and the best example of this is with a pinhole camera which has a large to infinite depth of field with no curvature of the lens - like a tiny piece of flat glass. With only a smaller area of any lens being utilized, that lens area is flatter (ie., less of a curve) portion of that lens. Microscope lenses are for very high magnification, and are made to be more spherical in shape, and these small lens shapes, much like a small bead of clear glass with a relatively high curvature, have a very small depth of field or focus distance. The lens must be very close the subject or object being viewed, and a cover slip (glass cover) is often placed over some objects being viewed so as the lens does not get debris on it. These objects to be viewed are initially placed onto a larger clear glass slide or plate, and a light source to brighten up the image seen can be placed beneath the glass slide and object, and-or above the object to reflect its light. Often with lenses, such as a magnifying lens, there is a small and minimum distance the lens must be from the object being viewed so as it will be in focus. In general, higher lens curvature results in more image distortion along its edges and-or sides, and due to that the light must be refracted or bent more so as to be at the focus, and this extra distance makes it effectively out of focus with image from the central area of the lens, and it will now have a another focus point nearby. This image and-or focus problem can be overcome by using a corrective lens. In a telescope or lens system:

f-ratio, f-number or f/number or f-stop = $\dfrac{\text{focal length mm}}{\text{diameter of lens opening mm}}$: objective lens opening = aperture.
Note: f-ratio does not mean focal ratio

Ex. (A 50 in. focal length / 5 in. diameter mirror) = 10 = fr = focal ratio = f10 = f/10 = hence: (focal length) (1/10)

Many "35mm sized" cameras, film or digital, have an adjustable lens opening, and which can effectively make a smaller objective lens, however, the total amount of light from the object will be reduced. As mentioned previously, the depth of field will increase when only the center part or area of the lens is being utilized, and this corresponds to a low f-number.

Since **power or magnification = (focal length of objective lens / focal length of eyepiece lens) = focal ratio**, if the focal length of the objective lens is short, the magnification will be small, and the f-ratio will also be small.

When magnification increases, such as with a longer focal length and-or bigger f-ratio, the part or area of the subject seen is actually less, but it will also appear as more magnified to the observer. What is happening is that the effective field of view decreases when magnification increases.

Lenses that let more light in are said as being "fast lenses" since a photographic image will be developed faster (ie., quicker, in less time) by the larger amount of light received than that of a smaller lens. Since a circle that is twice as big in diameter has 4 times the area, a lens that is twice as big can gather and focus 4 times as much light:

A1 = (pi) r^2 , if r doubles, A2 = (pi) (2r)^2 = (pi)4r^2 = 4 (pi) r^2 = 4 A1.

Concave and-or parabolic lenses, which if made flatter will have a low amount of curvature (ie., being less spherical, "more flat"), but will then have a longer focal length and greater magnification, however this will reduce the light intensity seen, and then some dimmer or fainter stars may not be visible. That is the "trade-off" size versus quality. Nonetheless, there is a solution to this problem using long exposure photography so as to effectively "build up" a better resulting or processed image. There are some computer programs to also do this, and you may look up **Registax** and similar programs that do "image stacking". These programs will process many images of the same scene and subject, and so as to produce an image having an effective longer exposure and with more sharpness or clarity (ie., with less noise [light pollution, sensor pixel issues] and blurring).

For more magnification, say double or triple, a special lens that you can purchase is called a **Barlow lens** that can be placed between the telescope and the eyepiece of the telescope. As indicated above, increasing magnification also decreases the intensity (id., amount of photons or energy per unit area) or apparent visible brightness of the light and-or object being viewed. Adding a Barlow lens and-or a diagonal viewing lens will also require you to refocus the telescope so that the focal length of the objective lens is kept, hence the viewing path will need to be shortened.

If the object you are viewing is high up in the sky, a diagonal mirror or special lens can be used to aid your viewing by making it more comfortable. Here, instead of viewing an object that is in the same direction as the length of the telescope, you can rather look perpendicular to that direction. This device can also be used to invert (ie., flip horizontally and-or vertically) the image of the object so as to have its proper (vertical) orientation, and especially when viewing objects "terrestrially" (ie., terrain, on Earth's surface. Optics and lenses is a fascinating scientific field of study, manufacturing and research, and a great hobby. Using a thick glass "blank [bulk, plate]" some people have made (ground, polished, and mirrored its surface) a main objective mirror for a Newtonian parabolic reflector telescope, but just like with solar-panels, it is typically much cheaper and less of a burden to buy it already made and assembled. It is very possible to make an inexpensive, thick (ex. 1 in. to 2 in.) glass mirror blank out of used scrap plate glass, perhaps 1/4 inch thick each, cut to a circle shape, and then slumping the layers in an electric kiln furnace to fuse them together as one. You will also have to research this kiln and mirror grinding process so as to have good results after some experimentation and practice. The longer the focal length of the mirror, the greater its (effective, parabolic) radius of curvature, and less glass will have to be removed, and in short, the mirror is "flatter" having less curvature.

The diameter of the human pupil averages from about 2mm to 3mm in bright light conditions, and up to 6mm to 8mm in the dark. Since telescopes are generally used at night, an eyepiece with a diameter of 3mm to 5mm are commonly used, and this will help maximize the magnification of a telescope system, however the trade-off or loss when using high magnification is lower light intensity or brightness. This should not be problem when viewing the Moon which is relatively bright. Note that a common size for a telescope eyepiece is 1.25", and this also indicates the size of the mounting tube it is in, however the final light output, exit lens or pupil lens on it is much less that this value. To focus an object, usually the eyepiece is moved slightly in or out along the light path (the direction(s) of the light)) through the telescope. The higher the magnification used for an object, the lower the (angular) field of view at the observers end. To see (ie., discern, resolve) the rings of Saturn, you will need a telescope with at least 50 power minimum or greater. Binoculars are generally lower power than this, perhaps 20 power, and are very useful for terrestrial use such as "bird watching" and astronomical use such as "star watching" where many more stars can be seen. Galileo is said to be the first to see a large, thin and flat "ring" surrounding the space about Saturn, and while using a telescope.

A general rule to expect for telescope magnification is that there can be up to a maximum power of 60 power per each inch (25.4mm) of objective lens, and before image quality is degraded. The focal lengths of the objective and eyepiece lenses used will determine the actual magnification. If the focal length doubles, the magnification doubles, however the image brightness or intensity decreases by the square of that value hence s^2 = 4. Note ex.: Given a 400 power telescope, note the square root of 400 is 20, as in 20 times (linear) wider, but given a 600 power telescope with 200 more power, the square root of it is slightly less than 25, hence an image about 25 times wider which is only a value of 5 units more, and this may not justify the price and-or reduction in image brightness. Today, with photographic techniques of long exposure duration photography and-or high quality (ie., resolution) digital cameras, many negative issues with brightness, magnification and resolution of a given telescope can be reduced.

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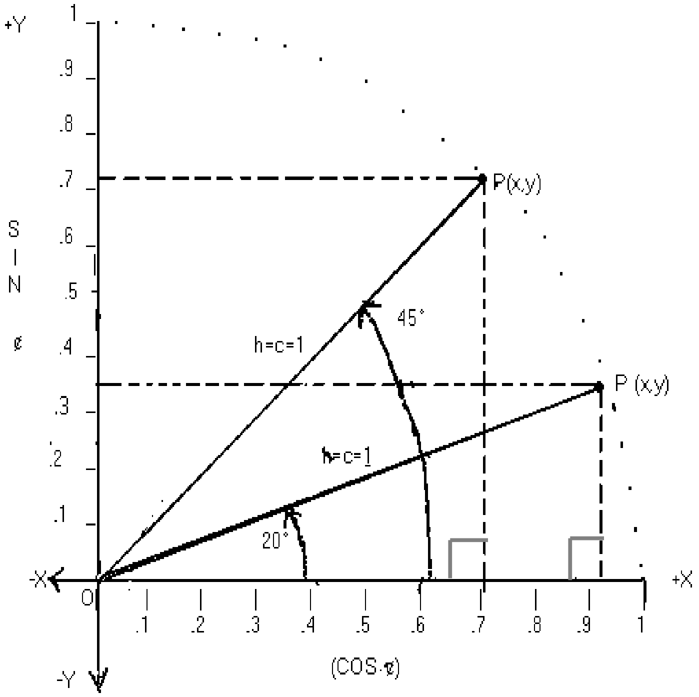
APPROXIMATING ANGLES AND TRIGONOMETRIC VALUES GRAPHICALLY

If a line segment, acting as a hypotenuse side of a right triangle, is given a length of one (1) unit (in "relative terms" or portions, this whole =1=100% value can represent any other whole or entire length value) and is rotated about the origin (O) of a rectangular coordinate measuring and location system, it will create an angle between that line and the X or horizontal axis. The Y coordinate of the endpoint (P) on the line will be equal to the sine of the angle, and the corresponding X coordinate of this endpoint will be equal to the cosine value of that angle. The point's location is therefore: P(x,y) = P(cos ¢, sin ¢). This is easily seen when lines are extended or "dropped" to both axis from that endpoint, and therefore creating a right-triangle construction that can be mathematically analyzed as a right- triangle. You may need a device that can measure (or create) angles such as a protractor. To "protract" is to extend outward as opposed to "retract" which means extend inward. In general, better or more accurate results are often easier to achieve by drawing a larger graph. [FIG 107]



Using the graph above, and the definition of SIN ¢:

$$\text{SIN } ¢ = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{y}{\text{hypotenuse}}$$

Since the hypotenuse here is equal to 1, or is considered as a relative value of 100% = 1.0 portion of some other length:

SIN ¢ = y : If the hypotenuse was a relative or percentage value of some length, then (y) and (x) will also be a relative values.

Likewise, $\text{COS } ¢ = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{x}{\text{hypotenuse}}$ Since the hypotenuse here is equal to 1:

COS ¢ = x : clearly in the graph, as the angle increases, the cosine of that angle decreases, and its sine value increases, but **not** linearly, such as for example that sin¢ + cos ¢ = would always some constant value

Also, TAN ¢ = $\frac{\text{opposite}}{\text{adjacent}} = \frac{y}{x}$: also note that y = (TAN ¢) x , an equation which has the form of: y = mx. We see that the larger the angle drawn, the larger TAN ¢ is, and TAN ¢ is basically a numerical value indicating the slope or steepness of the line. For the equation on the left, note that TAN ¢ is inversely related to x (the side adjacent to the angle) and directly related to y (the side opposite the angle). From the equation y=(TAN ¢) x, you would initially think that (y) is directly related to (x) from general mathematical or algebraic reasoning and from the concepts of linear equations. However, this is not always the case, as it is here, when it comes to the "non-linear" or "non-algebraic" functions such as the trigonometric functions. For example, clearly in the graph, as (x) increases, (y) decreases, and this is an inverse type of relationship. Trigonometric functions are "circular" or "periodic" (values eventually repeat in a continuous cycle) as opposed to being "linear" with values without limited bounds.

By use of the graph, SIN 20° is about 0.34, and COS 20° is about 0.94. SIN 45° is about 0.7 and COS 45° is about 0.7. Only at 45° (half of 90°) are the sine and cosine values equal. The SIN and COS values are identical in a reverse type of manner. While one value is increasing to 1, the other value is decreasing from 1. For example: SIN 0° = 0 and COS 0° = 1, SIN 90° = 1 and COS 90° = 0, and only at 45° are the values the same (at about 0.7) since 45° is halfway between 0° and 90°.

Note that SIN 0° = 0 and SIN 90° = 1, and this is verified below:

SIN 0° = $\frac{\text{opposite}}{\text{hypotenuse}} = \frac{y}{1} = \frac{0}{1} = 0$

SIN 90° = $\frac{\text{opposite}}{\text{hypotenuse}} = \frac{y}{1} = \frac{1}{1} = 1$: at here: opposite side = hypotenuse

Only at 90° is the sine value equal to the maximum possible value of 1. All other sine values will be less than one (1).

Likewise, in an opposite or reverse manner: COS 0° = 1 and COS 90° = 0:

COS 0° = $\frac{\text{adjacent}}{\text{hypotenuse}} = \frac{x}{1} = \frac{1}{1} = 1$: at here, adjacent side = hypotenuse side

COS 90° = $\frac{\text{adjacent}}{\text{hypotenuse}} = \frac{x}{1} = \frac{0}{1} = 0$

Like the SIN values, all COS values of an angle will be equal to or less than one (1). TAN values will rapidly get larger as the angle increases since the ratio of the opposite side (with respect or in reference) to the adjacent side, or (y) to (x), increases rapidly as (x) decreases when the angle increases. This increase in TAN values as the angle increases is practically an exponential relationship, and in fact, the value of TAN 90° approaches infinity and therefore it is considered unspecific or undefinable in value. A good TAN value to memorize is the tangent of 45° which is equal to one (1). Here, the values of the sides opposite and adjacent to the angle are exactly the same length, and therefore, their ratio is 1.

If the above graph is drawn in a similar manner with all four quadrants of a rectangular co-ordinate system, then all angles from 0° to 360° and their corresponding (signed) trigonometric values can be observed and approximated. Each angle can then be shown to have the same (unsigned, signless or absolute) trigonometric values as that of a trigonometrically corresponding angle that is less than or equal to 90° in the upper-right or "first quadrant" of the rectangular coordinate system. To understand this, consider that angles from 0° to 90° will have a specific trigonometric value, for example, such as SIN ¢ having the values of 0 at 0°, and up to 1 at 90°. Due to the cyclic (repetitive) or periodic nature of the trigonometric functions, then in a reverse type of manner as the angles get larger than 90°, the corresponding SIN ¢

values are decreasing from 1 to a value 0 at the angle of 180°. Therefore, an angle in the "second quadrant" having a value between 90° and 180° will have a corresponding angle and trigonometric value of an angle less than or equal to a 90° angle in the first quadrant.

By applying the Pythagorean Theorem to the triangle constructions that can be created in the graph, alternate expressions for SIN ¢, COS ¢, and TAN ¢ can now be derived:

h^2 = x^2 + y^2 or r^2 = x^2 + y^2, when h, the hypotenuse, is considered as the radius (r) of a circle.

Since x = COS ¢ and y = SIN ¢, h = 1, and substituting this into the above expression :

1 = (COS ¢)^2 + (SIN ¢)^2 : You can remember this as the trigonometric "Pythagorean identity".
An alternate syntax or notation form often used is:

1 = COS^2 ¢ + SIN^2 ¢ : For this indicated notation, only the trigonometric value is squared. Also in this special expression, (COS^2) and (SIN^2) are not being multiplied to the angle (¢).

Solving for SIN ¢ and COS ¢ from this expression:

$\text{SIN ¢} = \sqrt{1 - \text{COS^2 ¢}}$ and $\text{COS ¢} = \sqrt{1 - \text{SIN^2 ¢}}$: some TRIGONOMETRIC CONVERSIONS (Pythagorean-like equations)

The TAN ¢ in relation to the corresponding SIN ¢ and COS ¢ values can be found using:

from: $\text{TAN ¢} = \frac{\text{opposite}}{\text{adjacent}} = \frac{y}{x}$, and SIN ¢ = y , and COS ¢ = x

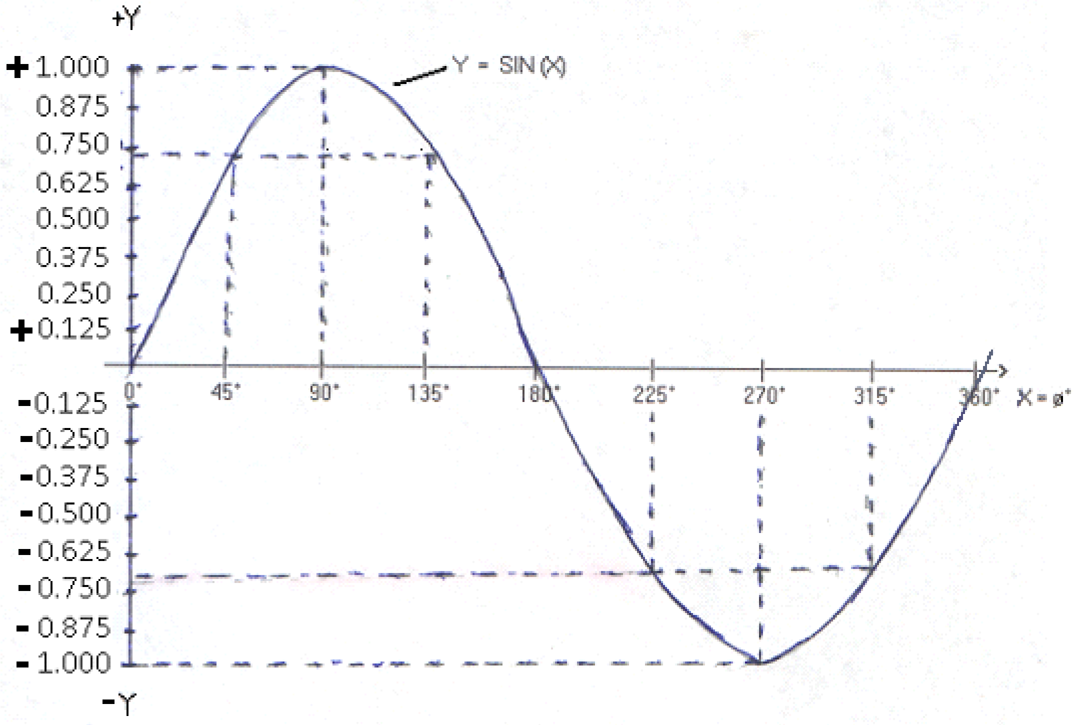
$$\text{TAN ¢} = \frac{\text{SIN ¢}}{\text{COS ¢}}$$

Another method for approximating trigonometric values is by drawing a graph of all the angles from 0° to 360° on the x-axis, and their corresponding trigonometric function values on the y-axis. This can be easily accomplished by selecting several of the angles between and including 0° and 360° and their corresponding trigonometric values, and then connecting the points with a smooth curve to represent and approximate all the other possible point values. The resulting curve is often called a periodic (repeating, repetitive, "oscillating" or cyclic) waveform. Below, the **sine waveform** is shown. Besides trigonometry where its' concepts originated, it is used to (graphically, visually) represent and analyze many other periodic events such as alternating (increasing in value, and direction) current (ie., ac electricity) and sound waves. Since a triangles trigonometric values are usually for angles less than or equal to 90°, and the values are cyclical or repeating for the larger angles, you may wish to only draw a graph with great detail or precision for only the angles between and including 0° and 90°.

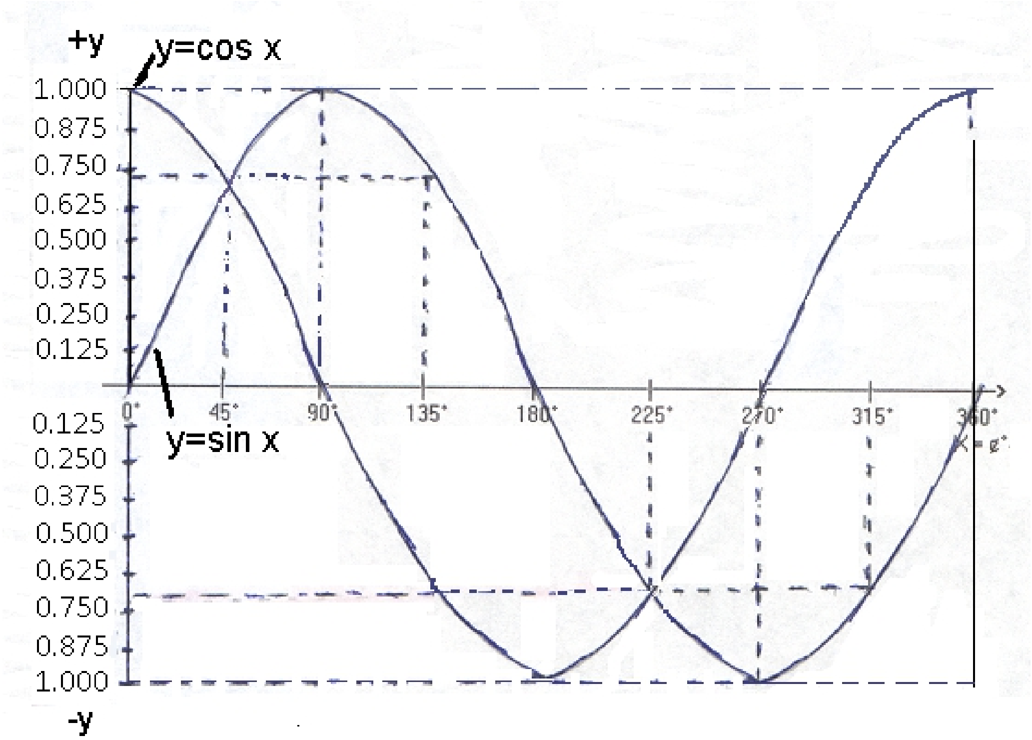
Selecting and plotting a few guiding, significant or "key" points (basically at the 45 degree increment for this specific curve) of this curve and drawing smooth line segments and-or curve segments so as to connect all the points of it, we have:

X = ¢	Y = SIN ¢	
0°	0	
45°	0.707	
90°	1	: the previous values in this set start repeating here. Trigonometric values of an angle (¢°) between 90° and up to 180° are equal to those of an angle from 90° down to 0°, and specifically: (¢° - 90°). For example, the trigonometric values of a 135° angle are similar to an angle of: (135° - 90°) = 45°
135°	0.707	
180°	0	
225°	-0.707	
270°	-1	

315° , -0.707
360° , 0 : The entire set of trigonometric values starts repeating here for angles >= 360°.
For any angle (¢) >= 360, the trigonometric equivalent angle is essentially: (¢-360°)
For example: sin 370° = sin (370° - 360°) = sin 10°.
In more technical terms it is: ¢ MOD 360. Though not needed in most circumstances in trigonometry, MOD is the modulus or "remainder" operator and it is discussed further ahead in this book. The result is that as the angle increases, the output of the MOD operation, the resulting, or "converted" (trigonometric equivalent) angle value will cycle from 0° up to 360° and then "restart" at 0°. In short, the "output" or "result" of a MOD mathematical operation are also cyclical. Most of the angles we will encounter will be less than or equal to 360°. An option to use if the angle is between 360° and 720° is that an ¢° in this range is trigonometrically equivalent to an angle of: (¢° - 360°). [FIG 108]



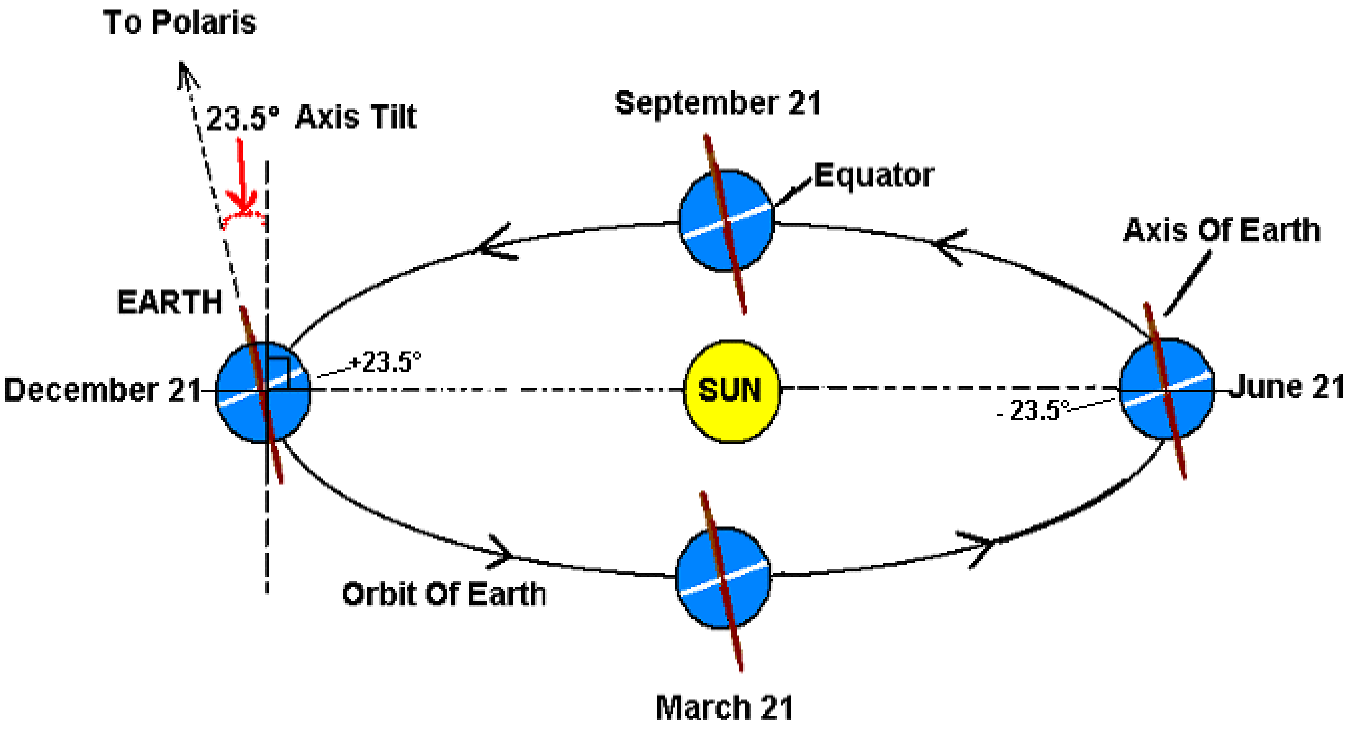
Notice how rapidly the SIN values change during the first 45° (half of 90°), where at 45°, it will have a value of about 0.707, and then the SIN values change much more slowly and only increase about 0.293 over the next 45° leading up to 90° where the SIN value is 1. As a graphical indication to how rapidly the initial SIN values change, notice how steep (steepness, or high slope values of) the SIN curve is near 0° and 180°. Also note at 30° (and 150°) , that the SIN value is already 0.50 which is already at half of the maximum SIN value of 90° which is 1.0, and that 30° is only at one-third of the way to 90°. In short, the relationship between the angles and their corresponding trigonometric values is non-linear (ie., non-proportional, or some other direct mathematical relationship), and non-algebraic (non-exact, no simple algebraic formula, irrational in value). Many algebraic-like methods have been developed that simulate or express their relationship so as to come as close as desired (usually in terms of practicality) to the true value or solution. Some of these will be presented further ahead in this book. Also notice that throughout the full 360° angle range, that the angles trigonometric (absolute) values (except at 0°, 90°, 180° and 360°) are repeated a total of four times during one cycle from 0° to 360°. Starting from the first quadrant (0° to 90°), each same trigonometric value instance corresponds to another quadrant of the rectangular coordinate system. The cosine curve is very similar to the sine curve above except that it effectively begins at the 90° mark of the sine curve and has a starting value of 1 instead of 0. This can also be thought of as like a sine curve being ("phase") shifted (in time and-or values) by 90°, and is often spoken as that the cosine curve (physically, in time) "leads" (ie., begins first) those of the sine curve. The sine curve is then often spoken as "lagging" the cosine curve by 90° in time and-or values. [FIG 109]



The SIN and-or the COS curves are known as periodic curves since the values will essentially keep repeating periodically as the angles grow larger over time. If the angle is greater than 360°, you can keep subtracting 360° from it to find its equivalent angle that is less than or equal to 360°. Since simple harmonic motion (ie., periodic motion or vibrations, that repeat after a certain amount of time, and are therefore said to have a frequency (ie. how often, or "hertz") or amount of cycles per amount of time, such as a second of time) such as the swing of a pendulum, sound and electro-mechanical generated electricity, and radio waves are periodic in nature, the SIN and-or COS curves are used to represent and mathematically analyze the physical creation, nature and motion or movements of this type. Here, the (x) axis becomes time, and the (y) axis becomes the voltage, current, or the height of a pendulum swinging which is its distance from an imaginary horizontal line directly equal to the height of the pendulum when it is not in motion.

As our planet Earth revolves around the Sun during a full year of time (indicated on the x-axis), the (apparent or effective) tilt of Earth (with respect to the solar (Sun) plane or "ecliptic", the direct line or orbital plane between the Earth and Sun) is (nearly) sinusoidal in nature. On about March 21 and September 21, the tilt (noted on the y-axis) of the Earth with respect to the Sun or solar plane is 0°. On about June 21 (toward the Sun) and December 21 (back or away from the Sun), the apparent or effective tilt is at it's maximum value of +,- 23.5°. If this was to be graphed, the x-axis would be the time (day, week or, month, etc.) of the year, and the y-axis would be the degrees of (apparent) tilt with respect to the Sun. Note that the rate of the tilt of the Earth is not consistent (a constant value) as indicated by the changing steepness or slope of the SIN curve or graph, and for the Earth to (apparently) tilt back in the other direction, it must first come to a brief stop after increasingly slowing down. This tilt was noted here as being apparent because Earth's axis, that the Earth rotates about in 1 day, always "points" (a direct line) to about (ie., very close to) the "North Star" (Polaris), and therefore the tilt of the Earth, or its axis, is only an apparent tilt, but still an "effective tilt" in terms of seasons due to Earths orbit (path of travel) around the Sun. More can be found about the tilt of the Earth in the topic of: An Equation For The Current Apparent Tilt Of The Earth. Since Polaris is many light-years away distant, then as Earths orbits about the Sun, and at opposite sides of this orbit which is equivalent to diameter of Earths orbit = 2 (radius of orbit) = 2 (93M miles) = 186M miles, the angular difference to Polaris is negligible, and approximately 0°.

[FIG 110]



Earth's equatorial (equator, shown as a white line in the above image) plane is constantly tilted about 23.5 degrees with respect to the (ecliptic or solar) plane of Earth's orbit about the Sun. On about Dec. 21, the Sun is directly overhead at noon at the tropic of Capricorn latitude (23.44°S) , and directly overhead at noon at the Tropic Of Cancer (23.44°N) latitude on about Jun 21 of the year. This can bee seen in the above image. It will take 6 months = 180 days of time to be overhead at these two maximum or extreme latitude locations mentioned, and overhead at noon at a slightly different latitude location for each different day that is between those two dates of time. The Earth's does not actually tilt, but it only appears (ie., the Sun and star angles) and feels (weather wise) as so, and this is due to the orbit of the (tilted) Earth. This effect also give the Earth its (real) seasons in the Northern and Southern hemispheres (half-sphere) of the Earth. When it is winter in the northern hemisphere, it is summer in the southern hemisphere, and vice-versa.

The above figure is an approximation showing linear, (24 hour) clock and-or calendar time imposed on an elliptical orbit with the Sun at the center - much like a circular orbit conceived by the yearly calendar time. Even though the orbit of the Earth is nearly circular about the Sun, it is infect an elliptical orbit with the Sun not in the exact center of the orbit, and which the velocity of the Earth increases as it gets closer to the Sun in the winter season of the northern hemisphere. It could be said that for half a year, the orbit is faster, and for the other half, it is slower, and the tilt angle of the Earth is not linear in nature and is constantly changing at a different rate. Nonetheless, it can be averaged like that of a circular orbit approximation as shown in this book further ahead, and so as to express the basic concept of Earths tilt angle with respect to the Sun. As mentioned, the Earth does not tilt with respect to the star Polaris, and is rather an apparent tilt due to the orbit of (tilted) Earth about the Sun.

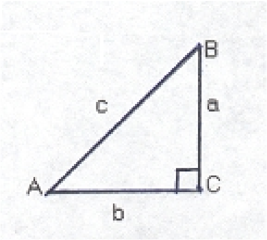
Etra: A Latin word for "Sun" that is still usd is "Sol". Greeks use "Helios" for "Sun". It is also used as part of the words "Solar" and "Solstice". Greeks use "Helios" for "Sun". A Latin word for "Earth" is "Terra". It is also used as part of the words "Terrain" and "Terrestrial". A Latin word for the "Moon" is "Luna". It is also usd as part of the word "Lunar".

COMPLEMENTARY ANGLES OF A RIGHT TRIANGLE

Since the right angle (C) of a right triangle is 90°, the other two angles of a right-triangle will always total to 90° since a triangles interior sum of angles is always a total of 180°. These other two angles that sum to 90° are called complementary angles, because they essentially complement each other so as their combined sum is equal to a (complete) 90° angle. Given two complementary angles, if one increases by N degrees, the other will decrease by N degrees so as to maintain their 90 degree sum.

$A + B + C = 180°$ Here, C=90° and is the right angle of a right triangle:
$A + B + 90° = 180°$ after transposing 90°:
$A + B = 90°$ **A and B, each less than 90°, are called the complementary angles of a right triangle and-or a right angle which is a 90° angle.**

[FIG 111]



For the figure above, Angles A and B, the non-right angles of a right triangle, are said to be complementary angles since the value of either angle always complements the other angle in order to create the required sum of 90° so as the internal angle sum of a triangle is 180°.

The trigonometric co-functions of complementary angles are equal. This is described below in mathematical terms:

SIN A = COS B and SIN B = COS A

$\frac{a}{c} = \frac{a}{c}$ $\frac{b}{c} = \frac{b}{c}$

If you had only a sine table, you would then also know the cosine of its complementary angle and vice-versa.

Ex. In a right triangle, if A = 50° , then the complementary angle, angle B = (90°- A) = (90°- 50°) = 40°

SIN A = COS B
SIN 50° = COS 40°
0.766 = 0.766

SIN B = COS A
SIN 40° = COS (90° - 40°) = COS 50°
$0.64\underline{3} = 0.64\underline{3}$

TAN A = COTAN B and TAN B = COTAN A

$\frac{a}{b} = \frac{a}{b}$ $\frac{b}{a} = \frac{b}{a}$

This equation above can also be expressed as:

$\frac{a}{b} = \frac{1}{\frac{b}{a}}$ and this can be expressed as:

$\text{TAN A} = \frac{1}{\text{TAN B}}$: Since the reciprocal of an angles TAN value is its COTAN value. Here, TAN B = 1/COTAN B, therefore mathematically, 1/TAN B = COTAN B

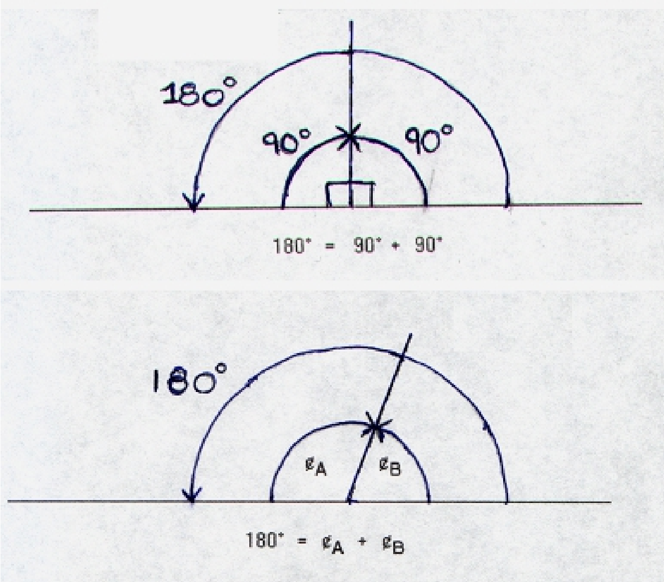
, and this can be used for solving for a part of a right triangle, or two angles that are complementary (that sum to 90°).

From the above equation, we mathematically have:

$(\text{TAN A})(\text{TAN B}) = 1$: the product of the TAN values of two complementary angles is always 1, and this is due to that they are reciprocals of each other

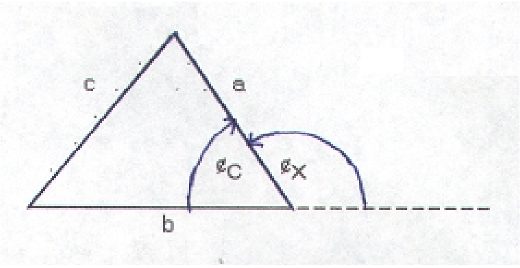
SUPPLEMENTARY ANGLES

Another important sum of angles not necessarily associated with triangles, but still within the practical realm of trigonometry, is the concept of supplementary angles. Any two angles that sum to 180° are said to supplement, or be supplementary to each other. An angle of 180° is very obvious since it forms a straight line segment. [FIG 112]



By knowing the concept of supplementary angles, it is easy to calculate angles such as the exterior angles to any triangle.

Ex. Below is a triangle with side (b) extended. Find angle X if angle C is 60° [FIG 113]



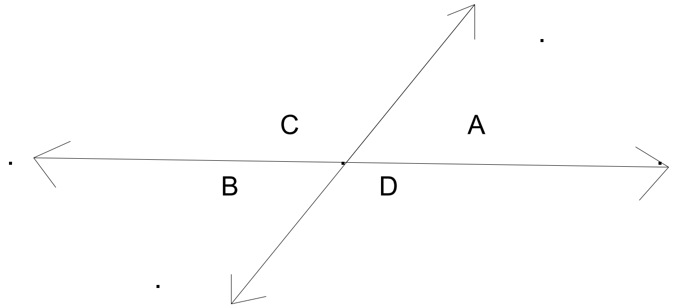
180° = ¢C + ¢X : by observation, ¢C and ¢X are supplementary angles that sum to 180°.
180° = 60° + ¢X Solving for ¢X by subtracting (or transposing) ¢C = 60° (the supplementary angle of ¢X) from each side of the equation, and then combining like terms:

¢X = 120° : also of note is that this value is also equal to the sum of the other two interior angles, here ¢A and ¢B, since all three interior angles of a triangle always sum to 180°.

The sum of all the exterior angles of a triangle is always 900° which is 5 times more than 180°.

VERTICAL ANGLES

Though not specifically related to triangles, the concept of vertical angles can play an important role in the further study of trigonometry, constructions and other fields of study. Vertical angles were previously briefly mentioned in this section. Whenever two lines cross each other, 4 angles are produced, all of which will have their vertex at the point of intersection of those two lines. 2 pairs of vertical angles are produced, and the angle for each corresponding pair will be equivalent. The equivalent pairs are directly "opposite", "across" or on the other side of the vertex point. In the drawing below, A and B are corresponding vertical angles, and C and D are corresponding vertical angles. [FIG 114]



What is the proof that vertical angles are the same (besides the somewhat obvious visual graphical proof)? In a worded form, for example, any and all supplementary angles to an angle in question are equivalent. If angle C is the supplementary angle of angle A, and angle D is also a supplementary angle (though along a different line that effectively creates another 180° straight-angle) to angle A, it is only logical to equate angles C and D. Here's an algebraic proof:

Due to supplementary angles:

$A + C = 180°$ and:
$A + D = 180°$ therefore, equating the two equations since they are both equal to 180° :

$A + C = A + D$ solving for C, transposing A:
$A + C - A = A + D - A$ combining:
$C = D$: C and D are vertical angles having the same value

Likewise:

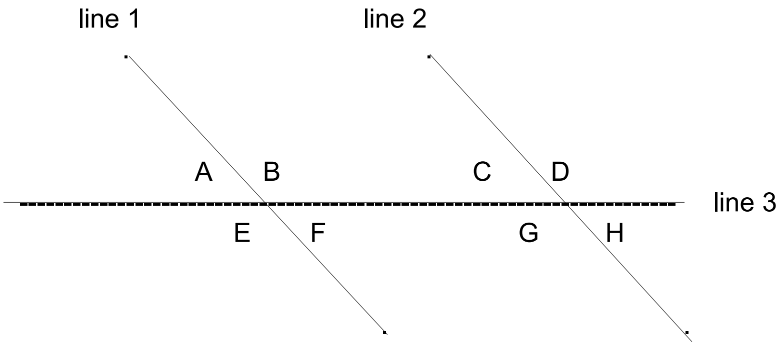
$A + D = 180° = D + B$

$A + D = D + B$
$A + D - D = D + B - D$

A = B : A and B are vertical angles having the same value

CORRESPONDING ANGLES CREATED BY A TRANSVERSAL LINE

A line that crosses two parallel lines is called a transversal line since it transverses (ie. crosses) between those two parallel lines. Below, line 1 and line 2 are parallel, and line 3 is the transversal line. [FIG 115]



At the intersection of each line it is obvious that pairs of vertical angles and supplementary angles are created. The corresponding angles at each intersection of the transversal line are also equivalent. That is:

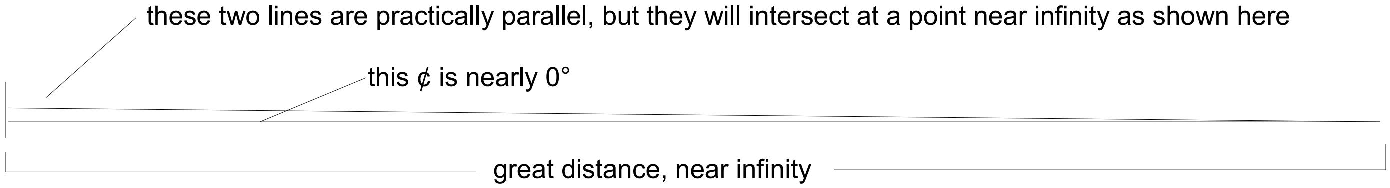
A = C and due to vertical angles:
A = F = C = H

B = D and due to vertical angles:
B = E = D = G

Perhaps it is not intuitive that corresponding angles created by a transversal line are equal. For example, what is a verification that angle A equals angle C?

One visual or graphical approach to answering this is to imagine the parallel lines as being so close that they practically overlap and yet are still distinct. Clearly, the angles A and C would clearly appear and be the same size. Consider that since the lines are parallel, they are exact images of each other, and all the angles created at each due to a transversal line will correspond and be equal.

Line 1 and line 2 extend to infinity without ever intersecting since they are parallel. Let's suppose that they meet at a point very far away or as "close to infinity" as can be. This can be accomplished when the lines are not quite parallel by only the very smallest amount possible, practically 0 degrees, a very minute value, but not equal to 0 degrees: [FIG 116]



The theoretical angle (¢) measured at this (imaginary) point of intersection is very close to a value to 0°. If we consider this (almost imperceptible) angle as the smallest of all angles possible, and angles B and C as the other two angles within this triangle construction, we get:

$180° = B + C + ¢$:180° = interior sum of angles of a triangle

Since the angle (¢), as mentioned above is nearly 0°, for all practical purposes, it can then be considered as 0°. Hence we get:

$180° = B + C + 0°$
$180° = B + C$ equating this to the supplementary angles of A and B, and C and D:

$180° = A + B = B + C = C + D$ therefore:

$A + B = B + C$ solving for A:
$A + B - B = B + C - B$
$A = C$

$B + C = C + D$ solving for B:
$B + C - C = C + D - C$
$B = D$

Also, though not quite obvious, angles B and C are supplementary angles due to that their sum is 180° as indicated above, and here is another verification that B and C sum to 180°:

Since: $(A + B) = 180°$ and $(C + D) = 180°$ adding both equations, or all the angles:
$(A + B) + (C + D) = 180° + 180°$
$A + B + C + D = 360°$

Since lines 1 and 2 are parallel, no matter how close (even if "on top" [coinciding] of each other) or far apart they are, the angles created by a transversal line at the intersection of any one parallel line will always be the same as those created at the intersection of another parallel line by that transversal line or another transversal line parallel to it. Since lines 1 and 2 are parallel or correspond to each other, the angles at each intersection of the transversal line will correspond and be the same value.

Since C corresponds to A: $C = A$
Since B corresponds to D: $B = D$

$A + B + C + D = 360°$ using algebraic substitution:
$C + B + C + B = 360°$ combining like terms:
$2B + 2C = 360°$ factoring 2 from each term on the left side:
$2(B + C) = 360°$ after dividing each side by 2:
$B + C = 180°$: verification

Perhaps a simpler way of expressing the above method and result is:

$A + B = 180°$ and since C corresponds to A, and as the same value:
$C + B = 180°$: sum of interior (between the parallel lines) angles on the same side of a transversal line is 180°, and this is equivalent to a "straight (line)", 180° angle.

Here is another derivation that the interior (between the parallel lines) angles on the same side of a transversal line will sum to 180°. Please refer to Fig 118.

Due to the concepts of vertical and corresponding angles created by a transversal line across two or more parallel lines we have:

A = C = F = H Since each is equal to angle C, and summing these angles, we have a sum of:
4C Likewise:

B = D = E = G Since each is equal to B, their sum is: 1B + 1B + 1B + 1B or:
4B

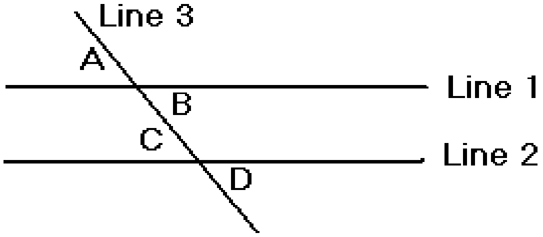
The total sum of angles around a point is 360°: A+B+E+F = 360° and C+D+G+H = 360°

sum of angles around a point+sum of angles around a point = sum of angles around a point+sum of angles around a point
sum of all angles around two points = sum of all angles around two points
A+B+E+F + C+D+G+H = A+C+F+H + B+D+E+H
A+B+E+F + C+D+G+H = 4C + 4B
360° + 360° = 720° hence:

4C + 4B = 720° factoring out 4 from each term:
4(C + B) = 720° after dividing both sides by 4:

B + C = 180° : verification that interior angles created on one side of a transversal line sum to 180°

Here is a verification that alternate (on opposite sides of the transversal line) interior (between the parallel lines) angles are equal. [FIG 117]

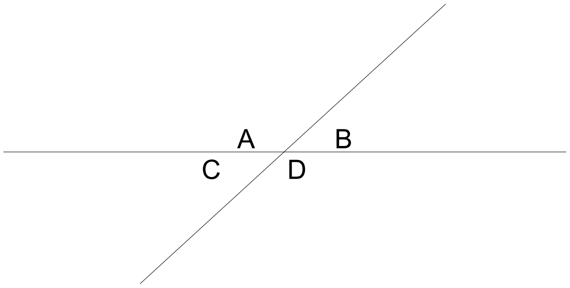


Line 1 and Line 2 are parallel lines. Line 3 is a transversal line that crosses both of the parallel lines. B and C are **alternate interior angles**. Due to vertical angles being equal:

A=B and C=D, Since A and C, and B and D are corresponding and equivalent angles, we have:
A = B = C = D hence:
B=C : a verification of alternate interior angles being equivalent
A=D : a verification of alternate exterior angles being equivalent

Angles A and D are examples of equivalent alternate exterior angles. A = D. Since vertical angles are equal:
A = B = D = C : all the angles shown in the figure above have the same value.

Now that transversal lines and the corresponding angles created have been discussed, a verification that the sum of all three interior angles of any triangle is always 180° can be described. First, observe the angles created by the intersection of two lines: [FIG 118]

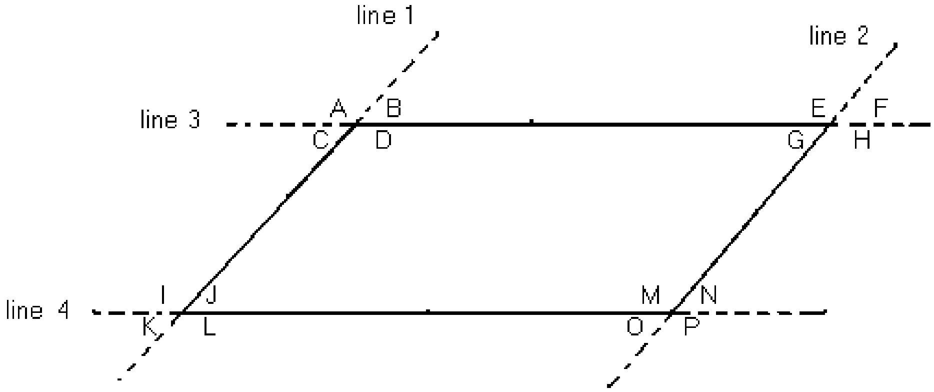


Since A and B are supplementary angles, and C and D are supplementary angles, the sum of angles around any point (as with the center of a circle) is:

$$180° + 180° = 360°$$

Verification Of The Sum Of A Triangles Inner Angles

It will now be shown that the sum of the interior angles of four sided (of any lengths) figures called quadrilaterals (any four sided portion of a plane; quad meaning four, and lateral meaning side) is always 360°. Squares and rectangles are common examples of quadrilaterals. A drawing of an quadrilateral called a parallelogram is shown below with its sides extended for the analysis. Note that a rectangle is a special instance of all possible parallelograms, and a square is a special instance of all possible rectangles. [FIG 119]



We see that a parallelogram can be drawn by intercepting two parallel lines by two parallel transversal lines.

Since there are 4 vertices or "corners" of the quadrilateral, and we know that the sum of the angles surrounding each corner or point is 360°, the total sum of all the angles indicated is:

$(360°)(4) = 1,440°$: internal and external sum of angles of a quadrilateral

We are to verify that the sum of interior angles of D, G, J, and M is (always) equal to 360°. By construction, line 1 and line 2 are parallel. Line 3 and line 4 are both parallel transversal lines to line 1 and line 2. From the concepts of vertical angles and corresponding angles, we can show some relationships of all the angles above (that total to 1,440°):

$A = D = E = H$: vertical angles, and since they all equal the same value of D, they sum to: 4D
$B = C = F = G$: vertical angles, and since they all equal the same value of G, they sum to: 4G
$N = O = J = K$: vertical angles, and since they all equal the same value of J, they sum to: 4J
$I = L = M = P$: vertical angles, and since they all equal the same value of M, they sum to: 4M

hence: $4D + 4G + 4J + 4M = 360°(4) = 1,440°$ factoring 4 from each term on the left side:

$4(D + G + J + M) = 360°(4) = 1{,}440°$ after dividing each side by 4:
$D + G + J + M = 360°$: verification of the inner sum of angles of a parallelogram

As mentioned previously, the internal or interior angles created at intersection of two parallel lines by a transversal line sum to 180° on one side of that transversal line:

Since: $D + G = 180°$: note that G = C, and D + C = 180°, and:
$J + M = 180°$ summing both sides of the equations:

$(D + G) + (J + M) = 180° + 180° = 360°$
$D + G + J + M = 360°$: verification

Continuing with the concepts of supplementary angles, it can be shown that the diagonal angles of a parallelogram are equivalent:

In the drawing above, if G is a supplementary (sum to 180°) angle to angle D, and if J is also a supplementary angle to angle D, then G=J. These angles are diagonal to each other. Likewise D=M. Considering any pair of the parallel lines, the internal angles created at their intersection of a line will sum to 180°. For two pair, the total is then 180° + 180° = 360°

$D + G = 180°$
$D + J = 180°$

$G = 180° - D$
$J = 180° - D$ Since both G and J are equal to the same value, they are equivalent to each other:

$G = J$: diagonal angles in a parallelogram are equal
$D = M$

If you were to divide the parallelogram in half, either vertically or horizontally, each half will contain half of the 360° internal sum or 360°/2 = 180°. This also shows that the internal angles created on one side of a (transversal) line that intersects two parallel lines have a sum of 180°.

As another simple verification::

Since A=D and B=C:

And $360° = A + B + C + D$ by substitution of equivalent angles:
$360° = D + B + B + D$ combining like terms:
$360° = 2D + 2B$ factoring 2 from each term:
$360° = 2(D + B)$ dividing each side by 2 and switching sides:
$(D + B) = 180°$ Since B=C=F=G:
$(D + G) = 180°$

Likewise:

$360° = 2J + 2L$:J=K, and I=L due to vertical angles
$360° = 2(J+L)$
$(J + L) = 180°$ Since I=L=M=P
$(J + M) = 180°$

$(D + G) + (J + M)$ removing grouping symbols:
$D + G + J + M$:sum of the interior angles of the parallelogram

180° + 180°
360° : checks

Here's another simple verification that uses all the angles on the "outside" (external) of the parallelogram to find those that are "inside" (internal) the parallelogram:

(A+B) = 180° , (E+F)=180, (K+L)=180° , (O+P)=180°, (I+C)=180° , (N+H)=180°

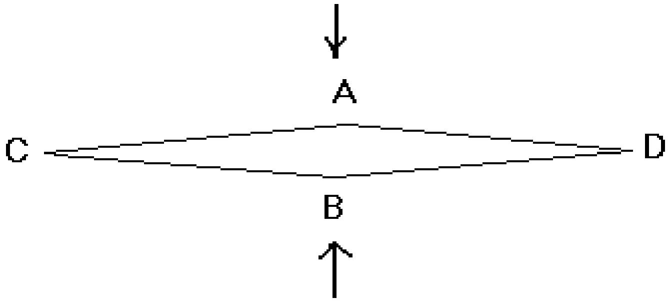
The sum of the above pairs is: 6(180°) = 1080°

Subtracting this specific external angle sum from the total possible vertice or "corner" angle sum of (360° x 4) = 1440° we have the sum of just the parallelograms internal angles:

1440° - 1080° = 360° : checks

Perhaps the simplest verification that the interior angles of a parallelogram sum to 360°:

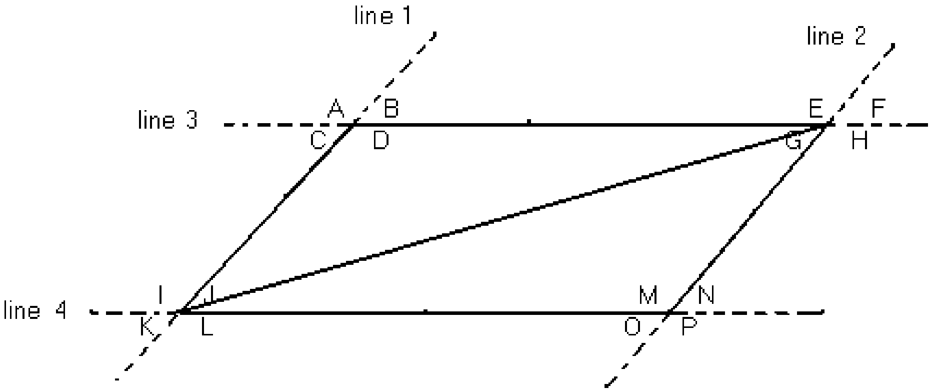
Here is a simple verification that the sum of the interior angles of a parallelogram is 360°. [FIG 120]



Consider A and B being pushed closer together as illustrated by the directional arrows. Those angles, A and B, will get larger and each will approach a value of a 180° "straight (line)" angle. At the same time, this will cause both angles of C and D to approach a value of 0 °. The net sum of internal angles of this parallelogram will be practically:

A + B + C + D
180° + 180° + 0° + 0° = 360° , or something as: 359.99999.... hence, approaching ----> 360°

When a (diagonal) line is drawn from two diagonal (ie. not on the same line) or "opposite" corners of a parallelogram, it will divide it into two congruent (identical) triangles. If the diagonal is extended for the analysis, and with the concept of supplementary angles, it is easy to see that the total vertice (ie. peak, corner, where two sides meet) sum of 1440° will be divided into two 720° (half of 1440°) sums. It is also easy to see that this line divides the interior sum of 360° into two 180° sums. Each 180° sum corresponds to the interior angles of a general or representative triangle.
[FIG 121]



Since the triangles are identical, their angles are identical, and therefore the sum of their angles must be the same:

(sum of angles of triangle 1) + (sum of angles of triangle 2) = 360°

Since the sum of angles of triangle 2 is the same as the sum of angles of triangle 1, using substitution:

(sum of angles of triangle 1) + (sum of angles of triangle 1) = 360°
2 (sum of angles of triangle 1) = 360° after dividing each side by 2:
sum of angles of triangle 1 = 180° : verification, and:
sum of angles of triangle 2 = 360° - sum of angles of triangle 1 = 360° - 180° = 180°

Since any quadrilateral can be divided into two (and not necessarily identical) triangles, the interior angle sum of any quadrilateral is twice that of a triangle or 2(180°) = 360°.

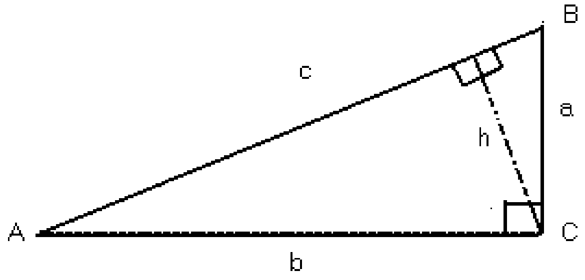
An interesting fact: Given the 3 vertex points of any triangle, the sum of angles around them is (3 x 360°) = 1080°. If you subtract the triangles 180° internal angular sum, we will have a triangles external angular sum of: (1080° - 180°) = 900°.

Given a corner (intersection of two straight line sides) of any construction such as a square or triangle, the external angle is 360° less the give angle of that corner: (360° - ¢°) = external or "opposite side" angle.

THE HEIGHT WITHIN A RIGHT TRIANGLE

Given any side of a triangle and considering it a base side, the perpendicular distance to the corresponding internal angle which is directly opposite from it is called the height or "altitude" at that base side or vertex. Sometimes the base side of a triangle needs to be considered as extended so as to create the perpendicular height line.

The two sides, other than the hypotenuse side, of a right triangle are equal to the height (or "altitude") from a vertex to the corresponding base side it intercepts perpendicularly (at 90° to that side). The third height (shown as (h) in the figure below) within a right triangle is the distance from the vertex of the 90° angle to the hypotenuse side it intercepts perpendicularly. [FIG 122]



As seen in the drawing of a right-triangle above, the height at (base) side (b) to the vertex B is equal to side (a). The height at side (a) is equal to side (b). A height line always intersects its corresponding base (of the height) line perpendicularly = (90°) . In the right triangle above: height at base side b = hb = a , and height at base side a = ha = b.

This height line to the hypotenuse (as a base) side will divide the right triangle into two smaller and similar triangles to the original outer triangle. This can easily be verified from the fact that each contains a right angle and that the complementary sum of the other two angles within each similar triangle is always 90°. In the drawing above, the pair of right triangles (the larger "outer" triangle, and the smallest interior triangle) have the same angle B. The other pair (the larger "outer" triangle, and the larger interior triangle) have the same angle A. Still, each of these similar triangles will and must have (so as to be similar) angles equivalent to A, B and C. Since we know that we have similar triangles, the simplest method to solve for the length of this height line (at base side c) is by using the concept of proportions (equivalent portions) of the sides of similar triangles sides, for example:

$$\frac{\text{hypotenuse}}{\text{adjacent}} = \frac{c}{b} = \frac{a}{h}$$ solving for h:

$$h = \frac{ab}{c}$$: INTERIOR HEIGHT FORMULA FOR A RIGHT TRIANGLE
: c = hypotenuse side

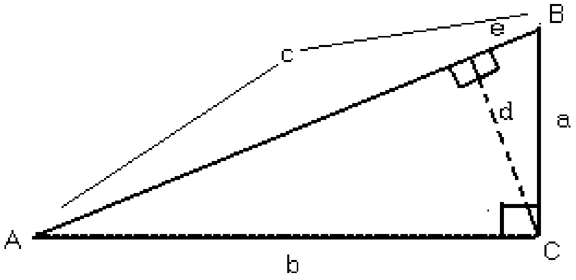
Ex. In a right triangle, a=3, b=4, c=5, find the altitude (height) from the right angle to the hypotenuse side.

$$h = \frac{(3)(4)}{5} = \frac{12}{5} = 2.4$$:height or altitude at side c

From the equation, also notice the possible methods to solve for a side if it is not known. For example, here is a simple formula for the hypotenuse (side c) if the height at side (c) is known:

$$c = \frac{ab}{h}$$: side (c) is the "base side" to altitude (line) of (h)

Using the concepts we now know, another derivation of the Pythagorean Theorem can be shown. Observe the following drawing of the right triangle with sides a, b, and c: [FIG 123]



Since the ratio of two sides of a triangle will equal the corresponding ratio of a similar triangle:

$\frac{a}{c} = \frac{e}{a}$ and $\frac{b}{c} = \frac{(c - e)}{b}$ simplifying:

$a^2 = ce$ and $b^2 = c(c - e) = c^2 - ce$

Using the two equations above, by adding both left-hand sides together, and adding both right-hand sides together, we have:

$a^2 + b^2 = ce + c^2 - ce$ combining like terms and switching sides:

$c^2 = a^2 + b^2$: PYTHAGOREAN THEOREM

Here is another way to solve for a part of a right triangle without needing to take a square root:

Considering $c = r =$ radius $= 1$, $\sin ¢ = \text{opp} / \text{hyp} = y / 1 = y$ and $\cos ¢ = \text{adj} / \text{hyp} = x / 1 = x$

$c^2 = 1 = x^2 + y^2 = \sin^2 ¢ + \cos^2 ¢$

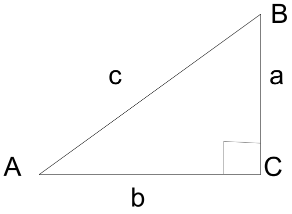
$\sin ¢ + \cos ¢ = x + y$

$\frac{c}{x + y} = \frac{1}{\sin ¢ + \cos ¢}$

$c = \frac{x + y}{\sin ¢ + \cos ¢}$ or $= \frac{a + b}{\sin ¢ + \cos ¢}$: ¢ = A or B in a right triangle.
Extra: $\sin A + \cos A = \cos B + \sin B = \sin B + \cos B$

LAW OF SINES

By using algebraic manipulation on the trigonometric ratio formulas for the angles, we can derive some other useful formulas. [FIG 124]



From: $\text{SIN A} = \frac{a}{c}$ and $\text{SIN B} = \frac{b}{c}$ therefore, mathematically:

$$c = \frac{a}{\text{SIN A}} \quad \text{and} \quad c = \frac{b}{\text{SIN B}}$$

, therefore, this is another method of calculating the hypotenuse without using the Pythagorean Theorem. The hypotenuse of a right triangle is equal to the ratio of any other side divided by the sine of its corresponding angle. For further practical use, the hypotenuse is also the diagonal line within a rectangle since a rectangle can be divided into two right triangles. In fact, the word hypotenuse seems to have originated first with rectangles since the root meaning of the word "hypotenuse" is to stretch or extend as we see with a diagonal line going (stretching or extending) from corner to opposite corner within a rectangle. Also notice that:

$$c = c$$ substituting the equivalent values for c:

$$\frac{a}{\text{SIN A}} = \frac{b}{\text{SIN B}}$$: alternate Hypotenuse Formulas

Since each fraction represents c, and SIN C = SIN 90° = 1, we can include side (c) into the equivalence above. This equivalence is known as the "law of sines".

$$\frac{a}{\text{SIN A}} = \frac{b}{\text{SIN B}} = \frac{c}{\text{SIN C}}$$: LAW OF SINES (here, verified for right triangles only)

From this we also have these equivalent fractions:

$$\frac{a}{b} = \frac{\text{SIN A}}{\text{SIN B}} = \text{TAN A}$$: Ratio Of Sines (derived for right triangles)

$$\frac{b}{a} = \frac{\text{SIN B}}{\text{SIN A}} = \text{TAN B}$$

Using the law of sines, if you have enough information to place into any two of the three equivalences shown, you can algebraically solve for a side or a trigonometric ratio value.

Ex. $a = \frac{b\ \text{SIN A}}{\text{SIN B}}$ also, $= a = \frac{b\ (\text{SIN A})}{(\text{COS A})} = b\ \text{TAN A}$:as expected since TAN A = a/b

This can also be expressed from what has been mentioned:

$\text{SIN A} = (\text{COS A})(\text{TAN A})$ likewise:

$\text{SIN B} = (\text{COS B})(\text{TAN B})$

Ex. $\text{SIN A} = \frac{a \text{ SIN B}}{b} = (\text{TAN A})(\text{SIN B})$: hence TAN A = SIN A / SIN B, and SIN B = SIN A / TAN A also:

Extra: Given a right triangle with C=90°:

$\text{TAN A} = \frac{a}{b}$ and $\text{TAN B} = \frac{b}{a}$ after dividing these two equations:

$$\frac{\text{TAN A}}{\text{TAN B}} = \frac{a^2}{b^2} = \frac{\text{SIN}^2 \text{ A}}{\text{SIN}^2 \text{ B}}$$: for right triangles only, where C=90°

Given a right triangle with C=90°:

$\text{COS A} = \frac{b}{c}$ and $\text{COS B} = \frac{a}{c}$ after dividing these two equations:

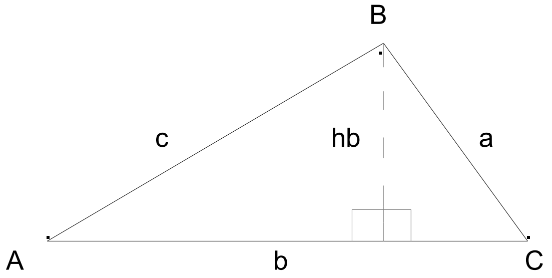
$$\frac{\text{COS A}}{\text{COS B}} = \frac{b}{a}$$: for right triangles only, where C=90° , note also: a COS A = b COS B

Inverting the above fraction, and squaring it:

$$\frac{\text{COS}^2 \text{ B}}{\text{COS}^2 \text{ A}} = \frac{a^2}{b^2} = \frac{\text{SIN}^2 \text{ A}}{\text{SIN}^2 \text{ B}} = \frac{\text{TAN A}}{\text{TAN B}}$$: for right triangles only, where C=90° taking the square root of each fraction:

$$\frac{a}{b} = \frac{\text{SIN A}}{\text{SIN B}} = \frac{\text{COS B}}{\text{COS A}} = \frac{\sqrt{\text{TAN A}}}{\sqrt{\text{TAN B}}} = \sqrt{\frac{\text{TAN A}}{\text{TAN B}}}$$: for right triangles only, where C=90°

All triangles can't be right-triangles and where two of the sides or "legs" are actually heights with respect to each other, so a more general verification of the law of sines is needed, and this is presented next. Observe the following drawing of a general (representative, and non-right) triangle: [FIG 125]



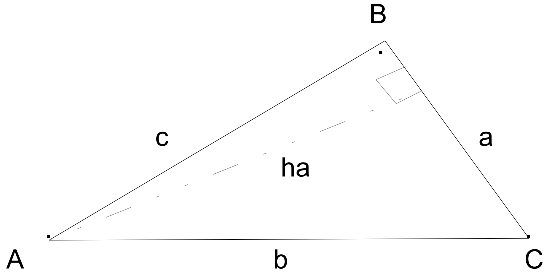
From: $\text{SIN A} = \frac{hb}{c}$, we get: $hb = c \text{ SIN A}$: hb = height at side b
For a right-triangle, hb is actually side a

From: $\text{SIN C} = \frac{hb}{a}$, we get: $hb = a \text{ SIN C}$ Equating the two expressions for hb:

$c\ SIN\ A = a\ SIN\ C$ algebraically:

$\frac{a}{SIN\ A} = \frac{c}{SIN\ C}$: for all triangles

Now observe the following drawing of the same triangle: [FIG 126]



From: $SIN\ B = \frac{ha}{c}$, we get: $ha = c\ SIN\ B$

From: $SIN\ C = \frac{ha}{b}$, we get: $ha = b\ SIN\ C$

Equating the two expressions for ha:

$c\ SIN\ B = b\ SIN\ C$ algebraically:

$\frac{c}{SIN\ C} = \frac{b}{SIN\ B}$: for all triangles

Equating this to the results of the previous drawing and equations (a use of what is known as the Transitive Law where if a=b, and b=c, then a=c. In short, a = b = c):

$\frac{a}{SIN\ A} = \frac{b}{SIN\ B} = \frac{c}{SIN\ C}$ **: LAW OF SINES (for all triangles)**

Some extra derivations from the above:

$\frac{side1}{side2} = \frac{SIN\ (corresponding\ angle\ to\ side1)}{SIN\ (corresponding\ angle\ to\ side2)}$, for example: $\frac{a}{b} = \frac{SIN\ A}{SIN\ B}$

Since SIN 90° = 1, for a right triangle:

$\frac{a}{SIN\ A} = \frac{c}{SIN\ 90°} = \frac{c}{1} = c$: where C is 90°, essentially, this is equivalent to SIN A = a/c
: for a right triangle only

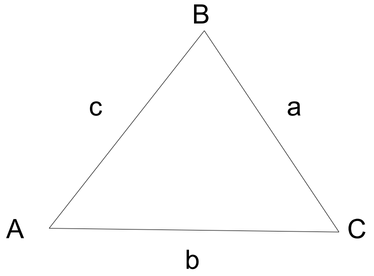
Likewise:

$\frac{b}{SIN\ B} = c$: where C is 90° , for a right triangle only

COMMON TYPES OF TRIANGLES

The most common type of triangle is the one that we have been studying so far, and that triangle is the **right-triangle**, however, there are other basic types or categories of triangles.

An equilateral (equal sided) triangle has all three of its sides equal in length, and all three interior angles have the same value of: (180°/3) = 60°: [FIG 127]



: EQUILATERAL TRIANGLE

In the above drawing of an equilateral triangle: $a = b = c$ and $A = B = C = 60°$

An **equilateral triangle** is always an acute (has no angle greater than 90°) triangle. If you were to divide an equilateral triangle along its' height (at exactly half of the base side) you can easily create two "30°, 60°, 90°" triangles.

You can divide an equilateral triangle into two right triangles to find a formula for the area of any equilateral triangle given just any one of its sides. In the drawing above, draw a perpendicular line from vertex B to half of side b. We will call this the height = h at side b. If we let s = side length, then from the Pythagorean theorem we have:

$s^2 = h^2 + (s/2)^2$: s = the lateral or side length, here, $s = a = b = c$, and $(s/2)$ is half that side length
$h^2 = s^2 - (s/2)^2$
$h^2 = s^2 - (s^2)/4$ combining fractions (LCD = 4), we have:

$$h^2 = \frac{4s^2 - 1s^2}{4}$$ combining like terms in the numerator:

$$h^2 = \frac{3s^2}{4}$$ taking the square root of both sides to isolate $h^1 = h$:

$$h = \sqrt{\frac{3s^2}{4}} = \sqrt{\frac{3}{4}}\ \sqrt{s^2} = \frac{\sqrt{3}}{2}\ s = 0.866025403\ s$$

: HEIGHT OF AN EQUILATERAL TRIANGLE.
The coefficient of (s) is also equal to SIN 60°
$\sin A = h/s$, $h = (\sin A)\ s$

$$A = \frac{b\,h}{2} = \frac{s\ \frac{\sqrt{3}\ s}{2}}{\frac{2}{1}} = \frac{s^2\ \sqrt{3}}{4}$$

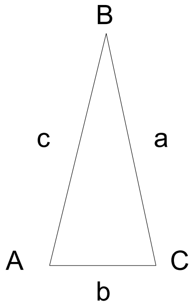
: algebraic formula for AREA OF AN EQUILATERAL TRIANGLE
It is based on, or directly related to, only one variable: s = side
Since $\sqrt{3}\ /4$ is a constant value:

$$A = 0.433012702\ s^2$$

: AREA OF AN EQUILATERAL TRIANGLE, (practical form)
The coefficient of (s) is equal to (sin 60°)/2

By observing the equation of the area of an isosceles triangle, it then becomes clear that its area is only a certain fraction of that of a square having the same side length. The area of a square is s^2. Also note that the height of the isosceles triangle does not equal its side length (s) or that of a square with side (s). Looking at the above equation for the height of an isosceles triangle, we see that it is less than (s). Also, by the Pythagorean Theorem, if (s) is considered a hypotenuse and the longest side of a triangle, the height must be less.

An **isosceles triangle** has only two of its sides equal in length. The third side of different length is often called the "base" (side), and the two equivalent angles created at the intersection of sides that base are called the base-angles of the isosceles triangle: [FIG 128]

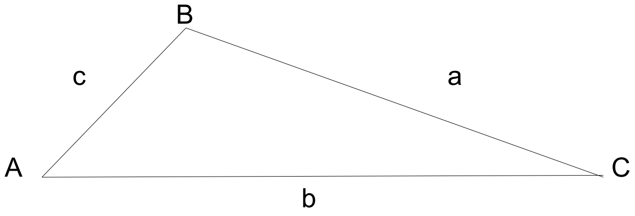


: ISOSCELES TRIANGLE

In the above drawing: $a = c$
$A = C$: "base angles" at b = "base side"

You can verify that A=C by drawing the height at side b and observing the two equivalent right triangles created. The height (h) of an isosceles triangle can be found using the Pythagorean theorem with: $c^2 = h^2 + (b/2)^2 = h^2 + b^2/4$
$h = \text{square-root}\,(c^2 - (b^2 / 4))$, For the isosceles triangle: Area = $bh / 2$ or= 2 (area of each inner right-triangle)

A **scalene triangle** has all three of its sides different in length: "Scalene" is a Latin-Greek word having a basic meaning of "odd scale or size", hence for a triangle it means that all sides of it are odd or different in size to each other. Note that a right triangle can be a scalene triangle. [FIG 129]



: SCALENE TRIANGLE

Congruent triangles are identical triangles. If all of the 6 parts (3 angles, and 3 sides) of two or more triangles have the same value, then those triangles are congruent or identical triangles. Even with a minimum of three specific corresponding parts having the same value is enough to determine that the triangles are congruent or identical, and all of the other corresponding parts will also be equivalent in value. Here is the minimum (3) number, and specific parts needed to determine if two triangles are congruent or identical:

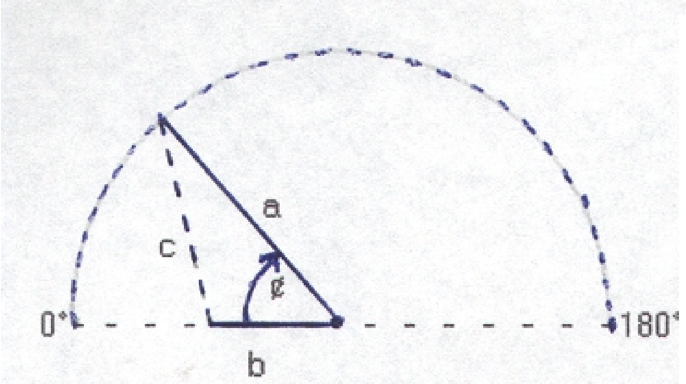
Three Sides : commonly known as the "SSS" test , S = side

An Angle And Its Two Adjacent Sides : commonly known as the "SAS" test, A = angle
(or "two sides and the included angle")

Two Angles And The Common Side Between Them : commonly known as the "ASA" test
(or "two angles and the included side")

AN OBSERVATION ON THE LENGTH OF A SIDE OF A TRIANGLE

Here is a general observation about the length of a side of a triangle in relation to the other two sides of that triangle. Observe the drawing below which shows a triangle being constructed: [FIG 130]



Sides (a) and (b) are shown and are representative of two sides of any angle of a triangle being constructed. When sides (a) and (b) are fixed in length, the length of side (c) then depends upon the angle (shown here at the intersection of sides (a) and (b)) created (usually called angle C since its opposite side c) when one of these sides is rotated about the other. Here, side (a) is being rotated about the intersection of sides (a) and (b). Since this angle can vary between 0° and 180° for the triangle, (c) can then vary in length, and it is therefore not possible to solve for side (c) given just the other two sides of the triangle alone (unless we know that it's a right triangle where C=90° and then the Pythagorean Theorem can be used to solve that triangle). Side (c) is then said to be ambiguous (undetermined, or can't be specifically determined) with just the limited amount of information given. For this not to be the case, more information about the triangle is needed such as an angle, a trigonometric ratio, a height, the perimeter of the triangle, or the area of the triangle. Even though the third side (side c) can vary in length, it does have a minimum and a maximum possible value. The minimum value for side (c) is when the corresponding angle (here, C) of it is small, practically or almost at 0° (there still needs to be an angle > 0° so as to have an actual angle and triangle construction). Here, the value of side (c) can be as low as (practically) 0 when the other sides (a and b) are near equal in length and the angle is near 0°. The size of (c) is always the distance between the endpoints of the two other sides (a and b) which intersect at and define its corresponding angle (C). The maximum possible value for side (c) is when the (corresponding) angle (here, angle C) is practically 180° (but not actually 180°, since a triangle must have 3 angles which sum to total value of 180°).

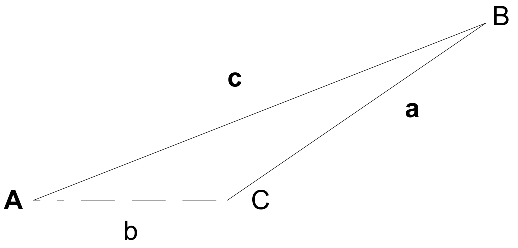
When the angle indicated in the above drawing is 180°, it is sometimes called a "straight angle". This is because the two sides (a and b) will form a (straight) line segment. For a triangle to exist, it must have three internal angles, and their sum is always 180°, and if one angle has that full value, the other two angles are 0° and non-existent, and therefore, the triangle is non-existent.

When the angle is fixed, the length of the side opposite that angle (side c above) depends upon the length of the two sides of that angle.

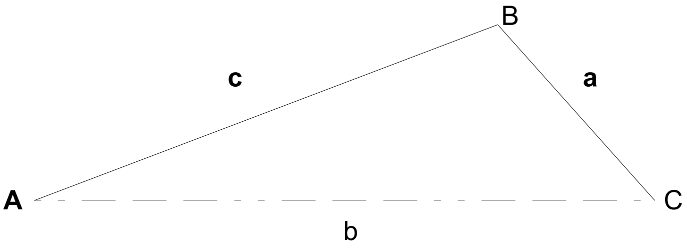
As hinted in the above discussions, even if you know the length of two sides of a triangle and an angle, a total of 3 parts of a triangle, it is possible that the triangle can't be completely solved for if the angle is not the angle between (ie., the "included" or vertical angle) those two given sides, and if no other information about the parts of the triangle are known. Such a condition is called the "ambiguous (undetermined) case". In the above drawing, side (a) and (b) are fixed in value, and side (c) then depends on the size of its corresponding angle, and if that angle is not known, then side (c) can't be determined if no further information about the triangle is known. Below is an example of this type of ambiguous situation.

For either or both triangles below, we are given these three parts: side (a)=3, side (c)=7, and ¢A=10° [FIG 131]

T1



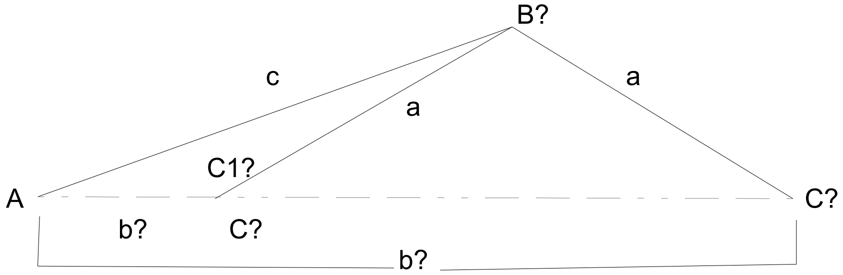
T2



Given the triangle information, first consider if it contains a minimum of 3 and specific parts needed, such as one part being a side, so as to solve that triangle, and then consider:

When given just two sides of a triangle and one angle, if the angle is not at the intersection of those two given sides, then a condition of an ambiguity exists, and even though the law of sines could be used (on each different ambiguous triangle), as it does with the limited information given in this example. We are given sides (a) and (c), and the angle that we need is at the intersection of these two sides, and that angle is B which was not given. If B, and not A, was given, we would of had enough information to use the law of cosines formulas to solve for some other parts of this triangle, such as:
sideb^2 = sidea^2 + sidec^2 - 2(sidea)(sidec)(cos ¢B) : a form of the law of cosines
Using the law of sines using: (a/Sin A) = (b/Sin B) = (c/Sin C), we can find C as about 23.90° OR (180° - 23.90°) = 156.01° due to the ambiguous condition being met. For both angles, their sine value is the same, here about: 0.405

Even though side (a) has the same length for both triangles, clearly, side (b) in T2 is longer than side (b) in T1. As mentioned previously, to continue solving either triangle, more information about the triangle must be known. This information can be found by measurement and-or by calculation using further known information about the triangle. Using the triangles above, here is a combined view of the two possible (ambiguous triangles): [FIG 132]



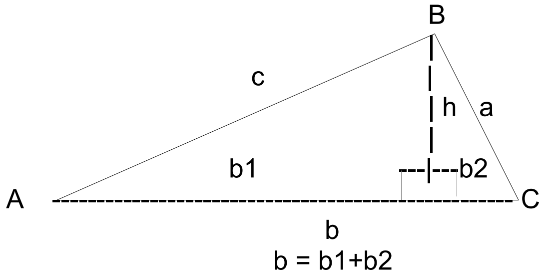
Given: a , A , c
Ambiguous: b , B , C
C + C1 = 180°
C1 = 180° - C

Notice the isosceles triangle indicated and having two sides equal to (a), and both base angles of it are equal to C. This fact will be of aid when solving for one of the two possible ambiguous triangle solutions. For one solution when ambiguity exists, C (which is the corresponding angle to the other given side, here side (c)) is greater than 90°, but if the given angle, here ¢A, is greater than or equal to 90°, there is no ambiguity since the other two angles will complete the 180° internal angle sum, and their sum will be equal to or less than 90°, and it is not then possible for another angle to be greater than 90°. If side (a), and which corresponds to the given angle A, was longer than side (c), then the two equal sides of the isosceles triangle indicated cannot exist, and therefore, the isosceles construction will not exist, and there will be one unique solution only since the ambiguous condition does not then exist.

When given 4 or more correct parts of a triangle, the triangle can be completely solved, that is, there is no chance for an ambiguous part(s). When given 3 parts, the minimum number of parts to solve a triangle, one part must be a side (to distinguish it from all other possible similar triangles that have the same angles), however, the triangle can still be ambiguous (have 2 possible solutions). A triangle is ambiguous when given only two sides and an angle (<90°) that corresponds to one of those sides (ie., the given angle was not the angle between and at the intersection of those two given sides). We were not given angle (B) at the intersection of sides (a) and (c). All other (the three remaining) parts of that ambiguous triangle will have two possible solutions. For the above example, the ambiguous parts are C, B and b. There are two possible solutions for these parts. Side (b) is an ambiguous side. Angles B and C are ambiguous angles.

METHODS FOR SOLVING ANY TRIANGLE

The discussion that follows is for solving triangles that are not necessarily right triangles, however, the formulas to be presented will also work for right triangles since a right triangle is also an instance of the infinite number of all possible triangles. Triangles that are not right triangles are also known as oblique (ie., tilted [from 90°]) triangles. The reason that these formulas to be presented are needed is that the Pythagorean Theorem mathematical relationship no longer applies to the 3 main sides of these triangles, and this should be somewhat obvious for equilateral triangles where all of its sides are equivalent in length, and all of its angles are equivalent but none are 90° which is required for a right triangle. Also, for an equilateral triangle, here is no hypotenuse, or longest side if all the sides are equivalent in length. Below is a representative triangle used for the derivation of THE LAW OF COSINES formula. [FIG 133]

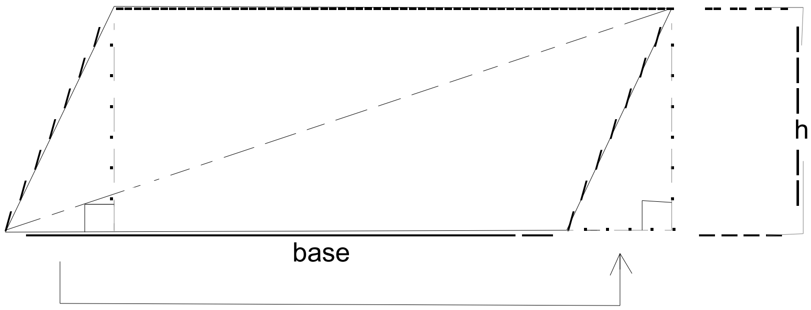


h = height is also known as the altitude of the vertice ("corner", peak, or vertex of a triangle, vertices is the plural of vertex) point where two lines, such as the sides of an angle or triangle meet or intersect. In a way, a vertex is like a "corner" of a triangle or construction. A triangle has three altitudes (either internal and-or external as when a (base) side needs to be extended so as to show the actual height of a vertex) since it has three vertices. The length of the altitude is the perpendicular distance from a vertice to the angles corresponding side or base. This base side is physically directly "opposite" or across from the angle and is sometimes said to "subtend" (meaning beneath) the angle. The three heights of a triangle will intersect or meet at a special point called the orthocenter. Note, for non-right-triangles that are obtuse ("large" or "wide", hence have an ¢>90°), the base line of two sides must be extended so as to show the altitude as perpendicular to the base. All the heights of an acute triangle (all ¢<=90°), such as an equilateral triangle, are within the triangle. The area of these triangles being discussed is also calculated from the same formula as that used for right-triangles.

$$\text{Area} = \frac{(\text{base})(\text{height})}{2}$$

: Area of any triangle
Height is the distance or altitude from the (perpendicular) base side to the corresponding vertice or angle.
For clarity: (base)(height) = (side of triangle)(height at that side)

This formula for area is so, since any triangles is equivalent to one-half of a parallelogram. A parallelogram, as discussed previously in this book, has 4 sides like a rectangle, but each pair of opposite sides need only be parallel, and adjacent (connecting) sides need not be perpendicular as required for rectangles. Actually, a rectangle is only one special case of an infinite number of possible parallelograms. If you imagine the corners of a square or rectangle as having hinges to let it tilt left or right, it is easy to conceive that a parallelogram and its associated area can be created. Given a parallelogram, it is easy to show the equivalent rectangular area when the indicated (as shown in the next figure) left triangular area of the parallelogram is moved over to the indicated equivalent triangular area adjoining the right side of the parallelogram and therefore creating a rectangle structure: [FIG 134]



Using the Pythagorean Theorem on the two internal right triangles that compose the general triangle indicated in Fig 133:

$a^2 = b2^2 + h^2$ and $c^2 = b1^2 + h^2$ solving for h^2 in both equations:

$h^2 = a^2 - b2^2 = c^2 - b1^2$

$\text{COS } C = \frac{b2}{a} = \frac{b - b1}{a}$, hence $b1 = (b - a \text{ COS } C)$, and $b2 = a \text{ COS } C$
therefore, $b2^2 = (a \text{ COS } C)^2 = a^2 \text{ COS}^2 C$

$a^2 - b^2 = c^2 - b1^2 = c^2 - (b - a \text{ COS } C)^2$ expanding the binomial:

$a^2 - b^2 = c^2 - (b^2 - 2ab \text{ COS } C + a^2 \text{ COS}^2 C)$ using substitution, and distribution:

$a^2 - b^2 = c^2 - (b^2 - 2ab \text{ COS } C + b^2) = c^2 - b^2 + 2ab \text{ COS } C - b^2$ combining terms:

$a^2 - b^2 = c^2 - 2b^2 + 2ab \text{ COS } C$ after solving for c^2 :

$\mathbf{c^2 = a^2 + b^2 - 2ab \text{ COS } C}$ **: LAW OF COSINES (for all triangles, side c)**
The formula is very similar to the Pythagorean Theorem, but has another term added to it. Below is an example.

$c = \sqrt{a^2 + b^2 - 2ab \text{ COS } C}$: side c, also:

$\mathbf{\text{COS } C = \frac{a^2 + b^2 - c^2}{2ab}}$ **: useful when only three sides are given and you need to find some angles**

¢C = ARCCOS (COS C) : ARC COS essentially means to find the corresponding angle given its corresponding COSine value. As cosine ¢ is a trigonometric function (can be thought of as an equation or process) that depends on the sides of that angle. arccos ¢ is a function that is essentially the inverse of the cos ¢ function.

Similar formulas for sides (a) and (b) can be derived as:

$a^2 = b^2 + c^2 - 2bc \text{ COS } A$: LAW OF COSINES (side a)

$b^2 = a^2 + c^2 - 2ac \text{ COS } B$: LAW OF COSINES (side b)

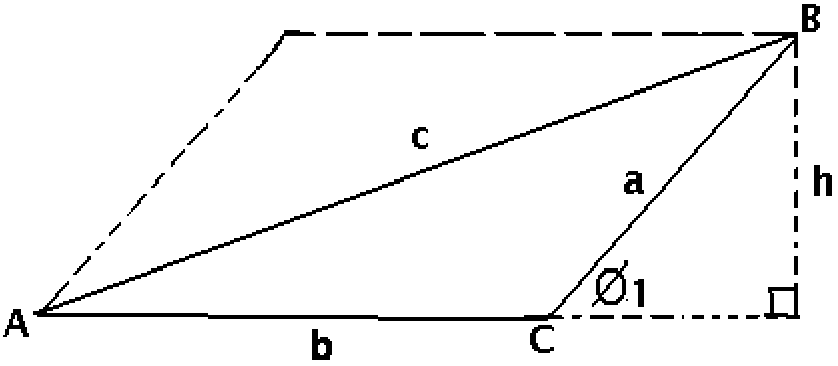
The basic format (which has a Pythagorean Theorem-like form) of each of the above equations for the law of cosines is therefore:

side^2 = (SUM of other two sides that are squared) - 2(PRODUCT of the other two sides)(COS corresponding angle)

Due to the fact that the LAW OF COSINES formula is so "cumbersome" or lengthy, if you like, you can always finish solving a triangle using the LAW OF SINES formula and the fact that the sum of a triangles interior angles is a constant of 180°. If the altitude that is completely within the triangle is known, it is also possible to solve for some of the triangles parts using the Pythagorean Theorem or trigonometric ratios. If the triangle is a right-triangle, the law of cosines formula reduces to the Pythagorean Theorem since COS 90° = 0. c^2 = a^2 + b^2 - 2ab COS 90° = a^2 + b^2 - 0 = a^2 + b^2. The fundamental basics of the law of cosines were known to early Greek mathematicians such as Euclid of Alexandria.

As an extra note, since c^2 = b1^2 + h^2 and a^2 = b2^2 + h^2 , we can get: h^2 = a^2 - b2^2
c^2 = a^2 + b1^2 - b2^2
b1^2 = c^2 + b2^2 - a^2 and b2^2 = a^2 + b1^2 - c^2
a^2 = c^2 + b2^2 - b1^2

Ex. Below is a triangle. Find side (c) given just two sides and (almost) the included angle between those sides. Side (a) = 5 , side (b) = 7 and ¢1 = 45°. Also indicated is that this triangle could be considered as half of a parallelogram and that (c) is equal to the diagonal within that parallelogram. [FIG. 135]



From the fact that supplementary angles sum to 180° :

C + ¢1 = 180° solving for C:
C = 180° - ¢1 substituting the known value for ¢1:
C = 180° - 45°
C = 135°

Now that we have two sides and the angle included between those two sides. We can use the law of cosines with this limited amount of information so as to find the third side which is side (c) which is also the length of the diagonal line in the parallelogram indicated.

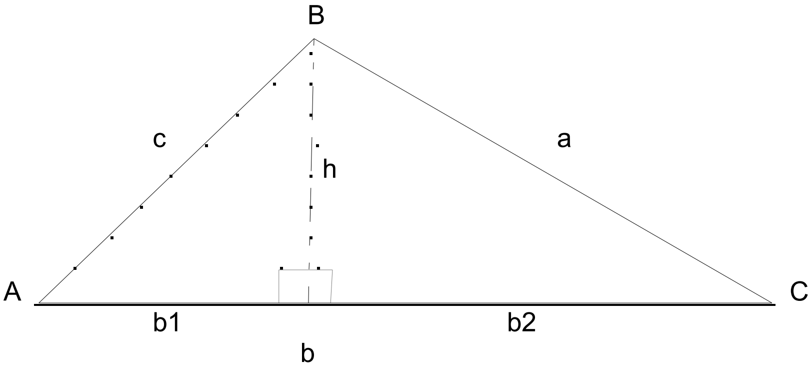
c^2 = a^2 + b^2 - 2ab COS C taking the square root of both sides and substituting the known values:

$$c = \sqrt{5^2 + 7^2 - 2(5)(7)(0.7071\underline{1})}$$

$$c = 11.11\underline{3}$$

CALCULATING ANY HEIGHT WITHIN A TRIANGLE

Here is a formula to calculate the (perpendicular) height (ie., "altitude") from any vertex of a triangle to its corresponding base side. The height between a side and the corresponding angle is also the shortest distance from that vertex to the opposite side. Though there are other methods, such as the law of cosines to solve for the height if enough information (triangle parts) about the triangle is known, the formula presented below can be helpful when a triangle construction is based upon a base side value and the (base) angles adjoining to that base side. [FIG 136]



$\text{TAN } A = \frac{h}{b1}$ $\text{TAN } C = \frac{h}{b2}$ therefore:

$b1 = \frac{h}{\text{TAN } A}$ $b2 = \frac{h}{\text{TAN } C}$

$b = b1 + b2$

$b = \frac{h}{\text{TAN } A} + \frac{h}{\text{TAN } C}$: An example of solving for a base side using the base angles and the height at that base side. After combining fractions:

$b = \frac{h\ (\text{TAN } A + \text{TAN } C)}{\text{TAN } A\ \text{TAN } C}$: an optional formula. Solving for h = hb:

$hb = \frac{b\ \text{TAN } A\ \text{TAN } C}{\text{TAN } A + \text{TAN } C}$: height at base side b given the adjacent angles. Likewise:

$ha = \frac{a\ \text{TAN } B\ \text{TAN } C}{\text{TAN } B + \text{TAN } C}$

$hc = \frac{c\ \text{TAN } A\ \text{TAN } B}{\text{TAN } A + \text{TAN } B}$

Observing the above formulas, here is a generalized or non specific HEIGHT FORMULA given a base side and the two adjoining angles to that base side:

$$\text{height} = \frac{\text{(base perpendicular to height) (product of the TAN values of both base angles)}}{\text{(sum of the TAN values of both base angles)}}$$

Here is a derivation of a helpful ratio, especially when the height is not known:

Given: $\tan A = \frac{h}{b1}$ and $\tan C = \frac{h}{b2}$ therefore:

$h = b1 \tan A = b2 \tan C$:

$\frac{b1}{b2} = \frac{\tan C}{\tan A}$: since b1 is closer to or adjacent to A, and b2 is closer to C, this is like a reverse ratio. A and C are the "base angles" adjacent or adjoining base side (b). b1 and b2 are segments of side (b) where the altitude or height intercepts that side.

A generalized formula for all sides is:

$\frac{\text{segment1 of base}}{\text{segment2 of base}} = \frac{\text{TAN of base angle at segment2}}{\text{TAN of base angle at segment1}}$: a base segments ratio, of how the height intercepts it and divides it into two segments

For example:

Between angle A and B is side (c). A=36.87°, B=53.13°, and (c)=5. The height at side (c) will divide that base side (c) into two segments (c1 and c2). What is the length of each segment?

$\frac{\tan A}{\tan B} = \frac{\tan 36.87°}{\tan 53.13°} = \frac{0.75}{1.3333} = 0.5625 = \frac{c2}{c1}$, hence: $c2 = 0.5625\, c1$, and: $c = c1 + c2 = 5$
also: $c1 = c2 / 0.5625$

Now we have two (simultaneous) equations. We can use substitution to create one solvable equation that contains only one variable:

$c1 + c2 = 5$ using substitution:
$1c1 + 0.5625c1 = 5$ combining like terms:
$1.5625\, c1 = 5$ after dividing both sides by 1.5625

$c1 = 3.2$: this segment is connected to, or nearer to (the vertice of) angle A

$c = c1 + c2$
$c2 = c - c1$
$c2 = 5 - 3.2 = 1.8$: this segment is connected to, or nearer to (the vertice of) angle B

Likewise:

Given: $\cos A = \frac{b1}{c}$ and $\cos C = \frac{b2}{a}$

$b1 = c \cos A$, $b2 = a \cos C$ dividing the left equation, by the right equation:
(dividing both sides by an equivalent (same) value)

$\frac{b1}{b2} = \frac{c \cos A}{a \cos C}$: another base segment ratio, of how the height intercepts it and divides it into two segments

$\frac{b1}{b2} = \frac{c \cos A}{a \cos C} = \frac{\tan C}{\tan A}$ hence: $c = \frac{a (\cos C)(\tan C)}{(\cos A)(\tan A)} = \frac{a \sin C}{\sin A}$, and: $\frac{c}{a} = \frac{\sin C}{\sin A}$
(law of sines)

There is a way to calculate the height of any triangle without using any trigonometric functions. This method utilizes a unique formula for the area of any triangle which was derived long ago in about the year 60AD by the Greek mathematician and inventor named **Heron** (aka Hero of Alexandria, Egypt). An easy to understand, modern version of the derivation of this formula is given in a book such as: **A Journey Through Genius, The Great Theorems Of Mathematics, by William Dunham**, and which could be described as an interesting, helpful, "good read", and which gives many mathematical topics a practical analysis for an easier understanding.

Given: $S = \frac{a + b + c}{2}$: half the triangles perimeter (variable S = Semi-perimeter distance), which is half the length of the sum of a triangle's' sides

TRIANGLE AREA = $At = \sqrt{S (S - a) (S - b) (S - c)}$: **HERON'S FORMULA** , for the area of a triangle. S = half the perimeter = semi-perimeter

As an interesting note for this unique formula and its structure, consider that half a square area with side (s) is equal to a triangle with half the area of that square, and if you square both sides of Heron's Formula , you will have this similarity:

(At^)2 = S (S-a) (S-b) (S-c) , and for a square: As = s^2 and therefore: (As)^2 = (s^2)^2 = s^4 = (s)(s)(s)(s)

Since: $At = \frac{(base)(height)}{2}$: base and height are perpendicular to each other. For example, if c is the base side, the corresponding height (h) is hc and its length is the perpendicular line from side c to the vertex of the two other sides (a and b) that define and are adjacent to angle C. (base)(height at that base) or= (side)(height at that side). Mathematically:

$h = \frac{2At}{base}$: another Height Formula (no TAN values needed)
: the height and base correspond and are perpendicular (right angle) to each other

$h = \frac{2 \sqrt{S (S - a)(S - b)(S - c)}}{base}$: a height formula based on Heron's formula for triangle area

From: $SIN\ A = \frac{hb}{c}$ and side b is perpendicular to hb = the height at side b, we can get:

$hb = \frac{2At}{b}$ using substitution:

$$SIN\ A = \frac{hb}{c} = \frac{\frac{2At}{b}}{c} = \frac{2At}{bc} = \frac{2 \sqrt{S (S - a) (S - b) (S - c)}}{bc}$$

¢A = ARCSIN (SIN A) : formal mathematical notation for finding the corresponding angle of the corresponding sine value of that angle

$SIN\ A = \frac{2\ At}{bc}$ $SIN\ B = \frac{2\ At}{ac}$ $SIN\ C = \frac{2\ At}{ab}$ and the general format is:

$SIN\ \text{¢} = \frac{2\ Area}{\text{product of both sides adjoining that angle}}$: adjoining=composing=creating=defining
From this, we can also derive another formula for the area of the triangle (At):

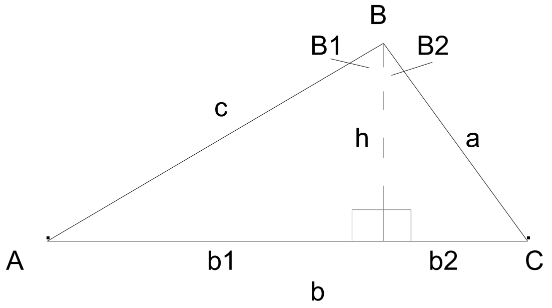
$At = \frac{(\text{product of both sides adjoining the angle})\ Sin\ \text{¢}}{2}$

Ex. $At = \frac{b\ c\ \sin A}{2}$: This can also be derived from A = (base)(height) / 2, where base = b, and Sin A = h/c, then h = c Sin A, From this we can also derive:

$\sin A = \frac{2\ At}{bc}$: this equation was also shown above

EXTRA EQUATIONS FOR THE SIDES OF NON RIGHT TRIANGLES

Observe this drawing below of a non-right triangle. To help solve for parts of it, the triangle is shown with two indicated internal right triangles which can be analyzed using the Pythagorean Theorem. The result is some helpful equations for your understanding, solving and utilization for non-right triangles: [FIG 137]



: h = hb (height at side b)
B = B1 + B2
b = b1 + b2

Using the Pythagorean Theorem:

c^2 = h^2 + b1^2 and a^2 = h^2 + b2^2

h^2 = c^2 - b1^2 h^2 = a^2 - b2^2

b1^2 = c^2 - h^2 b2^2 = a^2 - h^2 using substitution for h^2:

b2^2 = a^2 - (c^2 - b1^2) distributing (-1) to clear grouping symbols:

b2^2 = a^2 + b1^2 - c^2 solving for c^2 :

c^2 = a^2 + b1^2 - b2^2 : A Pythagorean-Like equation, where here, b2 is subtracted from the sum.

a^2 = c^2 + b2^2 - b1^2

Since: b = b1 + b2 ,

b2 = b - b1 and

b1 = b - b2

Here is a derivation of a helpful equivalence to remember so as to easily solve for a triangle height or side:

$At = \frac{(base)(height\ at\ that\ base)}{2} = \frac{(a)(ha)}{2} = \frac{(b)(hb)}{2} = \frac{(c)(hc)}{2} = 0.5\ a\ ha = 0.5\ b\ hb = 0.5\ c\ hc$:

Taking just the last three right side equivalences and "dividing through (all)" by (0.5) so as to "clear", "rid" or set all the numerical coefficients to 1, we have this equality or mathematical relationship among the sides and heights:

$a\ ha = b\ hb = c\ hc$: **EQUAL PRODUCTS OF BASE SIDES AND CORRESPONDING HEIGHTS**
All products of a side and the corresponding height at that side are equal.
Note also that twice the area = **(base) (height at base) = 2 At** = $a\ ha = b\ hb = c\ hc$
In general, (base) (height at base) is the area of a rectangle, hence this equals twice as much as the triangle area using the same (base) and (height at base).
Note the "reverse ratios" of, and for ex.: $(a / b) = (hb / ha)$

Many of the above values can also be derived and solved for if the other parts of the triangle are known. For example:

$a = \frac{b\ hb}{ha} = \frac{c\ hc}{ha} = At / ha$ These have the basic or general format of:

$$\textbf{side1} = \frac{\textbf{(side2) (height of side2)}}{\textbf{(height of side1)}} = \textbf{At / (height of side1)}$$

$ha = \frac{b\ hb}{a} = \frac{c\ hc}{a} = At / a$: in a right triangle, $(ha) = b$, and $(hb) = a$. * Basic format:

$$\textbf{(height side1)} = \frac{\textbf{(side2)(height of side2)}}{\textbf{(side1)}}$$ **: this can also be derived from the previous derivation above = At / (side1)**

Considering the above, and the law of sines, for example: $(a / b) = (\sin A / \sin B)$:

$a = \frac{b\ hb}{ha} = \frac{b \sin A}{\sin B}$ dividing each fraction by b:

$\frac{a}{b} = \frac{hb}{ha} = \frac{\sin A}{\sin B}$ The basic format here is:

$$\frac{\textbf{side1}}{\textbf{side2}} = \frac{\textbf{sin angle1}}{\textbf{sin angle2}} = \frac{\textbf{height of side2}}{\textbf{height of side1}}$$ **:** note that the ratio of the heights is in reverse ratio to both the sides and sines of the corresponding angles

* From the previously given, we also can derive: $hc = hb\ (b/c)$, and

If the triangle is a right triangle: $hc = hb \cos A$, therefore: $\cos A = hc / hb = b / c$: a and b are the "leg" sides of the hypotenuse side c

$hb = hc / \cos A = hc\ c / b$

$hc = ha\ a / c = ha \sin A$, therefore: $\sin A = hc / ha = a / c$

$ha = hc / \sin A = hc\ c / a$

$\tan A = a / b = \sin A / \cos A = (hc / ha) / (hc / hb) = hb / ha$

Extra: Since the products of any side and its corresponding height are all equal, their ratio is equal to 1, for example:

$1 = \frac{a\ ha}{b\ hb}$,

BASIC STEPS FOR SOLVING A TRIANGLE

1. In order to completely solve (find all 6 parts: 3 angles and 3 sides) a triangle, a minimum of 3 parts must initially be known and 1 part must be a side. Hence, we need, at a minimum, either 3 sides, or 2 sides and 1 angle, or 1 side and 2 angles to begin with. One side must be given so as to distinguish it from all other similar (ie. bigger or smaller) triangles which have the same angles, but different side lengths so as to be unique and-or non-congruent (same) triangle.

2. If an angle is 90°, the triangle is a right triangle, and the simplest method is to use the Pythagorean Theorem and the fact that angles $A + B + C = 180°$. If a triangle has an obtuse (>90°) angle, the triangle is an obtuse triangle, and the sum of its other two angles will be <90°.

3. If given just 3 sides, use Heron's formula to solve for the area of the triangle first, then use the formula shown previously to solve for the SIN of an angle. The law of cosines can also be used to solve for the cosine of an angle, and then the angle can be found using: angle = arccos (cos angle)

4. If given just 2 sides and 1 angle:

   A. Use the law of cosines if the two sides are the adjacent sides of the angle, that is, the angle is the included or vertical (at the vertex) angle. For example, given a, b, and C, use the law of cosines for side c. Angle C is at the intersection of sides (a) and (b).

   B. If the angle corresponds to one of the two sides given, that is, it is not the included angle between those two given sides, then the ambiguous triangle case exists when the angle is less than 90°. There are two possible triangles as the solution and you may have to choose either one depending upon the situation, measurements and calculations.

   For example, given (a), (c), and A that is less than 90°. Angle A is not the included angle, so the ambiguous case exists. The three remaining parts of the triangle, side b, angle B and angle C have two possible trigonometric solutions, resulting in two distinct triangles. By using the law of sines, you can find the sine of angle C, and therefore angle C. The law of sines, and the sine of the angle function, will return a sine value for an angle less than or equal to 90°, hence the corresponding angle to that sine value will be less than or equal to 90°. The other angle of this triangle, angle B, can be found from the fact that $A + B + C = 180°$. For the other value of angle C of the second ambiguous triangle, notice the isosceles construction shown previously in the discussion about the ambiguous case. Angle C of the second ambiguous triangle is the supplementary angle to both equivalent base angles of the isosceles triangle, hence this value of angle C being found for the second ambiguous triangle is 180° - angle C of the first ambiguous triangle. Angle B in this second ambiguous triangle can then be found from the fact that $A + B + C = 180°$.

   If the side given that corresponds to the given angle is greater than the other given side, then no ambiguity exists, and the triangle has only one solution. The internal isosceles construction, as shown previously and which shows the two possible solutions, does not exist. If this side equals the height (or altitude) at the ungiven third side, then the triangle is a right triangle.

5. If given just 2 angles and 1 side:

   The third angle can be found from $180° = A + B + C$. Use the law of sines to find a second side. For the third side, you can use either the simpler law of sines, or use the law of cosines.

6. If more than 3 parts of the triangle are known, the other parts can be found using either the law of sines, the law of cosines, the Pythagorean theorem, or the fact of: $A + B + C = 180°$.

7. If given just 3 angles, the triangle cannot be solved since it is indistinguishable from all the other similar triangles with the same exact angles, except that the triangle(s) may be magnified in size by magnifying all its sides by the same factor.

8. Sometimes you may be given some of the heights of a triangle and can solve for some of the 6 fundamental triangle parts from that. Here is a basic method for solving for a side or its corresponding height (ie., "altitude"), and this equation shows all the equal products of each base sides and its corresponding height:

   $a\,h_a = b\,h_b = c\,h_c$, from this, we can find some other mathematical relationships:
   : h_a = height at side (a)

   $\frac{a}{c} = \frac{h_c}{h_a} = \frac{\text{Sin } A}{\text{Sin } C}$, law of sines using the heights. Again, note the inverse ratios. From this we also have:

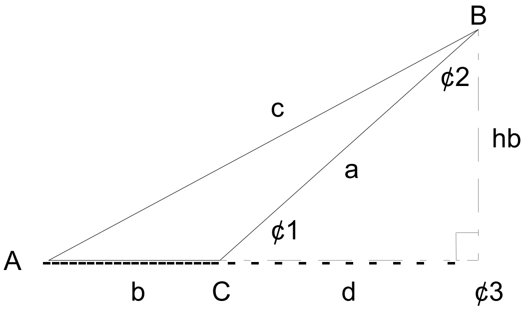
   $h_a \text{ Sin } A = h_c \text{ Sin } C$: $= h_b \text{ Sin } B$

9. You can make a check on all the given and-or found 6 parts by using the law of sines, and all the ratio values should be near in value to each other. If they are not near in value, then check the values used and the results. It always helps to draw the triangle during and for the analysis, and this may also prevent an ambiguous triangle:

$\frac{a}{\text{Sin } A} = \frac{b}{\text{Sin } B} = \frac{c}{\text{Sin } C}$: law of sines

FINDING THE HEIGHT USING SUPPLEMENTARY ANGLES

In the drawing below, there is a non-right triangle with side (b) extended to show the height from base (b) to the vertex of the other two sides (a and c) which defines the corresponding angle (here, angle B) to side (b). [FIG 138]



You are to find the value of the height (h) indicated (without actually measuring it), perhaps its' value needs to be known for some type of construction, or to simply calculate the triangle's area using the typical formula of: At=(base x height)/2. Note that the length of the base is always an actual (unextended) side of the triangle in question, that is, it never includes any extension added to that side so as to analyze and find the corresponding height. For example, in the triangle being analyzed above, the area is: ((side b) x height at (side (b))) / 2, and the area of the triangle is not: (side (b+d) x height at side (b)) / 2 , since that would be the area of a different triangle than the actual one shown.

From the concept of supplementary angles:

180° = ¢1 + C therefore:
¢1 = 180° - C

From SIN ¢1 = $\frac{h}{a}$: where h = hb

h = a SIN ¢1

Since ¢3 is a right angle, and from the concepts of complementary angles:

¢2 = 90° - ¢1

Angles A and (B+¢2) become complementary angles of a right triangle construction when the right angle (¢3) is considered, therefore:

90° = A + (B + ¢2)
(B + ¢2) = 90° - A which can be expressed as:
B + ¢2 = 90° - A after transposing ¢2:
B = 90° - A - ¢2
A = 90° - B - ¢2

Notice that side (c), and other parts of the triangle, can be found using the Pythagorean Theorem:

c^2 = (b+d)^2 + h^2 also:

From: SIN A = h/c, we can get:

$c = \frac{h}{\text{SIN A}}$ since $h = a \text{ SIN ¢}1$:

$c = \frac{a \text{ SIN ¢}1}{\text{SIN A}}$ algebraically:

$a = \frac{c \text{ SIN A}}{\text{SIN ¢}1}$

$\text{SIN ¢}1 = \frac{c \text{ SIN A}}{a}$ and $\text{SIN A} = \frac{a \text{ SIN ¢}1}{c}$

$\text{¢}1 = \text{ARC (SIN ¢}1) = \text{ARCSIN}\left(\frac{c \text{ SIN A}}{a}\right)$, and $C = 180° - \text{¢}1$, $B = 180° - (A + C)$

From the above equation for SIN ¢1, we can derive a "sort of" law of sines which can easily be memorized:

$\frac{a}{\text{SIN A}} = \frac{c}{\text{SIN ¢}1}$ from this we can also get: $\frac{a}{c} = \frac{\text{SIN A}}{\text{SIN ¢}1}$

Since ¢1 is the supplementary angle of: (180° - ¢1) = C, their trigonometric functions are equal: SIN ¢1 = SIN C, and the law of sines similarity above can now be expressed as the more familiar looking:

$\frac{a}{\text{SIN A}} = \frac{c}{\text{SIN C}}$ and: $\frac{a}{c} = \frac{\text{SIN A}}{\text{SIN C}} = \frac{hc}{ha}$: hc = height at side (c)
ha = height at side (a)

Can the relationship of side (b) and SIN B also be included into this law of sines for oblique triangles, therefore resulting into the same law of sines as shown for right triangles? Yes, and this will be shown ahead.

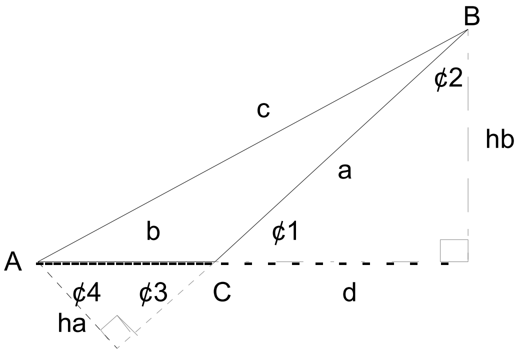
If your also looking for a Pythagorean-like relationship among this triangles sides, here is one which includes a "fourth" or extended side:

From: $c^2 = (b+d)^2 + h^2$, and $a^2 = d^2 + h^2$
$(b+d)^2 = c^2 - h^2$, and $d^2 = a^2 - h^2$ extending the left hand side:
$b^2 + 2bd + d^2 = c^2 - h^2$ after algebraic substitution for d^2:
$b^2 + 2bd + a^2 - h^2 = c^2 - h^2$ T. (-h^2), and switching sides:

$c^2 = a^2 + b^2 + 2bd$: PYTHAGOREAN-LIKE RELATIONSHIP
FOR AN OBTUSE TRIANGLE
(For when C > 90°, or B < the complementary
angle of A, or when b < c COS A since
COS A = adj./hyp., or when a > c SIN A since
SIN A = opp./hyp.)

$d = \frac{c^2 - a^2 - b^2}{2b}$

When side (a) is extended to show the height at side (a) (or base (a)), this will effectively create another right triangle construction: [FIG 139]



This right triangle construction is a similar right triangle (to the other newly created right triangle construction that is "attached" to the original triangle in question that is being analyzed) which can be verified by the concept of vertical angles:

¢3 = ¢1 : since they are vertical angles

Due to the concept of complementary angles that sum to a 90° right angle :

¢4 = ¢2 : (¢3 and ¢4) are complementary angles , and (¢1=¢3 and ¢2) are complementary angles

Since the angles are equivalent, their trigonometric values are therefore equivalent:

SIN ¢1 = $\frac{hb}{a}$ = SIN ¢3 = $\frac{ha}{b}$ therefore:

$\frac{hb}{a} = \frac{ha}{b}$: as should be expected for (magnified) similar triangles, the parts of each specific triangle have the same numeric portion or fractional value of that triangle

$\frac{a}{b} = \frac{hb}{ha}$: notice that the height and side ratios are expressed as an inverse ratio with respect to each other.

These can also verify, or can be likewise verified, by the fact of that the products of each base side and its corresponding height are equal in value: a ha = b hb = c hc

The above equation might not seem correct at first since each side appears to be a reverse or inverse type of ratio. One simple way to verify that this is correct is to consider a right triangle where the height at side (b), or hb, is equal to the other side (side a), and the height at side (a), or ha, is equivalent to the other side (side b), and all this is expressed above as the equivalent ratios.

Since ¢3 = ¢1 are supplementary angles to angle C, their trigonometric (absolute) values are identical.

SIN C = SIN ¢3 = $\frac{ha}{b}$ = SIN ¢1 = $\frac{hb}{a}$, therefore ha = b SIN C

By observing the new right triangle construction shown:

SIN B = $\frac{ha}{c}$, therefore ha = c SIN B

Equating the two expressions for ha:

$b\ \text{SIN C} = c\ \text{SIN B}$ algebraically:

$$\frac{b}{\text{SIN B}} = \frac{c}{\text{SIN C}}$$

expressing this equivalence to what was derived previously:

$$\boldsymbol{\frac{a}{\textbf{SIN A}} = \frac{b}{\textbf{SIN B}} = \frac{c}{\textbf{SIN C}}}$$

: LAW OF SINES (general, for all triangle)
Note also, mathematically, for ex.:

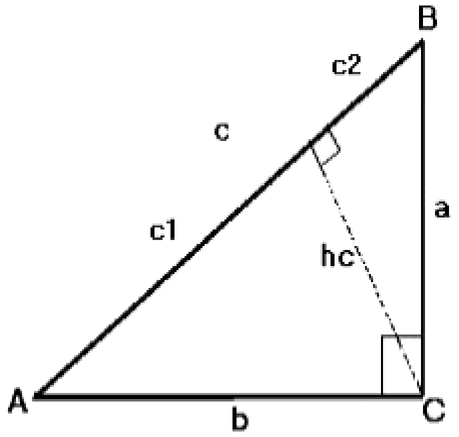
a / b = SIN A / SIN B = hb / ha

a = b (SIN A / SIN B) = b (hb / ha) = c (SIN A / SIN C) = c (hc / ha)

SIN A = (a/b) SIN B = (a/c) SIN C

EXAMPLES OF SOLVING FOR PARTS OF A TRIANGLE

[FIG 140]



The right angle, square, "turn" or "corner", in the above triangle indicates that this triangle is a right triangle. Here, C=90° = the right angle. By using the Pythagorean Theorem, the relationship among all the sides of a right triangle is:

$c^2 = a^2 + b^2$: note also that: $c = c1 + c2$

If a=3, and b =4: $c = \sqrt{a^2 + b^2} = \sqrt{3^2 + 4^2} = \sqrt{9 + 16} = \sqrt{25} = 5$

Note that $a = hb$ and $b = ha$. Due to equal products of the bases and their corresponding heights of any triangle:

$a\,ha = b\,hb = c\,hc$

$hc = \frac{a\,ha}{c} = \frac{b\,hb}{c}$: hc for a right triangle is therefore: $\frac{a\,b}{c}$

The height at side c of this right triangle will create 2 similar triangles to the original triangle, hence there is three similar triangles, and each will have the same three angles equivalent to: 90°, A, and B.

The height at side c will divide side c into two line segments (c1 and c2) that have a ratio that is equivalent to the ratio of the areas (A1 which corresponds to c1, and A2 which corresponds to c2) of each internal similar triangle:

$A1 = \frac{c1\,hc}{2}$ and $A2 = \frac{c2\,hc}{2}$ If you divide A1 by A2, you can find:

$\frac{A1}{A2} = \frac{c1}{c2}$

$\text{Tan } A = \frac{\text{opposite } A}{\text{adjacent } A} = \frac{a}{b} = \frac{3}{4} = 0.75$

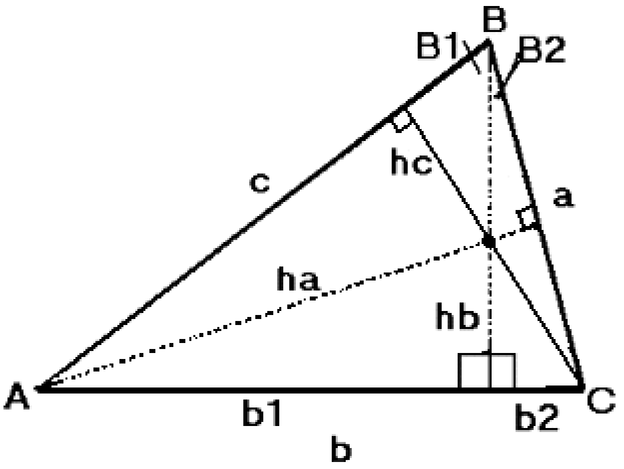
A = ArcTan(Tan A) = ArcTan(0.75) = 36.87°

Since B is a complementary angle to angle A:

A + B = 90°

B = 90° - A = 90° - 36.87° = 53.13°

Ex. Here is another triangle to analyze: [FIG 141]



For this non right triangle, hb=3, b1=4, c = 5. and b2=1. By the values given, the left side portion of this triangle is similar in some ways to the ("3, 4, 5") right triangle given above.

By the Pythagorean Theorem: $a = \sqrt{b2^2 + hb^2} = \sqrt{3^2 + 1^2} = \sqrt{9 + 1} = \sqrt{10} = 3.1622\underline{8}$

To find angle B2: $\tan B2 = \frac{\text{opposite B2}}{\text{adjacent B2}} = \frac{b2}{hb} = \frac{1}{3} = 0.3333$

B2 = arctan(tan B2) = arctan(0.3333) = 18.435°

$\text{Tan } A = \frac{\text{opposite A}}{\text{adjacent A}} = \frac{hb}{b1} = \frac{3}{4} = 0.75$

A = ArcTan(Tan A) = ArcTan(0.75) = 36.87°

By observing the internal right triangle construction, B1 is a complementary angle to angle A:

A + B1 = 90°

B1 = 90° - A = 90° - 36.87° = 53.13°

B = B1 + B2 = 53.13° + 18.435° = 71.565°

Since C is a supplementary angle to (A + B) when considered as one angle value:

180° = (A + B) + C

$C = 180° - (A + B) = 180° - A - B = 180° - 36.87° - 71.565° = 71.565$

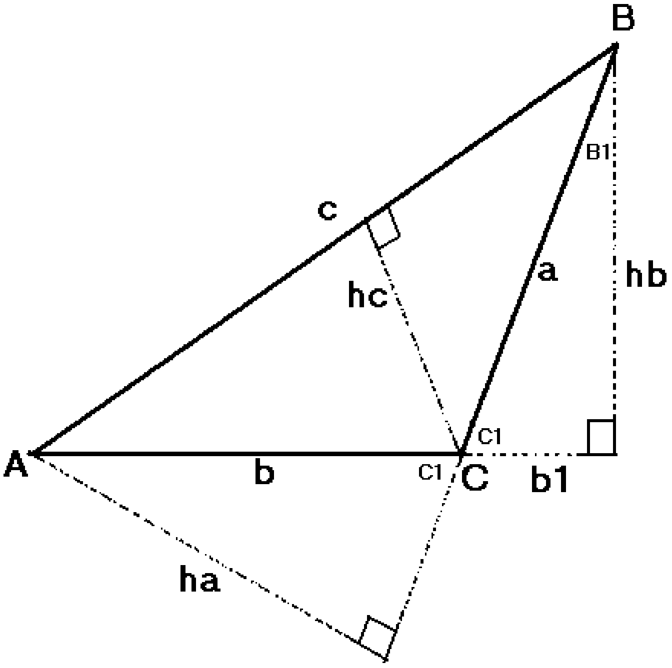
Since angle C = angle B, this is an equilateral triangle with B and C being the "base (side) angles". The base side of this equilateral triangle is side (a). The two equivalent sides are (c) and (b).

The indicated heights can be found from the **equal products of bases** and their corresponding heights:

$a\,ha = b\,hb = c\,hc$: note that (base)(corresponding height) = 2 At

The point of intersection of all three heights is called the **orthocenter**, and it is indicated in the above drawing.

Ex. Here is another triangle to analyze: [FIG 142]



For this triangle, hb=3, b=3, c = 5. and b1=1. By the values given, this triangle is similar in some ways to the ("3, 4, 5") right triangle given above. Since one angle is greater than 90° (here C>90°) two sides are extended to show their corresponding heights.

$$\text{Tan } A = \frac{\text{opposite } A}{\text{adjacent } A} = \frac{hb}{(b+b1)} = \frac{3}{(3+1)} = \frac{3}{4} = 0.75$$

$A = \text{ArcTan}(\text{Tan } A) = \text{ArcTan}(0.75) = 36.87°$

To find angle B1: $\tan B1 = \frac{\text{opposite } B1}{\text{adjacent } B1} = \frac{b1}{hb} = \frac{1}{3} = 0.3333$

$B1 = \arctan(\tan B1) = \arctan(0.3333) = 18.435°$

Since C1 is a complementary angle to angle B1:

B1 + C1 = 90°

C1 = 90° - B1 = 90° - 18.435° = 71.565°

Since C is a supplementary angle to C1:

180° = C + C1

C = 180° - C1 = 180° - 71.565 = 108.435°

Since the sum of a triangles interior angles is always 180°:

180° = A + B + C

B = 180° - A - C or = 180 - (A + C)

B = 180° - 36.87° - 108.435°
B = 34.695°

B + B1 = 34.695° + 18.435° = 53.13°

Notice that the extended sides will create two similar right triangle constructions. Each of these triangles contains angles of 90°, C1, and the complementary angle to c1 which is (90° - c1)° = B1

TO EASILY CONSTRUCT A 45°, 90°, 45° TRIANGLE:

First, before this discussion, a protractor is an angle gauge. It can be used to draw and-or measure angles. This book has an article on making your own protractor.

First draw a square, and then draw a diagonal line within it. This will create two triangles each being a 45°, 90°, 45° right triangle. Each side of this square, and the "leg" sides (a and b) of this triangle, will have the same measurement. As a check for "rightness" or correctness of the right angle, the hypotenuse side (c) of the triangle will be equal the square root of two (about 1.4142) times longer than the leg side; according to the Pythagorean Theorem.

TO EASILY CONSTRUCT A 30°, 90°, 60° TRIANGLE: (and possibly create smaller angles)

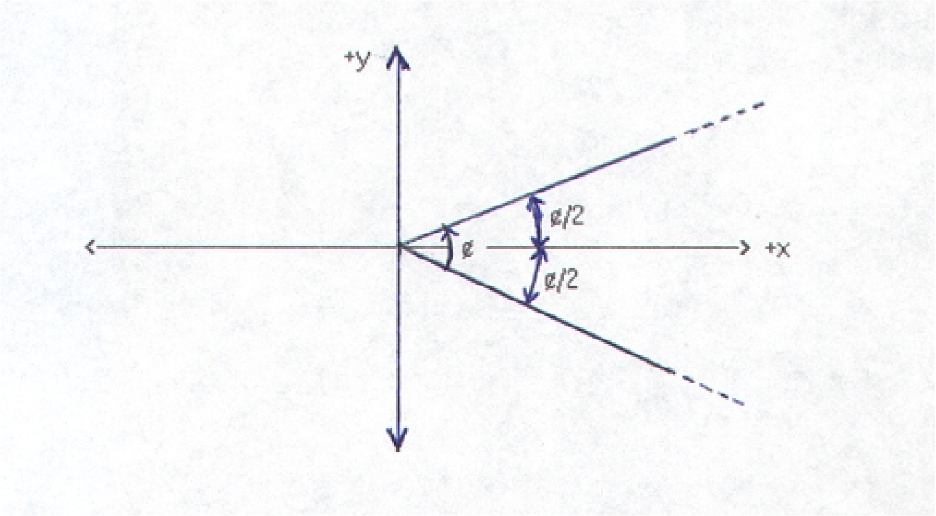
Draw an isosceles triangle. Each angle of an isosceles triangle is 60° Draw an altitude or height line from a base side to the corresponding opposite vertex. This line will divide that base side in half and create two 30°, 90°, 60° triangles. As a check to determine if the isosceles triangle was drawn or constructed right, the height should be about 0.866 times the hypotenuse of that right triangle, or any equivalent side of that isosceles triangle. This can be derived and checked with the Pythagorean Theorem. If smaller angles are needed: If a compass for drawing circles, or arcs of circles, is available, take the 30° angle created and create a 30° circular arc. Draw a ("chord") line at the endpoints of the arc. Divide this line into two, and this will be used to create an angle bisector line. This angle is 15°.

Before we move onto other topics of trigonometry, the concept of triangulation is of interest. **Triangulation** is very useful for when a location and-or direction of some object or event can be determined, such as on a map, by using two or more sensed (ex., eye witnesses) directions, angles, time, etc. Where each of these direction lines intersect in reality and-or on a map, is a very good estimate of a location, direction, angle and-or event. If the map has a scale, and-or distances between two observers are known, distances and angles to the object and-or location in question can be calculated or estimated.

BISECTING AN ANGLE

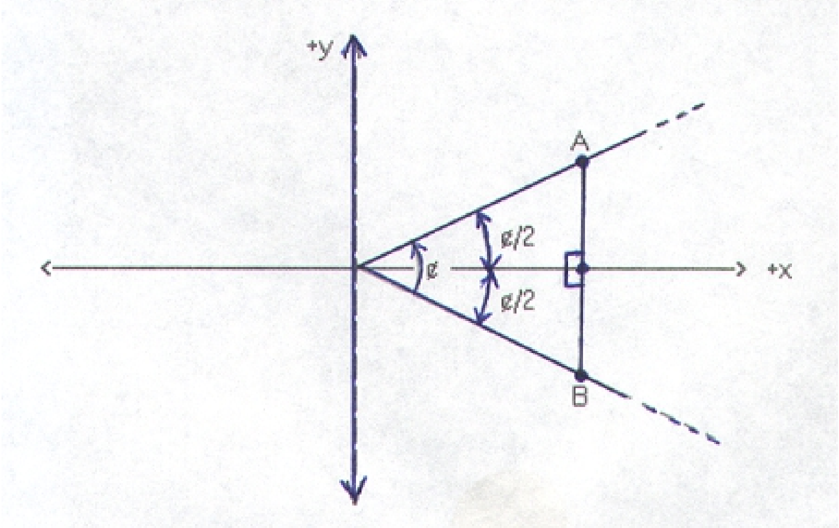
To bisect is to intersect and divide, usually into two parts or sections. The word prefix of: "bi" means two. To bisect an angle is to divide the angle into (usually) two equivalent angles. These two angles are also known as "half-angles" and are often simply expressed as: (¢/2). Below is an example of an angle that is bisected. A method to bisect and angle was previously given in this book during the discussion of how to construct a 30°, 60° and 90° triangle, and that uses an arc drawn by a compass, and half of the "chord" line at its endpoints. The discussion that follows is an analysis and another way to bisect an angle.

For simplicity in the following analysis, the horizontal axis indicates where the angle is bisected. A line that bisects an angle into two equivalent angles is called an angle bisector line. [FIG 143]



How can the angle bisector line or a point on it be found so that the angle can be bisected?

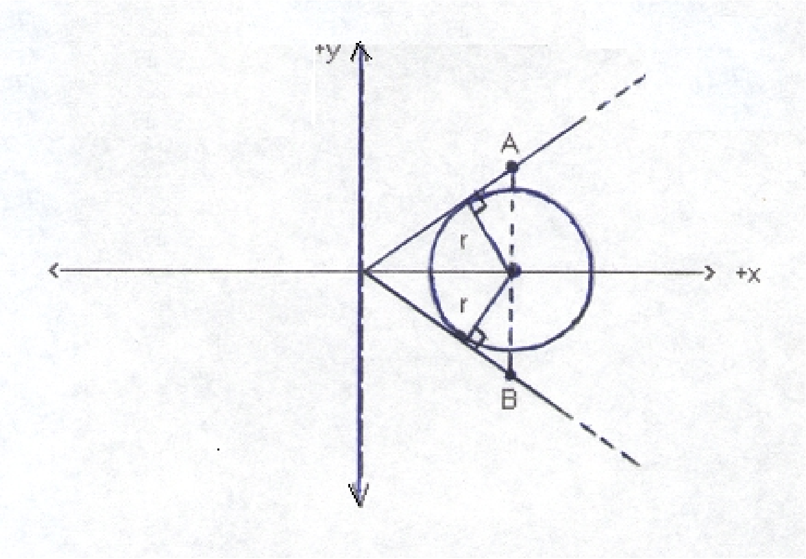
Observer the drawing below: [FIG 144]



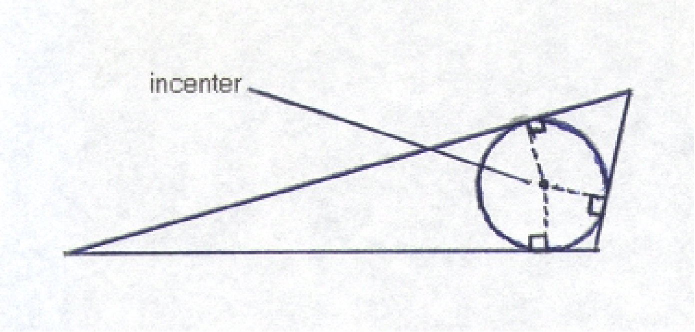
Points A and B are any equidistant (equivalent in distant) points from the vertex of the angle (¢). Here, each point has the same distance from the vertex at (0,0). The shortest distance between any two points, such as A and B, is always the straight-line distance between those points. This line is indicated and it crosses the angle bisector line perpendicularly creating two congruent (identical) right triangles having the same sides and angles. At half the distance on this perpendicular line is a point of angle bisection, that is, it is a point on the angle bisector line. The larger triangle construction is that of an isosceles triangle where the angle bisector line is the height of that triangle, and therefore, it intercepts the base side (here, between points A and B, which are of equal length) perpendicularly. If the triangle was not an isosceles triangle which has two equivalent sides, the angle bisector line would not intercept its opposite side perpendicularly.

In actual practice, if you are trying to find or draw an bisector line and a point on it away from the vertex point of the angle, and where an angle is to be bisected, the longer that the two equidistant adjoining sides of that angle are from the vertex, a greater accuracy can be achieved. Another point(s) of angle bisection can be used as a check of the angle bisector line.

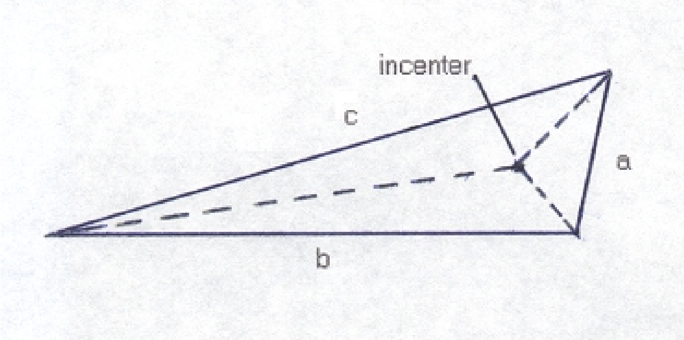
Now that we have found a point of angle bisection, it might be assumed, due to the above discussion, that the (perpendicular, from angle bisector line) indicated distance to point A or point B from the equidistant (from, or to the angles sides) angle bisector point is the shortest distance to the side of that angle. This is an incorrect assumption. In the drawing below, a circle is drawn between the two sides of the angle, and the center of this circle is at a point of angle bisection. The radius line of the circle will intersect the sides of the angle perpendicularly (at 90°), or vice-versa, since the angles sides will be "tangent" (meaning: slightly intercept at one point, but not go through) to the circle at the point of interception. The radius, being a straight line, is the shortest distance from the center of the circle to it's outer perimeter (ie. circumference), and therefore, it is also the shortest distance to the side of the angle. By observation, this perpendicular distance (ie. the radius) is the shortest distance since the other distance indicated is the hypotenuse (ie. the longest side) of the right triangle indicated. [FIG 145]



Any point along the angle bisector line can be the center of a circle whose perimeter (circumference) intersects the sides of the angle. Obviously, the farther that the center of the circle is from the vertex of the angle, the larger the circle, and therefore the larger the radius of that circle. At some point along the angle bisector line of an angle within a triangle, the corresponding circle's perimeter (circumference) will intercept the side opposite that angle. This is the largest possible circle that can be inscribed (scribed [write, draw, scribble, indicate]) within both a triangles angle and entire triangle with three angles. The center point of the largest inscribed circle within a triangle is called the **incenter** of the triangle. All three angle bisector lines will intersect at the (same) incenter point. The incenter is also (perpendicularly) equidistant (having the same distance) from all three sides of the triangle. The drawing below shows the incenter of a triangle:
[FIG 146]



The drawing below shows the angle bisector lines dividing the triangle up into three smaller triangles: [FIG 147]



The height of each of these smaller triangles is the value of the radius (r) of the largest inscribed circle. Remember that the height is perpendicular to the corresponding base (as seen in FIG. 109), and is the distance (here, it's equal to (r)) to the corresponding vertice. The incenter is the vertice point (and only common point) of all three angles that correspond to their base sides, which here, are the sides of the triangle. The total area of the triangle is equivalent to the sum of the areas of these three interior triangles:

$$A = A1 + A2 + A3$$

$$A = \frac{ar}{2} + \frac{br}{2} + \frac{cr}{2}$$: forms of At = bh/2 = br/2, combining fractions:

$$A = \frac{ar + br + cr}{2}$$ factoring out (r) from each term in the numerator:

$$A = \frac{r(a+b+c)}{2} = rs$$ s = (a+b+c)/2 = perimeter/2 = "semi perimeter", solving for (r):
(Extra note: This equation has an essence of Heron's formula for the area of a triangle when given its three sides.)

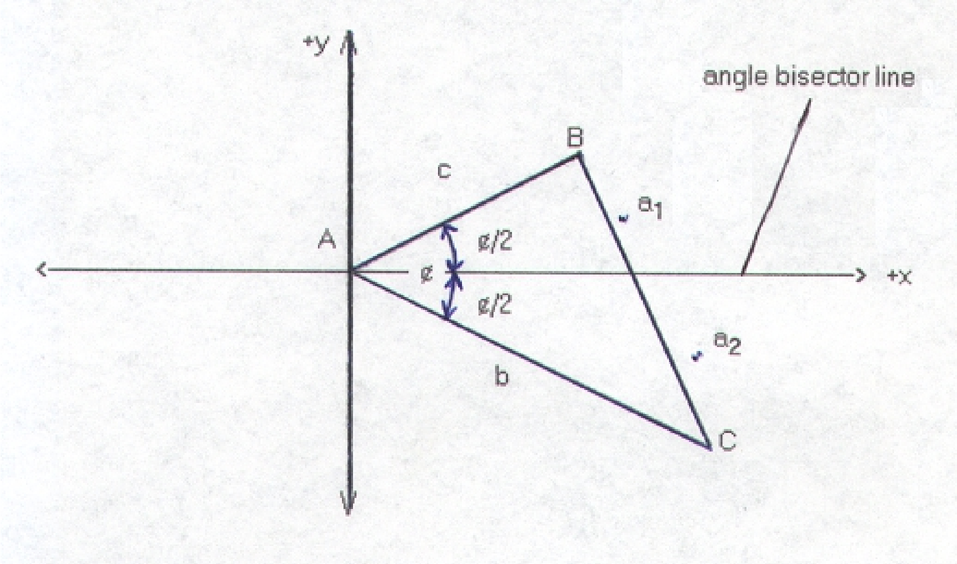
$$r = \frac{2A}{(a+b+c)} = \frac{2A}{\text{perimeter}} = \left(\frac{1}{s}\right)\frac{A}{1} = \frac{A}{s}$$: radius of largest inscribed circle in a triangle, and it's center is the incenter

Since (r) is effectively a height of a (internal) triangle as seen in the drawing, it can also be found using the height formula shown previously in this book. Be careful to use only use the half-angle values in the formula. The total area of a triangle can also be found using Heron's formula that requires just the three side lengths of triangle. Besides the angle bisector lines and incenter, there are several other well known defined lines and points within and outside a triangle.

[This space for book edits.]

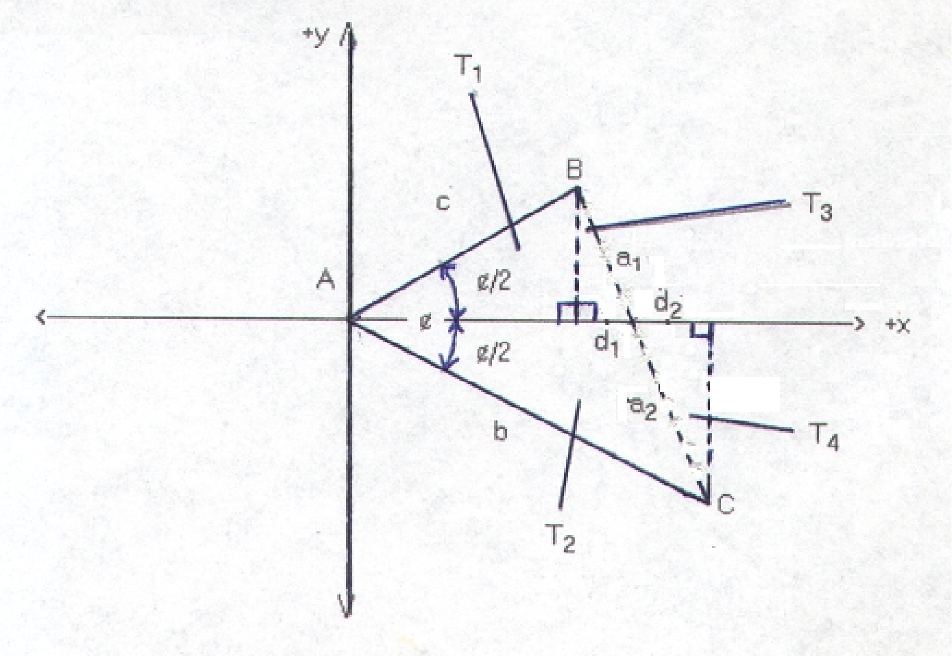
SOLVING A TRIANGLE USING AN ANGLE BISECTOR LINE

This topic shows that there are often some new, insightful and beneficial analysis methods to explore. At the point where an angle bisector line intersects (and not generally perpendicularly at 90°, and-or at half of) the third side of a triangle, it divides that side into two line segments whose ratio is equal to the ratio of the other two sides of the triangle. Observe the drawing below where (side a) = (a1+a2) is to be found: [FIG 148]



$\dfrac{a1}{a2} = \dfrac{c}{b}$ or $\dfrac{a1}{c} = \dfrac{a2}{b}$: note that (a1) and (c) are on the same side of the angle bisector line, and (a1) and (b) are on the same side also.

They are equivalent since these are ratios of corresponding sides of similar (right) triangles. This is verified below: [FIG 149]



Right triangles T3 and T4 are similar due to that they both have a 90° angle, and they also have the same vertical angle (across the angle bisector line), and therefore the third angle must be the same, and due to the complementary angle sum of 90°. They can also be verified as similar triangles due to the extension of the sides of one triangle, and the other parallel sides (some are coincidental or overlap, but they are still parallel lines).

Triangles T1 and T2 (which includes T4) are similar (right) triangles due to that they also have the same angles. Corresponding heights within similar triangles are known to have the same ratio as corresponding sides of those triangles. The heights, h1 of T1 ,and h2 of T2, are also the sides of T3 and T4 respectively. All corresponding sides of T3 and T4 are known to be proportional to each other since they are similar triangles. Summarizing this mathematically:

$$\frac{h1}{h2} = \frac{c}{b} = \frac{a1}{a2} = r$$

Side d1 of T3, and side d2 of T4 are corresponding sides of similar triangles and therefore have the same ratio.

$$\frac{d1}{d2} = \frac{a1}{a2} = \frac{c}{b} = r$$

You can solve for the value of the other "leg" (base side) of the two small right triangles (T3 and T4) by using ("right-angle") trigonometry. If you can find the value of d1 and d2, you can use the Pythagorean theorem to find the hypotenuse sides (a1 and a2) of these two small right triangles (T3 and T4). Summing up these two hypotenuse sides (a1 and a2) will yield the third side (side (a), between B and C) of the triangle being analyzed, and this is also the side opposite the bisected angle. The sum of d1 and d2 is equal to the base side (along the angle bisector) of T2 less the base side (also along the angle bisector) of T1. These base sides can be found using right triangle trigonometry. Once the sum of d1 and d2 is known, here's how to assign each it's proper value:

We now have the facts of:

Eq. 1: sum = d1 + d2 : sum = (base of T2) - (base of T1)

Eq. 2: $$\frac{d1}{d2} = r$$

Hence we have two simultaneous equations. From the second equation we can algebraically get:

$d1 = r\ d2$ creating a solvable equation by substituting this into the first equation:

$sum = r\,d2 + d2$ combining like terms, or factoring d2 from each term:
$sum = (r + 1)\,d2$ algebraically solving for d2:

$d2 = \frac{sum}{(r + 1)}$ d1 can now be found from the first equation:

$$d1 = sum - d2 = sum - \frac{sum}{(r+1)} = \frac{sum\,(r+1) - sum}{(r+1)} = \frac{sum\,(r+1-1)}{(r+1)} = \frac{r\ sum}{r+1}$$

The equations for (d1) and (d2) are nearly identical, except that the equation for (d1) includes a factor of (r), and this implies that d2 is longer, and it is since it is a side of a larger triangle construction as seen in the above figure.

You can also solve for the third side of a triangle using the **Distance Formula**. First find corresponding (x) and (y) coordinates of the endpoints of the third side of the triangle. These coordinates can be found easily by using right-angle trigonometry. In the drawing above, the coordinates of point B will be solved using trigonometry with T1, and the coordinates of point C will be solved using trigonometry with T2.

For point 1: From: $\cos ¢ = \text{adj.} / \text{hyp.} = x / h = x / r$ and $\sin ¢ = \text{opp.} / \text{hyp.} = y / h = y / r$:

$(x1 , y1) = (r1 \cos ¢ ,\ r1 \sin ¢)$: point 1, substituting the values for (r) which is the hypotenuse side (here, using side (b) or (c)) of the two right triangles:

Be sure to use only the half-angle values shown. (r), or radius, is equivalent to the hypotenuse of the right triangle, hence:

$(x1 , y1) = (b \cos ¢ ,\ b \sin ¢)$: point 1, b=length of side b

For point 2:

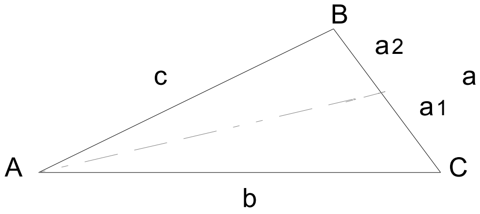
$(x2 , y2) = (r2 \cos ¢ ,\ -(r2 \sin ¢))$: point 2, therefore:
$(x2 , y2) = (c \cos ¢ ,\ -(c \sin ¢))$: point 2, c=length of side c

The negative sign is used since the angle is arbitrarily considered positive in sign or signless for this analysis (often, clockwise angles are considered negative in sign (ie. signed angles)), and (y) coordinates of points are negative in the fourth quadrant (or anywhere "below" the x-axis) Alternatively, you could remove the sign and consider the angle as a fourth quadrant angle (since the ending side of that angle is in the fourth quadrant) where the SIN ¢ values are known to be negative in value due to the signs of the co-ordinates of points in that quadrant. For example: p(5 , -2).

$$\text{distance} = \text{side } a = \sqrt{(x2 - x1)^2 + (y2 - y1)^2}$$

Below is an example of solving for the lengths of the line segments of a side (whose length is already known) intersected (and not necessarily perpendicularly) by an angle bisector line. This method shown does not directly utilize any angles, and it is essentially a simultaneous equation method.

In the drawing below, find the lengths of line segments a1 and a2 created at the intersection of side (a) due to the angle bisector line. [FIG 150]



$a = 3.606$, $b = 6$, $c = 5$

$3.606 = a = a1 + a2$ and from the facts about an angle bisector line and the bisected side:

$\frac{a1}{a2} = \frac{b}{c}$: side divided by an angle bisector line rule

$\frac{a1}{a2} = \frac{6}{5} = 1.2$ solving for a1:

$a1 = 1.2\ a2$ using substitution for a1

$3.606 = 1.2\ a2 + a2$: a solvable equation with one variable, combining like terms:
$3.606 = 2.2\ a2$ dividing both sides by 2.2 and switching sides:
$a2 = 1.64$

$3.606 = a1 + a2$
$3.606 = a1 + 1.64$
$a1 = 3.606 - 1.64$
$a1 = 1.966$

In about 350BC, an ancient and very famous Greek mathematician known as **Euclid** wrote the famous geometry book called "The Elements", in Alexandria Egypt. His book(s), or "volumes", which are still available today, translated to various languages, and it consisted of what was already known (but not yet derived, proved, or analyzed as much as Euclid provided) to the ancient world, and they also include his own monumental discoveries of which he is credited for such as the mathematical understanding of the length of an angle bisector line as shown below, and many other fundamental and useful concepts about geometry. It could be said that Euclid formalized and standardized mathematics and made it available for all others to learn and build upon.

The basic format for the length of an angle bisector line in a triangle is:

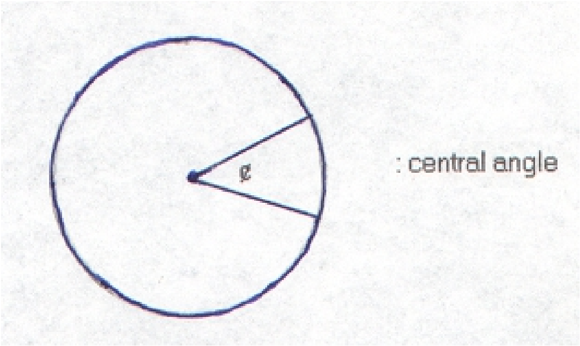
Given the 3 side lengths: side1, side2, side3 of a triangle: (ex. side_a, side_b, side_c),

And s = (side_a + side_b + side_c) / 2 : s = "semi-perimeter" = half-perimeter

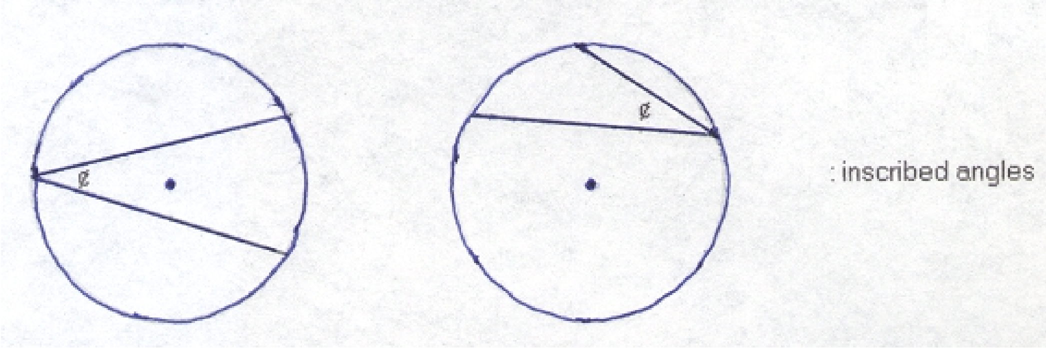
Length angle1 bisector line to side1 = $\frac{(2)\sqrt{s\ (s - side1)\ (side2)(side3)}}{(side2 + side3)}$: length of an angle bisector line to its corresponding side directly opposite that angle.

CENTRAL AND INSCRIBED ANGLES

Central and inscribed angles are formal names given to certain angles because of their position within a circle. A central angle has its vertex located at the center point within a circle: [FIG 151]



An inscribed angle has its vertex located at a point on the circumference of a circle: [FIG 152]



The portion of the circles circumference that subtends (being directly across or opposite to, or beneath) the angles vertex is called an arc (segment of the circles circumference). The angle and its corresponding arc are directly and proportionally (same fraction or portion of the whole or entirety) mathematically related. The greater the angle, the greater the corresponding arc length and vice-versa. For example, if the central angle doubles, its corresponding arc length doubles, This is very similar to that of any angle or triangle; the greater the angle, the greater the side length opposite that angle. Arc length can be calculated easily by setting up an equivalent portions (proportions) or equivalent fractions relationship:

From the ratio of the arc length (s) to the circumference (C, full arc), the corresponding angles have the same ratio value:

$$\frac{C}{360°} = \frac{S}{\text{¢}°}$$

: C = circumference of circle = an arc length of a 360° angle
: S = **ARC LENGTH** (of a central angle, or "centrally inscribed"). Letter S is used since it is technically a semi-perimeter or just a part of the entire circumference. See [FIG 153].
: ¢ is the central angle that subtends the arc S

Mathematically:

$$\frac{S}{C} = \frac{\text{¢}°}{360°}$$

An arc of a circle can be described in various ways such as:
A specific arc length, or the arc due to an certain angle. Ex. "an arc of 10°"

$$S = \frac{\text{¢}°\ C}{360°}$$

: **Arc Length**, this also shows that S is essentially a certain fraction of the circumference C.
Since radians = ((pi) /180°) (degrees) , degrees = (180° / (pi)) (radians) , mathematically:

S = ¢° C / 360° = (180°/(pi)) (radians) C / 360° = **S = C (radians) / 2 (pi)**
S =~ 6.281$\underline{8}$ (radians) C

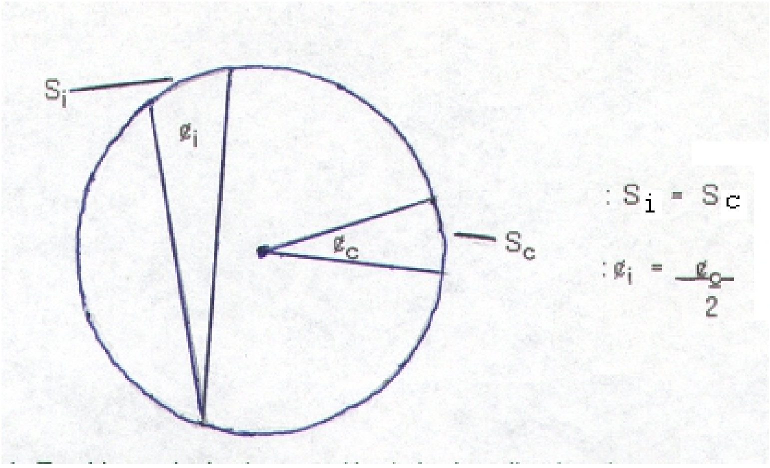
From the above formula for S we also have: S = ¢° / 360° C = ¢° (1 / 360°) C = 0.00027777$\underline{8}$ ¢° C

Extra: While at this topic, if the angle is 360°, then S = C. Since 360° = 2(pi) radians [radians are another unit angle measurement, besides degree units, that is discussed further ahead], and C = 2(pi)r, Substituting these values into the above formula, S = **arc length**, can also be shown to equal (¢r) radians.

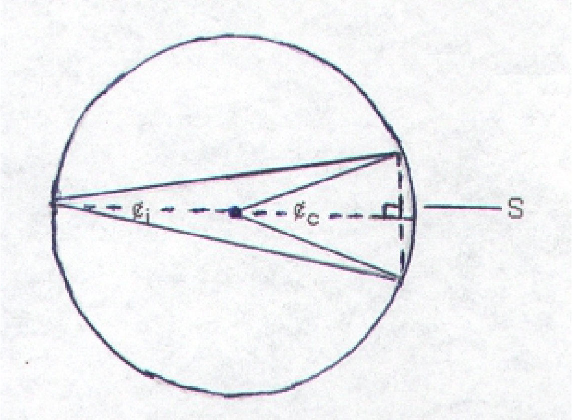
$S = \frac{¢\ (2(pi)r}{2(pi)} = (¢r)\ \text{radians}$

: r = radius length of the circle, ¢=radian angle (not a degrees angle)
: 1 radian or radian angle is the angle that creates an arc length equivalent to 1 radius length long. The larger the circle and its radius, for a certain or constant angle angle, the arc length will be larger. For a given circle with a certain radius, the greater the angle, the greater the corresponding arc length.

If the arc length (Si) subtending an inscribed angle (¢i) is equivalent to an arc length (Sc) subtending a central angle (¢c) of a circle with the same radius, diameter or circumference, that inscribed angle has a value of one-half (1/2) of that central angle: [FIG 153]

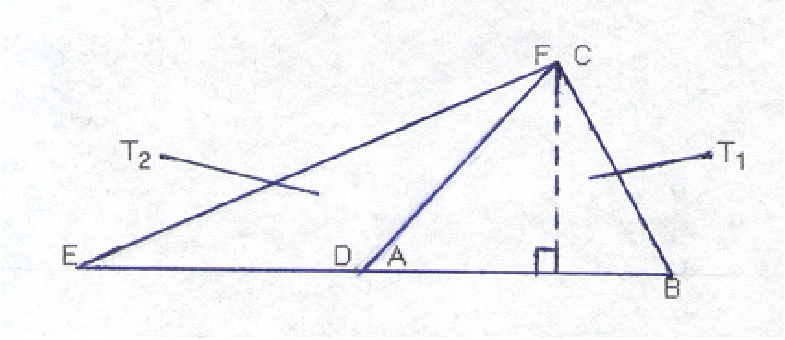


This will now be verified. For this analysis, the equivalent arc (length) of both the inscribed and central angle above are made to coincides at the same portion or arc length (S) of the circles circumference: [FIG 154]



Before the main analysis, if the vertex of each angle (¢i and ¢c) is considered as a center point of a circle, their circumferences will not actually coincide or overlap except for at 1 point only. The larger the radius of a circle, the slower the rate of change of the (y) values to the changes in the (x) values, and therefore, the larger the circle will be.

Also shown above is an angle bisector line that divides both angles in half, and a chord (a line that passes through two points on a circle's circumference; for example, the diameter is the longest chord possible for a circle) line that subtends (here, immediately below or across from) the arc of both angles. To broaden the analysis for use with triangles also, a more general representation using only the half angles is shown below where E = (¢i/2) and A = (¢c/2) : [FIG 155]



The vertex point of angles D and A is the center of the circle as shown in the previous figure, therefore lines ED = AB since they are radius lines of that same circle. It will be shown that angle E is half of the inscribed angle, angle A. By construction, line A to B, or line AB, equals line AC of T1 (triangle 1) since they are equivalent to the radius line of the same circle. Likewise, line DF and line DE of T2 (triangle 2) are also equal. Line EB is equivalent to the diameter of the circumscribing circle shown in the previous figure. Points B, C (and F, which coincides with point C) and E are points on the circumference of the same circle construction as shown in the previous figure above. Lines such as BC which pass through the circle and intersects it at two points on its circumference are often called a **chord line**. This word and term is based on that a piece of string was often called a chord, and of which can be pulled in opposite directions at both ends to made it straight like a line (ie., a chord line). The entire construction shown will create the two (equivalent in length) sides of angle D which adjoin or intersect to the base side EF (E to F) of an isosceles triangle (T2) construction. Note that angles B and C, and the line that joins their vertex point play no specific role in this analysis other than the fact that this line is a side opposite to both angles E and A whose numerical relationship is being found in this discussion. In short, D + A = 180° = E + F + D = 2E + D. Then: E + F = A or= 2E = A , and E = A/2.

Since $E = F$: base angles of T2 (an isosceles triangle)
$E + F = E + E = 2E$

$2E + D = 180°$: T2 interior angular sum of 180°

$D + A = 180°$: supplementary angle sum, therefore:
$D = 180° - A$ by algebraic substitution of D into T2 interior angular sum:

$2E + (180° - A) = 180°$ distribute (+1) to clear grouping symbols:
$2E + 180° - A = 180°$ after transposing +180° :
$2E - A = 0$ solving for E, transposing (-A): so add -(-A) = +A to each side :
$2E = A$ isolating E :

$E = \frac{A}{2}$: verification that an inscribed angle is half a central angle when the arc lengths have the same value

Since half-angles (of both the inscribed and central angles) were considered for this analysis, multiplying each angle by 2 to consider the full angle values, we find:

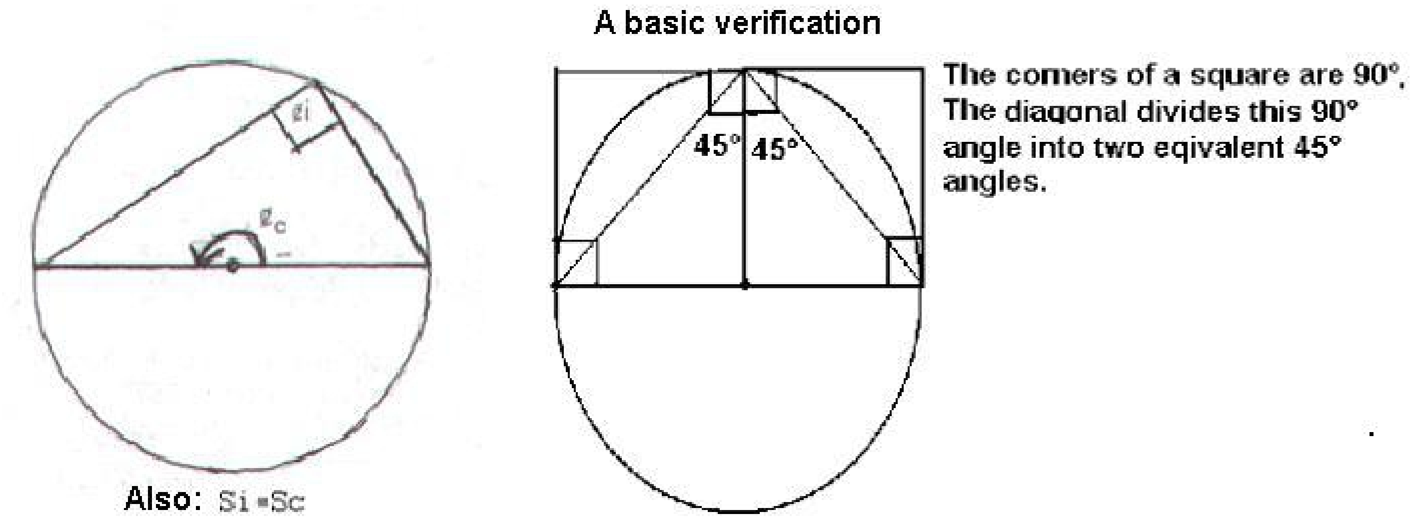
$(2E) = \frac{(2A)}{2} = A$ or: $¢i = \frac{¢c}{2}$: the central angle (¢c) is twice that of the inscribed angle (¢i) of a circle, and they both have the same value of arc length

Also, if not obvious in the above derivation, D is supplementary (sum to 180°) to A. The sum of (E + F) = (2E) is supplementary to D. Since 2E is supplementary to D, and A is supplementary to D, then 2E is equal to A (as expressed above where 2E = A). Extra note: Since the trigonometric functions are said to be non-algebraic (for example non linear mathematical relationship, non standard, not a standard algebraic equation), consider this example as to why:

sin (¢/2) /=/ (sin ¢)/2 : The sine of half an angle is not equal to half the sine of that whole angle.
For reference and to consider with the previous figure: sin ¢ = side opposite / hypotenuse

A general note is that if the base side of an angle doubles, that the corresponding height of it will also double, and this is the essence of making a similar and proportionally sided triangle. In both cases,the angle must remain the same value. For the analysis above, the inscribe angle is half that of the central angle, and this note will then consider the angle bisector line which happens to be the radius or diameter. If a base side of an angle doubles or increases by some factor (n) , then the corresponding height will also double or increase by that same factor (n) in a linear or proportional way, however, the trigonometric values of an angle do not increase in a linear or proportional way. If the base side increases by a factor of (n), but the height stays the same value, then the angle decreases by a factor of (n). In the above analysis for inscribed and central angles, the base side (here, the angle bisector line) of the central angle doubled to have the inscribed angle, but the height remained the same, and therefore, the angle is half that of the corresponding central angle.

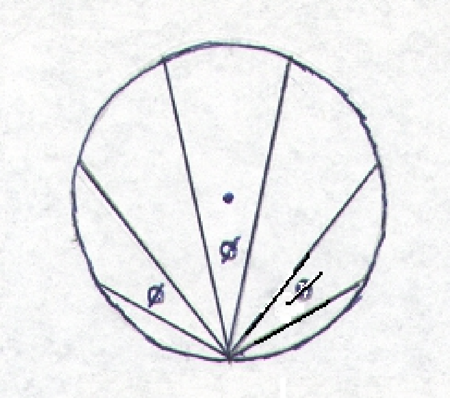
Because of the facts above, any angle inscribed within a semi-circle (half a circle) is a right (90°) angle: [FIG 156]



$¢i = \frac{¢c}{2} = \frac{180°}{2} = 90°$: ¢i and ¢c both have the same arc length of the circle, and: 45° + 45° = 90°
and here its equal to half the circumference

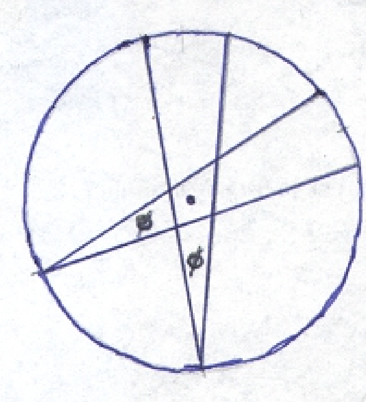
In the above drawing, another trigonometric fact can be easily pointed out here, and it is that the midpoint (shown as the center of the circle and which is equidistant from each point on the circumference) of the hypotenuse side of a right triangle is equidistant from all three vertices of that right triangle. This distance value is equal to the radius of the circumscribed (around, the triangle) circle. They hypotenuse of that right triangle is equal to the diameter of this circumscribing circle, and all three vertice (ie., "corners", where the sides meet or intersect) points of that triangle will coincide with points on the circles circumference.

It is now easy to understand that when you rotate an inscribed angle (or triangle) about the center of a circle that all the arc lengths will be identical. This is easily verified since the sides, chords and angles will still be the same values:
[FIG 158]

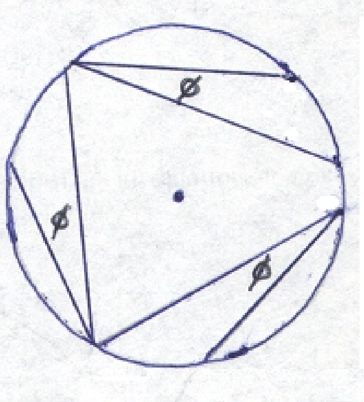


Likewise, the results would be the same if you rotated a circle which included one instance of that inscribed angle, as shown on the right side above.

It is easy to understand that if an angle is rotated about its vertex, that at any part in the rotation, the angle can only be the same constant value, however, as can be seen below, the sides of the angle are no longer the same. One might now ask, is the arc length of each (same) angle therefore different?. You might assume that it is bigger by what you know about angles in general; that the farther the opposite side is from the angle, then the larger that side (or chord of a circle) will be. For the circle construction shown below, this is not the case, and the arc and chord lengths are in fact identical when the inscribed angle is the same angle: [FIG 159]



These inscribed angles and arcs are equal.

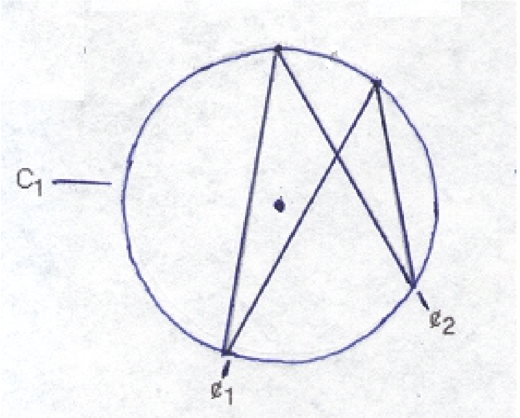


These inscribed angles and arcs are equal.

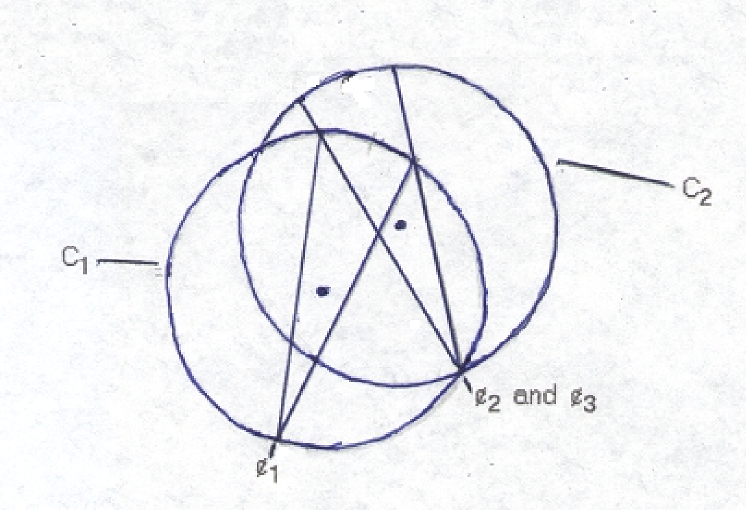
Here is a general verification that as long as the arc lengths remain the same, the corresponding (inscribed) angles are identical. Likewise, as long as the angles remain the same value, their corresponding arc lengths will be identical. For both cases, the circles must be equivalent in size.

Though chord length and its corresponding arc length are not proportional in value, chord length and the central angle are proportional in value. If one changes by a factor of (n), the other will also change by that same factor. Arc length and the central angle are also proportional in value.

In the drawing below, it will be shown that ¢1 = ¢2, and therefore their corresponding arc lengths S1 = S2. Also note that the angle bisector line of ¢1 happens to pass through the center of the circle. For the analysis, both arcs (arc lengths) were made to overlap exactly and can therefore only be the same length: [FIG 160]



In the drawing below, C2 is identical to C1 (as above) and contains the same exact angle as ¢1 of C1. This circle (and angle) is rotated slightly and is offset from C1 in such a way that it's angle (¢3) coincides exactly with ¢2 of C1. That is, ¢3 = ¢2: [FIG 161]



Since: ¢1 = ¢3
And: ¢2 = ¢3 we can conclude:

¢1 = ¢2 : verification : When the arc lengths are the same, the angles are the same. The angles being compared must be both central angles or both inscribed angles.

It must be noted that this verification is more of a "verification by visualization" rather than an "exact" algebraic method.

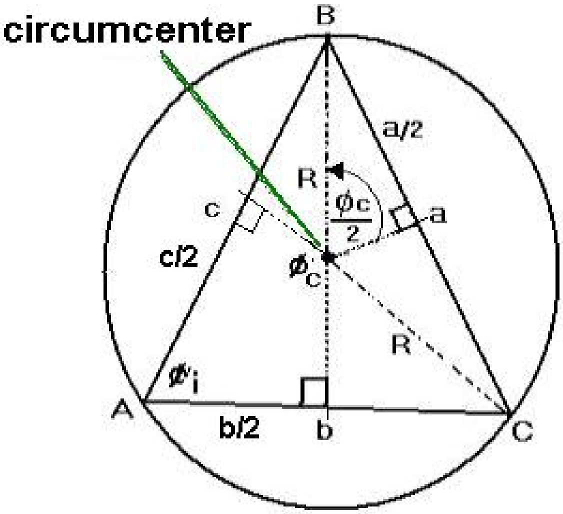
One method to tell if two angles are the same, without using a protractor, is that the distance between a selected point on one side of an angle, and a selected point on the other side of that angle, will be the same for both angles. These selected points along both angles are usually located to be at the same distance from the vertex of each angle. Though the angles might be found to be identical in size, we don't know their specific value without further calculation. As seen with inscribed and central angles having different angle values and yet having the same chord or arc length of a circle, you therefore cannot tell the actual size of any angles that have equivalent chord lengths, opposite [corresponding] side lengths or arc lengths. Other data ,such as side lengths corresponding to that angle must then be considered. Also, for a given circle, each arc length will have a unique corresponding central angle, and an inscribed angle equal to half of it. It is incorrect to consider that chord length and arc-length are (strict) mathematically proportional, other than that they will both increase or decrease by a different factor value each.

Since the incenter of a triangle was already discussed, this is a good place to introduce the **circumcenter** of a triangle. Notice that each side of the (circumscribed) triangle inscribed within a circle [as seen in FIG 157] is a chord of that circle. The perpendicular lines at half of each triangle side (or chord of a circle) will all pass through the center of this circle. This is similar to where tangent lines are always perpendicular to the radius line and vice-versa. This point is called the (or sometimes the "circumference") of the triangle. This point is equidistant from each vertex (located on the circles circumference) of the triangle. Since the excenter of a right triangle lays upon the hypotenuse and in the middle of it, it is relatively easy to see and-or understand that the radius (R) of the circumscribing circle for a right triangle is equal to half the hypotenuse side of a right triangle:

$$R = \frac{c}{2} = \frac{\sqrt{a^2 + b^2}}{2}$$

: radius of circumscribing circle for right triangles,
(equals half of the hypotenuse side, here, side (c))
The illustration is also one way to create and draw a right angle.

This formula below is be used for non-right triangles that are circumscribed by a circle. For acute triangles, the excenter will be within the triangle, and for obtuse triangles (have an ¢ > 90°), the excenter will be outside the triangle. Here is a derivation of a formula for the radius of the circumscribing circle of any triangle: [FIG 162]



The dotted lines indicated in the above image are (side) median lines that are perpendicular to the sides, and intersect at the circumcenter point. Perpendicular median lines do not generally intersect the opposite and corresponding vertex point in a triangle, and are therefore not the same concept as angle bisector lines. Median lines are given a more general discussion after this discussion.

$$A = \frac{¢c}{2}$$

: angle A is an inscribed angle and is equal to half of the corresponding central angle (¢c) which has the same arc length. These angles will have the same trigonometric function values.

$$\text{SIN } A = \text{SIN}\left(\frac{¢c}{2}\right) = \frac{\text{side opposite}}{\text{hypotenuse}} = \frac{\frac{a}{2}}{\frac{R}{1}} = \frac{a}{2R}$$

algebraically:

$$R = \frac{a}{2 \text{ SIN } A}$$

likewise:

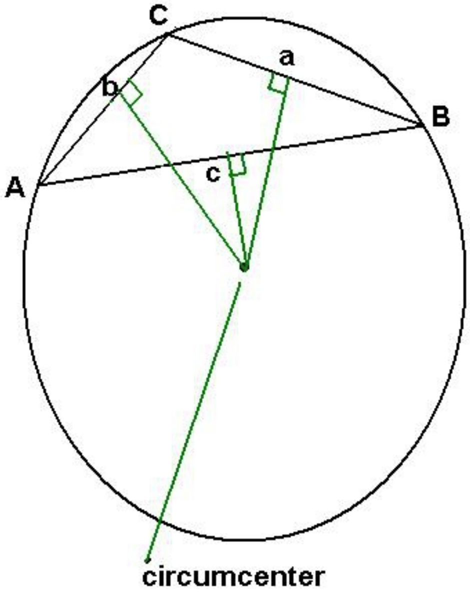
$$R = \frac{a}{2 \text{ SIN } A} = \frac{b}{2 \text{ SIN } B} = \frac{c}{2 \text{ SIN } C}$$

: Radius of a circumscribing circle of a triangle.
(when C=90°, SIN 90° = 1, and we see that R = c/2)

Note also, for example, that SIN A = $\frac{a}{2R} = \frac{a}{D}$: D = diameter of the circumscribing circle.
Also, for example: D = a / SIN A = b / SIN B = c / SIN C

A generalized formula is: $\text{SIN } ¢ = \frac{\text{corresponding side (or circles chord) to that angle}}{\text{diameter of the circumscribing circle}}$

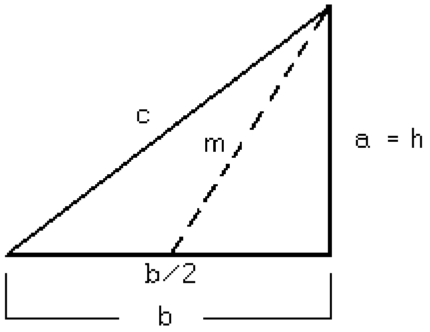
Another viewpoint of the circumcenter point of a triangle is this: If you have a circle and draw any triangle (or any other circular polygon [multi-sided structure]) within it that has all of its vertex points on the circumference of that circle then that circle is said to be a circumscribed circle to, around or about the inscribed triangle. The center point of this circumscribing circle is called the circumcenter of that triangle, and the radius of that circle is called the **circumradius** of that triangle. This circumcenter point can be found at the intersection of the perpendicular lines drawn from the center points of the three sides of that triangle. Due to this construction, each vertex point of the triangle will be at the same distance of: (r) = circumradius from the circumcenter of that triangle. [FIG 163]



Since the triangle in the above figure has an internal angle greater than 90°, it is an obtuse triangle and its circumcenter is a point outside of that triangle. For a right triangle, its hypotenuse side will coincide with the diameter of the circumcircle. For an acute triangle, the cicumcenter will be a point within the triangle.

MEDIAN LINES

A median (m) is a line within a triangle from half (ie. the center or middle (median)) of a given (base) side to the vertex of the corresponding angle opposite that side. In general, median lines are not perpendicular to the base side it intersects and-or originates from and goes to the corresponding vertex point. Any one of the three median lines of a triangle will divide the area of the triangle in half since a triangle with the same height but only half the base will have half as much area. [FIG 164]



The height and the base are the same for both of the two smaller triangles created. Since each has half the base of the original triangle and the same height as the original triangle, the area of each is half of that of the original triangle. Since area = A = (bh)/2 , area is directly related to the base side. If the base is divided by 2, area is divided by 2.

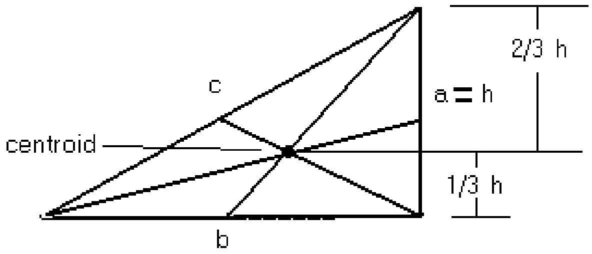
From: $A = \frac{bh}{2}$: Area of any triangle if we divide the base by 2, we need to do it to the other side of the equation also:

$$\frac{A}{2} = \frac{\frac{(b)\,h}{(2)}}{\frac{2}{1}} = \frac{bh}{4}$$

: area of each new triangle that was divided by a median line. This is the same as dividing the area of the original triangle (using it's values) by 2. If the base (b) or height (h) of a triangle is divided by multiplied by a factor (n), the area (A) will also be divided by multiplied by (n). In the formula shown here, (b) and (h) are the values of the original given triangle.

The three medians within a triangle will intercept at a point called the **centroid** of the triangle. Since each median line divides the area in half, the centroid is essentially a "balance point" or center of mass. That is, if a triangle shaped construction was physically constructed entirely of the same material and of the same thickness throughout its entirety, the triangle can be balanced at this centroid point. The centroid point divides a median line into a 2 to 1 ratio = 2:1

The three medians also divide the triangle into three smaller triangles, each being one-third (1/3) of the total area of the entire original triangle. Each of these smaller areas has a side of the original triangle. This side will then be the base side and the medians bounding or subtending that base side will define (be the sides of) the base angles at that side of that internal triangle. Also, do not assume that these are similar triangles. The centroid is located at one-third (1/3) of the way along the median line from each given (base) side of the original triangle. The other segment of each median line is then twice as long since: 1 - (1/3) = (3/3) - (1/3) = (2/3). [FIG 165]



Notice the area of the small triangle created which has (b) as its' base side. Since the base is the same as the base of the total area of the given triangle, to equal (1/3) of the overall area, the height side must then only be 1/3 of the total height. The same can also be said about the other two smaller triangles. Half the area of each of the three small internal triangle constructions is: (total area/3)/2 = (1A/6), or one-sixth of the total area of the original triangle construction, therefore, any two of these smaller areas sum to: (1/6)+(1/6) = (2/6) = (1/3) of the total area, and any three (in particular, those adjoining each other) smaller areas sum to (1/6 + 1/6 + 1/6) = (3/6) = (1/2) of the total area of the triangle.

An ancient mathematician known as Euclid is credited with mathematical understanding of the length of a median line by using the concepts of basic geometry with triangles, and here is a general format for the length of a median line:

$$\text{median length to side1} = 0.5 \sqrt{2\ \text{side2}^2 + 2\ \text{side3}^2 - \text{side1}^2}$$

: **length of a media line in a triangle**
Formula credited to Euclid
of Alexandria city in Egypt.
Note: Can also use 1/2 for 0.5

The formula for the median line is based on the Pythagorean Theorem, Apollonius Theorem, and Parallelogram Law. The derivation is not overly difficult and it is available elsewhere.

As a relatively easy verification to the above formula, you can use an equilateral triangle with sides say of 1 unit, and which also has there internal angles of 60°. Here s1 = s2 = s3 = s, and the above formula will simplify to:

$$\text{median length to any side s} = 0.5\sqrt{3\ s^2} = 0.5\ \sqrt{3}\ \sqrt{3^2} = 0.5\,(1.732051)\,s = 0.86602\underline{5}\,(1) = 0.86602\underline{5}$$

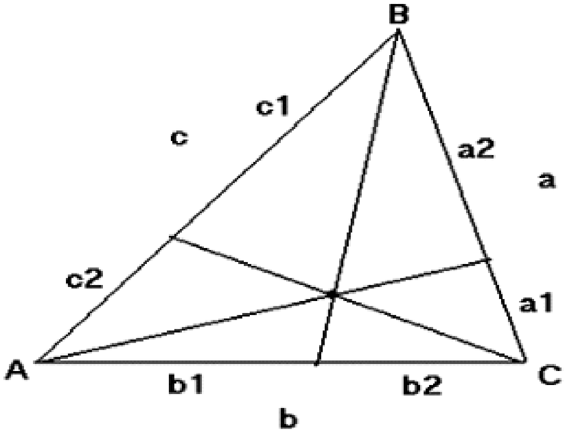
The **Apollonius Theorem** is usually something mentioned in more advanced geometric studies, and it states that given the three sides of a triangle, the sum of the squares of any two sides will equal the sum of the squares of half the third side, plus the square of the median line to that third side.

The **Parallelogram Law** states that the sum of the squares of all the sides of a (four sided) parallelogram will equal the sum of the squares of its two internal diagonal lengths. A parallelogram, being somewhat like a rectangle, will also have two pair of equal sides, and the two diagonals will also be an equal valued pair. If the parallelogram is a rectangle, and if we divide each side of this equation by 2, we could have the familiar Pythagorean Theorem.

With the concept of median lines, the lengths of both (1/3) and (2/3) of it can be found geometrically.

Even though many of these formally defined (interior and-or exterior) triangle lines such as medians, altitudes, and angle bisectors have been known about and studied for hundreds of years previously, any and all lines from a vertex of a triangle to the opposite side (even if extended) of that angle are more recently and formally called a **cevian** ("chevian") line. In 1678, **Giovanni Ceva** published and popularized a theorem and formula for the ratio of the lengths of a triangles side segments intercepted by cevians. **Ceva's Theorem** states that given a triangle with 3 interior and-or exterior lines that intersect at a concurrent (common), interior or exterior, (cevian) point, the product of all the 3 ratios of each two line segments on each side of that triangle is equal to 1. The derivation for this theorem is based on the ratios of the internal

triangle areas. The numerators and denominators of these line segments ratios can be determined by going in a clockwise or counter-clockwise direction. There are many other interesting facts and theorems created by others about cevian lines, unique points, triangles and circles, and are very useful for more advanced geometry studies. The following image shows three random cevian lines that intersect at a common point:
[FIG 166]



$$\frac{a1}{a2}\,\frac{b1}{b2}\,\frac{c1}{c2} = 1$$

$$(a1)(b1)(c1) = (a2)(b2)(c2)$$

$$\frac{a1}{a2} = \frac{(b2)\,(c2)}{(b1)\,(c1)}$$

$$a1 = a2\,\frac{(b2)\,(c2)}{(b1)\,(c1)}$$

$$a2 = a1\,\frac{(b1)(c1)}{(b2)(c2)}$$

If your making some kind of triangle-like analysis and-or construction with internal lengths or beams that intercept at some common point, the above equations will be very useful. For the first equation to be equal to 1, it implies that the numerator is equal to the denominator, and for here: (a1 b1 c1) = (a2 b2 c2)

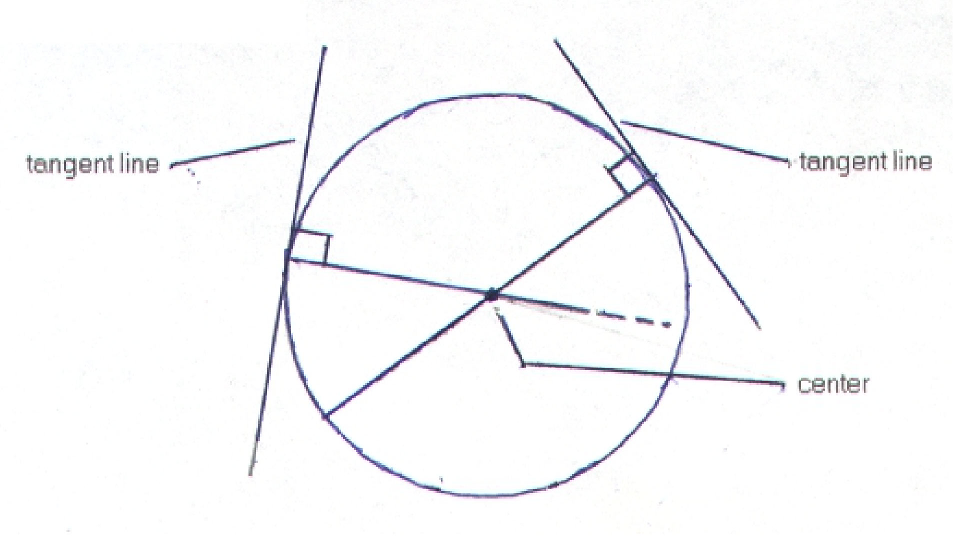
CONSTRUCTING A PROTRACTOR

Below are several methods that you can use as guides for constructing a homemade protractor, however, if you already have access to a protractor, you can use it as a basic template (pattern). The protractors below have a scale from 0° to 90°, hence a "single quadrant protractor". If you want, you can make one with a scale from 0° to 180° (a semi/half-circle protractor), or from 0° to 360° for a full circle protractor. Common protractors have indicated divisions or "graduations" on its' scale as small as 1°. A "rough or close estimate" protractor might only have divisions as small as 5°. You can draw your protractor on paper, cut it out, and then glue it onto a stiff paper or wood backing. Mentioned previously, if you draw an equilateral triangle and divide one side in half, you can easily construct a triangle with angles of 30°, 60°, and 90°. You can then, for example, bisect the 30° degree angle to 15°, and then the 15° angle into three 5° angles.

Method 1:

Draw a quadrant of a circle (typically) by using a device called a compass. A compass is basically anything that can assist you drawing various sized arcs or circles by adjusting the distance (essentially a radius distance) between its two indication or scribing points. To encompass something is to surround or encircle it. If you do not have a compass, you can use anything circular, such as a bowl or cup as a template to trace around. In general, the larger the radius of this circle the more accurate your protractor can possibly be, and therefore, the more accurate the drawn or measured angles The larger you make the protractor, it can also be more precise, perhaps allowing measurements as small as 1°. A radius of about 4 inches is good to start with. Measure the radius of this circle for it will be used in the calculations below. With a compass, the center of the circumscribed circle is clearly marked, otherwise, use something like the following method to find the circles center so as to then find the circles diameter and radius:

A line that intercepts a circle, or other curve, at only one point is called a tangent line, and the line is said to be tangent to the circle. The diameter line or radius line extended (perhaps with the aid of some right-angle, perpendicular guide, or just a plain ruler) to this (tangent) point will always be perpendicular (90°) to this tangent line. Likewise, tangent lines are always perpendicular to the radius line extended to the tangent lines point of interception with the circle. To find the center using a diameter line, draw a (or several) tangent line and then draw a perpendicular (diameter) line to it. Measure and divide the diameter line in half to find the value of the radius and the location of the center. To find the center using the radius line method (no measurement needed), draw another tangent and radius line. The point of interception of both radius lines is the center of the circle: [FIG 167]



Optionally, to find the center of a circle, you can find the circumcenter of a triangle that has all three of its vertices on the circumference of this circle. The circumcenter of the (circumscribed) triangle will coincide (be the same point) with the center of this circle. The circumcenterof a triangle was previous discussed in this book, and if the triangle is a right-triangle, the center of the circle is at half of the hypotenuse side of an inscribed right-triangle.

For each division on the protractor's angle scale, we need to first find the point on the circles arc (portion of its perimeter or circumference) where the ending or terminal side of the angle intercepts it. With this method, it takes two calculations to find the co-ordinates of each point. Also, in order to properly place an angels location point onto your drawing and angle scale, you will need a guide device (ie. something with "square" or perpendicular sides) to draw (perpendicular) lines at right angles with respect to each other.

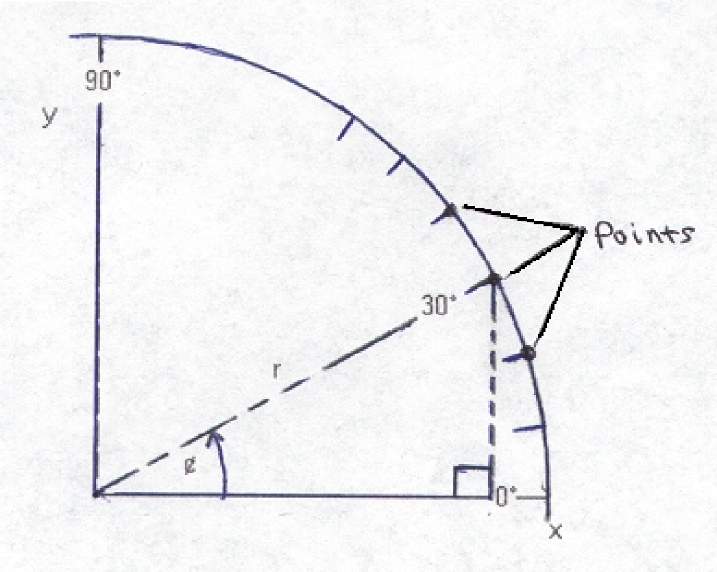
From : $\text{SIN } ¢ = \frac{x}{r}$

$x = r \text{ SIN } ¢$: x coordinate of point. If $r = 1$, then $x = \text{SIN } ¢$

From : $\text{COS } ¢ = \frac{y}{r}$

$y = r \text{ COS } ¢$: y coordinate of point. If $r = 1$, then $y = \text{COS } ¢$

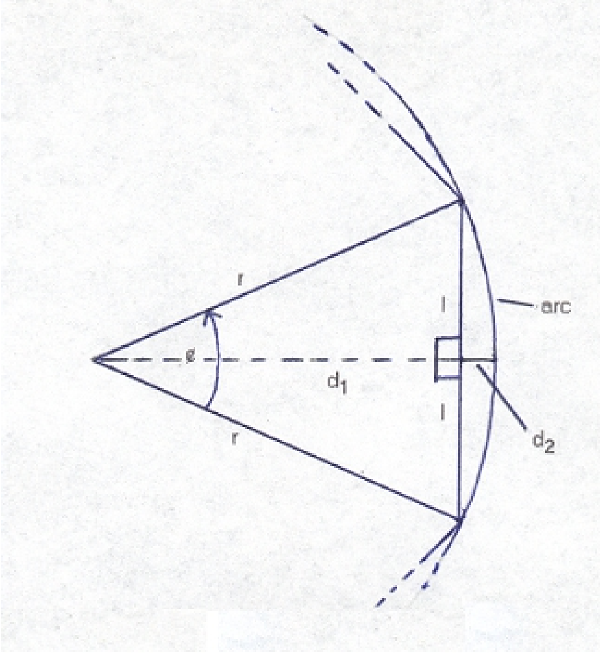
Once you find the co-ordinates of a point, draw a small line segment at the circle's or protractor's edge so as to indicate the angle. This line segment essentially lays on the radius line or angles terminal side at that point. [FIG 168]



Method 2:

If you already drew a small angle on your circular protractor, say 5° or 10°, you can trace it and use it as a template for the remaining angles on the scale of your circular protractor. Simply align the initial side of the template angle's with the terminal side of each successive angle drawn on your protractor, and then indicate (scribe) this new point. Another similar method is to simply trace the angles corresponding arc of the circle and use this "length of arc" as a guide to create similar arcs around your protractor. If you like, rather than use the arc, you can use the straight-line (ie. a chord line) distance between the end points of the arc, and drawing each next chord line segment to indicate additional angles.

While on the subject of arcs and chords, below is a description of how to calculate the chord length beneath the two endpoints of an arc of a circle: [FIG 169]



To make the analysis simple, the angle is bisected into two half-angles, creating two equivalent right-triangles. In the formulas below, use half the angle in question.

From : $\text{SIN ¢} = \frac{\text{opp.}}{\text{hyp.}} = \frac{l}{r}$: here, l is half the length of the chord

$l = r \text{ SIN ¢}$: be sure to use the half-angle value only

Twice the length of (l) is the (chord) length (L) of the line-segment between the endpoints on the arc:

$L = 2l$: Length of the entire line segment "beneath" the arc.
Such a line segment is called a "chord" of a circle.
Note that the (radius) line drawn from the center of the
circle to the midpoint of any chord will be perpendicular
to that chord line. If this line is extended to the circumference
of the circle, it divides the arc in half, and a line tangent to
the circle at this point will be parallel to the chord.

Additionally, if you want to know the length of d1 and d2:

From : $\text{COS ¢} = \frac{\text{adj.}}{\text{hyp.}} = \frac{d1}{r}$

$d1 = r \text{ COS ¢}$: be sure to use the half-angle value only

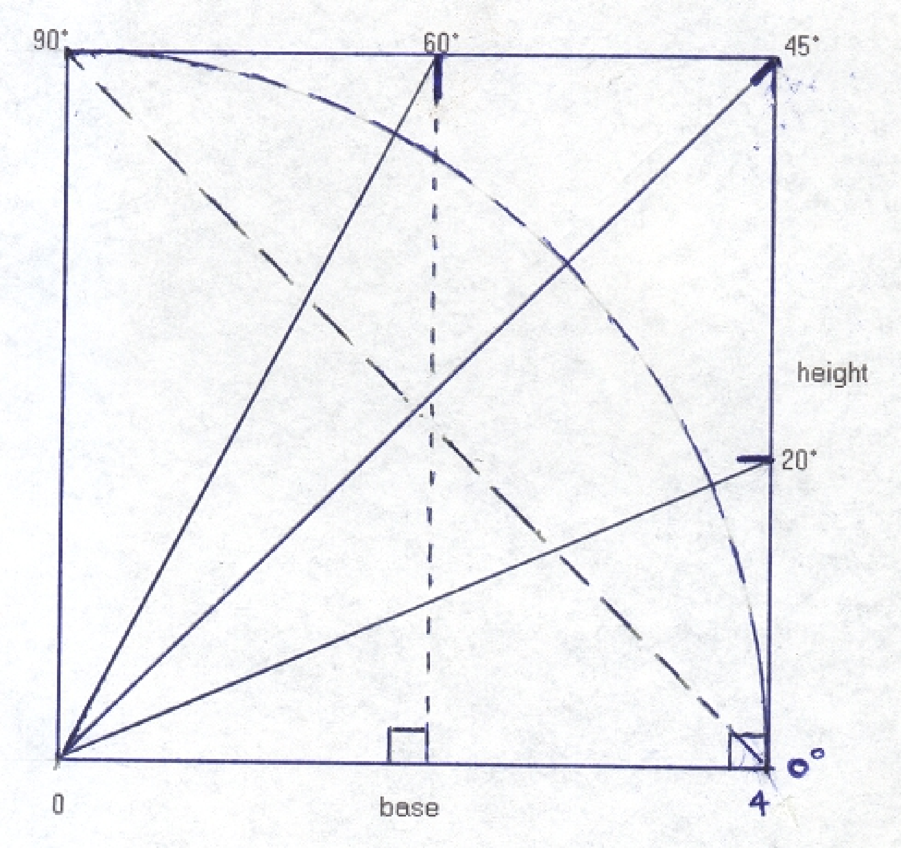
Since: $r = d1 + d2$ therefore:

$d2 = r - d1$

Method 3:

This method shows how to make a protractor with a straight scale rather than an arced or curved scale. However, due to that this scale would very large at the higher angle end past 45°, the scale will effectively be "bent" at a right angle to overcome this potential problem. Due to this modified construction, you now have two other design choices after creating this square scale, as indicated in the drawing below, of either a curved or circular scales, or a triangular scaled protractor.

To construct this protractor, draw a square, perhaps 4 inches on each side. The lower left corner will be the vertex of all the angles. The scale will be along the right side for angles from 0° to 45°, and along the top for angles from 45° to 90°. Each angle is calculated using the concepts of right triangles: [FIG 170]



For angles along the right side:

From : TAN ¢ = $\frac{\text{opp.}}{\text{adj.}} = \frac{\text{opp.}}{4"}$

opp. = 4in. TAN ¢

Ex. For an angle of 20°:

opp. = 4" TAN 20° : the side opposite is equivalent to the height of a right triangle, and 4" is the base side here.
opp. = 1.4559"

This must be done for each angle due to that there is no linear relationship between the side opposite or height of an angle and the adjacent or base side. The only exception to this is that say if you double the base side of a triangle, to make a similar triangle the height is likewise doubled or increased by the same factor. For similar triangles the ratio of all corresponding sides of the two triangles is the same value, and this is due to that each side is a factor increase or magnification of the other. Similar triangles are said to be proportional to each other.

If for example you find that 5° would be a height of 1inch, you cannot use a height of 2 inches for 10° and so on.

Converting the fractional part of this value to a number of 32nds or 64ths of an inch:

Since: $\frac{1"}{32} = 0.03125"$ dividing the fractional part of 1.4559" by this fraction (1/32) of an inch:

$$\frac{0.4559}{0.03125} = 14.6 \text{ , (one) thirty-two seconds of an inch}$$

If you need some more clarification on the result, consider this:

$$\frac{\frac{0.4559 \text{ in.}}{1}}{\frac{1\text{in.}}{32}} = \frac{0.4559 \text{ in } (32)}{1 \text{ in.}} = 0.4559\ (32) = 14.6$$

Or by using proportions:

$$\frac{1 \text{ inch}}{32,\ 32\text{-seconds}} = \frac{0.4599 \text{ inch}}{x\ \ 32 \text{ parts}} \quad : \ 1,\ 32 \text{ seconds} = 1 \text{ of } 32 \text{ parts} = 1/32$$

$$x\ \ 32\text{-seconds} = \frac{0.4599"\ (32\ 32\text{-seconds})}{1"} = 0.4599\ (32\ 32\text{-seconds}) = 14.6 \text{ thirty-two seconds}$$

To convert a number of inches to its equivalent number of thirty-two seconds, we see that you can simply multiply by 32. Using the same reasoning, to convert inches to sixty-fourths, simply multiply by 64.

Using the results above, indicate the 20° angle at 1 inch and 14.6 thirty-two seconds high on the right side.

For angles along the top side:

From : $\text{TAN ¢} = \frac{\text{opp.}}{\text{adj.}} = \frac{4}{\text{adj.}}$

$$\text{adj.} = \frac{4"}{\text{TAN ¢}}$$

Ex. For an angle of 60°:

$$\text{adj.} = \frac{4"}{\text{TAN } 60°} = 2.3094"$$

After converting the fractional portion to its equivalent value with units of thirty-two seconds of an inch, the angle should be indicated at 2 inches and 9.9 thirty-two-seconds rightward on the top side. This is possible on the top side since the top and bottom lines are parallel to each other and both are perpendicular to the left side line.

For a triangular shaped protractor, you can first create the square shaped protractor and draw a diagonal line from the opposite corners so as to create the triangle shaped protractor, otherwise, if it is directly created, you must still calculate each angle and the corresponding height since there is no linear relationship to use a specific amount of distance or length. In the case of the circular protractor, the chord lengths beneath an arc of a certain amount of degrees will all be the same length and distance from the angle vertex and-or center of the circle.

If needed: If the initial square shape is 4 inches, the diagonal (D) length within that square from corner to opposite corner can be found using the Pythagorean Theorem:

$D = \sqrt{4^2 + 4^2} = \sqrt{32} = 5.65\underline{7}$ inches : for a square, the diagonal is always: $D = \sqrt{2\ s^2}$

$D = \sqrt{2}\ \sqrt{s^2}$

$\mathbf{D = 1.41421\underline{4}\ s}$

1 sixteenth of an inch = $\frac{1 \text{ in}}{16} = \frac{(1)}{(16)}$ in = 0.01625 in

5.657 in = 5 in + 0.657 in = 5 in + (0.657in) (1 sixteenth / 0.0625 in) = 5 in + $\frac{(0.657)}{(0.0625)}$ sixteenths =

5 in + 10.512 sixteenths

Extra: Mentioned in this book is that an inscribed angle with its vertex on the edge (ie., it circumference) of a circle, and which has a chord opposite of it equal to the diameter of that circle, then that inscribed angle is a 90° or right angle. Half of this angle will create a 45° angle. A square also has four internal 90° angles.

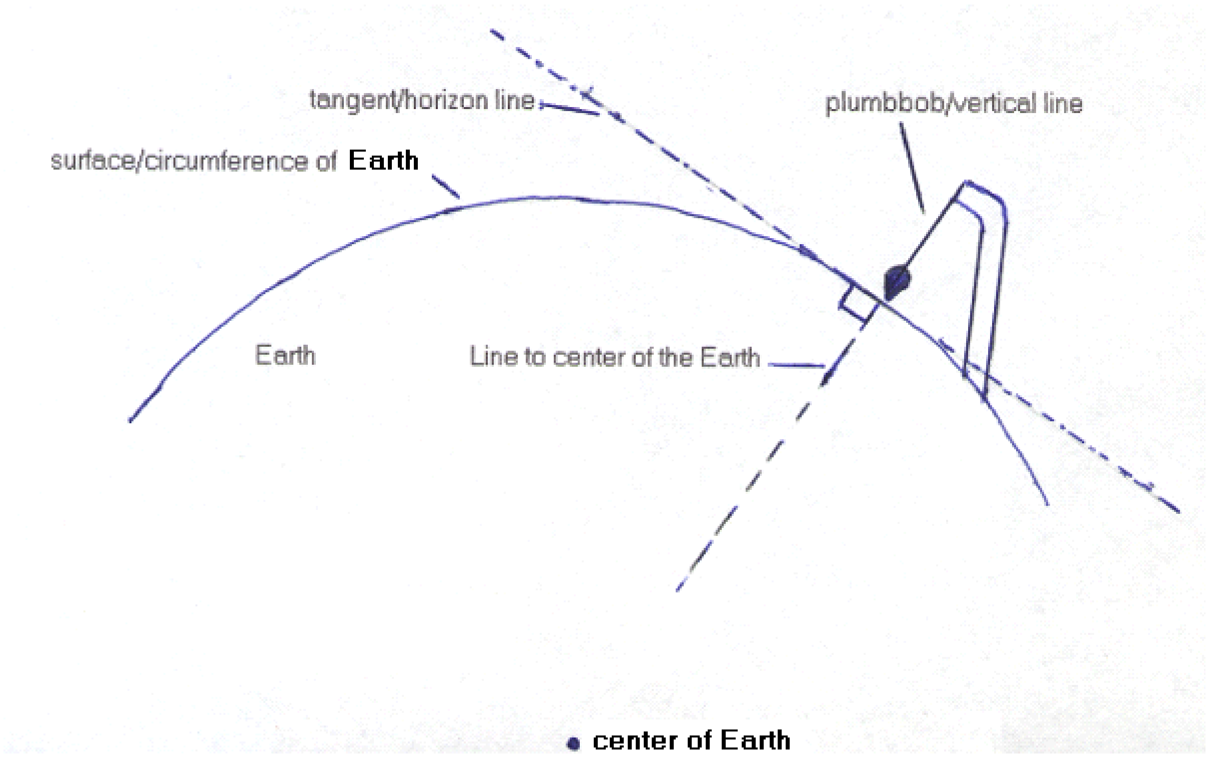
CONSTRUCTING A DEVICE TO MEASURE ANGLES FROM THE HORIZONTAL

A quadrant (a device to measure up to a 90 degree angle), inclinometer (an angle measuring device), astrolabe, sextant (often used for celestial (stars, space,"heavens") navigation which is often used for ship navigation by stars), theodolite, torquetum ["torketum", a device specifically made for celestial star coordinates mapping], or even a protractor-like device are ancient devices which can measure either vertical and-or horizontal angles such as the angle to the top of a building, a mountain, or the angle a star or other celestial object that is "above" or with respect to your (local) horizon or level (technically, a tangent line to the surface of the Earth at a location, and which is perpendicular to a radius line to the center of the Earth). As shown ahead, it basically consists of an angled scale, such as a protractor, and a pointer, aiming or sighting indicator. The sighting indicator device can be a straight and narrow flat surface. For the sextant to work properly it must always be level (parallel) to the ground since the angles will be measured with respect to the ground or any other line or surface near to, and essentially parallel to, the ground or horizon. For additional aid you can place the device on a level surface at the top of a sturdy tripod stand. Once the protractor scale is level (set, calibrated, or indicated to 0° when the sighting board or plane is parallel to the ground or horizon (horizontal line)), the actual object is "sighted" (aimed or pointed to) by the measuring instrument and the angle will be indicated on the protractor that is located alongside the edge of the sighting device. A figure of such a device will be shown further below in FIG 173.

With the aid of a plumb-bob device, see FIG 169, the proper positioning, calibrating or leveling of the protractor is not a concern, that is, it will effectively take place automatically after the calibration (proper initial setup for use) of the device has taken place. The sighting device will remain "fixed" (does not rotate about the vertex to measure the angle), and the protractor effectively rotates vertically. The word "plumb" (or "plum") in the context of a **plumb-bob** device means that which is vertical (ie. perpendicular) to the surface of the earth. If something is actually vertical it is said to be "plum" (meaning "true" to being vertical). A plumb-bob is hung from a thin cord called a plumb-line so as to indicate a true vertical line that is perpendicular (here, vertical) to the horizon line or surface of the earth. The word "plumb" in plumb-bob means a weight, and which was often a lead weight. An old world for lead is actually the word "plumb" (ie., for plumbers, a "true" vertical or horizontal lead pipe) of which now has a periodic table of elements entry identifying it as such: Pb. The word "bob" in plumb-bob means to be movie in an up and down manner as would a (fishing) "bobber" (with a "bobbing motion") used as a float and indicator when fishing. The plumb-line can be used to indicate and-or measure an angle with respect to the local horizontal 0° or vertical reference line. The plumb-bob weight usually has a point at the farthest end (near the ground) for maximum use of the device, however, for the final construction discussed, practically anything can be used as the weight at long as it does not interfere with the free movement of the plumb-line which will indicate the angle. The plumb-line can be made from common household sewing thread or fishing line. The common word "plumbers" who today, describes people who install and fix pipes.

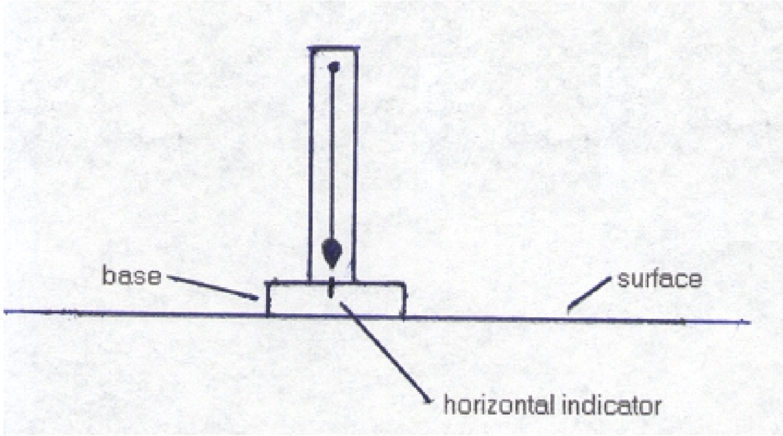
Why is the plumb-line automatically "true" vertically, or always perpendicular (90°) to the ground?

The force called gravity, originating from the matter (ie., materials) of the Earth and concentrated in the direction of its center, will pull the plumb-bob weight towards the center of Earth. If the plumb-bob is suspended by a plumb-line, the line will be made vertical due to gravity constantly pulling upon the plumb-bob and causing a tension force to be applied to the line. This line, if considered greatly extended, will always pass through the center of Earth just like a radius line, and radius lines are known to be are perpendicular to tangent lines. The (true, at the ground level) horizon is actually a tangent line at any point where a vertical line crosses above and below the surface of Earth. A horizontal line is perpendicular to this vertical line through the center of the Earth. As you go farther above the surface of Earth (ie. above sea level) the viewable portion of the surface of Earth will increase since the greatest possible "line of sight" distance between you and the (seen) edge of Earth is now larger. The max. viewable distance is where your line of sight is tangent to the surface of Earth. Since the circumference of the Earth is so large, it usually does not appear to be curved but appears flat when standing on it at a low height such as at sea-level (0ft altitude), and for most applications not involving high altitudes and-or distances, the curvature of the Earth does not need to be considered in any calculations. Due to these facts, any other line (such as the side of a wall or other construction such as a building) that is parallel to the plumb-line will also be vertical. Here is a magnified image of this discussion: [FIG 171]



In the above figure, the plumb-bob (ie., "vertically true" line and weight) is vertical (ie., "straight up and down") at the local area on Earth and the observer, and it is also perpendicular (ie., 90°, a right angle) to the local horizon which is tangent to the Earth at that location, hence the horizontal line is perpendicular to Earth's radius line and vice-versa.

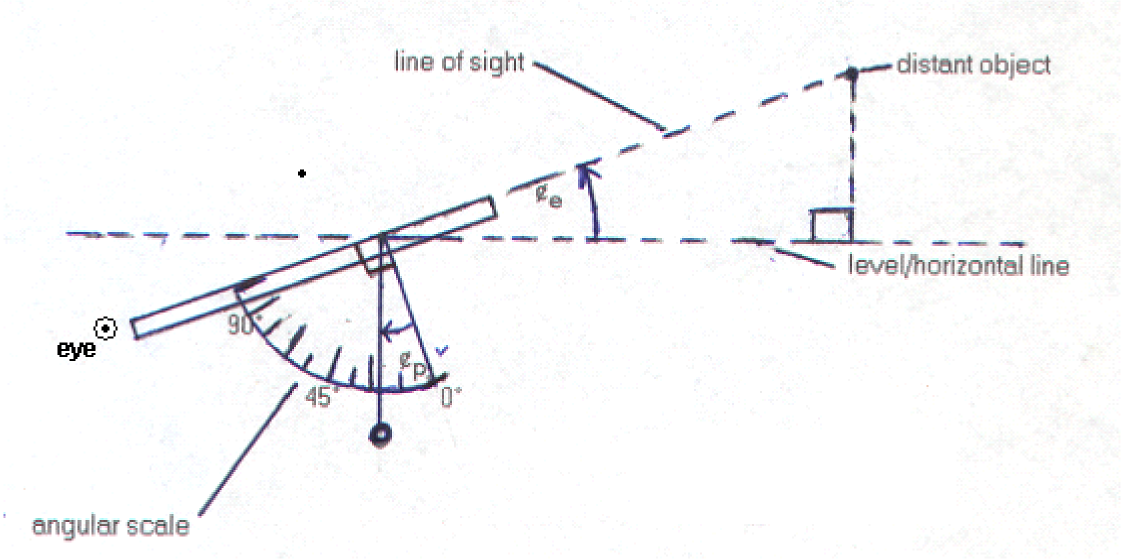
Below is a simple device that can indicate if a surface or line is level, horizontal, or parallel to the Earth's, local horizontal or surface. The working part of the device is a plumb-bob. You can construct such a device out of wood. The longer the plumb-line or string is, the better the accuracy of the device since the side opposite the angle between the plumb-line and true vertical line will be larger, and will then be more observable for small angles when the surface is not level (ie., a 0° angle or offset from the horizontal 0° reference angle). The dimensions of the device depends upon the application. For starters, you can construct a base with a length of about five inches, and a width of about one too six inches. The bottom side of the base should be perfectly flat, and it may need to be sanded flat The board suspending the plumb-bob can be about one foot high. Once constructed, the plumb-bob should swing freely with an arc of about one inch or more at the weighted end. With a stabilized (non-swinging) plumb-bob and the base being upon a surface known to be level or approximately level, calibrate (set-up, adjust for proper operation) the device by marking a point directly beneath the suspended plumb-bob whose pointer end is very close to the surface of the device. This point will be a natural calibrated horizontal or level indicator mark. It is even possible to make or modify the device with another flat surface (or "vertical base" if you will) and-or a vertical, drawn line, perpendicular to the bottom of its base side, so as to sense if another surface, such as a wall, is vertical. **A horizontal level indicator device**: [FIG 172]



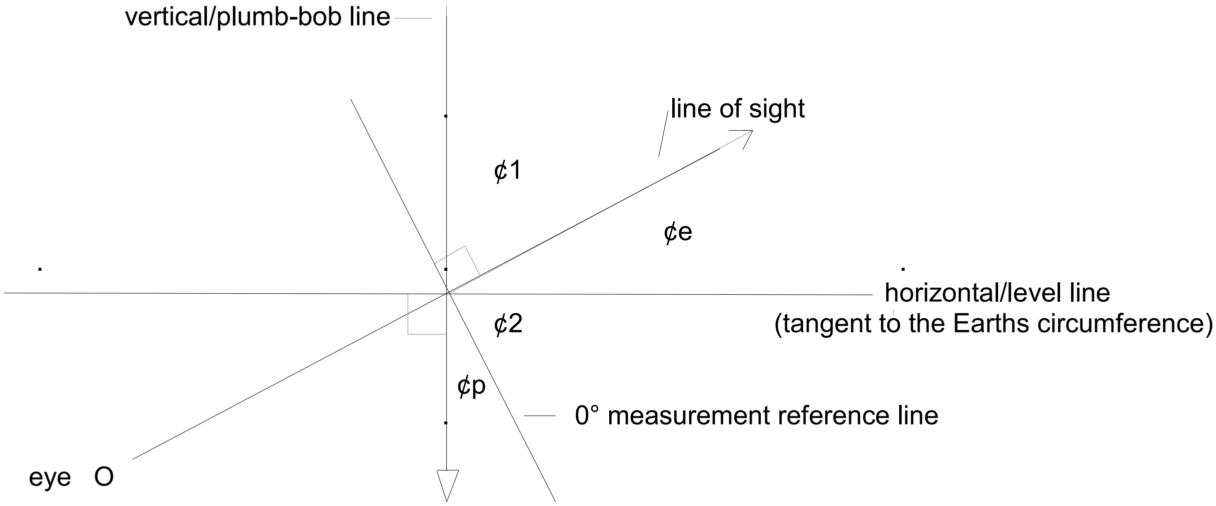
Now that we know the parts and construction of some basic angle measuring devices, the description of its operation is discussed and verified below.

To calibrate the sighting device, set the sighting device in parallel to the ground or horizontal level. The angle indicated should be set to 0° degrees. Note that a floating object of equal dimensions and having a flat surface on top can indicate a line or "level line" (no tilt or angle) that is parallel to the horizontal line at your location.

To use the angle measuring device, view ("line up", "aim") a distance object the "sighting (viewing) edge" of the device. The angle indicated on the angular scale or protractor is the angle between the horizontal (0°, "level" line of reference) and the line of sight to the distant object: [FIG 173]



Below is a verification that the angle indicated on the protractor (¢p) is equivalent to the actual angle (¢e) between the line of sight and the horizontal "level" line of the Earth or any other line parallel to this line. For this example, the angle is an (upward) angle of elevation since the object is above the horizon. To measure a downward angle (where the object is below the ("level") horizon or 0° reference line), you can simply turn the device around, that is, sight the object from the other end of the device, or construct the device to measure all possible (vertical, up and down) angles. [FIG 174]



¢e = angle of elevation above the reference line
¢p = angle indicated on protractor or angle measuring scale

Since the line of sight line is perpendicular to the 0° angle line:

¢e + ¢2 = 90° therefore:
¢2 = 90° - ¢e

Since the plumb-bob line is perpendicular to the horizontal/level line:

¢2 + ¢p = 90° algebraically substituting the equivalent of ¢2:
(90° - ¢e) + ¢p = 90° clearing the grouping symbols:
90° - ¢e + ¢p = 90° after transposing 90° :
- ¢e + ¢p = 0° after transposing (- ¢e) :
¢p = ¢e : angle indicated on protractor = angle of elevation (above the horizon) to object

In short, and as a verification, ¢p and ¢e are both complementary angles to ¢2, hence they must be the same value.

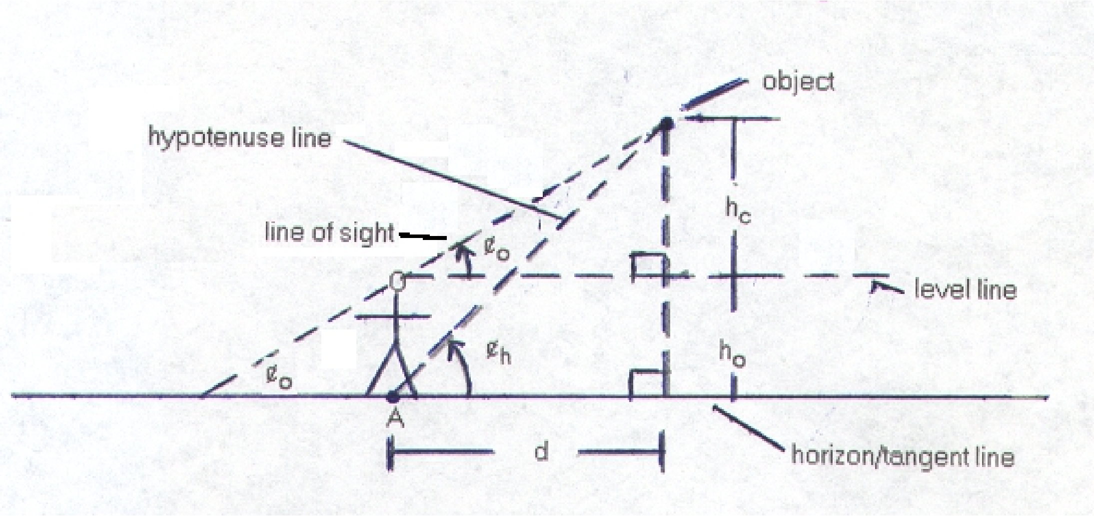
Extra: **Still water will naturally seek the local horizontal level.** A very flat object such as a rectangular block of wood that is floating in it can be used to create a level sighting pointer and-or horizontal reference. As shown previously was a plum-bob in the local force of gravity that be used to show a true, local vertical reference, and 90° from this value is a true, local horizontal reference.

A long (air) **bubble level,** which is also called a **(mineral, oil) spirit level** is often used to determine if a surface is horizontal or vertical. These are available in various lengths: 1 inch long to over 3 feet long. When the air bubble is centered between two indicator lines marked on the small glass cylindrical tube, the surface it is on is either horizontally level, or vertically perpendicular to the horizon, and-or parallel to the surface being checked.

Some modern computer phones can use an application (ie., app, computer program) so as it can function as a basic level indicating device. Another app can make the phone act as a magnetic compass. These apps are low cost or free.

SOME CAUTION WHEN CALCULATING AN OBSERVED HEIGHT

When you are close to an object and are solving for the height of it with the aid of an angle measuring device and (right-angled) trigonometry, you are actually measuring the height that the object is above the your observation point, (eye) level line (parallel to the horizon line) as shown in the drawing below. This topic was also briefly indicated previously in this book: [FIG 175]



The angle (¢o) measured by the observer (O) standing above point A will be the angle between the (eye) level line and the line of sight to the object. The actual angle (¢h), for the actual height of the object above the horizon line as seen at point A, is the angle between the horizontal line and the hypotenuse line intercepting point A as shown. Be sure to notice that ¢o and ¢h do not have the same value.

The calculated height (hc) by the observer will be smaller than the actual height of the object. To correct this, simply add the height (ho) that the observers eye is above the horizontal, base or ground line to the calculated height.

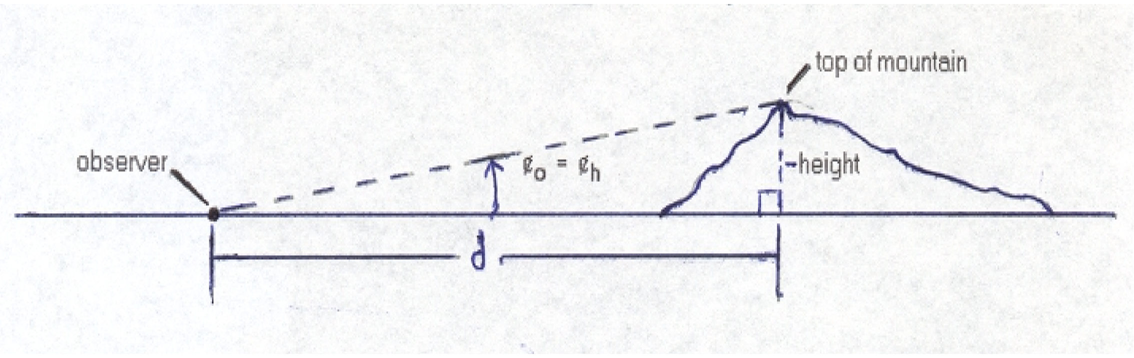
actual height = hc + ho

If needed, the actual angle (¢h) at point A can now be calculated using trigonometry:

$$\text{TAN ¢h} = \frac{\text{actual height}}{d} = \frac{(hc + ho)}{d}$$

¢h = ARC TAN (TAN ¢h)

If the object is very high (as compared to the observers relatively small height above the horizontal base or ground line) and distant, the observers height is practically irrelevant to the objects height and is therefore often omitted from the calculations. In this situation, the hypotenuse line and the line of sight become nearly parallel and along the same line, therefore, the angles (at ground level, and observed) will practically have the same value: [FIG 176]



SLOPE OF THE HYPOTENUSE LINE

The slope, as mentioned previously, is basically a numerical value for the "steepness" of the hypotenuse of a triangle or line being regarded as the hypotenuse of a level, right-triangle construction. Mathematically, in terms of two locations (points) on a graph, it is the vertical (height) change divided by the horizontal (base) change between any two points on a line. The concepts of slopes and-or rates of changes of variables are important and should not to be overlooked, especially if you need to understand basic calculus at some point in your study. Some other names for slope that are occasionally used are "grade", "incline", "inclination", and with the word "rate" (of changes of one variable in reference or respect to another).

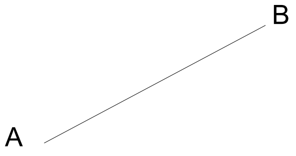
$$\text{slope} = \frac{\text{vertical change}}{\text{horizontal change}}$$

: Rate of the vertical change with respect to the corresponding horizontal change.
Or in more general terms:

$$\text{slope} = \frac{\text{vertical rise or distance}}{\text{horizontal run or distance}}$$

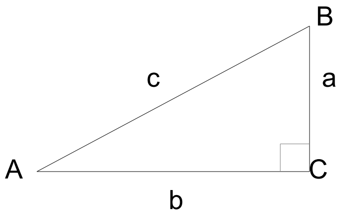
The slope value is meant to be purely numerical (ie., unitless), hence the units of the horizontal and vertical changes must be identical so as the result is unitless and just a numeric value, ratio or factor. As mentioned previously in this book, each line has it's own slope value, and it is a constant value.

Consider finding the slope of the following line segment between points A and B (this line segment may be indicated as: Line AB, or AB with a bar line over it). Let us consider Point A as level with the ground, and Point B as higher in elevation than point A: [FIG 177]



: line AB = $\overline{AB}$

For the slope discussion, consider these points as endpoints of a hypotenuse line and construct a right triangle around it: [FIG 178]



Let: a = vertical change, b = horizontal change, and c = line in question = hypotenuse of the right triangle analysis

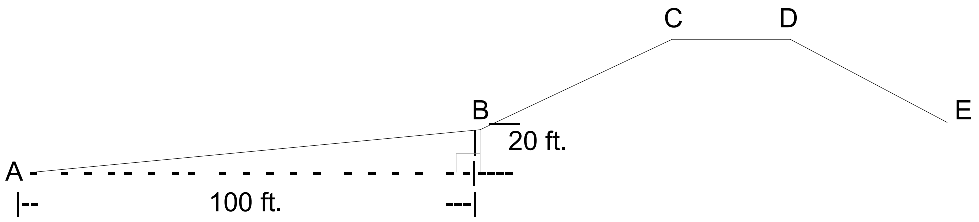
$$\text{slope} = \frac{\text{vertical change}}{\text{horizontal change}} = \frac{\text{rise}}{\text{run}} = \frac{\text{height}}{\text{base}} = \frac{a}{b} \quad : \text{slope of line c}$$

Clearly, this is equal to the TANgent of the angle that the line (c) makes with respect to the horizontal line (b). In this example, it is angle A and it is indicated at or as vertex point A of the triangle.

$$\text{slope} = \text{TAN A} = \text{TAN ¢A} = \frac{a}{b} \quad \text{and:} \quad \text{¢A} = \text{ARCTAN (TAN A)} = \text{ARCTAN } (a / b)$$

The angle and slope values are mathematically directly related. As one gets larger, the other does also. Slopes greater than 0 "slope" or rise upward at an angle with respect to the horizontal ("level-line", 0 slope) or base line. Slopes of 1 correspond to an ¢ of 45°, and the rate of change in the variables is 1 because their changes were the same value. Slopes less than 1, correspond to an ¢ less than 45°. Slopes of 0 indicate a horizontal line. Slopes less than 0 are of lines that slope downward below the horizontal or base line .

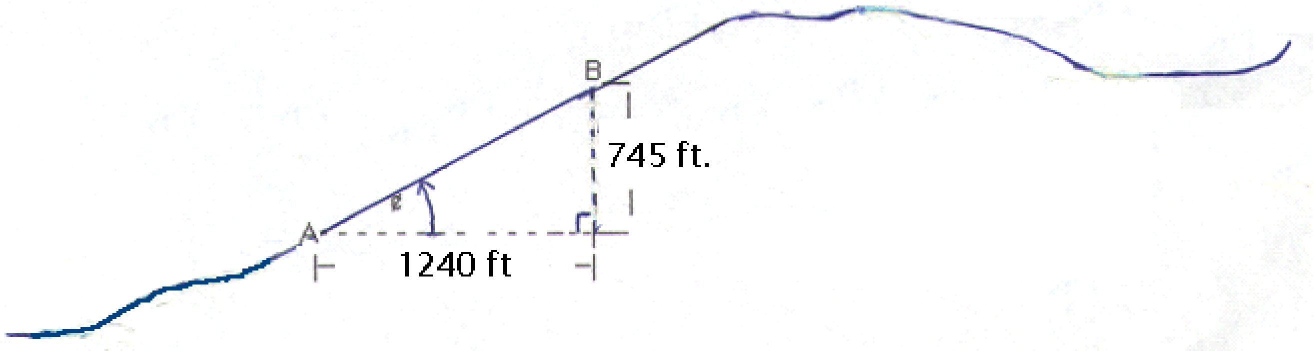
Ex. What is the slope of the line between points A and B, and what angle does it correspond to with respect to the level ground or horizon line? [FIG 179]



$$\text{slope} = \frac{\text{vert. change}}{\text{horz. change}} = \frac{20 \text{ ft.}}{100 \text{ ft.}} = \frac{0.2}{1} = 0.2 = \tan ¢ \quad : \text{the slope between points B and C is clearly more}$$

$$¢ = \arctan (\tan ¢) = \arctan 0.2 = 11.3°$$

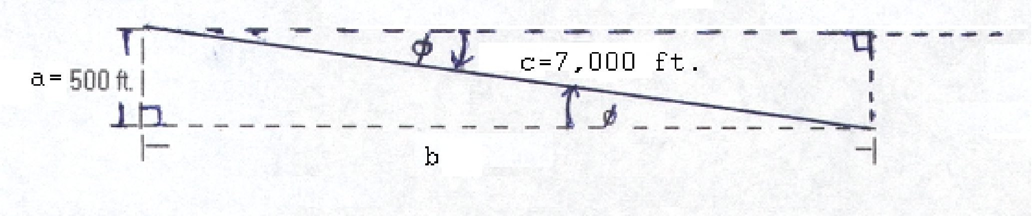
Ex. What is the slope and the angle of the mountain between points A and B with respect to a horizontal level line? [FIG 180]



$$\text{slope} = \text{TAN } ¢ = \frac{\text{vert. change}}{\text{horz. change}} = \frac{745 \text{ ft.}}{1240 \text{ ft.}} = 0.6$$

¢ = ARC TAN 0.6 = 30.96° : ~= 31° " approximately, or about 31° "

Ex. A vehicle went 7000 ft. in distance with a drop in elevation of 500 ft. to reach the bottom of a small mountain. Along its way down, the vehicle went across a few small level bridges and up a couple of small inclines. What was the vehicle's (average) slope over the entire dimensions given? [FIG 181]



Since the (level) bridges and inclines were relatively a small distance to travel, for all practical purposes, they will not change the average slope value by much, and hence those values will not be used for any calculation here.

First, we need to find the horizontal change (b). Considering the mountain structure as a side of a right triangle structure, it can be found using the Pythagorean Theorem:

$$c^2 = a^2 + b^2$$

$$b = \sqrt{c^2 - a^2}$$

$$b = \sqrt{(7000 \text{ ft.})^2 - (500 \text{ ft.})^2}$$

$b = 6982$ ft. : total horizontal distance traveled

Since there is a drop in elevation, this will be noted algebraically as a negative value, specifically (-500 ft).

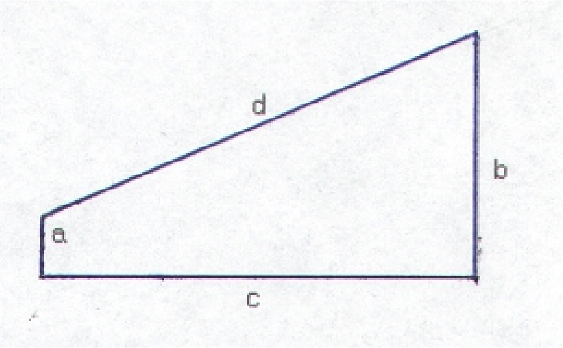
$$\text{slope} = \frac{\text{vert. change}}{\text{horiz. change}} = \frac{-500 \text{ ft.}}{6982 \text{ ft.}} = -0.0716$$

Since slope is equal to the tangent of the angle, slope = TAN ¢, we can find the vehicle's average angle of descent down the mountain.

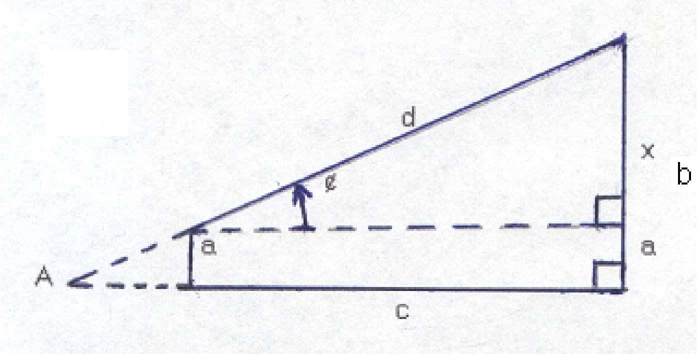
¢ = ARC TAN -0.0716
¢ = -4.095°

The negative result indicates that the angle is below the vehicles horizontal starting position. That is, it is equivalent to the vehicles angle of declination from the initial 0° horizontal or reference line. If the vehicle was going up the mountain, the angle would be a (positive valued) angle of elevation. The angles indicated on the graph are equal due to that they are corresponding angles of identical similar right triangles. The triangles are also identical in this case. The angles are also equal since they are alternate interior angles created at the transversal line across the two parallel "level" lines, and here with the hypotenuse line essentially being the transversal line.

Ex. Below, you are given a drawing of a construction with its' corresponding measurements. This construction happens to be a portion of a triangular shaped construction with an entire "corner" or vertex portion of the triangle removed. Solve for the angle at this removed or unseen vertex. [FIG 182]



Drawing a (dotted) line parallel to line (c), we create a small internal triangle (with sides equivalent to: c, x, and d) that is similar to the entire triangle construction that is being found. [FIG 183]



We also note that line (b) includes the same length of line (a) plus an unknown length that we will call (x) that we can solve for:

$b = a + x$ therefore:
$x = b - a$ also:

This angle can be found from:

From: COS ¢ = adj. / hypotenuse = c / d : when considering ¢ A , and:

¢ = ARC COS (COS ¢)

We could solve for (x) using the Pythagorean Theorem: $x = \sqrt{d^2 - c^2}$

We could also solve for (x) using a trigonometric function:

From: $\text{TAN } ¢ = \frac{\text{opp.}}{\text{adj.}} = \frac{x}{c}$

$x = c \text{ TAN } ¢$

The slope (m) of line (d) is equivalent to TAN ¢ which is also equivalent to TAN A due to corresponding angles of similar triangles:

$$md = \text{TAN ¢} = \frac{\text{vertical rise}}{\text{horizontal run}} = \frac{\text{opposite}}{\text{adjacent}} = \frac{x}{c}$$

: m is often the variable chosen for a slope.
md = the slope of line d

¢ = ¢A = ARCTAN (TAN ¢)

With this information you could reconstruct the entire triangle. If the triangle was not a right triangle, an analysis similar to the above example can be used after dividing the triangle into two right triangles.

The triangle portion removed in the above construction is actually a similar (but smaller) internal triangle, considering triangle (dxc), since its side is parallel to the larger triangle. The (de)magnification value of the smaller triangle is equal to the ratio (r) of any two corresponding sides of those similar triangles. For example: $a/x = r$. Each corresponding part (ie., sides only and not the angles) of the smaller triangle is essentially a (de)magnified construction but still has the same portions (ie., proportions) or fractional values with respect to each other and its whole, as that of the larger triangle and its parts.

The sides of the smaller triangle in the drawing are: $a = r(x)$, $r(d)$, and $r(c)$. These expressions are of the form:

$$\frac{\text{corresponding side of smaller or new triangle}}{\text{corresponding side of larger or reference triangle}} = (r)$$

: r = ratio = construction magnification factor

corresponding side of smaller or new triangle = (r) corresponding side of larger or reference triangle

Further ahead in this book, the slope value will be in terms of variables, and it is very similar this discussion shown above. Slope will be more of a mathematical relationship between the changing values of variables, and it is the rate (ie, a ratio) of change of a value with respect to, or in reference to another value. The slope value is how much the dependent variable (such as y) is currently changing when the independent variable (such as x) is changing. If you like, it could be called the rate or measure ("how fast" or "how slow" numerically) of changing or growing (of the value of the dependent variable). If the slope, the rate of growth or change, is 0, then there is no current growing of the dependent variable taking place even if the independent variable is changing and growing. An example of this is clearly seen for a horizontal line where as (x) is increasing, the corresponding (y) values are not changing. Slope is the amount or rate of the changes of two variables. Mathematically, the slope value is expressed as a (magnification, change, or increase) factor to the value of one variable when the other variable changes.

The change in a variable is the change in the values of that variable, and it is calculated as a difference:

change = a difference in values = (new_value - old_value)

$$\text{slope} = \frac{\text{(the corresponding change in y)}}{(\quad \text{a change in x} \quad)} = m = r = \text{ratio or rate of changes}$$

, mathematically, and showing the rate or slope as a magnification or factor value:

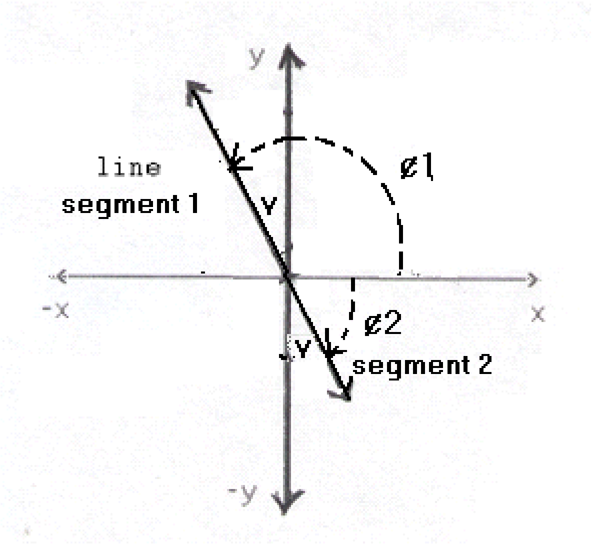
(the corresponding change in y) = r (a change in x)

If the (change in x) = 1, the corresponding (change in y) is (r)(1) = r.

For some equations, such as an exponential equation, where the dependent values are changing in value quickly as the independent variable changes , the slope is not constant and is also changing in value. Then the slope values are rather calculated at a specific instance or value of the independent variable, and therefore for a specific (small) section or point on the curve of the equation. The amount of change to consider for the independent variable when calculating the slope is then to be just a small amount, and if it is a very small change, it is called an instantaneous change, meaning that change happens quickly or in a very small amount or "instant of time".

SLOPES OF PERPENDICULAR LINES, DERIVATION

Before two perpendicular lines are shown, observe the following drawing of a single line that is also indicated as two line segments of that same line. [FIG 184]



v = equal vertical angles

First note that ¢1 and ¢2 are in reference to the exact same line with respect to the x reference axis of angle measurement. The right or positive side of this horizontal or x-axis is considered as the ¢° reference line for measuring angles.

The two angles indicated as (v) are vertical angles, and they are equal in value, and therefore have the same corresponding trigonometric values. ¢1 and ¢2 are supplementary angles, and therefore sum to 180°:

¢1 + ¢2 = 180° , therefore: ¢1 = 180° - ¢2 and ¢2 = 180° - ¢1

Any two angles that sum to 90° are called complementary angles. The trigonometric cofunction values of each complementary angle are equal. For example:

60° and 30° are complementary angles.

$$\tan 60^\circ = \text{cotan } 30^\circ = \frac{1}{\tan 30^\circ} = 1.732$$

Surely since ¢1 and ¢2 refer to the same line, the slopes of the indicated line segments are somehow similar, and it is that the line can be thought of as going in the leftward direction for line segment 1, and also as going in the rightward direction as for line segment 2. It is still the same line so these slopes must be the same value as either would have in reference to only one of those directions. Their slopes are negative values of each others slope value:

slope of any line segment of a line = - (slope of an other line segment considered in the opposite direction).

Lines that slope upward in one direction or angle (¢), will slope downward in the other direction or angle (180° - ¢). From this we can conclude that lines (or segments) that are separated by a 180°, and-or supplementary angles that sum to

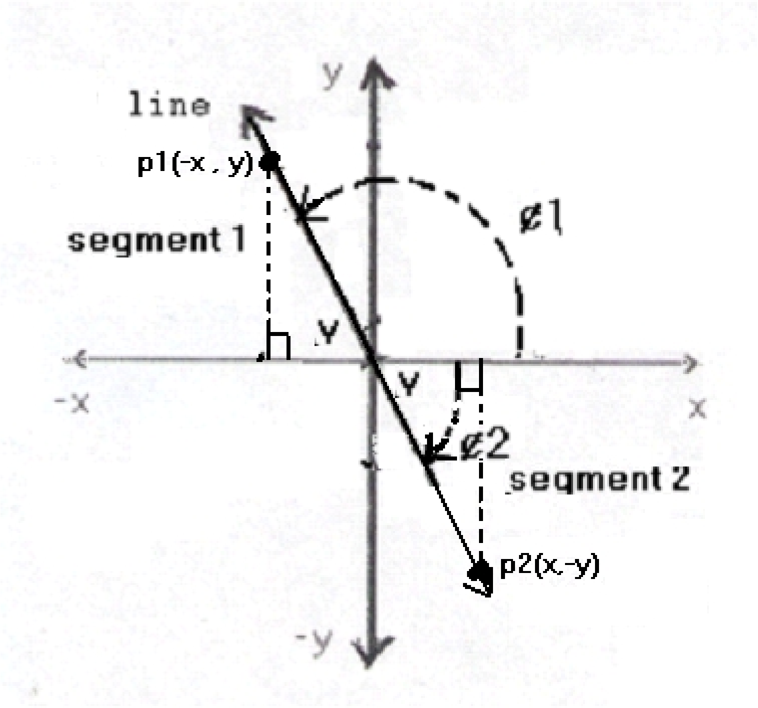
180°, have the same trigonometric function values with just a sign change between them because as one segment or side of the line is sloping upward in one direction, another line segment is also sloping downward in the other direction, and therefore, the slope will be the same value, but each will be negative (ie., "opposite") in sign with respect to each other. For example:

30° + 150° = 180° : 30° and 150° are supplementary angles

tan 30° = 0.577<u>4</u> and tan 150° = -0.577<u>4</u>

Notice that the angle of the line can be measured in two ways with respect to the x-reference axis. The line can be described as having an angle of either ¢1 or ¢2 with respect to the 0° reference axis, here the x-axis.

In the following drawing that is similar to the previous drawing, two points along the line, both being equidistant (equal in distance) from the center of the axis lines were selected. The coordinates of the points therefore have the same value except for their sign - that is, their absoluete values are the same. These coordinates will define an equivalent angle that the line or line segment has with respect to the x-axis. Since the angles are the same, their trigonometric values are the same, except for their sign. We see that the vertical (v) angles are the same, and that ¢2 is the same as the selected vertical angles indicated.
[FIG 185]



¢2 = v since ¢1 + v = 180° , v = 180 - ¢1 , using substitution:

¢2 = 180° - ¢1 : This shows that ¢1 and ¢2 are supplementary angles:

¢1 + ¢2 = 180° : This is verified by the straight line, and of which has a 180° angle about any point on it.

Trigonometric functions of supplementary angles are equal in value. One way to verify this is:

If ¢1=¢1 and ¢2 = (¢1 + 90°) and ¢3 = (¢2 + 90) , therefore: ¢3 = (¢1 + 90°) + 90° = ¢1 + 180°

Therefore, ¢1 and ¢3 are supplementary angles
¢2 is essentially perpendicular to ¢1, and ¢3 is essentially perpendicular to ¢2, and a (180°) straight angle to ¢1.

trig. function ¢1 = trig. cofunction ¢2 = trig. cofunction ¢3

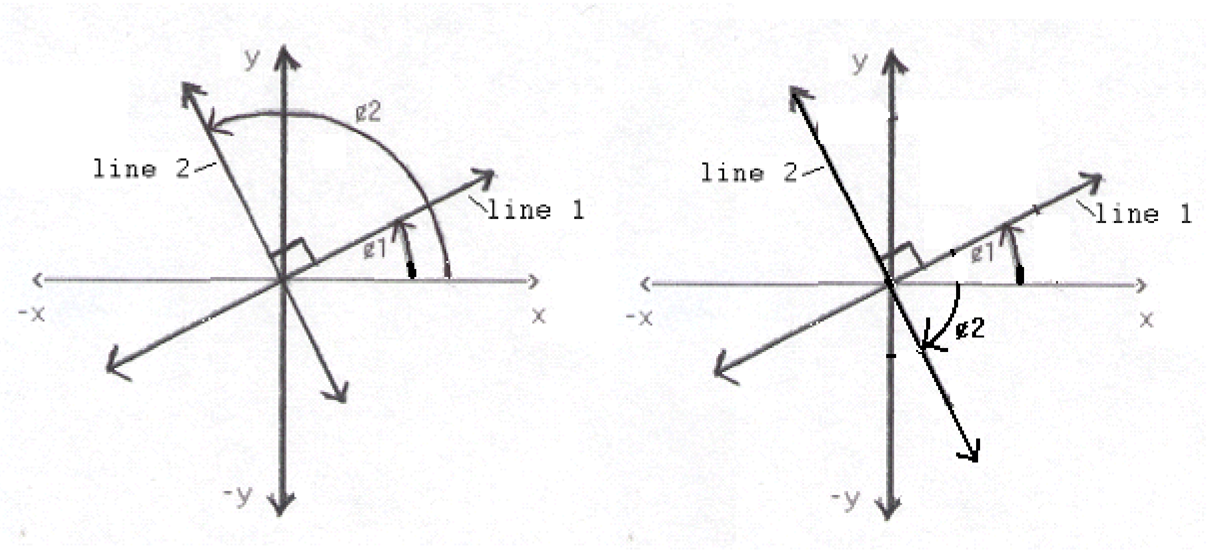
Ex. sin 30° = cos 120° = sin 210°

sin ¢ = sin (¢ + 180°) or sin ¢ = sin(¢ - 180°) : same absolute value, but the signs may not the same

Trigonometric cofunctions of complementary angles are equal in absolute value.

sin ¢ = cos (90° - ¢) or cos ¢ = sin (90° - ¢) : Ex. sin 60° = cos 30° and cos 60° = sin 30°

Now we will consider two lines which cross each other perpendicularly (at 90°) with respect to, or in reference to each other. The left and right images below show two of the many possible ways to analyze this type of situation.
[FIG 186]



By observation, any two lines that cross perpendicularly (at 90°, with respect or in reference to each other), one line will slope upward and have a positive slope value, and one line will slope downward and have a negative slope value. Don't assume that lines always cross at the axis of the coordinate system. The drawing above does not indicate the 0° reference value of the entire coordinate system, and the point of intersection of the two given lines could be any point in the coordinate system. Is there a certain mathematical relationship of the slope values of perpendicular lines? The following analysis will answer this question.

If the two angles (¢1 and ¢2) differ by 90° the corresponding lines are perpendicular. ¢1 and ¢2 are complementary angles that sum to 90°, and their trigonometric cofunctions are equivalent in value.

For example, if ¢1 = 30° and ¢2 = 60° , tan 30° = cotan 60° = $\frac{1}{\tan 60°}$ = 0.5774

Since ¢2 is a fourth quadrant angle, and is counter-clockwise from the x-axis reference line, the value of the angle should be noted as a negative valued angle: -¢2. Since points in this quadrant have a -y value, the tangent and slope values of any angle or line here will be negative in sign. Tan -60° = -0.5774 = slope of line2

Since ¢1 = (90° - ¢2) is the complementary angle of ¢2, and trigonometric cofunctions (complementary functions) of an angle and its' complementary angle are equivalent in value except for the sign as mentioned above:

TAN angle = - COT of the complementary angle:

TAN ¢2 = - COT ¢1 expressing the right side with its trigonometric equivalence:

TAN ¢2 = - $\frac{1}{\text{TAN ¢1}}$ since slope = TAN ¢ = m

$m2 = -\frac{1}{m1}$ multiplying each side by (-1), switching sides, and taking the reciprocal of each side:

$\mathbf{m1 = -\frac{1}{m2}}$ **: SLOPES OF PERPENDICULAR LINES ARE NEGATIVE RECIPROCALS**

SLOPE AS A MATHEMATICAL APPROXIMATION TOOL

Below are several basic methods to approximate a value of any equation where the value to be found is between two other close values that were previously calculated or from a table. An obvious choice of finding the approximate value would be to take the average of the two given values. This will work, but for better results, use something like the more advanced methods described below.

Let's say that you calculated, or found listed in a table, the value of two sine ¢ values or two square root values where the arguments (ie. the independent value or variable of the function) used are close in value. For example:

SIN 36°
SIN 37°

Let's assign (y) as the dependent variable equal to the value of the expression, and we then will have an equation which is often called a function since a variable(s) and its value(s) are essentially processed by the equation to produce a result. The equation is like a machine that applies a certain (preprogrammed, set, determined) process or function to the input values(s) so as to produce an output or result that was partially or significantly determined by that input values(s). Here, with an equation that is like a mathematical machine for an input value, the output is another mathematical value. Here are some ways to express what was just mentioned so as be helpful to its understanding:

input_value -----> process ---------> output value : the input value can be called the independent value
input_value -----> expression -----> output value : the output value can be called the dependent value
input_value -----> function ---------> output value : a function contains expressions that will use the input value
function(input_value) ---------------> output value = "(function) return value" : computer, pseudo or generic code
function(independent value) ------> dependent or return value : computer, pseudo or generic code
The independent value(s) is also called the
"function argument(s)" or "argument(s) sent to the function"

Ex. If the input value, or the value to be put into: (5 + x) is 3, we know that (5+x) is an expression, process or function, and the value of (x) will be set or assigned the value of 3. We can set (y) as equal to the output or resulting value of that equation or function:

input_value -----> expression ----->output value

3 ------> (5+x) ------------> y or expressed more commonly as, in a reverse direction:
y<------- (5+x) <------------ 3 and:
$y = (5+x)$: if, or letting, x=3, inputting, entering, setting, assigning or substituting 3 for x:
$y = (5+3)$
$y = 8$

Ex. $\mathrm{SIN}(x) = y$ or by switching sides:

$y = \mathrm{SIN}\ x$

Formally, (y) is said to be the result of and equal to this particular function of (x). The actual and-or specific value of (y) depends on the specific value of (x) and how it is processed by the entire expression that (x) is in. The mathematical relationship between (y) and (x) depends on the specific expression that (x) is in.

(x) is the independent variable (sometimes known as a declared, formal argument or parameter).
(y) is the dependent variable whose value depends upon the specific value of the independent variable (here, x) and then how it is processed by the equation to produce an output value based on both the equation and the input value (here, x).

We know that for any equation, other than linear (line) equations, that the relationship between the dependent variable and the independent variable is not a linear one. When the relationship is said to be linear, corresponding values of the dependent and independent variables are partially or wholly determined by a constant factor value called a slope (m):

$y = mx + b$: general line equation, m and b are constants, b can be 0.

For each integer increase (ie., 1) in the independent variable (x), the corresponding independent variable (y) can always be calculated from the previous value as (using a generalized notation of):

$y(n+1) = yn + m$: n is a subscript to express or indicate a specific instance of that variable, (here, y and its value). (n+1) indicates another instance, and here, the next instance and value of that variable. Let's begin to verify this expression:

If we took the given function and found two values of it: y1 from its corresponding value of x1 in the function, and then y2 from its corresponding value of x2 in the same function:

y1 = function(x1) and y2 = function(x2) we have found two point or corresponding values or locations. Note for example, if x1=3: y1 = function(x1) = y(x) = function(3) = f(3) = y(3)

point1=p1=(x1,y1) and point2=p2=(x2,y2)

Expressing the difference between the values, so as to find the change in the corresponding values:

(change in x) = (x2 - x1)
(change in y) = (y2 - y1) = function(x2) - function(x1) expressing this using the specific expressions or functions:

$y2 - y1 = (mx2 + b) - (mx1 + b)$: here n=2 in y2, and n=1 in y1. Distributing to clear grouping symbols:
$y2 - y1 = mx2 + b - mx1 - b$ combining the (b) terms:
$y2 - y1 = mx2 - mx1$ factoring (m) from each term on the right side:

$(y2 - y1) = m(x2 - x1)$: "point (and) slope" form of an equation of a line. This has the form of a linear equation: $y = mx$ and specifically: (change in y) = m (change in x) . Mathematically:

$m = \frac{(y2 - y1)}{(x2 - x1)}$: the formula for slope of a line, or the ratio of changes in the corresponding values of two variables. The result is a factor of how (y) will change in value when (x) first changes in value. This factor is therefore called the rate of change.

Now, if the change, or difference, in the (x) values was a constant value of: (x2-x1) = (1), the denominator becomes a constant value of (1), and it can effectively be eliminated, and we get:

$m = \frac{y2 - y1}{1}$ which can be expressed as:

$m = y2 - y1$: When (x) only changes by 1 in a linear equation, the change in (y) has a value of (m). Solving for y2:
$y2 = y1 + m$

That is, the next value of (y) can always be calculated using the previous value of (y) and adding a constant value of the slope (m) to it. It can also be said that when (x2 - x1) = 1, or the (change in x) = 1, the corresponding (change in y) = (m):

$yn+1 = yn + (\text{change in y})$: yn+1 is to be considered as y with (n+1) as a subscript

$yn+1 = yn + m$: when the (change in x) = 1

If (x) were to increment by 1 again (for a total increment or change of 2) the next corresponding value of (y) would be:

$y3 = y2 + m$ algebraically substituting the equivalent expression for y2:
$y3 = (y1 + m) + m$ clearing grouping symbols:
$y3 = y1 + m + m$ combining like terms:
$y3 = y1 + 2m$

This same substitution for y1 can be used to find the next value, y4, and so on. Clearly we see a pattern developing and it is that the coefficient of (m) is equivalent to the (change in x). In general, if (change in x) = c, the corresponding (change in y) equals (cm) and:

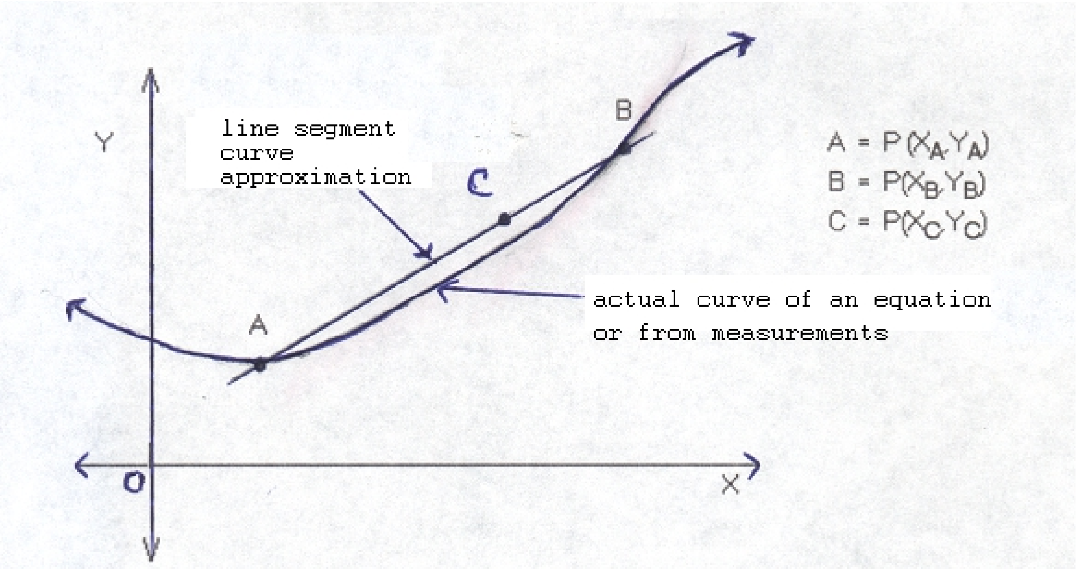
$yn+c = yn + cm$ or by using the commutative law:
$yn+c = mc + yn$: finding (y) given a previous value of (y). This still has the basic line form of: $y = mx + b$

This equation is a linear (line) equation and therefore it should hold valid for all values of (c), and not just for 1 and-or just for integers. This equation will be used ahead when calculating the square root of 10.7 when given the square roots of both 10 and 11.

INTERPOLATION TO FIND A CLOSE MATHEMATICAL ESTIMATE

Linear equations always resemble a line when graphed. Other types of equations resemble different curves when graphed. Depending on the type of equation, the basic shape of their corresponding curve may be predictable Even though they resemble curves when graphed, the closer the points are, the more that part of the curve between those points will resemble a (small and approximate) line segment of a line, hence in that region, the mathematical relationship of the points corresponding dependent and independent values (which determine those corresponding points) becomes more linear in nature. Its possible that a curve can be considered as a larger number of various sloped line segments. The formula shown above cannot be used in general since you will not necessarily be trying to find the value of the dependent variable given the next integer value of the independent variable. Even if you could possibly use it, you would have to know or calculate the slope between those successive values, and on some curves, it can possibly vary widely even between just two successive integer values (ex. between 1 and 2, or 7 and 8, etc.) of the independent variable, and therefore, the slope of the line segment is not even approximating a constant value at all.

If you know two close points on a curve (where the curve segment somewhat resembles a line segment), there is a method called (linear or line-method) interpolation (basically, a mathematical calculation and estimation since the line segment is an estimation of the curve from the given data) that will allow you to easily solve for the approximate corresponding values of a third point that is anywhere between those two points being considered on a (small, segment) line. **Interpolating is like a calculated guess or approximation, and the process can be thought of as being somewhat similar to the concept of finding an average of two values**. To verify this concept, the graphical aid of the concepts of similar triangles where the slope of a side (line) is constant for any given angle, and with the mathematical aid of proportions (equal portions, ratios or fractions of corresponding values) since the ratio of two sides of one triangle construction is equal to the ratio of the corresponding two sides of another similar triangle construction. [FIG 187]



Since we are dealing with a line (here, line AB, as a line segment) that approximates a small portion of the curve, the slope (M) between points A and B on the curve will equal the slope between points A and C, and-or C and B on that same line since the slope for a line is constant between any points on that line.

$M_{AC} = M_{AB} = M$: the slope (here, M) of a line is constant for any segment (between two points) of it.

$\frac{YC - YA}{XC - XA} = \frac{YB - YA}{XB - XA} = M$: equivalent rate of changes or "slope proportions" for the same line

For example, if the Y value of point A represents the square-root of 10, and the Y value of point B represents the square-root of 11, what is the square root of 10.7?

Given: $\sqrt{11} = 3.31662479$: B = point B = PB = Pb = Pb(Xb , Yb) = (Xb, Yb) = ($\sqrt{11}$, 3.31662479)

Find $\sqrt{10.7} =$? : C = point C = PC = Pc = Pc(Xc , Yc) = (Xc, Yc) = ($\sqrt{10.7}$, ?)

Given: $\sqrt{10} = 3.16227766$: A = point A = PA = Pa = Pa(Xa , Ya) = (Xa, Ya) = ($\sqrt{10}$, 3.16227766)

The above data of corresponding values of the square root of 10 and 11 may be from a calculation, a graph, or even from a table of square root values.

From: Y = function (X) : "Y is a function of X" or: "The value of Y is a function of X" or:
"Y is determined by this function of X".
Here, the function of X is the square root function: $y = f(x) = \sqrt{x}$

P = P(X, Y) : a general point notation for any function, and location on a graph.
point = position = location = address = corresponding pair of values
point = p(location) = p(independent value, dependent value) =
= p(input, output) = p(x, f(x)) = p(x,y)

PA = (XA, YA) = (10 , 3.16227766) : previously calculated, or given from a table
PC = (XC, YC) = (10.7, YC) : YC is to be found by calculation
PB = (XB, YB) = (11 , 3.31662479) : previously calculated, or given from a table

Solving for YC:

$$YC = YA + \frac{(YB - YA)(XC - XA)}{(XB - XA)}$$

: Interpolating (here, for a corresponding value of point C) using proportions (equivalent fractions) or rates (slopes) of the changes of two points).

$$YC = \sqrt{10.7} = 3.16227766 + \frac{(3.31662479 - 3.16227766)(10.7 - 10.0)}{(11.0 - 10.0)}$$

$YC = \sqrt{10.7} = 3.270320651$: accurate to 2 decimal places beyond the decimal point.
You might think the accuracy should be higher here, but this method assumes there is a linear or constant relationship between the points, and that is not so, since the equation was not a linear equation, and was only assumed to be one for that small line segment approximation for a portion or segment of that curve which actually has a varying slope (m), and therefore, it is not so line-like or linear as when the slope (m) is constant.

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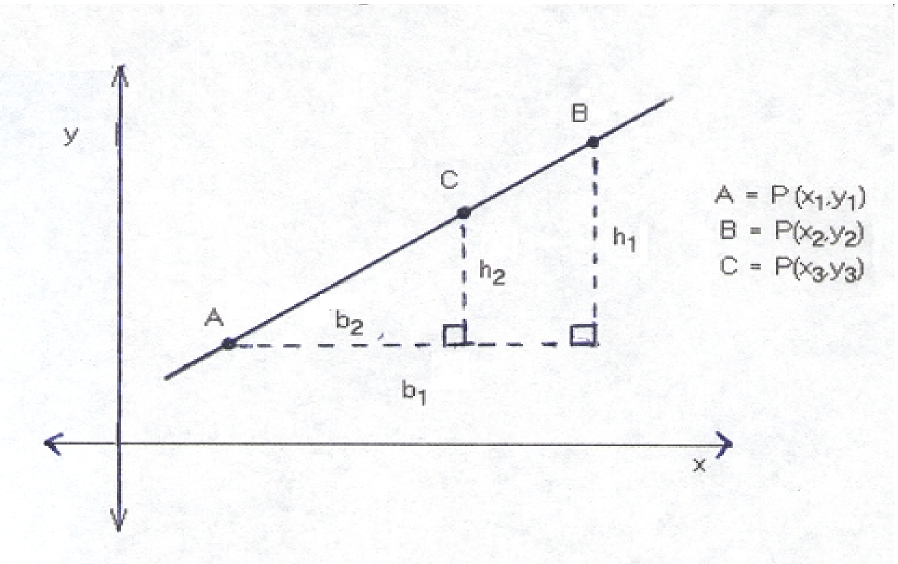
The concepts of similar triangles can also be applied. However, an adjustment method is used since the triangle, like any line, is not and should not be assumed to be (originate or have a point) at the origin of the system. The adjustment allows the triangles to be anywhere within the coordinate system. The resulting formula is very similar to that shown above. Here, the height (h) of each triangle is the difference of the (y) coordinates of the points, and the base value of each triangle is the difference of the (x) coordinates of the points: [FIG 188]



Since $x2 = x1 + b1$,we get: $b1 = (x2 - x1)$
Since $x3 = x1 + b2$,we get: $b2 = (x3 - x1)$

Since $y2 = y1 + h1$,we get: $h1 = (y2 - y1)$
Since $y3 = y1 + h2$,we get: $h2 = (y3 - y1)$

As like in the above example, we are solving for the coordinates of point C, specifically its (y) coordinate which is y3, however, using similar methods, the corresponding (approximate) (x) coordinate, for a given (y) coordinate, can also be solved for.

Since the triangles are similar triangles:

$\frac{h1}{b1} = \frac{h2}{b2}$: proportions of sides of each given triangle are equal in value to that of any other similar triangle. This value here is also equal to the tangent of the angle between the base and the hypotenuse side of that triangle construction, hence it is equal to the slope (m) of that hypotenuse side or line.

$h2 = \frac{b2\ h1}{b1} = \frac{(x3 - x1)(y2 - y1)}{(x2 - x1)}$: Interpolating using side proportions, and as noted above:

$y3 = y1 + h2$ for the above example:

$y3 = \sqrt{10.7} = y1 + h2$

Since percentages (discussed previously in this book, and sometimes described as "relative values" of the whole = 100% = 1) are a linear concept, they can also be applied to the problem of finding a (approximate) value between two other close values. Given the two points (numerically, the two sets of corresponding values), consider:

If the minimum value is to 0%, its corresponding value is to 0%.
If the maximum value is to 100%, its corresponding value is to 100%.

Given a value that is somewhere between the minimum and maximum value, it is common sense that its corresponding value will be >0% and <100%, or between 0% and 100%, and that it will also be a percentage.

$$\% = \frac{\text{length 1}}{\text{length 2}} = \frac{(\text{max. 1} - \text{min. 1})}{(\text{max. 2} - \text{min. 2})}$$

$$\% = \frac{b2}{b1} = \frac{(x3 - x1)}{(x2 - x1)}$$

Using the last examples values:

$$\% = \frac{(10.7 - 10)}{(11 - 10)} = \frac{0.7}{1} = 0.7 = 70\%$$: 10.7 is 70% of the way from 10 to 11

From the concepts of similar triangles, where to construct a similar triangle, all the parts are magnified by the same value, resulting in that the ratio of corresponding parts of the two constructions is constant and is equivalent to that magnifying or factor value:

If b2 is 70% of b1, (b2/b1) = 0.70, h2 should also be 70% of h1. The resulting equations are similar to the last method.

$$\frac{h1}{b1} = \frac{nh1}{nb1} = \frac{h2}{b2}$$: here, n=0.7 = 70%

$$h2 = \frac{b2h1}{b1} = \frac{(x3 - x1)(y2 - y1)}{(x2 - x1)}$$: Interpolating by equivalent fractions

$$y3 = y1 + h2$$

By having a segment of a line, we can find the equation of that entire line given just a small piece or segment of it. We can find the equation of this line since we are essentially given two points on this line, which will satisfy the minimum number of points needed to define that line and make it corresponding equation. Once we have this equation, any other or "third point" on the line can then be found, and specifically for this discussion, the point must be between the two points which bound (ie. are endpoints of) that line segment approximation of any curve.

The equation we are to write is a simple linear (line) equation of the form:

$$y = mx + b$$: "slope-intercept" form of a line, algebraically:

$$m = \frac{y - b}{x}$$: (m) is a constant for a given line, algebraically:

$$b = y - mx$$: as mentioned previously, (b) is defined as a (arbitrary) constant for a given line. (It could be thought of as the "(initial) height" or starting value of the line when x=0.)

We don't not know yet the value of (b) (technically the "y axis intercept" where x=0, this is point(0,b)) in order to calculate (m). Since (m) is equivalent to the slope of the line, and that we can calculate the slope of any line given two points on it:

$$m = \frac{(y2 - y1)}{(x2 - x1)} = \frac{\text{corresponding change in y values}}{\text{corresponding change in x values}}$$: slope of the line segment between p(x1,y1) and p(x2,y2)

Using the above values:

$$m = \frac{\sqrt{11} - \sqrt{10}}{11 - 10} = 0.15434713$$

$b = \sqrt{10} - (0.15434713)(10)$: from $b = y - mx$

$b = 1.618806358$ constructing the equation of the line:

$y = 0.15434713\,x + 1.618806358$: Equation of the line and a (simple) formula for approximating the square root of values from 10.0 to 11.0 where x=radicand, and y=root. We see that this is a very powerful formula since many approximate values of the square roots of values between 10 and 11 can be calculated from this single formula.

Finding the square root of 10.7, x=10.7, and the corresponding (y) value which is the approximate square root of 10.7 is:

$y = 0.15434713\,(10.7) + 1.618806358$: Interpolating using a linear equation

$y = \sqrt{10.7} = 3.270320649$: accurate to 2 decimal places beyond the decimal point.

For comparison, the actual square root of 10.7 is 3.271085447 and the difference between the calculated (or measured) value and the actual (or reference) value is:

calculated value - actual value = difference
3.270320649 - 3.271085447 = - 0.000764797

The negative sign of the difference indicates that the calculated value was "low", or less than the true (actual) value.

Taking the ratio of the calculated value to the actual value will give us a relative numerical representation for how close the calculated value is to the actual value:

$\frac{3.270320649}{3.271085447} = 0.999766194$: roughly 99.97 % correct

Taking the ratio of the difference value to the actual value will give us a relative numerical representation for the amount of difference from the true or actual value. Since its a relative value, absolute (ie. signless) values will be used:

$\frac{0.000764797}{3.271085447} = 0.000233805$: roughly 0.023 % error, or "off", or percent different.
Note: percent correct + percent different = 1.0 = 100%

As another check to the square root of 10.7, let's use the previously derived formula of:

yn+c = mc + yn : finding (y) given a previous value of (y)

$\sqrt{10.7} = (0.15434713)(0.7) + \sqrt{10}$
$\sqrt{10.7} = (0.15434713)(0.7) + 3.16227766$
$\sqrt{10.7} = 0.108042991 + 3.16227766$
$\sqrt{10.7} = 3.270320651$: correct to 2 decimal places

Here is a relatively easy to understand method of (linear) interpolation (estimating or calculating an unknown value or point that is between two other given values or points), and its based on differences, and it's very similar to one method already shown. This method assumes that the ratio of differences between the same members of two sets (each set (or corresponding pair) has a member of (x) and its corresponding value of (y)) of data values is equal to that of any other two sets of that data. In general, it is assumed that each set of data or points lays upon the same line, and specifically for this type of analysis, a line segment when the points are close so as any other or "third point" between them will also be close in value.

For example, you are given a chart or table of data and are to find a (corresponding) set of data that is between two other sets of data, specifically, you are to find the corresponding (y) value for a given (x) value. You can identify or assign variables to the given sets of data or points, such as: (x1, y1) and (x3, y3), and you are to find the "third, middle or unknown" set of data of: (x2, y2): [FIG 189]

TABLE OF DATA ,OR "LOOK-UP" CHART

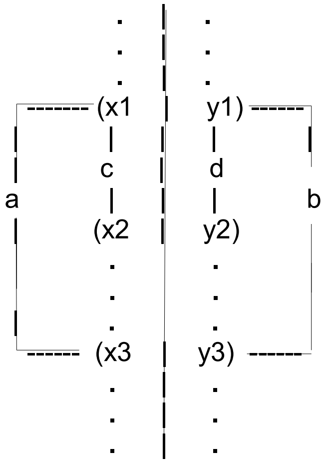
x	y	: corresponding values, or coordinates (locations) of points, of successive data of a curve
.	.	
.	.	
(x1	y1)	: point 1 = P1 , a pair or corresponding set of data
.	.	: the dots represent any possible intermediate values such as between (x1,y1) and (x2,y2)
.	.	
(x2	y2)	: point 2 = P2, which is to be found. x2 is given, and its corresponding value of y2 is to be found
.	.	
.	.	
(x3	y3)	: point 3 = P3
.	.	
.	.	

To help simplify the description and analysis of this method, we will identify the differences between corresponding members of two sets, and we will assign them (the differences) identifiers or variable names. For assistance, look at the drawing below of a portion of a "look-up" or reference table of values:

The differences ((d), "lengths", or "changes") are:

a = x3 - x1 : could also use something like: d1 for a, d2 for b, d3 for c, d4 for d
b = y3 - y1
c = x2 - x1
d = y2 - y1 : y2 is unknown and being solved for. Note that: y2 = y1 + d , [FIG 190]

(x , y) : A LIST OR TABLE OF CORRESPONDING DATA VALUES OR POINTS ON A LINE GRAPH



: pairs of corresponding values, or coordinates (locations) of points, of successive data of a curve, equation, mathematical relationship, or table of "lookup" values

: A typical reference table-list of (increasing or decreasing) data values. We are given two points or sets of corresponding data: (x1,y1) and (x3,y3). Find (x2,y2), the unlisted value being found that is "in-between" the given data.

According to this method of (linear [constant mathematical or numerical relationship] , line, or "equivalent slope") interpolation, and considering the data values, or changes, as sides of similar triangles:

$\frac{a}{c} = \frac{b}{d}$ or-from mathematically: $\frac{a}{b} = \frac{c}{d}$: corresponding ratios, or equivalent fractions, of the differences or changes

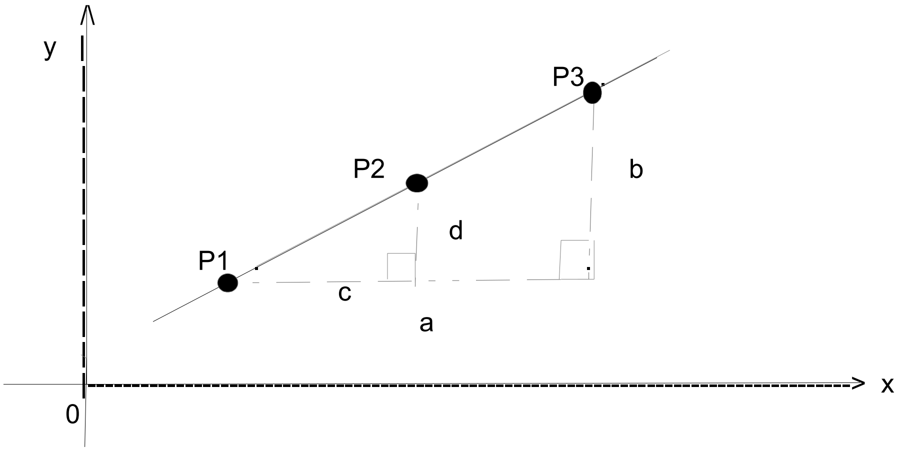
We have equivalent ratios. Each value in the numerator is said to be proportional to the value in the denominator. That is, the value in the numerator represents the same part, portion or fraction of the denominator value. Two or more equivalent fractions are therefore often called "proportions", where the "pro" prefix means to the effect of being or having the proper (equivalent, same) portions.

Here, we are solving for (d) so that it can be added to y1, so as to have y2.

$$d = \frac{bc}{a}$$

$$y2 = y1 + d$$

Here is a general representation of the points (pairs or sets of corresponding data) on a line. The differences calculated above are shown as sides of similar right triangles and can be analyzed as such. The result of this interpolation process is very similar to ones previously shown. [FIG 191]



Expressing some (equivalent) ratios of corresponding sides we have the following proportion:

$\frac{a}{c} = \frac{b}{d}$: the same proportion as above

Notice that the proportion can be algebraically expressed as:

$$\frac{b}{a} = \frac{d}{c}$$

Expressed this way, it is the ratio of the differences of corresponding members of any two sets, and it is equal to that of any other two sets of points on that line, or sides of a similar triangle.

Here, the value of each ratio is actually equal to the slope (m) of the line:

$m = \frac{\text{change in } y}{\text{change in } x} = \frac{b}{a} = \frac{y3 - y1}{x3 - x1}$: slope of all line segments of, or on a line will all have the same value

$m = \frac{\text{change in } y}{\text{change in } x} = \frac{d}{c} = \frac{y2 - y1}{x2 - x1}$: slope of all line segments of, or on a line will all have the same value

Therefore, if you can remember this general formula for slope (m), you can also remember this simple formula for (d), which you can think of as: "the difference":

d = (change in y) = m (change in x) and then:

y2 = y1 + (change in y)

y2 = y1 + d

Or showing this in a more algebraic sense:

$\frac{y2 - y1}{x2 - x1} = m$ therefore:

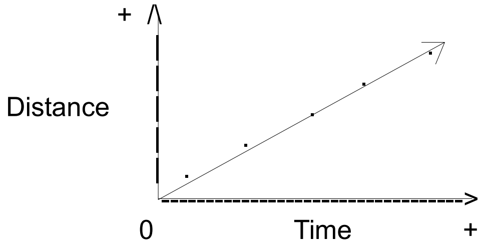
y2 - y1 = m (x2 - x1) : has the basic linear form of: y = mx + b, Transposing y1:

y2 = y1 + m (x2 - x1) : or y2 = y1 + m(change in x) = y1 + d
This has the basic linear form of: y = mx + b , y2 = m(x2-x1) + y1

SLOPES FOR ADVANCED MATHEMATICS

If the rate of change, essentially the slope value, of one variable, say (y), in reference or respect to another variable, say (x), is constant, a straight line will represent this data on a graph or in physical reality.

Ex. Distance = Speed x Time. This is a linear equation when the Speed remains constant. Below is a graph of this equation, and we also see that Distance also increases at a steady or constant rate because of the Speed being a constant value. Time increases at a constant rate automatically: [FIG 192]



; distance is the change or difference in length that an object that has moved [traversed] between two locations or points during a change in time

The rate of a change of distance with respect (in reference) to its corresponding value of change of time is mathematically equal to speed:

$$\frac{\text{change in distance}}{\text{change in time}} = \text{velocity}$$

: we see that this is essentially an equation for the slope of a line:
distance = Speed x Time = velocity x Time
velocity is nearly the same concept as speed, Velocity is basically a certain speed at a certain time, and speed is more like an average value.

$$\frac{\text{total distance}}{\text{total time}} = \text{(average) speed}$$

If the rate was constant, this is an exact value, rather than an average value where the rate value may have changed due to small stops, accelerations (speeding up) or decelerations (ie., decelerating, slowing down, not increasing speed, opposite of accelerating).

$$\frac{\text{distance}}{\text{time}} = \text{speed}$$

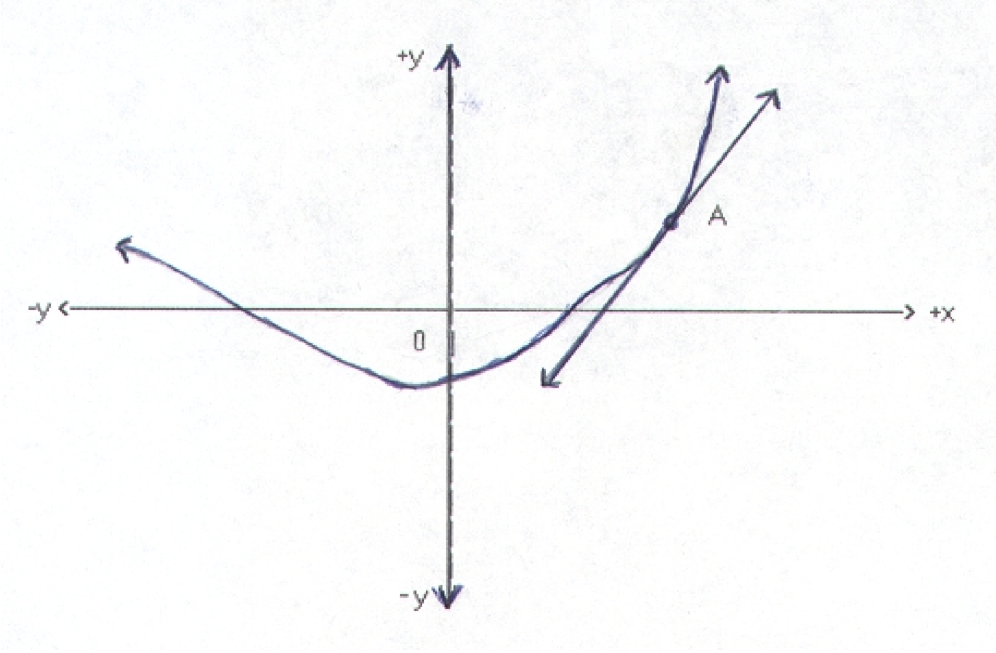
: the common formula for speed
distance = speed x time,
time = distance / speed

We see that speed (ie., "(a measure of) how fast something is going, moving, or traveling") is calculated as a ratio of two changes, and its value is therefore a ratio or rate of changes and speed is essentially equivalent to a slope value of a line. If this rate or slope value is constant, the graph showing the relationship of the variables (here, distance with respect to time, or more correctly due to the equation as distance with respect to both speed and time, however with the slope representing or being indicative of the speed) will be a straight line, and the mathematical relationship of those two values or variables is said to be constant or linear (line-like) relationship. In a linear mathematical relationship, as one value or variable changes by some factor, the other changes by that same factor.

It should also be noted that since the slope is constant, the tangent value of the angle created (between the corresponding (y) and (x) values, or Distance and Time values), is also constant since mathematically, slope = TAN ¢, and if this angle is constant, due to the rate of the changes in those values being constant, the slope is constant, and vice-versa. If the rate (speed for this example above) varies, the slope varies, and a straight line will no longer be produced on the graph of the equation, but a new line or curve (with no constant slope) will. This is true for all curves, even circles where the slope constantly varies (changes). The faster the object goes (ie., its speed), the greater the distance it will travel per unit of time, mathematically giving a greater slope or rate of change of distance with respect to time, and the line will then be steeper on the graph. When an object has a change in speed, this change is called acceleration. It could be either positive, or negative in value such as when it slowed down due to some force acting upon or being applied to it.

The concept of slope is used extensively in understanding the fundamental concepts of a branch of mathematics called calculus, where essentially a whole entire value can be solved for by accounting for all the (infinite number of infinitesimally small) smaller values or parts that make up or define that whole value. Here, the slope value is generally not between any two points on a curve (or a line which can be considered a special curve and-or graph), but between successive (adjacent, infinitely close, "instantaneous", in an instant in time) points. Therefore, these very close (as possible) points essentially (or practically) define very small line segments which compose the entire curve. This change (of the independent values or variables, and then the corresponding change in dependent values or variables) in variables under this condition is very small, which usually occur in a very small change or an "instant" of time. Therefore, the slope value of an equation in calculus is considered as the "instantaneous slope" or "instantaneous rate of change" of one variable with respect to, or in reference to, another variable, rather than an overall average slope value. This slope value is only the slope of the curve at only, or essentially at one particular point on that curve, and is generally not the (variable, changing) slope value for the entire curve. Except for lines with a constant slope, since the slope of a curve is constantly changing, it is variable in value and it could be said that the slope is variable. A constant can not then be used to define a (changing) slope of a curve, but a special and constant (slope) equation can be derived and used for the slope value.

For this discussion, observe the drawing below which indicates the slope of a curve at one point, here point A: [FIG 193]

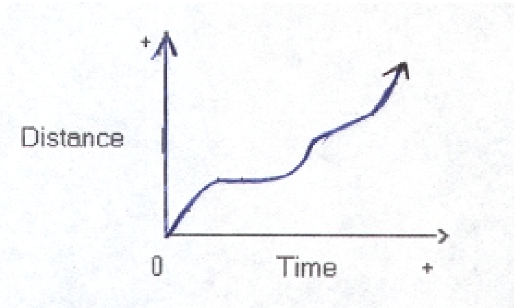


You are to find the slope of the curve at point A. As an aid to the analysis, the line indicated, if properly drawn, is simply an extension of the very small line (segment) created by the two adjacent points located at point A. You could say this is a "tangent line" to the curve at that point, location or section used for this analysis. To find the slope at this point on the curve, any two points on this line, or extended line, can then be utilized to calculate it since the slope of a line is constant at all points along it. In this example, since (x) is the horizontal or independent variable, and (y) is the vertical or dependent variable, the slope at any point on this curve is equal to the corresponding change in (y) divided by its corresponding change in (x):

$$\text{slope} = \frac{\text{change in (y) of the (small) line}}{\text{change in (x) of the (small) line}}$$: The slope at a particular point on the curve is the instantaneous rate of change of (y) with respect to (x)

Since a line was used in this example to find the slope (m) of the curve at a particular point, and that TANgent (of the angle) values are directly related to slope values, lines which intersect a curve at a point are often called "tangent lines". Previously mentioned in this book were lines which intercepted a circles circumference at just one point, hence these lines are also called tangent lines. For a circle, tangent lines will always be perpendicular (90°) to a radius line from the center of the circle to the point (on the circumference of the circle) where intersects the tangent line at the circumference or curve of the circle.

Ex. You are traveling in an automobile of which is constantly going faster (accelerating) or slower (deceleration), that is, its speed is changing and is not some fixed or constant value. The rate of a change of distance with respect to its' corresponding values, or change of time is not a constant value, and the representative curve or graph of this situation may look something like this figure: [FIG 194]



The faster you go, that part of the curve will be more vertical since the slope of that section of the curve, which represents speed, is higher in or during that section of time. If your speed stopped, any increase in distance will stop, and if the value of time keeps increasing, the graphing of this situation would be just a straight, horizontal line

You can find your average speed for the entire time and-or distance traveled using the above graph and-or equation (where change in distance = total distance, and change in time = total time), but for between any specific and-or instantaneous time period, this overall average value may then not even be a close enough representation. For example, there may be times where you are not moving or perhaps going at a very slow speed (hence, probably a much different or lower value than an average (calculated) speed) and the distance traversed is therefore very small. You can, however, find the average speed just between any specific time period or time interval (difference or length of time):

$$\frac{\text{change in distance}}{\text{change in time}} = \frac{\Lambda d}{\Lambda t} = \text{average speed for the specific time interval}$$

As the change in time becomes very, very small, the corresponding change in distance likewise becomes very, very small (regardless of how fast you are actually moving). The speed value at this specific instant of time is often referred to as the "instantaneous speed", or simply as the velocity at that point, location and-or time.

Just as distance can change with respect to speed or time, speed can also change or vary with respect to time. The change in speed (or the slope value) with respect to time is called acceleration. For example if you are going 30 miles and hour, and want to go 50 miles and hour, you must first change (here, increase by 20 miles per hour) or accelerate (ie.,change, increase) your speed by using some amount of energy, force or power until you are then moving 50 miles and hour. Once you reach your desired speed, you must then cease (none) accelerating if you want to stay at that certain speed. Once going that 50 miles an hour in speed, the rate of change of speed with respect to time, of which is called "acceleration", acceleration rate, or rate of acceleration, will be 0, but your speed will still be 50 miles per hour as you travel normally without any extra added energy applied to your motion.

DERIVATIVE

Given a mathematical relationship (expressed as an equation) or function of variables, a single and special equation called a derivative can be mathematically created by deriving it out from that initial equation. The derivative equation will mathematically represent the value of the instantaneous rate of change, or "slope", of variables between any two consecutive (adjacent, near identical) points in question of the function or curve.

For an example of a derivative, velocity is the name often used for the derivative of the speed equation:

$\text{speed} = \frac{\text{distance}}{\text{time}}$, Velocity is the (instantaneous) rate of the change of distance with respect to time: In short, velocity is the speed at only a certain instant of time.

$\text{velocity} = \frac{\text{(instantaneous change in distance)}}{\text{(instantaneous change in time)}}$: clearly , this equation is much like that for the slope of a line, and velocity can be thought of as an instantaneous slope, speed or rate.

Since velocity (or think of speed) can also change with respect to time, another derivative formally called acceleration can be derived which is the rate of change of velocity (or speed) with respect to time. A derivative equation of another derivative equation is mathematically called a "second derivative" since it will be a derivative of the first derivative. The derivative of velocity is called acceleration:

$\text{acceleration} = \frac{\text{(instantaneous change in velocity)}}{\text{(instantaneous change in time)}}$: acceleration could be thought of as the slope of velocity

Whenever an object slows down or speeds up due to some force applied to that object, its velocity will therefore change, and hence there is an acceleration or change in motion and speed. If the object increases speed, the change in speed is called an acceleration, or a deceleration when an object "slows down" or reduces is speed or velocity. When speed or velocity is not changing (ie., not increasing or decreasing) and is staying the same constant value, there is therefore no (0) acceleration happening.

Considering just lines where the slope (ie., "speed" rate) is a constant value, it is not too difficult to realize that the derivative of any line (linear) equation is a constant value equivalent to that slope.

To give you a brief insight on derivatives and inverse derivatives (often called anti-derivatives which are associated with the topic of calculus known as integration or integral calculus which sums all the infinitely small parts of a value, so as to find the entire value, a short, but very powerful discussion related to basic calculus is given below. The discussion will help you understand and give verification to some of the topics discussed further ahead in this book. Be sure to also review the brief discussion about derivatives in the Squares Example previously shown in the Basic Algebra section of this book.

UNDERSTANDING BASIC CALCULUS

The word "calculus" is a form of the word "calculate". Calculus essentially means calculating (mathematically reasoning, quantifying and solving, to calculate, and those who did it often were sometimes called calculators) by using just the small parts or amounts of the whole or entire part. Here, the small parts considered are infinitesimally small ("infinitely small"), and are practically, but not actually equal to 0 so as to be still meaningful. Isaac Newton (1643-1727) from England, and Gottfried Leibniz (1646-1716) from Germany, independently discovered the initial and-or practical mathematical concepts and understandings of the subject formally called Calculus in about the years from 1665 to 1700. With those two people developing these initial concepts, after several hundred years of developments by many, we today have a combined, single and better concept of the important subject of Calculus.

Some of the following concepts and the notation to express them have been given a reasonable mention in a previous discussion in this book about using slope as a mathematical approximation tool. They will be briefly mentioned here also as part of this topic.

If (y) is a function (f) of (x), this can be noted mathematically as: y = f(x). The word function means that the value of (y) is related to or depends upon the value of (x) inside the expression that contains it. A mathematical function is often thought of as a logical (as opposed to physical) device or machine that applies or performs the specified operation(s) upon the variable input to it, and returns, outputs or gives a corresponding or associated value as the result. If the expression containing the (x) variable is 3x , that is f(x) = 3x, and if we set (y), as like a basket or placeholder for the result, equal to this expression we get the equation: y = 3x.

The general processes of finding derivatives is called differentiating which is a somewhat lengthy, but mathematically simple process. The derivative of the function and-or expression of: y = 3x is a constant value equal to 3. The notation for the derivative of a function is often noted as: y' ("y prime") or f'(x), hence y' = f'(x) = the derivative of (y) = the derivative of the function of (x) of which y is equal to. Another notation is dy/dx which means the derivative of (y) (or the equivalent function to which it is set equal to) with respect to the (x) variable in the function. It does not mean "d times y" or (d)(x). More formally or technically, dy or dx mean a differential of (y) or a differential of (x). The word differential means an infinitely small (ie. a very small, nearly 0) difference or change, and this change is mathematically calculated as, and equal to a difference:

$\frac{dy}{dx}$ = "the differential of y, with respect to the corresponding differential in x".
It is expressed as a ratio, and the result is a rate value which is very similar to a (instantaneous) slope value.

(y2 - y1) = difference of the y values = change in y value
(x2 - x1) = difference of the x values = change in x value

You could say dx is the limit of (change in x) as it approaches 0. (ie. (x2 - x1)-->0 , which is necessary for two side by side points so as to find the slope at a particular location or point on a curve, which is the mathematical rate of changes between two values or variables). This small difference is noted as: dx which could be considered as an (infinitesimally) small bit of x. The corresponding (change in y) = (y2 - y1) will then be automatically and correspondingly as small as possible, but not necessarily the same exact value, between those two points, hence it is noted as: dy.

$\frac{dy}{dx}$ = "instantaneous slope" = slope at a specific point on a curve, or the mathematical relationship of two variables at one instance (instant) or specific value, location, or place in time values.

$\frac{dy}{dx} = \frac{\text{very small change in (y)}}{\text{very small change in (x)}}$ = Slope at a particular (single) point on, or instance of the curve.
This would be equal the slope (m) of a tangent line to the curve at
at that specific point.

Since y = f(x), this can also be expressed with function notation as:

$$\frac{dy}{dx} = \frac{y2 - y1}{\text{very small change in (x)}} = \frac{f(x + \text{change in x}) - f(x)}{\text{very small change in (x)}}$$

: This is usually called the "delta process". You can think of delta as meaning a very small change or differential (difference = d).

Above, the words "very small" also mean instantaneous, or infinitely small. For even the smallest change in a variable to occur it will take at least a very small amount of time, and hence the concept of an "instant in time" or something being "instantaneous". An instant of time is a small or "short" or "quick" amount of time. An instantaneous change in (x) will produce a corresponding instantaneous change in (y), however as mentioned, do not assume that this corresponding change in (y) is always equal to that of the change in (x). If they were always equal, their ratio (ie. slope) would always be the constant value of 1 which is not very useful to describe all the various possible curves or mathematical relationships with varying slopes and-or rates of changes between the variables.

If y = 5x^1 , the derivative of this function is: 5x^0 = 5(1) = 5, and this result will be clarified below.

Notation(s) for this derivative: y' = f '(x) = dy/dx = d(5x)/dx = 5. , Also note mathematically: dy = f'(x) dx

Notice that y=5x is a linear equation, and therefore its slope and derivative must be a constant numerical value, and in this example, it's 5. If the equation had an arbitrary constant (such as b) added in to it, the slope and derivative would still be the same. The constant simply shifts (moves) an existing curve "up or down" vertically, and all the corresponding slope values still remain the same. In simple words, the mathematical relationship of the variables and shape of the curve is still the same regardless if it's shifted (moved) up or down vertically.

Below is an example of differentiating the formal (strictly algebraic, ie. with variables only) linear equation of:

y = m x + b : basic linear equation

For showing corresponding pairs of the variables, such as the corresponding coordinates of a point, this equation can be expressed with subscript notation as:

$y_n = m x_n + b$: Basic linear equation, here, n is a subscript value and used to distinguish each instance of pair of corresponding (x,y) variables. Ex. (xn, yn) , (x5,y5), and so on.

The process below will produce the same results if values were used instead of variables, for example:

y = 2x + 3

, however, we are looking for a formal or general expression for the derivative of any linear equation, rather than a specific derivative expression or value of one specific linear equation.

Considering any two pairs or sets of corresponding values of (x) and (y), which defines two points on any curve:

y1 = m x1 + b : for one point, ex.: p1(x1, y1)
y2 = m x2 + b : for another point, ex.: p2(x2, y2) on the same curve from the same equation

Since: (x2 - x1) = (change in x) we have:
x2 = x1 + (change in x)

And: (y2 - y1) = (change in y) we have:
y2 = y1 + (change in y)

With: y2 = m x2 + b using algebraic substitution of the above expressions:

(y1 + change in y) = m (x1 + change in x) + b

Now, the coordinates of the second point are expressed in relation to, or in terms of the first point. This eliminates some variables and will help us below.

Since: y1 = m x1 + b substituting this expression for y1 into the above, and with some simplification:

m x1 + b + (change in y) = mx1 + (m) change in x + b

Solving for (change in y):

change in y = (m)(change in x) dividing each side by (change in x) :

$$\frac{\text{change in y}}{\text{change in x}} = m$$

: Greek letter "delta", is often used to indicate a change, which is essentially a difference. Delta = Δ is sometimes used represent this:
Also note mathematically: (change in y) = m (change in x)

$$\frac{\Delta y}{\Delta x} = m$$

: basic slope equation using "delta-notation"

As the change in (x) approaches the specific limit of 0, being as small as possible but never actually equal to 0 (remember, division by 0 is not allowed, and a value of 0 also means that there wasn't a change at all), the corresponding change in (y) will consequently approach a value of 0 also. The above notation modified to reflect this special condition (where the change is as small as possible) is:

$$\frac{dy}{dx} = m$$

: The ratio or rate of the differentials is equal to the derivative of the line function:
y = mx + b, and is equal to m.
m, the rate of changes, is a constant value. The derivative of a linear equation is equal to the (constant) slope value of the line.

This may also be spoken as: "the differential of, or in, y with respect to the differential of, or in, in x". As indicated above, differential means a very small change (ie. a difference) in a value. You may also read it as: "the derivative of y with respect to x" or "the differential of y with respect to the differential of x". Notice that the constant term (b or = bx^0) of a linear equations plays no role in the derivative other than perhaps the addition of 0 in the derivative which has no effect. The value of b will essentially shift the curve, here, actually a line or "line curve", vertically on the graph by the specific value of b, but the mathematical or numerical relationship between (y) and (x) will remain the same regardless of the specific value of the constant b.

Often when solving for the general or formal derivatives of functions, other than linear equations, using a similar process as shown above, you will arrive at something like the following:

(change in x)^2

Terms containing powers of (change in x) greater than (1), are considered as practically having a value of 0, and are therefore removed when they are encountered. This is so since for derivative purposes, the (change in x) is considered a very small value (ie., a fractional value, and much less than 1), and if it raised to a power higher than 1, the result is yet an even much smaller value since a fraction of a fraction (like some kind of a "second order fraction") is a much smaller value that is insignificant and can be eliminated and-or considered as having a value of practically 0. Even though dy and dx are very small values near 0, they are not removed since they are not raised to a power higher than 1, and their values are still enough to be meaningful, and since they are actually what we are solving for.

Continuing onward with some derivatives of other functions:

If y = 3x^2 , its derivative is 6x^1 = 6x. Checking using the "change" or "delta" process:

$$\frac{dy}{dx} = \frac{\text{(change in y)}}{\text{(change in x)}} = \frac{y2 - y1}{\text{(change in x)}} = \frac{f(x + \text{change in x}) - f(x)}{\text{(change in x)}}$$

(Where for the delta process, and for finding derivatives (a processes called differentiation), (change in x) and hence the corresponding (change in y) is as minute or infinitesimally small as possible. This essentially will give the slope value at a particular point or instance on the curve which visually or graphically shows the mathematical relationship between two variables at that specific location, point or instance).

The right hand expression is all in "terms (expressed as, or with) of the variable", here its (x). Substituting the given values:

$$\frac{dy}{dx} = \frac{3\,(x + \text{(change in x)})^2 - 3x^2}{\text{(change in x)}} \qquad \text{: extending the binomial in parenthesis:}$$

$$\frac{dy}{dx} = \frac{3\,(x^2 + 2x\text{(change in x)} + \text{(change in x)}^2) - 3x^2}{\text{(change in x)}} \qquad \text{distributing 3 :}$$

$$\frac{dy}{dx} = \frac{3x^2 + 6x\text{(change in x)} + 3\text{(change in x)}^2 - 3x^2}{\text{(change in x)}}$$

After canceling like terms, and the fact that (change in x)^2 is a very, very small insignificant value compared to (change in x)^1 which itself is already as small as can be, (change in x)^2 can be considered as practically meaningless and a value of 0:

$$\frac{dy}{dx} = \frac{6x\text{(change in x)} + 3(0)}{\text{(change in x)}} = \frac{6x\cancel{\text{(change in x)}}}{\cancel{\text{(change in x)}}}$$

$$\frac{dy}{dx} = 6x \qquad \text{: the derivative of the function } 3x^2 = dy/dx = 6x \qquad \text{: The (instantaneous, at a specific point) rate of change of (y) with respect to (x) is 6x.}$$

Note that the value of the derivative above is not a constant value like it would be for a linear (line) equation where the slope (= an expressed numerical or mathematical relationship between the variables) is the constant value of (m), but with the above derivative expression of 6x, the slope can vary and now depends upon the specific value of (x). The result on a graph of: y=3x^2 is that the curve does not have a constant slope like a line does, and therefore, the curve will not look straight like a line. The slope of the curve when x=2 is: 6(2) = 12. The slope of the curve when x=3 is: 6(3)=18. Clearly this is not a constant value.

First, another example description of a derivative for further understanding of these things:

Let: y = mx + b = 2x + 1 : this is a linear equation and-or function, and that y is a function of x.

When there is a change in the value of x, there will be a change in the value of y. To find out how (y) changes when there is a change in (x), we can divide the two, and which is essentially a rate since changes take some amount of time, even if infinitely quick, a short or instantaneous amount of time.

$$\frac{\text{(change in y)}}{\text{(change in x)}} = \frac{(y2 - y1)}{(x2 - x1)} \;,\quad \text{if (x) changes by 1: } = \frac{(y2 - y1)}{1} = \frac{(2\,x2 + 1) - (2\,x1 + 1)}{1} =$$

$$\frac{2\,x2 + 1 - 2\,x1 - 1}{1} = \frac{2\,x2 - 2\,x1}{1} = 2\,(x2 - x1) = 2(\text{change in } x)$$, and here = 2(1) = 2 , a constant , and a constant rate

Expressing all this in an equation:

(change in y) = 2 (change in x) and $\frac{(\text{change in } y)}{(\text{change in } x)} = 2$

Regardless of the actual or specific value of (x) or the corresponding (change in x), the corresponding (change in y) will be a factor of 2 times more since the rate of the changes is 2. This can also be thought of as: if (x) changes by 1, (y) will change by 2.

Here are the derivatives of some other functions, and there is a note below this that describes this process:

If $y = x^2$, its derivative is $2x$. , and this can be considered as:

$\frac{(\text{change in } y)}{(\text{change in } x)} = 2x$, and (change in y) = 2x (change in x)

Here, the rate of changes is no longer a constant value, but depends upon the value of (x) considered when a change is applied or added to it. Here, when (x) changes = (change in x) , the corresponding (change in y) will be a factor of (2x) more, and the value of this factor always depends upon the specific value of (x) being considered, hence a better way to think of the above ratio is that this equation and rate applies to only a specific or "instantaneous" value of (x) and or where the change in (x) is so small or "instantaneously small" and-or "infinitesimally small") that the difference or result is practically that same value of (x) being considered:

$\frac{(\text{the instantaneous change in } y)}{(\text{the instantaneous change in } x)} = 2x$: 2x is not a constant, but an equation and variable value that depends upon the value of (x) being considered. The rate of change, here (2x), is not a constant value, but here, the rate of change actually increase as (x) increases in value.

(the instantaneous change in y) = 2x (the instantaneous change in x) : x = the current value of (x)

Or in general: dy / dx = (derivative of the function) = (instantaneous slope) , and mathematically :

(dy) = (derivative of the function) (dx) : a very small or instantaneous change is also called a differential (d)

When the derivative or "instantaneous slope" = ("instantaneous m") is large in value, indicating a high rate of change, and it depends upon the specific value of (x), the (change in y) values will be getting larger, and on a graph, the drawing or curve of the relationship between the variables will be getting more steeper or vertical. For a linear or line equation, the rate of changes is constant.

Ex. When (x) is at or near 1, the corresponding (y) values are changing by a rate, factor or multiplier of: (2)(x) = (2)(1) = 2.
When (x) is at or near 2, the corresponding (y) values are changing by: (2)(x) = (2)(2) = 4.
When (x) is at or near 3, the corresponding (y) values are changing by: (2)(x) = (2)(3) = 6.

If $y = 7x^2$, its derivative is $14x^1$.
If $y = 8x^2$, its derivative is $16x^1$.
If $y = 5x^3$, its derivative is $15x^2$.

If $y = 6x^3$, its derivative is $18x^2$.

If $y = x = x^1$, its derivative is $1x^0 = 1(1) = 1$, a constant. $y = x = x + 0$, a linear equation, with a=m=1, and b=0.

If $y = 3x = 3x^1$, its derivative is $3x^0 = 3(1) = 3$, a constant. Since the derivative is constant, the slope of the curve (here, actually a line) is constant. Since the mathematical relationship between the variables is a constant mathematical relationship, the equation is called a linear equation which also graphs as a line. Note that here, the anti-derivative (ie. finding the source function (F) of which the derivative expression was derived from) of 3 is: $3(1) = 3x^{(0+1)} = 3x^1$

If $y = f(x) = \frac{x^3}{3}$, its derivative is: $\frac{d\,f(x)}{d\,x} = \frac{dy}{dx} = y\,' = f\,'(x) = \frac{3x^2}{3} = x^2$: optionally showing much notation.

Observing the pattern seen in the derivatives of the above type expressions, we find: Multiply the exponent of (x) to the numerical co-efficient of (x), and then reduce the exponent of (x) by one. Knowing such patterns is very practical and commonly used in calculus so as to find quick and practical solutions. This specific pattern just shown is commonly known as the (derivative) "power rule".

In algebraic notation, the derivative of a function of this form or type: ax^n

is: $nax^{(n-1)}$: **DERIVATIVE POWER RULE**

This can be expressed as: Given a function that is a power of a variable: (coefficient) variable^exponent

It's derivative is: (coefficient)(exponent) variable^(exponent-1)

Taking the derivative of this first derivative will give the second derivative of that same function of: ax^n:

$n(n-1)ax^{((n-1)-1)}$

is: $n(n-1)ax^{(n-2)}$

Taking the derivative of the second derivative will give the third derivative of that same function of: ax^n:

$n(n-1)(n-2)ax^{(n-3)}$

Ex. The derivative of the square root, or even the cube root, of (x) can be found using the derivative power rule:

$$\frac{d\sqrt{x}}{d\,x} = \frac{d\,x^{0.5}}{d\,x} = 0.5\,x^{(0.5-1)} = 0.5\,x^{-0.5} = \frac{0.5}{(1)\,x^{0.5}} = \frac{1}{2\,x^{0.5}} =$$

$$\frac{d\sqrt{x}}{d\,x} = \frac{1}{2\sqrt{x}}$$

: this is also the slope at any point (x,y) on the curve of: $y = \sqrt{x}$

For this derivative of a square root equation, you can see that as (x) gets larger, the slope will decrease, hence, here the slope is inversely related to (x). As (x) gets very large, the slope approaches a value of 0 indicating that there is not much of a change in the corresponding (y) values when (x) changes.

You can then use this derivative in the interpolation formula derived previously so as to find a close approximation for the square root of values near to (x). This method of interpolation using a derivative is also demonstrated in some tables shown in this book so as to find an unlisted value that is between two listed, calculated values.

Let d = (change in y) = m (change in x) replacing (m), the slope value, with the above derivative value:
d = a difference in the (y) values of two points.

$$d = \frac{1}{2\sqrt{x}}\text{(change in x)} = \frac{\text{(change in x)}}{2\sqrt{x}}$$

$$y2 = y1 + \text{change in } y1 = y1 + \text{change in } y = y1 + d$$

If you were given any of the derivatives and were to find the **anti-derivative** (sometimes noted as F for the anti or reverse-derivative Function) or expression from which that derivative itself can be or was derived from, there is a reverse pattern: increase the exponent of (x) by one (even if the exponent of (x) is 0, and-or (x) is not even shown since $x^0 = 1$) and then divide the numerical coefficient of (x) by the "new" exponent of x. Showing this algebraically, if given the derivative or function of:

$$nax^{(n-1)}$$

The reverse derivative is:

$$\frac{nax^{((n-1)+1)}}{(n-1)+1} = ax^n$$

The following notation may bring some additional clarity to this process:

Given a function that is a power of a variable: (coefficient) variable^exponent

It's derivative is: (coefficient)(exponent) variable^(exponent-1)

Therefore, when given a function or derivative, to find the anti-derivative, the only indication of what that given coefficient was multiplied by is the value of the given exponent. We know that this exponent is now one less than the original function or anti-derivative, and that the coefficient was multiplied by this given exponent plus one. Therefore we must divide the given coefficient of the variable by the given exponent plus one, and then increase the exponent of the variable by one, so as to have the anti-derivative:

Essentially given: ((coefficient)(exponent)) variable^(exponent-1) its anti-derivative is, after increasing the exponent: by 1, and dividing by the exponent plus 1:

$$\frac{\text{(coefficient)(exponent) variable}^{\text{(exponent-1+1)}}}{\text{(exponent-1) + 1}} = \frac{\text{(coefficient)}\cancel{\text{(exponent)}}\text{ variable}^{\text{(exponent)}}}{\cancel{\text{(exponent)}}} =$$

(coefficient) variable^exponent : the original given function (ie., F, the anti-derivative)

Ex. Given a function or derivative that is: $y=5$, the anti-derivative is: $\frac{5x^{(0+1)}}{1} = 5x^1 = 5x$

Ex. Given: $14x^1$

The reverse derivative is: (or anti-derivative) $\frac{14x^{(1+1)}}{(1+1)} = \frac{14x^2}{2} = 7x^2$

You can check this by taking the derivative of $7x^2$ using the derivative power rule.

In the field of calculus, you will usually see something like this expressed:

$\frac{dy}{dx} = 14x^1$: 14^1 is the derivative of some function of x. Mathematically:

$dy = 14x^1 \ dx$

Expressing the sum of all these bits, or very small changes or parts of the "whole thing", we can find the anti-derivative and the "whole thing" from which all the bits came from. This process is also called integration. To **integrate** is to combine things, and for math, that then means to add things so as to produce a new thing or sum. An integral part of a whole thing is a part that composes that whole thing. We can say that the part has been integrated into that whole thing. In math, the whole thing is called the sum or **integral** (as in calculus) since it is composed of all the smaller parts than have been integrate into it. In calculus, all the parts are generally similar and infinitely small in size and-or value, and this then cause an infinite-like sum since there are so many. A line segment has an infinite number of points, but we know it still has a definite or specific length, unlike a general line of infinite length which can be called an indefinite integral and can only be represented as an equation. The line segment is then analogous to a definite integral, and which can be thought of an isolated or bound part of a line. A line segment is a specific amount of line.

$\int dy = \int 14x^1 \ dx$: $\int$: is a summation symbol, it looks like the letter S and is the first letter of the the word Summation and is the act to, and-or the result of adding and-or combining things and-or numeric values.

$Y = 7x^2$

You might now ask what happened to dx in this process. Just like the sum of all the dy bits is equal to (y), the sum of all the dx or bits of (x) is equal to the whole (x), and this (multiplier) is what increases the power of (x) when it goes through the integration process. The appendix of this book includes more discussion about integration called: Basic Concepts Of Integration. The small bits have been summed and have produced a non-specific sum and-or another equation, as can bee seen due to the variable (x). This is the result of what is called an **indefinite integral** that is unbounded or non-specific and is rather a general representation of the resulting (infinite) sum of terms.

Ex. What is the anti-derivative of $y = f(a) = a^2$?

$F(a) = \frac{a^{(2+1)}}{(2+1)} = \frac{a^3}{3}$ Taking the derivative of this as a check: $\frac{3\ a^{(3-1)}}{3} = a^2 = f(a)$: checks

The derivative of a sum of functions of (x) is equal to the sum of the derivatives of each function of (x). This is known as the **derivative sum rule** and can easily be verified with an example:

If function 1 = $2x + 3 = f1(x) = y$ and:
If function 2 = $5x + 7 = f2(x) = y$

$\frac{dy}{dx} = \frac{d\ (f1)}{d\ x} = \frac{d\ (2x + 3)}{d\ x} = 2$ and:

$\frac{dy}{dx} = \frac{d\ (f2)}{d\ x} = \frac{d\ (5x + 7)}{d\ x} = 5$

$f3 = f1 + f2 = (2x + 3) + (5x + 7)$
$f3 = f1 + f2 = 2x + 3 + 5x + 7$ combining terms:
$f3 = f1 + f2 = 7x + 10$: the sum of the two given functions

$$\frac{dy}{dx} = \frac{d\,(f3)}{d\,x} = \frac{d\,(f1 + f2)}{d\,x} = \frac{d\,(7x + 10)}{d\,x} = 7 = \frac{d\,(f1)}{dx} + \frac{d\,(f2)}{d\,x} = 2 + 5 = 7$$: checks

Ex. Given the general or formal quadratic equation of: $ax^2 + bx^1 + cx^0 = 0$

,with the concepts of derivatives, we can find the equation for the minimum or maximum point of a parabola curve. This min. or max. point is often referred to as the extreme point of the parabola. At the extreme point on a parabola, the slope is 0, hence the derivative at that point is 0 (think of a small line, or tangent line to the curve at that point, and its slope is 0. A line at this point will be neither sloping up or down, but it would be a horizontal line). We can find the derivative of the general quadratic equation above by taking the derivative of each term when considering the function as a sum of two or more other functions:

$$\frac{dy}{dx} = 2ax^1 + 1bx^0 + 0$$

$$\frac{dy}{dx} = 2ax + b$$: notice that the derivative of a second degree equation is a first degree equation

Setting this "instantaneous slope" equation (ie. setting y) equal to 0 since we want to find out at what corresponding value of (x) will be where the slope equals 0, and which is where the location of the extreme point of this parabola is:

$2ax + b = 0$: Setting the slope equal to 0. After solving for x, we find:

$x = \frac{-b}{2a}$: (x) coordinate of the extreme point of a parabola, the corresponding y coordinate of that point can be found by substituting this value of (x) into the original quadratic equation given. This book previously shown an algebraic expression for the corresponding (y) value.

Another very useful rule is for the derivative of a function of another function. This rule is commonly known as the **chain rule**. Let's begin with some notation. If we let (y) equal a function of (x), its' notation can be expressed as:

$y = f(x)$: standard typical function notation

If there is more than one function of (x) being considered, a more compact notation is commonly used:

$y(x)$: compact function notation indicating "y is a function of x": $y = f(x) = y(x)$

Using compact notation, if (u) and (v) are other functions of (x), their notation is:

$u=f(x) = u(x)$ and $v=f(x) = v(x)$: note f(x) is an unspecific notation to represent a function, and here, the function or expression contains the (x) variable. This is similar to how p(x,y) is the general notation for point, or any point.

Then if (v) also happens to be a function of (u), that is, (v) is a function of another function, its' notation is:

1. $v = f(u)$: v is a function of u, and since $u = f(x)$, using substitution:
2. $v = f(f(x))$: u is the "inner function", and v is the "outer function" which is also function of the inner function u: v(u)

(v) is now essentially like the dependent variable of the independent variable (u). Just the same, (u) is essentially the dependent variable to the independent variable (x). Therefore, (v) is also dependent upon (x).

By the first equation, the derivative of (v) is the derivative of the entire equation given with respect to (u) since (v) is a function of (u). By the second equation (which is equivalent to the first equation), it explicitly implies or indicates that the derivative of (v) with respect to (x) is the same as the derivative of (v) with respect to (u), but will also include a factor that is the derivative of (u) with respect to (x) since (u) itself is a function of (x). If we let variable name (y) = (v) for notation conformity to what we have been using up to this point:

1. $y = f(u)$: y is a function of another function (here, function u), and since u = f(x), therefore

$y = f(f(x))$: y is a function of u which is a function of x, y is therefore also a function of x:

$$\frac{dy}{dx} = \frac{dy}{du}\,\frac{du}{dx}$$: **Derivative Chain Rule , The Derivative Of A Function Of Another Function Rule, or the derivative of a function that contains or has another function**

As a simple verification to this formula consider:

Let y = v
v is a function of u, and
u is a function of x

$$y = v = v(u) = v(\,f(x)\,)$$

The derivative of y with respect to x is therefore:

First let:

$$\frac{dy}{dx} = \frac{dv}{dx}$$ continuing:

Since (du/du)=1 (anything divided by itself equals 1), and multiplying the numerator and denominator on the right hand side by du (creating an equivalent fraction), we have:

$$\frac{dy}{dx} = \frac{dv}{dx}\,\frac{du}{du}$$ which can be mathematically expressed as:

$$\frac{dy}{dx} = \frac{dv}{du}\,\frac{du}{dx}$$ Since y = v, this can be expressed as:

$$\frac{dv}{dx} = \frac{dy}{du}\,\frac{du}{dx}$$: derivative chain rule

Ex. If $u = x^2$ and $y = (x^2)^3$: both (u) and (y) are functions of x. u=f(x) and y=f(x)

Clearly, we see that: $y = f(u) = u^3$: y=f(x) = f(u) or y=f(f(x))

$$\frac{dy}{dx} = \frac{dy}{du}\,\frac{du}{dx}$$

$$\frac{dy}{dx} = \frac{d\,(u^3)}{du}\,\frac{d\,(x^2)}{dx}$$

$$\frac{dy}{dx} = \frac{d((x^2)^3)}{du} \quad \frac{d(x^2)}{dx}$$

$$\frac{dy}{dx} = 3(x^2)^2 \quad (2x^1) \qquad : \text{or} = (3u^2)(2x^1)$$

$$\frac{dy}{dx} = 3x^4 \quad 2x^1 = 6x^5$$

Checking: Since $y = (x^2)^3 = x^6$, their derivatives must be the same:

$$\frac{dy}{dx} = \frac{d(x^6)}{dx} = 6x^5 \qquad : \text{checks}$$

Using this example above, another verification to the chain rule is shown which can be further generalized and verified in algebraic terms:

$$\frac{dy}{du} = 3(x^2)^2 = 3x^4$$

$\frac{du}{dx} = 2x$ therefore: $du = 2x\ dx$, using substitution:

$\frac{dy}{du} = \frac{dy}{2x\ dx} = 3x^4$ solving for dy / dx: :

$\frac{dy}{dx} = (3x^4)(2x) = 6x^5$ showing this expression in pure algebraic terms:

$$\frac{dy}{dx} = \left(\frac{dy}{du}\right)\left(\frac{du}{dx}\right)$$

: chain rule
If a function contains a factor that is another function, the chain rule can be utilized.
In the example, the function of (y) contained a factor that is the function of (u).

Ex. We are given the function of: $y = x^3$ and are to find dy/dx. For this example, we can solve this using the derivative chain rule, but could also solve this using the derivative power rule. If this given function contains a factor that is equivalent to another function of (x), we can also use the derivative chain rule to find the derivative (dy/dx) of that initial given function.

If $(x^1)(x^1)(x^1) = x^3$, then x^3 contains a factor of x^1.

Letting: $y = f(x) = x^3$ and $u=f(x)=x^1$, therefore, (y) contains a factor of (u), and $y= f(u) = u^3 = x^1\ x^1\ x^1$

$(y) = f(u)$, and $(u) = f(x)$, $y = f(u) = f(x)$

Since (y) is a function of (u), the derivative of (y) with respect to (x) will then be equivalent to a derivative of (y) with respect to (u), and since (u) is also a function of (x), and that (y) contains a factor of (u), this derivative of (y) will then actually consist of a factor that is the derivative of (u) with respect to (x):

The derivative of (y) with respect to (x) = **(**derivative of (y) with respect to (u)**)** **(**derivative of (u) with respect to (x)**)**

$$\frac{dv}{dx} = \frac{d\,(f(f(x)))}{dx}$$

$$\frac{dv}{dx} = \frac{dy}{du}\,\frac{du}{dx}$$

As an aid to understand this, consider these two functions which are both functions of (x), and that u = f(x) = x^1:

1. y = x^2 , y = f(x) = f(u)
2. z = x^3 z = f(x) = f(u)

Both functions contain a factor of x^1, hence both contain a factor of (u)=f(x) = x^1

Since u=x^1 is the same for both functions, du/dx is therefore the same for both functions, and we also know that the derivatives of both functions can't possibly be the same since those two given functions are different. The derivative of each function with respect to (x) will actually contain (as a factor) the derivative of that function with respect to (u) that is a function of (x).

Consider that since (u) contains a factor of (x), and that (u) is a factor of (y), a derivative of (y) with respect to (x), will contain a derivative of that factor (u) with respect to (x), and a derivative of (y) with respect to (u) will obviously contain a derivative of (y) with respect to (u).

Consider if function2 = (f2) is itself a function of another function, say function1 = (f1). In other words, f2 contains a part of f1, and is dependent upon f1. Since the value of f1 depends on the variables and equation used in that specific function, the values of f2 depends on the variables and equation in both itself, f2, and f1.

A small change in (x) will first cause a small change in (u). The derivative of (u) with respect to (x) will be a certain value: du/dx = slope1 = value1 This change in (u) will cause a small change in (y). The derivative of (y) with respect to (u) will be a certain value: dy/du = slope2 = value2. Therefore, the rate of change of (y) with respect to (x), is also determined by the rate of change of (u) with respect to (x). Therefore, the derivative of (y) with respect to (x) is also partly determined by the derivative of (u) with respect to (x). and value2 is partly determined by value1. One rate of change is party determined by the value of another rate of change. This almost seems like a derivative of another derivative concept here, but as mentioned, the derivative of (y) with respect to (x) is partly determined by both (u) with respect to (x), and (y) with respect to (u), and hence it is partly determined by each.

Considering that a derivative is a form of a rate. Here is a simple way to explain why one derivative is being multiplied by another derivative. and that each is a factor of the result.

value1 = rate1 value2 : note for example, mathematically: value1/value2 = rate1
value2 = rate2 value3

Since value2 is partly determined by value3, and if value1 is partly determined by value2, essentially then, value 1 is partly determined by value3. Because of the rates shown, of which can vary upon each given situation, the values are not completely determined by the other values alone, but also by the (varying or current) rates.

value1 = (rate1 value2) = (rate1) ((rate2) value3) this can be expressed as:
value1 = (rate1) (rate2) value3

$\frac{\text{value1}}{\text{value3}} = \text{rate1 rate2}$: the rate of value1 with respect or in reference to value3 is the product of rates.

Ex. value1 = 50% value2 or: value1 = 0.50 value2
value2 = 30% value3 or: value2 = 0.30 value3

value1 = 0.50 value2 = 0.50 (0.30 value3) = (0.50)(0.30) value3 = 0.15 value3

$$\frac{\text{value1}}{\text{value3}} = (0.50)(0.30)$$

: the ratio of value1 to value3 is much like a fraction of a fraction concept, and here it is expressed as a product of two ratios, or a ratio of a ratio

Or:

$$\frac{\text{value1}}{\text{value2}} = \text{rate1} = \frac{\text{value1}}{\text{rate2 value3}}$$

: value2 = rate2 value3 :

therefore:

$$\frac{\text{value1}}{\text{value3}} = \text{rate1 rate2}$$

With just some basic knowledge of calculus, you can find things such as the area beneath a portion of a curve, or the arc length between two points on a curve. In the Advanced Topics, and Appendix sections of this book, derivatives and anti-derivatives were used as part of the derivation process for many of the very important mathematical series to be presented. The appendix section contains very useful examples of using a derivative as part of interpolation to mathematically calculate an estimate of a value not listed in a table of values.

Here are some more examples of derivatives:

Ex. Find the derivative of the circles circumference (C) formula with respect to a change in the radius (r). This basically will find the equation or constant value of how the circumference will changes when the radius changes by one unit.

C = 2(pi) r Note that this has the form of a linear equation: y = mx , 2(pi) is equivalent to the slope = m variable. C corresponds to y, and r corresponds to x. C = 2 (pi) r = (pi) d

$$\frac{dy}{dx} = \frac{dC}{dr} = \frac{(1)(2)(pi)r^{(1-1)}}{1} = \frac{2(pi)r^0}{1} = 2(pi)(1) = 2(pi)$$

: = about 6.28, a constant

As (r) increases or changes by 1, C will increase or change by 2(pi) = 6.28... :

	r =	1	2	3	4	. . .
C =	2(pi)r =	2(pi)	4(pi)	6(pi)	8(pi)	. . .
C ~=		6.283	12.566	18.85	25.13	. . .

: the difference or change between terms is 6.28... = 2 (pi) , when (r) changes by (1). This is an example of an **arithmetic series** where the difference is constant.

Because the equation had the variable, here (r) raised to just the first power, the ratio of C to (r) also equal to the derivative, and which is a constant numeric value, and as expected for an equation having a linear form.

C / r = 2(pi) : a circles circumference is always larger than its radius by a constant factor of 2(pi) = 6.28...

Ex. From the last equation of a circles circumference, given we can solve for r, and then find the derivative of a circles radius with respect to its circumference:

$$r = \frac{C}{2(pi)}$$

or = ((2)(pi))^-1 C and dr / dC = 1 / (2(pi)) = 1/6.28... = 0.15915... , a constant

This means that when C increases or changes by 1 unit, the radius of that circle has, or must increase or changes by

0.15915...

C =	4	5	6	. . .
r =	0.6366	0.7958	0.9549	. . .

Ex. Find the derivative of the area (A) of a circle, with respect to its radius (r):

A = (pi)r^2 , " dy/dx " = dA / dr = 2(pi)r : = about 6.28r , also: A / r^2 = (pi) =~ 3.14..., a constant
2(pi)r = (pi) D = circumference of a circle

This derivative value is not a constant since it contains a variable, hence the derivative or rate of the increase or change of A with respect to a increase or change in (r) depends on the specific or current value of (r). This derivative is a "first order" or linear equation where the independent variable has a power of 1, and was derived from a "second order" equation where the independent variable has a power of 2.

As (r) increases: When (r) = 1unit, A = 6.28 square-units, and A is increasing or changing by 6.28(1) = 6.28sq-units
When (r) = 2units, A = 12.56 square-units, and A is increasing or changing by 6.28(2) = 12.56sq-un. for each increase in (r). The area, like a distance, is now increasing at a much faster rate (rate) or speed which has been accelerated or increased over 1 time unit which is similar to increasing (r) by 1. In a pseudo-distance equivalence representation: A = (pi)r^2 = distance = (speed)(time^2).

Notice that the derivative for a linear or first-order equation was a number, a constant, and this corresponds to a dimensionless point. We just seen that the derivative of a length, such as a circumference is a constant (here, (2)(pi)). The derivative of a second-order equation was a first-order equation. We just seen that the derivative of an area with two dimensions was a one-dimensional linear equation. Likewise, the derivative of a volume which has three dimensions, is a second-order equation:

V = three-dimensions = (1 dimension)^3 = (1 unit)^3 ,

dV / d (1 dimension) = d(1 unit)^3 / d (1 dimension) = $\frac{\text{2 (1 unit)^2}}{3}$

The equation for the volume of a sphere is: Vs = $\frac{\text{4 (pi) r^3}}{3}$, The derivative of Vs with respect to (r) is:

dVs / dr = (3)(4)(pi)(r^(3-1)) / 3 = 12(pi)r^2 / 3 = **4(pi)r^2** , This value is also equal to the **surface area of a sphere**
It is also equal to **4 times the area of a circle with same(r).**
Extra: **As** = 4 (pi) r^2 = 12.56637061 r^2 =~ **12.5$\underline{7}$ r^2**

If we let r = d/2, As = 4(pi)(d/2)^2 4(pi)(d^2/4) = **(pi) d^2** : Alternate surface area of a sphere formula
Another alternate surface are of a sphere formula is: **As = C D = (2 (pi) R) (2R) = 4 (pi) R^2 = (pi) D^2 = 4 (pi) R^2**
This As = CD derivation of the formula for the surface area of a sphere is mentioned further in this book. This value of 4 (pi) =~ 12.57 is also the number of **steradian** or "three dimensional", solid" angles from the center of a sphere. On a sphere's or spherical surface at a distance of R from the center point, each steradian angle intersects that surface and defines an area of: (length)(width) = (R)(R) = R^2, and where here each length is actually an arc segment of the circles curved surface, but is still equal to R in length.

The derivative of the surface area, (Ss), of a sphere with respect to its radius (r) is:

dSs / dr = d 4(pi)r^2 / d r = 8(pi)r = 25.13274123 r

A fundamental part or lower dimension of a line is a point. A fundamental part or lower dimension of an area (two dimensions) is a line (one dimension). A fundamental part or lower dimension of a volume (three dimensions) is an area (two dimensions).

Extra: The ratio of a spheres volume to its surface area is not a constant value, but depends on its specific radius value:

$\frac{Vs}{As}$ = (4(pi)r^3 / 3) / 4(pi)r^2 = (r / 3) : the volume of a sphere is greater than its surface area by r / 3 = 0.33333 r , also, from this we find:

$Vs = \frac{As\ r}{3}$ = 0.3333 (As r) also:

Extra: **C = 2 (pi) r = (pi) 2r = (pi) d** : **Circumference of a circle** = 3.141592654... d , C = 6.283185307... r

Ac = (pi) r^2 = (pi) (d/2)^2 = (pi) (d^2 / 4) = 0.785398163 d^2 =~ **0.7854 d^2** : **Area of a circle**

$\mathbf{Vs = \frac{4\,(pi)\,r^3}{3}}$ = 4.188790205 r^3 **=~** $\mathbf{4.188\underline{8}\ r^3}$ **=** $\frac{4\,(pi)\,(d/2)^3}{3}$ = $\frac{4\,(pi)\,(d^3/8)}{3}$ = 0.523598775 d^3 **=~** $\mathbf{0.523\underline{6}\ d^3}$: **Volume of a sphere**

Ss = 4 πr^2 ~= $12.5663706\underline{1}$ r^2 :**Surface area of a sphere** , if r doubles to (2r), , (2r)^2 = 4r , hence the surface area will increase by 4.

Also: **Ss** = **4 (pi) r^2** = **C D** = 2 (pi) R (2r) = **(pi) D^2**

Before such formulas as shown above were created, it would have been very practical to measure several examples of circles and spheres so as to come up with some general ideas, conclusions and formulas to consider. If a person was to measure the circumference of a circle, it would always be slightly more than 3 diameter (d) length units long:

C =~ 3 d

With the same reasoning as mentioned above, the area of a circle, volume of a sphere, and surface area of a sphere could be measured just the same so as to have some general ideas, and that some constant value, fraction or multiple of (PI) and power of (r) is in the measured result.

Some thoughts on the surface area of a sphere:

It is now known that the **surface area of a sphere** is 4 times the area of the largest disk or surface area of the circle. Consider looking at a flat disk or sphere from a distance that appears to be a disk, and it has two sides (front and back) of equal area. The surface area of a sphere is at least twice the area of this disk shape seen, but it is actually larger since the sphere is curved and is therefore longer than the diameter seen from a distance, and the value of this distance on the sphere is actually half of the circumference (2 (pi) r) of that sphere, and which is then (C/d) = [2 (pi) r] / 2 r = (pi) times longer. According to the surface of the area of a sphere formula, the "flat" area ((pi) r^2) of each side seen as a flat disk area from a distance is actually twice that same area on the actual and larger curved surface of that side of the sphere, and that value is: 2 (pi) r^2. To then account for the other side of the sphere, twice this value is the surface area of the sphere: 4 (pi) r^2.

> You can also consider the surface area of a sphere as being an infinite number of thin rectangular segments or strips having a length of (C) fully around the circle, and over a width of (D) and so as not to overlap and include any area twice. If the sphere is cut along one side and laid flat, a rectangle can also be conceived at having and infinite number of thin strips having a height of (D), and over a width of (C). The results are also that: **Ss = CD**

Extras: Ac = (pi) r^2 = (pi) (d/2)^2 = (pi) d^2 / 4 = $\frac{(pi)}{4}$ d^2 = $0.78539816\underline{3}$ d^2 = $0.785\underline{4}$ (4 r^2)

From this we can also conclude: d^2 = 4 r^2 , and that the surface area of each side of the sphere is (pi) / 2 times more than this value, or that the total surface area of the sphere is (pi) times more.

A / C = (pi) r^2 / 2 (pi) r = r / 2 , mathematically: A = C r / 2 = C d / 4 = (3.14 d) d / 4 = 3.14 (d^2 / 4)
A = 3.14 (0.25 d^2) = (pi / 4) d^2

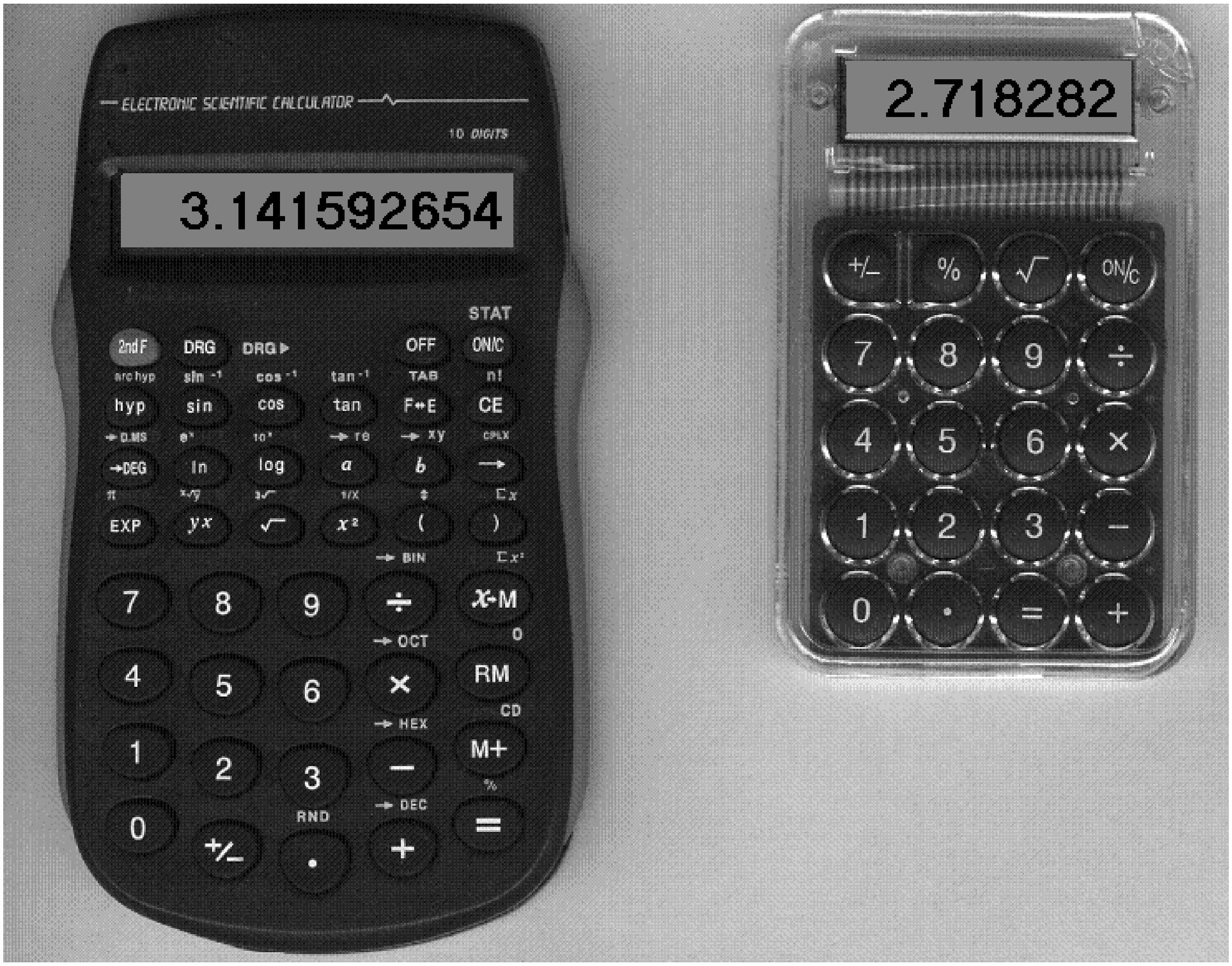
C = 2 (A / r) = 2 (pi) r = 2 (pi) (d / 2) = (pi) d and r = 2 (A / C) and d = 2r = 2 [2 (A / C)] = 4 (A / C)

Extra: If the radius or diameter of a circle changes by a factor of (n), the circumference will change by that same factor value of (n), and the area will increase by (n)^2. C = 2 (pi) r = (pi) D , and if we multiply each side by (n): (n) C = (n) 2 (pi) r = (n) (pi) D. Given: Ac = (pi) r^2 , (n^2) Ac = (pi) (nr)^2 = (pi) n^2 r^2 = n^2 (pi) r^2

If the radius or diameter of a circle, or side of a square increases by sqrt (n) , Ex: $\sqrt{2}$, its area will increase by (n):
Ac = (pi) r^2 , 2Ac = (pi) ((sqrt 2) r)^2 = (pi) 2 r^2 = 2 (pi) r^2 , also : As = s^2 , 2As = ((sqrt 2) s)^2 = 2 s^2

SECTION 4: ADVANCED TOPICS

This section is about making some important calculations when a scientific calculator or reference ("look-up") table(s) is not available. Specifically, this section shows you how to calculate logarithms, antilogarithms (inverse logarithms), trigonometric functions values of an angle, and inverse trigonometric functions, that is, calculating the corresponding angle given its corresponding trigonometric function value. Actually, this section could be called "Required Topics" since how could you perform these crucial mathematical operations out of "thin air", without a "look-up" table, or some direction as to how to proceed in their calculation? Since these calculations (especially division) could become tedious when repeatedly performed with hand, pen and paper, something such as a modernly inexpensive 4-function, home-use type of electronic calculator can be utilized to its fullest potential as if it were practically a full-function scientific calculator. Many of these types of electronic calculators also have a generous square-root, "fifth-function", key which is of great aid, especially for calculating some trigonometric values. Much of this section can also be adapted for computer programming languages that may or may not have these preprogrammed functions. In the image below, on the right is a typical 5-function "home-use calculator". On the left is an example of a scientific or multifunction calculator. Today, calculators are often inexpensively available at thrift stores, and some have a solar power option. There are also calculator apps for phones. [Image magnified about 25%]
[FIG 195]



Before the invention of electric or electro-mechanical calculators there were helpful mechanical devices such as the **abacus**. The abacus was used by most people who had one, so as to add and subtract, but it could also be used for multiplicatoin and division. The main concept of the abacus is using moveable or slideable beads on a rod or in a slot. It is a very old device, first used in about 2500BC in Sumeria - Mesopotamia (ie., the ancient Irak region), and was then used in some form in many parts of the "old-world" (the eastern hemisphere of Earth). It is still in minimual use, and even as a "converstation piece".

RADIAN ANGLE MEASUREMENT

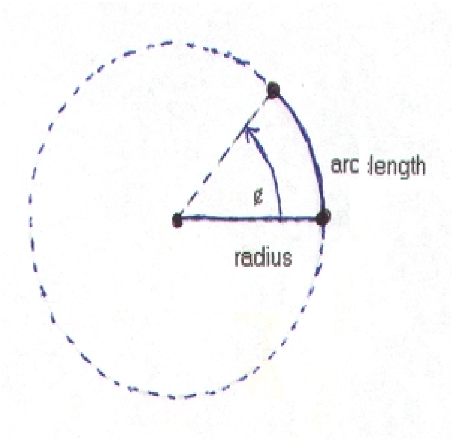
Just like degrees is a unit of measurement for angles, radians is also a unit of measurement for angles. The word radian is derived from the word "radius" of a circle. A radian angle measurement is natural or inherent to every circle's construction and is therefore a way to measure angles in contrast to the various man made values and units such as degrees (360), grads (400), revolutions (revs or turns), percent of a rotation, etc.

A radian angle is defined and measured using only the natural physical properties of all circles. All circles have the same shape and are all similar in many ways mathematically, and specifically, the ratios of parts within each circle's construction is the same as that for any other circular construction. For example, since C = circumference = 2(pi)(radius), the ratio of the circumference of a circle to the radius of that circle is always: (circumference)/(radius) = 2(pi) =~ 6.28 , hence the circumference is always about 6.28 times longer than its radius.

A degree angle measurement is the amount of parts, or the fractional value, of a full rotation that is defined as equal to 360 parts or degrees. 1 degree = 1° = the portion or fraction of: (1/360 = 0.0027777...) of a full or complete circle or rotation having a total angle of 360° A radian angle measurement is the ratio (ie., or the "number of times") of the length of an arc on the circumference of a circle with respect or in reference to the radius length of that circle.

An angle made when one point rotates about another "fixed" point, such that the point being moved describes (defines) an arc that is equal to that of the distance (ie., radius of rotation = radius of the circle) between those two points, is defined as an angle of one radian (or rad). For a give circle, the greater the arc length, the greater the angle, and vice-versa.

For this discussion about radians, consider that the point rotating or orbiting about another point, is located at the end of the radius line of a circle, and that the point is rotating about the center point of that circle. An arc is a portion (or "segment") of a circles circumference, and has an arc length, "length of arc", or distance associated with it. [FIG 196]



Expressing this using a mathematical equation:

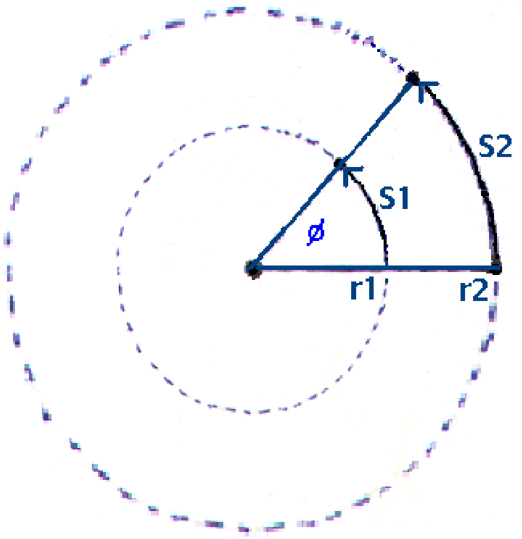
$$\text{¢} = \text{¢ radians} = \frac{\text{arc length}}{\text{distance between points}}$$

: arc length is often noted as the letter S
S = (¢ in radians) (r)

Radius length (r) is often substituted for distance from the center of rotation to a point on the circumference or path of rotation, and this is where the word "radians" originates:

$$\text{¢} = \text{¢ radians} = \frac{\text{arc length}}{\text{radius}}$$: When S = r, the angle is 1 radian.

Note that a radian angle generally has nothing to do with the actual size of a circle or the specific distance (ie., radius) from the center point of rotation. All the possible and different arc lengths of concentric (have the same center) paths or orbits of points with each having their own corresponding distance (ie., radius) from the center point of rotation, can describe the exact same angle. For the same fixed or given angle, the farther a point is from the center of rotation, the greater the corresponding arc length corresponding to that angle, and this is similar to how the corresponding side opposite an angle in a triangle also depends on the size of that angle and the distance from it's vertex: [FIG 197]



This figure above shows that different arc lengths and radius values can correspond to the same angle, and vice-versa. This is due to that all circle constructions are similar and the internal proportions of one is the same as that as another. Specifically, as an example for this discussion of concentric circles:

(arc length) / (radius) = a constant ratio value equal to the (radian) angle.

Since arc length and radius length have the same units, their units will be canceled during the division, hence radian angle measurement, in a strict sense, is a (strict) "numeric only" or unitless quantity, and is often indicated that way as just a (ratio) value without any indicated units. Generally, if an angle measurement does not have a degree symbol (°) or some other indication that the units are degrees, it could then be considered as a radian angle measurement. Many scientific calculators have a radian or degree angle, working mode setting (button, key, switch) for entering (as operands) angles, and displaying angles (the results). From the above equation we mathematically have this handy formula:

arc length = ¢ (radius) : ¢ is a radian angle , If the arclength = C , ¢ = 2(pi) = 6.28... radians

s = ¢ r

We see that arc length is directly related to the angle and the radius. If either the angle or radius changes by a factor (n), arc length also change by that same factor. For example, if the angle doubles (multiplied by 2) the corresponding arc length will also double. If the radius, orbit, or rotation distance doubles, then the corresponding arc length will double:

(n) arc length = ¢ (n) radius : for example, for a given angle, if the radius is halved by multiplying by 0.5, the corresponding arc length will be halved. This equation can also be expressed as:

(n) arc length = (n) ¢ radius : for example, for a fixed (in value, constant) specific radius, if the angle doubles, the corresponding arc length doubles, or vice-versa

We see that arc length is directly related and proportional (ie., same portions of parts in each similar construction, and therefore, the same ratio) to both the angle and the radius.

Given an arc length, the corresponding sector area can be calculated, and vice-versa, and a discussion of this has been included as part of the topic of: Surface Area Of A Cone, and it is given further ahead in this book.

The circumference (C) of all circles can be divided by its respective diameter (D) a total of π ("pi") times. pi is a value of about 3.14. π is a constant value and classified as an irrational value since this value cannot be fully or rationally represented mathematically as the quotient of one value divided by another, as like a rational value could be represented. For example, 5 is a rational value since 5/1 , 10/2 , and 2.5/0.5 can be used to represent it, or in other words, these division problems have a definite quotient which is 5.0. Irrational values such as π have an endless and non-repeating digit pattern to the right of the decimal point. The approximate value of π correct to fifteen decimal places is:

$$\pi = 3.141592653589793...$$

For any and all circles, the ratio of its circumference to its diameter is a constant value equal to pi:

$\frac{C}{D} = \pi$ therefore mathematically: $C = \pi D$ and $D = \frac{C}{\pi}$

Since the diameter of a circle equals twice the radius of a circle (D=2R):

$C = \pi D = 2\pi R$ therefore, $\frac{C}{R} = 2\pi$: ~ $2\pi = (2)(3.14) = 6.28...$

If an arc length is equal to the circumference of the circle, there are 2π radians in a circle:

$¢ = \frac{\text{arc length}}{\text{radius}} = \frac{C}{R} = 2\pi$ (understood as 2π radians or "rads") : equivalent to a 360° angle, or full rotation. 360° = 2π , 360 degrees = 2(pi) radians

We are now at a point where we can convert between radian and degree angle measurements.

Given 360° = 2π radians after dividing both sides by 2π :

57.29577051° = 1 radian after dividing both sides by 1 radian :

57.29577051° / 1 radian : 57.29577051 degrees per radian

Or given: 2π = 360° (remember, the "radian units" are not often indicated since the values are "numeric" or unitless)

Solving for (1) radian(s):

$\frac{2\pi \text{ rads}}{2\pi} = \frac{360°}{2\pi}$ after canceling:

$1 \text{ rad} = \frac{180°}{\pi}$: about 57.29577951° , or ~ 57.3° as indicated above

Hence, to convert radian angle measurement to its equivalent degree angle measurement, it's simply a matter of repeated addition, or multiplication (to both sides of the equation as shown below), multiply the radian angle measurement by (180°/π) or = 57.29577951°. For example, to convert 0.5 rads (radian units of angle measurement) to it's equivalent amount of degrees, multiply each side of the above equation by 0.5:

Ex. $0.5\ (1) \text{ rads} = 0.5\ \frac{(180°)}{(\pi)} =$ about 28.648° : For a radians to degrees conversion, use the "degrees on top" from the fraction (180°/π) for the multiplication. This is easier to remember than 57.29577951

Given 2π = 360°, solving for (1) degree:

$360° = 2\pi \text{ rads}$ dividing each side by 360:

$\frac{360°}{360} = \frac{2(pi) \text{ rads}}{360°}$ after canceling:

$1° = \frac{\pi}{180} \text{ radians} =$ about 0.017453292 radians. Here, it is obviously easier to remember the numerator and denominator of this fraction, rather than remember this quotient value. If using this quotient value: radians = (0.017453292) (degrees angle)

To convert a degree angle measurement to its equivalent radian angle measurement, multiply the degree angle measurement by (π /180°) = 0.017453292.

Ex. $10°\ (1°) = \frac{10°}{1}\ \frac{(\ \pi\ )}{(\ 180°\ )} = \frac{\pi}{18} =$ about 0.1745 radians : For a radian to degree conversion use "degrees on top"

This is equivalent to multiplying both sides of the previous equation by the same value (here, 10°).

¢ degrees = $\frac{180°}{(pi)}$ (¢ radians) : **radians to degrees conversion**, use "degrees on top".
degrees = 57.29577951 (radians)

¢ radians = $\frac{(pi)}{180°}$ (¢ degrees) : **degrees to radian conversion**, use "pi on top".
radians = 0.017453292 degrees , degrees = radians / 0.017453292

Using a full circle as a starting reference, here's how you can get a perspective of corresponding radian and degree values:

From: 360° = 2(180°) = 2π (about 6.28 radians) : full circle, an angle of a "full turn" or rotation

Dividing by 2 to find out the amount of radians equivalent to 180°:

180° = 1 π = π (about 3.14 radians) : half a circle : = "two quarters" = 50 percent of a circle
π , "pi", as an (radian) angle, can be thought of as half a "full turn" or rotation.

Dividing by 2 to find the radian angle equivalent of 90°:

$90° = \frac{\pi}{2} = 0.5\,\pi$ (about 3.14 / 2 = 1.57 radians) : "one quarter" (25 percent) of a circle

Since 270° = 90° x 3 , and 90° = π/2 we have:

$$270^\circ = \frac{3(\pi)}{(2)} = 1.5\,\pi$$ (about (3)(1.57) = 4.71 radians) : "three quarters" (75 percent) of a circle

Below is a table that summarizes these values:

A TABLE OF CORRESPONDING DEGREE AND RADIAN ANGLE MEASURMENT VALUES

degrees angle = radian angle = (approximate value)

$$90^\circ = \frac{\pi}{2} = \pi/2 = 1.57$$

$$(2)90^\circ = 180^\circ = 2\,\frac{\pi}{2} = 1\pi = 3.14$$

$$(3)90^\circ = 270^\circ = 3\,\frac{\pi}{2} = 1.57\pi = 4.71$$

$$(4)90^\circ = 360^\circ = 4\,\frac{\pi}{2} = 2\pi = 6.28$$

A non-radian analysis of an arc length of a circumference:

Ex. Given a circle, the arc length of a quarter (90°) of its circumference is obviously a quarter (1/4) of that circumference length, hence (1C/4), and since: 360°/90° = 4 :

$S = C/4 = 2(\text{pi})\,r/4 = (\text{pi})\,r/2 = 1.570796327\,r \;=^{\sim}\; 1.570\underline{8}\,r$: for a 90° rotation

For other angles that are not 90° , what is their arc length? We can put that value in terms of what we know with the 90° angle or rotation. For example if the angle is 45° which is half of 90° , we should expect that its corresponding arc length is also half. This is a linear relationship. Expressing this concept in a formula, we have:

$$S = \frac{1.5708\,r}{\left(\frac{90^\circ}{\text{¢}^\circ}\right)}$$

The (straight) diagonal line and-or chord length near and within a 90° arc length part of the circle obviously has a (slightly) shorter length than that of the arc length or portion of the circles circumference:

$$\text{diagonal length} = \sqrt{r^2 + r^2} = \sqrt{2r^2} = \sqrt{2}\,\sqrt{r^2} = 1.414213562\,r$$

SERIES

A mathematical, or numerical series or progression is a sequence (or "string") of numbers or terms that are mathematically related in some way, where either or both the next term or previous term in that specific series can be calculated by knowing the mathematical relationship (ie., a mathematical pattern) between, or of those ("adjacent") numbers or terms.

Every and all terms in a series can be formally represented or expressed algebraically by a single representative term that expresses the general format of each successive term in that series.

Formulas can also be developed to calculate the value of any specific term of a series, or for the sum of a certain number of terms in that series. Later, some very important series will be developed to allow us to calculate things such as trigonometric, natural logarithm, and natural anti-logarithm values from the sum of the terms of these special series.

Here is a pseudo-algebraic description of a series (of mathematically related terms or values):

term1
term2 = mathematical operation(s) applied to term1 : Though the mathematical operation(s) is constant, the
term3 = mathematical operation(s) applied to term2 result of that mathematical operation(s) that is to be
term4 = mathematical operation(s) applied to term3 applied to each term is not necessarily a constant value.
(termN or term(N) using a special notation (and which is not multiplication) and so on for other terms of the series.)

We see that the general format of each successive, or next, term is:

term(n+1) = mathematical operation applied to term(n)

For example, if given a term value of 5, and we keep adding 2 to each term so as to find the value of the next term, we will have this series:

successive term numbers = n: 1 , 2 , 3 , 4 , . . .

successive term expressions: 5 , 5+2 , (5+2)+2 , ((5+2)+2)+2 , ... : (term n+1) = (term n)+ 2 , or:
"next term" = "current term" + 2
Mathematically:
(term n) = (term n+1) - 2 , or:
"previous term" = term - 2

Simplifying the expressions:

successive term values: 5 , 7 , 9 , 11 , , . . . : a portion of an odd numbers series

A certain mathematical operation was applied to a term so as to find the next term. If given that value of the next term, and so as to find the value of the previous term, the inverse, "reverse" or "undoing" operation of that given operation is to be applied to that given term value. The operand will be the same value as that first applied. This can be expressed as:

(term)----> operation -----> (next term) therefore: (next term) -----> inverse operation ------> term
or=: term ------> inverse operation ------> previous term

Ex. 5 -------> +2 -------> 7 therefore: 7 ---------> -2 --------> 5 : checking: 5 + 2 = 7, and 7-2 = 5.
Subtraction and addition are inverse
operations with respect to each other

Since the expressed operation indicated above included both the operand and mathematical operation to apply to the term (which could be either a single value or possibly an expression considered as a single value), another notation is:

(term)-----> function -----> (next term)   therefore: (term) <------ inverse function <------- (next term)
or =:   (next term) -------> inverse function ------> (term)

ARITHMETIC SERIES

In an arithmetic series, successive term values are separated by a value called the difference (d). We are all familiar with the arithmetic series formally known as the counting (whole and integer) numbers where the difference (d) between successive terms is 1:

. . . -3, -2, -1, 0, +1, +2, +3 **. . .**   : integer number series where each term differs by (d=1)

In the "even series": 2, 4, 6, 8, 10, and so on, the change or difference between successive terms and-or between the value of a term and the previous term is always 2. Likewise, for the "odd series": 1, 3, 5, 7, 9, etc. (and so on), as was previously shown, the difference is also always 2. If the difference is not known, it can be found from subtracting the current term from the next term in the series since:

from: current term value + difference = next term.   and   previous term = current term - difference.

difference = next term - current term   difference = current term - previous term

Since a series might not be an ascending (term values increasing as the term number increases) series, but a descending (values decreasing) series, perhaps a better name to use than difference, would be (a signed) change (c) since the value to be added to find the next term may be either positive for an ascending series, or negative as for a descending series. The concept of a difference can still be used, but it must be considered as a "signed difference" so as to be more algebraically and mathematically useful:

next term = current term + change

The actual value of the variable, here called: (change) can be either positive or negative in sign. Any variable(s) such as the ones shown here should be considered as having either a negative or positive sign. This is so because a placeholder is not the actual value, and that what it holds or contains is, and the sign is actually a part of that unique number. Solving for change:

change = next term - current term

Since a value (either positive or negative in sign) is added to a given term so as to find the next term of an arithmetic series, an arithmetic series can be considered or thought of as a "basic addition series".

When the next term becomes, or is now considered as the "new" current term in question, the "old" current term effectively becomes the previous term. Substituting this into the above expression:

change = current term - previous term   solving for previous term:

previous term = current term - change

This can be expressed in a more general way as:

If you were to add the same value (here, d) to each new term or sum created, this can be expressed as:

$\text{value}_{(n+1)} = \text{value}_{(n)} + d;$: n = the corresponding term number , or since the values are the values of the terms, (n+1) would correspond to the next term, and (n-1) would correspond to the previous term. (n) would correspond to the current term:

. . . , "previous terms" , term , "next terms" , . . .
. . . , value(n-1) , value(n) , value(n+1) , . . . or:
. . . , term(n-1) , term(n) , term(n+1) , . . .

Solving for the added in constant value, or (d):

value(n+1) - value(n) = d Given a term in this series and solving for the previous term value:
or: value(n) - value(n-1) = d

value(n+1) - d = value(n) : since value represents the value of a term, the word "term" can be used instead
or: value(n) - d = value(n-1) of the word value:

term(n+1) = term(n) + d

term(n)
term(n+1) = term(n) + 1d
term(n+2) = term(n+1) + d = (term(n) + d) + d = term(n) + 2d

Ex. The whole number series where d = +1:

0, 1, 2, 3, 4, 5, . . . : an ascending (increasing in value) series

Ex. The integer number series where d = +1:

. . . , -5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5, . . .

Ex. In this series, d = +2 : :the "even series"

0, +2, +4, +6, 8, . . .

Ex. In this series, (d) is also = +2 : :the "odd series"

+1, +3 , + 5 , +7, . . . : with d = +2, this very similar to the "even series", except that the first term here is 1

Ex. In this series, d = -2 :

. . . , +8, +6, +4, +2, 0, -2, -4, -6, . . . : a descending (decreasing in value) series

Ex. In the series below, d = +2.5 :

. . . , 0, 2.5, 5.0, 7.5, 10.0, 12.5, . . .

For the series above, what is the next term after 12.5?

next term = current term + change

next term = 12.5 + (+2.5)
next term = 12.5 + 2.5 after combining:
next term = 15.0

For the series above, what is the next term after 30?

next term = current term + change
next term = 30.0 + (+2.5)
next term = 30.0 + 2.5 after combining:
next term = 32.5

Let's express the terms of an arithmetic series in a pure (without using actual number values) algebraic format:

term(n), term(n+1), term(n+2), term(n+4), . . . or:

term1, term2, term3, term4, . . . using algebraic substitution:

term1, term1+d, term2 + d, term3 + d

term1, (term1+d), (term1+d)+d, ((term1+d)+d)+d simplifying:

term1, term1+1d, term1+2d, term1+3d

Note that all the terms now contain the initial, reference or (considered as the) "first" term of: (term1), the first term, as part of the expression that represents the value of that specific or current term in question. Each expression for the next term also contains another term added in that is the product of the term number (N) less one, and the difference. Expressing this:

termN = term1 + (N-1)d or: termN = firstterm + (N-1)d : first_term is the value of any term of a series that is being considered as the start, current, or reference term of that (infinite) series of terms

Ex. Here is the expression for the value of the 10th term of an arithmetic series, given the first term:

term10 = term1 + (10-1)d simplifying:
term10 = term1 + 9d

As an example, here is a series where the difference is +3, and if the first term is +2, what is the value of the 10th term?

term10 = term1 + (10-1)d = +2 + (10-1)(3) = 2 + (9)(3) = 2 + 27 = 29 checking:

N = 1 , 2 , 3 , 4 , 5 , 6 , 7 , 8 , 9 , 10
term = 2 , 5 , 8 , 11 , 14 , 17 , 20 , 23 , 26 , 29

Below is another, perhaps similar and simpler form of the above simplification:

Here is a simple method to find the difference (d) between each term of an arithmetic series when two consecutive terms are not even known so as to quickly find the common difference value of that series. Now, rather than have been given two consecutive terms, you are given any two terms of that series and their term numbers Since the difference (d) is added to each term to find the next term and so forth, the total sum of these differences can be expressed with multiplication since multiplication is repeated addition:

(number of terms from an initial term)(d) = (total difference in the terms values) therefore:

$$d = \frac{\text{total difference in terms values}}{\text{number of terms from an initial term}}$$

Ex. What is the common difference (d) of a series if the first term is (considered as) 6 and the fourth term is 10.5?

$d = \frac{10.5 - 6}{4 - 1} = \frac{4.5}{3} = 1.5$, A partial (fragment, portion, section, or segment (of the)) series with this is shown below:

. . . 6, 7.5, 9, 10.5 , **. . .** : d = 1.5

You can also find (d) by writing an equation. If the first term is 6, then the second term is (6 + d), the third term is ((6 + d) + d) = 6 + d + d , or after combining like terms: third term = 6 + 2d. Since we know that the value of the fourth term is 10.5, we get:

term in question - first term = total difference to term in question algebraically:

first term + total difference to term in question = term in question which can be expressed as:
first term + (number of terms from initial term)d = term in question

$6 + 3d = 10.5$ solving for (d), we get:

$3d = 10.5 - 6$
$3d = 4.5$
$d = 4.5/3 = 1.5$

Considering what has just been discussed, a simple formula to find the nth term of an arithmetic series is derived as follows:

Let n = nth term in question
Let F = value of the first or initial term taken, given or considered as the first term of the series

n	=	1	2	3	4	. . .	n	: term number
Term	=	F	(F+d)	(F+d)+d	((F+d)+d)+d	. . .		clearing grouping symbols :
Term	=	F	F+1d	F+2d	F+3d	. . .	F + (n-1)d	: after combining like terms

By observing the above sequence for any patterns, we can arrive at a general (algebraic) formula for the nth term. Notice that the numerical coefficient of d is 1 less than the term in question, hence:

nth term = F + (n-1)d : **FORMULA FOR THE nth TERM OF AN ARITHMETIC SERIES**
Since a series can have an infinite number of terms, there may not be a first term or start of that series, so (n=1) is the number of the term that is taken or considered as the first term of this particular string or segment of terms under consideration.

Using the last example's series, what is the value of the 5th term?

4th term = 6 + (4-1)1.5
4th term = 6 + (3)(1.5) distributing:

4th term = 6 + 4.5 combining:
4th term = 10.5

Checking by adding d=1.5 to the previous or fourth term:

5th term = 4rd term + difference
12.0 = 10.5 + 1.5 :checks

Finding the 5th term directly from the equation:

5th term = 6 + (5-1)(1.5) = 6 + (4)(1.5) = 6 + 6 = 12

Checking:	(n):	1	2	3	4	5	. . .
	term:	6 ,	7.5 ,	9 ,	10.5 ,	12 ,	. . .

Now that we know the formula for the nth term of an arithmetic series, we can also use this with the concept of simultaneous equations so as to find the value of the difference (d) between terms, even though a formula for (d) has already been shown.

Ex. Find the difference (d) if the third term is 6 and the tenth term is 20.

This information can be expressed as a system of simultaneous equations:

third term: F + 2d = 6 : First term + total difference to the 3rd term = third term
tenth term: F + 9d = 20 : First term + total difference to the 10th term = tenth term

Algebraically solving for F in the third term equation:

F = 6 - 2d algebraically substituting this value of (F) into the equation for the tenth term:

F + 9d = 20
(6 - 2d) + 9d = 20 removing parentheses (distribute +1):
6 - 2d + 9d = 20 combining like terms:
6 + 7d = 20 transposing terms which do not contain the variable (d) in question to the other side of the equation:

7d = 14 dividing the left and right side by 7 to isolate (d):
d = 2

The two simultaneous equations above were solved using an algebraic substitution method, however, the addition and subtraction method, or any other possible method could also have been used.

At this point, you can also find the value of the first term (F) by substituting (d) into any of the original simultaneous equations:

F + 2d = 6 F = value of the first timer. Solving for F, after transposing (2d):
F = 6 - 2d substituting the calculated value for (d):
F = 6 - 2(2) distribute to clear grouping symbols:
F = 6 - 4 combining:
F = 2

Here is the partial series:

Term n :	1	2	3	4	5	6	7	8	9	10 . . .
Term value:	. . . 2 ,	4 ,	6 ,	8 ,	10 ,	12 ,	14 ,	16 ,	18 ,	20 , . . .

Though less used, to solve for previous terms in an arithmetic series, the change (regardless if its positive or negative in value) is not added, but subtracted instead since:

From: next term = previous term + change , and if the next term becomes the new current term of reference:

current term = previous term + change therefore:
previous term = current term - change or:

current term = previous term + d therefore:
previous term = current term - d

With F still considered as the first term, this leads to a slightly rewritten formula for the nth (previous) term of an arithmetic series:

first term = F : n=0
first previous term = (F - d) : n=1
second previous term = (F - d) - d = F - 2d : n=2

nth previous term = F - nd : **FORMULA FOR THE nth PREVIOUS TERM OF AN ARITHMETIC SERIES**

As a final note, considering any three consecutive terms of an arithmetic series, the second term, or "middle term", can be found without knowing the difference (d) between any terms.

Term #:	n	n+1	n+1	considering term n as the first term for this specific analysis:
Term:	F	F+d	F+2d	

Adding or "summing" the first term and third term (essentially, the previous term and next term considering if they were in reference to the second or middle term):

Sum = F + (F + 2d) clearing grouping symbols:
Sum = F + F + 2d combining like terms:
Sum = 2F + 2d factoring 2:
Sum = 2(F + d) dividing both sides by 2:
$\frac{\text{Sum}}{2} = (F + d)$: We arrive at the value of the second term. This derivation indicates, that the middle term can simply be found using the arithmetic mean of the two given terms.

Note also that (d) can be found by:

$d = \frac{\text{Sum}}{2} - F$: due to the derivation, as noted above, Sum = previous term + next term, when considering the (middle) term between those two terms in the series. Essentially, three consecutive terms of the series will be found: first, second or middle, third or terms: n , (n+1), (n+2)

Subtracting the first term from the third term, (2d) and also (d) can be found:

2d = (F + 2d) - (F) Let 2d equal the total difference between the first and third terms. Distributing (-1):

$2d = F + 2d - F$ combining like terms:
$2d = 2d$ after solving for (d) by dividing by 2:
$d = d$

Ex. If the first term (or any term considered as the first) is 5 and the third term is 15, what is (d)?

$15 - 5 = 10$: $10 = 2d$

$$d = \frac{10}{2} = 5$$

The above concepts can be expanded into another general formula to find (d) :

Considering one term as the first term (F) and the nth term as $F_n = F + (n-1)d$ and subtracting the two values to find the total difference:

nth term - First Term = total difference

total difference = $F + (n-1)d - F$ combining like terms:
total difference = $(n-1)d$ isolate (d) by dividing by its multiplying factor of (n-1):

$$d = \frac{\text{total difference}}{(n-1)} = \frac{(n-1)d}{(n-1)} = d$$: or in general:

$$\mathbf{d = \frac{Fn - F}{n-1}}$$: this is similar to an average formula, here, an average difference or change from F to Fn, and (n-1) is the "number of differences". (Fn - F) is the total difference or change.

Ex. If the first term is 3 and the fifth term is 15, find (d).

$$d = \frac{15 - 3}{5-1} = \frac{12}{4} = 3$$ here is the partial series:

. . ., 3, 6, 9, 12, 15, . . . : $d = 3$

A method to find the nth previous term of an arithmetic series can be developed which uses the same formula to find the nth next term of an arithmetic series, however, the change or difference used in the formula will be noted as the "inverse", "reverse", or negative of the difference = $-(d) = -d$:

next term = term + d or next term = previous term + d mathematically:
previous term = next term - d or previous term = term - d the next previous term is:
previous term - d = (term - d) - d = term - d -d = term - 2d = F - 2d observing the pattern:

nth previous term = $F + n(-d) = F + n(-1)(d) = F + (-1)nd =$ **F - nd : nth previous term of an arithmetic series**

The appendix contains a discussion about summing consecutive terms of an arithmetic series.

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GEOMETRIC SERIES

In a geometric mathematical or number series, successive or adjacent terms or numbers differ or vary by a constant factor value. This factor is simply called the ratio (r). To find the next term of a geometric series, multiply the current term by the common ratio value between the terms. In a reverse type of manner, to find the previous term of a geometric series, divide the current term by the ratio value. If the ratio for a geometric series is not known, it can be found by dividing the next term by the current term in the series, or dividing the current term by the previous term.

(term) x (ratio or multiplier) = (next term) : letting (term) = (current term) in question or as reference to both the (previous term) and (next term)

$$\text{ratio} = \frac{\text{next term}}{\text{term}} = \frac{\text{term}}{\text{previous term}}$$

If the next term is now considered as the current term, the "old" current term effectively becomes the previous term:

previous term x ratio = term solving for previous term:

$\text{previous term} = \frac{\text{term}}{\text{ratio}}$ or = term/ratio = (term) (1/r) = term or current term times the reciprocal of the ratio. In one direction, to find the next term, we multiply by (r), and to find the next term in the other direction, we multiply by (1/r), the reciprocal value, which is the same as dividing by that ratio value. Still, in either way or direction, the multiplier is the essentially the reciprocal of the value used for the other direction.

This can be expressed in a more general way as:

value(n+1) = (r) value(n) : n = the corresponding term number , or since the values are the values of the terms:

term(n+1) = (r) term(n) mathematically:

$\frac{\text{value2}}{\text{value1}} = \frac{\text{term(n+1)}}{\text{term(n)}} = r$ mathematically:

$$\frac{\text{term(n+1)}}{r} = \text{term(n)}$$

Ex. In a series, the first term is 3, and r = +2:

For one direction: multiplier = factor = r = 2

term x multiplier = next term

3
3 x 2 = 6
6 x 2 = 12
12 x 2 = 24
24 x 2 = 48

. . . , 3, 6, 12, 24, 48, **. . .** : an ascending (in value) series, since: (r > 1)

For the other direction: multiplier = factor = (1/r) = 1/2 = 0.5 : multiplying by (1/r) is the same as dividing by (r)

term x multiplier = next term

48 : here, considering 48 as the first term, but any term in the (ascending) series could be used
48 x 0.5 = 24
24 x 0.5 = 12
12 x 0.5 = 6
6 x 0.5 = 3

. . ., 48, 24, 12, 6, 3**, . . .** : a descending (in value) series, since: (r < 1)

Ex. Here is another series where r=+2:

1, 2, 4, 8, 10, **. . .** : r = +2

Ex. 1.5, 3, 4.5, 6.75, 10.125, **. . .** : r = +2

Ex. In the series below, r = +0.5

. . . , 100, 50, 25, 12.5, 6.25, **. . .** : a descending series, r < 1 , here the multiplier = +0.5, but in the other direction the multiplier to find the next term is (1/r) = (1/0.5) = 2 , since 2 > 1, the value of the terms will increase or "ascend".

Ex. In the series below, r = -2

. . . , +100, -200, +400, -800, **. . .** : an "alternating series". This description is in reference to the alternating sign of the terms, where the terms alternate or change in positive or negative sign. Considering just the absolute or signless value of each term, it is an ascending series. This is because the absolute value of (r) is still greater than 1.

Here is a method to find the common ratio (r) between terms of a geometric series when two successive or consecutive terms are not known:

Let n equal the term number in question. Let F equal the value of the first, or any term considered, taken or given as the first or reference term of the series in question.

n	=	1	2	3	4	**. . .**	
Term	=	F	Fr	(Fr)r	((FR)r)r	**. . .**	after simplification:
Term	=	F	Fr	Fr^2	Fr^3	**. . .**	

By observing the above sequence for any patterns, we can derive or develop a general formula for the nth term of a geometric series. Notice that the indicated power (the exponent) of (r) is 1 less than the term number in question, hence:

nth term = Fr^(n-1) : **FORMULA FOR THE nth TERM OF A GEOMETRIC SERIES**

Ex. If the first term of a geometric series is 2 and the third term is 18, what is the second term?

To solve, first find the value of (r) for this particular series. We know:

F = 2 : first term, and where n=1: Fr^(n-1) = 2r^(1-1) = 2r^0 = 2(1) = 2

Fr^(3-1) = Fr^2 = 18 : third term

Choosing to solve these simultaneous equations for (r) by algebraic substitution:

Fr^2 = 18 substituting the known value of the first term:
(2)r^2 = 18 solving for (r)

r^2 = $\frac{18}{2}$

r^2 = 9 taking the square root of both sides of the equation:

r = $\sqrt{9}$ = 3

The second term is therefore:

Fr^(2-1) = Fr^1 = 2(3^1) = 6

And the (partial) series is shown below:

. . ., 2, 6, 18, . . . : r=3

When r<1 and positive (usually indicated as 0<r<1, ie. a fraction of 1, since it is less than 1), that the values of the next terms will get smaller and smaller till they are effectively meaningless or practically equal to zero. If the terms of this type of series are summed from any point (ie. a term) in that series, the sum of those terms will approach a definite or specific value. The sum is said to converge or "zero-in" (having only a small negligible difference, practically 0) to a certain value. This type of series is called a converging series. Even with this said, it is possible for the terms of some geometric series to get smaller and smaller and yet the sum does not converge to a specific value. These types of series are called diverging or divergent series. When making calculations where only one specific value is being sought, converging series will obviously be utilized.

There are many other useful formulas for working with series, such as the sum of the first (n) terms of a series that is shown in the appendix section of this book. This basic discussion on series is meant to prepare you for the discussions that follow in this book. It should also be mentioned that the series, such as the trigonometric series, to be discussed further ahead, do not have a constant difference or ratio between successive terms. Each term is still predictable due to a pattern and can be calculated (ie. via an algebraic formula), and that the sum of these series still converges to a specific value. This specific value of the sum of terms depends upon the specific value of the independent value or variable used in the terms that are to be summed. This calculation using a series of terms is still an approximation (ie., not "algebraic" or exact as with a simple formula) of the true result, and as close (ie., acceptable precision, fineness) as desired.

How do I solve for a previous term of a geometric series?

Though less used, the method to solve for the previous term of a series is based on the same method to find the next term of a series:

From: (term) (r) = (next term) mathematically:

$$r = \frac{\text{next term}}{\text{term}} \quad \text{or=} \quad \frac{\text{next term}}{\text{current term}} \quad \text{or=} \quad \frac{\text{current term}}{\text{previous term}} \qquad \text{mathematically:}$$

$$\text{term} = \frac{\text{next term}}{r} \quad \text{or:} \quad \text{previous term} = \frac{\text{term}}{r} \quad \text{or:} \quad \text{previous term} = \frac{\text{current term}}{r} = \text{current term } (1/r) \text{ , therefore:}$$

The ratio of the previous term to the current term is (algebraically solving for it):

$$\frac{1}{r} = \frac{\text{previous term}}{\text{current term}} = r\text{^-1} = (1/r)$$

This is essentially the inverse or "reverse" of the ratio of that of when finding the value of the next term in a geometric series. This reciprocal of (r) can still be >1 or <1, and it depends if the initial series was an ascending (r>1), or descending (r<1) series.

$$\text{previous term} = \text{current term} \; \frac{(\,1\,)}{(\,r\,)} = \text{current term } (r\text{^-1}) = \frac{\text{current term}}{r}$$

Ex. If the 7th term of a series is known to have a value of 64, and (r) is known to have a value of 2, what is the 3rd term?

Since we are to find a previous term, we can essentially construct a new series using the inverse or "reverse" ratio:

$$\frac{1}{r} = \frac{1}{2} = 0.5 \qquad \text{or:} \quad \text{new r} = 1/\text{old r} = r\text{^-1} = 1/2 = 0.5$$

Since (r) is now less than 1, the value of the next following terms in this new series will decrease.

Considering F as the first term of this new (reverse) series, the second term in this new series is Fr. The next or third term is the value of the second term times (r): (Fr)r = Fr^2. Observe the pattern: the exponent of (r) corresponds to the nth previous term of the original series.

nth previous term = Fr^n : **FORMULA FOR THE nth PREVIOUS TERM OF A GEOMETRIC SERIES**
(Note: here r is the "inverse", reciprocal or reverse ratio of the original series)

From the given series, the 3rd term is (7 - 3) = 4 terms lower (ie. the 4th previous term) than the 7th term that is considered as the first term in this analysis:

nth previous term = Fr^n considering 64 as the first term:
4th previous term = 64(0.5^4)
4th previous term = 64(0.0625)
4th previous term = 4

Another analysis to find the nth previous term of a geometric series is as follows:

Since: (term) (r) = next term , or: (previous term) (r) = term , Considering term = first term = (F) :

$$\text{previous term} = \frac{F}{r}$$

Dividing the previous term by (r) so as to find the term that preceded it (ie., the 2nd term in this series):

$$\frac{\frac{F}{r}}{\frac{r}{1}} = \frac{F}{r^2}$$

Continuing this process of dividing by (r) so as to find the next previous term, we note a pattern that the resulting or indicated power of (r) and the term in question are the same, and we arrive at a formula for the nth previous term:

nth previous term $= \frac{F}{r^n} = Fr^{-n}$: **FORMULA FOR THE nth PREVIOUS TERM OF A GEOMETRIC SERIES**
(r is the normal ratio of the series, n corresponds to the nth previous term)

Here is a simple verification to this formula: Consider the "formal" or algebraic expressions for the terms of a geometric series:

$\ldots,\ Fn,\ Fr^{(n+1)},\ Fr^{(n+2)},\ Fr^{(n+3)},\ \ldots$ for our purposes here, letting n=0:

$\ldots,\ F,\ Fr^1,\ Fr^2,\ Fr^3,\ \ldots$

Now, consider any term as the first term. It is quite obvious that the power of (r) of the next term is one higher, and that the power of (r) in the previous term is one less. Letting **Fr^0 = F** be the first term or reference term:

previous terms $\ldots,\ Fr^{-3},\ Fr^{-2},\ Fr^{-1},\ \mathbf{Fr^0},\ Fr^1,\ Fr^2,\ Fr^3,\ \ldots$ next terms

$$\ldots,\ \frac{F}{r^3},\ \frac{F}{r^2},\ \frac{F}{r^1},\ \mathbf{F},\ Fr^1,\ Fr^2,\ Fr^3,\ \ldots$$

We see that the first previous term has the divisor of (r) to the first power, the second has (r) raised to the second power and so on.

From the last example given, the 3rd term is (7 - 3) = 4th previous term:

4th previous term $= \frac{F}{r^4} = \frac{64}{2^4} = \frac{64}{16} = 4$:checks

As a final note for this discussion about geometric series, if given any three consecutive terms of a geometric series, the second term, or "middle term", is the geometric (as opposed to arithmetic) mean of the first and third terms. Mathematically, a geometric mean of two values is the square root of the product of those two (factor) values:

Term #:	1	2	3
Term:	F	Fr	Fr^2

Multiplying the first and third terms:

$(F)(Fr^2) = F^2r^2$

Taking the square root of this value, we arrive at the value of the "middle" term, and we verify the definition:

$$\sqrt{F^2 r^2} = \sqrt{F^2}\,\sqrt{r^2} = Fr$$

Mathematically, we also see that a geometric mean is also the product of the square roots of both factors or terms.

Ex. Suppose that there are squares with areas of: 2, 4, and 8. This is a geometric series with $r=2$. The geometric mean of 2 and 8 is:

$\sqrt{(2)(8)} = \sqrt{2}\,\sqrt{8} = \sqrt{16} = 4$: the geometric mean of the areas is 4

$\sqrt{(2)(8)} = \sqrt{16} = 4$

Notice that the value of 4, or the ratio (here, $r=2$) between terms was not needed in this calculation.

If given a value (considered as the first term) and another value (considered as the third term) you can find the value of the ratio by the following fact:

$\frac{\text{third term}}{\text{first term}} = \frac{Fr^2}{F} = r^2$ and:

$$r = \sqrt{r^2} = \sqrt{\frac{\text{third term}}{\text{first term}}}$$

Ex. If the first term is 3 and the third term is 75, what is the second or "middle" term, and what is (r) ?

$\text{second term} = \text{geometric mean} = \sqrt{(3)(75)} = \sqrt{225} = 15$

$r = \sqrt{\frac{75}{3}} = \sqrt{25} = 5$: = third term / second term = 75/15 = second term / first term = 15/3

The second term can also be found from: $Fr^1 = (3)(5) = 15$

Using these facts, it is now possible to solve for (r) given any two terms, that are not necessarily separated by only one term, of a geometric series (without using simultaneous equations as previously shown):

Ex. If the first term is considered F, and the nth term is $Fr^{(n-1)}$, we have:

$$\frac{Fr^{(n-1)}}{F} = r^{(n-1)}$$

To isolate (r), that is, the base of this indicated power value, take the (n-1) root of this power:

$$r = \sqrt[(n-1)]{r^{(n-1)}} = r^{((n-1)/(n-1))} = r^{(1/1)} = r$$

Ex. If the first term is 1000 and the thirteenth term is 2000 find (r).

$\frac{2000}{1000} = 2 = r^{(n-1)}$ since the nth term is term 13, and $(n - 1) = 12$, taking the 12th root of both sides:

$r = \sqrt[12]{2} = 1.059463094$: An important musical constant, as shown previously in this book

Ex. Here is a series where r=2:

. . . , 2, 4, 8, 16, 32**, . . .**

Note here that you can find the value of (r) by several ways. You can divide any term by the previous term, or consider the values somewhat like the corresponding values in the equation to find the slope (m) of a line since the ratio (r) is the (constant) rate of change for a line and for a geometric series of terms. That is, r=m:

First let: $y1 = m\,x1$ and $y2 = m\,x2$: linear equations, $m = r = yn/xn$

$y1 = r\,x1$ and $y2 = r\,x2$

Ex. $4 = (2)\,2$ and $32 = (2)16$: these values in the series above
p1(x1,y1) = p(2,4) and p2(x2,y2) = p(16,32)

$$m = r = \frac{y2 - y1}{x2 - x1}$$

Ex. $$m = r = \frac{16 - 4}{8 - 2} = \frac{12}{6} = 2$$

Checking: $$\frac{(n\,r\,r\,r) - nr}{(n\,r\,r) - n} = \frac{nr^3 - nr}{nr^2 - n} = \frac{r\cancel{(nr^2 - n)}}{\cancel{(nr^2 - n)}} = r$$: For the specific example: F = n = 2

Extra: If the sum of 4 consecutive terms of this series is 30, what is the first term of those 4 consecutive terms if r=2?

$n1 + n2 + n3 + n4 = 30$ considering n1 as the first term, we can express each term in terms of n1:
$n1 + n1r + n1r\,r + n1\,r\,r\,r = 30$
$n1 + n1r + n1r^2 + n1r^3 = 30$ dividing each term by n1 :
$n1\,(1 + r + r^2 + r^3) = 30$ and if r = 2:
$n1\,(1 + 2 + 2^2 + 2^3) = 30$
$n1\,(15) = 30$ solving for n1:
$n1 = 30/15 = 2$

The appendix contains a discussion about summing consecutive terms of a geometric series.

FIBONACCI SERIES

The Fibonacci number series was known to the mathematicians in India long ago, but today, it is usually credited to Leonardo (Fibonacci) of Pisa due to his work with the Golden Ratio (a special value of about 1.6180...) and the fact that this simple series has many properties that involve this ratio value. Fibonacci is also credited to spreading the knowledge of the decimal number system (first developed and used by the people in India, and then into the refined and popular symbols and some basic algebraic use with the Arabs) into Europe in about 1202AD as a better alternative to the commonly used and awkward (for advanced calculations and mathematical advancements) Roman numeral system.

The Fibonacci series is not a simple arithmetic series where there is a common (constant) difference between successive terms, and it is not initially a geometric-like series with a common (constant) ratio between successive terms. The terms of this series are from the whole, counting or "natural (as often found in nature)" numbers (ie., 0, 1, 2, 3, 4, 5 ... up to infinity). The specific values actually used in this series are often found in nature, such as the number of petals on flowers, naturally arranged rows of fruits and seeds, and other natural arrangements. Each next term of the series is quite simple to calculate and it is simply the sum the two previous terms. The first term of the Fibonacci series is 1. Here are some of the initial terms of the Fibonacci series:

0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987, 1597, 2584, . . .

An expression for each next number (or term n) in this series is: $Fn = F(n-1) + F(n-2)$

Ex. What is the next term after 2584 in the above series?

From: $Fn = F(n-1) + F(n-2)$
$Fn = 2584 + 1597$
$Fn = 4181$

The higher the term number, the ratio of the value of a term to the previous term approaches the value of the Golden Ratio (about 1.6180...).

Ex. $\frac{4181}{2584} = 1.618034056$ we will use G to represent this value here:

The higher the term number, this equation approaches a value of G:

From: $\frac{F(n+1)}{Fn} = G$, mathematically:

$F(n+1) = G\,Fn$: as (n) increases, and for practical purposes as (n>=10), this approaches a geometric series where Fn is essentially the previous term of the term F(n+1), and that the common ratio becomes a value that approaches G = the golden ratio =~ 1.618...

The Fibonacci series found in nature can be described as that it expresses the essence of undying, ideal life, and which reproduces and-or divides into 2, and then summing up that which was previously born so as to have the total.

SIN ¢ SERIES

The SIN ¢ series is presented below. Of all the various trigonometric series, it is a good idea, at some point in your mathematical journey, to try memorize the basic format of the SIN ¢ series, and so as to always have a mathematical foundation to find the various trigonometric values of an angle. The COS ¢ and TAN ¢ values can also be calculated from the SIN ¢ value. This is so, since any angle has a specific set of corresponding (SIN, COS, and TAN) trigonometric values associated with it. The appendix also contains a few more series such as the COS ¢ series, and a derivation of the SIN ¢ series. Here is the common "formula" for the SIN ¢ series:

$$\text{SIN } X = \sum_{n=0}^{n=\infty} \frac{(-1)^n \; X^{(2n+1)}}{(2n+1)!}$$

: X is a radian angle. The ∞ symbol means "infinity". Though infinity does not have a specific value, it indicates an endless process, and here, an endless number of terms to be summed together, in theory, are needed for the true result, In practice, the number of terms used is relatively few due to that the sum is converging or becoming equal to the true value very quickly with only a few terms with just a small difference between them, and therefore, only a small difference between the sum and the true result. The expression on the right is the general format for each term that is to be summed together.

To properly use this series, X must be a radian angle measurement. Radians (units) is a natural or inherent (to the circle, or rotating points) measurement for angles, but degrees (units) are not totally natural since they are man made units for the measurement for angles, specifically with the near random value of 360.

The symbol that looks like a large S alphabetical letter is called "Sigma" and is to be considered as the symbol and first letter for summation. Here, mathematically related terms are to be summed, and the basic format for each term is shown to its right side. (n) is a variable value that will change in the expression for each next term by an increment of 1. That is, n is the term number, and for the first term n=0 (as indicated immediately below the summation symbol). For the second term, n=1, for the third term, n=2, and so on till (n) is equal to infinity (∞) as indicated at the top side of the summation symbol.

In short, the notation means: "evaluate the terms from n=0 to n=∞, and sum them". Usually, far fewer terms than an infinite number of terms will be utilized for the sum since the value of the "next" terms get significantly smaller and rapidly converges or approaches to being a value close to 0 and meaningless, and the sum of those limited number of terms then rapidly converges or approaches a specific numerical value. The actual number of terms to be utilized for the sum depends on the number of correct (ie., accurate) decimal places (ie., the "precision", preciseness or fineness) desired.

As a basic rule for this series, for (x) decimal places of accuracy, sum (x+2) terms. When X < (π /4), (about 0.79), equivalent to an angle of: ¢<45°, fewer terms are usually required. The value of the factor (-1)^n will produce alternating (positive and negative) signs or values of the terms, hence an "alternating series". **!** (an exclamation point) is called the **factorial** symbol. It is the accepted symbol used to indicate the product of all integers from 1 up to and including the indicated value that precedes it:

n! = product of all integers from 1 up to and including n : n! is read as "n factorial"

Ex. 4! = 1 x 2 x 3 x 4 = 24 : "4 factorial"

Ex. 5! = 1 x 2 x 3 x 4 x 5 = 120

A factorial of a number (N) can easily be found by taking the previous integer factorial and multiplying it by that number:

$$N! = (N-1)!\,N$$

Ex. $5! = (5 - 1)!\,5$
$5! = (4!)\,5 = 24\,(5) = 120$

Checking: $5! = 1 \times 2 \times 3 \times 4 \times 5 = 120$ which can be expressed as:
$5! = (1 \times 2 \times 3 \times 4)\,5$ the value within the grouping symbols can be expressed as:
$5! = (4!)\,5$: this checks or corresponds with that shown above

Note: $(A!)(B!) = A!\,B!$ does not equal $(AB)!$ or $(A + B)!$.

For example: $2!\,3!$ /=/ $(2 \times 3)!$ /=/ $(3 + 2)!$
$(2)(1)\,(3)(2)(1)$ /=/ $6!$ /=/ $5!$
$(2)\,(6)$ /=/ 720 /=/ 120
12

However since 3! contains 2! as a factor:

$$2!\,3! = 2!\,(2! \times 3) = (2!)(2!)(3) = (2!)^2\,(3) = 2^2\,(3) = 4(3) = 12$$

As you can see, you can treat the entire factorial expression (and not just the numerical coefficient of the ! symbol) as like a variable, for example, Let $3! = A = 6$, and then: $A! = (3!)! = (6)! = 6! = 720$.

Here are the first 20 factorials:

1! = 1
2! = 2
3! = 6
4! = 24
5! = 120
6! = 720
7! = 5,040
8! = 40,320
9! = 362,880
10! = 3,628,800 : = 9! (10)
11! = 39,916,800
12! = 479,001,600
13! = 6,227,020,800 : = (62,270,208)(100) , can use this for some calculators with a limited number of digits
14! = 87,178,291,200 for each factor or entry.
15! = 1,307,674,368,000 : = (12!)(13)(14)(15) = (12!)(2730) = (479,001,600)(2730)
16! = 20,922,789,888,000
17! = 355,687,428,096,000
18! = 6,402,373,705,728,000
19! = 121,645,100,408,832,000
20! = 2,432,902,008,176,640,000 = 19! (20)

"Expanding" the SIN ¢ series by substituting values for (n) and expressing the sum of each term:

$$\text{SIN } X = X^1 - \frac{X^3}{3!} + \frac{X^5}{5!} - \frac{X^7}{!} + \frac{X^9}{9!} - \frac{X^{11}}{11!} + \frac{X^{13}}{13!} - \frac{X^{15}}{15!} \ldots$$

Notice that terms of the expanded series can easily be memorized as having this general form:

$\pm \frac{X^{odd}}{odd!} \ldots$: A simplified formula for each term of the SIN X series.
odd = odd numbers starting at number 1.
The first term is positive.

To assist calculations, the factorials were evaluated (simplified) in the series of terms below:

$$\text{SIN } X = X - \frac{X^3}{6} + \frac{X^5}{120} - \frac{X^7}{5{,}040} + \frac{X^9}{362{,}880} - \frac{X^{11}}{39{,}916{,}800} + \frac{X^{13}}{6{,}227{,}020{,}800} - \ldots$$

For the 7 terms shown, an accuracy of 9 correct significant decimal digits for angles between 0 and 90 degrees can be achieved by using a computer that can perform computations with 16 digits. Further terms used in the sum will result in the next accurate (correct) significant digit(s) with a descending or less decimal weight of which you may then compare to the previous sum. The smaller the angle, the fewer the terms required for a given precision. It is also possible to use the reciprocals of the denominators as factors (multipliers) in the numerators; (1/denominator), if you prefer multiplication rather than division.

Ex. What are the sine, cosine, and tangent values of 60°.

Converting 60° to its equivalent radian angle measurement required for the SIN X series or formula we find:

60° = about 1.047 (radians or rads)

Since only 3 decimal places are utilized for the angle's approximation, the best accuracy, and practically regardless of the number of terms evaluated, that can be expected for the SIN 60° result using the SIN ¢ series (or even from a calculator) is only about 3 decimal places.

Substituting 1.047 for X in the SIN ¢ formula, and using only the first 4 terms of its expansion (ie., as a sum of terms):

$$\text{SIN } 1.047 = \frac{1.047}{1} - \frac{(1.047)^3}{6} + \frac{(1.047)^5}{120} - \frac{(1.047)^7}{5040} + \ldots$$

Here, it is easier to simplify each fraction rather than trying to add them all using a common denominator.

SIN 1.047 = 1.047 - 0.1913 + 0.0105 - 0.0003 : using approximate values of the terms

SIN 1.047 = 0.8659

Hence, SIN 60° = SIN 1.047 = $0.86\underline{6}$

From COS ¢ = $\sqrt{1 - \text{SIN}^2 ¢}$: finding the COS ¢ using the corresponding SIN ¢

COS 60° = $\sqrt{1 - (\text{SIN } 60°)^2}$

COS 60° = $\sqrt{1 - (0.866)^2}$

$\text{COS } 60° = \sqrt{0.25}$ (rounded)

$\text{COS } 60° = 0.5$:actually, this happens to be exact (due to the rounding), but in general, the result should be considered as only an approximation as noted above)

An alternate method to find COS 60° is from the fact that complementary angles have the same trigonometric co-function values.

$\text{COS } ¢ = \text{SIN } (90° - ¢)$: ¢ and (90° - ¢) are complementary angles. They sum to 90°.
$\text{COS } 60° = \text{SIN } (90° - 60°)$
$\text{COS } 60° = \text{SIN } 30°$: This indicates that you can calculate SIN 30° so as to find COS 60°. Since 30° is a smaller angle value than 60°is, it will cause the series to converge to a specific value (result) faster (fewer terms required).

From $\text{TAN } ¢ = \frac{\text{SIN } ¢}{\text{COS } ¢}$: Finding TAN ¢ given the angles corresponding SIN ¢ and COS ¢ values.

$\text{TAN } 60° = \frac{\text{SIN } 60°}{\text{COS } 60°} = \frac{\text{SIN } 60°}{\text{SIN } 30°}$: optionally finding Tan ¢ using only SIN ¢ values

$\text{TAN } 60° = \frac{0.866}{0.5} = 1.732$: approximately

If you have an 8-digit display calculator, or are performing the calculation completely by hand (where multiplication may be preferred over division), you may prefer to use the SIN X series expressed in this form:

$$\text{SIN } X = X - \frac{X^3}{6} + \frac{X^5}{120} - \frac{X^7}{5{,}040} + \frac{X^9}{362{,}880} - \frac{X^{11}}{39{,}916{,}800} + \frac{X^{13}}{6{,}227{,}020{,}800} - \ldots$$

$$\text{SIN } X = X - \frac{X^3}{6} + \frac{X^5}{120} - \frac{X^7}{504\ (10)} + \frac{X^9}{36{,}288\ (10)} - \frac{X^{11}}{399{,}168\ (100)} + \frac{X^{13}}{62{,}270{,}208\ (100)} - \ldots$$

$$\text{SIN } X = X - 0.1666667\ X^3 + 0.0083333\ X^5 - 0.0019841\ (0.1)\ X^7 + 0.000{,}027{,}6\ (0.1)\ X^9 - 0.000{,}002{,}5\ (0.01)\ X^{11} + 0.000{,}000{,}02\ (0.01)\ X^{13}$$

For an angle of 90° you can expect about 6 decimal places of accuracy. As the angles get smaller, you can expect better accuracy. For example, with all the terms shown, you can expect about 8 decimal places of accuracy for angles around 45°, and 4 digits of accuracy using only three terms. Don't expect too much more accuracy for smaller angles since only 8 digits of the reciprocals of the denominators of the original series are shown and will be used in the calculations. Some of the reciprocals were also rounded. If you have a 10-digit display calculator, you may wish to rewrite the SIN X formula in a similar manner using 10 digit values.

The appendix contains a computer program to calculate the sine of an angle.

TRIGONOMETRY OF ANGLES GREATER THAN 90°

One might logically think that since SIN 90° = 1, that the SIN value of angles greater than 90° is greater than 1. This is however not so since the maximum SIN value, regardless of the angle, is always 1.

$$\text{SIN } ¢ = \frac{\text{side opposite the } ¢}{\text{hypotenuse}}$$: Trigonometric values are formally defined using a right triangle.

And we know that in a right triangle:

hypotenuse > side opposite and hypotenuse > side adjacent ,

that is, the hypotenuse is always the longest side, and using this fact in the SIN ¢ definition above, the numerator will always be smaller than or equal to the denominator, and therefore, this ratio is always less than or equal to 1 mathematically. Only when the angle is 90°, the side opposite this angle will then be equal to the length of the hypotenuse side.

From 90° to 180°, the SIN values start to decline in a corresponding "reverse" (or "mirror image") type of manner and are equivalent to what would be for the (declining) SIN values of angles from 90° back to 0°. This is indicated in the figure below as the angle of (90°- Q1), which is also the complementary angle value to Q1. That is, each obtuse (>90°) angle has an equivalent SIN value of an (corresponding or trigonometrically equivalent) acute angle between 0° and 90°. After calculating this corresponding acute angle, you can use it as an (trigonometrically) equivalent angle of reference, or reference angle, to calculate the trigonometric values of angles greater than 90°.

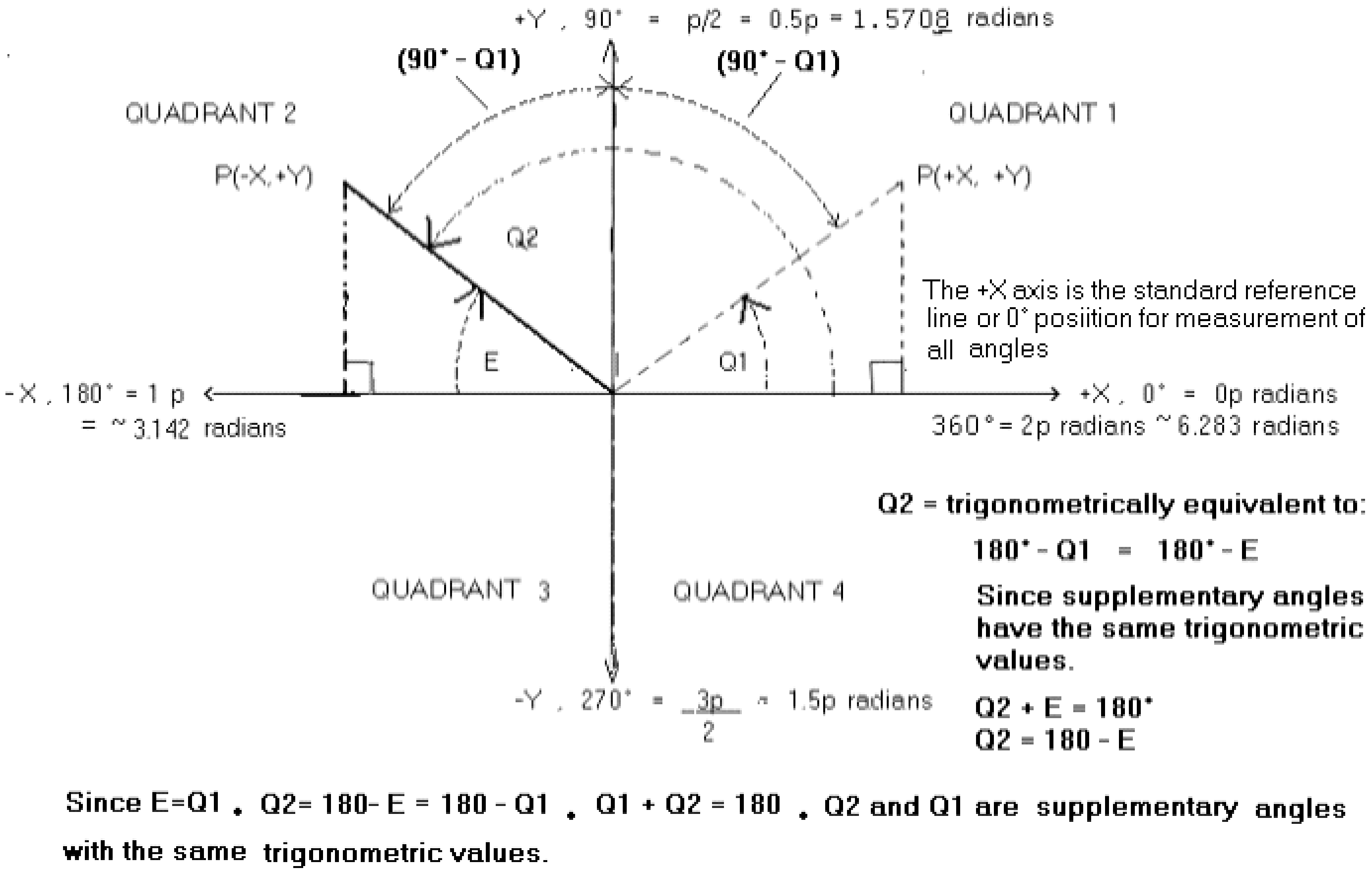
In the full range of angles from 0° to 360°, the trigonometric values will cycle or repeat within 90° (but not necessarily every 90, due to the nature of the SIN values or waveform, as will be seen in a graph below) , hence they will repeat a total of four times. Therefore, the trigonometric values for angles in the range of angles from 0° to 90°, can simply be reused (as a reference) for the trigonometric values of all the remaining angles between 90° and 360°.

In this discussion, Qn is used indicate a quadrant of the rectangular coordinate measuring system, where (n) will have a value of 1 to 4. A quadrant angle is an angle measured in reference to the x-axis, 0° reference line or angle. If the ending or terminal side of that angle is in a specific quadrant, the angle is associated and indicated as an angle of, or in that quadrant. For example, a quadrant 2 angle would be indicated or expressed as a Q2 ¢, "a quadrant two angle", or simply as Q2.

Quadrant	Angles	
1	0° to 90°	: Q1
2	>90° to 180°	: Q2
3	>180° to 270°	: Q3
4	>270° to 360°	: Q4

Below is a method to find a trigonometrically equivalent (E) or corresponding acute angle that is less than or equal to 90° when given an angle greater than a right-angle (90°). Angles greater than 90° are often called obtuse (wide, large) angles, and angles less than or equal to 90° are often called acute (narrow, thin) angles. As shown below, these obtuse angles greater than 90° have their "ending (after rotating)" side within another quadrant (quarter) other than the first quadrant of a circle or a rectangular co-ordinate measuring system. By creating a right triangle with the side adjacent to the angle on the horizontal or (x) axis of the quadrant in which the angles ending side is in, and the hypotenuse side of that angle created from the ending side of the (obtuse) quadrant angle, a trigonometrically equivalent or corresponding acute angle is created. The trigonometric functions of this corresponding acute (< 90°) angle are equal to that of the given obtuse (> 90°) angle.

When using a calculator to find the angle given a trigonometric value, care must be taken since it may return (ie. the output, indicated or displayed) the trigonometrically equivalent or corresponding acute angle (ie., an angle less than or equal to 90°, hence it would also be trigonometrically equivalent to a first quadrant angle) of which you must then convert to the proper angle if needed. In the figure shown below, the angle in question is a second quadrant angle, (Q2), since it is an angle between 90° and 180°. [FIG 198]



The +X axis is the standard reference line or 0° posiition for measurement of all angles

Q2 = trigonometrically equivalent to:

180° - Q1 = 180° - E

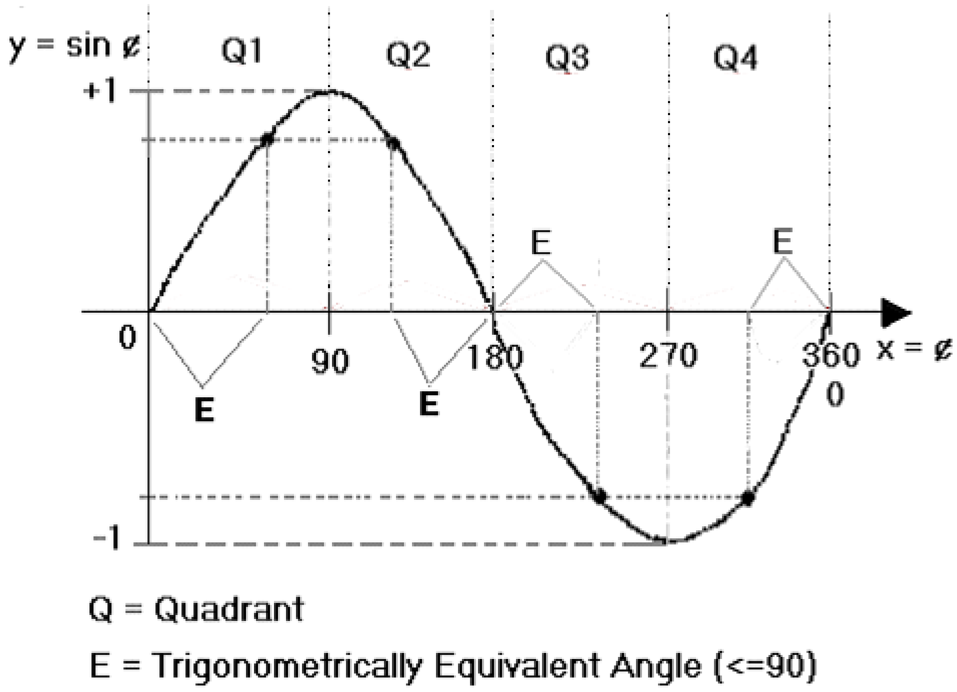
Since supplementary angles have the same trigonometric values.

Q2 + E = 180°
Q2 = 180 - E

Since E=Q1 , Q2= 180- E = 180 - Q1 , Q1 + Q2 = 180 , Q2 and Q1 are supplementary angles with the same trigonometric values.

Using the ending or terminal side of this angle and the X-axis of the quadrant in which it is in, the trigonometrically equivalent or corresponding acute angle (E) is shown. Clearly, this angle (E) is equal to 180° less the value of the quadrant angle (Q2) since they are supplementary (sum to 180°) angles. Notice the ("mirror image") identical triangle (with the same side lengths) created in the first quadrant, and its trigonometric values will be equivalent to those of any other trigonometrically equivalent (E) or corresponding acute (<=90°) angle in any quadrant, except perhaps for the mathematical signs (+,-) of those trigonometric values.

Below is a graph-plotting of the SIN of the angle waveform with respect or in reference to the (linear) angle values along the (x) or horizontal axis. It shows that after the first quadrant (Q1) of and from 0 to 90 degrees which defines the acute trigonometric equivalent reference angle (E), that the trigonometric values will essentially repeat in the other quadrants (Q2, Q3 and Q4). It could be said that the other trigonometric values (here, for SIN ¢) for angles greater than 90 degrees are essentially like "mirror images" since they will have the same (side lengths, and trigonometric) values. That is, angles greater than 90 degrees have a corresponding trigonometrically equivalent (E) angle in the 0 to 90 degree range, and which has the same trigonometric (absolute) values. In short, supplementary angles which sum to 180° have the same trigonometric values. [FIG 199]



Clearly, each trigonometric value (here in this example, SIN ¢ was used) indicated as points on the curve, will occur 4 times in the range of 0 to 360 degrees. Each occurrence will be in one of the 4 quadrants where each quadrant has a range of (360°/4) = 90 degrees.

From 0°, as the angle increases, the corresponding SIN ¢ values will increase up to an absolute value of 1 at the peaks (of the curve or waveform at 90° and 270°), and then decrease in the exact reverse (or mirror-like) manner from this value of 1 to the troughs (of the curve or waveform at 0°=360° and 180°) and have a value of 0.

Another way to state the above graph is that within 0° to 360°, 4 angle values will have the same SIN value as that of an angle (between 0° and 90°, in Q1) that has that same SIN value. This SIN value will occur 4 times during 360°. Notice that the corresponding SIN values are mirror images and symmetrical at and about the 90° mark, and the 270° mark (which is 180° more from 90°). As seen in the image, clearly these angles and their equivalent sine values are not generally separated equally by 90° (ie., not complementary angle values). Take note of the first point indicated, that the amount of angle from here up to 90° is equivalent to its complementary angle (an angle and its complementary angle sum to 90°). 90° plus this same complementary angle value is the corresponding trigonometric angle of the second point and angle with the same trigonometric values. Point 1 and point 2 are equidistant (or "symmetrical") from the 90° mark where SIN 90°=1. For example, they could both be 20° from the 90° mark, when point 1 corresponds to 70°, and point 2 corresponds to 90° + (complementary angle to Q1) = 90°+ (90° - Q1) = 90 + (90° - 70°) = 90° + 20° = 110°. SIN 70° = SIN 110°. The SIN values will decrease on each side of this peak till 0° or 180° is reached where SIN 0°= 0. The sine of 89° is the same as the sine of 91°. The sine of 70° is the same as the sine of 110°. The sine of 1° is the same as the sine of 179°.

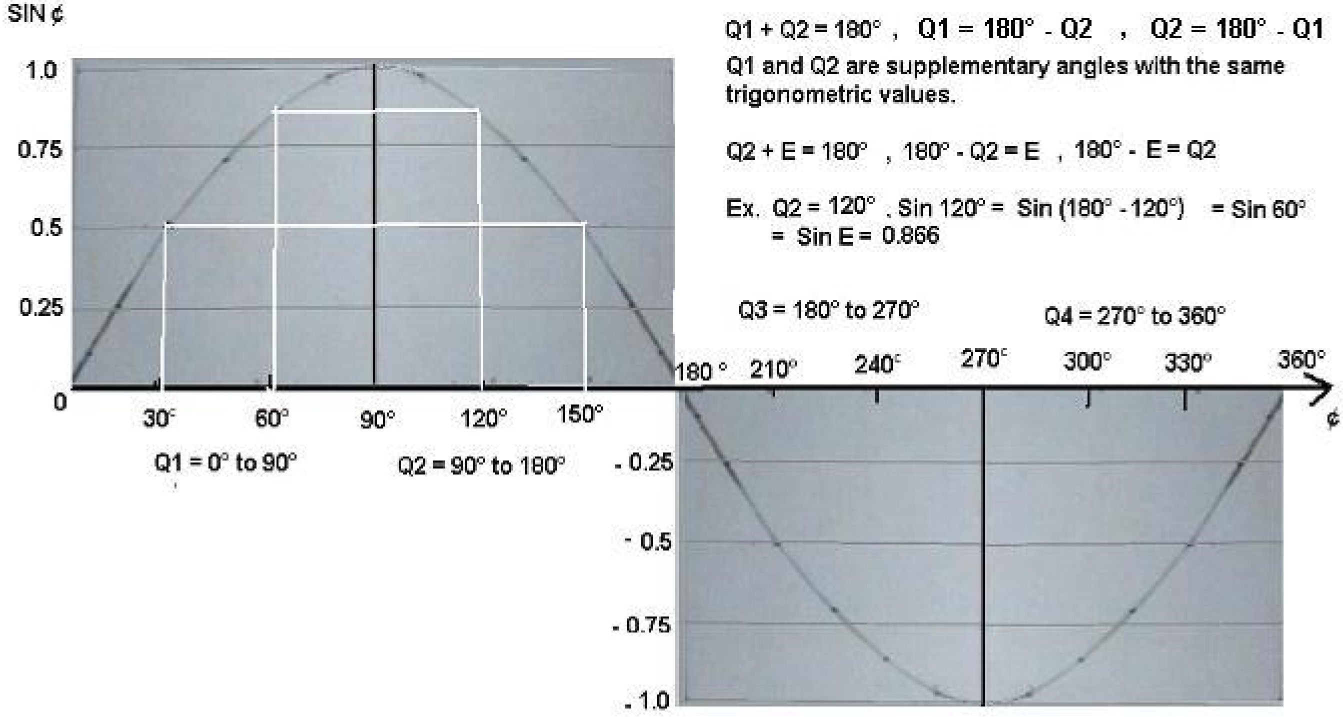
Each point, or angle and its corresponding SIN value on the SIN curve, has an angular distance from the peak (ie. high values, where the SIN values are 1) that is a value equal to the complementary angle of the first quadrant angle = Q1 = trigonometrically equivalent angle (E). Just the same, it could be said that each point (angle and its corresponding SIN value) has an angular distance from or to the "troughs" (ie. low values, where SIN values are 0), and this value is equal to the first quadrant angle = Q1 = trigonometrically equivalent reference angle (E).

Observe the following as an example, and it will be verified in an example below after a discussion. The following 4 points (with coordinates: (x=¢ , y=sin ¢)), with each having the same absolute "y" = sin ¢ coordinate-location value, are indicated on the graph above:

Q1 Q2 Q3 Q4 : showing which quadrant the angle is within.
sin 50° = sin 130° = sin 230° = sin 310° Q1, Q2, Q3, and Q4 all have the same trigonometrically equivalent (E) corresponding angle of Q1 = 50°
Each has the same SIN value of SIN 50° = about 0.766

Notice for example that SIN values are negative in the third and fourth quadrant, this is because the y coordinate values of points in the third and fourth quadrant are negative in sign, and SIN ¢ = opp/adj = y/x. As previously shown, when the radius line was considered as 1 or 100%, SIN ¢ = y/x = y/1 = y. The SIN values in Q2 are a mirror image of the SIN values in Q1. Except for the sign, the SIN values in Q3 are the same as the SIN values in Q1. The SIN values in Q3 are a mirror image of those in Q1. As seen in the above graph, it is an error to think that angles must have a separation of 90° to have similar trigonometric values. For angles to have the same trigonometric values, they will have the same trigonometrically equivalent (E) acute (<90°) angle.

Though not drawn to "linear scale" on the vertical axis, here is sine-like graph that can help illustrate the trigonometric equivalences of angles that are symmetrical in their amount of degrees or angle value from 90°: [FIG 200]



Q1 + Q2 = 180° , Q1 = 180° - Q2 , Q2 = 180° - Q1

Q1 and Q2 are supplementary angles with the same trigonometric values.

Q2 + E = 180° , 180° - Q2 = E , 180° - E = Q2

Ex. Q2 = 120° . Sin 120° = Sin (180° - 120°) = Sin 60°
= Sin E = 0.866

If you were to switch the corresponding angle values shown, the graph would resemble the standard rectangular coordinate system with the angles measured from the positive (x) reference (0°) axis.

Here is a method to calculate the trigonometrically equivalent angle given any angle, or angle in any quadrant:

For an angle that is less than or equal to 90°, it is in the first quadrant and the trigonometrically equivalent (E) angle (<90°) is obviously equal in value to that angle:

E = Q1 : trigonometrically equivalent angle for an angle in the first quadrant

For an angle that is greater than 90° and less than or equal to 180°, it is in the second quadrant, and its trigonometrically equivalent angle sum to 180° :

180° = Q2 + E : supplementary angles sum to 180°, therefore supplementary angles have equivalent trig. values
E = 180° - Q2 : trigonometrically equivalent (E) angle for an angle in the second quadrant,
Q2 = 180° - E

For an angle that is greater than 180° and less than or equal to 270°, it is in the third quadrant and is equal to 180° plus its' trigonometrically equivalent angle:

Q3 = 180° + E mathematically:
E = Q3 - 180° : trigonometrically equivalent angle for an angle in the third quadrant

As a special note: A triangle can not contain a single angle equal to or greater than 180° since the interior sum of all three angles within a triangle is always 180°.

For an angle that is greater than 270° and less than or equal to 360°, it is in the fourth quadrant and is equal to 360° less its trigonometrically equivalent angle.

360° = Q4 + E
Q4 = 360° - E
E = 360° - Q4 : trigonometrically equivalent angle for an angle in the fourth quadrant

Because of the facts stated above, the trigonometric (absolute) values of two corresponding supplementary angles (that sum to 180°) are the same value.

Ex. SIN 10° = 0.173648177

The supplementary angle of 10° is (180° - 10°) = 170°

SIN 170° = SIN (180° - 170°) = SIN 10° = 0.173648177

Ex. Q1 Q2 Q3 Q4 : Qn = "Quadrant angles"
sin 50° = sin (180° - 50°) = sin (180° + 50°) = sin (360° - 50°)
sin 50° = sin 130° = sin 230° = sin 310° = about 0.766 : absolute value

Note above, that for this example, Q1 and Q2 are supplementary angles. Q1 and Q3 are separated by 180°. Q3 and Q4 are separated by 180°

Here is a helpful chart to express these values:

Quadrant	Angles		Trigonometrically Equivalent (E) Angle
1	0° to 90°	: Q1	E = Q1
2	>90° to 180°	: Q2	E = 180° - Q2
3	>180° to 270°	: Q3	E = Q3 - 180°
4	>270° to 360°	: Q4	E = 360° - Q4

Ex. Find SIN 135° = SIN 2.35619449 (rads)

The angle is a second quadrant angle since 135° is between 90° and 180°. Its' corresponding acute angle is:

E = 180° - Q2 : E = acute (<=90°), trigonometrically equivalent angle
E = 180° - 135°
E = 45° : Therefore, 45° is trigonometrically equivalent to 135°, and vice-versa. 45° is also the supplementary angle to (180° - 45°) = 135° :

SIN 135° = SIN 45° = SIN 0.785398163 (rads) = 0.707106781

Likewise: | COS 135° | = | COS 45° | : same values, but using absolute (or "signless") values here
| TAN 135° | = | TAN 45° |

The absolute value symbols are needed due to the fact that cosine and tangent values are negative in sign in the second quadrant and positive in the first quadrant. Even though the numerical signs may differ, the angles trigonometric (absolute) values are equivalent. The trigonometric (ratio of side lengths of an angle) values were initially defined for right triangles which has side lengths all being sign-less in value, hence essentially positive in value. It may then seem somewhat odd to then assign a (pos. or neg.) sign to these trigonometric values. The sign is now due to the fact that the axis of the rectangular coordinate system is signed in value, therefore lengths and values will be signed, hence assigning a direction to those lengths:

QUADRANT 1 (0° to 90°)

$$\text{COS ¢E} = \frac{\text{opp.}}{\text{adj.}} = \frac{Y}{X} = \frac{\text{pos. value}}{\text{pos. value}} = \text{pos. value}$$

COS Q1 ¢ will be positive.

QUADRANT 2 (90° to 180°)

$$\text{COS ¢E} = \frac{\text{opp.}}{\text{adj.}} = \frac{Y}{X} = \frac{\text{pos. value}}{\text{neg. value}} = \text{neg. value}$$

COS Q2 ¢ will be negative.

Ex. Here of a Q2 angle:

COS 135° = - (COS 180° - 135°) = - (COS 45°) :after affixing the proper sign to the result, as defined above
COS 135° = - (0.707106781)
COS 135° = - 0.707106781

Sinusoidal Oscillations

Many things in nature and science (such as "ac", alternating (back and forth movement or flow, with the measurement values then having an opposite polarity, hence positive and negative) voltage and current electricity) are said have a sinusoidal or sinusoidal-like quality about them because they do something repetitively of which can be mathematically expressed with a sinusoidal equation and curve or waveform that is repetitious. As indicated, a good example of this is with the voltage and current output of an electricity generator. When its rotating coil of wire effectively becomes perpendicular to the magnetic field of a magnet, then maximum voltage is induced (ie., produced) which causes a corresponding maximum current to be produced. The output voltage and-or current will have a sinusoidal (sine) waveform or shape with respect to the rotation angle and-or time. The actual amount of voltage produced depends of several factors such as the number of turns of wire to make the coil, and the speed of the rotation of that coil and-or magnet (field), and the magnetic field strength of which also includes how close the coil and magnet can become so as to have maximum electricity generation effect.

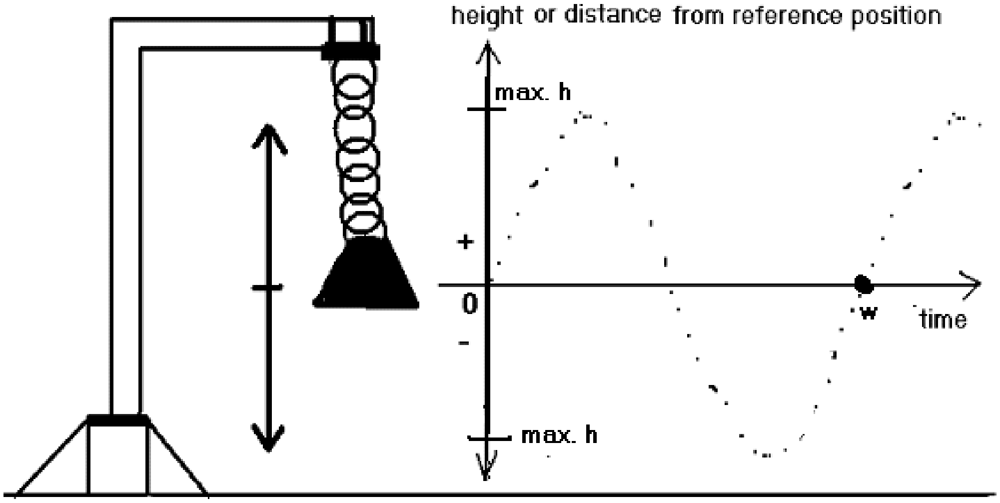
Besides the word "sinusoidal", other common words for repetitious behavior are "vibrate", "oscillate", "cyclic", and "periodic" (repeating over the same time period). All of these words are essentially a form and meaning of the word "repetition" (repeating).

Below is a mechanical oscillation system. A weight is suspended by a spring. If this string is either stretched or compressed, storing that input energy, and then released, its movement and corresponding position will be sinusoidal (sine like). Note that for the weight to change direction, it must first slow down and then actually stop for an instant of time, before changing direction. The weight will repeatedly accelerate and decelerate. Its speed or velocity will change, hence on the graph, the slope of the curve is changing and has a value of 0 when the weight stops and changes direction. Because of some (fractional, small) losses (such as due to friction) of energy during each cycle in this system, the maximum height or "amplitude" value will also decrease or decay in value as time increases.

The length or amount of time taken for this system to complete one cycle of this repetitious motion is called the wavelength (as shown in the image below between 0 and w on the time-line axis). The reciprocal of this wavelength value is the number of repetitions per second, and is usually called the frequency (of repetitions or cycles) and having units of (cycles/second) = hertz. Frequency is "how often" something occurs during some length or period of time. The faster something vibrates, repeats or oscillates, the less time taken for each one of those cycles or waveforms to complete, and therefore frequency and wavelength (time) are mathematically inversely related.

Ex. For a frequency of 1000 hertz or cycles per second. A 1000cps = 1000hz signal has a wavelength of 1/(1000 c/1s) = 0.001s/c = 0.001 seconds per cycle = 1 millisecond per cycle, or simply 1 millisecond.

In the figure below, the motion, particularly the height of the vibrating or oscillating object held by the spring will be sinusoidal in nature. When the spring stretches or compresses, it stores energy and the object will decelerate and loose kinetic energy and the object will eventually be at its maximum, positive or negative height or distance from its rest position where no energy was applied to it. If there is no energy to replace any energy losses in this system, such as due to mechanical friction and the resulting heat, then the oscillations will decrease in height until they eventually stop. This is sometimes called as the oscillations and the amplitude being dampened over time. [FIG 201]



Since the height waveform corresponds to, and can be mathematically represented as a SIN waveform with respect to time, we can therefore associate the maximum (100%=1) height (also called the amplitude of the vibration, waveform, change or movement) with the maximum value of the SIN ¢ value which is 1 and = 100% = 1.0 as a relative or fractional value. During the period of time equal to the wavelength, we can calculate the corresponding height at any instant or point in time. The SIN value will simply be the fraction of, and factor to the maximum height value. Since this graph is in relation to time rather than the angle as required for the SIN ¢ trigonometric function, we can convert this specific time value to its specific and corresponding angle or (relative, fractional) part of 360°. The time length of 1 complete wavelength or cycle will correspond to 360°. The ratio of the specific time during that wavelength or period of time to the entire wavelength (ie., time duration of the complete cycle or wave) will give us what specific fraction or portion of the wavelength is being considered and therefore, what specific fraction of a full cycle, rotation, or 360°, and hence what specific angle corresponds to that specific time value:

Letting max. value = maximum value = amplitude of the waveform:

y = SIN ¢ = (max. value)(fraction of max. value) = (max. height)(SIN ¢) , and: ¢ in relation or respect to time is:
y = (max. height) SIN ((fraction of 1 cycle or 360°, being considered) (360°)) : max. height = max. amplitude

T = period time = total time of 1 cycle of the waveform , and or the rotation of something
Ts = period = 1 / frequency = 1 / F = 1 / (cycles / second) = 1 / (c/s) = seconds / 1 cycle
F = 1 / T = 1 / period : note that frequency and time are reciprocals: FT = 1 , (c/s) (s/c) = 1

$$\text{height} = (\text{maximum height}) \times \text{SIN} \left(\left(\frac{\text{time}}{T}\right)\left(\frac{360°}{1}\right)\right)$$

: wavelength = distance = vT , T = period = 1 / frequency
: *

Extra: wavelength = distance the (energy) wave will travel through a material after 1 cycle is completed and-or the time of 1 period has elapsed (completed).

wavelength = distance = vt = vT = v / frequency = v / f

If the frequency increases, the wavelength will decrease. This in an inverse mathematical relationship. When frequency (F) increases, the period (T) time will decreases, and the wavelength will decreases.

Extra: 1 revolution = 1 cycle = 360° = 2(pi) radians : 2(pi) = 2(3.14159265..) =~ 6.2818

Note that: T = 1/F = seconds / cycle = s / **rev**olution = s / **rot**ation

(Also considering the above formula, *):

height = current or instantaneous value = SIN ((t / T) 360°));
height = value = (max. amplitude) SIN ((t) (1 / T) 360°)) = (max. amplitude) SIN ((time)(F) 360°));

Or since: 360° = 2(pi) = about 6.28 radians , and some rearranging of the factors:

value = (max. amplitude) SIN (2 (pi) F t)

The value of: (2(pi)F) is sometimes called the "angular velocity" per unit of time, and is given a symbol that looks like: W
Angular velocity is the amount of rotation per second , letting W = angular velocity , W = angle / time = ¢ / t

value = amplitude SIN (W t) , if t = T, the angle = 360° , and:

W = "angular velocity" or "angular speed" = "rotational speed" = $\frac{\text{revs}}{t} = \frac{\text{cycles}}{t}$ = frequency of rotation = F

$W = \frac{360^\circ}{T} = 2(pi) (1/T) = 2(pi) F$: here we see that F is essentially a multiplier to the angle, cycles or revs, and determines and is needed to express the actual angular or rotational velocity. With this equation, the units of W are rps (rotations per second) since the units of frequency are cycles per second of time, and that 360°= 2(pi) radians of rotation is being done each second of time. If you multiply the (rps) value by 60, you will have the equivalent value of revolutions per minute (rpm). 60 sec. = 1 min.

W = angle / time = ¢ / t = (s / r) / t = s / (r t) = (distance / t) / r = v / r : v = linear and=or= tangential velocity
: s = arc length , ¢ rads = s / r , s = ¢r = r ¢

W = change in rotation or angle / change in time : angular or rotational velocity

v = change in distance / change in time = r W = ds / dt = r d¢ / dt = r W = W r , Extra: r = v / W

From: arc length = s = ¢ r , and where the angle is in radians, r = radius

A full 360° rotation has an arc length of: s = C = Circumference = 2 (pi) (r) , then

W = ¢ / t = (s / r) / t = s / (r t) = C / (r t) = 2 (pi) (r) / (r t) = $\frac{2\,(pi)}{t}$ =~ $\frac{6.28 \text{ radians}}{t}$: for a full orbit here, t = P = period of orbit

A body in rotation, and even if not having translational or linear motion (with its corresponding KE value) and-or velocity, still has kinetic energy and it is proportional to its mass, the square of the angular or rotational velocity (W), and the square of the average radius (r) of all the points of that mass: **KE = mv^2 / 2 = m (W r)^2 / 2 = m W^2 r^2 / 2**

In terms of torque: Pw = energy / time = J / s = (Torque)(rotational velocity) = (T nm)(rotational velocity) = T W

Considering an angle velocity of W = 1 rotation or 1 revolution per second = 2 (pi) radians / s:
P = (Torque)(angular velocity) = T W = T (¢ / s) , If we let the rotational velocity expressed with units of revolutions per second = revs / s = rps , then ¢ = 2 (pi) / 1 rev , W = (n rps) 2 (pi) radians / s
If given a rpm value, we can divide that value by 60 so as to find the corresponding rps:
P = T W = T (rpm) / 60 = T (2 (pi) rps) / s

Extra: Consider: C = 2 (pi) radius = 2 (pi) r = a linear length , and: C / r = 2 (pi) r / r = 2 (pi) , and that

360° = 2 (pi) radians = an angle of 2 (pi) =~ 6.28

AN EQUATION FOR THE APPARENT TILT OF THE EARTH

The apparent tilt and-or angle of the Earth with respect to the Sun or solar plane is sinusoidal in nature. The Sun or solar plane is essentially the imaginary disk or plane created between the Earth and the Sun as Earth revolves (orbits, travels, moves, goes) around the Sun. Because of the orbit of Earth about the Sun being a slight ellipse shape, and not exactly circular, the apparent tilt of the Earth with respect to the Sun throughout the year is actually not exactly sinusoidal in nature, but it is close enough to be considered and analyzed as such in terms of practicality.

Earth's axis is tilted, and it does not change in value. Earth's axis does not change in value, but actually remains constant at about 66.5° with respect to the Solar Plane in the direction of the Sun. The true north to south line, axis or pole line, constantly points (direction) to (nearly) the star called Polaris or the "pole star" that is directly overhead at the north polar axis. With respect to the perpendicular (90°) to this solar plane, the axis is therefore often noted as constantly being: (90° - 66.5°) = 23.5° tilted form it. What is tilting throughout the year then? It is the angle between the Earth's standardized, geographical equator line and the solar plane - say toward the direction of the Sun, that changes throughout the year. As the Earth orbits the Sun throughout the year, in the summer time of the northern hemisphere, it is tilted toward the Sun during the day, and the equator line and-or region of the Earth is tilted below the solar plane by a maximum of 23.5°, and in the winter time, it is tilted above the solar plane by a maximum of 23.5°. This apparent axis tilt of the Earth throughout the year is generally said as what actually causes the climate seasons of the year.

Throughout the yearly orbit of Earth about the Sun, the eastward to westward path of the Sun will change slightly every day. The Sun will rise and set in a slightly different location and time every day. The Sun also has an effective or apparent angle with respect to the local horizon throughout the day as Earth revolves on its axis from eastward to westward. The tilt of the Earth, such as at the equator, with respect to the solar plane (or "ecliptic") is at 0° on March 21 and Sept 21 during the year.

Earth's axial tilt with respect to the solar plane, and for every day of the year is about $23.4\underline{4}$° as of the year 2020AD. We can express this value in a formula to find the tilt of the Earth at a specific day number from March 21 which can be considered as day number 0 for this yearly or 365 day cycle, hence the sinusoidal wave's wavelength is 365 days (365.25 to be more exact).

Lets begin this analysis with a simple liner equation expressing the mathematical relationship between two variables.

$y = x$: a linear equation, if x is multiplied by a value (m):
$y = mx$: y will be a multiple of the values of x

If m increases from 0 to a maximum value, y will also vary from 0 to this maximum value = amplitude:
If x is an angle:

y = m ¢

If m is set as a constant or maximum value, and the angle increases from 0, y will also increase from a value of 0. Here, y would be equal to a multiple of the angle.

If the angle is made to cycle or repeat from 0 up to a maximum value, y will also cycle or repeat from a value of 0 to some maximum value determined by both factors of (m) and the and the maximum angle value.

If the angle cycles from 0 up to a maximum value, and then descends from this maximum value to back to 0, the corresponding y values will also repeat or cycle in the same manner, and this can be mathematically represented using sine values with respect to, or of, the angle values, and these sine values also vary in the same cyclic manner from 0 up to a maximum value of 1, and then from 1 back down to 0. In short, these sin values then become as a relative or fractional value of the maximum value of the (specific) sine waveform (with a maximum amplitude, and having a certain wavelength and-or frequency):

y = (max. value) sin ¢ : for the initial discussion about the (apparent) tilt of the Earth throughout the year, we can use these corresponding values in this equation:

tilt angle = (maximum tilt ¢) x (sin ¢) : sin ¢ yields values 0 to 1, hence the seasonal, apparent tilt angle will be less than or equal to the maximum tilt angle. The tilt angle value will be a sinusoidal curve of values, and whose maximum value or amplitude is 23.5°

At a certain day number of the year, this represents a certain fractional portion of the 1 year or wavelength of the complete cycle, and-or it represents the same fractional portion of a complete 360° rotation. This (simpler) formula below considers the (average) orbit of Earth as a circle, but it is actually slightly elliptical. Applying these facts to the above formula:

tilt angle = 23.4$\underline{4}$° x SIN (day number from March 21 (360°/365.25)) : apparent tilt of the Earth throughout the year. A basic equation. 360° / 365.25 days is a constant of: ~ 0.985626283° / 1 day when averaged, however, the rate of tilt actually changes in a sinusoidal manner.

tilt angle = 23.4$\underline{4}$° x SIN (day number from March 21 x 0.985626283) : averaged, circular orbit, apparent tilt angle for Earth

The apparent tilt of the Earth (axis, pole or equator, reference position) with respect to the solar plane (which is considered the 0° reference angle and position) will be 23.5 degrees, max. "tilt" forward or toward the Sun during the daytime in the summer (at about June 21) for the northern hemisphere, and 23.5 degrees, max. "tilt" back away from the 0° reference position, solar plane or Sun during the daytime in the winter (at Dec 21, which is 6 months or half the yearly cycle past Jun 21). This angle is usually expressed as -23.5 since the (apparent) tilt is in the opposite direction from the 0° solar plane or reference angle. When the Earth (apparently or effectively) tilts away from the Sun, the northern hemisphere of the Earth will experience winter, and the southern hemisphere will experience summer.

The total tilt is then the amount of angle between these two angles: 23.5 - (-23.5) = 23.5 + 23.5 = 47°.

The tilt of the Earth is actually an apparent tilt, or illusion, since the imaginary line extending from Earths daily axis of rotation about itself always extends or "points to" the "north star" which is also called Polaris which is many "light years" (time of light travel) away.. The axis of the Earth is constantly tilted 23.5 degrees with respect to the solar plane (or "ecliptic"), but due to the orbit of Earth around the Sun throughout the year, it will have the effect that the one pole get closer to the Sun and the other pole gets farther away, hence creating the weather changes or seasons throughout the year.

The maximum apparent "height" that the Sun will appear above the horizon (0° reference line) at the observers local noon (true 12:00 pm) will be 90° less the latitude ("lateral" or side location, measured in degrees from the equator plane which is the imaginary line between north and south on the Earth) of the observer plus the current tilt angle. You can find **the latitude of your location** from some maps, or you can measure the angle between your local horizon and the north star (Polaris). The reference latitude at the equator is considered as 0°, and latitude is the angle created from the point at the center of the Earth to a location on the surface of the Earth. When the equator is located at the solar plane during the year, a person at the north pole of the Earth will see Polaris, the North Star, nearly directly overhead, and the angle between the horizon line and the line to Polaris is 90°, and this is the latitude at Earths north pole (axis). A person at the Equator will see Polaris as being at the horizon, and the angle between the horizon and Polaris will be 0°, and which is the latitude at the equator. The word "latitude" is based on the word lateral and slats (ie., side lateral, location), like the horizontal side shingles on houses that go from the ground level area to the top area of the house. In relationship to Earth, latitude is your location between north and south, and a line of latitude extends around the earth in the east and-or west directions.

Longitude is how far east or west a point on Earth is. The word "longitude" is based on the word "long". In relationship to Earth, longitude is your location eastward and-or westward, and a longitude line actually extends from the north pole to the south pole.

Greenwich, a city that is about 5 miles east of London, England along the Thames River. Greenwich has an astronomical observatory at a location that was standardized as the 0° reference position of longitude. This longitude line is called the **Prime Meridian** and goes in a northern and southern direction from the north pole to the south pole. This line is also the reference meridian and-or time zone of the world.

The time of an event anyplace or location in the world can be expressed in terms of the time at this reference longitude, and it is called: Greenwich Mean Time (GMT, mean=average) and-or Universal Coordinated Time (UTC or UT1), and loosely as "World Time" Each of the 24 (longitudinal) and hourly "time zones" is (in theory) approximately 360°/24 time zones = 15°/ timezone. Each longitude line separated by an angle of 15° can be said as being 1 hour and-or 1 timezone apart. For each and every location along the same longitude line, the true local time will be the same and the Sunlight will be directly overhead at the same time of noon which is 12PM. 15° of Earth's rotation and-or longitude separation corresponds to 1 hour = 60 minutes of time. 5° or Earth's rotation and-or longitude separation will correspond to 20 minutes. For 1 minute, setting up a equivalent fraction and-or proportion equation:

15°/60min as = x°/1min , x° = (15°)(1min) / 60min = 0.25° , therefore 1 min is to 0.25° , and
1s is to $0.004166\underline{7}°$, 4 min is to °1

OR: 15° / 1hr = 15° / 60 min = 0.25° / min = 0.25° / 60s = $0.004166\underline{7}°$ / 1s
The Moon appears from an observer on Earth to be 0.5° wide in diameter , and it will appear to move this diameter length in 2 minutes.

United States decided to keep large areas of the country's population and schedules at the same time, and made 4 time-zones that have roughly a north to south imaginary boundary line separating them. The 4 times-zones in the continental United States are the Eastern, Central, Mountain and Pacific time-zones. Starting from the Eastern boarder of the United States with its Eastern time-zone, and going in a westward in direction, each other time zone is an hour behind its eastern neighbor time-zone. Juneau, SE Alaska, USA is GMT-8. Hawaii Islands, USA is GMT-10. North-east Canada is also in what is called the Atlandic time-zone, and this is +1 hour ahead of the Eastern time-zone.

Many countries have adopted a concept called **Daylight Savings Time (DST)** where all clocks are set an hour ahead (ex. from 2AM to 3AM) later during some months so as to effectively or apparently have another hour of daylight each day - particularly in early (AM) waking and start of work hours during the winter months when the northern hemisphere of Earth is effectively tilted away from the solar plane and receives less daylight hours. When not in the months when DST is applied, it is called **Standard Time** which is the normal time according or in reference to GMT. DST in the USA is currently from 2AM on the second Sunday in March, to 2AM on the first Sunday in November, and on that day in March, the time is moved ahead one hour to normal true time. In the winter months in locations where DST is considered, the Sun will then appear to rise an hour earlier in clock time, and it will therefore, likewise set an hour earlier. The concepts of DST are naturally somewhat confusing to many, and is even debated by people.

Without considering the time-zones of the United States, New York City is 4 hours behind in time than that of the GMT. Since 15° about the Earth corresponds to 1 hour of time, New York City being -74° is 74°/15° =~ 4.93 hours from Greenwich city in England, and this may be expressed as GMT-4. 4.93 hours is nearly 5 hours, but is still only within the -4 to -5 hour longitude lines from GMT and its hourly longitude lines. If it is 1pm in New York City, the GMT is 4 hours ahead and that time is (1pm + 4hours) = 5pm such as in Greenwich England and all locations north and south in that hourly time zone or region bounded by longitude lines separated by 15° apart about the Earth.

Total Minutes = (hours)(60min/hour) + (remaining minutes of an hour) and:

Longitude Degrees from Greenwich = (Total Minutes)(0.25°/min)

There is an article in the Extra's section at the end of this book, and which is about calculating your local noon time from your known longitude.

How To Find Your Local Longitude. At noon time which is 12PM and where the Sun is the highest angle in the sky, calculate the difference from the Greenwich mean time at the prime (ie., reference, initial) meridian which is considered at the 0° longitude reference line. If the Sun is not shining that day, if your clock was set to the true local time, you could then find the difference that way. For each hour = 60 min. difference between the two time values, there is a 15° change (increase or decrease) in longitude from Greenwich. To calculate the total number of minutes, multiply the hours value by 60, and add any remaining minutes of an hour to that value.

Ex. If the Sun is at its highest angle in the sky for the day, and-or as indicated as 12pm = noon on a sundial, and it is known to be 5hr:30min "world time " in Greenwich England, your local longitude from Greenwich is:

Local longitude = (total hours)(15°/1hr)

First, converting the fraction (here minutes) of an hour to hours: 60 min / 1hr = 30 min / x hr , xhr = 0.5hr

5:30 = 5hr:30m = 5hr + 30 min = 5hr + 0.5 hr = 5.5hr

Local longitude = (total hours)(15°/1hr) = (5.5hr)(15°/hr) = 82.5° West of Greenwich

Approximate Latitude And Longitude Of Some Selected Cities And Areas

The longitude of a location is its (eastward (+, ahead in time) or westward (-, lower in time) angle from the Greenwich, England longitude line which is considered as the reference longitudinal line having a value of 0°. This location is also the accepted reference time for the world.

The latitude of a location is its central angle from the equator line. The location of a point or place on Earth has coordinates that are the latitude and longitude of that point:

A point (p) and-or location on the surface of Earth's sphere = p(latitude° , longitude°).

City	Latitude	Longitude	: close, approximate values
Greenwich , England	51.5°N	0°	: the reference (eastward and westward) longitude angle location, coordinate and world time value
New York City , USA	40.7°N	74°W	: N = north of the equator , W = west of Greenwich or: -40.7° lat, and -74° lon.
Kansas City , Missouri, USA	39.1°N	94.58°W	: central USA
Los Angeles , California, USA	30.05°N	118.2°W	
Anchorage , Alaska, USA	61.22°N	150°W	
Chicago , Illinois, USA	41.86°N	87.64°W	
Moscow , Russia	55.7°N	37.6°E	
Beijing , China	39.9°N	116.4°E	
Tokyo , Japan	35.7°N	139.8°E	
Sao Luis , Brazil	2.53°S	44.26°W	
Mecca , Saudi-Arabia	21.39°N	39.86°E	
Capetown , South Africa	33.9°S	18.4°E	: S = south of the equator, (-) sign also means south of equator or west of Greenwich 33.9°S , 18.4°E = -33.9° Lat, -18.4 Lon.

Nagercoil , India's	8.18°N	77.4°E	: India's most southern city:
Rome , Italy	41.9°N	12.5°E	
Honolulu , Hawaii, USA	21.31°N	157.86°W	: near the middle of the Pacific Ocean
Easter Island	27.11°S	109.35°W	: in Pacific ocean, 2180 miles west of Caldera, Chili
Miami , Florida, USA	25.76°N	80.2°W	
Jerusalem , Israel	31.77°N	35.2°E	
Cairo , Egypt	30.06°N	31.25°E	
Perth , Australia	31.95°S	115.86°E	
Mexico City , Mexico	19.43°N	99.13°W	: capital city of Mexico
Brasilia , Brazil	15.79°S	47.88°W	: capital city of Brazil
Ushuaia , Chili	54.8°S	68.3°W	: In Patagonia, South America Region, Chili's most southern city. relatively close to Antarctica
Georgetown Island or Ascension Island	7.93°S	14.41°W	: In middle of south Atlantic ocean, and between Angola, Africa and central Brazil. Northwest of Tristan Island
Tristan Island	37.11°S	12.28°W	: In middle of the south Atlantic ocean, and between Capetown, South Africa , and Southern Brazil
Mt. Everest , Nepal	27.98°N	86.93°E	: Highest mountain on Earth, in Himalayas Mts, north of India
Manila , Philippines	14.6°S	120.98°W	
Quito , Ecuador	0.18°S	78.47°W	: "ecuador" is a Spanish word for "equator"
Hithadhoo , Maldives	0.61°S	73.1°E	
Islamabad , Pakistan	33.74°N	73.08°E	
Montreal , Canada	45.51°N	73.59°W	
Quebec , Canada	46.81°N	71.21°W	
London , England	51.51°N	0.118°W	: very close to Greenwich, England, and GMT=**GMT+0**=UTC+0
Dublin , Ireland	53.35°N	6.266°W	
Stockholm , Sweden	59.33°N	18.06°E	
Gibraltar , Spain	36.14°N	5.354W	: southern extended tip of Spain, a peninsula, a Britain Territory
Accra , Ghana , Africa	5.615°N	0.206°W	: in north-west Africa along the Atlantic ocean to the south
Kathmandu , Nepal	27.7°N	85.3°E	
New Delhi , India	28.61°N	77.22°E	
Horta , Azores Islands	38.66°N	28.07°W	: approximately in the central, north atlantic ocean. These 9 islands are autonomouse territories of Portugal.
Arctic Circle	66.5°N		: or= (90° - 66.5°) = 23.5° south of the north geographical pole
Tropic Of Cancer	23.5°N		: more accurately as 23.44°N , north of equator
Tropic Of Capricorn	23.5°S		: more accurately as 23.44°S , south of equator
North Pole	90°N		
South Pole	90°S		
Equator	0° N and 0° S		: the reference of latitude, halfway between Earth's poles

A **magnetic compass** is a lightweight, horizontally balanced needle (thin, lite-weight, narrow, "pointer") made of iron that can freely rotate till the needle points in the direction of the north magnetic pole of the Earth, hence indicating which direction is (roughly) north, and also indirectly indicating the other directions of south, east and west. This compass functions on the basic principle that iron or steel is attracted to a magnet and-or can also be influenced by the north to south magnetic field lines. The magnetic pole of the Earth moves and varies by a few miles or degrees or up to 20 miles a year since Earths molten iron core is moving slightly due to various forces such as the gravity of the Moon that orbits around the Earth, and the change in the Sun's gravity throughout Earth's elliptical orbit about the Sun.. In short, the geographical (true) north pole of the Earth is generally not at the same geographical location or direction as Earth's north magnetic pole (and magnetic compass direction), and is only a close approximation.

As of the year 2020, the coordinates of Earths north magnetic pole are about: 164° E longitude and 87° N latitude. The coordinates of Earth's south magnetic pole are about 136° E longitude and 64° S latitude.

If you are seeking to travel in a certain direction, the true north (geographical, not magnetic) pole direction can be found by locating the North-Star, also called Polaris (the pole or polar star), and traveling along the surface of the Earth in that direction. The angle between true north and magnetic north is an error or (magnetic) offset angle of the compass in terms of direction, and the value of this angle depends on where you are locate on Earths surface. This angle could be 0° and up to 15°. If the direction of a compass and magnetic north is compared to that of the direction to the North-star, this error angle can be found and recorded for your local area, and so as traveling in a (true) geographical direction is then more accurate when using a compass. It is also useful for various astronomical and yearly events such as the directions of sunrise and sunset locations. Each city, park and or other locations should have a permanently installed, fixed direction pointer as a decorative and functional device. Due to that the magnetic pole of Earth can change in location, and thereby direction, every few years, and that locations with a high amount of ferrous (ex. iron ore) can also cause a compass to point in the wrong direction, a **solar compass** was invented in 1831 by James Clark Ross Ross was the first to find the location of Earth's north magnetic pole, and during an expedition in the arctic region. This solar compass will find true or geographical north by observing the true east sunrise and-or sunset in the west. If you do not have a solar compass, the direction of the North Star, Polaris, at night-time can be used, and-or by knowing the eastward to westward apparent path of the Sun across the sky during the daytime.

As the Earth rotates on its polar axis, at every new hour, the Sun will be overhead in the next westward time zone as the Earth rotates in the eastward direction. Some "shortwave, AM" radio stations continuously broadcast the current UTC. The military sometimes calls GMT as "Zulu Time" ("zero meridian time"). The first westward time zone from Greenwich is sometimes noted as having a time of: -1 GMT, meaning "an hour behind" (in Earths rotation eastward and-or also is used for sunrise or sunset times in reference to GMT time), and the first eastward time zone from the Prime Meridian is sometime noted as having a time of: +1 GMT, meaning "an hour ahead" than the time at the Prime Meridian time zone. In reference to GMT, Paris, France is GMT+2. New York City, USA is GMT-5 in the summer months.

Time zones are based on an angle, and are not based on some fixed distance. The maximum distance of a 15° time zone is nonetheless at the equator where an angle represents a much longer (circular, east-west direction) arc than locations near Earths poles. At the equator, the Earths arc, disk or circumference is about 25000 miles which corresponds to 360°. Using proportions or equivalent fractions: 1° is to x miles, as 360° is to 25000 miles, and we find 1° corresponds to about 70 miles (at the equator line only), and 15° corresponds to about (70miles)(15) = 1047 miles.

During the equinox days where the length of day and night are equivalent to 12 hours each at any location on Earth, locations that are 70 miles apart eastward and westward will have a solar shadow that is 1° in difference, and this can be measured on a fine (ie., precision) scaled solar clock or sundial. For locations north and south of the equator, this distance corresponding to 1° will get less and less toward the poles. Using a proportions or an equivalent fraction equation for linear relationships: 70 miles is to 1°, as is X miles or the circumference of the Earth is to 360°, and we find X equal to about: (360)(70mi) = 25200 mi.

Each season is defined as one-quarter (1/4) of a year, or 365.25 days/4 = $91.3\underline{1}$ days = roughly 90 days ~ 3 months:

Time Of Year Date	Season	Days From March 21	Tilt	
March 21 to June 21	"spring"	0 to 90	0° to +23.5°	:March 21 is an equinox day
June 21 to September 21	"summer"	90 to 180	+23.5 to 0°	
September 21 to December 21	"fall"	180 to 270	0° to -23.5°	:September 21 is an equinox day
December 21 to March 21	"winter"	270 to 360	-23.5 to 0°	

As was previously indicated, the tilt of the Earth throughout the year is nearly, or not exactly, sinusoidal. This is due to that the orbit of Earth around the Sun is not exactly circular and is rather slightly elliptical or "egg shaped". The Sun is therefore not at the geometric center of either an ideal circular orbit or its actual elliptical orbit, but is offset slightly some distance from the center point of its elliptical orbit. During the northern hemispheres winter season and a slightly elliptical orbit, Earth is actually slightly closer (about 3 million miles closer than the average of 93 million miles) to the Sun and the increased gravity is causing it to orbit the Sun slightly faster, and during the summer, Earth is slightly farther from the Sun and the lessened gravity is causing it to orbit the Sun slightly slower. This has the effect of changing the (apparent) tilt angle slightly faster (than a yearly average or rate) than expected in the winter, and slightly slower than (the average) expected in the summer. Note that the time of each Earth day or rotation about is axis is still the same time of about 24 hours, and rather the Suns position or (angle of) elevation in the sky is slightly different (more or less) than expected, and this should be considered for (more accurate) Sundial clocks so as to make a correction to the time physically displayed, as a shadow line, and so as to still use the standard, yearly, 24 hour time system. You can look into the topic of: "Equation Of Time" which is the understanding and calculation to account for this apparent "error" or difference from what (ie., average) was to be expected. At the extreme error points, the maximum difference between true solar noon where the Sun is at the highest angle (with respect to your local horizon) overhead and where it will be in the sky throughout the day, and localized (due to time zones, so everyone in a zone has the same time; hour and minute, to avoid confusion) noon time on the clock, will be "off" (in error) by about +- about 15 minutes. Since 1 hour or 60 minutes corresponds to an apparent 15 degrees of Sun movement, the maximum time error of 15 minutes will correspond to an apparent 3.75 degrees of maximum angle error from its commonly expected "average position".

Because of the apparent tilt of the Earth throughout the year, the locations of the sunrise and sunset along the horizon will change in the same (nearly) sinusoidal periodic manner and cycle. The locations of the moonrise and moonset will also be affected in a similar manner. The Moon also has a constant, 5° inclined (or "tilted") orbit about the Earth, and therefore it goes above and below the solar plane during its orbit around the Earth. This is a visible, total apparent change of 10° during its orbit, and this is the apparent diameter of 20 moons since it appears at about 0.5° in diameter or wide to an observer on Earth. The apparent tilt of the Earth throughout the year will also make the Moon's angle above the horizon appear to change. The total apparent tilt of the Earth throughout the year is about 23.5° + 23.5° = 47°.

As the Earth tilts throughout the year, the apparent location of the Sun up in the sky as seen at the same location on Earth and time every day will have a locus or path that has the shape of a "figure eight = 8" throughout the year. As the northern hemisphere of Earth tilts southward and toward the Sun in the summertime, the Sun will appear to travel and set in a slightly more northern direction and results in the Sun setting in a north-west direction along the horizon, and is actually 23.5° northward or clockwise from true west on about July 21. Likewise, in the winter, the Sun will set in the south-western direction along the horizon, and is actually 23.5° southward or counter-clockwise from true west on about December 21. This process and annual (yearly) apparent path of the Sun as seen at the same time of day, such as local noon or 12pm or 0pm on a 24hour time system, is known as the **Analemma** of the Sun. Because the orbit of Earth is closer to the Sun in the Winter and traveling faster, and then farther from the Sun in the Summer, one end of this figure eight (or somewhat sinusoidal or elliptical in shape) position or path up in the sky will be longer than the other.

Equation of a sine wave at particular frequency:

The general meaning of the word frequency is how frequent something is or how often it occurs, such as in the number of times or occurrences per unit of time such as a second.

Since some of these concepts below were also discussed in the previous topic, you may wish to review it so as to possibly have a greater understanding of these concepts.

For something that is spinning, such as a point on the circumference of a rotating wheel, or a periodic wave (such as a sine wave) that is repeated over and over, what is the basic formula for the height or amplitude (ie. max height) of that waveform?

For 1 revolution, the angle is 2(pi) radians =360°.

y = sin (¢) = sin (x) , : Here the amplitude is 1: y = 1 sin (x), but if the maximum amplitude is no longer 1, but some other multiplying or magnifying factor value, then it must be included in the equation as a multiplier to sin (¢):

y = height = amplitude = (maximum amplitude) sin (¢)

For something rotating in a periodic manner, it will have a certain number of rotations per second, and this is the same as cycles per second (cps =hertz = hz) or frequency (f). The time (t) duration or (time) period of each cycle is: time = (1/frequency). Frequency of the cycles and the time of each cycle are inversely related, and are in fact, mathematical reciprocals of each other:

$\text{Fhz} = \frac{1}{\text{Ts}}$ and $\text{Ts} = \frac{1}{\text{Fhz}}$ and: (F)(T) = 1 , Ex. $1000\text{hz} = \frac{1}{0.001\text{s}}$

For example, the higher the frequency, the lower the time value or period of each cycle.

The amount of angle that has happened after a time value while rotating depends both on the frequency of the cycles and the time elapsed from 0° which is considered as the starting or reference position and angle. From the above "reciprocal equation":

(frequency)(time elapsed during the cycle) = 1 = (cycles / second)(seconds) ,
the result is a fraction or relative value of the complete rotation (2p rads = 6.28 radians = 360°) or cycle in question.

y = (max. amplitude) sin (¢) : general sin equation, and (y) is with respect to the value of the angle

y = (max. amplitude) sin ((2pi) f t) : (2)(pi) = a complete rotation or (radian) angle of 360°. (2)(pi) radians = 360°. (2)(pi)(f) = total degrees of rotation per second which is sometimes called the angular velocity, and multiplying by (t) we can find both the angle or degrees rotated and-or the amplitude at a certain value of time.

y =~ (max. amplitude) sin (6.28 f t) : f = frequency in cycles per second = hertz (hz) , t = time in seconds

CONVERSIONS OF AN ANGLE'S TRIGONOMETRIC VALUES

Each trigonometric value has a corresponding angle (<=90°) associated with it, and therefore, that trigonometric value corresponds to that angles other corresponding trigonometric values. Each angle (<=90°) has a corresponding SIN, COS and TAN value. Given just one value, you can calculate the other two values. Most of the conversion equations or formulas below can be derived from the trigonometric **identity** ("identical", a mathematical relationship of trigonometric values which will hold true for any and all values of the angle) of:

$\text{SIN}^2 ¢ + \text{COS}^2 ¢ = 1$: this is sometimes called the "Pythagorean Trigonometric Identity"

The equation above can be derived by observing the drawing, given previously in this book, of approximating the trigonometric values graphically, and using the Pythagorean Theorem with it. Some of the identities derived below were previously given.

From: $h^2 = r^2 = x^2 + y^2 = 1 = (\text{SIN } ¢)^2 + (\text{COS } ¢)^2$: when r = radius line = 1, or hypotenuse line = 1

$\text{SIN } ¢ = \sqrt{1 - \text{COS}^2 ¢}$

$\text{COS } ¢ = \sqrt{1 - \text{SIN}^2 ¢}$

And from $\text{TAN } ¢ = \frac{\text{SIN } ¢}{\text{COS } ¢}$: also $\text{COS } ¢ = \text{SIN } ¢ / \text{TAN } ¢$ and $\text{SIN } ¢ = \text{COS } ¢ \ \text{TAN } ¢$

$\text{TAN } ¢ = \frac{\text{SIN } ¢}{\sqrt{1 - \text{SIN}^2 ¢}} = \frac{\sqrt{1 - \text{COS}^2 ¢}}{\text{COS } ¢}$: calculating TAN ¢ using either the SIN or COS of the angle

Here is the trigonometric identity for cofunctions of complementary angles:

Given an angle (¢< 90°) its complementary angle is (90° - ¢). The trigonometric cofunctions of these angles have the same value:

$\text{SIN } ¢ = \text{COS } (90° - ¢)$ also: $\text{TAN } ¢ = \text{COTAN}(90° - ¢)$

$\text{COS } ¢ = \text{SIN } (90° - ¢)$ Ex. $\text{COS } 60° = \text{SIN } (90° - 60°) = \text{SIN } 30°$

$\text{TAN } ¢ = \frac{\text{SIN } ¢}{\text{COS } ¢} = \frac{\text{SIN } ¢}{\text{SIN } (90° - ¢)}$: calculating TAN ¢ using the SIN values of two angles

Here is how to solve for either sin ¢ or cos ¢ using tan ¢.

From: $1 = \sin^2 ¢ + \cos^2 ¢$ since $\tan ¢ = \sin ¢/\cos ¢$, $\sin ¢ = \cos ¢ \tan ¢$:

$1 = \cos^2 ¢ \tan^2 ¢ + \cos^2 ¢$ factoring out $\cos^2 ¢$:

$1 = \cos^2 ¢ \, (1 + \tan^2 ¢)$ solving for $\cos^2 ¢$:

$\cos^2 ¢ = \frac{1}{1 + \tan^2 ¢}$ solving for cos ¢ by taking the square-root of both sides:

$\cos ¢ = \sqrt{\frac{1}{1 + \tan^2 ¢}} = \frac{1}{\sqrt{1 + \tan^2 ¢}}$

Since $\sin ¢ = \tan ¢ \cos ¢$ and substituting the above expression for cos ¢ :

$$\sin \text{¢} = (\tan \text{¢}) \quad (\cos \text{¢})$$

$$\sin \text{¢} = (\tan \text{¢}) \left(\frac{1}{\sqrt{1 + \tan^2 \text{¢}}} \right)$$

$$\sin \text{¢} = \frac{\tan \text{¢}}{\sqrt{1 + \tan^2 \text{¢}}}$$

Also, $(\sin \text{¢}) (\cos \text{¢}) = \dfrac{\tan \text{¢}}{(1 + \tan^2 \text{¢})^{0.5}} \; \dfrac{1}{(1 + \tan^2 \text{¢})^{0.5}} = \dfrac{\tan}{1 + \tan^2 \text{¢}}$

Here are some identities (ie., trigonometric equivalences) of the complementary angles (A and B) of a right triangle :

From $c^2 = a^2 + b^2$ and $\mathrm{SIN}\,A = \dfrac{a}{c}$ and $\mathrm{SIN}\,B = \dfrac{b}{c}$, $c = b / \mathrm{SIN}\,B = a / \mathrm{SIN}\,A$, algebraically:

$a = c\,\mathrm{SIN}\,A$ squaring both sides of the equation:
$a^2 = (c\,\mathrm{SIN}\,A)^2$
$a^2 = c^2\,(\mathrm{SIN}\,A)^2$

$b = c\,\mathrm{SIN}\,B$ squaring both sides of the equation:
$b^2 = (c\,\mathrm{SIN}\,B)^2$
$b^2 = c^2\,(\mathrm{SIN}\,B)^2$

$c^2 = a^2 + b^2$ using algebraic substitution:
$c^2 = c^2\,\mathrm{SIN}^2 A + c^2\,\mathrm{SIN}^2 B$ since c^2 is a common factor to both terms:
$c^2 = c^2\,(\mathrm{SIN}^2 A + \mathrm{SIN}^2 B)$ dividing both sides by c^2 and canceling:

$1 = \mathrm{SIN}^2 A + \mathrm{SIN}^2 B$ therefore:

$\mathrm{SIN}\,A = \sqrt{1 - \mathrm{SIN}^2 B}$: B = (90° - A), B is the complementary angle to A, and:

$\mathrm{SIN}\,B = \sqrt{1 - \mathrm{SIN}^2 A}$: A = (90° - B), A is the complementary angle to B:

Substituting $a = (c\,\mathrm{COS}\,B)$ and $b = (c\,\mathrm{COS}\,A)$ into the initial derivation above, it can also be shown that:

$\mathrm{COS}\,A = \sqrt{1 - \mathrm{COS}^2 B}$: A and B are complementary angles, and :

$\mathrm{COS}\,B = \sqrt{1 - \mathrm{COS}^2 A}$: A and B are complementary angles

As an alternate derivation, consider that we already know:

$\mathrm{SIN}\,B = \mathrm{COS}\,A$ squaring both sides:
$\mathrm{SIN}^2 B = \mathrm{COS}^2 A$ using substitution into the following:

$\mathrm{SIN}^2 \text{¢} + \mathrm{COS}^2 \text{¢} = 1$: Pythagorean-like relationship of SIN and COS, using angle A:
$\mathrm{SIN}^2 A + \mathrm{COS}^2 A = 1$ substituting for $\mathrm{COS}^2 A$:
$\mathrm{SIN}^2 A + \mathrm{SIN}^2 B = 1$: Pythagorean-like relationship of complementary angles ,
(and with the same trigonometric function; here it's SIN)

SIN A = √1 - SIN^2 B = √1 - SIN^2 (90° - A) = \/ 1 - COS^2 A

Ex. The sine of 70° can be calculated using the sine of its complementary angle of 20° :

SIN 70° = \/ 1 - SIN^2 20° : 90° - 70° = 20°

As mentioned previously, the smaller the angle, the fewer the terms that need to be calculated for a given precision using the SIN ¢ series. 20° is a much smaller angle than 70°

From SIN^2 A + COS^2 B = 1 : where A and B are complementary angles

The angles above can also be supplementary to each other since if ¢ > 90° (a second quadrant angle):

SIN ¢ = SIN (180° - ¢) and: COS ¢ = COS (180° - ¢)

Ex. (SIN 160°)^2 + (COS 20°)^2 = (SIN 20°)^2 + (COS 20°)^2 = 1

This leads to a general conclusion that the COS^2 of an angle of a triangle, plus the SIN^2 of the sum of the other two angles (which sum to a supplementary angle of the third angle) of that triangle equals 1.

SIN^2 (¢1 + ¢2) + COS^2 ¢3 = 1 : Trigonometric Identity Of All Three Angles Of A Triangle

Since COS ¢ = \/ 1 - SIN^2 ¢ , after substituting this into the above equation and simplifying:

SIN^2 (¢1 + ¢2) = SIN^2 ¢3 taking the square root of both sides and switching sides:

SIN ¢3 = SIN (¢1 + ¢2) : Trigonometric Identity Of All Three Angles Of A Triangle

Consider that since ¢3 and the sum of (¢1 + ¢2) are supplementary angles which have the same trigonometric values.

Likewise:

COS^2 (¢1 + ¢2) + SIN^2 ¢3 = 1 : Trigonometric Identity Of All Three Angles

COS^2 (¢1 + ¢2) = COS^2 ¢3 taking the square root of both sides:

COS ¢3 = COS (¢1 + ¢2) : Trigonometric Identity Of All Three Angles Of A Triangle
(use absolute (positive sign) trigonometric values)

TAN ¢3 = SIN (¢1 + ¢2) / COS (¢1 + ¢2) : Trigonometric Identity Of All Three Angles Of A Triangle,
Angle ¢3 is supplementary to (¢1 + ¢2).

Extra: Given a circle, From: sin ¢ = opp. / hyp. = y / r and cos ¢ = adj. / hyp. = x / r

y = r sin ¢ and x = r cos ¢ r = \/ x^2 + y^2
r = y / sin ¢ and r = x / cos ¢ , hence: y / sin ¢ = x / cos ¢ , y / x = sin ¢ / cos ¢
x / y = cos ¢ / sin ¢
x = r cos ¢ = y / sin ¢ cos ¢ = y cos ¢ / sin ¢ = y / tan ¢ : when r is not initially known
y = r sin ¢ = x / cos ¢ sin ¢ = x sin ¢ / cos ¢ = x tan ¢ : when r is not initially known
y / x = tan ¢ = opp. / adj.

From the above derivations, is can also be shown that: 1 = (cos ¢ / sin ¢) (y / x) , hence a constant equal to 1
Also: y cos ¢ = x sin ¢ = r

[This space for edits.]

Double And Half Angle Trigonometric Identities

These trigonometric identities are commonly known today in the mathematics community and they were progressively derived during a span of about two-thousand years starting from the Babylonian, Egyptian, Greek, Indian, Asian, Arab, and European-Western and other mathematicians who took mathematics to new heights. The derivations are typically built upon the other more basic trigonometric identities mentioned in this book. These identities were commonly used to make trigonometric tables and still can be used if you prefer not to calculate them by other means such as the series methods. These equations presented here for are only for basic familiarity purposes if you happen to encounter them or other similar types of equations. These equations are not required for further use in this book and may be skipped over for now and viewed at another time.

$\sin(2x) = 2\sin x\cos x = \frac{2\tan x}{1 + \tan^2 x}$: sine of twice the angle , x = the angle

$\cos(2x) = \cos^2 x - \sin^2 x = 2\cos^2 x - 1 = 1 - 2\sin^2 x = \frac{1 - \tan^2 x}{1 + \tan^2 x}$

Note that, from: $2\cos^2 x - 1 = 1 - 2\sin^2 x$, we have the "Pythagorean-like trigonometric identity" :
$\sin^2 x + \cos^2 x = 1 = (\sin x)^2 + (\cos x)^2$

$\tan(2x) = \frac{2\tan x}{1 - \tan^2 x}$

Note that from: $\tan x = \sin x / \cos x$, therefore,
$\tan(2x) = \sin(2x) / \cos(2x)$ and $\tan(x/2) = \sin(x/2) / \cos(x/2)$

$\cos(x/2) = \pm\sqrt{(1 + \cos x) / 2}$: This can be derived from solving for : $\cos(2x) = 2\cos^2 x - 1$ and considering that if $(2x)$ is the angle, then (x) is half the angle. Therefore: $\cos x = 2\cos^2(x/2) - 1$. Solving for $\cos(x/2) = \cos$ (half angle), we have this identity shown. $(x/2)$ represents the half an angle value.

$\sin(x/2) = \pm\sqrt{(1 - \cos x) / 2}$: Credited to (Claudius) Ptolemy of Alexandria

Ex. Find Sin 0.5°

If we let $x = 0.5°$, then twice the angle is: $2x = 2(0.5°) = 1°$, and half the angle is: $(2x)/2 = 1°/2 = 0.5°$:

$\text{Sin } 1° = 0.017452406$

$\text{Cos } 1° = \text{SIN } 89° = 0.999847695$

$\text{Sin } 0.5° = \text{SIN }(1°/2) = \sqrt{(1 - 0.999847695) / 2} = 0.008726535$

If we solve the last formula given for the cosine of the angle of which is actually twice of that specific angle value of $(x/2)$ on the left side, we can derive the above equation for: $\cos(2x) = 1 - 2\sin^2 x$

From: $\sin^2 x + \cos^2 x = 1$, $\sin^2 x = 1 - \cos^2 x$, and substituting this value into: $\cos(2x) = 1 - 2\sin^2 x$, we have another identity as shown above for: $\cos(2x) = 2\cos^2 x - 1$. If we solve this last equation for $\cos^x$ we have:

$\cos x = \sqrt{(\cos(2x) + 1) / 2}$, and since the angle on the left hand side is half of that on the right hand side, we have the above half-angle identity for $\cos(x/2)$.

$\tan x = \dfrac{2 (\tan x/2)}{1 - (\tan x/2)^2}$: from tan (2x) = ¢ identity , and using ¢ = (x/2). Similar to the identity for: tan 2x

$\tan (x + y) = \dfrac{\tan x + \tan y}{1 - \tan x \tan y}$: if y=x, tan (x + y) = tan (x + x) = tan(2x) and the formula is shown above.

Using the above identity, if you have a table of tangent values of angles from 0° to 45°, you can use it to solve for the tangent values of angles from 45° to 90°:

If x = 45°: $\tan (45° + y) = \dfrac{\tan 45° + \tan y}{1 - \tan 45° + \tan y} = \dfrac{1 + \tan y}{1 - 1 + \tan y} = \dfrac{1 + \tan y}{\tan y}$, here, y° /=/ 0°

$\tan (x - y) = \dfrac{\tan x - \tan y}{1 + \tan x \tan y}$, $\tan(45° - y) = \dfrac{1 - \tan y}{1 + \tan y}$ Ex: tan(30°) = tan(45° - 15°) , here, use: y=15° in the formula.

FINDING THE ANGLE GIVEN A TRIGONOMETRIC VALUE

The formula for the series expansion of ARCSIN X is given below and its derivation can be found in the appendix. This is one of the possible formulas to keep near a basic "home-use", 4 function calculator. There are similar formulas for ARCCOS X and ARCTAN X but they are not presented right here since it is easier to work with, and possibly remember only one formula. A series for ARCCOS X and ARCTAN X can be found in the appendix.

$$\text{ARC SIN } X = X + \sum_{n=1}^{n=\infty} \frac{(1)\ldots(2n-1)}{(2)\ldots(2n)} \cdot \frac{X^{(2n+1)}}{(2n+1)}$$

: ARC SIN Series expansion of the function SIN x. This expansion, and some others, were developed by **Colin Maclaurin** when he used a (Brook) Taylor series with the functions derivatives evaluated at x=0.

The angle calculated by this series is a radian angle. Notice that some initial factors (on the left) are indicated along with the common expression for those factors. This indicates that all the previously calculated factors of:

$\frac{(2n-1)}{(2n)}$ up to and including this terms value are to be also included in each new term.

n is an integer (regular counting number) from 1 to infinity (∞) for this series: 1, 2, 3, 4, 5, 6, 7, . . .

Expanding the first eight terms of the series:

$$\text{ARCSIN } X = X^1 + \frac{(1)X^3}{(2)(3)} + \frac{(1)(3)X^5}{(2)(4)(5)} + \frac{(1)(3)(5)X^7}{(2)(4)(6)(7)} +$$

$$+ \frac{(1)(3)(5)(7)X^9}{(2)(4)(6)(8)(9)} + \frac{(1)(3)(5)(7)(9)X^{11}}{(2)(4)(6)(8)(10)(11)} + \frac{(1)(3)(5)(7)(9)(11)X^{13}}{(2)(4)(6)(8)(10)(12)(13)} +$$

$$+ \frac{(1)(3)(5)(7)(9)(11)(13)X^{15}}{(2)(4)(6)(8)(10)(12)(14)(15)} + \frac{(1)(3)(5)(7)(9)(11)(13)(15)X^{17}}{(2)(4)(6)(8)(10)(12)(14)(16)(17)} + \ldots$$

Notice the pattern of which can be memorized as: The coefficients of X start with one in the numerator and increase by one alternately in every numerator and denominator until it equals the exponent of X that increases by 2 for each term.

As a memory aid, the general format for each term can be memorized as:

$$\frac{(\text{products of odd series up to } n-2)\ X^n}{(\text{products of even series up to } n-1)\ \ n}$$

: where the exponent of X is the odd series: 1, 3, 5, 7, 9, . . . You can view the topic of: Products Of First Terms Of The Odd and Even Series for some formulas that could be used to find and-or express these indicated products.

Simplifying most of the terms given above:

$$\text{ARCSIN } X = X + \frac{X^3}{6} + \frac{3X^5}{40} + \frac{15X^7}{336} + \frac{105X^9}{3{,}456} + \frac{945X^{11}}{42{,}240} + \frac{10{,}395X^{13}}{599{,}040} + \frac{135{,}135\,X^{15}}{9{,}676{,}800} + \ldots$$

The smaller the sine value (generally due to a smaller trigonometrically equivalent reference angle (<=90°)), the faster (ie. fewer terms needed to be evaluated) the sum of the terms converges to a given value since the value of each next term is more rapidly approaching a value of 0. Hence, for a given precision (essentially, the number of correct significant digits), the smaller the angle, the fewer number of terms that need to be evaluated. Remember that a power of a fraction ($X<1$) is actually a fraction of a fraction and so on which results in a very small value. To give you an idea of the results using only the 8 terms given above: when $x=1$ (the sine value of 90° angle), precision (ie., lowest decimal numerical weight) correct (ie., accuracy) to the 1st or 2nd decimal place (ie. weight) is achieved, when $x=0.5$, a precision correct to the 5th or 6th decimal place is achieved, and when $x=0.1$, a precision correct to the 16th or 17th decimal place is achieved. One method to increase accuracy without utilizing more terms when X is greater than say about 0.5 is to convert the angles sine value to its (lower) corresponding cosine value and then use the ARCCOS X series.

Ex. Given SIN ¢ = 0.425, find ¢.

$$\text{ARCSIN } 0.425 = 0.425 + \frac{(0.425)^3}{6} + \frac{3(0.425)^5}{40} + \frac{15(0.425)^7}{336} +$$

$$+ \frac{105(0.425)^9}{3{,}456} + \frac{945(0.425)^{11}}{42{,}240} + \frac{10{,}395(0.425)^{13}}{599{,}040}$$

$$+ \frac{135{,}135(0{,}425)^{15}}{9{,}676{,}800} + \ldots$$

$$\text{ARCSIN } 0.425 = 0.425000000 + 0.01279427 + 0.001039934 + 0.000111808 + 0.000013744 + 0.000001828 + 0.000000256 + 0.000000037$$

ARCSIN 0.425 = about 0.438961877 radians which is about 25.15066292°.

Checking by using a calculator:

ARCSIN 0.425 = 0.438961885 (radians) = 25.15066341°. : If a few more terms were utilized, the result shown above would match this correct value.

When the sine value given is high, here is a method that utilizes the complementary angle's sine value which is a lower value to use in the series, and where fewer terms are then required for a certain precision:

$90° = A + B$: sum of the complementary angles A and B

$A = 90° - B$ which can be expressed as:

$\text{ARCSIN (SIN A)} = 90° - \text{ARCSIN (SIN B)}$: note, A = ARCSIN(SIN A) and B = ARCSIN (SIN B)

Since: $\text{SIN B} = \sqrt{1 - \text{SIN}^2\text{ A}}$: as shown previously, therefore:

$\text{ARCSIN (SIN A)} = 90° - \text{ARCSIN}\left(\sqrt{1 - \text{SIN}^2\text{ A}}\right)$: here, optionally using degree angles as indicated with the 90° angle value.

Ex. Evaluate ARCSIN 0.93969262

Let SIN A = 0.93969262

$\text{SIN B} = \sqrt{1 - 0.93969262^2} = 0.342020143$

B = ARCSIN (SIN B) = ARCSIN 0.342020143 = 0.349067219 rads = 20° Therefore,

A = ARCSIN (SIN A) = ARCSIN 0.93969262 = A = 90° - B = 90° - 20° = 70°

If you want to use all radian angles instead of degree angles:

Since 90° = 1.570796327 rads:

$1.570796327 = A + B$: here, A and B are complementary angles with units of radians, that sum to 1.570$\underline{8}$ radians mathematically:

$A = 1.570796327 - B$
$B = 1.570796327 - A$

Modifying the general formula so as to correspond to radian angles:

$\text{ARCSIN}\ (\text{SIN}\ A) = 1.570796327 - \text{ARCSIN}\ (\sqrt{1 - \text{SIN}^2 A})$: using radian angles

If given an angles COS or TAN value and you want to use the ARC SIN series, convert the given value to the angles corresponding SIN value.

Ex. Given COS ¢ = 0.82 find the angle.

First convert the value given to the (same) angles corresponding SIN value.

$\text{SIN}\ ¢ = \sqrt{1 - \text{COS}^2\ ¢}$

$\text{SIN}\ ¢ = \sqrt{1 - (0.82)^2}$

$\text{SIN}\ ¢ = 0.57236352$

Placing this SIN ¢ value into the ARC SIN expansion, we find that the angle is 34.91520625°.

Hence: ARCCOS 0.82 = ARCSIN 0.57236352 = 34.91520625°

Ex. Given TAN ¢ = 1.732 find the angle.

First convert the value given to the corresponding SIN value of the (same) angle.

$$\text{SIN}\ ¢ = \frac{\text{TAN}\ ¢}{\sqrt{1 + \text{TAN}^2\ ¢}} = \frac{1.732}{\sqrt{1 + (1.732)^2}} = \frac{1.732}{1.999956} = 0.866019052$$

Placing this SIN value into the ARCSIN series, we find that the angle is about 60°.

Hence: ARCTAN 1.732 = ARCSIN 0.866 = 60°

Here are some other trigonometric identity methods to evaluate ARCSIN X:

From: $\text{COS}\ ¢ = \sqrt{1 - \text{SIN}^2\ ¢}$ and letting:

$\text{ARCSIN}\ X = ¢ = \text{ARCCOS}\ (\text{COS}\ ¢)$: their trigonometric results is the same angle, therefore:

$\text{ARCSIN}\ X = \text{ARCCOS}\ \sqrt{1 - \text{SIN}^2\ ¢}$ Since $X = \text{SIN}\ ¢$:

$\text{ARCSIN}\ X = \text{ARCCOS}\ \sqrt{1 - X^2}$: $X = \text{SIN}\ ¢$

From: $\text{TAN}\ ¢ = \dfrac{\text{SIN}\ ¢}{\sqrt{1 - \text{SIN}^2\ ¢}}$ and letting:

ARCSIN X = ¢ = ARCTAN (TAN ¢) therefore:

$$\text{ARCSIN } X = \text{ARCTAN}\left(\frac{\text{SIN } \text{¢}}{\sqrt{1 - \text{SIN}^2 \text{¢}}}\right)$$

Since X = SIN ¢ :

$$\text{ARCSIN } X = \text{ARCTAN}\left(\frac{X}{\sqrt{1 - X^2}}\right)$$

: X = SIN ¢

Some methods for evaluating ARCCOS and ARCTAN will be given further ahead.

EPSILON (e)

The symbol (e) represents a constant that is an irrational value, like pi=3.14159...is, and is approximately 2.718281828. This value is used to represent the limiting (ie. maximum) factor value of constantly (ie. infinitely) compounded growth (growth with respect to the current size, or "growth upon (previous) growth - which includes any growth due to interest growth") over a (one) period of predetermined time. It is mathematically defined to be a very natural number and is frequently used when representing growth or decay (loss) processes in nature, therefore, the constant (e) is often found in many mathematical formulas. You could say that (e) and (pi) are natural constants. (e) is equivalent to the expression below, and **a derivation of (e) is given in the appendix section of this book**.

$$e = \left(\frac{1}{1} + \frac{1}{n}\right)^n = \text{about } 2.71828 \text{ as } n \dashrightarrow \infty$$

: (n -->∞) means: "as n approaches infinity". Remember, infinity has no specific value or end, and therefore, it cannot be mathematically represented, but only "approached", or approximated for practical purposes. In a way, infinity is an irrational value. Infinity is a concept and general description for a non-ending or unending process. "As n approaches infinity" could be thought of as: "as (n) gets (unendingly) higher in value". (e) is also called Euler's (pronounced as "Oiler's") Number.

Mathematically, combining the fractions:

$$e = \left(\frac{n+1}{n}\right)^n = \frac{(n+1)^n}{n^n} \qquad : \sim 2.71828 \text{ as } n \dashrightarrow \infty$$

As (n) approaches infinity, the entire base of this indicated power approaches: 1.0000....1 which is practically 1. It is as if the added 1 in the numerator becomes less and less meaningful as (n) increases, and it is a mathematical fact when you consider that 1 part of an increasing number of total parts, becomes less of a portion or fraction of the whole part or entire parts. The result will always be slightly over 1 as (n) becomes high n value:

For example, consider these fractions where (n) increases in value:

$$\frac{(n+1)}{n}\ ,\ \frac{8}{7} = 1.142857143\ ,\ \frac{15}{14} \sim 1.071428571\ ,\ \frac{1500}{1499} = 1.000667111\ ,\ \frac{999999}{999998} = 1.000001$$

Also, using the mathematically equivalent form of the base value:

$$\left(\frac{1}{1} + \frac{1}{n}\right)\ ,\ \left(\frac{1}{1} + \frac{1}{1499}\right) = 1 + 0.000667111\ ,\ \left(\frac{1}{1} + \frac{1}{999999}\right) = 1 + 0.000001 = 1.000001$$

Raising this value to a very high power (such as with an exponent that is high value of (n)):

$$(1.0000...1)^{\infty} = e \sim 2.71828...$$

Now it will be shown in a simple manner where this value of 2.71... , and not some other value, comes from.

If we let $x = \frac{1}{n}$, and substituting this value into the above power of a binomial, and expanding the indicated power:

$$\left(\frac{1}{1} + \frac{1}{n}\right)^n = (1 + x)^n$$

also notice that since $x = 1/n$, $n = 1/x$, hence: $(1+x)^n = (1+x)^{(1/x)} = x\sqrt{1+x}$
:when n is high, as n ----> infinity, x---->0, and the result approaches 2.71828.... = e

$(1 + x)^1 = 1 + 1x$: $n=1$

$(1 + x)^2 = (1+x)^1(1+x)^1 = 1 + 2x + x^2$: $n=2$

$(1 + x)^3 = (1+x)^2(1+x)^1 = 1 + 3x + 3x^2 + x^3$: $n=3$

$(1 + x)^4 = (1+x)^3(1+x)^1 = 1 + 4x + 6x^2 + 4x^3 + 1x^4$: $n=4$

We observe a pattern that the second terms numerical coefficient for (x) is equal to the value of (n). So if (n) is some high value, as indicated below, the first two terms would be for example:

$(1 + x)^{50} = 1 + 50x + \ldots$: when $n=50$

$(1 + x)^{999999} = 1 + 999999x + \ldots$:when $n=999999$

Since $x=1/n$, $x = 1/999999 = 0.000001$, (x) approaches an infinitely small value as (n) approaches infinity. The second term in this series (for e) is always equal to 1 regardless of the value of (n). For example if (n) = 999999, (x)= 1/n = 1/999999 = 0.000001. The product of (n) and (x) is always 1, and this is so since (n) and x) are reciprocals.

$999999 (0.000001) = 1$

The value of the first two terms is always: $1 + 1 = 2$

The remaining (infinite number of) terms will sum up to 0.71828... for a total of: $2 + 0.7182\ldots = 2.71828\ldots = e$

The remaining terms will get smaller and smaller in value. This is due to the increasing power of (x) in the terms. Given a value less than 1, the higher the indicated power (ie., the exponent) it is raised to, the smaller the resulting power value. From basic multiplication of decimals, a fraction of a fraction, or a power (>1) of a fraction, will result in a smaller value.

From $x = \frac{(1)}{(n)}$

$x^2 = \left(\frac{1}{(n)}\right)^2 = \frac{1}{n^2} > x^3 = \left(\frac{1}{(n)}\right)^3 = \frac{1}{n^3}$: $n > 1$

Also, a quotient is inversely related to the divisor. The higher the divisor, the smaller the quotient.

Whereas (e) is used to represent a naturally or "continuously" growing or increasing value,

e^{-1} or= $\frac{1}{e}$ is frequently used when representing a naturally or "continuously" decaying value.

(e) and-or (e^{-1}) is sometimes used in some electronic formulas, such as for capacitors (essentially electric charge holders, potential energy), to express the natural like changes and values of voltages and-or currents (minute, nearly infinitesimally small, electrons having an electric charge).

Using a relative value of 1 = 100%, if this value of 1 grows compoundedly with an infinite (nearly endless, continuous, unbroken) number of growing steps over 1 period of time, it will grow to a maximum or limit of (e) = 2.718... The time it takes for the value of 1 to grow to (e) is the period time of the given situation. If then given an initial value of 5 and it had naturally compounded growth during the same time period, that value of 5 will grow to: $5e = (5)(2.71828) =\sim 13.\underline{6}$ at the end of one period of time. If the initial was 13.6 and naturally decays compoundedly during the same amount of time, it would become a value of: $13.6 / e^1 = 13.6 / 2.71828 = 5$

A negative power, such as a negative power of e, of can be calculated by using the reciprocal of the corresponding positive power:

$$b^{-x} = \frac{1}{b^x}$$ If the base of the power is (e):

$$e^{-x} = \frac{1}{e^x} \approx \frac{1}{2.7182818^x}$$

$$e^{-2} = (e^{-1})^2 = \left(\frac{1}{2.7182818}\right)^2 = 0.367879441^2 = 0.135335283$$

And generalizing the above**, we can solve for a negative power of a base, here (e), using the same indicated power (here, x) of the reciprocal of that base.**

Since $(1/e) = (1/2.718281828) = 0.367879441 = e^{-1}$ and raising each side to the (x) power, and switching sides:

$(e^{-1})^x = 0.367879441^x$ after using the power to a power rule on the left side:

$\mathbf{e^{-x} = 0.367879441^x}$ and $\mathbf{e^x = 2.718281828^x}$

$\mathbf{b^{-x} = (1/b)^x}$ and-or $\mathbf{b^{-x} = \frac{1}{b^x} = (b^{-1})^x = (1/b)^x}$ **: A negative power of a base is equal to the same positive power of the reciprocal of that base.**

As discussed previously in this book, (e) is the base for natural logarithms. Fortunately, a series for calculating powers of (e) has been developed. It will provide great computational use. This series for (e) is shown next, and its derivation is shown in the appendix.

SERIES FOR POWERS OF (e)

$$e^x = \sum_{n=0}^{n=\infty} \frac{x^n}{n!} \quad : \; 0! = 1 \text{ and } x^0 = 1$$

Expanding the terms of this series:

$$e^x = 1 + x^1 + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \frac{x^7}{7!} + \dots$$

: a derivation of this is shown in the appendix section

Simplifying the factorials:

$$e^x = 1 + x^1 + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + \frac{x^6}{720} + \frac{x^7}{5{,}040} + \frac{x^8}{40{,}320} + \frac{x^9}{362{,}880} + \frac{x^{10}}{3{,}628{,}880} + \dots$$

When the value of x is less than or equal to 1, the series rapidly converges toward 0. The smaller x is, fewer terms need to be evaluated for a given accuracy (ie., number of correct significant digits) of the power of e. To give you a general idea of the results of using the 11 terms given above, when x=1, you can expect precision (decimal place or position) correct (accuracy) to the 6th decimal digit, when x=0.5, you can expect precision correct to the 9th digit, and when x=0.1, you can expect a precision correct to the 16th digit.

When x is greater than 1, rather than use more terms for a greater precision and accuracy, you can factor (e^x) to where each exponent of (e) is less than or equal to one. The result is that of repeated multiplication. For example:

$e^2 = (e^1)(e^1)$ or perhaps:

$e^2 = (e^{0.5})^4 = (e^{0.5})(e^{0.5})(e^{0.5})(e^{0.5})$: that is, use the series to evaluate $e^{0.5}$, and then raise that value to the 4th power. You can an use multiplication to do this.

Here are some useful powers of (e):

$e^1 = 2.71828182845904\ldots$

$e^{0.5} = \sqrt{e} = 1.64872127070012\ldots$

$e^2 = e^1 e^1 = 7.38905609893064\ldots$

$e^{-1} = \frac{1}{e^1} = 0.3678794411714\ldots$

$e^{-2} = \frac{1}{e^2} = \frac{1}{e^1}\,\frac{1}{e^1} = 0.135335283\ldots$

$e^{10} = 22{,}026.46579481\ldots$

Ex. e^3.74 = e^(2 + 1.74)
e^3.74 = (e^2)(e^1.74)
e^3.74 = (e^2)(e^1)(e^0.74)
e^3.74 = (7.389056098)(2.718281828)(2.095935514)
e^3.74 = 42.09799016

Ex. e^-1.47 = (e^-2)(e^0.53) : from -2 + x = -1.47 , therefore x = 2 + (-1.47) = 2 - 1.47 = 0.53
e^-1.47 = (0.135335283)(1.698932309)
e^-1.47 = 1/e^1.47 = 0.229925485**...**

Ex. Evaluate e^21

N = e^21 = (e^20)(e^1)
N = (e^10)^2 (e^1)
N = (22,026.46579)^2 (2.718281828)
N = (485165195.4)(2.718281828) : Using a 10-digit calculator
N = 1,318,815,734

Ex. Evaluate 5^2.3 using a power of (e). To do this, lets first convert the base of 5, to its equivalent power of (e):

Let 5 = (e^x) raising each side to the 2.3 power: 5^2.3 = (e^x)^2.3

Given: 5 = e^x let's solve for the necessary and proper exponent of (e) by taking the logarithm of both sides:

ln 5 = ln e^x using the log of an indicated power rule, and this can be expressed as:
ln 5 = x ln e dividing both sides by ln e:

$x = \frac{\ln 5}{\ln e}$ = ln 5 = 1.609437912 therefore:

5^2.3 = (e^1.609437912)^2.3 using the power to a power rule; multiply the exponents:

5^2.3 = e^3.701707199 = ~ 40.5164

For any power of the same base, here 5, the expression to evaluate would then have the form of:

5^x = (e^1.609437912)^x = e^(1.609437912 x) : using a power of (e) to evaluate any indicated power of 5.

Using the above expressions, a general formula for evaluating any indicated power of a value by using a power of (e), is:

First we would arrive at:

b^x = (e^(ln b))^x which can be expressed using the power to a power rule as:

b^x = e^(x ln b) : **A GENERAL FORMULA FOR EVALUATING ANY INDICATED POWER OF A NUMBER WITH A POWER OF (e)**.
Since indicated roots can be expressed as an indicated power of a number, this same formula can be used to solve for roots.
Consider this derivation also:

b^a = e^x , therefore: ln b^a = ln e^x therefore:
a ln b = x ln e , therefore: a ln b = x therefore, substituting this value of (x) in e^x, we have:

b^a = e^(a ln b) Ex. 5^2 = e^(2 ln 5)

Find: $\sqrt{2}$ converting this root or radical expression to its corresponding exponential notation:

$\sqrt{2}$ = 2^(1/2) = 2^0.5 Using a power of (e) to solve this:

$\sqrt{2}$ = b^x = 2^0.5 = e^(x ln b) = e^(0.5 ln 2)

$\sqrt{2}$ = e^((0.5)(0.69314718)) = e^0.34657359

$\sqrt{2}$ = 1.414213562

EVALUATING POWERS OF (e) USING A SMALL TABLE OF CONSTANTS

Besides using the series for (e) to directly calculate or solve powers of (e), you can use the method described below which utilizes multiplication of powers of (e) that have already been solved for. This book has strived to promote using formulas instead of using tables. The small table below with only 100 entries can effectively replace a table with thousands of entries. Tables may still be necessary for various reasons, and therefore, some are given in the appendix section of this book, and they are still very useful for checking formula calculations when a calculator is unavailable.

Ex. Find e^0.35

Factoring this into powers of (e) where the exponent of each factor corresponds to each different decimal number and position of the given exponent:

e^0.35 = e^0.3 e^0.05 Now factoring these exponents into a power of a power:
Ex: e^0.3 = (e^1)^0.3 = (e^(1/10)^(0.3 x 10) = (e^0.1)^3

e^0.35 = (e^0.1)^3 (e^0.01)^5 : essentially repeated multiplication, ex: (e^0.1)^3 = (e^0.1)(e^0.1)(e^0.1)

You must calculate at least one of the powers of (e), usually (e^0.1), by using the series for powers of e, or use a table to find the corresponding value.

e^0.1 = 1.105170918076

Or, if you don't want to use the series for (e), you can calculate the value of (e^1) based on its definition and take its' tenth root to get the value of e^0.1.

$\sqrt[10]{e}$ = e^1/10 = e^0.1

Likewise, e^0.01 can be calculated using the series, or it can be "derived" by taking the tenth root of e^0.1:

e^0.01 = e^(0.1/10) = (e^0.1)^(1/10) = $\sqrt[10]{e^{0.1}}$

e^0.1 = 1.105170918076
e^0.01 = 1.010050167084
e^0.001 = 1.001000500167
e^0.0001 = 1.000100005
e^0.00001 = 1.00001000005
e^0.000001 = 1.000001000001 : notice as the exponent approaches 0, the value of the power approaches 1

Rather than perform repeated multiplication of the above constants, you can make a small, but powerful table of all the integer powers, from 1 to 9, of these constants . For example:

x	e^x	e^0.x	e^0.0x	e^0.00x	e^0.000x
0	1.000000000000000	1.000000000000000	1.000000000000000	1.000000000000000	1.000000000000000
1	2.718281828459045	1.105170918075648	1.010050167084168	1.001000500166708	1.000100005000167
2	7.38905609893065	1.22140275816017	1.020201340026756	1.002002001334	1.000200020001334
3	20.08553692318767	1.349858807576003	1.030454533953517	1.003004504503377	1.0003000450045
4	54.59815003314424	1.49182469764127	1.040810774192388	1.004008010677342	1.000400080010668
5	148.4131591025766	1.648721270700128	1.051271096376024	1.005012520859401	1.000500125020836
6	403.4287934927351	1.822118800390509	1.06183654654536	1.006018036054065	1.000600180036005
7	1096.633158428459	2.013752707470477	1.072508181254217	1.007024557266849	1.000700245057177
8	2980.957987041728	2.225540928492468	1.083287067674959	1.008032085504274	1.00080032008535
9	8103.083927575384	2.45960311115695	1.09417428370521	1.009040621773868	1.000900405121527

x	e^0.0000x	e^0.00000x	e^0.000000x	e^0.0000000x	e^0.00000000x
0	1.000000000000000	1.0000000000000	1.000000000000000	1.000000000000000	1.000000000
1	1.00001000005	1.0000010000005	1.000000100000005	1.00000001	1.000000001
2	1.000020000200001	1.000002000002	1.00000020000002	1.00000002	1.000000002
3	1.000030000450005	1.0000030000045	1.000000300000045	1.000000030000001	1.000000003
4	1.000040000800011	1.000004000008	1.00000040000008	1.000000040000001	1.000000004
5	1.000050001250021	1.0000050000125	1.000000500000125	1.000000050000001	1.000000005
6	1.000060001800036	1.000006000018	1.00000060000018	1.000000060000002	1.000000006
7	1.000070002450057	1.0000070000245	1.000000700000245	1.000000070000003	1.000000007
8	1.000080003200085	1.000008000032	1.00000080000032	1.000000080000003	1.000000008
9	1.000090004050122	1.0000090000405	1.000000900000405	1.000000090000004	1.000000009

e^0.35 = e^0.3 e^0.05 : e^x factored to indicated powers that have one digit greater than 0. For using the table, this can be expressed as:

e^0.35 = e^((0.1)3) e^((0.01)5)

In the table above, use:

(e^0.x) where x = 0.1 , and choose the corresponding row 3 which is the 3rd multiple of e^0.1
(e^0.0x) where x = 0.01 , and choose the corresponding row 5 which is the 5th multiple of e^0.01

e^0.35 = (1.349858808)(1.051271096)
e^0.35 = 1.419067549

If the indicated power of (e) is negative, the same method can be used if you place it into a form with a positive exponent and then take its reciprocal:

$$e^{-0.35} = \frac{1}{e^{0.35}} = \frac{1}{1.419067549} = 0.704688089$$

By using a reverse type of manner to that described above you can solve for the (indicated exponent) power of (e) that will be equal to the power value presented. In essence, you will be solving for a logarithm (an exponent), and here, with this table with powers of (e), it will be a natural logarithm. To find the next digit of the logarithm, divide the remainder (ie.

remaining factor) of the number by the highest corresponding power of (e) in the table that corresponds to that digit position. This process is also discussed further ahead in the topic of: CALCULATING LOGARITHMS USING A SMALL TABLE OF CONSTANTS .

Ex. Solve for x (the exponent), given e^x = 5 from the table above.

First note: e^(x.xx...) = (e^x)(e^0.x)(e^0.0x) . . . :(Note, x is used here only as a position indicator for the number being found, that is, after the first x is found, x will be recalculated for each new positional value)

Another, perhaps **clearer**, way to express this concept is:

e^(a.bcd...) = e^(a + 0.b + 0.0c...) = (e^a)(e^0.b)(e^0.0c) . . .

The basic process is to factor the number, here in this example it is 5, into factors that are powers of e:

5 = e^(x.xx...) = (e^x)(e^0.x)(e^0.0x) . . . : the last expression is the individual factors of 5

If you divide 5 by one of these factors, you will have the remaining value of the other factors, clearly, for example, after dividing both sides of the equation by (e^x), we have the total value of the remaining factors:

$$\frac{5}{e^x} = (e^{0.x})(e^{0.0x}) \ldots$$

After finding the value of (5/e^x), (ie. dividing the factor and value of e^x out of 5), we keep dividing each successive remaining quotient (here, of the initial given value of 5) by each successive total remaining factor value:

Starting with 5:

Looking at the table we see that e to the power of x that is nearest to 5 (without exceeding it) is 1.

e^1 = 2.718281828 Solving for how much of the 5 is left to continue factoring, we factor it by each new value:

5 / 2.718281828 = 1.839397205

Looking at the table we see that e to the power of 0.x that is nearest to 1.839397205 (without exceeding it) it is 0.6. Factoring to find what value is left of the 5 for the new factor to consider:

1.839397205 / 1.822118800 = 1.0094687843

Looking at the table we see that e to the power of 0.0x that is nearest to 1.0094687843 (without exceeding it) is 0, which means that 0 will be in that position of the exponent. e^0.00 = e^0 = 1, and multiplying by 1 will not change the result.

What we have up to this point in the process is: (e^1)(e^0.6)(e^0.00) = e^1.60

Factoring what is left of the 5 for the new factor to consider is the same value we had previously. We are to find the closest value to it for the position: e^0.00x , and its value is: e^0.009 .

What we have up to this point in the process is: 5 = (e^1)(e^0.6)(e^0.00)(e^0.009) = e^(1 + 0.6 + 0.00 + 0.009) = = e^1.609 :

Hence x is approximately 1.609 , and e^1.609 = about 5 : note also that 1.609 = ln 5 = $\log_e 5$

The table of powers of (e) can also be used to find the natural logarithm (base e) of a number.

The above concepts are powerful, and can even be applied to powers of any other base, for example, powers of ten = = 10^x as shown here:

Evaluating Powers Of 10 Using A Small Table Of Constants

Ex. 10^0.752 = 10^(0.7 + 0.05 + 0.002) =
= (10^0.7) (10^0.05) (10^0.002) , Converting these exponents to integers we have:
= (10^0.1)^7 (10^0.01)^5 (10^0.001)^2

10^0.1 = 1.258925411794
10^0.01 = 1.023292992281
10^0.001 = 1.002305238078
10^0.0001 = 1.000230285021

Again, rather than perform repeated multiplication, a small table is presented here:

x	10^x	10^0.x	10^0.0x	10^0.00x	10^0.000x
0	1.000000000000000	1.000000000000000	1.000000000000000	1.00000000000000	1.000000000000000
1	10.00000000000000	1.258925411794167	1.023292992280754	1.0023052380779	1.000230285020825
2	100.0000000000000	1.584893192461114	1.0471285480509	1.004615790278395	1.00046062307284
3	1000.000000000000	1.99526231496888	1.071519305237606	1.006931668851804	1.000691014168259
4	10000.00000000000	2.51188643150958	1.096478196143185	1.009252886076685	1.000921458319296
5	100000.0000000000	3.162277660168379	1.122018454301963	1.011579454259899	1.001151955538169
6	1000000.000000000	3.981071705534972	1.148153621496883	1.01391138573668	1.001382505837099
7	10000000.00000000	5.011872336272722	1.17489755493953	1.016248692870696	1.001613109228309
8	100000000.0000000	6.309573444801933	1.202264434617413	1.018591388054117	1.001843765724026
9	1000000000.000000	7.943282347242816	1.230268770812382	1.02093948370768	1.002074475336479

x	10^0.0000x	10^0.00000x	10^0.000000x	10^0.0000000x	10^0.00000000x
0	1.000000000000000	1.000000000000000	1.000000000000000	1.00000000000000	1.000000000000000
1	1.000023026116027	1.000002302587744	1.000000230258536	1.000000023025851	1.000000002302585
2	1.000046052762256	1.00000460518079	1.000000460517125	1.000000046051703	1.00000000460517
3	1.000069079938699	1.000006907779138	1.000000690775767	1.000000069077555	1.000000006907755
4	1.000092107645368	1.000009210382787	1.000000921034461	1.000000092103408	1.000000009210341
5	1.000115135882277	1.000011512991739	1.000001151293209	1.000000115129261	1.000000011512926
6	1.000138164649436	1.000013815605993	1.00000138155201	1.000000138155115	1.000000013815511
7	1.000161193946858	1.000016118225548	1.000001611810864	1.00000016118097	1.000000016118096
8	1.000184223774555	1.000018420850406	1.000001842069771	1.000000184206825	1.000000018420681
9	1.00020725413254	1.000020723480565	1.000002072328731	1.00000020723268	1.000000020723266

Ex. 10^1.79 = (10^1) (10^0.7) (10^0.09)
10^1.79 = (10) (5.011872336) (1.230268771)
10^1.79 = 61.65950019

Ex, 10^-3.7 = 10^(-4 + 0.3) =
10^-3.7 = (10^0.3)(10^-4)
10^-3.7 = (1.995262315)(10^-4) By observation; moving the decimal point 4 places leftward:

10^-3.7 = 0.0001995262315

We can also solve the above example as: 10^-3.7 = 1/10^3.7 = 1/((10^3)(10^0.7))

This table for powers of 10 can also be used to find the common (base 10) logarithms. For example, in a previous example we calculated:

10^1.79 = 61.65950019 , hence $\text{Log}_{10}\ 61.65950019 = 1.79$

NATURAL LOGARITHMS

Natural logarithms are logarithms where the base is the ("natural") constant of (e) = 2.71828**... ,** and which has been discussed previously in this book. Several series for natural logarithms have been developed and most are similar, and they are of great and necessary computational aid, especially before the age of electronic calculators. The following discussion about logarithms is fairly extensive, and the reader may skip over most of the material till some other time as long as they have a basic understanding of what a logarithm is.

Here is a simple explanation of what a logarithm, and specifically what a natural logarithm means:

N = e^x : e^x will calculate the growth (numerical) factor that a value will effectively have after continuously (with infinitely compounded growth) growing after x periods of time, where after each 1 period of time, it grows by a factor of e^1 = about 2.718 times more.

With x=2, e^2 = (e^1)(e^1) , a value will grow by a factor of (e^1) twice, There will be two periods of natural or infinitely compounded growth. After the first period it will grow by a factor of (e^1) and produce a resulting growth, say growth1, and then after another period of this type of growth, the growth1 value will also grow by a factor of (e^1) resulting in the value of growth2. The initial value will grow or increase by the factor of: (e^1)(e^1) = e^2 times more.

Time: 0 , period 1 of growth , period 2 of growth
Value: value , value (e^1) = value2 , value2(e^1) = value3 by substitution:

value3 = value2(e^1) = (value1(e^1))(e^1) = value1(e^2)

After 2 periods of infinitely compounded growth, value 1 will grow by a factor of (e^2). In general, after x periods of infinitely compounded growth, an initial starting value will grow by a factor of (e^x).

ending value = initial value (e^x) : after x periods of infinitely compounded growth

x = Ln (N) : will solve for how many periods of time (ie. the exponent x, in e^x) it will take a value to grow continuously (with infinitely compounded growth) so as to be equal to the value of N.

If you know the starting value of growth (c), it will grow to N after x periods of infinitely compounded growth:

N = c e^x : after x periods of growth. If you divide both sides by (c) we can still find the number of time periods of infinitely compounded growth for (c) so as to be equal to N. Note, (c) can equal 1 also, and then the equation is simply: N = 1 e^x = e^x, but you you understand that 1 is now the starting value that the natural growth will be applied to.

$\frac{N}{c}$ = e^x = (1)e^x solving for x, by taking the natural logarithm (ln) of both sides:

$\ln (N/c) = \ln e^x = x \ln_e e = x (1) = x$

x = ln (N/c)

Ex. 36.945 = 5e^x

We see that 5 is the initial value that will undergo continuous or natural growth, and that this value of 5 has, or will, grow to 36.954 after some time (ie., some number of time periods) of growth. We can find the number of time periods of growth by solving for x:

$x = \ln (36.945 / 5)$
$x = \ln 7.345 = 2$: after a time length of 2 periods of natural growth, a value of 5 will "grow" to 36.945

The most common series for natural logarithm calculations is presented below and it is easy to remember. The derivation of this series can be found in the APPENDIX section of this book.

First, a simple equation must first be solved since the series is for the solution of LN (1 + x), where (1 + x) must be <=2, therefore (x) must be <=1 since (1+1)=2. Checking:

$1 + x \le 2$ solving for x by transposing (+1):
$1 + x - 1 \le 2 - 1$ combining:
$x \le 1$: x must be less than or equal to 1

Even though (x) must be less than or equal to 1 for the series to converge to a specific value, the series actually evaluates the natural logarithms of values between 0 and 2. The terms will get smaller and smaller in absolute value. Though 2 is a limited or low value, there are mathematical methods so as to use the series for any general value.

Ex. When evaluating LN N given N = 0.5, what should (x) be for use in this natural logarithm series?

Equating LN (1+x) = LN N : N <=2 since N=0.5
LN (1+x) = LN 0.5

Therefore, $1 + x = 0.5$
$x = 0.5 - 1$: note, the general formula for x is then: $x = (N - 1)$
$x = -0.5$

Here is the series for the natural logarithm of a number (N):

$$\text{LN } N = \text{LN } (1+x) = \sum_{n=1}^{n=\infty} (-1)^{(n+1)} \cdot \frac{x^n}{n} \qquad : N \le 2 \text{ , } x \le 1 \quad : x=N-1$$

: A NATURAL LOGARITHM SERIES
(Credited to Nicolaus Mercator)

Expanding some initial terms:

$$\text{LN } N = \text{LN } (1+x) = \frac{x^1}{1} - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \frac{x^6}{6} + \;.\;.\;. \qquad : N \le 2 \text{ , } x \le 1 \quad x=N-1$$

Generally, for an accuracy of (n) significant digits when N is about 0.7 (then x is about: (0.7 - 1) = -0.3), use 2n terms. The smaller (x) is in absolute value, fewer terms need to be evaluated for a given precision. If you create a computer program or algorithm (method(s), step(s) or procedure(s) to solve a problem such as the series shown) to perform or calculate this expression, a large amount of terms can easily be evaluated, and-or the program can be made to stop immediately when the difference between successive terms is at some acceptable level (ie., close to 0.0), and the value of all further terms is considered negligible and need not be calculated. For example, when the difference is less than or equal to 0.000,001 (ie., a millionth), there will be at least 6 accurate decimal digits or "places" in the result.

Here are the corresponding values of (x) for ln 0.7 and ln 1.3:

ln 0.7 = ln (1 - 0.3) use x = 0.7 - 1 = - 0.3
ln 1.3 = ln (1 + 0.3) use x = 1.3 - 1 = + 0.3 : note that this has the same absolute value as when N=0.7

Here are the corresponding values of (x) for ln 0.2 and ln 1.8:

ln 0.2 = ln (1 - 0.8) : x = 0.2 - 1 = - 0.8
ln 1.8 = ln (1 + 0,8) : x = 1.8 - 1 = 0.8 ; the same in absolute value as above

By observing the two results above, the closer the value of N is to 1, as the first instance is, the smaller the absolute value of (x). When N is less than 1, (x) will be negative and all terms of the series, and the resulting sum, will be negative in sign. Logarithms of values between 0 and 1 are negative in value.

The natural logarithm of 2 can be useful when calculating some natural logarithms by first factoring the number and taking the sum of their logarithms so as to find the actual logarithm. Other methods will be presented below for calculating any natural logarithm.

ln 2 = 0.6931471805599451. . .

Ex. $\ln 5 = \ln\left(\frac{10}{2}\right) = \ln 10 - \ln 2 = 2.302585093 - 0.693147181 = 1.609437912$

Here's another example that shows what to do when the value (ie.,N, the number of the logarithm) is greater than or equal to 2 when trying to find its logarithm with the series:

Ex. Evaluate ln 21

ln 21 = ln (10^2)(0.21) : expressing the number (here 21) in scientific notation
ln 21 = ln (10^2) + ln (0.21)
ln 21 = 2 ln 10 + ln 0.21
ln 21 = 2 (2.302585093) + ln 0.21 : the important point to remember (or at least have available) here is that (ln 10) is always equal to the constant of 2.302585092994..., and that the natural logarithm to be found is in the required range (0<N<2) for the logarithm series formula given.

ln 21 = 4.605170186 + (-1.560647748)
ln 21 = 4.605170186 - 1.560647748
ln 21 = 3.044522438

Or if you want slightly better accuracy using the same number of terms, you can use a different logarithm factor that is a higher power of 10, for example, here is a small reference table that is very useful:

ln 10 = 2.302585092994)
ln 100 = ln 10^2 = 2 ln 10 = 2 (2.302585092994) = 4.605170185988
ln 1,000 = ln 10^3 = 3 ln 10 = 3 (2.302585092994) = 6.907755278982
ln 10,000 = ln 10^4 = 4 ln 10 = 9.210340371976
ln 100,000 = ln 10^5 = 5 ln 10 = 11.51292546497
ln 1,000,000 = ln 10^6 = 6 ln 10 = 13.81551055796

Ex. Using the last example: ln 21

ln 21 = ln (10^3)(0.021)
ln 21 = ln 1000 + ln 0.021
ln 21 = $6.9077\underline{6}$ + ln 0.021 = $3.0445\underline{3}$: in the series for LN N, use x = (N - 1) = (0.021 - 1) = -0.979

There are other methods to evaluate the logarithm of a value greater than 2, but the method above is perhaps the one to use since the number of the logarithm to be found can be placed into a value not only less than 2, but less than 1. Below are some examples of some other methods to use to solve a logarithm. Some people may say something like: "Why bother with these methods when we have calculators to do this". The answer is why bother with learning any math at all then? This book tries to show some of the "in and outs", or "nuts and bolts" of mathematics so that you may have more of a grasp of these topics, and so as to apply in some way.

Ex. Evaluate ln 5 (other than the method shown above that uses the basic log rules)

Taking the third root of 5, we get: $\sqrt[3]{5}$ = 1.709975947

Therefore: 1.709975947^3 = 5

Actually, any root could have been taken as long as the final logarithm to be evaluated (using the series) has N, the number, less than or equal to 2.

hence: ln 5 = ln 1.709975947^3
ln 5 = ln (1.709975947) (1.709975947) (1.709975947)
ln 5 = ln 1.709975947 + ln 1.709975947 + ln 1.709975947
ln 5 = 3 ln 1.709975947
ln 5 = 3 (0.536479304)
ln 5 = 1.609437912

Another method is to factor the number for factors less than 2 as required for the series given, and then sum the logarithms of each of the factors for the resulting logarithm:

ln 5 = ln (3 x 1.666666667) : after factoring 5
ln 5 = ln (3) (1.666666667) factoring 3:
ln 5 = ln (1.7) (1.764705882) (1.666666667)
ln 5 = ln 1.7 + ln 1.764705882 + ln 1.666666667

Below is a method that uses a special factorization of N. The result is very similar to that of first placing the number into scientific notation.

Ex. ln 11

ln 11 = ln (10 + 1) : since we already know ln 10. Factoring 10 from each term:

ln 11 = ln 10 $\left(1 + \frac{1}{10} \right)$

ln 11 = ln 10 (1 + 0.1)

ln 11 = ln 10 (1.1) : here N is basically 11 factored by 10. Using the log product rule:
(11 / 10) = 1.1, hence 11 = (10)(1.1) = N

ln 11 = ln 10 + ln 1.1

Ex. $\ln 9$

$\ln 9 = \ln (10 - 1)$: since we already know $\ln 10$. Factoring 10 from each term:

$\ln 9 = \ln 10 \left(1 - \frac{1}{10} \right)$

$\ln 9 = \ln 10 (1 - 0.1)$

$\ln 9 = \ln 10 (0.9)$ using the log product rule:

$\ln 9 = \ln 10 + \ln 0.9$: Note, since $\ln 10$ is obviously greater than $\ln 9$, there must be something to reduce this, and this value is the negative value of logarithms of values less than 1, such as $\ln 0.9$ shown.

A perhaps simpler method to get the same results as above is to factor the number by the highest positional product (ie., the weight of that digit position, times the digit value) of that number. For example:

$\ln 537$

$\frac{537}{500} = 1.074$, hence: $537 = (500)(1.074)$ using substitution:

$\ln 537$
$\ln (500)(1.074)$
$\ln (5)(100)(1.074)$
$\ln 5 + \ln 100 + \ln 1.074$

If 5 is factored, you would have to evaluate the logarithm of another number, hence with this type of factorization method, it is handy to have the logarithms of the first 9 integers available:

$\ln 1 = 0$
$\ln 2 = 0.693147180559$
$\ln 3 = 1.098612288668$
$\ln 4 = 1.386294361119$
$\ln 5 = 1.609437912434$
$\ln 6 = 1.791759469228$
$\ln 7 = 1.945910149055$
$\ln 8 = 2.079441541679$
$\ln 9 = 2.197224577336$

$\ln 10 = 2.302585092994$
$\ln 100 = 4.6051701859881$: same as $\ln 10^2 = 2 \ln 10$
$\ln 1{,}000 = 6.907755278982$: same as $\ln 10^3 = 3 \ln 10$
$\ln 10{,}000 = 9.210340371976$: same as $\ln 10^4 = 4 \ln 10$

Here is another method to evaluate logarithms:

$\ln \left(\frac{1}{x} \right) = \ln \left(\frac{1x^0}{x^1} \right) = \ln \left(\frac{1 x^{0-1}}{1} \right) = \ln 1x^{-1} = -1 \ln x$

$\ln \left(\frac{1}{x}\right) = -\ln x$

or by multiplying each side by -1, and then switching sides:

$\ln x = -\ln \left(\frac{1}{x}\right)$: "LOG RECIPROCAL FORMULA"
: The log of a number is equal to the negative of the log of its reciprocal.

Also notice that the larger (x) is, the lower N = (1/x) is. On the right hand side, when N is less than 1 (when x is greater than 1) we see that (since its equal to ln x on the left hand side) the logarithm of N actually gets higher in value, though with a negative sign, as N approaches 0 (when x gets very large, approaching infinity). In short, logarithms between 0 and 1 are negative in sign. Since log 1 = 0, it is not difficult to imagine that when the number is less than 1, that the log will be less than 0, or negative in value.

Using the log quotient rule as a check:

$\log \frac{1}{x} = \log 1 - \log x = 0 - \log x = -\log x$, multiplying by (-1) and switching sides:

$\log x = -\log \frac{1}{x}$: checks

Ex. Find ln 52

$\ln 52 = -\ln \left(\frac{1}{52}\right)$

ln 52 = - ln 0.019230769
ln 52 = - (- 3.951243719)
ln 52 = + 3.951243719

Here are some examples of using logarithms to evaluate expressions:

Ex. Evaluate 3.25^1.3 : find the 1.3 indicated power value of 3.25

Let N = 3.25^1.3

From: N = e^(x ln b) : derived in the discussion about calculating powers (N=b^x) with logarithms.
b is the base of the power (here, it's 3.25):

N = e^(1.3 ln 3.25) = e^((1.3)(ln 2 + ln 1.625)) : after factoring 3.25 for factors <= 2
N = e^((1.3)(1.178654996))
N = e^1.532251495
N = 4.628586339

Ex. Evaluate $\sqrt[2.3]{5}$

Let N = $\sqrt[2.3]{5}$

From: N = e^((ln b) / x) : derived in the discussion about calculating roots with logarithms.
b = radicand (or base in 5^(1/2.3) = $\sqrt[2.3]{5}$)

N = e^((ln 5) / 2.3)
N = e^(1.609437912 / 2.3)
N = e^0.699755614
N = 2.013260635

Ex. Show that N^x = e^(x ln N) : a good way to calculate powers of N using the series for powers of e.

Taking the log of both sides:

ln N^x = ln e^(x ln N) Using the log rules:

x ln N = (x ln N) ln e since ln e = 1:

x ln N = x ln N : since the results are the same, the values were the same and therefore equal

Ex. Given 10^x = e^1 solve for x.

Taking the natural logarithm of both sides:

ln e^1 = ln 10^x using the log rules:
1 ln e = x ln 10
1 = x ln 10 solving for x and switching sides:

$x = \frac{1}{\ln 10}$: = about 0.4342944819
: ln 10 = about 2.302585093

Ex. Given 10^x = e^2 solve for x.

Taking the natural logarithm of both sides:

ln e^2 = ln 10^x using the log rules:

2 ln e = x ln 10

2 = x ln 10 solving for x and switching sides:

$x = \frac{2}{\ln 10}$: = approximately (2) (0.4342944819) = 0.8685889638
Or: two times the reciprocal of ln 10

Ex. Given 10^3 = e^x solve for x:

Taking the natural logarithm of both sides:

ln e^x = ln 10^3
x ln e = 3 ln 10
x = 3 ln 10 : about 3 (2.302585093) = 6.90775527898
Or: three times (ln 10)

Above, we found that 10^0.4342944819 = e^1. If we want to find the value of the second power of (e), we can raise each side to the second power or any other indicated power of (e) in question, and so as to also have its equivalent power of 10, and perhaps to express that power of (e) as a power of 10:

(e^1)^x = (10^0.4342944819)^x , with powers to a power, distribute the exponent:

e^x = 10^(0.4342944819 x) If we want to find the value of the second power of (e), set x=2:
For reference: **10^x = e^(2.302585093 x)** :
The coefficients of (x) in each equation are reciprocals, and this will be verified later in this book.

e^2 = 10^((0.4342944819)(2)) = 10^0.868588962 =~ 7.389056068

A method to calculate a logarithm is to factor the number into a power of (e) and a remaining factor that is less than (2) for the series. Due to that many more terms are required for a given accuracy for values greater than 1 for the logarithm series, it is best that this remaining factor be less than or equal to 1.

$$\frac{N}{e^x} = \text{(remaining factor)}$$: (remaining factor) = (remaining factor of N) = quotient

N = (power of e) (remaining factor) = (e^x) (remaining factor)

ln N = ln ((e^x) (remaining factor))
ln N = ln e^x + ln (remaining factor)
ln N = x + ln (remaining factor)

N can be divided by a constant that is powers of (e), or by repeated division by e^1 (= 2.718281828...) and counting the number of these divisions which is equivalent to the power of (e):

$$\frac{N}{e^1} = Q1 = \text{quotient}$$: if this quotient or remaining factor of N is >1, keep dividing by e^1:

$$\frac{Q1}{e^1} = Q2$$ This is equivalent as dividing N by e^1 repeatedly, which is the same as dividing N by the next integer power of (e) as shown here:

$$\frac{Q1}{e^1} = \frac{\;\frac{N}{e^1}\;}{e^1} = \frac{N}{e^2}$$

Ex. Find or calculate: ln 5

5 / e^2 = 0.676676416 therefore: 5 = (e^2)(0.676676416) , taking the ln of both sides of this equation:

ln 5 = ln ((e^2) (0.676676416))
ln 5 = ln e^2 + ln 0.676676416
ln 5 = 2 + (-0.390562087)
ln 5 = 1.609437913

CALCULATING LOGARITHMS USING A SMALL TABLE OF CONSTANTS

In general, this is the reverse process of finding powers of (e) using a table of constants as shown previously in this book.

Ex. Find x = ln 3.2

The general working method, or formula in relation to the table (shown previously) is:

(Note, (x) below represents a place holder or digit position in the number, and is not a regular variable):

ln N = ln(e^x.xxxx...) : or perhaps for more clarity as: ln (e^a.bcde...)
ln N = ln(e^x.0 e^0.x e^0.0x e^0.00x e^0.000x . . .)
ln N = ln(e^x.0) + ln(e^0.x) + ln(e^0.0x) + ln(e^0.00x) + ln(e^0.000x) + . . .
ln N = x + 0.x + 0.0x + 0.00x + 0.000x + . . .

Here is a helpful description of this process:

ln N = ln N factoring N:
ln N = ln (factor1)(factor2) factoring factor2 into factor3 and factor4:
ln N = ln (factor1)(factor3)(factor4) factoring factor4 into factor5 and factor6:
ln N = ln (factor1)(factor3)(factor5)(factor6) : and so on, the factorization can continue with the "remaining factor"

ln N = ln (factor1)(factor3)(factor5)(factor6) = ln factor1 + ln factor2 + ln factor3 + ln factor4 + ln factor5 + . . .

Using the table, when x=2 , e^x = e^2 and this value is larger than the given value of 3.2 , so therefore, x=1 is where to start the factorization process:

$$\frac{3.2}{e^1} = \frac{3.2}{2.718281828} = 1.177214212$$

: factoring 3.2, here by using a power of (e)
Therefore:

ln 3.2 = ln ((2.718281828)(1.177214212)) : ln N = ln (factors of N)
ln 3.2 = ln (2.718281828) + ln (1.177214212)
ln 3.2 = ln (e^1) + ln(e^0.?) : now we are to find the value of the exponent in e^0.x

From the table for the e^0.x constants, e^0.2 is larger than the new value of "N" we are considering, specifically the remaining factor of: (1.177214212) that we are now comparing powers of (e) to, so use e^0.1 to factor this new "N" factor in question:

$$\frac{1.177214212}{e^{0.1}} = \frac{1.177214212}{1.105170918} = 1.065187468$$

Now we have these factors of N=3.2::

ln 3.2 = ln (2.718281828) + ln (1.177214212) continuing the factorization process:
ln 3.2 = ln (2.718281828) + ln (1.105170918) + ln(1.065187468) expressing these factors as a power of (e):
ln 3.2 = ln(e^1) + ln(e^0.1) + ln(e^0.0?) A ln results in the indicated power (ie. exponent):
ln 3.2 = 1.0 + 0.1 + ?

From the table, the greatest value of (x) for e^0.0x is 6 for the new "N" (1.065187468), hence the updated (more correct) logarithm is:

ln 3.2 = ln(e^1) + ln(e^0.1) + ln(e^0.06) + . . .
ln 3.2 = 1.0 + 0.1 + 0.06 + . . .
ln 3.2 = 1.16 + ? + . . .

Continuing this process by factoring the latest factors, we have:

ln 3.2 = 1.0 + 0.1 + 0.06 + 0.003 + 0.0001 + . . .
ln 3.2 = 1.1631 + . . .

Don't forget, you can also solve for this as:

$$\ln 3.2 = -\ln \frac{1}{3.2} = -\ln 0.3125$$: here, N<1, see below.

Note that when N=1, ln N = ln 1 = 0, and give any other value (such as 0.3125) of N that is less than 1 and between 0 and 1, the result will be negative (<0) in value. However, using the table that was created for positive powers of (e) would be a bit difficult to use for negative powers of (e), so here is a similar table with negative exponents that can be utilized when N is less than 1.0. Actually, each value in this table below is the reciprocal of its corresponding value in the table of positive powers of (e) given previously. To avoid accessing both tables, just keep factoring N and adding in each new digit, where each new digit corresponds to the equivalent or next higher value of N.

For example, when finding ln 0.3125, the first sum to be added in to the exponent of (e) is -1 (not -2) which corresponds to 0.367879441, which is the closest higher value. Therefore, 0.3125 factors to 0.367879441 and 0.849463071, then continue the process with this "new" N, specifically the remaining factor(s) of N.

The basic reason that the next highest value is chosen is that factoring (ie. dividing) by using the next lower value will yield a factor >1, and hence the other table must also be utilized for this "new" N.

x	e^ - x	e^ - 0.x	e^ - 0.0x	e^ - 0.00x	e^ - 0.000x
0	1.0000000000000000	1.0000000000000000	1.0000000000000000	1.0000000000000000	1.0000000000000000
1	0.3678794411714423	0.9048374180359595	0.9900498337491681	0.9990004998333749	0.9999000049998333
2	0.1353352832366127	0.8187307530779818	0.9801986733067554	0.9980019986673331	0.9998000199986666
3	0.04978706836786394	0.7408182206817179	0.9704455335485082	0.9970044955033729	0.9997000449955004
4	0.01831563888873418	0.6703200460356393	0.9607894391523232	0.9960079893439915	0.9996000799893343
5	0.006737946999085467	0.6065306597126334	0.9512294245007139	0.9950124791926824	0.9995001249791694
6	0.002478752176666359	0.5488116360940265	0.9417645335842487	0.9940179640539352	0.9994001799640054
7	0.000911881965554516	0.4965853037914095	0.9323938199059482	0.9930244429332351	0.9993002449428433
8	0.000335462627902512	0.4493289641172216	0.9231163463866358	0.9920319148370606	0.9992003199146837
9	0.000123409804086679	0.4065696597405991	0.9139311852712282	0.9910403787728837	0.9991004048785274

x	e^-0.0000x	e^-0.00000x	e^-0.000000x	e^-0.0000000x	e^-0.00000000x
0	1.0000000000000000	1.0000000000000	1.0000000000000000	1.0000000000000000	1.000000000
1	0.9999900000499999	0.9999990000005	0.9999999000000051	0.9999999900000002	0.9999999989999999
2	0.9999800001999988	0.999998000002	0.9999998000000199	0.9999999800000003	0.9999999980000001
3	0.9999700004499955	0.9999970000045001	0.999999700000045	0.9999999700000004	0.999999997
4	0.9999600007999893	0.999996000008	0.99999960000008	0.9999999600000007	0.9999999959999999
5	0.9999500012499791	0.9999950000125	0.9999995000001249	0.9999999500000013	0.999999995
6	0.9999400017999639	0.999994000018	0.9999994000001801	0.9999999400000017	0.999999994
7	0.9999300024499429	0.9999930000245	0.999999300000245	0.9999999300000024	0.9999999929999999
8	0.9999200031999146	0.9999920000319998	0.9999992000003201	0.9999999200000032	0.9999999920000001
9	0.9999100040498785	0.9999910000404998	0.9999991000004049	0.999999910000004	0.999999991

Here is an example of using the above table to calculate negative powers of (e):

Ex. Evaluate e^-5.03

e^-5.03 = e^(-5 (+) -0.0 (+) -0.03) = e^-5 e^-0.0 e^-0.03
e^-5.03 = (0.006737946)(1.0)(0.970445533)
e^-5.03 = 0.006538809

You could always calculate the result using the table for positive powers of (e) when (e) is first expressed with a positive exponent as:

$$e^{-5.03} = \frac{1}{e^{5.03}} = (\text{value})$$

, and since (value) is the reciprocal of the actual result, we simply take the reciprocal of (value):

$$e^{-5.03} = \frac{1}{(\text{value})} = \frac{1}{152.933} = 0.006538811$$

CALCULATING POWERS BY CONVERTING THE BASE OF THE POWER TO (e)

By converting a base of a power to (e) we can solve powers, common antilogs (essentially powers of the common base of 10), and roots since we have a series for e^x. For this power of (e) to represent a given value correctly, the exponent of (e) must be determined or set to its correct or corresponding value.

Given N, what is the exponent of (e) required so that e^a = N ?

From e^a = N from the definition of logarithms:

a = $\log_e$ N = loge N = ln N : remember that a log value is essentially an exponent value

N = e^a raising each side to some indicated power expressed with the exponent of x:

(N)^x = (e^a)^x using the power to a power rule:

N^x = e^(ax)

This also shows that e^a is a constant or specific value (equivalent to N which is a constant value) when calculating all the power values (indicated with a exponent of x) of a given value of base N.

Ex. For calculating anti-logarithms (essentially calculating a power of a value) with a base of 10 (the common anti-logarithm base):

e^a = 10 therefore, according to the log definition:

a = ln 10 = 2.302585093 therefore:

e^2.302585093 = 10 raising each side to the (x) indicated power:

10^x = e^(2.302585093 x) : This is an equation or "formula" for calculating all powers of 10, or all common anti-logarithms, using a power of (e).

For reference, shown previously in this book was:
e^x = 10^(0.4342944819 x)
Note that the multipliers to the indicated power of (x) in the
two equations reciprocals, and this can be verified as:
When (x) = 1 for either equation we have:
10 = e^2.302585093 and e = 10^0.4342944819
Taking the ln for the left equation, and the log of the right equation:
ln 10 = 2.302585093 and log e = 0.4342944819 , mathematically:
(ln 10)(log e) = (2.302585093)(0.4342944819) = 1 : the product of reciprocals is 1
This also shows that: (ln 10) and (log e) are reciprocals:
(ln 10) = 1 / (log e) and (log e) = 1 / (ln 10)

Here is a derivation of a general formula for this concept :

N = b^x taking the (natural) logarithm of both sides:
ln N = ln b^x
ln N = x ln b hence by the log definition:
N = e^(x ln b)

N = b^x = e^(x ln b) : A GENERAL FORMULA TO EVALUATE ANY INDICATED POWER USING A POWER OF (e)

Ex. If the base of the power is: b=10:

N = 10^x = e^(x ln 10) = x e^((x)(2.302585093) = e^(2.302585093 x) : as just shown above

Using almost the same derivation as above, a formula can be derived for evaluating powers of any number via powers of 10. Evaluated powers of 10 are often found in common antilogarithm tables:

N = b^x taking the (common) logarithm of both sides:
log N = log b^x : USING base 10 so as the result is a power of 10
log N = x log b hence by the definition of logarithms:
N = 10^(x log b) which can be expressed as:
N = b^x = 10^(x log b) :A formula to evaluate any power via a power of 10.

Ex. 30^2 = 10^(2 log 30) = 10^((2)(1.4771$\underline{2}$)) = 10^2.9542$\underline{4}$ = 900 : for comparison, 10^3 = 1000

If the base of an indicated power is (e) and you want to calculate or express that power value using powers of 10, and using the formula above:

N = b^x = e^x = 10^(x log e) = 10^(0.4342944819033 x) : POWERS OF (e) VIA POWERS OF 10

Ex. e^1 = 10^(0.4342944819033 (1)) = 10^(0.4342944819033) = 2.718281828**...**

Here is a more general or algebraic method for calculating any power using any base:

N = b^x taking the log (using any base) of both sides:
Log N = Log b^x : this can be expressed as:
Log N = x Log b from the definition of logarithms:

N = b^x = logbase^(x $\log_{\text{logbase}}$ b) : A GENERAL FORMULA FOR CALCULATING ANY POWER BY USING A DIFFERENT BASE.
b = base of the given power, x = exponent or indicated power,
use a common logbase for calculations, N=power value

Ex. N=5^2 , calculate this power using a power of (e) and then a power of (10):

N = 5^2 = e^(2 $\ln_e$ 5) : here, base (e) was used as expected for a natural logarithm

Or by using any other base for the logarithms, such as base 10:

N = 5^2 = 10^(2 $\log_{10}$ 5) : e^(2 $\ln_e$ 5) = 10^(2 $\log_{10}$ 5) : both equal b^x = 5^2

N = 25

EQUATING POWERS

Here is a general formula for equating powers. A preliminary discussion of this was already mentioned during the topic of: SOLVING FOR AN EXPONENT USING LOGARITHMS.

N1^x1 = N2^x2 Taking logs (with any base you choose) of both sides:

log N1^x1 = log N2^x2 From the log of an indicated power rule:

x1 log N1 = x2 log N2 Solving for the exponent variable x2:

$$x2 = \frac{x1 \log N1}{\log N2}$$

: **GENERAL FORMULA FOR EQUATING POWERS: N1^x1 = N2^x2**
(Finding The Proper Exponent Of The Base)
Use the same base for both of the logarithms.

Ex. What is the equivalent of 7^2 using a base (or power value of) of 2 ?

N1^x1 = N2^x2

7^2 = 2^x2

$$x2 = \frac{x1 \log N1}{\log N2} = \frac{2 \log 7}{\log 2} \qquad \text{or by using a base of (e):} \quad \frac{2 \ln 7}{\ln 2}$$

x2 = 5.614709844

Hence, 2^5.6147049844 = 7^2 :

Ex. Write a formula for expressing all powers of 7 in terms of a power of (e).

7^x = e^x2 Hence, we need to find the exponent x2 in terms of (a function of) x.

$$x2 = \frac{x \ln 7}{\ln e}$$

With natural logarithms (have base e), the denominator becomes a value of 1.

x2 = x ln 7 Substituting this value of x2 into e^x2 above, we have:

7^x = e^(x ln 7) : or= e^(ln 7^x), now evaluating the constant of: ln 7 we have:

7^x = e^(1.945910149 x) : e^1.945910149 = 7 , 7^x = (e^(1.945910149))^x = e^(1.945910149 x)

Using the above example, it is not to difficult to realize that when the exponent of (e) is a natural logarithm, that the resultant value is equal to the number of that logarithm. This is easily verified below:

N2 = ln N1 Hence the antilog notation to solve for N1 is:

e^N2 = e^(ln N1) = N1

For example: $e^{(\ln_e 7)} = e^{1.945910149} = 7$

Hence: $7^x = (e^{\ln 7})^x = e^{(x \ln 7)}$: using the power to a power rule of multiplying exponents

The same can also be said whenever the base of the power and the base of the logarithm are the same:

$N2 = \log N1$: Choose any base (b) for the logarithm. By the log definition:

$b^{N2} = b^{(\log_b N1)} = N1$

For example: $10^{\log 100} = 10^2 = 100$: same as first taking a log on N, and then taking the antilog of that exponent that equals the log of N.

CALCULATING ROOTS USING (e)

When solving for a root of a value using the constant (e), first place the radical into its equivalent exponential form (ie. an indicated power form), and then equate it to a power of (e).

Ex. $\sqrt{10} = 10^{(1/2)} = 10^{0.5}$

$$
\begin{aligned}
\text{from: } 10^x &= e^{(2.302585093\, x)} \\
10^{0.5} &= e^{(2.302585093(0.5))} \\
10^{0.5} &= e^{1.151292546} \\
10^{0.5} &= 3.16227766
\end{aligned}
$$

$\sqrt{10} = 3.16227766$: the square root of a value calculated by its equivalent power value of (e)

Ex. $\sqrt[3]{7}$

$\ln 7 = 1.945910149$, hence $e^{1.945910149} = 7$, therefore, raising each side to the same indicated power:

$7^x = (e^{1.945910149})^x = e^{(1.945910149x)}$: any power of 7 converted to a power of (e).

$\sqrt[3]{7} = 7^{(1/3)} = 7^{0.333333333...}$

$$
\begin{aligned}
7^{0.333333333} = \sqrt[3]{7} &= e^{(1.945910149(0.333333333))} \\
\sqrt[3]{7} &= e^{0.648636715} \\
\sqrt[3]{7} &= 1.912931181
\end{aligned}
$$

Here is another general formula for calculating square-roots using (e):

Find $\sqrt{N}$

Let $\sqrt{N} = e^x$: converting the radical to its equivalent exponential form:

$N^{0.5} = e^x$ taking the natural logarithm of both sides of the equation:

$\text{LN } N^{0.5} = \text{LN } e^x$ using the log exponential rule:

$0.5\ \mathrm{LN}\ N = x\ \mathrm{LN}\ e$
$0.5\ \mathrm{LN}\ N = x$ substitution this value for (x) in a previous equation:

$\sqrt{N} = e^x = e^{(0.5\ \mathrm{LN}\ N)}$ (or= $e^{((\mathrm{LN}\ N)/2)}$) : **A SQUARE ROOT FORMULA (Using a power of e)**

Ex. $\sqrt{2} = e^{(0.5 \ln 2)} = e^{(0.5 (0.69314718))} = e^{0.34657359} = 1.414213562$

Ex. Solve $\sqrt[5]{76}$

$\text{root} = \sqrt[5]{76}$ placing this radical into its equivalent exponential form:
$\text{root} = 76^{(1/5)}$ taking the logarithm of both sides, using any base consistently:
$\text{Log root} = \text{Log } 76^{(1/5)}$

$\text{Log root} = \frac{(1)\ \text{Log } 76}{(5)}$ by the definition of logarithms:

$\text{root} = b^{((\log_b 76)/5)}$ if the base is (e):

$\text{root} = e^{((\ln 76)/5)} = 2.377730992$ and in general:

$r = \sqrt[i]{N} = b^{((\log_b N)/i)}$: **A GENERAL FORMULA FOR ANY INDICATED ROOT**
r=root, N=radicand, i=index, b=base of choice for the calculations

If you are using (e) for the base, perhaps to use the assistance of a "look-up" (ie., to match corresponding values) or reference table:

$r = e^{(\ln N / i)}$: **FINDING ROOTS USING (e)**
r=root, N=radicand, i=index

It is best that N be small when using the series for evaluating powers of (e). Here is an example of how to do this:

$\sqrt{47}$: expressing the radicand using a scientific notation (or power of 10) form:

$\sqrt{(0.0047)(10^4)}$

$\sqrt{10000}\ \sqrt{0.0047}$

$100\ e^{(0.5\ \mathrm{LN}\ 0.0047)}$

$\sqrt{47} = 6.8556546$

Here is a related equation that can be useful:

$\mathrm{LN}\ AB = N$: here AB is the product of A and B. By the definition of logarithms:
(or simply raising (e) to the (equivalent) value of both sides:)
$e^N = e^{(\mathrm{LN}\ AB)} = AB$ switching sides:

$AB = e^{(\mathrm{LN}\ AB)}$ the log of an indicated product can be expressed as:
$AB = e^{(\mathrm{LN}\ A + \mathrm{LN}\ B)}$: A PRODUCT EXPRESSED AS A POWER OF (e) or:
$AB = e^{(\mathrm{LN}\ A)}\ e^{(\mathrm{LN}\ B)}$: A PRODUCT EXPRESSED AS POWERS OF (e)

A generalization of the above, using any base (b), can be expressed as:

let N = (factor1)(factor2)   : or as many factors used,

N = b^(logb ((factor1)(factor2))) = b^ ((logb factor1) + b^(logb factor2)) = b^(logb factor1) b^(logb factor2)

SOLVING A LOGARITHM USING ANOTHER BASE

Since a series for logarithms which have a base other than (e) have not been developed, the natural logarithm of the same number can be found and then adjusted so that it equals the actual logarithm in question by using the constant of proportionality (c) between the two logarithms. This constant between two log systems is often called a modulus.

Ex. What is the constant of proportionality between common logarithms (base 10) and natural logarithms (base e) ?

First choose a value for N. A value of 10 was chosen for N in the example given below since the common logarithm of 10 is easily found to be 1. Then divide the two logarithms to find the quotient which is the constant of proportionality (ratio, factor) between the systems (ie., with a different log base) of logarithms in question.

$$\frac{\log N}{\ln N} = c1 \qquad \text{or} \qquad \frac{\ln N}{\log N} = c2$$

Notice that c1 and c2 are algebraically reciprocals (c1 = 1/c2 and c2 = 1/c1), and that log N or ln are logarithms having a different base which can be found from either equation:

$$\log N = c1 \ln N = \frac{\ln N}{c2} \qquad \text{and} \qquad \ln N = c2 \log N = \frac{\log N}{c1}$$

$$\frac{\log 10}{\ln 10} = \frac{1}{2.302585093} = 0.4342944819033 = c1 = (1/c2)$$

$$\frac{\ln 10}{\log 10} = \frac{2.302585093}{1} = 2.302585092994 = c2 = (1/c1)$$

Expressing the above common and natural logarithms in a general or basic terms for any value:

log N = (0.4342944819033) ln N   : SOLVING COMMON LOGS USING NATURAL LOGS

ln N = (2.302585092994) log N   : SOLVING NATURAL LOGS USING COMMON LOGS

Both of the above equations have a linear equation form of : y = m x. A discussion and graph of this will be given at the end of this current discussion.

Ex. Find log 15 using the natural log base (e).

We already found that the constant of proportionality when evaluating a common log with a natural log is always:
(log N / ln N) = (0.4342944819033) :

log 15 = (0.4342944819033) ln 15

But remember, for the natural log series shown previously, (x) must be greater than 0 and less than 2: (0<x<2). A simple method to convert N to be within this range is by placing N into a scientific notation (SN) form.

log 15
log (10^1)(1.5)
log 10 + log 1. 5
1 + log 1.5
1 + (0.434294482) ln 1.5

or:

log 15
log (100)(0.15) or= log (10^2)(0.15)
log 100 + log 0.15
2 + log 0.15
2 + (0.434294482) ln 0.15

or:

log 15
(0.434294482) ln 15
(0.434294482) (ln (10^2)(0.15))
(0.434294482) (2 ln 10 + ln 0.15)
(0.434294482) ((2)(2.302585093) + ln 0.15)
(0.434294482) (4.605170186 + ln 0.15)

log 15 = 1.176091259

Ex. Find log 15. This method will find a new number (N) value by repeatedly dividing by (e), which is essentially dividing by a power of (e^1) = 2.302585093 :

$$\frac{N1}{e} = N2 , \quad \text{therefore,} \quad N1 = N2\, e\text{^}1$$

$$\frac{\frac{N1}{e}}{\frac{e}{1}} = \frac{N1}{e\text{^}2} = N2, \quad \text{therefore,} \quad N2 = N1e\text{^}2$$

By repeatedly dividing 15 by e^1=2.302585093, we find:

15 = 0.037181282 e^6

log 15 = log (0.037181282 e^6)
log 15 = (0.434294482) (ln (0.037181282 e^6))
log 15 = (0.434294482) (ln 0.037181282 + ln e^6)
log 15 = (0.434294482) (ln 0.037181282 + 6 ln e)
log 15 = (0.434294482) (ln 0.037181282 + 6)
log 15 = 1.176091259

There is nothing intuitive that there should be a constant between two systems of logarithms. We can try other examples to help verify that this value is repeated over and over (using the same bases (ex. 10 and e) of the logarithms) and therefore, it is most likely a constant, but the best method to prove that it is indeed a constant is to have an algebraic verification (whose variables generally represent any value):

Given: N = b2^x by the definition of logarithms:

$x = \log_{b2} N$ taking logs of both sides of the above equation using another base (b1):

$\log_{b1} N = \log_{b1} b2^x$ using the log power rule:

$\log_{b1} N = x \log_{b1} b2$ since $x = \log_{b2} N$:

$\log_{b1} N = \log_{b1} b2 \ \log_{b2} N$: **A GENERAL FORMULA OF EVALUATING LOGS WITH A DIFFERENT BASE**
Mathematically:

$\dfrac{\log_{b1} N}{\log_{b2} N} = \log_{b1} b2 = r$: (constant) ratio of logarithms of N using different bases
Here, the second equivalence can be thought of as:
$\log_{b1} b2 = \log_{\text{old base}} \text{new base}$

Since b1 and b2 are both constants, $\log_{b1} b2$ can only be a constant, and the ratio above is a constant value.

When a base of a log is set to a lower value, the value of this log expression will naturally be higher, therefore in order to equal the original log of the number, this higher value must be reduced. It gets reduced to a fraction of itself by multiplying it by a value less than 1 (ie., a fraction), and this will be the value of the constant.

Ex. Evaluate $\log_{10} 20$ using natural logs. Since base (e) is a lower value than base 10, the result will be a higher value than it would be using a base of 10. This higher value will be reduced by the constant known between the two bases. Using the above general formula:

$\log 20 = \log_{10} e \ \ln 20$

$\log 20 = 0.434294481 (2.995732274)$
$\log 20 = 1.301029996$

Using the above concepts we can also derive another formula for equating powers:

Given: $b1^x = N = b2^y$ Solving for y: By the definition of logarithms:

$x = \log_{b1} N$ and $y = \log_{b2} N$ from the concepts discussed above:

$\log_{b1} N = \log_{b1} b2 \ \log_{b2} N$ using algebraic substitution:

$x = \log_{b1} b2 \ y$ solving for y:

$y = \dfrac{x}{\log_{b1} b2} = \dfrac{x}{r}$ equating this to the previous derivation given in the **GENERAL FORMULA FOR EQUATING POWERS:**

$y = \dfrac{x \log b1}{\log b2} = x\,c$: use any, and the same consistent base for the logs
(c) is also a constant equal to the ratio (r) of the two logs of N.

(c) = (1/r).
$y = x\,c$ and $x = y/c$ and $(y/x) = c$ and $x/y = (1/c)$

Given $y = \log_b x$, and if you want to use your own base (for the logarithm) instead of base (b), here is what to do. It is a form of the "change of base" formula:

$y = \log_b x$: log_b = logb is an alternative notation for a log, here with base value of (b).

From the definition of logarithms:

$b^y = x$ taking logs of both sides with your choice of the base; perhaps you may want to use (e):
$\log b^y = \log x$
$y \log b = \log x$ solving for y:

$$y = \log_b x = \frac{\log x}{\log b} = \frac{\ln x}{\ln b}$$

: A method to evaluate a logarithm using a "change of base".
: The base used must be the same in the numerator and denominator.
As a basic verification, consider this example:

$$\frac{\log 5}{\log 3} = \frac{\ln 5}{\ln 3} = 1.4649\underline{7}$$

Ex. $7^y = 49$, find y.

$$y = \frac{\log 49}{\log 7} = \frac{\ln 49}{\ln 7} = 2$$

: the simple verification to this is shown above with $b^y = x$, and then taking the log of both sides and solving for (y).

Expressing the above or a more generalized formula:

$$\mathbf{b^x = N}$$

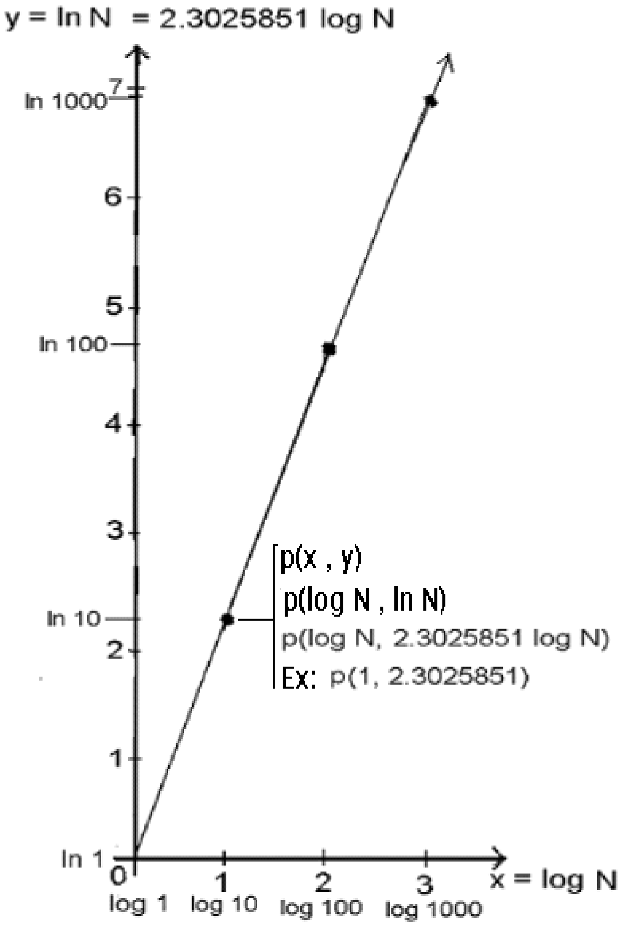
$$\mathbf{x = \frac{\ln N}{\ln b}}$$

: use any base of the logarithm consistently, that is, in both the numerator and denominator mathematically and considering if natural logarithms were optionally used:

$\ln N = x \ln b$ by the log definition: $\mathbf{N = e^{(x \ln b)} = b^x}$, the actual Number or power value , and:

$\ln b = \frac{\ln N}{x}$ by the log definition: $\mathbf{b = e^{((\ln N) / x)} = \sqrt[x]{N}}$

Here is a graph showing the linear, or constant mathematical relationship between two logarithms of the same value (here, N), where each logarithm has a different base. This graph specifically shows the mathematical relationship of, or between, ln N and log N, hence the mathematical relationship of, or between, base 10 and base (e) logarithms.
[FIG 202]



ln N = (2.302585092994) log N : The relationship between natural and common logarithms (or any other two logarithms having different bases) of N is a linear or constant mathematical relationship. You can also write an equation and make a graph of their mathematical relationships and the main difference in the graphs of the lines would be the value or steepness or slope of the line.

This has the form of a linear equation:

y = m x : where y = mx = ln N = (2.302585092994) log N , where m= 2.302585092994 and x = log N

The mathematical relationship between the log of a power of a value, and the indicated exponent is linear if N, the base of the indicated power, is a constant value:

y = log N^x = x log N = log N x : has the general linear equation form of: y = mx
We see that (y) is simply a multiple of the log of the base of the indicated power, and this multiple or factor is the value of (x).

Before moving on, here is a related thing to know about logarithms. It is somewhat surprising, and It may help fill in some gaps when trying to understand logarithms to the fullest.

$\log_{b1} b2$ and $\log_{b2} b1$ are reciprocals

Let's give some algebraic verification to this fact. From what we know about reciprocals, their product should equal 1.

$(\log_{b1} b2)\ (\log_{b2} b1) = 1$ this could be represented and expressed as this simplified form as their respective log value:

$x \quad y = 1$: $x = 1/y$ and $y = 1/x$, From the log definition:

$b1^x = b2$ and $b2^y = b1$ By substitution or replacement:

$b1^x = b2$

$(b2^y)^x = b2$

$b2^{(yx)} = b2^1$

Clearly, by observational matching or comparison of the exponents, we find that (x) and (y) are reciprocals:

$yx = 1$, and therefore, $x = 1/y$ and $y = 1/x$, and the logarithm expression that equal (x) and (y) are therefore, also reciprocals.

For another verification of the above:

$b1^x = b2$ taking the x root of both sides:

$\sqrt[x]{b1^x} = \sqrt[x]{b2}$ converting the radicals to their equivalent exponential form:

$b1^{(x/x)} = b2^{(1/x)}$

$b1 = b2^{(1/x)}$ Since $(y) = (1/x)$, by substitution:

$b2^y = b2^{(1/x)}$ Again, by observation of the exponents, $y = 1/x$, hence (x) and (y) are reciprocals. If we continue further:

$b2^y = \sqrt[x]{b2} = b2^{(1/x)}$: when (x) and (y) are reciprocals

Perhaps a simpler verification of what is expressed above would be:

$b1^x = b2$, according to the definition of a power or repeated multiplication of the same value a certain number of times that is equal to the exponent value, and then doing the reverse or inverse would be repeated division by that same base value a certain number of times equal to the same exponent. The quotient of this repeated division would be equal to the original base value. For example:

$2^3 = (2)(2)(2) = 8$ and: (1): $8/2 = 4$, (2): $4/2 = 2$, (3): $2/2 = 1$

This is 3 complete (ie., no remainder) divisions of 8 by 2.

2 is therefore the 3rd root of 8:

$\sqrt[3]{8} = 2$, and in general:

If b1^x = b2, then b1 = $\sqrt[x]{b2}$ = b2^(1/x) = b2^y : after assigning variable (y_ as being equal (1/x)

Ex. b1^5 = b2 then b2^(1/5) = $\sqrt[5]{b2}$

Ex. 100^0.5 = $\sqrt[2]{100}$: 0.5 and 2 are reciprocals of each other: 0.5 = 1/2 and 2 = 1/0.5

10 = 10 : checks

b1^x = b2 and b2^y = b1 : this condition is true when x and y are reciprocals

100^0.5 = 10 10^(1/0.5)=10^2 = 100

Here is the mathematical relationship between b1 and b2 expressed algebraically:

From: b1^x = b2 and b2^y = b1

Dividing both sides of the first equation by the same or equivalent value of b1:

$\frac{b1^{x}}{b1^{1}} = \frac{b2^{1}}{b2^{y}}$ by using the exponential rules:

b1^(x-1) = $\frac{1}{b2^{(y-1)}}$ or=:

b1^(x-1) = b2^(0 - (y-1)) = b2^(1-y) : The mathematical relationship between b1 and b2. As shown above, the relationship between (x) and (y) is that they are reciprocals and where: $x=\log_{b1} b2$ and $y=\log_{b2} b1$

Ex. Given $\log_{e} 10 = 2.302585093 = x$

We now know that the reciprocal of the above expression is:

$\log_{10} e = 0.434294481 = y$ checking: 2.302585093 = 1 / 0.434294481 , or= x = (1/y)

If we let b1=e and b2=10 their mathematical relationship is:

b1^(x-1) = b2^(1-y)

e^(2.3026 - 1) ~ 10^(1 - 0.4343)
e^1.3026 ~ 10^0.5657
3.6788 = 3.6788 : checks

Ex. This is a basic example: **$1.414213\underline{6}$ = 2^x** , by the log definition: x = log2 $1.414213\underline{6}$ = $\log_{2} 1.414213\underline{6}$

Taking logs of both sides: log $1.414213\underline{6}$ = log 2^x = x log 2 , therefore, after solving for x:

$x = \log_{2} 1.414213\underline{6} = \frac{\log 1.4142136}{\log 2} = \frac{\log \text{Number}}{\log \text{Base}}$: using any consistent base, say 10 or e, we find: x = 0.5

DIFFERENT BASES

Here, the logarithms (essentially the exponents) of two log expressions having different bases are set to be equal in value to (x). Since the bases are different, the number of one of the log expressions must of course be properly adjusted (by solving for it).

Equating two logarithms:

$x = \log_{b1} N1 = \log_{b2} N2$ therefore, according to the definition of logarithms:

$N1 = b1^x = b1^{(\log_{b2} N2)}$

$N2 = b2^x = b2^{(\log_{b1} N1)}$

Ex. Find the equivalent logarithm equation of $\log_{10} 100$ using a base of 5.

Expressing this in a mathematical equation form, solve:

$\log_{10} 100 = \log_5 N$ considering N as the second number or N2:

$N2 = b2^{(\log_{b1} N1)}$

$N2 = 5^{(\log_{10} 100)}$

$N2 = 5^2$

$N2 = 25$

Hence: $\log_{10} 100 = \log_5 25$

$2 = 2$: checks

EQUATING POWERS OF (e) AND NATURAL LOGARITHM EXPRESSIONS

Sometimes you may prefer to convert between expressions containing a power of (e) and a natural logarithm for various reasons. One reason for the conversion may be to use a series that you are more comfortable with. Below, the results are interesting and easy to remember, but the result is such that a series must be used twice.

Let $e^x = \ln N$

N can be solved simply by the definition of logarithms:

$N = e^{(e^x)}$: Note, this is not a "power to a power",
that is, it is not: $N = (e^e)^x = e^{(ex)}$

Ex. $e^2 = \ln N$ Solving for N:

$N = e^{e^x}$

$N = e^{e^2}$

$N = e^{7.389056099}$

$N = 1618.177992$ Hence:

$e^2 = \ln 1618.177992$: both sides are equal to $7.389056\underline{1}$

Solving for (x) given $e^x = \ln N$

Taking logarithms of both sides of the equation:

$\ln e^x = \ln(\ln N)$ Using the log exponential rule:

$x \ln e = \ln(\ln N)$ Dividing both sides by (ln e) to isolate (x):

$x = \dfrac{\ln(\ln N)}{\ln e}$ Since $(\ln e) = 1$:

$x = \ln(\ln N)$

Ex. $e^x = \ln 1.5$

$x = \ln(\ln 1.5)$
$x = \ln(0.405465108)$
$x = -0.902720455$

Hence $e^{-0.902720455} = \dfrac{1}{e^{0.902720455}} = \ln 1.5$: both sides equal 0.405465108

As like all other equations, if either expression is multiplied by some value (known, such as a constant, or unknown, such as a variable), simply multiply the other side of the equation by the same value so as to keep the expressions and-or values on both sides of the equations equivalent:

Given $e^x = \ln N$ Multiplying both sides by some constant (c):
$ce^x = c \ln N$: or= $\ln N^c$

Ex. Using the last example, and multiplying both sides by 3 we have:

$$3 e^x = 3 \ln 1.5$$

Still, $x = \ln (\ln N)$, this can be easily seen by ridding (in this case, dividing) both sides of the equation by the constant (3 in this example) when solving for (x), and we then have:

$$e^x = \ln 1.5$$
$$x = \ln (\ln 1.5)$$
$$x = -0.902720453$$ hence, after multiplying both sides by 3:

$$3 e^{-0.902720453} = 3 \ln 1.5$$

Checking: $3(0.405465108) = 3(0.405465108)$
$1.216395328 = 1.216395328$

If a multiplying constant is to play a part of the variable being solved for, the methods are almost identical to the original derivations shown:

Given $e^x = c \ln N$ taking the logarithm of both sides:

$$\ln e^x = \ln (c \ln N)$$
$$x \ln e = \ln (c \ln N)$$ since $\ln e = 1$:
$$x = \ln (c \ln N)$$ Hence, to find the value of (x), simply take the logarithm of the opposite side of the equation.

Given $\ln N = ce^x$ by the definition of logarithms:

$$N = e^{(ce^{x})}$$

Hence, the value of N is simply (e) raised to an indicated power equivalent to opposite side of the equation.

SOLVING FOR RECIPROCALS USING (e) AND NATURAL LOGARITHMS

Solving for reciprocals is easily done on a calculator or computer, otherwise, you must solve for it using "long" division or some other method. One reason to solve for a reciprocal is that division of a dividend (or numerator) value can effectively be performed by multiplying that dividend value to the reciprocal value of the divisor (or denominator). For example, multiplication is usually performed in les time (ie., "faster") than division in the realm of computers, and especially if many calculations by "pen and hand" have to be made which could essentially slow the performance or results. Even though it may be overly complex and the long way of doing things, it is still helpful to know how to manipulate equations of all sorts when needed and-or to express them in a certain way. Here is one possible method to solve for a reciprocal:

let $y = \frac{1}{x}$ taking the natural logarithm of both sides:

$\ln y = \ln \left(\frac{1}{x}\right)$ using the "log reciprocal rule":

$\ln y = -\ln x$ by the definition of logarithms:

$y = e^{(-\ln x)}$ and since $(1/x) = y$, we have:

$\frac{1}{x} = e^{(-\ln x)} = \frac{1}{e^{(\ln x)}}$: A RECIPROCAL FORMULA USING (e)

Ex. $\frac{1}{53} = e^{-\ln 53} = \frac{1}{e^{\ln 53}} = e^{-3.970291914} = 0.018867924$

Clearly, by taking the reciprocal of each fraction above, or by simply equating numerators and denominators visually, $e^{\ln x} = x$. That is, this power of (e) simply equals (x) since it is the general form of the natural antilog notation for the number (here it's x). Also, as a check:

If $x = \ln 53 = \log_e 53$, then $e^x = 53 = e^{\ln 53}$

LOGARITHMS OF SIMILAR DIGITS

An interesting note about common logarithms is that when the number (N) varies by some power of 10 (essentially the result of moving the decimal point through the number's digits), that the whole or integer portion (frequently called the "characteristic") of the logarithms of N will increase or decrease by an integer value, and the fractional portion (frequently called the "mantissa") of the logarithm will be the same. This concept can also be applied to creating and using a table of logarithms.

Ex. $\log 10.00 = 1.0$ $\log 100.0 = 2.0$ $\log 1000 = 3.0$: When the base is 10, and N increases by 10, the log increases or increments by 1.

Ex. $\log 1.50 = 0.176091$**...** $\log 15.0 = 1.176091...$ $\log 150 = 2.176091$**...**

As indicated, when N increases by a factor of 10^1 or 10, that its' logarithm is increased by 1. When N increases by a factor of 10^2 or 100, that its' logarithm is increased by 2. In general, given N, the logarithm of: N 10^x will be a value of (x) higher, or more specifically its logarithm is: x + (logarithm of N):

$$\log (10^x) N = x + \log N$$

This is easily verified using the log rules:

$\log (10^x)N$
$\log 10^x + \log N$
$x \log 10 + \log N$: $\log 10 = 1$
$x + \log N$: checks

Ex. Find log 15 with the aid of natural logarithms.

First of all, for the natural logarithm series given previously, N, the Number of which the ln is to be found, must be $0<N<2$.

$15 = 1.5(10^1)$

From : $\log (10^x)N = x + \log N$

$\log 15 = \log (10^1)(1.5) = 1 + \log 1.5$ checking using the log product rule:
$\log (10^1)(1.5) = \log 10^1 + \log 1.5 = (1) \log 10 + \log 1.5 = 1 + \log 1.5$

$\log 15 = 1 + \log 1.5 = 1 + (0.434294481) \ln 1.5 = 1.176091259$

Or perhaps:

$\log 15 = \log (10^2)(0.15) = 2 + \log 0.15$

$\log 15 = 2 + \log 0.15 = 2 + (0.434294481) \ln 0.15$

$\log 15 = 2 + (0.434294481)(- 1.897119985)$

$\log 15 = 1.176091259$

We see that when N is divided (ie., decreases) by a power of 10, (x) will be negative and the logarithm of N will therefore decrease by this value (the indicated power of 10).

Ex. Find log 1.5 : N = 1.5

$\log 1.5 = \log (10^{-1})\, 15 = \log \left(\frac{15}{10^1}\right)$: the right side shows the equivalent of N, and it's being divided by 10^1
From the log quotient rule, this is equal to: log 15 - log 10^1 = log 15 - 1

$\log 1.5 = \log (10^{-1})15 = \log 10^{-1} + \log 15 = -1 + \log 15$ or= $\log 15 - 1$

$\log 1.5 = -1 + 1.176091259$: (or: 1.176091259 -1)

$\log 1.5 = 0.176091259$

This concept can also be verified directly from the log "quotient rule":

$\log (N)(10^{-x}) = \log \left(\frac{N}{10^x}\right) = \log N - \log 10^x = \log N - x$: Since N was increased by a factor of 10^1, from 1.5 to 15, the logarithm will then need to be reduced by that same power of 10, which is 1 here.

CATENARY CURVE

Besides the basic curves shown previously in this book and the (somewhat similar equation) "bell curve" shown ahead in this book, there is a somewhat lesser known, but still an important curve to be mentioned. This curve is called the catenary curve. The word catenary is a derivative word from the Latin word for chain. A catenary curve is somewhat similar in shape to a parabola curve near its vertex or narrow part, and the formula for the shape of the catenary curve is based upon the upward pull or tension due to the downward pull of gravity upon a freely suspended (ie., no initial tension or force placed upon its ends and-or length) chain, thin flexible metal, or a rope. Because the single length of chain or rope is hung from the same height on two poles or rods, the curve will have two symmetrical sides. An inverted (ie. "up-side down") catenary curve is the strongest arch shape for a bridge, and if the "legs" of it are near vertical, then the weight of the bridge and anything on the bridge is then directed vertically into the ground that supports the bridge on the Earth's surface.

Since the value of (e) is used to mathematically express natural events, it is then of no surprise that the equation of the catenary curve includes this value of (e). The derivation (usually credited to Johann Heinrich Lambert) of the catenary equation is not presented below, but it is based on equations for the horizontal and vertical forces of gravity and tension acting upon the curve.

Though not usually important to a practical mathematician, a catenary curve is also the locus (all the locations or points of the (continuous) path to be traveled or traced out) of the focus point of a parabola curve that is made to "roll" along a line. A catenary curve is therefore related to a parabola curve or vice-versa.

Some examples of catenary curves are: a suspended chain link fence, a simple suspension bridge, a hammock or clothes line. The gigantic stainless steel Gateway ("gateway to the west") Arch in St. Louis, MO. was built to have the pleasing shape of an inverted catenary curve.

Consider a catenary curve made of a freely (ie., no horizontal forces applied to influence its shape) hanging length of chain links with its upper end links suspended or held in a firm position at the same height at both ends. The links at the "bottom" in the center of the curve do not have to support (hold up with tension (force)) any other links or weight applying a downward force, and therefore these links are practically parallel to the ground. As you then go up along the "sides" of the catenary curve, the links further up must (upwardly) support more and more (downward) weight, and therefore, this part of the curve or chain is nearing vertical due to the increasing weight pulling on it. This makes the sides of the catenary curve are slightly more narrow or closer together than that of a parabola.

A good thought experiment or actual experiment to help with the understanding of this catenary curve is to lay a length of rope on the ground or table, and then slowly pick the two ends up to the same height, and you will see the curve develop in shape, and with the center portion of it being "flat" or horizontal until a sufficient height is reached so as to have a complete curve shape, and this depends upon the height needed to raise the endpoints. The minimum possible height for the catenary curve is half the length of the rope. The rope can be longer than the distance between the poles, but it cannot be shorter because there would be no possible complete curve between the poles endpoints. Another thought experiment is to imagine a thin strip of material (paper, wood, plastic), perhaps 1 meter long and at least 1 cm wide. At its ends, it will not sag or curve downward due to gravity because its ends are essentially being pushed up by the poles and-or that the "surface tension (ie., a force)" is not allowing that area of the material to move downward and-or bend downward. Where the material and its weight is not supported, the material will begin to bend downward from each pole and it will be at its lowest in the center.

The basic equation of a catenary curve is:

$$y = \frac{e^x + e^{-x}}{2} \quad \text{or} = \frac{\frac{e^x}{1} + \frac{1}{e^x}}{2} \quad \text{or} = 0.5\,(e^x + e^{-x})$$

: this expression form is that of a number and its reciprocal, and here the number is a power of a value; here (e).

: this equation nearly has an exponential equation form.

When an exponent is negative (as in the above equation), it essentially means the reciprocal of that power, hence only one power value of (e) actually needs to be found for each instance (ie., of x) or use of this equation.

Note, as (x) approaches a value of infinity (and much lower for practical purposes of the curve), the reciprocal of e^x approaches a value of 0, and (y) approaches a value of: (e^x)/2. When x=0, y=1. This is also the minimum point of the catenary curve.

The denominator of 2 simply shifts the curve vertically downward by dividing each (y) value by 2, and even if this denominator was not present, the curve still has the same shape. This value of 2 is essentially due to that the derivation of this equation considered that each side supports half the weight (or force) of the chain or material used. That is, the (y) value corresponded to the force needed to supporting each side of the suspended chain at that value of (x).

The equation of this curve rather seems to be relative in nature for values between 0 and 1. Given the same length of chain or rope, and the same distance between the vertical supports, the height of the poles and the curve above the ground plays no actual role in the shape of the catenary curve. Even if the poles are at different heights, the curve is still a catenary curve, and that all catenary curves are similar, but are effectively, magnified versions of one another.

Hyperbolic Trigonometric Functions

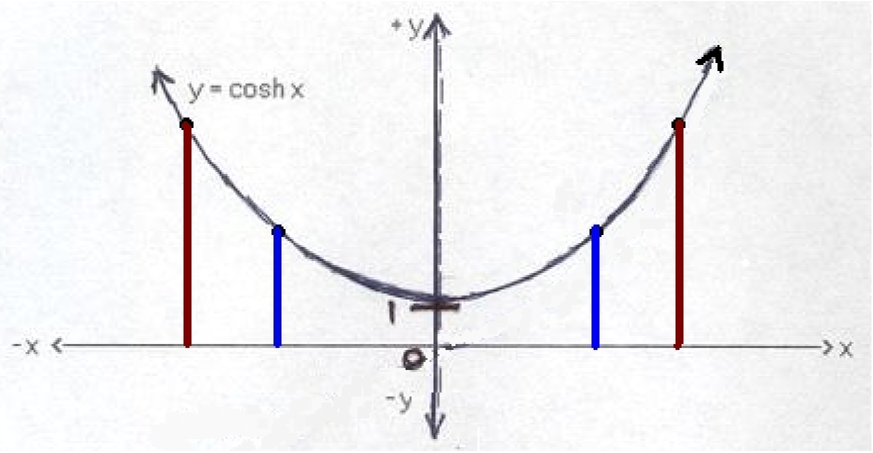
The right-hand side of the equation above is formally called the hyperbolic cosine (here, of x) function. The words hyperbolic cosine are often expressed or noted in the simple form of: cosh. Expressing this with an arbitrary numerical coefficient (a) for plotting on a graph:

$$y = a \cosh \left(\frac{x}{a}\right)$$

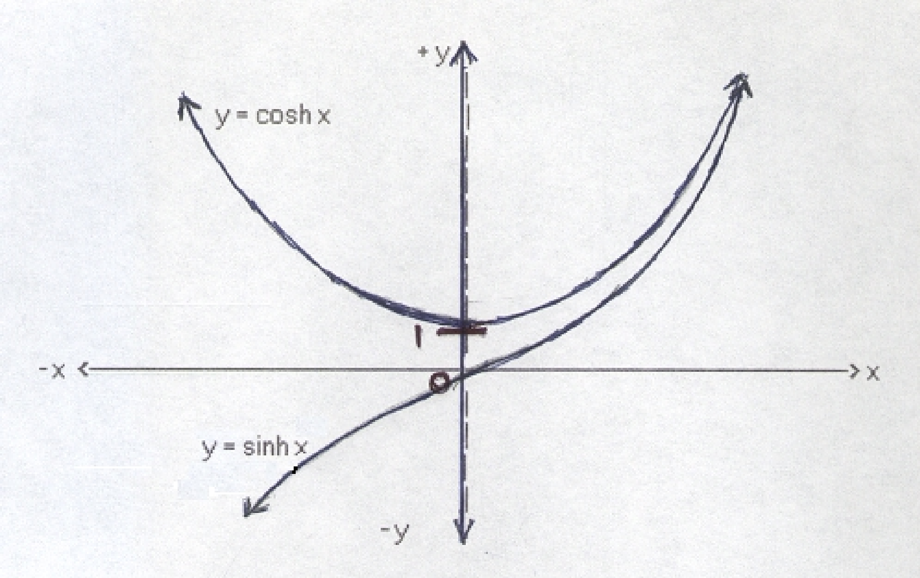
: a catenary curve, the greater (a) is, the "steeper" the sides of the curve
If a=1, the equation reduces to simply: y = cosh x. The lower (a) is, the more "narrow" the curve, and its sides do not expand or open wider as fast when (x) increases.

Remember, this equation is symmetrical (-x can be substituted for x without any problems) about the y-axis and in actual physical reproductions of the basic form of the defined catenary curve where both ends of the material used to construct the curve are suspended at the same height and under no pull, tension or force which would then distort, alter or "straighten" the natural catenary curve. The less the distance between the points of suspension given a fixed length of material, the "steeper" the sides of the catenary curve. The minimum distance possible between the points of suspension is practically 0, and the maximum distance is the length of the material used. Regardless of the material used to construct the curve, it has no effect on the basic equation of the catenary curve.

Given a constant length of rope being supported and creating a catenary curve, if the equidistant distance between the supports is shortened, the curve shape would still be the same as shown here: [FIG 203]



Here is the basic shape of a catenary curve, and it could be describe as having a shape between that of a circle and parabola: [FIG 203A]



The three common hyperbolic trigonometric functions are given below: [Note, these not actually trigonometric.]

$\sinh x = \frac{e^x - e^{-x}}{2}$: this somewhat resembles e^(sin x) which is periodic (repeats) every 2pi radians (about 6.28 radians), since (sin x) cycles through values from 0 to 1. Also: You may also look up Euler's Formulas for sin x.

$\cosh x = \frac{e^x + e^{-x}}{2}$: this is the curve (as seen in the above graph) of a commonly seen catenary curve

These functions are often preprogrammed into many scientific calculators, and have a corresponding input key (button), and therefore, some explanation is needed. These ("non-circular" or non-periodic) hyperbolic functions are somewhat similar in value to the trigonometric identities of the standard "circular" (values are periodic, or repeating) trigonometric functions of SIN, COS, and TAN described in the TRIGONOMETRY section of this book. For example:

(cosh x)^2 - (sinh x)^2 = 1 is similar to the familiar: (sin x)^2 + (cos x)^2 = 1 , and

sinh 0 = 0, cosh 0 = 1 is similar to the familiar: sin 0 = 0, cos 0 = 1

When (x) is small in value (around 10 or less), the hyperbolic functions defined above will make this simple hyperbolic equation shown below to be true by substituting (cosh x) for (x), and (sinh x) for (y) :

x^2 - y^2 = 1 : the std eq. of a hyperbola is : (x^2/a^2) - (y^2/b^2) = 1
For this equation of the catenary curve shown, a=1 and b=1.
Here: x = square-root (y2 + 1) and y = square-root (x^2 + 1)

This is why sinh and cosh are known as hyperbolic (trigonometric) functions. Notice on the graph that the greater (x) is, that (sinh x) becomes closer in value to (cosh x) and therefore, the above equation cannot be satisfied since their difference, being close to zero, does not equal 1.

When x is high, the second term (e^-x) in the numerator gets smaller (practically 0) and it has less and less effect on the curve), and sinh and cosh become closer to the same value of:

$\sinh x = \frac{e^x}{2} = 0.5\, e^x$: essentially this curve becomes that of the e^x curve

$\cosh x = \frac{e^x}{2} = 0.5\, e^x$

From: $(\cosh x)^2 - (\sinh x)^2 = 1$, we mathematically have:

$\cosh x = \sqrt{(\sinh x)^2 + 1}$

$\sinh x = \sqrt{(\cosh x)^2 - 1}$

For completeness, an alternate method to calculate sinh x and cosh x is a series method that is very similar to finding (sin x) and (cos x) with a series:

$$\sinh x = \sum_{n=0}^{n=\infty} \frac{x^{(2n+1)}}{(2n+1)!}$$

$$\cosh x = \sum_{n=0}^{n=\infty} \frac{x^{(2n)}}{(2n)!}$$

: this is very similar to the sinh expansion above, but here, 1 is not added to (2n).

(x) is a radian angle measurement. For both the (sinh x) and (cosh x) series above, you can expect about 8 decimal places of accuracy when (x) is near 1, and 8 terms are summed. The lower (x) is, the greater the accuracy for a given number of terms evaluated. Actually, the series for sinh x and cosh are almost identical to the (sin x) and (cos x) series, except that the terms of these hyperbolic series do not have the $(-1)^n$ factor that produces an alternating (signed terms) series, hence all the terms for sinh and cosh are formally defined as being positive in sign.

Ex. $\sinh 1 = 1.175201194$
Ex. $\cosh 1 = 1.543080635$

Since $\text{TAN } x = \frac{\text{SIN } x}{\text{COS } x}$, and applying the above hyperbolic expressions to this, tanh is defined as:

$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{\frac{e^x - e^{-x}}{2}}{\frac{e^x + e^{-x}}{2}} = \frac{e^x - e^{-x}}{e^x + e^{-x}} = e^x - (1/e^x)\ /\ e^x + (1/e^x)$$

The greater (x) is, the values of sinh (x) and cosh (x) become nearer and nearer, e^{-x} becomes very small, and therefore, the value of tanh (x) approaches $e^x / e^x = 1$.

Below, some generalities about the cosh equation will discussed.

$$y = \cosh u = \frac{e^x + e^{-x}}{2}$$

Why is the denominator 2? Like a numerical coefficient, the denominator for this equation shifts the curve of: $(e^x + e^{-x})$ along the y-axis. To further simplify this analysis, we will let the denominator equal 1, and let $N = e^x$. It is clear to see that the equation reduces to that of a number plus the reciprocal of that number:

$y = e^x + e^{-x} = e^x + \frac{1}{e^x} = N + \frac{1}{N} = \frac{N^2 + 1}{N}$: the numerator shows that the catenary curve is related to a quadratic or parabola curve.

When (x) is negative in value, the terms effectively switch position, resulting in the same general equation:

$$y = e^{-x} + e^{-(-x)} = \frac{1}{e^x} + e^x = \frac{1}{N} + N$$

This simply indicates that the curve of this equation is symmetrical about the y-axis. It also indicates that (y) can never be negative in value (assuming N is always a positive value). When x=0, we will find the minimum value of (y):

$y = e^0 + \frac{1}{e^0} = 1 + 1 = 2$: though with the full equation above, the min. is this value divided by 2, hence: $2/2 = 1$

To verify this, first consider $y = 0$:

$0 = N + \frac{1}{N}$ Multiplying both sides by N to solve for N, we get: (Or by just combining fractions on the right hand side)

$0 = N^2 + 1$: a quadratic equation where a=1, x=N, b=0, and c=1. Solving for N^2:

$N^2 = -1$ taking the square root of both sides of this equation:

$N = \sqrt{-1}$: the solution is imaginary (non-real)

Now consider $y = 1$:

$1 = N + \frac{1}{N}$ After multiplying both sides by N to solve for N, we get:

$N = N^2 + 1$: Here, the solution(s) for N cannot be real since any value squared and added to 1 is always greater than that value. Continuing to solve for N:

$0 = N^2 - N + 1$: a quadratic equation where a=1, x=N, b=-1, and c=1

The result is imaginary since when a=1 and c=1, |b| must be greater than or equal to 2 to produce a positive radicand in the quadratic formula:

$0 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ when a=1 and c=1, the radicand reduces to:

$b^2 - 4$ setting this expression equal to 0 (the smallest positive value for a radicand) and solving for b:

$0 = b^2 - 4$

$b^2 = 4$

$b = \sqrt{4} = 2$

We see that | b | must be equal to 2 just to "break even" into the positive realm for a positive value radicand. If | b | is less than 2 (given a=1, and c=1) the radicand will be negative and the solutions for N are imaginary. Due to these facts, the equation is now:

$y = 2 = N + \frac{1}{N}$: 2 is the min. value for real solutions of N, therefore:

$y = 2 = e^x + e^{-x}$: this is true when x=0. Dividing both sides by 2 we have:

$y = 1 = \frac{e^x + e^{-x}}{2}$: Hence 1 is the minimum value for cosh. This happens when x=0

If both sides of the equation were initially subtracted by 2, rather than divided by 2, and so as to set y=0, this would be incorrect since the minimum value of y, as noted above, is always 1 (when x=0) and not 0. This helps answer the question as to why the denominator is 2 and not some other number.

The actual amount of gravity, specifically gravitational acceleration (g), due to a mass such as a planet, moon, etc.,, will not affect the (natural forces) shape of a catenary curve. If the distance between the supports are the same that catenary curve will have the same identical shape on both the Moon and Earth. (g) of the (local) mass or body is not included as a variable or part of the catenary curve equation. The forces acting upon the chain or catenary curve will be proportional to the amount of (local) gravity upon the chain. In short, all catenary curves are similar in shape and one is simply a magnification of the other. The catenary curve equation is expressed as a very natural mathematical representation of a very natural effect of the (downward) force of gravity upon the entire curve and (changing, force) the tension needed to support the weight of the curve material at points along it. The shape of a catenary curve is somewhat like an "upside-down" bell curve. In short, the catenary curve is the curve of the value of the natural tension forces along the curve. Each end of the catenary curve is supporting half the weight of that material used, and therefore, the tension is half (1/2) for each side, and this is also a reason that the denominator of the equation is 2.

Regardless of the height of the curve above the surface of the ground, the catenary curve will still have the same shape and equation describing it. Excluding the 2 in the denominator, here is an analysis to consider of the sum of a power value of (N) and the reciprocal of that same power value:

$N^x + N^{-x} = N^x + \frac{1}{N^x}$ combining these fractions:

$\frac{N^{(2x)} + 1}{N^x}$, when N and-or x is high, the added 1 becomes more meaningless and the equation reduces to:

$N^{(2x)} / N^x = N^{(2x - x)} = N^x$, and this can also be seen above when $(1 / N^x)$ becomes a small value and is practically equal to 0 and negligible.

Also consider other equations where when x=0, the corresponding value of y is 1:

$y = \cos x$, if $x = 0$, $y = \cos 0 = 1$: a trigonometric equation

y = n^x , if x = 0 , y = n^0 = 1 : an exponential equation

At what value will y=2 for the cosh function or equation?

y = 2 = 0.5 (e^x + e^-x) mathematically dividing each side by 0.5, we have:
4 = (e^x + e^-x) rather than evaluating e^x, we can let N = e^x, then we have:
4 = N + 1/N =
4 = (N^2 + 1) / N it was found by manual trial and error that N is about equal to 4.236068 , therefore:

N = e^x =~ 4.236068 : This can also be found using a program with the fast equation solver algorithm
as shown in this book. Search this book for FANA or FESA and update what variables
to now utilize , and mainly for this example: 4 and use ((x^2 - 1) / x

x =~ ln 4.236068 =~ 1.44363548

If the curve was nearly at the ground, then at x =~ 1.44363548 , y =~ 1

The cosh curve is a result of only the natural tension in the rope due to gravity. The weight of the rope to uphold at the bottom center of the curve is 0 and the tension forces from each side of the rope are equal in value and opposite in direction, hence a net force of 0. Each supporting end of the curve will support (1/2) the weight of the rope. It is also of note that the weight of the rope is proportional to its own length and regardless of how it is straight, knotted, bent or curved.

MODULUS OPERATION

Though one of the less used operations, the modulus operation is still worth mentioning due to its use in computer programming, and occasional use elsewhere, and is unrelated to logarithms, etc.. The modulus operation is also known as the "remainder operation" since this is essentially what it does or returns (the result). The result of a modulus operation is the (whole number) remainder of the division of its two operands. The commonly used symbols for a modulus operation are MOD and the percent symbol (%); as opposed to using the standard division symbol of: (/).

The modulus operation was briefly mentioned before in this book during a discussion of converting any given angle value so as to be an equivalent value between 0° and 360°. As the value of an angle increases to greater than 360° and beyond, the output or resulting angle value of the MOD operation will cycle from 0° up to 360° and then "restart" or "reset" at 0°. In short, the "output" or "result" of a MOD mathematical operation is a range of values that are cyclical (ie., repeats).

The format of a modulus operation is:

operand_1 MOD operand_2 : MODULUS OPERATION

, where operand 1 is essentially the numerator, and operand 2 is essentially the denominator. Since division by 0 is unacceptable, operand 2 cannot be 0.

Ex. 10 MOD 5 = 0 since: : 10%5 would be the notation or "syntax" used in some computer programs

since $\frac{10}{5} = 2$ with a remainder of 0

Ex. 10 MOD 3 = 1 and:

since $\frac{10}{3} = 3$ with a remainder of 1

checking: 3 x 3 = 9 and 10 - 9 = 1

Ex. 7 MOD 7 = 0

since $\frac{7}{7} = 1$ with a remainder of 0: 7 x 1 = 7, and 7 - 7 = 0

Whenever the numerator is greater than or equal to the denominator, the result of the MOD operation will always be between, and including: 0, and 1 less than the denominator operand:

Ex. 4 MOD 4 = 0

since $\frac{4}{4} = 1$ with a remainder of 0

Ex. 5 MOD 4 = 1

since $\frac{5}{4} = 1$ with a remainder of 1

Ex. 6 MOD 4 = 2

since $\frac{6}{4}$ = 1 with a remainder of 2

Ex. 7 MOD 4 = 3

since $\frac{7}{4}$ = 1 with a remainder of 3

Ex. 8 MOD 4 = 0

since $\frac{8}{4}$ = 2 with a remainder of 0

Whenever the numerator is less than the denominator, the result of the MOD operation will always be equal to the that numerator.

Ex. 6 MOD 8 = 6

since $\frac{6}{8}$ = 0 with a remainder of 6

Here is how you can perform a MOD operation by hand or with a calculator:

1. Divide the two operands and obtain the whole portion of the quotient.
2. Subtract the product of the whole portion of the quotient and the denominator from the numerator.

Ex. Evaluate: 17 MOD 8

$$\begin{array}{r} 2. \\ 8\,\overline{)\,17.0} \\ -\,\underline{16} \\ 1 \end{array}$$

: The division can stop here since the entire whole portion of the quotient has been found.

(17 MOD 8) = 17 - (8)(2) = 17 - 16 = 1 : 1 is the remainder, hence (17 MOD 8) = 1

In computer programming, the bits of a byte generally do not have a unique (ie. integer) memory address (location) assigned to them. Still, you can find out the status (set=1, not set=0) or numerical value of a bit by using a small amount of computer code or a program. This results in accessing (reading or writing) the entire (8-bit) byte a bit is part of, and then use various forms of logical or "bitwise (bit for corresponding bit)" operations or bit-shifting (such as with the C-programming language, binary value (bit) shift operators: << and >> which can be used to essentially rid or remove unwanted bits from a byte), etc. to find the specific value (0 or 1) of a bit. These methods are forms of "software addressing". Given the bit address value of a certain bit in memory, how do you find which actual byte address in memory that bit is within or part of? We can use the MOD operation to find what the byte number or byte address is. For example, and optionally using logical (you can think of this as referring to offset values from the start location of 0) rather than physical address values which start at 1 as for counting things.

Byte address:	0	1	2	3 . . .	: each byte has 8 bits
Bit address:	0 1 2 3 4 5 6 7,	8 9 10 11 12 13 14 15,	16 17 18 19 20 21 22 23,	24 . . .	"8 bits wide"

Below, the corresponding byte of which a bit is in is being found. An operation, somewhat like the concepts of a MOD operation is to be used, but here the quotient, and not the remainder, is to be used. Essentially then, the MOD operation is not needed here, but rather just a division or "quotient" operation, and we will use the notation of QUO for this. Though the logical bit position or offset of a bit within a byte do in fact cycle from 0 to 7, byte numbers do not cycle in such a manner.

Logical bit #5, or the bit at logical bit address 5 is at the byte address of : 5 QUO 8 = 0
Logical bit #14, or the bit at logical bit address 14 is at the byte address of : 14 QUO 8 = 1
Logical bit #16, or the bit at logical bit address 16 is at the byte address of : 16 QUO 8 = 2
Logical bit #21, or the bit at logical bit address 21 is at the byte address of : 21 QUO 8 = 2
Logical bit #25, or the bit at logical bit address 25 is at the byte address of : 25 QUO 8 = 3

To access bit #21 so as to find and-or change its value, first access the byte and its value at address 2 that contains that bit. Then find the value of the proper bit (logical 0 to 7) in that byte. 21/ 8 = 2 with a remainder of 5, ie: 2 + 5/8

For example, to access bit #21, we need to first access the byte (ie. find its value) at the logical byte address which also contains that bit as part of that byte value or data:

21 QUO 8 = logical byte number 2. The bit in question is at the logical (bit) offset of 5, as calculated here:

(logical bit offset in the byte containing bit # N) = (logical bit # N being accessed) - (8 x logical byte number)

5 = 21 - (8 x (21 QUO 8))
5 = 21 - (8 x 2)
5 = 21 - 16 = 5

To access (ie., read and-or write) a bit value of a byte, we will need to manipulate the value of that byte, such as with a (byte) shift operator, or with an (logical) XOR operation.

SOME ADVANCED FUNCTION CONCEPTS

COMPOSITE FUNCTIONS

A function that is a function of another function is called a composite function. A composite function is then composed of at least two functions.

If function (g) is a function of variable (y), this can be expressed with the notation of: g(y). If g(y) = 7y^2 + 2, and if (y) is a function (ie. depends upon) of another function (f) where (y) = f(x) = 3x, this composition can be expressed as the composite function gf. This composite function, for example, if set equal to (z), might be used to plot points: P(x, y, z), or perhaps better expressed as: P(x, y(x), z(y)) in a three dimensional coordinate system. Note that gf is the (function) notation for a composite function, and it does not mean the sum or product of function (g) and function (f). Here is an example:

y = f(x) = 3x : y is set equal to a function, and here, it is a function of x.
y is then simply described as a function of x.

z = g(y) = 7y^2 + 2 : z is set equal to function g of which is a function of y. And since y = f(x):

z = g(y) = 7(f(x))^2 + 2 : function g is now explicitly shown as a function of x on the right hand side:

z = g(f(x)) = 7(3x)^2 + 2 Using simplified notation:

z = gf(x) = 7(9x^2) + 2 : gf is a composite function, a function composed of two or more functions.
gf can be noted as a "function that contains another function".

z = gf(x) = 63x^2 + 2 : variable z is a function of the composite function gf

INVERSE FUNCTIONS

If (y) is a function of (x), you may want to solve for the corresponding (x) values which determined those (y) values. This can be accomplished by the substitution of the (y) value into the inverse equation and then solving for (x). However, here we are interested in a general or algebraic method in the form of a formula. The method to do this is to create an inverse function from the known function to essentially undo or "work backwards" to find the value of the variable (x). A given function and its corresponding inverse function are both inverse functions of each other. It should also be stated that these inverse functions are also not generally the mathematical reciprocal of each other.

If (g) is the inverse function of y=f(x), then evaluating (g) with this specific value of (y) will produce its corresponding value of (x):

y = f(x) : f(x), (f) is a function, here evaluated using or containing (x), hence f(x) is called a function of (x).
The actual function is an expression containing variable (x).

x = g(y) : g(y), (g) is a function, here evaluated using or containing (y), hence (g) would be called a function of (y).
If (g) is the inverse function of (f), and if (g) evaluates the output of function (f). which is the (y) value, it
it will find its corresponding value of (x).

The relationship of f(x) and g(y) can be considered as like the corresponding relationship of the (x) and (y) values of a point, except that (f) and (g) are used to find all the specific corresponding points. That is, (f) and (g) actually represent all the possible corresponding pairs of two values, rather than just a single specific pair. This can be expressed as:

p(x, y) = p(g(y) , f(x)) : a point is both a location and a corresponding pair of values, here, (x) and (y).
Here, y=f(x) and x=g(y) , f(x) and g(y) are inverse functions of each other.

Ex. Let f(x) = 7x

If g(x) is the inverse of f(x), then $g(x) = \frac{x}{7}$

This is so since x was multiplied by 7 in f(x). To make the inverse function, (x) must be divided by 7 as expressed for g(x) so as to remove its multiplying factor:

$y = f(x) = 7x$ therefore algebraically:

$x = \frac{f(x)}{7}$ since y = f(x):

$x = \frac{y}{7} = g(x)$

The above is mathematically correct, however for conformity, letting variable (y) represent the dependent variable, and variable (x) represent the independent variable, we will rewrite the above equation as:

$y = \frac{x}{7}$ letting function (g) represent this equation:

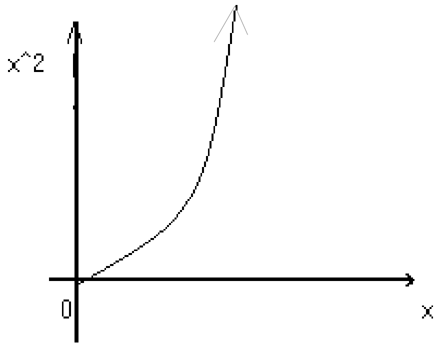
$y = g(x) = \frac{x}{7}$

Ex. If y = f(x) = x^2, what is the inverse function?

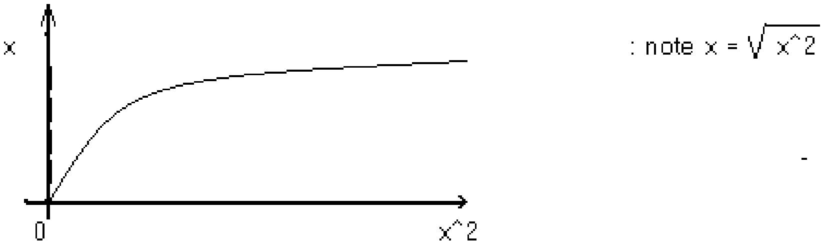
Let g(x) equal the inverse function of f(x). This may sometimes be mathematically noted as:

g(x) = f^-1(x)

Observe the two graphs below for a visualization of these inverse functions: [FIG 204]



Switching the axis, we have: [FIG 205]



However labeled, still the expressions on the horizontal axis represent the horizontal distance (ie. "x") from 0, and the expressions on the vertical axis represent the vertical distance (ie. "y") from 0.

To find the inverse function of f(x), solve for x given that function:

y = x^2 taking the square root

$\sqrt{x^2}$ simplifying:

g = x

Since the process only involved taking a square root, the inverse function is therefore a square root:

g(x) = $\sqrt{x^2}$ = x : that is, if x was squared, you would need to take the square root of that resulting value so as to find what x was

Since: y=f(x) and x=g(y) , f(x) and g(y) are inverse functions

Ex. If f(x) = n^x , find the inverse of this function where base (n) is a constant value.

n^x solving for x by taking a logarithm:
log$_b$ n^x

Or something as the following where y or z is set equal to the given function:

y = n^x taking the log of both sides:
log y = log n^x using the log exponent rule:
log y = x log n using a log with a base of n: logn n = 1 , and switching sides:
x = logn y since y = n^x :
x = logn n^x

Hence to solve for x, only a logarithm needs be taken. For example:

y = n^x : here, n is a constant, not a variable
y = 2^3 : if n=2
y = 8 expressing these steps with a general formula:

x = g(y) = log$_2$ y : Note the base is expressed as written right next to the word log, here it's 2, rather than as a subscript or other form such as log_2. It's a simple text alternative form of expressing logarithm expressions.

$x = g(y) = \log_2 8$
$x = 3$

$y=f(x) = 2^x = 2^3 = 8$
$g(y) = \log_2 y = \log_2 8 = 3$

Using standard functional notation where x is the independent variable and y is the dependent variable:

$y = g(x) = \log_2 x$
$y = g(8) = \log_2 x = 3$

Ex. $y = f(x) = 2x + 3$ find the inverse function.

$y - 3 = 2x + 3 - 3$
$y - 3 = 2x$ after dividing both sides by 2, canceling, and switching sides:

$$x = \frac{y - 3}{2}$$

Therefore, the inverse function will contain a subtraction of 3 and a division by 2. In standard function notation where y is set to the dependent variable, and x is set to the independent variable:

$$y = g(x) = \frac{x - 3}{2}$$

$f(6) = 2(6) + 3 = 12 + 3 = 15$

$$g(15) = \frac{15 - 3}{2} = \frac{12}{2} = 6$$

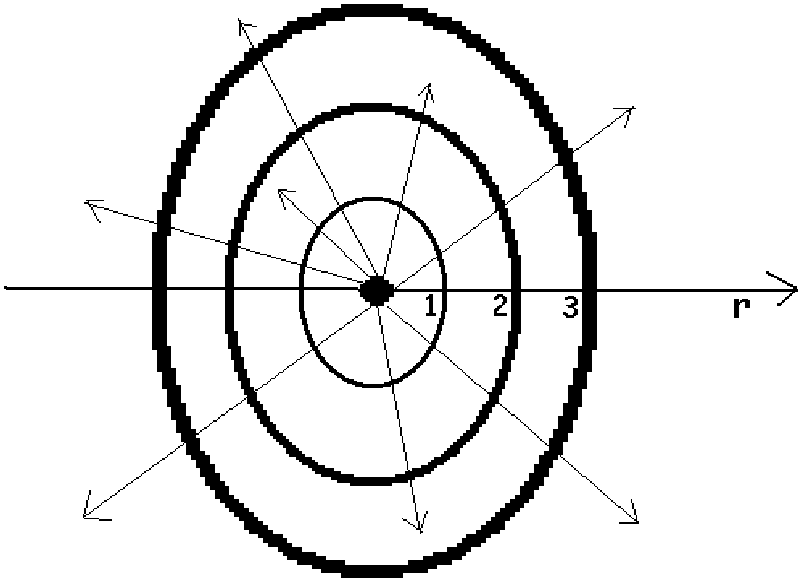
INVERSE SQUARE LAW

The inverse square law is a common name given to a relationship when values or variables are not only related to the reciprocal or inverse (ie. inversely related) of another value or variable, but related to the mathematical square (ie. value^2) of that reciprocal value.

$A = \frac{1}{B}$: Here, A is related to the inverse or reciprocal of B, and if given:

$A = \left(\frac{1}{B}\right)\left(\frac{1}{B}\right) = \left[\frac{1}{B}\right]^2 = \frac{1}{B^2}$: Here, A is related to the square of the inverse or reciprocal of B, and may be commonly spoken as: "A is related to the inverse square of B", or "A is the inverse square of B".

Common examples of this relationship are found when dealing with energy (or the measured intensity of it) as it expands outward in all directions from the (point) source. For example, light, sound, radiation, gravity fields and so on, often expand this way and their corresponding intensity or strength reduces dramatically by the square of the distance from that source. Below in the drawing we see a "point-source" (infinitely small and uniform source or distributor) of energy at the center of the (imaginary) concentric spheres drawn to be used for this analysis. This point could actually be an infinitesimally small object or a large object so far away as to be considered practically as a point (such as a star in the sky). Since the energy is uniformly distributed outward from the source, a surrounding "shell" or (surface of a) sphere at various distances or radius values from that source will be used for the analysis. [FIG 206]



$S_s = 4\pi r^2 \approx 12.56637061\underline{1}\, r^2$:Formula for the **SURFACE AREA OF A SPHERE** (essentially a ball) as derived by Archimedes is approximately: $12.566\underline{4}\, r^2$. Units are those used for (r).
If we let r = radius = diameter/2 = d/2, we have this for surface area of a sphere:
Ss = **4(pi)r^2** = 4(pi)(d/2)^2 = 4(pi)d^2/4 = **(pi) d^2** = (pi)(area of a square) = **CD**

$V_s = \frac{4\pi r^3}{3} \approx 4.1887902\, r^3$: **VOLUME OF A SPHERE (with units being the same as those used with r)** is approximately: Vs =~ 1.333... (pi) r^3 = $4.18879020\underline{5}\, r^3$ =~ 4.1888 r^3
The formula is credited to Archimedes. **Vs = (pi) d^3 / 6 = (pi) Vcube / 6** = **V sphere = 0.52398776 V cube** having the same diameter.

For inspiration and verification for the volume of a sphere formula, Archimedes may have used a sphere in container of water with a known volume, and then measured how much the sphere caused an increase in its volume when immersed in it. This increase in the volume of water is equal to the volume of the sphere.

In the drawing above, you can see (energy) ray or traversal lines coming from the point source indicating the direction of energy, and here it's in all directions outward or away from that point or source. Clearly, the total amount of energy coming from the source will then be distributed throughout a larger surface area (or volume) as the distance (indicated as a radius = r value) from that point increases. Though the total value or sum of energy available at each shell or sphere surface area remains the same as it expands outward, the measured value or intensity of energy will actually get or be less per unit area (of measurement) since the total amount of energy must remain the same (not more or less) as before. Though the area of each sphere increases, the total energy itself does not increase or decrease, and only the energy or intensity measured per unit area decreases. At this point in this discussion, we now can understand that the intensity (I) level of the energy per unit of area is inversely related to the distance (r) from the point source. Now note the ratios of surface areas of the spheres as (r) increases:

$S = 4(Pi)r^2$: basic equation for the surface area of a sphere , and if r increases by 1, area increases by 3 to be a total of 4 times more:

$\frac{S2}{S1} = \frac{4 (Pi) 2^2}{4 (PI)\ 1} = 4$: $\frac{\text{surface area of sphere at r=2}}{\text{surface area of sphere at r=1}}$, ie., related to the square of the radius , and if r increases by 1 again, the area increases by 5 to be a total of 9 times more than the original area.

$\frac{S3}{S1} = \frac{4 (Pi) 3^2}{4 (Pi)\ 1} = 9$: $\frac{\text{surface area of sphere at r=3}}{\text{surface area of sphere at r=1}}$

According to the basic equation for the surface area of a sphere, we see that the relationship of surface area to the radius is a squared relationship. Surface area is mathematically directly related to the square of the radius from the source. If we express the energy level or intensity (I) per unit area (A) we can call this the density (D) or concentration of energy per unit area, or "energy density". More technically, since energy has no form, it has no actual density either, and "energy density" could rather be considered as the "energy concentration" or "concentration of energy".

$D = \frac{I}{A}$: Let D = energy density or intensity (I) per unit area (A).
: For the calculation, use A = surface area of the sphere.

$D = \frac{It}{4(Pi)r^2}$: It = total energy or intensity emitted from the source.
r = distance from the source.

As the distance (r) from the source increases, the surface area (A) of the shell or sphere will increase. Surface area of a sphere is directly related to the square of the radius.

We see that for example, D, energy density or level, at r=2, will be only one-fourth (1/4) of that at r=1. This is so, since the (limited or constant amount of) energy (at r=1) will expand or be distributed into four times that area at r=1. At r=3, the density will be only one-ninth (1/9) of that at r=1, since the energy at r=1 will expand into 9 times the area at r=3. We can conclude that as (r) increases, the intensity per unit area decreases, and its value is inversely related to the square of the radius from the point-source.

Ex. At r = 1, if the measured intensity is 100 units.

At r = 2, (twice the distance from the source) the intensity is less by a factor of: $2^2 = 4$,
or $D1/4 = 100/4 = D2 = 25$

At $r = 2.5$, the intensity will be less by a factor of: $2.5^2 = 6.25$,
or $D1/6.25 = 100/6.25 = D = 16$

At r=2 from the source, for example, even though the intensity decreases by $r^2 = 2^2 = 4$, the surface area actually increases by 4, and the total energy passing through or received by each entire shell or sphere from the source is constant:

Total Energy In = Total Energy Out : considering no losses along the way
Intensity In = Intensity Out

From $D = \frac{I}{A}$: energy or intensity per unit area, we have $I = DA$: total energy at that area

Ex. If the density (D) or intensity per unit area was measured to be 100, and if A increases by 4, D will also decrease by a factor of 4, such as when r=2:

$I1 = I2$: total intensity or energy at surfaces at r=1 and r=2 respectively
$D1A1 = D2A2$: You may sometimes see proportions like: $D1/A2 = D2/A1$ when dealing with inverse square relationships. an alternate expression of these values is:
DnAn is Dn Rn^2 : R = radius , n = corresponding values of D and A
$100(1) = 25(4)$

$100 = 100$: showing that, the total energy at each (increased in size) shell or sphere is the same.

Ex. If r is now 3, what is the calculated density or intensity per unit area?

From: $D1A1 = D2\,A2$ mathematically:

$$D2 = \frac{D1A1}{A2} \text{ or= } \frac{D1R1^2}{R2^2} = (100)(1) / 3^3 = 100/9 =\sim 11.1$$

You probably have noticed that the intensity or level of sound and-or light changes as the distance between you and that source changes. A sound can be very loud if you are close to it, or barely heard if you are far from the source of that sound. As your distance doubles (at r=2) from that source (or possibly a reflection [and redirection]), the intensity of the energy received will be reduced (ie. divided by the factor) by $r^2 = 4$, or multiplied by factor of (1/4) = 0.25. Just the same, in a reverse type of manner, if the distance between you and the source is halved ($r/2 = 0.5r$), the intensity of the energy will increase or multiplied by the factor of 4.

An article in the appendix section of this book will contain another verification, ideas and examples; see
More About The Inverse Square Law

If the intensity (I) and-or measurement is inversely related to the square of the distance (r) from the source or a measurement location, we can express this as:

$$I = \frac{1}{r^2} = r^{-2} \qquad \text{solving for r:}$$

$$r = \sqrt{\frac{1}{I}}$$

If the distance (r) from the source has increased, the intensity (I) will decrease due to the inverse real (physical) and mathematical relationship. and if the intensity is measured to be a certain percentage (n) as that of the first measurement, this can be expressed as: nI , and for example, if I is now at its 50% level, n = 0.5 and nI = 0.5I. The corresponding distance (r) can be found by using this value in the above equation. Considering I as a relative value of 100%=1 value:

$$r = \sqrt{\frac{1}{n(I)}} = \sqrt{\frac{1}{0.5(1)}} = \sqrt{\frac{1}{0.5}} = \sqrt{2} = 1.4142...$$

: when (r) increases by the factor of 1.4142, the intensity (will) be decrease by the factor of 0.5

Checking:

From: $I = \frac{1}{r^2}$ If (r) increases by the factor of 1.4142, (I) will increase by the factor of 0.5, hence decrease by 2:

$$I = \frac{1}{r^2} = \frac{1}{(1.4142\ r)^2} = \frac{1}{(\sqrt{2}\ r)^2} = \frac{1}{2\ r^2} = (0.5)\frac{1}{r^2}$$

For the equation to be in balance, if we have performed mathematical operations of one side, such as essentially multiplying one side by a value, here (0.5), we must do the same to the other side of the equation, and we then have (0.5)I = 0.5I when (r) increases by the factor of 1.4142. I is then at half of its measured value at (r), when (r) changes by the factor of 1.4142. In short, and as shown in the above equation, if (r) changes by a factor of (n), (I) will then change by the factor of: 1/(n^2).

Given : $I = \frac{1}{r^2} = \left(\frac{1}{r}\right)^2 = r^{-2}$: It could be said that (I) is inversely related to the square of the reciprocal of (r).

Given : $I = \frac{1}{r^2} = r^{-2}$

, the instantaneous rate of the corresponding changes in (I) to the changes in (r) is the derivative of (I) with respect, or in reference to (r) :

$$\frac{dI}{dr} = -2\,r^{-3} = -\frac{2}{r^3}$$

: Clearly, as r increases the slope decreases rapidly till it is practically 0, and the curve resembles a line. That is, for large changes in (r), when (r) is about five times or more than that of the first measured distance, the intensity, though it is very small in value, it will not decrease by a larger and larger factor past these high values of (r). This situation allows us to still receive, see or detect extremely distant (but weak) starlight and radio signals.

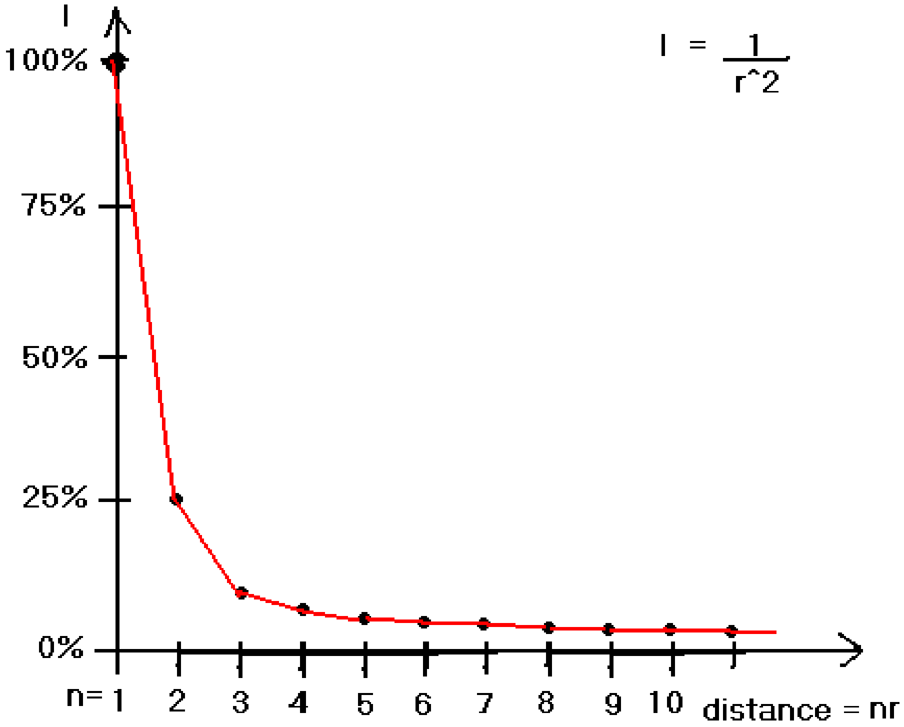
Given : $I = \frac{1}{r^2} = \left(\frac{1}{r^1}\right)^2 = r^{-2}$: It could be said that (I) is inversely related to the square of the reciprocal of (r^1).

This is a much steeper slope than if (r) was not squared:

$$\text{value2} = \frac{1}{\text{value1}^1} = \text{value1}^{-1} \quad \text{and} \quad \frac{d\,(\text{value2})}{d\,(\text{value1})} = -1\,\text{value1}^{-2} = -\frac{1}{\text{value1}^2}$$

This is essentially a "reciprocal equation" since value2 is the reciprocal of value1.

Here is a graph of: $I = \frac{1}{r^2} = r^{-2}$, and it is essentially the inverse of: $I = r^2$. [FIG 207]



(ie., multiples of (r))

Notice that at high values of (r), say past r=5, or 5r, that the intensity is a low value, but it remains relatively stable in value after that for very high values of (r), and this allows us to still receive and see enough light from a very distant star. A telescope with a large main (or "objective") lens or mirror can gather very small amounts of light and collect and concentrate them (into a smaller area or image) so as to effectively increase the intensity of that received light per unit area, and therefore, produce a brighter and more visible image.

PRODUCTS OF THE FIRST TERMS OF THE ODD, AND EVEN SERIES

Integer Series: 1 , 2 , 3 , 4 , 5 , 6 , 7 , 8 , 9, . . . : an arithmetic series where d=1
The value of the nth term is: 1(n) = n

Even Integer Series: 2, 4 , 6 , 8 , 10 , 12 , 14 , 16, . . . : the value of the nth term is: 2n
Odd Integer Series: 1, 3 , 5 , 7 , 9 , 11 , 13 , 15, . . . : the value of the nth term is: (2n - 1)

The following is a practical example of working with factorials. Lets begin with the product of the first (n) terms of the even series:

Product = (2)(4)(6)(8)(10)....(2n) : 2n is used and not just (n). This is since if we want to find the n=10 first terms that are even in value, we must go through 2(n) = 2(10) =20 total terms since there will also be n=10 odd terms in the series to go through to reach the 10th even valued term. (2n) is always an even number. (2n)-1 and (2n)+1 are odd.

By dividing each factor by 2, each factor can be expressed as a multiple of 2. For example: Since 8/2 = 4, factor 8 can be expressed as 8 = (2x4):

Product = (2x1)(2x2)(2x3)(2x4)(2x5)...(2n) Each factor has a factor of 2

(2)(1)(2)(2)(2)(3)(2)(4)(2)(5)...(2n) All these factors of 2 can be expressed as a repeated multiplication of 2:

(2)(2)(2)(2)(2)x(1x2x3x4x5...n) which can be expressed with a power of 2 as:

Product = 2^n (1x2x3x4x5.....n) : The second factor is equal to n**!**

Product = $(2^n)(n!)$: Product of the first (n) terms of the even arithmetic series.
This value will be used in the following derivation:

Now let us find the product of the first n terms of the odd series:

If we take the factorial of the first (2n) integers (both odd and even), and divide out the unneeded factors such as the even series, we should have the product of the first (n) factors of the odd series:

$$\frac{1 \times 2 \times 3 \times 4 \times 5 \times 6 \times 7 \times 8 \ldots (2n)}{2 \times 4 \times 6 \times 8 \ldots (2n)} = \frac{(2n)!}{(2^n)(n!)}$$

: Product of the first (n) terms of the odd series.

As a check, multiplying the odd and even products (ie. the divisor and quotient) of the first (2n) terms, we get the quotient of :

$$(\text{even products})\ (\text{odd products}) = (2^n)(n!)\ \frac{(2n)!}{(2^n)(n!)} = 2n!$$

, and dividing by 2 to have half of the values we get: (n**!**)

Ex. Find the product of the first 3 odd terms, or the first 3 terms of the odd integer series: (here, n=3):

$$\frac{(2n)!}{(2^n)\,n!} = \frac{(2 \times 3)!}{(2^3)3!} = \frac{6!}{(8)\,(3!)} = \frac{720}{(8)(6)} = \frac{720}{48} = 15 = (1)(3)(5)$$

[This space for edits.]

THE BINARY NUMBER SYSTEM

The reader may skip the section below about binary numbers and read it at a later time if they become interested in this topic. The binary number system will not be discussed further in this book, except that it may possibly help with the included computer programs and with further study about some methods of computer programming.

Before this discussion begins, some people may be worried as to what great of an extent they may need to understand binary numbers as to be a computer programmer. With most modern computer languages, working directly with binary bits and bytes and-or their signed value computer representation is often not needed as rather their equivalent and common decimal values are used directly in computer programs.

The word "binary" has a "bi" prefix which means double, two or 2. "Unary" is a word meaning single, unit or 1. The binary number system uses only two counting symbols (0 and 1). Hence, a "highest count" of only one (1) can be reached before positional notation, just like that used with the decimal number system, is required to represent larger quantities or numeric values. That is, in binary addition, 1+1 = 0 with a carry of 1. This can be shown as 1+1=10. As an aid to avoid ambiguity and to be certain which system is being used, letter b for binary (base 2), or letter d for decimal (base 10) can be explicitly expressed or indicated. For example: 1b+1b = 10b. 10 ("one, zero") is the binary representation or equivalent of the decimal notation for 2: 10b = 2d.

Lets increment a binary value of 0 by 1, using the binary system and positional notation:

```
   0
 + 0
   0                 : 0b = 0d

   0b      0d            :  Same as:   1b     : using the commutative law.
 + 1b    + 1d                        + 0b
   1b  =   1d                          1b    ,    1b = 1d


  1 <------------------: This is a carry of 1 from the previous column or position sum.
   1b      1d        : As in the decimal system, leading zeros are also possible, for
 + 1b    + 1d           example:  1 + 1  =   0001 + 0001  =  0010
  10b      2d        : "1 plus 1, equals zero with a carry of 1".  The result is "one, zero" in
                        binary, or 2 in decimal.  10b = 2d

  10b          10b       2d
 + 1b    or:  + 01b     + 1d
  11b          11b   =   3d            : 11b = 3d

 11            11
  11           011          3d
 +  1    or:  + 001        + 1d
 100b          100b   =    4d          : 100b = 4d
```

Decimal Equivalent	=	Binary Equivalent	
0d		0b	After incrementing each by 1 successively :
1		1	
2		10	
3		11	
4		100	
5		101	
6		110	
7		111	
8		1000	
9		1001	
10d	=	1010b	: 10d = 1010b

Notice in the above table of binary values, that you can immediately tell if the binary value is odd or even. Odd binary values will have the least significant bit or position of the entire number value set to 1. Even (exactly divisible by 2) binary values will have the least significant bit or position of the entire number set to 0. Binary values can also have fractional (a part of 1, and that are less than 1) values just like decimal values do following the decimal point, but for the binary system, the correct terminology is the binary point.

The numeric base of a number system is the number of unique counting symbols for each integer and before positional notation is required. The base in the decimal system is "ten" = 10d, and the base in the binary system is "two" = 10b = 2d. A more formal word for base of a number system is the radix of the number system. "Radix" is a word based on the common word "root" and which for this discussion means the root or base (ie. start, based upon) of the numeric system. The word "radical" is also a form of the word radix, and is the notation and operation to find a root (ie.source, common factor) of a value. These words and others such as "radius" (from the center of a circle) and "radicle" (pre-root or first root of a plant) are based upon the very old word "wrad" which means a root or branch (from a source or tree which extends outward).

From the letter "b" in the word binary, and the letters "it" from the word digit, binary digits are frequently called "bits". A binary number will have a series of these digits or bits and their corresponding bit positions in a binary number consisting of several bits. A bit can have a numeric value of only 1 or 0. A binary number composed of a series of 8 bits together is called a byte.

The maximum numeric value for a byte is 255, that is, $11111111_b = 255_d$. Since the binary number system uses only two values, it is easily implemented with mechanical, electro-mechanical (ie., electrical and physical-motional), and high speed electronic systems such as with electronic computers. A simple on-off switch or settings is a simple binary system. Something set to "on" can be represented logically (ie. the truth=1 or falseness=0) and mathematically in binary as 1, and something set to "off" can be represented logically and mathematically as 0. Computer circuits such as electronic memory (storage of information), the data bus (communication wires or lines), and the micro-processor (electronically processes data or information and programs) are usually set up to work with whole bytes of data (information) at once rather than a bit at a time as for serial communication which is usually slower (although it might be simpler and cheaper). This is accomplished by using 8 wires or "data lines", one for each bit of the byte, and this method is called parallel data and-or communication. For the internal computer system, data is placed in the form of a binary number representation (which may be optionally coded) which is implemented electronically where a high voltage signal represents a bit set to 1 ("a 1 bit"), and a low voltage signal represents a bit set to 0 ("a 0 bit"). For example, with the American Standard Code for Information Interchange (ASCII), letter A is numerically $65_d = 01000001_b$, letter B is $66_d = 01000010_b$, and so on. It must be noted that fundamentally, data storage media such as hard-disks (magnetic), floppy-disks (magnetic), CD-DVD (optical) drives, and many portable "memory sticks or cards" often use only 1 data line, hence serial communication, and therefore 8 bits, one after the other (ie., serially or sequentially), must be accessed to essentially build an 8 bit byte that can then be accessed (read or written) for internal processing by the computer. Therefore, accessing a byte on disk is often at least 8 times slower than accessing a byte at a time in the computers electronic (byte sized) memory. This time can be reduced in many cases by using an electronic memory cache (storage) where extra bytes, just beyond those requested, are automatically read from the disk and stored in this cache for possible immediate use.

Here are the binary number system weights of the bits of a byte which is a binary encoded number or value, having a "width" (digits or bit count) of 8 bits:

most significant part of the number   least significant part of the number

2^7 , 2^6 , 2^5 , 2^4 , 2^3 , 2^2 , 2^1 , 2^0 : binary weights in powers of 2 (binary)
128 , 64 , 32 , 16 , 8 , 4 , 2 , 1 : equivalent decimal values

Notice that in the binary number system that each weight is twice (2) that of the previous bit position weight, rather than ten (10) times that of the previous position weight as with the decimal number system.

For either the decimal or binary number system, a positional product is the product of the digit or bit value and the weight of the position it is in. This is so since each digit or bit value is a multiple of the weight value of that position.

Since a bit can only be 0 or 1, each positional product of a binary number can only be either 0, since (0 x any_weight) = 0, or (1 x weight) = weight. This makes it easy to find the decimal equivalent of a binary value since no multiplication needs to be done, and essentially, just the addition of the binary weights (ie., positional significance, value) is needed.

Ex. What is the decimal equivalent of the byte: 00100011 , or without the leading 0's as: 100011 ?

0 0 1 0 0 0 1 1 : 00100011b

0 + 0 + 32 + 0 + 0 + 0 + 2 + 1 : position products =
bit value x weight value or=:
position value x position weight

By simply "skipping over the 0 bits", that have no value:
32 + 2 + 1 = 35b : positional sum
00100011b = 35d

Ex. What is the binary equivalent of 35 = 35 decimal = 35d ?

First, since the value is less than or equal to 255d, only one byte, or= 8 bits, is needed to represent this value in binary. Actually, only 6 bits are needed to represent this value since the leading two bits are 0.

Since the binary weights of 128 and 64 are larger than 35, they will not be included or set (to 1) in the binary representation. If building a (8-bit) full byte they should rather be indicated as 0, and since leading 0's wont add to the value. It is possible in some electronic systems that if they are not deliberately set to zero (or "cleared"), that they may still contain a previous ("garbage", incorrect) value that could result in an error if those bits are possibly considered and included as part of the value.

0 0 . . . . . . : first two bits of the byte set to 0. Since these bits actually represent a larger part of the number, they are called most significant bits. The logical bit numbers of these are bit #7 and bit #6 from left to right. The first physical bit has a logical bit number of bit #0.

The bit corresponding to the weight of 32 can be set to 1 since 32 is less than or equal to 35, that is 32 is part of, or within 35, and therefore it needs to be included or represented in the binary number:

001 . . . . .

The remainder of decimal value must now be considered so as to keep building the equivalent binary value.

35 - 32 = 3 : Subtracting 32 out from the given decimal value since it is now accounted for in the binary number. 3 is the remainder of the decimal value left that must be converted to its equivalent binary value and added into the binary value being built.

The three next lower value binary weights of 16, 8 and 4 are larger than the remainder of 3, and therefore, their corresponding bits will be set to 0::

001000 . .

Continuing the process, we find that we can use the second bit, which has a weight of 2, since it is less than or equal the decimal value of 3. The remainder left is then 3 - 2 = 1. The first bit has a weight of 1 and can be used to represent this value. Since the remainder is now 1 - 1 = 0 , the conversion process stops, and the binary equivalent of 35d is found to be:

$35_d = 00100011_b$

With the binary number system, you can quickly multiply a number by 2, by moving all the bits one position leftward. Likewise, You can divide a binary number by 2, by moving all the bits one position rightward. This is the same basic procedure when multiplying and dividing a number by 10 as with the decimal number system.

Repeatedly shifting this byte, 1 position leftward:

Binary System Numeric Value	Decimal System Numeric Value
0000 0001	1
0000 0010	2
0000 0100	4
0000 1000	8
0001 0000	16
0010 0000	32
0100 0000	64
1000 0000	128

If each bit of a 8 bit byte is set, the equivalent maximum decimal numeric value of this is found to be:

11111111b = 255d : = (2^8) - 1 = 255d : 2^8=256d, notice that the exponent of 2 is the number of bits

If 0 binary = 0b = 00000000b = 0d is considered, a byte can actually represent 256 unique, whole values. 2^8 = 256

More about binary numbers to consider:

Given any number (n) of binary digits or "bits", the maximum value it can represent is 2^n -1 in the decimal number system. If just another binary digit is included, the maximum value becomes twice that value less one:
((2)(2^n)) -1 = ((2^1)(2^n)) -1 = 2^(n+1) -1.

Given any number (n) of binary digits, the maximum value is when they are all set to 1, and this value is 2^n -1. If this value is multiplied by itself, it will yield a maximum value of the product of two binary values, such as say two 8 bit bytes shown here: (2^n)(2^n) -1 = (2^n)^2 -1 = 2^(2n) -1 : and this product will require a maximum of (2n) or twice the number of bits to hold that value as shown here:

Ex. With 8 bits = 1 byte numbers: (11111111b)(11111111b) -1 = (256d)(256d) -1 = 256^2 -1 = 65535d is the maximum value. (2^8)(2^8) = 2^(8+8) = 2^((2)(8)) = 2^16 = (256)(256) = 256^2 = 65536. The product of two bytes will require a maximum of (2n) bits, here (2)(8) = 16 bits = 16bits / (8bits / byte) = 2 bytes

NEGATIVE BINARY VALUES

When not explicitly specified, binary values are assumed as being positive values. It is possible to have a system of signed (positive and-or negative) binary values. The usual way to accomplish this is to assign the first or most significant bit as the "sign bit". If the bit is set to, or has the value of 0, the entire binary value is a positive value. If the bit is set to, or has the value of 1, the entire binary value is a positive value. This following discussion about how negative binary values are expressed is generally only for technical, near circuit level, or for computer programmers only, and may be skipped over if the reader does not need it.

Since a bit of the byte is used for the sign, there is now a "trade off" or loss, and that is that the signed binary value can now only have an absolute value of about half of what would normally be possible with that given number of bits in the binary number. For an unsigned 8 bit binary number, the maximum value is 255d:

According to the above discussion, if the first bit is the sign bit:

10000001 = -1
10000010 = -2
11111111 = -127 : 127 is nearly half of 255 which is the decimal value of a full byte where each bit is set to 1.
-127 is the maximum negative value of a 8 bit byte. The maximum positive value of a
8 bit byte is : 01111111 = +127.

Though the above is the practical method to express negative numbers, the actual way a computer does it is slightly different, and it is to encode the negative value using a well defined manner so as it can then be used with the computers (preexisting) addition (or "combine") circuitry and produce the correct result of a signed number addition operation. Essentially then, the (coded) negative value will be considered as an operand in a subtraction operation of which is then performed using the addition circuitry of the computer.

Here is how is a negative value encoded so as to be used by the computers addition circuitry:

11111111b = 255d : maximum value for an 8 bit unsigned binary value

If you were to then add 1 to this binary value, all of the 8 bits will "roll over" (as if on a wheel, such in a cars mechanical mileage counter mechanism) to 0, and the (final) carry will be discarded ("dropped", or "lost" if its unused).

11111111b = 255d
+ 00000001b = + 1d
100000000b 256d if this last or final carry is dropped (unused) we have this value remaining:

00000000b = 0d when the carry bit is dropped.

A decimal system analogy to this would be for example: if given a 4 decimal digit system, when the max. value of 9999d is reached, and if 1 is then added to it, the value of 9999 will "roll over" to all 0's, and the (final) carry is dropped:

111 <--------- carrys from the previous column or digit position. Each carry is a value that is 10 times the previous
9999d weight value.
+ 0001d
10000d = 0000d = 0d when the last or final carry is dropped (unused)

Now consider, given 0000000b, if you were to "go back" one or subtract 1 from this given value of 0bso as to have the previous value before 1 was added to it, the result would be the value just before this "roll over" value of 0, and that value is: 11111111b. You can also consider this verification:

1 00000000b : A 9 bit binary value if the carry was kept, and which will help for any possible borrowing.
\- <u>00000001b</u> : Subtracting 1 to "go back" to the previous value just before rollover to all "zeros" or 0 bits:
11111111b

Now observe how negative binary numbers are usually expressed:

011111111b = +127 : max. signed positive value for an 8 bit binary number, now repeatedly subtracting 1:
.
.
.
00000001b = +4d
00000011b = +3d
00000010b = +2d
00000001b = +1d

00000000b = 0d Repeatedly subtracting 1:

11111111b = -1d : consider that this is the value just before "roll over" when adding 1 to it.
11111110b = -2d : = -1 -1 = -2 : subtracting 1 is the same as adding negative 1
11111101b = -3d
11111100b = -4d
.
.
10000000b = -128 : max. signed negative value for an 8 bit binary number

As a check on this system, consider that when you add (-1) and (+1) the result is 0:

+1d = 00000001b
\+ <u>-1d</u> = + <u>11111111</u>b
0d 100000000b : the carry is once again dropped, and the result is:

= 00000000b : as expected

Because of the way negative binary values are expressed, it also allows the existing electronic addition circuitry of the computer to easily add signed binary values, and the result will be properly signed automatically during the process.

For most computer programmers, it is interesting to know how negative numbers are usually implement within a computer, however there is relatively few times when there is a need for a person to convert these negative value representations to their equivalent decimal value.

A method to convert a negative binary value to its corresponding unsigned or positive equivalent value is to invert or change all the 0 valued bits to 1, and all the 1 valued bits to 0, and then add 1 to the resulting inverted value. A name for an inverted or "flipped" bit or byte is the complement of that bit or byte. Two corresponding complements will sum to 0 when the final carry is unused.

Here is an example that converts a coded or signed binary negative value to its corresponding signed decimal value.

Ex. 11111101b = -3d , with inverted bits:
00000010b, adding 1 to this, we have:
00000011b = +3d

If we know the binary equivalent of 3d, we can do the same as the above process to show the computers binary number representation of -3d:

00000011b = +3d Inverting each bit:
11111100b adding 1:
11111101b = -3d

On a technical or computer programming note, for various reasons and usually other than the concepts of a computers representation of negative binary numbers, each bit of a binary value can be inverted by XORing (with the XOR binary "bitwise" operation; bit by bit, or bit to corresponding bit position) it with a bit set to 1. The resulting value is sometimes called the (ones) complement of the given binary value. An entire 8 bit binary value can also be complemented in just one step by XORing it with an 8 bit binary value with all of its bits set to 1. 255d = 11111111b which has all 8 bits set to 1. Subtracting a binary value by a value with each byte set will also create the complement of that byte value.

Ex. 10000001b XOR 11111111b = 01111110b

The C-computer language also has a special bitwise operator to easily complement all the bits of a byte value, and this uses the "tilde" symbol of (~). If variable (a) was set equal to 1d = 00000001b , then if variable (b) is set equal to the complement of variable (a): b = ~a , then b = ~1d = ~00000001b = 11111110b = 254d when non-signed or unsigned values are only being considered.

The inverse of each bit creates the 1's or binary complement of the value. The sum of a byte value and its 1's complement will result in a sum of 0 with a carry of 1 which can be discarded. The 2's complement of a binary value is equal to the binary complement and add 1 to it. A binary value and its 1's complement will be equal to a value with each bit set. The computers addition circuit can be used to subtract a value by converting the value to be subtracted to its 2's complement value before adding it and discarding the carry. For example, 15d - 5d = 1111b - 0101b = 1010b = 10d this bit by bit subtraction is correct since it is the reverse of addition bit by bit, however, for the computer to utilize the existing addition circuitry to subtract, we must use: 15d + (2's complement of 5d) = 15d + (1's complement of 5d + 1) = 1111b + (1010b + 0001b) = 1111b + 1011b = 1010b = 10d. For another example: 1d - 1d = 0d = 1d + (2's complement of 1d) = 1 + (1's complement of 1d + 1) = 0001b + (1110b + 0001b) = 0001b + 1111b = 0b = 0d when the carry is discarded.

Extra: What is a nibble?

A **nibble** is a play on the word byte (which sounds like bite) which is defined as 8 bits of data and-or 8 bits in width or wide. A nibble is a "small byte" being just 4 bits wide: 0000b = 0d through 1111b = 15d ,and this is a total of 16 possible values (0d to 15d). Since the hexadecimal numbering system has 16 digits and-or counting symbols: (00h = 0d through FFh = 16d), a 4 bit nibble number can be expressed as 1 hexadecimal value, and-vice versa. Ex. 1111b = Fh An 8 bit binary value or byte, can be expressed as two nibbles or two hexadecimal digits. Ex. 11111111b = 1111 1111 = FFh , Ex. 10100010b = 1010 0010 = A2h

FRACTIONAL BINARY VALUES

The concepts of expressing fractional (<1) binary values is nearly identical to that of expressing fractional decimal values, except that for the binary system which is based on 2, will use negative integer powers of 2 instead of negative integer powers of 10 as it is for the decimal number system. Here are the binary weights and their equivalent decimal equivalents, and note for example that 2^-1d = 1/2d = 0.5d , that is a negative indicated power is equivalent the reciprocal of the corresponding positive indicated power. Here are the decimal equivalents of the bit positions of a binary number:

2^7 , 2^6 , 2^5 , 2^4 , 2^3 , 2^2 , 2^1 , 2^0 . 2^-1 , 2^-2 , 2^-3 , 2^-4 , 2^-5 , 2^-6
128 , 64 , 32 , 16 , 8 , 4 , 2 , 1 . 0.5 , 0.250 , 0.125 , 0.0625 , 0.03125 , 0.015625

Notice that going from left to right, or from the most significant weights to the least significant weights, that each is half of the previous. Going in the other direction, each is double (2 times, twice) the previous.

Convert 1.1b and 1.01b to their corresponding decimal equivalent value and representation:

Showing many intermediary steps:

1.1b = 1(2^0)d + 1(2^-1)d = 1d + (1/2^1)d = 1d + 0.5d = 1.5d

1.01b = 1(2^0)d + 0(2^-1)d + 1(2^-2)d = 1d + 0(1/2^1)d + 1(1/2^2)1d = 1d + 0(1/2)d + 1(1/4)d =
= 1d + 0(0.5)d + 1(0.25)d =
= 1d + 0.0d + 0.25d
= 1.25d

To convert a binary value that has a both a whole part and a fractional part, simply convert the whole part and then the fractional part to their corresponding and equivalent decimal values, and place the (.) symbol between them. This same method can be used to convert a decimal value to its corresponding and equivalent binary value.

How To Make A Random Number Using The Concepts Of Binary Numbers

Use a two-sided object such as a coin. You can label or consider one side as 0 and the other side as 1. Flip it into the air and see what side lands upward. You can write this value down onto paper. Repeat the process to create as many binary digits or bit-positions as needed. You would usually then convert this binary value to its equivalent decimal value.

If your tying to create a random number between a certain range of numbers, you will have to use a certain minimum number of bits to create that value:

bits	binary	decimal
1	1	1
2	11	3
3	111	7
4	1111	15
5	11111	31
6	111111	63
7	1111111	127
8	11111111	255

Each next value will be: (2 x current value)+1 :

Ex. To create a random decimal number between 0 and 100, we will need a binary value composed of 7 binary digits, or bits.

A set of 7 randomly created binary bit values are:

0100101b after converting this to its equivalent decimal notation value:

0100101b = 37d

If the random number generated was greater than 100, you can redo the process.

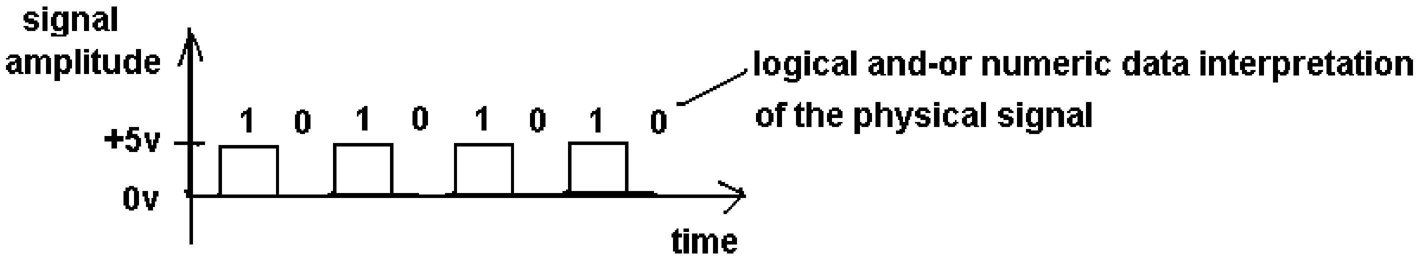
If you only want to consider only decimal values when creating or generating a random number, you can put numbers 0 through 9 on similar objects, such as paper or balls into a bag and pick out a ball to create a digit for a random number. If you do not put each randomly chosen ball back into the bag after recording its value, that value will not be available and possible for the next digit in the random number. If this is the case, and If you took out 3 balls, the minimum random number generated or possible would be 012, and the maximum random number would be 987. If you want to give each digit the same chance or possibility of having a value of 0 through 9, then you must put the ball back into the bag to have all the balls available for the next random selection.

This book includes a computer program to create random values, that can be viewed, searched, saved and loaded into a computer file.

The **Morse Code** system of "dots and dashes" is actually a simple and efficient binary or two-state(s) of information system. Two different signals are used to represent each smallest bit of information such as the coded series used for each letter. A series or more of these pieces of information can be used to construct a letter, and many can then be used to construct a sentence of a message. Consider graphing paper with many horizontal and vertical lines creating many uniform (ie., similar to) and regularly spaced small squares, boxes, or picture elements (tiny "bits" or "pieces" of, "**pixels**" = "picture elements") on that paper. Each box can be set to be either one of two states, either filled with a color or left empty. Each box can be a pixel element of an entire two color or black and white picture such as seen in some books and newspapers. The more pixels, the finer or more detailed an image can be, and this "fineness" or "image precision" is given a measurement called its image resolution which can be summarized here as pixels per distance on the photo. We can communicate this image by letting number 1 mean to fill a box, and 0 to keep it unfilled. A longer binary number say for example: 10101010 might numerically represent the upper row of a text character (ie., the image of a letter) or a picture being drawn. We see that a binary, two state system and-or numbers is the simplest way to represent information, and it can be transmitted as very fast electronic pulses through a wire of an electric circuit, including the telegraph or radio communication system. It should be also of note that each letter and-or text character is usually given a unique 8 bit code or binary number such as with the ASCII code system. A computerized, "8 by 8, dot-matrix" printer will effectively have in its memory storage a matrix or array of picture elements for each ASCII text character to be printed, say printed or displayed using 8 horizontal bits in each of 8 rows. Each row of pixel element information can be stored as if it is an 8-bit binary number or information. A pixel of an image is also the smallest discernible or resolvable (ie., resolution) piece of an image.

With binary picture elements being either 1 or 0, how can colors or gray scales be used since we can not have half a bit of information and we can only have "full bits", each with just two possible states only? The answer for example would be to use a binary number coded (gray or "color intensity") scale of values. We can create a 256 shades of gray using an 8 bit binary number. We can let 00000000b represent no gray in the box, hence clear or white, and 11111111b represent full black. Values in between 0 and 255 will represent a particular shade of gray, say increasing in darkness as the binary value increases. This system can display what is called a "black and white" picture, but the image or picture is technically seen as a "**gray scale**" picture that has been printed with just one available color, usually black. Now that an image can become better and-or more accurate or representative, we see that more information must be used, and here we see that each picture element now requires 8 bits now instead of just 1 bit. The entire image will require 8 times the information or data and its storage, and 8 times longer to transmit it.

As mentioned previously, 1's and 0's are numeric and-or logic values, that can be thought of as either a signal or presence, and a lack of signal or presence. For example the signal or presence can be a voltage value say +5 volts to represent a "high" or 1 value, and 0 volts to represent a "low" or 0 value. The signal could be a charge storage or lack of it such as in a capacitor, or it could be a magnetized spot or location on a metallic hard-disk drive, or it could be a transistor setting considered either on or off. Here is a basic graphical image of 1's and 0's such as that which could also be visualized using an electronic oscilloscope that can display a visualization of a signal and-or waveform: [FIG 208]



When a computer program receives and senses 8 bits of data at its communication port or input, it will convert this to a byte of binary data that the computer can store it in a file and-or process it further, such as displaying it as a text character on the screen.

CHANCE

If given a certain range of values from say 0 to 99, hence one-hundred known and unique values, and when you randomly ("blindly", non-specifically) pick or choose just one of those values, that value chosen is then called a random value. If you were hoping that the value was a specific or certain valued number between the given range, this is a concept of the larger topic of "chance", "odds" or "probability". If the range was smaller, say from 0 to 9, you will have a higher chance or success, on average after many attempts of picking a certain value. As the range increases, say from 0 to 999, you will have a lower chance or success, on average, of picking a certain value randomly.

The word probability is related to the word "probable" and the concept of something that is likely to, or may happen. It should be fairly obvious that the more digits a number has, the larger in value or numerically it is, and the more difficult to guess, select or pick a certain valued number among that complete set of many possible random values from 0 up to the maximum number that all those digits could represent. There is a higher chance, success or probability of you selecting a 1 digit number out of 10 numbers: 0 to 9, than there is of selecting a 2 digit number out of 100 numbers: 0 to 99.

Ex. A 3 digit decimal (each digit being a one value from 0 to 9) number can represent any value, or random-like value, from 000 to 999. That is a total of 1,000 = 10^3 possible values. For the probability or numerically predicted average of 1,000 guesses or chances (but could actually be more or less guesses in reality and current luck), and to have the 1 correct guess of a random value in that range is 1 in 1000 = 1/1000 = 0.001 , hence a fairly low probability, and commonly said to be as a "low or small chance(s)", and-or "a low probability", and-or "low odds".

Digits	Total Possible Values Or "Combinations"	Chances of picking a certain value, expressed mathematically.
1	10 , 0 through 9	1 in 10 = 1/10 = 0.1 : "one in ten" , "one out of ten" , "a tenth"
2	100 , 0 through 99	1 in 100 = 1/100 = 0.01 This could also be thought of as since each digit has a 1 in 10 chance, the new or total chance becomes less, and now becomes equal to only a fraction of a fraction, and the new net result or chance is equivalent to the product of the chances, odds or probabilities of each new chance, step or digit: Even if the first digit is right, there is still only a 1 in 10 chance that the next, or second, digit is right. The "combined chances" of each digit can be expressed as one new total chance equal to their product: $(1/10) / 10 = 1/100 = (1/10)(1/10) = (0.1)(0.1) = 0.01$ "a tenth of a tenth" = "a hundredth" = "one in a hundred"
3	1,000 , 0 through 999	1 in 1,000 = 1/1,000 = 0.001
4	10,000 , 0 through 9,999	1 in 10,000 = 1/10,000 = 0.0001

The **probability** of an event or outcome happening and being considered a successful (calculated) prediction can be numerically expressed as the ratio of it happening (success) to the total number of possible events or outcomes:

$$\textbf{probability}\text{ of event happening} = \frac{\text{total number of favorable events or outcomes happening}}{\text{total possible events or outcomes}}$$

Since the numerator is always less than the denominator in a probability ratio, the ratio value will be between 0 and 1.

If there is a set of 10 values labeled from 0 through 9 in a bag, and if only one is being picked out from, the probability of each different number, or picking a certain number, is: 1/10 = 0.1 : "one in ten" = 1 in 10 = 1 out of 10

If a second set of the same 10 values: 0 to 9 was initially combined into the first set of 10 values, the probability of each different number, or picking a certain number, is: 2/20 = 1/10 = 0.1 , hence the same probability as before

If an event or outcome is 10% probable, then it could be said that it is (100% - 10%) = 90% "not likely" or "not probable":

"odds" of something happening = "probability" odds $= \dfrac{\text{probable}}{\text{not probable}}$, ex: $\dfrac{10\%}{90\%} = \dfrac{0.1}{0.9} = 0.1111 = 11.11\%$

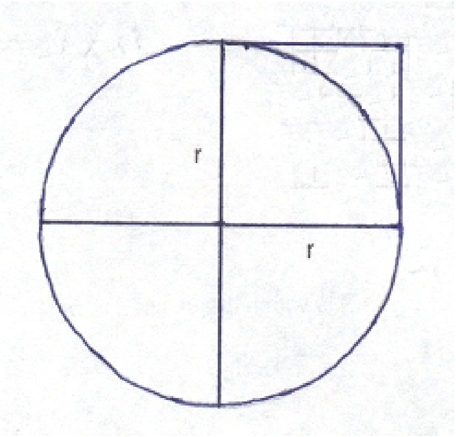
[This space for edits.]

SCIENCE AND APPENDIX SECTION

This book section contains some helpful derivations of the formulas previously mentioned in this book, and it was initially written for the initial basic math, algebra, trigonometry, and advanced topics sections of this book before the science topics were placed in it. This section also contains other useful supplemental material and it is therefore a mix of topics, and yet it could still be considered as a natural continuation or extension of much of the previous material in this book.

A DERIVATION OF PI, AREA AND CIRCUMFERENCE OF A CIRCLE

Below is a drawing of a circle and in the upper right quadrant, a square is overlaid whose side is equal to the radius length of the circle. The area of the this square is therefore: As = (r)(r) = r^2. [FIG 209]
If you were to draw or imagine an identical square in each of the remaining three quadrants, the area of the circle (Ac) would be clearly seen to be less than that of the total area of those four squares:



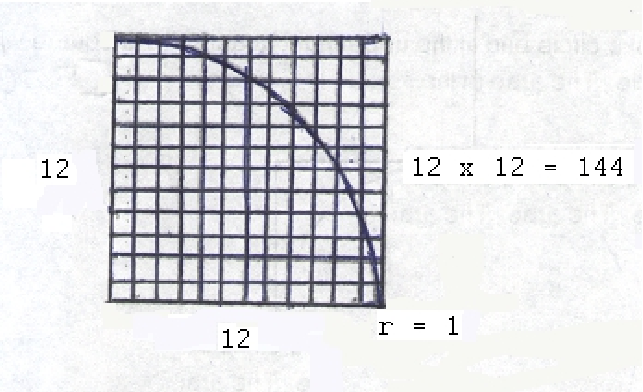
Below is a drawing of a circle and in the upper right quadrant, a square is overlaid whose side is equal to the radius length of the circle. The area of the this square is therefore: As = (r)(r) = r^2. If you were to draw or imagine an identical square in each of the remaining three quadrants, the area of the circle (Ac) would be clearly seen to be less than that of the total area of those four squares:

Ac < r^2 + r^2 + r^2 + r^2 combining like terms:
Ac < 4 r^2 : Each side of this larger square is 2r, hence its area is: (2r)^2 = 4r^2

Comparing the (smaller) area of the circle to the larger square area of 4r^2, we find that the coefficient of r^2, which is 4, effectively causes that square area to be slightly more than that of the circle. If were to then write an equation for the area of the circle, a value less than 4 would be the proper numerical coefficient of r^2. We can set a variable (n) equal to the actual or proper numerical coefficient of r^2 so as to have an equation that represents the exact value of the circle = Ac:

Ac = n r^2 : n < 4

All sorts of geometric methods can be used to find a formula for the area of circle. Most divide the circle into smaller areas, such as a large number similar triangles all having the same vertex at the center, and then these are summed to find a close approximation of the area of a circle. Perhaps the simplest method is to divide the square (shown in the upper right quadrant) up into many small identical squares or square units, and the more used, the better the approximation of the area of the circle will be. More squares can also be drawn by starting with a large drawing of the square and quadrant of the circle so as to have a closer approximation for Ac. For example: [FIG 210]



Obviously, in the figure directly above, the portion of the area of the circle that is within the square (ie. a quadrant, or quarter of the circle) is only some fraction (or percentage) of that square's area. The area of this portion of the circle is less than 1 r^2, but greater than half of it or $r^2/2 = 0.5\ r^2$, as can be seen by drawing a diagonal line from the upper-left to the lower-right corner in the drawing above. Letting variable (x) represent this portion, we algebraically get:

$Ac = xr^2 + xr^2 + xr^2 + xr^2$: $x<1$, combining like terms:
$Ac = 4 \times r^2$: Since $x<1$, therefore for Ac, the actual numerical coefficient (n) of r^2 is a fraction of this value of 4, and can be expressed as $n<4$:
$Ac = nr^2$: where $(n<4)$

(x) can be calculated from:

$$x = \frac{\text{Area of the circle portion in the square}}{\text{Area of the square}} = \frac{Aq}{As}$$: x is a ratio, Aq = area of a quadrant of the circle

Using the above graph, x can be geometrically (approximately) calculated without using a specialized formula as:

$$x = \frac{Aq}{As} \approx \frac{\text{number of small squares within the circle's portion of the square}}{\text{total number of small squares in the square}}$$

This ratio value can also be quickly estimated by observation. Obviously, the area of the circle that is within the square is clearly greater than 1/2 = 50% of the square (can check this by drawing a diagonal within the square from upper-left to lower-right), and less than 1 = 100% of the square. It appears to be a value about midway between these two values, and this is three-quarters of the square = 3/4 = 0.75 = 75%. Taking the average of these two values we have:

$$x = \frac{50\% + 100\%}{2} = \frac{150\%}{2} = 75\% = 0.75 \text{ in decimal form}$$

Substituting this value of (x) into Ac:

$Ac = nr^2 < 4 \times r^2$
$Ac = nr^2 \approx 4\,(0.75)\,r^2$
$Ac = nr^2 \approx 3\,r^2$: approximation, and this is similar to the approximation of the length of a circles circumference being about 3 times its diameter length.

If you solve for (x) by the geometric methods, you will see that this ratio value is slightly greater than 0.75 and therefore, the numerical coefficient (3) of r^2 shown is slightly lower than it should be. For a simple example, using the drawing shown and considering a small square in the circle's portion if half or more of it is within the circle's portion:

$$x = \frac{\underline{\text{number of small squares within circle's portion of the square}}}{\text{total number of small squares in the square}} = \frac{\underline{110}}{144} = 0.764$$

We find that this calculated value or estimation is slightly more than 0.75 , and is actually: 0.785398163 ... =~ $0.785\underline{4}$

Adjusting the formula, we have:

$Ac = nr^2 = (3 + c)\ r^2$: c = a slight adjustment or correction. $0<c<1$ since the coefficient of r^2 was shown to be less than 4. This means (c) is both greater than 0 and less than 1. Here, (c) is a constant and not a variable.

Verse 4:2 in the Second Book Of Chronicles in the (about 4000 year old) Old Testament book gives the first written indications that the circumference of a circle is 3 times greater than the diameter (widest) length of the circle. Today, the exact value of the constant $n = (3 + c)$ is formally called and identified as the constant of: "Pi" , and is also given the symbol of: π, but this may be also noted as using just common text characters as (Pi) if the (π) symbol is not available to be displayed.

$n = \pi = (3 + c)$: Pi is slightly greater than 3

$Ac = \pi r^2$: AREA OF A CIRCLE , (π = "pi" =~ 3.14159265...)

If you estimated (x) as 0.78 , PI = π can be expressed with three significant digits:

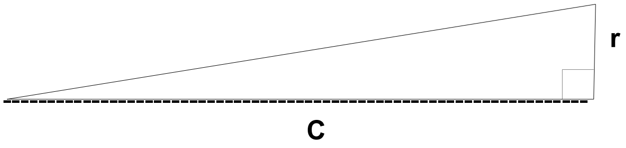
$\pi = 4x \sim= 4\,(0.78) = 3.12 = (3 + 0.12)$: 3.12 has 2 correct significant digits; the 3 and the 1

If you again estimated (x) as 0.785, π can be expressed with three significant digits, and with the two most significant being shown to be accurate when compared to the above value. All three digits shown below for (x) are accurate.

$\pi = 4x \sim= 4\,(0.785) = 3.14 = (3 + 0.14)$: 3.14 is the common approximation of Pi which is given the π symbol.

$Ac = 4x\, r^2 = \pi r^2 =\sim 3.14\, r^2$: formula for the area of a circle, and the approximate area of a circle

Now, let's try to find a formula for the circumference of a circle by using the formula for the area of a circle. Since circumference is a linear (as in a distance, one-dimensional) concept, let's convert the area value that represents the circular shape into an equivalent value of area that is represented by a triangular shape. As shown previously in this book, the area of a circle can be conceived and represented as the area of a triangle whose base is equal to the circumference (C) of the circle and whose height is equal to the radius (r) of the circle: [FIG 211]



Equating the area of a circle to that of this representative triangle:

$At = Ac$: $At = \dfrac{(\text{base})(\text{height})}{2}$ using substitution:

$At = (\text{base})\,(\text{height}) / 2 = Ac = (\text{pi})\, r^2$: set equivalent for this analysis only since the basic formulas for these area shapes are different, but it is sill possible to have a triangle and circle having the same exact area.

$\frac{C\,r}{2} = \pi r^2$: **AREA OF A CIRCLE** , r = radius of the circle = half the circles diameter (d or D).
r = distance from center point of the circle to its (curved-line) edge or circumference (C).

Solving for C:

$C = \frac{2\pi r^2}{r}$ canceling or dividing out common factors, here it's r^1:

$C = 2\pi r$: **CIRCUMFERENCE OF A CIRCLE**

Since 2r = D: : D = diameter = longest length, or width, across ("dia-") or through and measurement ("meter") of the circle. r = D / 2 , D / r = 2 ,

$C = \pi D$: hence, the circumference of a circle is about 3.14 times larger or longer than its diameter.

$\pi = \frac{C}{D}$: **"pi" = π** , is usually defined as the (constant) ratio of any circles circumference to its diameter
(pi) =~ 3.13159265... , $r = C / 2\pi = C / 6.283\underline{2}$... , $C / r = 2\pi = 6.2832$...

We could have found π if we knew before-hand that it was equal to the circumference of a circle divided by its corresponding diameter. Logically, or "truly", Pi is mathematically expressed as, and equal to exactly: C/D, however the value of Pi calculated by any means possible will always be an approximation no matter how "close" it is to the true value of C/D. Because of this, the value of Pi is said to be an irrational value or number. With the calculation of π using small squares as shown previously, the result will always an approximation. With advanced mathematical formulas, π can be calculated to a great and accurate precision ("preciseness", fineness, fraction) but it is still an approximation since Pi will always be an irrational value with endless digits.

It is not easy to measure the circumference of a circle using a typical and inflexible ruler. One method to measure the circumference of a circle is to mark a point on the circumference of the circle where it is tangent (ie. coincides, touches) to a flat surface, and also mark a corresponding point on the flat surface. Roll the circle along a straight, line-like path on that flat surface for one complete rotation and then mark another point on that flat surface where the point on its circumference is again tangent to the flat surface. The measure of this distance between these two points on that flat surface will equal the circumference (distance is equivalent to corresponding "straight" linear distance when "flattened") of that circle.

If the concept of Pi is already known, a basic verification to the above derivation(s) can be made by considering a more well known derivation for the area of a circle, and that is to divide the circle into an infinite number of similar isosceles triangles that have the same height and base side values. The height of each triangle will be equal to the radius (r) of that circle, and the base side of each triangle will be an infinitely small segment portion of the circumference. As the number of triangles gets larger in value, the base side of each will approach a value of 0, but will never actually be zero. Ac = (area of the sum of each triangle), and since all the triangles are the same, their areas are the same, and the circle area can then be expressed with multiplication of the area of just one similar triangle by the total number (n)of triangles. Ac=n(area of triangle). one might quickly say that the area of that circle will be infinite if the number of triangles to be summed is infinite, and this is a good thought, but we must also consider that the area of each triangle is infinitely small in value or close to 0. A practical version of this derivation is to then construct a rectangle from all these triangles by alternately placing upside-down next to each other in a row. The area of this rectangle construction will be:
Ar = (length)(width), and where the (length) = (circumference / 2), and (width) = (r). Ar = (length) (width) = (2(pi)r / 2) (r) = (pi) r^2 = Ac

A **disk** is a flat-like volume much like a circle but also having a thickness, hence 3-dimensional) structure that has circular shaped area, and its surface area on the top or bottom is equal to that of a circle (ie., two dimensional): Adisk = Acircle = (pi)r^2. A coin has a disk structure or shape, and technically, a disk is also a **cylinder shape** that has a low height compared to its width, radius and- or diameter.

If a two-dimensional (hollow) circle or (solid, filled circle) disk is moved a height upward, into another or the third dimension, it will create a three-dimensional volume called a (hollow, empty, non-solid) tube or cylinder, or solid tube or cylinder which is sometimes called a rod. The volume of this construction is simply the product of the two dimensional circle or disk area, often called the base area, times the distance or length that it was moved along the third dimension or "axis".

The formula for a cylinder is very similar to the formula of a cube shape which is the base area times the height that the base area is moved vertically: Vcube = (base area) x (height). and for a cube, the base area is that of a square, hence base area = s^2, and since the height is also equal to (s), Vcube = (s^2)(s^1) = Vcube = s^3. Other (rectangular solid, prism) volumes have a rectangular base areas and various heights, but the formula is still the same:

V = (base area) x (height).

Vcylinder = (circular base area) x (height) and:

Vcylinder = (Acircle) x (height) = ((pi)r^2) h = **(pi) r^2 h** **: VOLUME OF A TUBE, CYLINDER, OR ROD**
A rod is a long and solid cylinder construction.

A **tube** (ie., usually a flexible pipe, like a garden watering hose) or **pipe** is essentially a hollow cylinder shape and having an outer surface or wall having a thickness to it, and therefore, there is an inner and outer diameter to consider in your calculations:

outer diameter = inner diameter + wall thickness
inner diameter = outer diameter - wall thickness
wall thickness = outer diameter - inner diameter :cross sectional wall area = cross sectional area of the cylinder if it were solid - cross sectional area of the hollow region

As indicated previously, the formula above is very similar to that derived when moving a square area along a third dimension so as to construct a volume called a rectangular prism. The word "prism" is a fancy word for an object having plane (flat, planar) shaped sides or "faces". A solid rectangular prism is also called a block. A hollow rectangular prism is called a box. A cube is a rectangular prism with all dimensions or sides having equal lengths. If all the side lengths are 1 unit in length, it is called a cubic unit or cubic unit of (volume) reference or measurement for any other volume.

V rectangular prism = (V rectangle) x (height moved) = (LW)H = **LWH** **: VOLUME OF A RECTANGULAR PRISM, BLOCK, OR BOX**

Or:

V rectangular prism = (rectangular base area) x (height moved) = (base area)(height) = LWH
In your calculations, be sure to consider or take account of any wall thicknesses, for this will reduce its holding capacity or volume of some material placed into it.

A triangle shape or surface moved a height will create a volume called a triangular prism.

$$V_{\text{triangular prism}} = (\text{base area})(\text{height}) = \left(\frac{bh}{2}\right)\left(\frac{h}{1}\right) = \frac{bh^2}{2}$$

: VOLUME OF A TRIANGLULAR PRISM

A pyramid is a polygonal (flat, multi-straight sided) shape or volume that diminishes (gets smaller, reduces) to a single point (called the apex) at a height above it. The formula for the volume of a triangular pyramid or cone is:

$$V = \frac{(\text{base area})(\text{height})}{3}$$

: VOLUME OF A PYRAMID OR CONE
= 1/3 VOLUME OF ITS CORRESPONDING, OUTER CYLINDER

This formula for the volume (Vp) of a pyramid can be verified by using the volume of a cube. A cube has 6 sides that we can consider as base areas of internal pyramids. We can construct 6 internal pyramids within that cube with each having its apex at the center point of this cube. Therefore, the volume of each pyramid is 1/6 that of volume of that cube (Vc):

$$Vp = \frac{Vc}{6} = \frac{LWH}{6} = \frac{(\text{base area of each pyramid})\ H}{6}$$

: Vc = volume of cube, Vp = volume of pyramid.
base area of each pyramid = area of each side of cube

The height of each pyramid is only half of that of the cube. h = H/2. Therefore, we cannot use just H in the formula for the volume of each internal pyramid. We can create an equivalent fraction by dividing both the numerator and denominator by 2:

$$Vp = \frac{\dfrac{(\text{base area of each pyramid})\ H}{2}}{\dfrac{6}{2}}$$

Expressing (H/2) as the height of the internal pyramid:

$$Vp = \frac{(\text{base area of pyramid})(\text{height of pyramid})}{3}$$

:**VOLUME OF A PYRAMID OR CONE**

This formula also works for the volume of a cone since it can be considered as a unique pyramid with an infinite number of very thin (approaching 0 in width) sides extending up from its polygon base area. The word "cone" is rooted in the word of "point", "combine" and "concentric" which means all having the same center or point of reference. A cone can also be thought of as an infinite number of concentric circles or disks that get smaller and smaller until the radius is just the size of a point. The rate at which the diameters or radius get smaller and smaller as the height or base value changes (ie., increases) will determine the specific shape (ie., side slant or "slope") of that cone.

For the surface area of a cylinder or rod shape, imagine taking the curved surface area portion of a cylinder or rod and making a straight cut along its length, and then flattening that surface into a rectangle plane shape.

A = lateral or side area of the rod + circular area at both ends of the rod
A = area of curved portion + circular area of both ends

A = area of rectangle created from the curved portion + area of both circles at the ends

$A = (\text{length})\ (\text{height}) + 2(\text{pi})(r^2)$: r = d/2 = (outer width or thickness of the cylinder or rod) / 2

The height (h) of this rectangle corresponds to that of the circumference (c) of the cylinder or rod:

$A = (\text{length})\ (2)(\text{pi})(r) + 2(\text{pi})(r^2)$ factoring $2(\text{pi})r$ from each term:

$A = (2)(\text{pi})(r)\ (\text{length} + r)$

$A = (c)\ (l + r)$: SURFACE AREA OF A CYLINDER OR ROD SHAPE
c = circumference of cylinder or rod , r = radius
l = length of the rectangle = height or length of the cylinder or rod

If a regular, circular or right (created by rotating a rectangle [having right or 90° angle sides] about its side) cylinder is cut at an angle which is not perpendicular (90°) to its axis and-or sides, it will produce a cross-sectional shape and-or surface area of an ellipse, and a cylinder with this constant cross sectional shape is called an **elliptical cylinder**. The full or longest length of the diagonal cut will be the major-axis (a) of the ellipse, and twice the radius of the cylinder with be the minor-axis (b).

SURFACE AREA OF A CONE

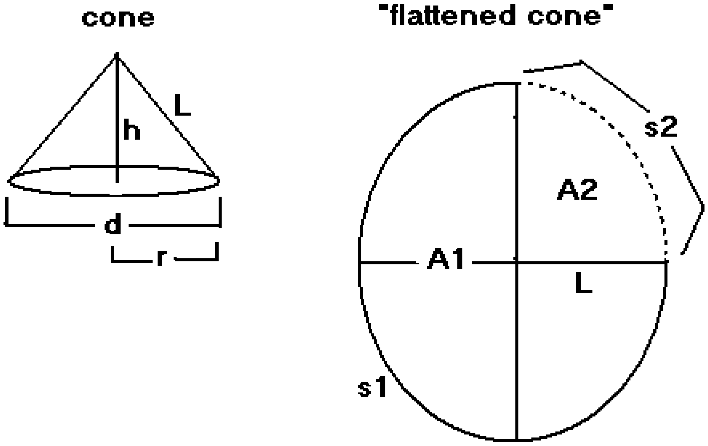
When a two-dimensional right triangle is rotated about one of its "leg" sides, a three-dimensional cone shape is created. For the surface area of a (right) cone shape:

A = surface area of the (lateral or side, "pointy part" of the) cone shape + area of the base or flat side of the cone

The length or height of the cone, and the radius of the base are the major factors for both the surface area and volume of a cone. The length of the lateral side of the cone can be found from the Pythagorean Theorem:

lateral side or "slant" length of a cone = L = $\sqrt{r^2 + h^2}$: r = radius of base of cone , h = height of cone
r = (diameter of base of cone) / 2
L is essentially a hypotenuse length

If you were to cut the side straight, unfold, expand, "roll-out" or "flatten" a hollow cone shape or surface area, it would look like a circle with a triangular-like portion or sector of it missing as shown below as sector A2: [FIG 212]



The area of the circle segment that is actually there is equal to the area of the lateral surface area of the cone. The "slant" length (s or L) is equal to the radius of this created circle, and is generally not equal to the radius of the base of the cone. The length of the circumference that is actually there is equal to the circumference of the base of the cone. The length of the circumference (Ct) of the entire possible circumference including the portion of the circumference of the missing sector, is equal to:

Ct = 2(pi) r = 2(pi)L = s1 + s2 , s1 = arc length 1 and it corresponds to the circumference of the base of the cone. s2 is the arc length of the circles segment area not present.

At = (pi)r ^2 = (pi)L^2 = A1 + A2 : At = total area of this circle created from the "flattened" cone.
: A1 = corresponding area of the lateral surface of the cone
: A2 = sector area created that was not part of the cones lateral surface area.

The figure in the particular example above clearly shows that a quarter (here A2) of the circle is not present, and it will help verify that the ratio or fraction of the arc length (s2) to the circumference (C = St, total sector length in a circle) of the circle, is equal to the ratio of the corresponding sector area (A2) to that of the total area of the circle (Ac = At = A1 + A2). Clearly, s2 is a quarter ("one-fourth" = 1/4) of the circle's circumference (Ct), and A2 = As = sector area is a quarter of the circle's area (At). The angle the arc length suspends or corresponds to is also 1/4 of the total angle of a circle which is 360° Expressing this, we have:

s2 = C / 4 = 0.25C and A2 = Ac / 4 = 0.25At and C / s2 = 1/4 = 0.25 and sector ¢ = **¢s** = 360° (C / s)

$S2 / C = 0.25$ and $A2 / A c = 0.25$ and $\frac{s\text{¢}^\circ}{360^\circ} = \frac{s}{C} = \frac{As}{Ac}$ **: equivalent fractions of a circle when considering sectors of it**

* Extra: $s2 / A2 = 0.25Ct / 0.25At = C / At$: the ratio of the arc length of a (area) sector, to the corresponding area of that sector, is equal to the same ratio of the circumference and corresponding total area of the circle. The circumference of a circle is also the largest arc length of a circle. Technically these types of fractions are incorrect due to that the units are different in the numerator and denominator, and therefore are not a proper fraction in terms of units, however it is algebraically correct. **This ratio is also not a constant:**
$C / A = 2 (\pi)r / (\pi) r^2 = 2 / r = 1 / 0.5r =$ and therefore, this type of ratio value depends on the value of the radius (r). As the value of (r) increases, this ratio then decreases in value.

When the ratio of s1 and A1 are also considered:

$\frac{s2}{A2} = \frac{s1}{A1}$ mathematically: $\frac{s1}{s2} = \frac{A1}{A2}$: the ratio of are lengths of a sector of a circle is equal to the corresponding ratio of their sector areas

$A1 = (s1 / s2) A2$: sector area A1, this is the cones lateral (or "side") surface area.
$A2 = (s2 / s1) A1$: sector area A2

Expressing the above (*) into a more generalized formula:

From: $s2 / A2 = C / At$: At = total area of a circle = Ac, mathematically:
: C = circumference can be thought of as total arc length in a circle = St

$$\frac{\textbf{arc length of circle segment}}{\textbf{circumference of circle}} = \frac{\textbf{area of circle segment}}{\textbf{area of the circle}} = \frac{s2}{C} = \frac{A2}{At} = \frac{\textbf{sector angle}^\circ}{360^\circ}$$

We find that the ratios of arc lengths (s) is equal to the ratio of their corresponding sector areas (As). Arc lengths and sector areas are proportional. From these equations, other helpful equations can be derived:

$s2/A2 = C/At$ and: $As / Ac = \text{¢}s / 360^\circ$: As = sector area , Ac = At , ¢s = sector angle
$As = Ac (\text{¢}s / 360^\circ)$

$A2 = (s2/C) At$: A2 is a sector area of a circle , and in general for any circle where Ac = At :

sector area = $As = (s / C) Ac$: As = sector area and s = arc length of the corresponding sector area,
Ac = total area of the circle , C = circumference or arc length of the entire circle.
Extra: **$As = Ac (s\text{¢} / 360^\circ)$**

or=: $As = s (Ac / C)$, substituting the known formulas for these variables:

sector area = As $= s (((\pi)r^2) / ((2)(\pi)(r))) = s (r / 2) = s$ **$r / 2$, $s = 2 As / r$**

sector area $= \frac{s\, r}{2}$: **SECTOR AREA OF A CIRCLE** , with units of a square unit of the length units used.
(r) = radius of the circle , s = S = arc length of the sector area
Another derivation here that is easy to remember is: **$s\, r = 2As$**

As a check on this formula, if the arc length (s) is equal to the circumference C):

sector area = $\frac{C\, r}{2} = \frac{2(\pi)\, r\, r}{2} = (\pi) r^2$: as expected since the corresponding sector area is that of the entire circle

When using radian angles: s = ¢r , then: **sector area** = s r / 2 = (¢r) r / 2 = **¢r^ / 2** **: ¢ in radians**

Continuing onward with the area of a cone discussion:

The radius (R) of this circle created from the "flattened" cone is equal to the slant length (L) of that cone:

R = L = $\sqrt{\text{(height of cone)^2 + (radius of cone base)^2}}$: **length of the slant length of a cone**

The circumference of this complete circle is equal to: C = 2(pi)R = 2(pi)L

arc length s1 = circumference of the base of the cone = 2(pi)(r)
arc length s2 = circumference of the complete circle - s1 = 2(pi)L - 2(pi)r = 2(pi) (L - r)

The total area of this circle is equal to: At = (pi)R^2 = (pi)L^2

The total area of this circle minus the segment portion missing is equal to the lateral surface area of the cone:

Acone = (Acircle - Asegment missing)

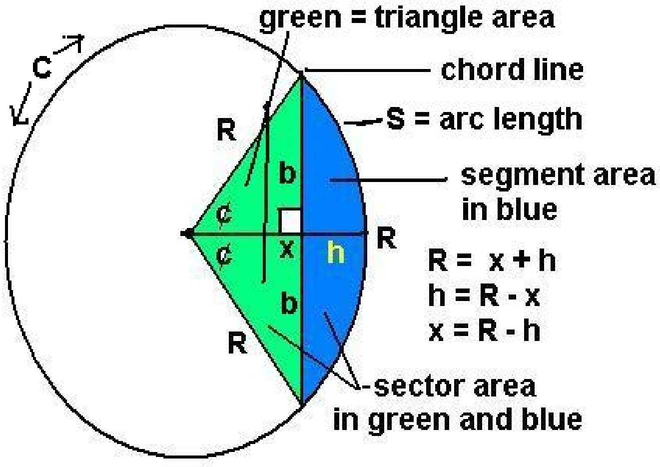
By now knowing the concepts of equal ratios for segment areas and arc-lengths, and observing the above figure:

$\frac{s1}{A1} = \frac{Ct}{At}$: A1 = lateral or side surface area of the cone, and mathematically:

$\frac{s1}{Ct} = \frac{A1}{At}$, mathematically:

A1 = $\left(\frac{s1}{Ct}\right)$ At = $\frac{2(pi)r}{2(pi)L}$ (pi)L^2 = **(pi)(r)(L)** **:(Lateral) SURFACE AREA OF A CONE**
L="slant length", r = radius of the base circle.
The base area is: (pi)(r^2)

Extra: A **segment area of a circle** is the area between the chord and the circumference arc of that circle. The area of this circle segment, or segment area part of the circle is the sector area defined by the chords endpoints on the circle, and the inner equal-lateral triangle created with its vertex at the center of the circle and whose base side is that chord. The segment area is a smaller part of the total sector area minus the area of the isosceles triangle which is basically two identical right triangles connected . [FIG 212B]



sin ¢ = opp. / hyp = b / R , b = R sin ¢
2b = chord length = 2R sin ¢

tan ¢ = opp. / adj. = b / x
x = R cos ¢ = b / tan ¢

Atriangle = (base)(height) / 2 = (2b)(x) / 2 = bx
= (chord length) (x) / 2

As/Ac = ¢s/360° , As = Sector area = (Ac) (¢/360) =
As = (pi) R^2 (¢/360°)

Sector Area = Triangle Area + Segment Area

S / C = ¢ / 360 , S = C (¢ / 360) = (2)(pi)(R) (¢ / 360)

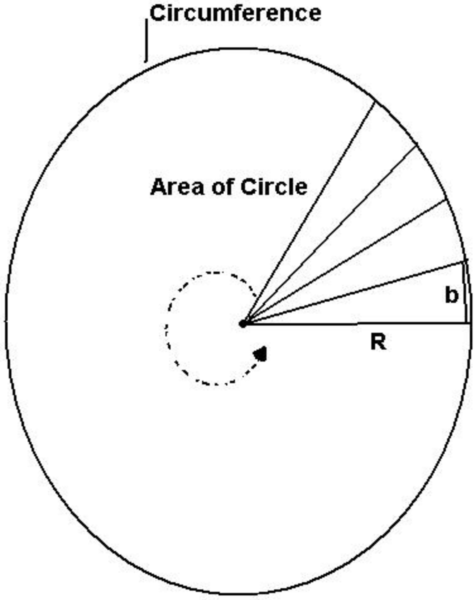
With respect to the central angle, S=arc length and sector area are linear, however, segment area and the isosceles

triangle area are not even though these two values always sum up to the sector area. From the above equations:
Segment Area = Sector Area - Triangle Area

TRIANGLE SEGMENT METHOD TO DERIVE FORMULAS FOR CIRCUMFERENCE, PI AND AREA OF A CIRCLE

Since all circles are similar, and-or are essentially just magnifications of each other, this analysis considers any circle of any size, and the value of (Pi) will always be the same result. (Pi) is therefore a constant for each circle.

Given a circle, you can divide its area into many triangular segment areas with each being the same area (At) and each having its vertex at the center of that circle, and its base (b) along the circumference (C) of that circle. When the number (N) of triangles is large, the base (b) of that triangle gets smaller and approaches (nearly becomes equal to in value) that of a straight line segment, the height of the triangle approaches R, and the triangle shape and area approaches that of a right triangle (however more exactly, it is an isosceles triangle which can be considered as two right triangles connected - hence an automatic doubling of the right triangles considered), and the area between the chord line (b, the triangle base) and the corresponding arc on the circumference becomes negligible. The arc length will approach the value of (b).
(Pi) = C / D = C / (2R) for any circle. [FIG 212C]



$C = b1 + b2 + ... + bn$: the more triangles, the smaller (b) is, and the more accurate the sum equals C. Geometrically, (b) is a chord beneath the arc segment of the circle, and therefore, it is slightly shorter than the arc or curve segment since (b) is the straight-line, shortest distance. Since each base (b) is the same length, this can be expressed as:

$C = N b$: N = n , is the number of triangles considered , If N increases by a factor, (b) decreases by that same factor, and the formula for C remains consistent, and C gets more accurate.

For each triangle: tan ¢ = opp / adj , and if we let R=1=adjacent side, we have: tan ¢ = b / 1 , and b = tan ¢

The smaller the triangle area, the smaller (b) is, and to do this, the angle must be smaller. If the area is exactly half, then the base (b), or height of a triangle, is exactly half. If we let the angle be 1°:

b = tan 1° = 0.017455065

How many (b) values or terms are along the circumference will be considered into the sum?

N = 360° / (angle of each triangle segment) = 360° / 1° = 360 : **¢= 360° / N , where N and-or angle is an arbitrary amount, and the higher the better accuracy of the result.**

C = N b = 360 (tan 1°) = 360 (0.017455065) = 6.283823374 : *** , N = C / b**

Here, the calculated ratio of: C / R = 6.283823374 / 1 = 6.283823374 , since D = 2R , D = 2 (1) = 2
C / D = 6.283823374 / 2 = 3.14191167 : = **(Pi)** approximation, for all circles.
: Mathematically: **C=(Pi)D = (pi) 2R = 2 (pi) R**
Circumference of a circle

If the number of triangles was N=1000, (Pi) would approximate as: 3.141633996
If the number of triangles was N=10000, (Pi) would approximate as: 3.141593067
If the number of triangles was N=100000, (Pi) would approximate as: 3.141592658
Here : ¢ = 360 / N = 360 / 100000 = 0.000360° ,
b = tan 0.000360° = 0.00006283185307 = 6.283185307 (10^-5)
C = N b = 100000 (0.00006283185307) = 6.283185315 = C / R = C / 1 ,
C / D = Pi = C / 2R = 6.283185315 / 2 = 3.141592658
If the number of triangles was N=1000000, (Pi) would approximate as: 3.141592654 =~ **Pi ,** to 9 decimal places of precision and accuracy.

Using the expressions derived above, we can now derive the formula for the area of a circle:

Ac = (N) At : At = area of each similar sector areas or central triangle , N = number of similar triangles

Ac = N At = N (b h / 2) = N (b R) / 2 = (C / b) (b R) / 2 = ((2 Pi R) / b) ((b R) / 2) = (**Pi) R^2 : * , Area of a circle**

A METHOD TO REMEMBER PI AND EPSILON (e)

Below is a non-mathematical method to obtain the first ten digits of π = "Pi". To do so, you must first know that it is approximately equal to a value of 3.0 and that it is slightly higher in value, say someplace between 3.0 and 4.0, but closer to 3. The next closest integer value greater than 0 is 1, hence we now have:

3.1 , For the third digit, add the first two digits to produce 4:
3.14 , The next integer value greater than the digits of 14 is 15:
3.1415 , Summing the last digits of 14 and 15 we have 9:
3.14159 , Summing the first two digits of the 14 and 15 we have 2:
3.141592

These last two digits above could have also been found by summing 14 and 15, producing 29, and then reversing these digits to have 92.

So far, we have used the integers: 1, 2, 3, 4, and 5, but have not used 6 yet, and this is also the sum of the 1, 4, and 1 digits, or the next number after 5:

3.1415926 , The next two descending integers after 6 are 5 and then 4:
3.141592654 **π** = (pi) rounded to ten total digits (or 9 decimal places)

As a check, the sum of the first ten digits of π is 40. In this close approximation, there was no digit used here with a value of 7 or 8.

In a similar manner, below is a method to obtain epsilon (e) to 10 digits. First you must know that epsilon is approximately equal to 2.7 and is slightly larger in value. Given 2.70, the next higher integer value greater than 0 is 1:

2.71 , Summing the digits of 7 and 1, we have 8:
2.718 , The first and the fourth digits are 28:
2.71828 , Repeating the last four digits of 1828 we have:

2.718281828 : Epsilon (e) to ten digits (or 9 decimal places)

As a check, the sum of the first 10 digits of (e) is 47. The digit values used are: 1, 2, 7 an 8.

Here is a mnemonic (a shorthand or a way to memorized) where the number of letters for each word corresponds to a digit value of the number (e):

2. 7 1 8 2 8 1 8 2 8 4
"If natural e constant is pleasing, a circular Pi inspires math"

Some other options or choices: 8 : pleasing , tasteful, fruitful, delivers, flattens, embraces, inspires
4 : food, hope, good, love, math

A METHOD FOR CALCULATING POWERS AND ROOTS

The process shown below for finding integer roots is an iterative (repeat, repetitive, iteration, do again) process. You simply end the process when you have obtained enough accurate or correct digits of the true result.

A root of a number is a single and constant factor value, that when it is repeatedly multiplied by itself, it produces a power value equal to the given number of which the root is being found. The square or second root of 4 is 2 since (2)(2) = 2^2 = 4. The square root of 9 is 3 since (3)(3) = 3^2 = 9. The cube or third root of 8 is 2 since (2)(2)(2) = 2^3 = 8. The fourth root of 10000 = 10 since (10)(10(10)(10) = 10^4 = 10000.

The square root process below is based of the fact that if the square of R is equal to N, that is, if (R)(R) = R^2 = N , then:

R is the square root of N, and this is usually expressed as: $R = \sqrt{N}$

N = (R)(R) Since the factors of N are identical, R is the root of N.
Mathematically, after dividing both sides by R:

$R = \frac{N}{R}$: This implies a method so a to find the value of R.

For this repetitive process to find the root of N, we will let R1 be the initial guess or estimate of the square root of N, and let R2 be the next closer and calculated value of the actual root. Expressing this concept in a formula:

$R2 = \frac{N}{R1}$: a square root iteration formula

To begin to find $\sqrt{N}$, first guess or estimate (R1) the root. The estimate can actually be any value (except 0, of course), but the closer it is to the actual root, a less number of iterations (repetition) of this process are needed for a given precision (fineness, minuteness) and accuracy (correctness). For values between 1 and 10 you can use a value of 2 or 3 as the first estimate. For values between 0.1 and 1, you can use the same given number modified so that it has a value of 2 greater in the most significant digit (the tenths position). Actually, when the given number is around 0.8 to 1, the square root approaches the same value as the given number, hence you may not want to add in the recommended value. Consider that since a fraction of a fraction results in a lower value, the root of a fraction is actually a higher value, symbolically:

$$(\text{fraction})(\text{fraction}) = (\text{fraction}^2) = (\text{lesser fraction}) \quad \text{, therefore:} \quad \sqrt{(\text{lesser fraction})} = \text{fraction}$$

Ex. An estimate of $\sqrt{0.5}$ would be 0.7, the actual root is about 0.7071.

To verify this, consider expressing the radicand with scientific notation to have an idea of the square root.

Ex. $\sqrt{0.5} = \sqrt{(50)(10^{-2})} = \sqrt{50}\ \sqrt{10^{-2}}$

$$\sqrt{10^{-2}} = 10^{(-2/2)} = 10^{-1} = (1)(10^{-1}) = 0.1 \quad \text{or:}$$

$$\sqrt{10^{-2}} = \sqrt{(1)(10^{-2})} = \sqrt{\frac{1}{10^2}} = \frac{\sqrt{1}}{\sqrt{100}} = \frac{1}{10} \quad \text{or=} \quad 1(10^{-1}) = 0.1$$

checking: $(10^{-1})(10^{-1}) = (10^{-1})^2 = 10^{(-1(+)-1)} = 10^{(-1-1)} = 10^{-2}$

$\sqrt{0.5} = \sqrt{50}\ \sqrt{10^{-2}}$ And the estimate of the root is therefore:

$\sqrt{0.5} \simeq 7\ (10^{-1}) = 0.7$: checking, $(7)(7) = 49$, $49\ (10^{-2}) = 0.49$
or: $(0.7)(0.7) = 0.49$, (almost 0.5)

Ex. $\sqrt{0.00267} = \sqrt{26.7}\ \sqrt{10^{-4}}$

And the estimate of the root is therefore:

$\sqrt{0.00267}$ = about: $5\ (10^{-2}) = 0.05$: $(5)(5) = 25$ which is almost 26 , and
$(0.05)(0.05) = 0.0025$

A square root formula. Here are the iterative (repetitive) steps to find a square root:

1. Since N = R^2 = RR , a check of the root estimate can be made by dividing the radicand by an estimate of the root:

$$R2 = \frac{N}{R1}$$

: N = the number to find the square root of, hence it is the radicand. When R2 = R1 , you have found the exact value of the square root, and this process can stop

When the first or leading digits of R2 match the corresponding digits of R1 you have found correct or accurate significant digits of the root. Note that if you "go high" in the estimate value of (R1), then R2 will "be lower" than the actual root, or vice-versa, due to the inverse mathematical relationship between a quotient and divisor. Since R2 is not usually equal to R1, to get closer, your estimate (ie., R1) must be corrected and reused. Too send R1 and R2 into closer agreement, the average of these two values can be taken for the next estimate. Note also, that if you did not take their average, but just assign R1 equal to the value of R2, that the next result of the division will be the original value of R1 and the process will be endlessly "circular" (repetitive) without getting closer in value to the true square root.

2. Since root $= \frac{R + R}{2} = \frac{2R}{2} = R$

, a new closer estimate can now be made by taking the average of the previous estimate and check values:

$$R3 = \frac{R1 + R2}{2}$$

: new modified (calculated and adjusted) estimate

3. Now, R3 is a modified estimate which is closer to the actual root. Set R1 equal to R3 and repeat the iterative steps (1,2 and 3). This process of repeatedly reusing the value output of an equation or function as the new input for the same equation or function is called recursion.

Summarizing the above steps into a single square root approximation formula:

Since $R2 = \frac{N}{R1}$, using algebraic substitution of R2 in: $R3 = \frac{R1 + R2}{2}$, we have:

$$R3 = \frac{R1 + \frac{N}{R1}}{2}$$

:R1 is an approximation, and R3 is the next closer approximation. Summarizing this into a more general formula for iteration:

$$Rn+1 = \frac{Rn + \frac{N}{Rn}}{2}$$

Mathematically, this can also be expressed as::

$$Rn+1 = \frac{Rn^2 + N}{2Rn} = 0.5\left(\frac{Rn^2 + N}{Rn}\right)$$

: **SQUARE ROOT SUCCESSIVE APPROXIMATION (RECURSION = repetition and reusing) FORMULA**

Ex. Find $\sqrt{10}$

With an initial estimate of Rn = R1 = 3, and using the above formula:

Iteration	Result = Rn+1
1	3.166666666
2	3.162280702
3	3.16227766
4	3.16227766

: for computer programs, an iteration, cycle or repetition of a process is called a loop

For the fourth iteration, since there were no new changes in any of the digits, we can assume that this is the accurate or correct (at least for the limited number of digits, or precision shown) square root.

Using a similar method to finding square roots, cube and even higher integer roots can be found.

A description of the cube root successive approximation formula is: The coefficient of the first term in the numerator is 1 less than the indicated root (x), and the denominator of the second term in the numerator is also raised to an indicated power of 1 less than the indicated root. The entire numerator is divided by a value equal to the indicated root. Summarizing this into a general formula for iteration:

The (x) root of N is expressed as $R = \sqrt[x]{N}$, and R can be calculated using:

$$R_{n+1} = \frac{(x-1)R_n + \frac{N}{(R_n)^{(x-1)}}}{x}$$

: **A FORMULA FOR CALCULATING ANY INTEGER ROOT**
Here x = indicated root , such as 2, 3, etc.
The appendix section of this book contains a computer program that utilizes this formula to find any root or power.

Ex. For the 3rd, or cube root of N, use:

$$R_{n+1} = \frac{\frac{2R_n}{1} + \frac{N}{(R_n)^2}}{3}$$

Not only can this general formula be applied for solving integer roots, but it can just as easily be applied for solving for a root or power of any real number. The appendix section of this book contains a computer program using this method to solve for a root. Real is a general mathematical name given to all the integers and mixed number values which are values with a possible whole portion and-or a possible fractional ("decimal") portion whose digits terminate at some point, or where some or all of its digits repeat endlessly to be classified as a "rational" (real) number. Any rational (real) number can be mathematically expressed as the division of two integers. "Irrational" (real) numbers are values that are not rational, such as "pi" and roots where the decimal portion is non-ending and non-repeating. Real numbers or values are sometimes called "floating (adjustable, position) point" values in computer programming, where point essentially means the decimal point that separates the whole portion from the fractional portion that is less than 1.

Ex. Here is how to find the 2.7th root of N:

$$R_{n+1} = \frac{\frac{1.7R_n}{1} + \frac{N}{(R_n)^{1.7}}}{2.7}$$

: For this example: (x-1) = (2.7 - 1) = 1.7

However, as you can now see, you must raise Rn to a non-integer power. Computer languages and machines, or scientific calculators, can handle such computations easily. It appears almost impossible to calculate by hand, but there are several methods. Here is a simple method besides the others presented in this book:

Given N^(x+y) , this can be factored into:

N^(x+y) = N^x N^y

If (x) represents the whole portion of an exponent, and (y) represents the fractional portion of an exponent, we can solve for real or fractional powers using only integer roots and powers.

Ex. Find N^1.7

N^1.7 = N^1 N^0.7
N^1.7 = N^1 N^(7/10)
N^1.7 = N^1 10√ N^7 or= N^1 (10√ N)^7

Ex. Find 4^0.3. This example also shows a similar, but alternate method.

Let: X = N^0.a raising each side to the 10th power:
X^10 = (N^0.a)^10
X^10 = N^a

Hence to solve for X, the power value we are looking for, take the 10th root:

X = 10√ X^10

Since X^10 = N^a, this can be written as:

X = 10√ N^a = N^0.a

Solving the initial example:

X = 4^0.3 = 4^(3/10) = 10 √ 4^3 = 10√ 64 = 1.515716566

You would usually calculate integer powers by repeated multiplication or with the aid of a calculator. For calculating very high powers, rather than perform an enormous amount of repeated multiplication or using other computational methods, you can simply use the concepts of a "power to a power". Actually, many calculators will display an error message (usually as "E") if the power to be calculated is too high, and hence, you must resort to calculating it the old fashioned way "by hand" using pen and paper.

Here are some examples of factoring the exponent,that you may utilize:

N^2 = N^1 N^1
N^10 = (N^5)^2 : hence for example, to find the 10 power of a value, you can find the 5th power and then square it.
N^100 = (N^10)^10 = ((N^5)^2)^10
N^1000 = ((N^10)^10)^10

N^735 = N^(700 + 30 + 5) = N^700 N^30 N^5
N^735 = (N^7)^100 (N^3)^10 N^5
N^735 = ((N^7)^10)^10 (N^3)^10 N^5

As a calculating aid, many calculators are programmed so that by repeatedly pressing the enter (=) key, the last operation and operand will be applied to the current displayed value. The current displayed value will then act as if it's the first operand of the last operation entered. Integer powers can be easily found this way,and without the need for entering a factor (here the base of the power) repeatedly. For example, to calculate N^3, you would first enter N x N = to get N^2 (actually on many calculators you can also simply enter: N [x] [=], that is, first enter the number, then press the multiplication or "times" button, and then press the "equals" buttons to get the square of N, and this is of great aid when N has many awkward digits). Pressing the [enter] key or [=] again will yield N^3, pressing [enter] again will yield N^4, and finally pressing [enter] again will yield N^5 and so on.

Here is another note on solving for integer powers using repeated multiplication. The (portable or "hand-held") calculator is a limited precision device, that is, it is built to display only a specific number of maximum digits (usually 8 or 10). Any least significant values that cannot be entered or displayed are lost. However insignificant those digits are, it can eventually result in less accuracy in the displayed results. For example, if you have an 8-digit calculator, we know that multiplying two 8-digit operands usually yields a 16 or 17 digit product. Since the calculator can only enter, process and display 8 digits, the other 8 or 9 least significant digits are essentially lost and not displayed on the calculator. If this result is used "as is" for an operand of further operations, errors or differences from the actual or true values can possibly start to "show up" beginning in the least significant digits of the result(s) displayed. These errors can grow in value, and become a significant error or difference from the true value, when these values are used for operands of further operations. For most practical cases, where say being accurate to 3 to 6 decimal places or a "thousandth", this will not be of a concern. In general, when the calculator returns (ie., the result or output) a value of one of the function key operations such as for the square root of the operand, that value is usually accurate for all of the digits displayed except perhaps the last digit if it was rounded. The calculator generally does not know if any entered and-or reused is accurate or not, and essentially considers any entered and-or reused value as a user desired operand.

The same formula for roots can be applied for calculating any power of N. You might ask what root of N results in a value equal to a certain indicated power of N? The answer is easily determined if you express this question in a mathematical form as:

$$\sqrt[x]{N} = N^y$$

Solving for (x), the indicated root, by placing the radical into exponential form:

$N^{(1/x)} = N^y$ equating the exponents by observation (or by taking the log of both sides and dividing by (log N)):

$\frac{1}{x} = y$ solving for x:

$x = \frac{1}{y}$: we see that the index and exponent of N are reciprocals of each other

For example, to calculate the second power of N using the root formula shown, you would calculate the 1/2 = 0.5 root of N since $\sqrt[0.5]{N} = N^{1/0.5} = N^2$.

It is possible to calculate odd roots, such as cube roots, by using square roots. Here is a method:

$R = \sqrt[3]{N}$: R = cube root of N, therefore (raising each side to the 3rd power):

$R^3 = N$ after dividing each side by R:

$R^2 = \frac{N}{R}$ taking the square root of both sides:

$R = \sqrt{\frac{N}{R}}$ expressing this with successive approximation notation:

$$R_{n+1} = \sqrt{\frac{N}{R_n}}$$

: **A METHOD TO FIND CUBE ROOTS USING SQUARE ROOTS** (This works, but it is not the most efficient method, ie., it is "slower" meaning more iterations are usually needed to have a convergence to the true result).

In general, for (d) correct digits, repeat the formula (3d+1) times.

Though not obvious, it is incorrect to solve for R, the cube root, using the iteration formula below since it does not converge to any result:

From, if (R)(R)(R) = (R^1)(R^2) = R^3 = N , mathematically: R = N / R^2 and:

$$R_{n+1} = \frac{N}{(R_n)^2}$$

: Though algebraically correct, this is an incorrect iteration method, however, there is a computer method that can be used to solve for cube roots using this equation. The method shown can be adapted to solve many difficult equations. See A METHOD TO SOLVE DIFFICULT EQUATIONS further ahead in this book.

A GENERAL METHOD FOR FINDING ANY POWER

Previously, a method was shown to calculate a power of a base value where the exponent had only one digit in the fractional portion. Here is a similar method which is generalized to calculate any possible power. Though you may generally not use these types of methods to find a power or a root, the knowledge or "know-how" aspect could have potential use in various mathematical situations and understanding.

Consider this:

$N = X^{a.bcd}$ "factoring" the right hand side:

$N = X^{(a + 0.b + 0.0c + 0.00d)} = X^a \quad X^{0.b} \quad X^{0.0c} \quad X^{0.00d}$ OR:

$N = X^a \quad (X^{0.1})^b \quad (X^{0.01})^c \quad (X^{0.001})^d$

$N = X^a \quad X^{0.1b} \quad X^{0.01c} \quad X^{0.001d}$

$N = X^a \quad X^{(b/10)} \quad X^{(c/100)} \quad X^{(d/1000)}$ expressed in radical notation:

$N = X^a \quad (\sqrt[10]{X^b}) \quad (\sqrt[100]{X^c}) \quad (\sqrt[1000]{X^d})$ or for the purpose here:

$N = X^a \quad (\sqrt[10]{X})^b \quad (\sqrt[100]{X})^c \quad (\sqrt[1000]{X})^d$:GENERAL METHOD

$N = X^a \quad (\sqrt[10]{X})^b \quad (\sqrt[10]{\sqrt[10]{X}})^c \quad (\sqrt[10]{\sqrt[10]{\sqrt[10]{X}}})^d$: a working method

The best way to solve the last equation is to first find the 10th root of X and then you can reuse this result to find the 100th root of X and so on since:

$$\sqrt[10]{\sqrt[10]{X}} = \sqrt[100]{X}$$

$$\sqrt[10]{\sqrt[100]{X}} = \sqrt[1000]{X}$$

If the exponent is negative, the simplest method to evaluate the power is to first find the power with an equivalent positive exponent and then take the reciprocal of this power since:

$N^{-x} = \frac{1}{N^x}$, also taking the square root of both sides: $\sqrt{N^{-x}} = \sqrt{\frac{1}{N^x}}$ or= $\frac{1}{\sqrt{N^x}}$

If the base of the indicated power is negative, things can get somewhat cloudy. For example:

$(-2)^2 = (-2)(-2) = +4$

$(-2)^3 = (-2)(-2)(-2) = -8$

So far so good, you can check these with a (scientific) calculator, and get the same results. By observing the above examples, it seems quite logical that if you make the exponent equal to a value anywhere between 2 and 3 then you should also be able to have an answer (the evaluated power of the base -2). After all, for example, $+2^2 = 4$, $+2^{2.5} = 5.6585...$, and $+2^3 = 8$. Now take notice if the base is negative and the indicated power is a mixed number:

(-2)^2.5 = **HAS NO SOLUTION**, unless you consider the concept of **imaginary numbers**. (-2)^(-2.5) also has no solution. The reason is that the exponent of 2.5 contains a whole portion and a fractional portion. When the base value is negative, to be able to solve for the indicated power value, the exponent must be a whole number. If you try this expression on a (scientific) calculator, you will get an error message. Here is a verification why:

First: Only odd roots, not even roots, can be taken for negative values. The root will also be negative in sign.
Ex. $\sqrt[3]{-8} = -2$:since (-2) (-2) (-2) = (-2)^3 = -8

Let's find out just what an imaginary number is. Even indicated (with the index) roots of negative numbers are formally known as imaginary numbers. First consider that when the square root of a number is multiplied by itself, it will be equal to that number or radicand.

root = $\sqrt[2]{\text{radicand}}$

root x root x . . . = root^2 = radicand

Ex. $\sqrt{100}\ \sqrt{100} = 100$ or= ($\sqrt{100}$)^2 = (100^1/2)^2 = 100^(2/2) = 100^1 = 100

checking: (10) (10) = 100

Ex. $\sqrt{-4}\ \sqrt{-4} = -4$: has the form of: root x root = (root of radicand) x (root of radicand) = radicand

Though the above should be true, the square root of -4 itself cannot be evaluated in the standard or normal way, so therefore, it can only be indicated or represented with an expression since any number or root, either positive or negative in sign, multiplied by itself is always positive in sign according to the basic rules for signed numbers. Factoring the radicand in the above example we find:

$\sqrt{-4} = \sqrt{(-1)(4)} = \sqrt{-1}\ \sqrt{4}$ simplifying further, we find this equal to:

$2\sqrt{-1}$: note also that $\sqrt{-4}\ \sqrt{-4} = -4 = 2\sqrt{-1}\quad 2\sqrt{-1}$: checking

$= (2)\ (2)\ \sqrt{-1}\ \sqrt{-1}$

$= 4\ (-1) = -4$: checks with the given radicand

The square root of -1 is called the "**imaginary unit**" since its rationalized and specific numeric value can only be imagined to exist, and in the following example, we see that there are 2 imaginary units being summed or combined:

$\sqrt{-1} + \sqrt{-1}$ each value can be expressed as having a numerical coefficient of 1:

$1\sqrt{-1} + 1\sqrt{-1}$: here, showing the understood numerical coefficients of 1. Adding like radicals, values or variables, we combine or add their numerical coefficients and keep that same value:

$2\sqrt{-1}$

You can imagine this value as a distance of 2 units along or into an imaginary zone or imaginary number line or axis that is perpendicular to the normal or real number line that we are all familiar with. We know the effect of multiplying a number by -1 is essentially a 180° shift or rotation of its position on the number line. This produces the negative, number line inverse or counterpart of a number. For example, (+5)(-1) = -5, and (-5)(-1) = +5. The effect of multiplying a number by $\sqrt{-1}$ is that only half of that 180°, or= a 90° degree rotation, or "half-way to a sign change" on this **complex** (both real and imaginary) number plane. If you were to multiply a value by $\sqrt{-1}\ \sqrt{-1}$, it would result in a 180° rotation since the square root of a number times the square root of that same number is equal to that number: $\sqrt{-1}\ \sqrt{-1} = -1$, and that is 180°

from +1 on the real number line. The square root of -1 is called the **imaginary unit** (which we can imagine as equal to some constant value) and is often given the identifying symbol of: (**i** or j) so that you can work with **i**maginary numbers in a manner similar to that with working with typical algebraic variables. Therefore:

$2\sqrt{-1} = 2i$: sometimes the letter j is used instead of i, for example $2i = 2j$

What is the connection between real numbers and imaginary numbers? As already shown above, a real number can be a coefficient of an imaginary number. Another important fact is that the product of two imaginary numbers is a real number. First consider the square roots of -1:

$\sqrt{-1}\ \sqrt{-1} = -1$ or:

$i \quad i = i^2 = -1$: -1 is a real number. Negative numbers are 180° from the positive side of the real axis, and this could be thought of as being in the opposite direction. Imaginary numbers on the imaginary number line could be thought of as being in a sideways (90°, right angle) direction from the real number line.

Ex. $(2i)(5i) = (2)(5)(i^2) = 10(-1) = -10$: multiplying by i^2 is the same as multiplying by (-1)

Also note: $i^3 = i^2\, i^1 = -1i = -i$: Numbers 180° from positive side of imaginary axis. This is equivalent to a 270° rotation of a number from the positive real number axis.

$i^4 = (i^2)^2 = (i^2)(i^2) = (-1)(-1) = +1$: Numbers on the positive side of the real axis.

A **complex number** is a number that is part real and part imaginary. You might have already encountered a root of a quadratic equation that is a complex number. The pseudo and formal algebraic forms of a complex number are:

$(\text{real} + \text{imaginary}) = (a + bi)$: Basic form of a complex number. (a) represents the real part, and (b) is a real number numerical coefficient of the entire imaginary number part. We see that a complex number is expressed as the sum or union of its real and imaginary part,

Ex. $5 + 2i$: 5 is the real number, and 2i is the imaginary number. Commonly said as 5 is the real part, and 2 is the imaginary part.

The sum or product of a complex number and its corresponding conjugate (ie., with a "sign change") complex number is a real number.

Ex. $(5 + 2i) + (5 - 2i)$ removing parentheses or grouping symbols:
$5 + 2i + 5 - 2i$ after combining like terms:
$5 + 5$
10

Ex. $(3 + 5i)(3 - 5i)$: the product of a complex number and its conjugate. Distributing:
$(3)(3) + (3)(-5i) + (5i)(3) + (5i)(-5i)$
$3^2 + (-15i) + (15i) + (-25i^2)$ since $i^2 = -1$:
$3^2 - 15i + 15i + (-25(-1))$ combining like terms:

9 + 25 : the result is a binomial (two terms) of the squares of the numerical parts of the complex number.
34 Special thanks to Mr. Barry Gale from Australia for improving this here.

Now we will continue with the discussion on finding any power.

Ex. $\sqrt[2]{-8}$ = CANNOT BE SOLVED normally unless the imaginary solution is considered since there is no (root) value (positive or negative) that can be multiplied by itself any number of even times (such as 2 indicated here) that can ever be negative in sign. Essentially, even indexed roots for negative radicands (numbers) are not allowed.

Ex. $\sqrt[10]{-23}$ = CANNOT BE SOLVED normally, as indicated above, since the radicand is negative and the index (the indicated root to be taken) is even.

Given (-2)^2.5 , which as indicated above, cannot be solved normally since 2.5 is not an integer. It can be solved if an imaginary solution is acceptable:

Factoring (-2):

(-2)^2.5 = (-2)^(2 + 0.5) = (-2)^2 (-2)^0.5 : Now take note of this second factor just shown :

(-2)^0.5 = (-2)^(5/10) = (-2)^(1/2) = $\sqrt{-2} = \sqrt{2}\ \sqrt{-1} = \sqrt{2\ i}$: again, an even root of a negative radicand, or :

(-2)^2.5 = (-2)^2 (-2)^0.5 = (-2)^2 ((-2)^0.1)^5 = (-2)^2 ((-2)^(1/10))^5

(-2)^2.5 = 4 ($\sqrt[10]{-2}$)^5 : There is no real number solution to any indicated even root (here, 10th) of any negative value number (here, -2).

The basic fact from the above discussions is that the square root (or any other even root) of -1 has no real solution, and is rather an imagined or imaginary solution. Consider the basic fact that any real or physical length can only be positive in value, and that a negative length can then only be imagined, hence an imaginary length having an imaginary value. Once signed numbers and negative (or reverse) directions are considered, then it is possible to describe negative directions and-or lengths, but still, on a fundamental, real or physical level, there is no such thing as a negative length.

Also shown in the above discussion is that N^x can not be solved when N is negative and x is not an integer.

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A METHOD TO FIND A LINEAR (LINE) EQUATION GIVEN A SET OF POINTS

There are many methods to find a (representative) linear equation given a set of points or data that appear to be linear and could be graphed using a single (representative or approximate) line . A simple method is given below that will provide very reasonable results. This method is what can be called an "upper and lower average method" and-or the "**average line method**". Many "scientific" calculators have some statistical (the key is often labeled as "stat") functions such as averaging a series of input data values.

If there is an excessive amount of data values or points to consider, then simply choose a manageable and representative sample amount to use, and in particular, discard any extreme values that appear as very high or low as compared to most of the data values since these "extreme [too high,or too low] and-or erroneous values" can affect the accuracy and effectiveness of the entire representative line being created. Sometimes it may help to visually plot the points so as to determine if there area any "extreme" values.

Another quick test to determine if there is a linear or linear-like relationship is to take some of the data values (to be plotted or graphed as points) and find a rough or basic slope (m) value by dividing a few dependent (y) values by its corresponding independent (x) value of that point, and then using this (roughly constant) value for comparing and determining if a data value or point is "out of (the linear or line) range", possibly due to some anomaly or measurement error. These "extreme values" can then be discarded and-or eliminated from this line analysis.

1. Given a set of acceptable points (that are graphed and appear to be linear in nature), find the average of the x coordinates of all the points, which can be expressed as x with a short line over it, and then find the average of all the y coordinates of all the points. An **average value** of a set of values can be found by summing up all those values and then dividing by the total number of the values used or included in that sum.

   $Pa = (\bar{x} , \bar{y})$: main average point, at the average x value and average y value of the given points, the more the better the line equation will represent and-or predict the data.

   On a point plotting graph and-or coordinate (location) system of points, you may wish to draw a horizontal and vertical line at this point (Pa). All points to the right of Pa will have greater x values and will be used to find the upper-x average value. All points to the left of Pa will have lower (x) values and will be used to find the lower-x average value. This same reasoning will be used to find the upper and lower y average values.

2. Given the set of all points whose coordinates are less than the average, find the lower-x and lower-y average point:

   $Pl = (\overline{x_l} , \overline{y_l})$: average point of points lower than the main average

3. Given the set of all the points whose coordinates are greater than the average, find the upper-x and the upper-y average point:

   $Pu = (\overline{x_u} , \overline{y_u})$: average point of points greater than the main average

We now have three points on the line: Pa , Pl , Pu These points can be connected to display the **average line**. To make an equation of this line so as to plot any general point on that line given an (x) value, we need to write the equation for that specific line, and we will need to find the slope value (m) of that line, and the y-axis intercept point: Pi (0 , b)

4. With the upper and lower average points, these two points can be used to determine the slope (m) of the corresponding line of which those points are on:

$$m = \frac{\overline{yu} - \overline{yl}}{\overline{xu} - \overline{xl}}$$: slope of the line between the upper and lower average points

5. (b), the y-axis intercept where x=0, can now be found knowing the basic equation of a line. This will also give us another point on that so as to draw that line using a ruler and pen:

From: $y = mx + b$ mathematically solving for (b):

$b = y - mx$: b = the "y-axis" intercept of the line and where x=0 , p(x=0 , b) = p(0 , b)

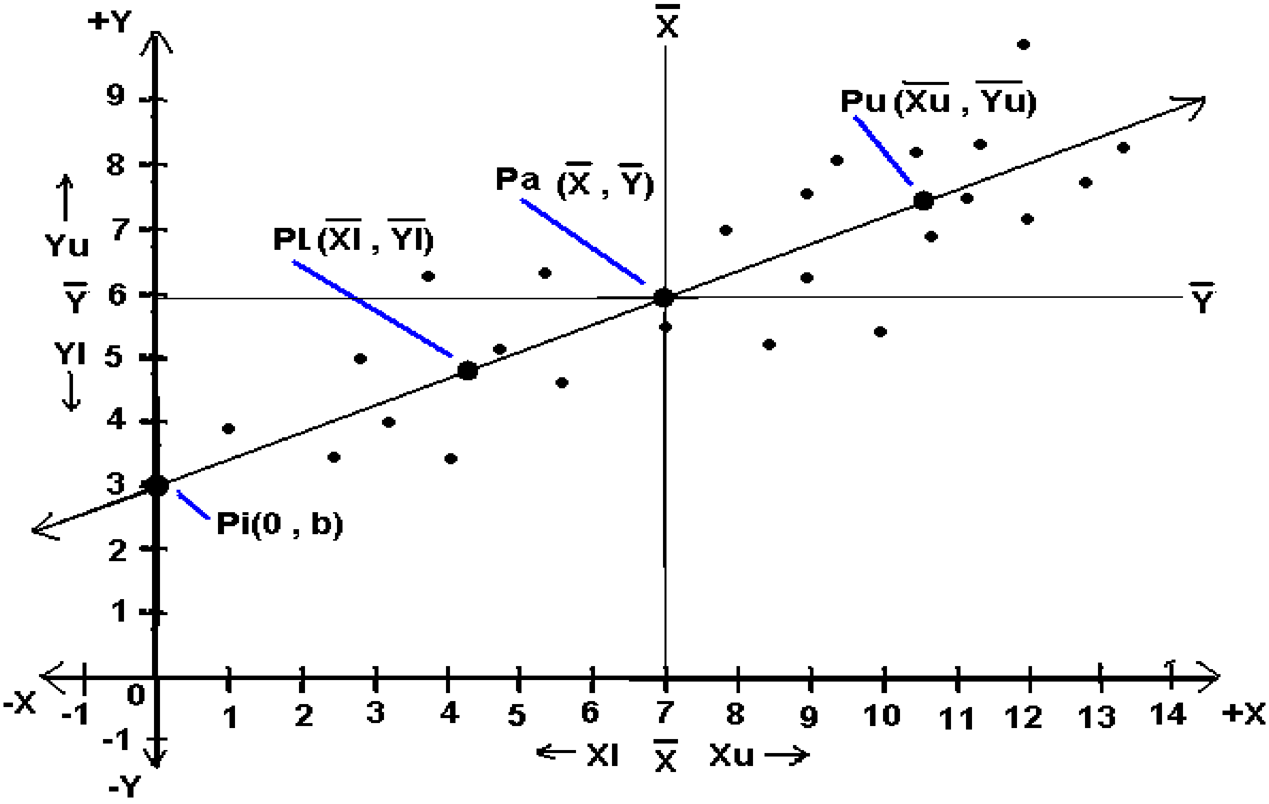
Clearly, we need the corresponding values of (x) and (y), and the slope (m) so as to solve for (b) of that line.

To find a value for (b), we can substitute the coordinates of the average point (Pa) of the entire set for (x) and (y):

$b = \overline{y} - m\overline{x}$: b = the "y-axis" intercept of the line and where x=0

6. The equation of a representative line for the data is: $y = mx + b$: where (m) and (b) are constants for the particular line

You can now plot all the given and new points from both the data, and line (from the created linear equation) on a graph and see how close this line is to the actual data, and vice-versa. This line which represents the known data values can now be used for predicting other data values and-or for comparison to other similar lines. If the data and-or points change due to some reason, perhaps the points are now having a lesser or greater (y) value on average, the line needs to be recalculated so as it can properly and-or truly represent that data. [FIG 213]



A DERIVATION OF SIN ¢ SERIES

In order to give you validity and confidence in using the SIN ¢ and other similar series, the fundamental steps of its' derivation are presented below. It is recommended to first view the previous topic of: Understanding Basic Calculus.

In theory, many functions (the mathematical relationship of variables) can be expressed as a series of terms. For a very simple example, consider this:

$y = f(x) = 5x$ Expressing the function 5x using repeated addition which is a series of terms:
$y = f(x) = 1x + 1x + 1x + 1x + 1x$: You can solve for the value of $y = 5x$ by using this sum of terms.
Expressing this series of terms using "summing notation":

$$y = f(x) = \overset{5}{\underset{1}{S}} \; 1x$$

: sum the terms, where each is term is 1x, and for a total of 5 terms of these terms

Here is the general format for a **Power Series** representation of a function:

$$y = \text{a function of } x = f(x) = \sum_{n=0}^{n=\infty} Cn\, x^n$$

: General format for terms of a power series.
C or c are constants that are specific to the function.
Sometimes "inf." is used in place of the infinity symbol: ∞ ,
n is a variable, here starting at 0, and is to be incremented by 1 for each and every successive term. For Cn, n is a subscript corresponding to the term number less 1. For example: c0 = "constant zero", which is actually the first constant in the series. c1 = "constant 1".

Expanding some of the terms of the general power-series format shown above:

$$y = f(x) = c0\, x^0 + c1\, x^1 + c2\, x^2 + c3\, x^3 + \ldots$$

: Power Series (in, or of variable x) general format. $c0\, x^0 = c0\,(1) = c0$

Here is a simple verification of the power series, however, the terms used are finite (not infinite) in number since you should expect a definite result for an algebraic function (as opposed to the non-algebraic functions such as the trigonometric functions):

$$y = f(x) = (x + 2)^3$$

$$y = f(x) = (x + 2)^3 = (x + 2)^2\,(x + 2)^1 = (x^2 + 4x + 4)\,(x + 2) = x^3 + 6x^2 + 12x^1 + 8$$

Arranging these terms into an ascending series of expressed or indicated power of x:

$$y = f(x) = 8 + 12x^1 + 6x^2 + x^3$$

: note the ascending exponents or powers of the variable

Note that the first constant (c0) can be evaluated by setting the value of the variable (x) to 0, effectively eliminating all the terms containing that variable (x), after all, 0 is often a possible value for any variable in a function. In the above expression, $C_0 = 8$. In mathematical notation:

$c0 = f(0)$: f(0) = evaluate the function after setting the value of the variable (here, x) to 0.
This value is also the "y-axis intercept" of the curve of the equation, and where the corresponding value of (x) is 0. y intercept point = point(x, y) = p (0 , 8).

By taking the derivative (term by term, of the terms that represent an equivalent [equal] in value sum expression of that same function) of the formal or general Power-Series expression above, and its resulting derivatives, we can find a formula for the constants (cn) of the general Power-Series. For example, the first derivative of:

$y = f(x) = c0\,x^0 + c1\,x^1 + c2\,x^2 + c3\,x^3 + \ldots$ is:

$\frac{dy}{dx} = y' = f'(x) = c1 + 2c2x^1 + 3c3x^2 + \ldots$: a derivative of a sum of terms is equal to the sum of the derivatives of each term. The derivative of the term: $c0\,x^0$ is 0.

Now, the (logical) second constant (here, c1) can be evaluated by setting the value of the variable (x) to 0 in this derivative:

$c1 = f'(0)$: c1 is the numerical coefficient of x^1

Taking the derivative of the first derivative will give us the second derivative of the original function:

$y'' = f''(x) = (1)(2)c2 + (2)(3)c3x1 + \ldots$

Now, the third constant (c2) can be solved for by setting the value of the variable (x) to 0 in this derivative of the previous derivative:

$f''(0) = (1)(2)c2 = (2!)\,c2$ hence mathematically: $c2 = \frac{f''(0)}{2!}$

We see that the pattern and general format for the constant term in each derivative is:

$cn = \frac{f^n(0)}{n!}$: where f^n is the notation for the nth derivative of the function, and not the nth power of the function. Some other mathematical notations for the third derivative, may be expressed as: $y''' = f''' = f^3 = f'3$

By placing the above formula for the constants into the general Power-Series, we have:

$$y = f(x) = \sum_{n=0}^{n=\infty} \frac{f^n(0)}{n!}\,x^n$$

: The Power Series expressed this way is called a (Colin) **Maclaurin Series**. **Brook Taylor** made the initial progress for these types of Taylor series expansions of functions in about 1715. Note that "derivative 0" = f^0 which is the function itself, and 0**!** is defined to be equal to 1**!** = 1 A Maclaurin series is a successive approximation series or expansion of a function, and where x is set to 0 when using a Taylor series expansion of that same function. Maclaurin created his method and series in about 1730.

Expanding some initial terms:

$$y = f(x) = f(0) + \frac{f^1(0)\,x^1}{1!} + \frac{f^2(0)\,x^2}{2!} + \frac{f^3(0)\,x^3}{3!} + \frac{f^4(0)\,x^4}{4!} + \ldots$$

Notice that in each term, the indicated derivative number, the indicated (exponent) power of (x), and the indicated factorial have the same basic numerical value equal to (n).

Let's make a check on this series using the basic formats of the linear and quadratic equations.

y = f(x) = ax + b : basic linear equation and function

y(0) = f(0) = a(0) + b = b : evaluating the function with (x) set equal to 0, and this is actually the y-axis intercept where x=0. point(x , y) = p(0 , b)

The first derivative is:

y' = f'(x) = (1)ax^0 = a(1) = a

The second and all further derivatives are 0, that is, the derivative of any constant (such as (a), or even 0) is equal to 0. If you consider the line y=a, where (a) is a constant, it is a horizontal line which has no slope, hence 0. After substituting this into the general Maclaurin Series:

y = f(x) = b + ax^1 + 0 + 0 . . .

y = f(x) = b + ax : This is actually the same original function except that the terms are rearranged.

Now consider the basic form of a quadratic equation:

y = f(x) = ax^2 + bx + c : basic quadratic equation, or by rearranging terms for an ascending order of increasing powers of the variable (x):

y = f(x) = cx^0 + bx^1 + ax^2 : in relation to the general power series, we see that constant c0 = c, c1 = b and c2 = a

y(0) = f(0) = a(0^2) + b(0) + c = c

The first derivative of this quadratic equation is:

y' = f'(x) = 2ax^1 + b : this is a linear equation

The second derivative of the quadratic equation, and which is the derivative of this linear equation is:

y'' = 2ax^0 = 2a : this is a constant, and this is common for the derivative of linear or line equations.

The first derivative where x is set equal to 0 is:

y'(0) = f'(0) = 2a(0^1) + b = b : 0^1 = 0

The second derivative where x is set equal to 0 is:

y''(0) = f''(0) = 2a(0^0) = 2a(1) = 2a : Note that: 0^0 = 1

0^0 = 1, and to verify this, remember that any value raised to the 0 power is equal to 1. The value can be very huge or infinitely so small that it is practically considered as equal to 0. Be sure to also remember 0^1 = 0, that is, any value raised to the first (1) power is equal to that value. It does seem natural or intuitive to think that a value with a higher exponent should result in a greater value than the base value (here 0). 0^1 might be thought of as being greater than 0^0, but it is not. Consider that a number very close to 0, or infinitely small, raised to the 0 power is still equal to 1. For example: (0.000,000,000,001)^0 = 1. One point argued about this is "nothing (0) raised to the nothing (0), or no (0) power should still be equal that nothing (0)". But then with that same mindset, in division, a value that is divided by nothing (0) should still be equal to that undivided value since no (0) division has actually taken place. Formally, division by 0 is undefined (not defined, having no definite or specific result, hence it is considered an error to even try to divide by 0).

Substituting the above information into the general Maclaurin series:

$$y = f(x) = \frac{c}{0!} + \frac{bx^1}{1!} + \frac{2ax^2}{2!} + 0 + 0 + \dots$$: 0! is defined as equal to 1

$$y = f(x) = c + bx + ax^2$$: This is actually the same original function except that the terms are rearranged into ascending powers of (x).

Writing a (algebraic-like) Maclaurin Series (approximation) for the (non-algebraic) SIN x function:

(Note, the derivatives of the SIN and COS functions shown below are verified in the next discussion.)

f (x) = SIN x : the given function, here a trigonometric function of SIN x, its derivative is:
f1(x) = COS x : the first derivative of the SIN x function is equivalent in value of the COS x function
f2(x) = -SIN x : the second derivative of the SIN x function is equal to the derivative of the COS x function
f3(x) = -(COS x) = -COS x : the third derivative of the SIN x function

c0 = f(0) = SIN 0 = 0 : after evaluating the function: SIN x, at x=0
c1 = f1(0) = COS 0 = 1 : after evaluating the first derivative of SIN x, at x=0
c2 = f2(0) = - SIN 0 = 0 : after evaluating the second derivative of SIN x, at x=0
c3 = f3(0) = - COS 0 = -1 : after evaluating the third derivative of SIN x, at x=0
c4 = f4(0) = SIN 0 = 0 : after evaluating the fourth derivative of SIN x, at x=0

At the fourth derivative of SIN x, we see that it is the same as the original function given (SIN x) and therefore, the derivatives and constants (cn) are starting to repeat:

f4(x) = -(-SIN x) = SIN x
c4 = f4(0) = SIN 0 = 0 Placing these constants into the SIN x series:

$$\text{SIN } x = f(x) = 0 + 1x^1 + \frac{0x^2}{2!} + \frac{-1x^3}{3!} + \frac{0x^4}{4!} + \frac{1x^5}{5!} + \dots$$

Since any variable or value multiplied by 0 is 0 or "nothing", those terms, here which are the even powers of the variable x, are effectively eliminated from the sum of terms, and the expression reduces or simplifies to:

$$\text{SIN } x = x^1 - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots$$: x is a radian angle value

This, and the other trigonometric series shown below were first developed in the late 1600's and early 1700's by Newton (1642-1727) from England, Gottfried Leibniz (1646-1716) from Germany, Brook Taylor (1685-1731) from England, Colin Maclaurin (1698-1746) from Scotland, and James Gregory (1638-1675) from Scotland. **Madhava Of Sangamagrama** (1340-1425) was a mathematician and astronomer from India and had previously made some independent progress in series and calculations before Newton and Leibniz made their own independent and discoveries, proofs and the formalization (ie., set standards, awareness and usefulness) of calculus. Some modern texts indicate a joint discovery credit for some mathematical series to both the western discoverers and Madhava from the eastern culture of India. Before this time frame of discoveries and advancements, ancient Greek mathematicians such as **Archimedes** (287 BC - 212 BC), of Syracuse, Sicily-Italy, have found some solutions to geometric problems by using what could be described today as infinite sums and pre-calculus, and this must of inspired Newton. **Srinivasa Ramanujan** (1887-1920), a mathematician from India, made some surprising and useful mathematical algorithms, series and curiosities of which are being studied until this day so as to be applied and make some other advancements in mathematics. This book has a discussion and computer program about his amazing ellipse perimeter approximation formula(s).

COS ¢ SERIES

$$\text{COS } X = \sum_{n=0}^{n=\infty} (-1)^n \frac{X^{(2n)}}{(2n)!} \qquad : X \text{ is a radian angle value}$$

Expanding the terms:

$$\text{COS } X = \frac{1}{0!} - \frac{X^2}{2!} + \frac{X^4}{4!} - \frac{X^6}{6!} + \frac{X^8}{8!} - \frac{X^{10}}{10!} + \frac{X^{12}}{12!} - \ . \ . \ .$$

Simplifying the factorials in the denominators:

$$\text{COS } X = 1 - \frac{X^2}{2} + \frac{X^4}{24} - \frac{X^6}{720} + \frac{X^8}{40{,}320} - \frac{X^{10}}{3{,}628{,}800} + \frac{X^{12}}{479{,}001{,}600} - \ . \ . \ .$$

The results to be expected are similar to those of the SIN X series. When the angle decreases, the greater the accuracy achieved for a given number of terms. When X=1 (about 57°) you can expect an accuracy of 6 digits when using only 6 of the terms shown. Note that when the angle X is 0, that sin 0 = 0, and cos 0 = 1, and hence as to why the cos x series has the initial "1 term". The main difference between the sin and cos series is in the use of all odd or even powers of (x).

Notice the last term given for the series, if you only have an 8 digit calculator, you cannot enter the 9 digit denominator of this fraction. The method to overcome this limitation is to factor the denominator, resulting in numbers (factors) with less digits. For a quick method to simplify or process the resulting fraction, the numerator will be divided by one of the factors in the denominator, and then the resulting quotient will be divided by the other factor in the denominator. Perform the division as shown below:

$$\frac{X^{12}}{479{,}001{,}600} = \frac{X^{12}}{(10)(47{,}900{,}160)} = \frac{\ \frac{X^{12}}{47{,}900{,}160}\ }{\frac{10}{1}} \qquad \text{or:} \qquad \frac{\ \frac{X^{12}}{10}\ }{\frac{47{,}900{,}160}{1}}$$

You can also evaluate COS X from one of the other trigonometric identities previously shown, for example:

$$\text{COS } X = \frac{\text{SIN } X}{\text{TAN } X} = \sqrt{1 - \text{SIN}^2 X}$$

The series above for COS X is derived from the SIN X series since the derivative of SIN X and its equivalent series is the COS X series. Taking the derivative of the expansion of the SIN X series term by term will yield the COS X series:

$$\text{COS } x = \frac{d\,(\text{SIN } x)}{d\,x} = \frac{1\,x^{(1-1)}}{1} - \frac{3x^{(3-1)}}{3!} + \frac{5x^{(5-1)}}{5!} - \ . \ . \ .$$

$$\text{COS } x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ . \ . \ .$$

As previously indicated, taking the derivative of this series, term by term, will yield the negative of the SIN series:

$$\frac{d\,(\text{COS } x)}{d\,x} \;=\; -\,\text{SIN } x$$

A FORMULA FOR EVALUATING TAN ¢

Like the other trigonometric functions, there are several series and methods to evaluate TAN ¢. Below is a method that involves continued fractions which are basically "never-ending" compound fractions where each successive denominator is predictable via a formula. The advantage of using the continued fraction below is that many "levels" of it can be simplified to a single fraction, hence only one division operation needs to be performed. Showing this simplification process algebraically using a generic representative or symbolic continued fraction:

$$\cfrac{A}{B + \cfrac{C}{D + \cfrac{E}{F + \;.\;.\;.}}} \qquad \text{OR=:} \qquad \cfrac{A}{B + \cfrac{C}{D + \cfrac{E}{F + \;.\;.\;.}}}$$

To simplify continued fractions, begin by combining the fractions in the denominators, starting with the "lowest denominator" or fraction and working "upwards" to the first numerator:

$$D + \frac{E}{F} \;=\; \frac{FD + E}{F}$$

and the partially simplified continued fraction is:

$$\cfrac{A}{B + \cfrac{C}{\cfrac{FD + E}{F}}}$$

continuing the simplification process:

$$\cfrac{A}{B + \cfrac{CF}{FD + E}}$$

$$\cfrac{A}{\cfrac{BFD + BE + CF}{FD + E}}$$

$$\frac{AFD + AE}{BFD + BE + CF}$$

: a simplified form of several levels of the continued fraction shown above.
This simplification has only one numerator, and one denominator.

The continued fraction shown below for TAN ¢ was developed by a Swiss-German mathematician named **Johann Heinrich Lambert** (1728-1777) born in Switzerland, and it based upon **Euler's Continued Fraction Formula** made in 1748 and of which a (converging in value) compound fraction can sometimes be made to solve for certain values. These compound fractions are actually another form or notation of the more standard common series of terms used to find the same value, such as for e^x, log(1+x), sin x, cos x, arcsin x, arctan x, etc. Continued, compound and converging fractions can be an interesting mathematical study in itself.

Below, in the continue fraction, the angle (X) must be a radian angle since degree units for angles are rather arbitrary man-made values, whereas radian units are essentially defined naturally with the dimensions of any circle.

$$\text{TAN } X = \cfrac{X}{1 - \cfrac{X^2}{3 - \cfrac{X^2}{5 - \cfrac{X^2}{7 - \cfrac{X^2}{9 - \cfrac{X^2}{11 \; . \; . \; . \text{ (and so on)}}}}}}}$$

: TAN X continued fraction

The simplified form of all the 6 "tiers", "levels" or divisions shown is:

$$\text{TAN } X = \frac{10395X - 1260X^3 + 21X^5}{10395 - 4725X^2 + 210X^4 - X^6}$$

: **A FORMULA FOR EVALUATING TAN ¢**

With the expression above, you can expect almost 7 decimal places for an angle of 70° (about 1.2217 radians). Expect 9 decimal places for an angle of 57.3° (about 1 radian), and more for lower angles when using the same number of "tiers" or "levels" (ie. denominators or divisions).

Note for example that you can quickly calculate powers of a value (here, such as X) by "reusing" precalculated values:

$5^4 = (5^2)(5^2) = (25)(25) = 625$ then:
$5^5 = (5^4)(5^1) = (625)(5) = 3{,}125$

This can be algebraically expressed as: $X^n = (X^{(n-1)})(X^1) = X^{(n-1)}X$

For angles greater than 45°, in particular those closer to 90°, you can use the derivation below for improved accuracy.

From the trigonometric co-functions of angles A and B of a right triangle:

$$\text{TAN } A = \frac{\text{SIN } A}{\text{COS } A} = \frac{\text{COS } B}{\text{COS } A} = \frac{\text{COS } B}{\text{SIN } B} = \frac{\text{SIN } A}{\text{SIN } B} = \text{COT } B = \text{COT }(90° - A) = \frac{1}{\text{TAN } B} = \frac{1}{\text{TAN }(90° - A)}$$

, and $90° = 1.570796327$ radians $= \pi/2$ radians

$$\text{TAN } X = \frac{1}{\text{TAN }(\pi/2 - X)} = \frac{1}{\text{TAN }(1.570796327 - X)}$$

: X is a radian angle

Here is an identity and a short table which will allow you to calculate TAN X by evaluating the tangent of a smaller angle, in particular as shown below, an angle of 10° or less where you can expect an accuracy of 9 or more significant digits when using the simplified continued fraction shown above:

$$\text{TAN }(\text{¢}1 + \text{¢}2) = \frac{\text{TAN ¢}1 + \text{TAN ¢}2}{1 - (\text{TAN ¢}1)(\text{TAN ¢}2)}$$

: TANGENT OF THE SUM OF TWO ANGLES FORMULA

A small table is presented below to aid your calculations. The second column contains the radian equivalent of the degrees angle. The appendix contains a table with more values entered.

TAN degrees = TAN radians = TAN ¢

TAN 1° = TAN 0.017453292519943 = 0.0174550649282
TAN 2° = TAN 0.034906585039887 = 0.0349207694917
TAN 3° = TAN 0.05235987755983 = 0.0524077792830
TAN 4 ° = TAN 0.069813170079773 = 0.0699268119435
TAN 5° = TAN 0.087266462599716 = 0.0874886635259
TAN 6° = TAN 0.10471975511966 = 0.1051042352657
TAN 7° = TAN 0.1221730476396 = 0.1227845609029
TAN 8° = TAN 0.13962634015955 = 0.1405408347024
TAN 9° = TAN 0.15707963267949 = 0.1583844403245

TAN 10° = TAN 0.1745329251994 = 0.1763269807085
TAN 20° = TAN 0.3490658503989 = 0.3639702342662
TAN 30° = TAN 0.5235987755983 = 0.5773502691896
TAN 40° = TAN 0.6981317007977 = 0.8390996311773
TAN 50° = TAN 0.8726646259972 = 1.191753592594
TAN 60° = TAN 1.047197551197 = 1.732050807569
TAN 70° = TAN 1.221730476396 = 2.747477419455
TAN 80° = TAN 1.396263401595 = 5.671281819618

Ex. Evaluate TAN 74°

We can look-up (find) the value of TAN 70° from the above table, and then use the continued fraction to calculate the value of TAN 4°:

$$\text{TAN } 74^\circ = \text{TAN } (70^\circ + 4^\circ) = \frac{2.747477419 + \text{TAN } 4^\circ}{1 - (2.747477419)(\text{TAN } 4^\circ)}$$

: only one division needed, especially when done "by hand" without a calculator.

TAN 4° = TAN 0.06981317 = 0.069926811

$$\text{TAN } 74^\circ = \frac{2.747477419 + 0.069926811}{1 - (2.747477419)(0.069926811)}$$

TAN 74° = 3.487414444

When one of the angles used is 45°, the tangent of the sum of two angles formula reduces to simply :

$$\text{TAN } (45^\circ + ¢) = \frac{1 + \text{TAN } ¢}{1 - \text{TAN } ¢}$$

: special tangent rule for the sum of two angles

You can also evaluate TAN X indirectly as indicated in the above trigonometric identities for TAN X, from one of the other trigonometric identities, and vice-versa. For example:

$$\text{TAN X} = \frac{\text{SIN X}}{\text{COS X}}$$

A DERIVATION OF THE ARCSIN SERIES

To give you validity and confidence using the ARCSIN X and other similar series, the fundamental steps of the derivation of the ACRSIN X series are presented below.

First, the Binomial Series is a series expansion for expressions of the form: $(x+1)^a$, or its equivalent: $(1+x)^a$. We see that this expression is a power of a binomial or two terms expression. The Binomial Series can be derived from a Maclaurin Series expansion of $(1+x)^a$, or from the Binomial Formula that is discussed ahead. Remember, binomial means two terms, and for the binomial series below, one term must be the value of 1. The binomial series will be used further ahead in this derivation of the series for ARCSIN X.

Binomial Series general format:

$$(1 + x)^a = 1 + \frac{ax^1}{1!} + \frac{a(a-1)x^2}{2!} + \frac{a(a-1)(a-2)x^3}{3!} + \frac{a(a-1)(a-2)(a-3)x^4}{4!} + \ldots \qquad : \textbf{Binomial Series}$$

Here is a simple example to verify the Binomial Series by solving $(1 + 4)^3$. First, this could easily be solved by using the order of operations or by "extending" the binomial by using distribution. The result is $5^3 = 125$. Letting a=3 and x=4:

$$(1 + 4)^3 = 1 + \frac{(3)4^1}{1} + \frac{(3)(2)4^2}{2} + \frac{(3)(2)(1)4^3}{6} + \frac{(3)(2)(1)(0)4^4}{24} + \quad \text{(all further terms will also include the factor of 0).}$$

$$(1 + 4)^3 = 1 + 12 + \frac{96}{2} + \frac{384}{6} + 0 + 0 + \ldots$$

$$(1 + 4)^3 = 1 + 12 + 48 + 64$$

$$(1 + 4)^3 = 125$$

The derivative of ARCSIN X is: $\frac{1}{\sqrt{1 - X^2}}$ or= $(1 - X^2)^{(-1/2)}$, and this is derived below:

Since ARCSIN X is the corresponding angle (¢) associated with the sine value that is equal to X, the derivative of ARCSIN X is the instantaneous rate of change of the angle (¢) with respect to the SIN value = X, of that angle. It was previously shown that the derivative of SIN ¢ with respect to the ¢ is equal to COS ¢. If we let X represent the angle here, this can be expressed as:

$$\frac{d(\text{SIN } X)}{d(X)} = \text{COS } X$$

There is a simple way to verify this: notice on the graphing of the SIN X curve, that at 0°, the slope of the curve, and hence derivative, appears to be 1 (such as for the tangent value of 45°), and that is what COS 0° equals. At 90°, the slope of the (now horizontal) curve is obviously 0, and that is what COS 90° equals. This leads to the fact that the COS and SIN curves are essentially a derivative curve of each other. The actually numeric values of the SIN and COS derivatives are also negative in value to each other, and this can bee seen on the curves where as one curve is sloping upward with a positive value, the other curve is sloping downward with a negative value for a given angle in question.

Given ¢ = Y = ARCSIN X = SIN^{-1} X, therefore, X = SIN Y

We are to find $\frac{dY}{dX}$: the derivative of Y with respect (in reference) to X.
Or in common words: The derivative of the angle with respect to the SIN of that angle: d¢ / d SIN ¢

Since $\text{¢} = Y = \text{ARCSIN } X$, the above is equivalent to:

$$\frac{dY}{dX} = \frac{d(\text{ARCSIN } X)}{dX} \quad \text{or: } = \frac{d\,\text{¢}}{d\,\text{SIN } \text{¢}} \quad \text{, therefore:}$$

Since $X = \text{SIN } Y$, then $d(X) = dX = d(\text{SIN } Y)$: expressing the derivative or differential (infinitely small change) of both sides

And we know now: $\frac{dX}{dY} = \frac{d(\text{SIN } Y)}{dY} = \text{COS } Y$: here, Y is an angle

From the "Pythagorean-like" trigonometric identities:

$\text{SIN}^2 Y + \text{COS}^2 Y = 1$ mathematically:

$\text{COS}^2 Y = 1 - \text{SIN}^2 Y$

$\text{COS } Y = \sqrt{1 - \text{SIN}^2 Y}$ therefore:

$\frac{dX}{dY} = \text{COS } Y = \sqrt{1 - \text{SIN}^2 Y} = \sqrt{1 - X^2}$: since as noted above, $X = \text{SIN } Y$

By taking the reciprocal of the above:

$$\frac{dY}{dX} = \frac{d(\text{ARCSIN } X)}{dX} = \frac{1}{\sqrt{1 - X^2}} \quad \text{or=} \quad (1 - X^2)^{-1/2}$$

It can also be shown that $\frac{d(\text{ARCCOS } X)}{dX} = -\frac{1}{\sqrt{1 - X^2}}$: nearly the same expression, but is negative in sign

As a verification to this derivative, let's consider using the chain rule for finding derivatives.

Let $x = \sin \text{¢}$ then $\text{¢} = \arcsin x$

Taking the derivative of both sides:

$$\frac{d(x)}{dx} = \frac{d(\sin \text{¢})}{dx}$$

Using substitution, we can show that we have a function of a function, and that the chain rule should be employed:

$$\frac{d(x)}{dx} = \frac{d(\sin(\arcsin x))}{dx}$$

We see that sin ¢ is the outer function which is then a function of the inner function of: $\text{¢} = \arcsin x$. That is, sin ¢ is a function of the angle. The derivative of x with respect to x is 1. The derivative of sin ¢ is cos ¢. Continuing with the derivative, we find:

$1 = \cos \text{¢} \left(\frac{d\,\text{¢}}{d\,x}\right)$ using the trigonometric identity for cos ¢, and $\text{¢} = \arcsin x$:

$1 = \sqrt{1 - \sin^2 ¢}\ \frac{(d \arcsin x)}{(dx)}$ since x = sin ¢:

$1 = \sqrt{1 - x^2}\ \frac{(d \arcsin x)}{(dx)}$ which can be expressed mathematically as:

$$\frac{d \arcsin x}{dx} = \frac{1}{\sqrt{1 - x^2}} = \frac{1}{(1 - x^2)^{(1/2)}} = (1 - x^2)^{(-1/2)}$$

Note also that the derivative of this trigonometric-transcendental function is a definite algebraic function.

Expanding the ARCSIN X derivative using the Binomial Series, letting $a = -\frac{1}{2}$, and $x = -X^2$:

$$1 + \frac{\left(\frac{-1}{2}\right)\left(\frac{-X^2}{1}\right)^1}{1!} + \frac{\left(\frac{-1}{2}\right)\left(\frac{-3}{2}\right)\left(\frac{-X^2}{1}\right)^2}{2!} + \dots$$

Note: $(-X^2)^2 = (-1X^2)(-1X^2) =$
$(-1)(X^2)(-1)(X^2) = +1\,X^{(2+2)} = X^4$

$$1 + \frac{X^2}{(2)} + \frac{3X^4}{(2)(4)} + \dots$$

Integrating (essentially the reverse of the derivative process, that is, finding the anti-derivative or function that the derivative was derived from) this term by term and keeping the products in their factored form so that a mathematical pattern can be seen, we get the actual function from which the derivative given above was derived:

$$\text{ARCSIN } X = \frac{X}{1} + \frac{1X^3}{(2)(3)} + \frac{3X^5}{(2)(4)(5)} + \dots$$

$$\text{ARCSIN } X = X + \frac{1X^3}{3!} + \frac{3X^5}{5!} + \dots$$: ARCSIN X SERIES

FINDING ARCCOS X

Rather than use a new series or some continued fraction to solve for ARCCOS X, you can reuse the ARCSIN X series in a simple formula. The verification of this formula will be presented. Remember, the formula returns a radian angle and then you can convert it to its equivalent degree angle if needed.

ARCCOS X = $\frac{\pi}{2}$ - ARCSIN X : X is a radian angle, this is equivalent to: ARCCOS ¢ = 90° - ARCSIN ¢. This trigonometric identity is due to complementary angles which sum to 90°.

$\frac{\pi}{2}$ = 1.570796327...

ARCCOS X = 1.570796327 - ARCSIN X

The above method to solve for ARCCOS X using the ARCSIN X series is derived from the concept of the complementary angles of a right triangle:

COS A = SIN B : the trigonometric cofunctions of complementary angles that sum to 90° are equivalent in value

and: A = 90° - B
A = 90° - ARCSIN (SIN B) in terms of radians:
A = $\frac{\pi}{2}$ - ARCSIN (SIN B)

A = 1.570796327 - ARCSIN (SIN B) Due to the cofunctions of complementary angles: COS A = SIN B:
A = 1.570796327 - ARCSIN (COS A) or in a more general form:
¢ = 1.570796327 - ARCSIN (COS ¢) : using radians for the units of the ¢

Ex. If COS ¢ = 0.3 , what is the angle?

¢ = 1.570796327 - ARCSIN (0.3)
¢ = 1.570796327 - 0.304692654
¢ = 1.266103673 = 72.54239688°

Here are some other trigonometric identities:

From: COS ¢ = $\sqrt{1 - \text{SIN}^2 ¢}$

ARCSIN X = ¢ = ARCCOS (COS ¢) = ARCCOS $\sqrt{1 - \text{SIN}^2 ¢}$: also, ¢ = ARCTAN (TAN ¢)

Since X = SIN ¢:

ARCSIN X = ARCCOS $\sqrt{1 - X^2}$

From: SIN ¢ = $\sqrt{1 - \text{COS}^2 ¢}$

ARCCOS X = ¢ = ARCSIN (SIN ¢) = ARCSIN $\sqrt{1 - \text{COS}^2 ¢}$

Since $X = \text{COS}$

$$\text{ARCCOS } X = \text{ARCSIN } \sqrt{1 - X^2}$$

If you want to use the ARCTAN X series to evaluate ARCCOS X:

From: $\text{TAN } ¢ = \dfrac{\sqrt{1 - \text{COS}^2 ¢}}{\text{COS } ¢}$

Therefore: $\text{ARCCOS } X = ¢ = \text{ARCTAN } (\text{TAN } ¢) = \text{ARCTAN } \dfrac{\sqrt{1 - \text{COS}^2 ¢}}{\text{COS } ¢}$

Since $X = \text{COS } ¢$

$$\text{ARCCOS } X = \text{ARCTAN } \frac{\sqrt{1 - X^2}}{X}$$

SERIES FOR ARCTAN ¢

$$\text{ARCTAN } X = \sum_{n=0}^{n=\infty} (-1)^n \frac{X^{(2n+1)}}{2n+1} \qquad : |X| \le 1$$

Expanding the terms:

$$\text{ARCTAN } X = X - \frac{X^3}{3} + \frac{X^5}{5} - \frac{X^7}{7} + \frac{X^9}{9} - \frac{X^{11}}{11} + \frac{X^{13}}{13} - \frac{X^{15}}{15} + \frac{X^{17}}{17} - \ldots$$

When X is greater than one you can convert X to the angles equivalent sine value and evaluate this value in the ARCSIN series. This series above is based on the term by term expansion of the derivative of TAN X, and then taking the anti-derivative of each term. The ARCTAN X series shown above, created by Leonhard Euler, converges to the true value so slowly that it is practically useless to use. There are some other similar ARCTAN series that converge to a specific (true) value using much fewer terms and their derivation is beyond the scope of this book. Here are some more practical ways or formulas to evaluate ARCTAN X :

From: $\text{SIN } \text{¢} = \dfrac{\text{TAN } \text{¢}}{\sqrt{1 + \text{TAN}^2 \text{¢}}}$

$$\text{ARCTAN } X = \text{¢} = \text{ARCSIN}(\text{SIN } \text{¢}) = \text{ARCSIN}\left(\frac{\text{TAN } \text{¢}}{\sqrt{1 + \text{TAN}^2 \text{¢}}}\right)$$

But since X = TAN ¢,

$$\text{ARCTAN } X = \text{¢} = \text{ARCSIN}\left(\frac{X}{\sqrt{1 + X^2}}\right)$$

These equations at first do look a bit odd, but given any angle with its unique SIN, COS, and TAN values, the INVerse or ARC of each resolves back to the exact same angle:

$$\text{¢} = \text{ARCTAN}(\text{TAN } \text{¢}) = \text{ARCSIN}(\text{SIN } \text{¢}) = \text{ARCCOS}(\text{COS } \text{¢})$$

Here is another derivation of a formula for ARCTAN X:

From $\text{COS } \text{¢} = \dfrac{1}{\sqrt{1 + \text{TAN}^2 \text{¢}}}$

$$\text{ARCTAN } X = \text{¢} = \text{ARCCOS}(\text{COS } \text{¢}) = \text{ARCCOS}\left(\frac{1}{\sqrt{1 + \text{TAN}^2 \text{¢}}}\right)$$

Since X = TAN ¢,

$$\text{ARCTAN } X = \text{ARCCOS}\left(\frac{1}{\sqrt{1 + X^2}}\right)$$

: this is almost identical to the ARCIN X equivalence for ARCTAN X shown above except that it has X in the numerator.

Here is a continued fraction developed by a mathematician named **Johann Lambert** (1728-1777), from Switzerland-Germany-France, so as to evaluate ARCTAN X. The result is a radian angle. The result here is much easier to obtain than by using the series previously shown:

$$\text{ARCTAN } X = \cfrac{X}{1 + \cfrac{1X^2}{3 + \cfrac{(2X)^2}{5 + \cfrac{(3X)^2}{7 + \cfrac{(4X)^2}{9 + \cfrac{(5X)^2}{11}}}}}}$$

: **ARCTAN continued fraction**

This continued fraction is similar to the one for TAN X. This fraction also converges slower, but it is still practical. The pattern is not too difficult to recognize. Each denominator is the next odd integer plus a term that contains the square of the product of the next integer and X. This method is good for values of X less than or equal to 1 (X<=1). The lower X is, the less "tiers" or "levels" (essentially divisions) of the continued fraction required for a specific number of correct digits.

Here is a simplified form of the 7 levels of the continued fraction shown above:

$$\text{ARC TAN } X = \frac{387660\,X + 300680\,X^3 + 50332\,X^5}{387660 + 429900\,X^2 + 116100\,X^4 + 8100\,X^6}$$

To give you an idea of the results using the above expression, when X is close in value to 1, you can expect 3 decimal places of accuracy. When X = 0.5, you can expect 5 decimal digits of accuracy, and when X = 0.2, you can expect almost 9 decimal digits of accuracy. Using this expression, all results will be slightly lower than the true value.

Here is a method you can use when x>1:

From the trigonometric co-functions of the complementary angles being equal:

$$\text{TAN } A = \text{COT } B = \frac{1}{\text{TAN } B}$$

: where (TAN B) has a value > 1,
then COT B and TAN A will be < 1

And from the sum of complementary angles:

$$B = 90° - A$$

in terms of radians:

$$B = \frac{\pi}{2} - A$$

$$B = 1.570796327 - A$$
$$B = 1.570796327 - \text{ARCTAN}(\text{TAN } A)$$
$$B = 1.570796327 - \text{ARCTAN}(1/\text{TAN } B)$$

or in a more general form:

$$¢ = 1.570796327 - \text{ARCTAN}(1/\text{TAN } ¢)$$

Ex. If the tangent value in question is 1.333333333, the angle is:

Since this tangent value is greater than 1, lets convert it to its corresponding tangent value of its complementary angle:

If we let 1.33333333 = TAN B , then for its complementary angle, angle A, TAN A, is:

$$\text{TAN A} = \frac{1}{\text{TAN B}} = \frac{1}{1.333333333} = 0.75$$

$\text{¢} = 90° - \text{ARCTAN (TAN complementary angle)}$ converting 90° to its equivalent radian units angle:
$\text{¢} = (\text{Pi}/2) - \text{ARCTAN } (0.75)$
$\text{¢} = 1.570796327 - \text{ARCTAN } (0.75)$
$\text{¢} = 1.570796327 - 0.643501108$
$\text{¢} = 0.927295218 = 53.13010235°$

The ARCTAN X formula can also be utilized to evaluate ARCSIN X and ARCCOS X. Given the angles sine or cosine value, convert this value to the angles corresponding tangent value.

$$\text{From TAN ¢} = \frac{\text{SIN ¢}}{\text{COS ¢}} = \frac{\text{SIN ¢}}{\sqrt{1 - \text{SIN}^2\text{ ¢}}} = \frac{\sqrt{1 - \text{COS}^2\text{ ¢}}}{\text{COS ¢}}$$

Ex. Evaluate ARCSIN 0.8 using ARCTAN X

$$\text{TAN ¢} = \frac{0.8}{\sqrt{1 - 0.8^2}} = 1.33333333$$

$\text{¢} = \text{ARCSIN } 0.8 = \text{ARCTAN } 1.33333333 = 0.927295217 = 53.13010235°$:as shown above

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THE BASIC FORMAT OF A C LANGUAGE COMPUTER PROGRAM

Below is a basic format of a c language program. The next program after this one contains some more notes about the C programming language. This book is not about C programming, but contains some helpful math computer programs written in the C programming language, and so some minimal discussion about it is needed for people who have not seen and-or used it before. The reader is also encouraged to obtain some entry level C programming books if they wish to know more and to create and-or edit their own programs which can do something useful such as calculate a value. The program addnumbers.c is shown further below and it utilizes this basic and common format discussed next.

```
/*-------------------------------------------------------------------------------------------
   myprogram.c

   C source and text code, and saved as myprogram.c

   (c) date,  name     Ex.  (c)  June 10, 2021 by J.P.A.

   Compiler and-or version used.

   Program description and or notes, edits and updates.

   Use a plain text editor to write a C program. This source
   code which is quite readable with practice, will be then
   converted to an .exe or executable code that the computer
   can utilize.  A C language compile program will do this conversion.

  This program below shows a basic format of a C programming
   language program.  Every C program must start with the default
   starting or entry point that is called: main and this is actually
   the name of an instance of a C programming structure called a
   function.  C can have many functions that are called by their
   uniquely assigned  name such as:   addnumbers()  or
   get_text_string(), etc.  A function is usually terminated from
   being run or executed when it encounters the return keyword
   and program flow command.  The program will resume after the
   point where the function  was called.  Data can be sent to a
   function to work with, and a function can also create a value
   within it and return it to the calling point in a program.  Once a
   function is written, it can be used over and over millions of
   times and with different data values sent to it, accessed by it,
   calculated by it, and-or returned from it.  When a function and-or
   its program code is called to be  ran or processed, it will have a
   format of:

   function_name(arguments to be sent to the function separated by a comma);
   or simply:

   function();  if the function is void (ie., absent) of any arguments
   such as the value(s) of the variable(s) to be sent to it, and so as to be
   used by it.

   For the actual function program itself, its syntax will also include the
   data type of each of the arguments or variables sent to it, with each
   separated by a comma:
```

function_name(data_type argument1 , data_type argument2 , . . .)
Ex: function_name(char c, int n, float height, float width);

Generally unseen, even to the programmer, is that when a function is called to be ran or processed, the current location and local variables and their values of the current function being ran or processed are first stored in a memory area called a "stack", and so that when that location, directly after the function call is returned to, that function or program code can continue directly after at where it was interrupted by the function call. Sometimes it is said that the "(local) environment" data is stored on the (memory) stack. In ("low-level, near pure machine language) Assembly language computer programming, it uses instructions or commands such as: PUSH to place data onto the stack, and POP to read and remove that data from the (memory) stack, in a reverse type of manner. The stack process is handled by the computer's operating system (such as DOS or Windows) which handles or oversees the .EXE ("executable") program being ran or processed.

The program below just has one function, the main function, and this can be expressed here as: main()

The "body" (program code) of a function or loop, etc., is placed within or "delimited" (defined, marked) by the formal syntax using curly braces: { };

This program was compiled in the same directory as the tcc.exe program. It can be compiled using a DOS or system command such as: TCC myprogram.c

A batchfile of commands is a list of DOS or system commands to be run, executed or performed one after the other, and you can use something like this text and save it as something like:

Compile_myprogram.bat
Double click on a batchfile to run it, much like how any other computer program is started or run:

```
REM  myprogram.bat to compile myprogram.c
@echo off
tcc ./myprogram.c
rem Or perhaps tcc myprogram.c
echo press a key
pause
```

The compiled program can be run by clicking on its icon, its filename.exe and-or by typing its name with or without the .exe filename extension on a new DOS or system command line.

If your computer system is in the DOS or command line mode, you can type EXIT to go back to the Windows system mode.

A simple batchfile to go to the current directory or folder of which it is in is:

```
REM CurrentDiretory.bat to goto the current directory
cmd.exe
EXIT
```

Type the above batch commands into a text editor as a plain, pure textfile and save and-or rename it with a **.**bat filename extension and not the **.**txt filename extension.

To see a list of files in a directory, enter DIR. To see a list of some or most DOS commands, enter HELP . To get information on using a DOS command enter the command name followed by /?. Books are available about creating batchfiles of DOS commands. A batchfile is essentially a program written with the DOS language (ie., commands).

Here is an actual basic C program example below that you may compile:

```
/*--------------------------------------------------------------------------------------------------
addnumbers.c

A program to add two user input numbers and displays the result or sum.

 (c) JPA  2021
  Version 1.0 , Compiled with the tinyC compiler.
--------------------------------------------------------------------------------------------------*/
void main(void)                /* : Required or default starting function of a C program. */
{                              /* : The function body is placed within braces */
double number1=0.0;            /* : Declares a variable to hold a data type that
                                    is a double precision floating-point data value
                                    or number,  and initialize or assign it with 0.0  */
double number2=0.0;            /* A float means floating-point, adjustable or movable */
                               /* decimal point location within the number  */
double result=0.0;

printf("Input Number 1 :  ");     /* : A statement to  display a line of text.       */
scanf("%lf",&number1);           /*  : Let the user input the value of number 1. */
                                 /*   Statements and other C programming
                                       structures are ended with a semicolon.  */

printf("\nInput Number 2 : ");   /* : Display a line of text on the next line */
scanf("%lf",&number2);          /* : Let the user input the value of number 2 */
                                /*   %f is a format specifier of the type of data
                                     that will be entered, and it will be stored
                                     in the address assigned to hold the data of
                                     the variable identified as  number2.        */
/* To add explicit numbers or "hard-coded numbers" and-or variables with
   any value they may be,  use the + = add operator.  Use - to subtract.
   Use * to multiply.  Use / to divide.                                       */

result = number1 + number2;   /* Add the value of number1 and number2
                                 and store the result in variable called
                                 result. */

/* Display the values and the result.  Note below that %lf format specifier
can be used but %g is used to eliminate any possible trailing 0's from being
displayed.  Each format specifier must have an argument associated with it,
and those are expressed one after the other with a comma. Format specifiers
```

```
are not displayed, but are placeholders and will be substituted with the print
function () argument or data associated with it in function call (ie., use). */

printf("\n%g plus %g is %g",number1,number2,result);

printf("\n\nPress A Key To End The Program: ");
getch();
return;
};        /*  : notice the semicolon syntax used here,and at the end of each full statement */
/*------------------------------------------------------------------------------------------------------------------------*/
```

For the C computer programming language, the syntax for the common math operations is:

Operation	Syntax	
addition	+	: this was shown in the example above, having the basic format of: operand1 operation operand2 = result operand1 + operand2 = sum Ex. 5 + 2 , and the result is 7
subtraction	-	: Ex. 5 - 2 , and the result is 3
multiplication	*	: Ex. 5 * 2 , and the result is 10
division	/	: Ex. 10 / 5 , and the result is 2

A C language, float data type formally means a floating (decimal) point data type. You can imagine a float as an integer value (ie., C's int data type) with some of the data being used as an indicated position where the decimal point should be located in that value. Each data type in C is declared in the program and is then assigned or allotted a fixed number of bytes so as to be large enough to hold that type of data (usually in the form of a binary number(s)).

In the program above, the data type of the numbers to be input by the user was declared as a double (ie., double float), and this is a more advanced form of the simpler float data type which has only about 6 decimal digits maximum of precision possible due to the limited number of bytes assigned to the float data type. A double (or double float) data type has twice the number of bytes assigned to it as compared to that of a float data type, and therefore essentially has twice the number of digits possible. In general, 15 to 16 digits of precision are possible with a double float (binary) data type, and this is good for nearly all practical mathematical applications. It must be remembered that the standard way to save numeric numbers in memory, etc, is in a binary (1's and 0's) numeric form. Numbers are displayed as graphic text characters that have an ASCII code and which can be converted back into numeric form with some of the (text, ASCII) string to numeric conversion functions available in the C programming language. It is also very educational and useful to write your own costume functions to do this such as your own function to input a string in stead of using gets().

The C language and other computer programming languages also have what is called as **unary operations** where the operand is just a single value such as a byte (0 to 255d). C's unary operators are:

Increment: ++   ,   ex: n++   : so as to increment the value of n by 1.0 , hence the same as adding 1 to n.
Decrement: --   ,   ex: n--   : so as to decrement or reduce the value of n by 1.0
Negation: -   ,   ex: -n   : so as to have the negative of the value of n, this is the same as multiplying n by (-1) = negative one
complement: ~   ,   ex: ~n   :so as to invert the state of each bit of n, 1 becomes 0, and 0 becomes 1 Also called the binary or ones complement.
logical not or negation: !   , ex: !n   : inverts the logical result of n , this is like an electric not gate , 0 to 1, or 1 to 0.

[This space for edits.]

A COMPUTER PROGRAM TO MAKE A TRIGONOMETRIC TABLE

```
/* -----------------------------------------------------------------------------------------------------------------------------------*/
TrigonometricTable.c

   This program displays the trigonometric values of angles from 0 to 90 degrees using the C-language
   (input) source code compiler's "built in", or prewritten mathematics trigonometric functions.  Programs
   are shown in this book as to how to create and-or modify your own "homemade" trigonometric and
   other basic math functions for use with a computer.  You may use a text editor to copy the source code
   of these programs, so as to enter it into a compiler program that converts it into a machine or
   computer code ("running" or "executable" [.exe]) program which is basically a list of numbers which
   represent the data, instructions, and memory addresses that will be needed for that program.

   With a DOS compatible (PC) computer, you can redirect the screen output to a file for some programs.
   This method uses something such as shown here in symbolic form :  program_name.exe  >  textfile.txt

   (c) JPA July 20, 2017, March 2019

   The next program in this book briefly discusses some of the basics of the C computer programming
   language.
/*-----------------------------------------------------------------------------------------------------------------------------------*/
#include "stdio.h"
#include "math.h"    /* Used to access C's built in trigonometric functions that were prewritten. */

/*------------------------------------------------------------------------------------------------------------------------*/
void main(void)
{
int n=0;

double degreeangle=0.0;
double radianangle=0.0;

long double sin1=0.0;    /* long double float or floating point values have more precision available than a double or float */
long double cos1=0.0;
long double tan1=0.0;

printf("\nDeg       Rad              Sin              Cos              Tan");

printf("\n\n");                                                        /* display two blank spaces */

for(;n<=90;){

   if(n==90){  degreeangle=89.99999999999999; };  /* not generally needed, but keeps tan 90° = undefined,  from
                                                    rolling over to some negative value with the programming language,
                                                    here C, and remains a high value  */

   radianangle=degreeangle * 0.0174532925199433;  /* convert the degree angle to the equivalent radian
                                                      angle measurement needed for the std. sin() in the C language  */

    sin1=sin(radianangle);  /* Calculate the sine of a radian angle using the ANSI standard C's built in math function sin() */
    cos1=cos(radianangle);  if(n==90){ cos1=0.0; };       /* to correct a slight error, garbage; a very small negative value */
    tan1=tan(radianangle);
```

```
/* Lets display the trigonometric values for each angle: */

    printf("%2d  %16.14g  %16.14g  %16.14g  %16.14g",n,radianangle,sin1,cos1,tan1);
    printf("\n");              /* Above, %16.14g is the format specifier for a floating point number with a
                                  display  width of 16 spaces wide, and 14 maximum fractional decimal digits. */
    n=n+1;
    degreeangle=degreeangle+1.0;
    };


printf("\n\nPress A Key");  getch();             /* This keeps the displayed table on the screen and waits
                                                    for a key press by the user to end the program. */
return;
};
/*-------------------------------------------------------------------------------------------------------------------*/
```

Here is the output of the computer program as displayed on the viewing screen:

```
Deg          Rad                          Sin                          Cos                    Tan

 0               0                0                1                0
 1  0.017453292519943   0.017452406437284   0.99984769515639   0.017455064928218
 2  0.034906585039887   0.034899496702501   0.9993908270191    0.034920769491748
 3  0.05235987755983    0.052335956242944   0.99862953475457   0.052407779283041
 4  0.069813170079773   0.069756473744125   0.99756405025982  0.06992681194351
 5  0.087266462599716  0.087155742747658    0.99619469809175  0.087488663525924
 6  0.10471975511966    0.10452846326765    0.99452189536827    0.10510423526568
 7   0.1221730476396    0.12186934340515    0.99254615164132   0.1227845609029
 8  0.13962634015955    0.13917310096007    0.99026806874157   0.14054083470239
 9  0.15707963267949    0.15643446504023    0.98768834059514   0.15838444032454
10   0.17453292519943   0.17364817766693    0.98480775301221   0.17632698070847
11   0.19198621771938   0.19080899537654    0.98162718344766   0.19438030913772
12   0.20943951023932   0.20791169081776    0.97814760073381   0.21255656167002
13   0.22689280275926   0.22495105434387    0.97437006478524   0.23086819112556
14   0.24434609527921   0.24192189559967    0.970295726276     0.24932800284318
15   0.26179938779915   0.25881904510252    0.96592582628907   0.26794919243112
16   0.27925268031909   0.275637355817      0.96126169593832   0.28674538575881
17   0.29670597283904   0.29237170472274    0.95630475596304   0.30573068145866
18   0.31415926535898   0.30901699437495    0.95105651629515   0.32491969623291
19   0.33161255787892   0.32556815445716    0.94551857559932   0.34432761328967
20   0.34906585039887   0.34202014332567    0.93969262078591   0.3639702342662
21   0.36651914291881   0.3583679495453     0.9335804264972    0.38386403503542
22   0.38397243543875   0.37460659341591    0.92718385456679   0.40402622583516
23    0.4014257279587   0.39073112848927    0.92050485345244   0.4244748162096
24   0.41887902047864   0.4067366430758     0.9135454576426    0.44522868530854
25   0.43633231299858   0.4226182617407     0.90630778703665   0.466307658155
26   0.45378560551853   0.43837114678908    0.89879404629917   0.48773258856586
27   0.47123889803847   0.45399049973955    0.89100652418837   0.50952544949443
28   0.48869219055841   0.46947156278589   0.88294759285893    0.53170943166148
29   0.50614548307836   0.48480962024634   0.8746197071394     0.55430905145277
30    0.5235987755983           0.5        0.86602540378444    0.57735026918963
31   0.54105206811824   0.51503807491005   0.85716730070211   0.60086061902756
32   0.55850536063819   0.52991926423321   0.84804809615643   0.62486935190933
```

33	0.57595865315813	0.54463903501503	0.83867056794542	0.64940759319751
34	0.59341194567807	0.55919290347075	0.82903757255504	0.67450851684243
35	0.61086523819802	0.57357643635105	0.81915204428899	0.70020753820971
36	0.62831853071796	0.58778525229247	0.80901699437495	0.72654252800536
37	0.6457718232379	0.60181502315205	0.79863551004729	0.75355405010279
38	0.66322511575785	0.61566147532566	0.78801075360672	0.78128562650672
39	0.68067840827779	0.62932039104984	0.77714596145697	0.80978403319501
40	0.69813170079773	0.64278760968654	0.76604444311898	0.83909963117728
41	0.71558499331768	0.65605902899051	0.75470958022277	0.86928673781623
42	0.73303828583762	0.66913060635886	0.74314482547739	0.90040404429784
43	0.75049157835756	0.6819983600625	0.73135370161917	0.93251508613766
44	0.76794487087751	0.694658370459	0.71933980033865	0.96568877480707
45	0.78539816339745	0.70710678118655	0.70710678118655	1
46	0.80285145591739	0.71933980033865	0.694658370459	1.0355303137906
47	0.82030474843734	0.73135370161917	0.6819983600625	1.0723687100247
48	0.83775804095728	0.74314482547739	0.66913060635886	1.1106125148292
49	0.85521133347722	0.75470958022277	0.65605902899051	1.150368407221
50	0.87266462599716	0.76604444311898	0.64278760968654	1.1917535925942
51	0.89011791851711	0.77714596145697	0.62932039104984	1.2348971565351
52	0.90757121103705	0.78801075360672	0.61566147532566	1.2799416321931
53	0.92502450355699	0.79863551004729	0.60181502315205	1.3270448216204
54	0.94247779607694	0.80901699437495	0.58778525229247	1.3763819204712
55	0.95993108859688	0.81915204428899	0.57357643635105	1.4281480067421
56	0.97738438111682	0.82903757255504	0.55919290347075	1.4825609685127
57	0.99483767363677	0.83867056794542	0.54463903501503	1.5398649638146
58	1.0122909661567	0.84804809615643	0.5299192642332	1.6003345290411
59	1.0297442586767	0.85716730070211	0.51503807491005	1.6642794823505
60	1.0471975511966	0.86602540378444	0.5	1.7320508075689
61	1.0646508437165	0.8746197071394	0.48480962024634	1.8040477552714
62	1.0821041362365	0.88294759285893	0.46947156278589	1.8807264653463
63	1.0995574287564	0.89100652418837	0.45399049973955	1.9626105055052
64	1.1170107212764	0.89879404629917	0.43837114678908	2.0503038415793
65	1.1344640137963	0.90630778703665	0.4226182617407	2.1445069205096
66	1.1519173063163	0.9135454576426	0.4067366430758	2.2460367739042
67	1.1693705988362	0.92050485345244	0.39073112848927	2.3558523658238
68	1.1868238913561	0.92718385456679	0.37460659341591	2.4750868534163
69	1.2042771838761	0.9335804264972	0.3583679495453	2.6050890646938
70	1.221730476396	0.93969262078591	0.34202014332567	2.7474774194546
71	1.239183768916	0.94551857559932	0.32556815445716	2.9042108776758
72	1.2566370614359	0.95105651629515	0.30901699437495	3.0776835371753
73	1.2740903539559	0.95630475596304	0.29237170472274	3.2708526184841
74	1.2915436464758	0.96126169593832	0.275637355817	3.4874144438409
75	1.3089969389957	0.96592582628907	0.25881904510252	3.7320508075689
76	1.3264502315157	0.970295726276	0.24192189559967	4.0107809335358
77	1.3439035240356	0.97437006478524	0.22495105434386	4.3314758742842
78	1.3613568165556	0.97814760073381	0.20791169081776	4.7046301094785
79	1.3788101090755	0.98162718344766	0.19080899537654	5.1445540159703
80	1.3962634015955	0.98480775301221	0.17364817766693	5.6712818196177
81	1.4137166941154	0.98768834059514	0.15643446504023	6.3137515146751
82	1.4311699866354	0.99026806874157	0.13917310096007	7.1153697223842
83	1.4486232791553	0.99254615164132	0.12186934340515	8.1443464279746
84	1.4660765716752	0.99452189536827	0.10452846326765	9.5143644542226

```
85   1.4835298641952   0.99619469809175   0.087155742747658   11.430052302761
86   1.5009831567151  0.99756405025982   0.069756473744125   14.300666256712
87   1.5184364492351  0.99862953475457   0.052335956242944   19.081136687728
88   1.535889741755    0.9993908270191     0.034899496702501   28.636253282916
89   1.553343034275    0.99984769515639   0.017452406437283   57.289961630761
90   1.5707963267949             1                          0                  1.6331778728384e+016

Press A Key
```

Before modern electric component, high-speed computers to do many calculations, there were, and still are hand operated calculators to do these calculations for you. A "scientific calculator" is an advanced calculator capable of doing many more mathematical operations than a standard 4-function (mathematical operations) "home calculator", and which came several years later. Before these electronic calculators, there were mechanical calculators that had often had just a few mathematical operations, and which were made of various gears and various levers (such as "cams" for mechanical functions) and other mechanical parts. A mechanical "cash register" is a good example of a mechanical calculator. A slide-rule is a mechanical calculator which could be thought of as an advanced ruler for some low precision calculations. As of the year 2021, slide-rules are rarely used and-or seen, and are more of a collectors item for people who collect all types of calculator machines, computers and devices. Before any type of calculators, look-up tables of previously (ie., pre) calculated values were made and available in books. Look-up tables and-or lists of many types are still used everywhere in modern society, except that they are generally not for mathematical calculations but for sets of corresponding data values that the many people in public can utilize.

A modern electronic computer being a high speed (fast), information processing (electric) machine is internally instructed with, and processes simple binary data ("1's and 0's"), numeric values and results. Decimal values, numbers and-or text can be entered or input at the keyboard, however it will be eventually converted to its binary representation so as to be utilized or processed by the computer, and then perhaps a result will be displayed as a decimal number and-or text character (visual alphabetical letter, number, symbol, etc) on the output screen or printer. In short, the internal language of the computer is binary values being either 1's or 0's (ie, [electronically] "on or off", "set or not-set" as like a switch), and can be thought of as a numeric language.

A COMPUTER PROGRAM TO CALCULATE THE SINE OF AN ANGLE

Below is a computer program to calculate the sine value of a degrees angle. The computer program is shown below and is written using the popular C programming language syntax (essentially the required grammar of the programming language used). By observation, it should not be too difficult to understand and convert it to the BASIC or any other computer language syntax. Some explanation about the program and its syntax will also be given. You may skip over this section if you have no need of learning or understanding any computer language that is used to program a computer to help do things. Most general computer users except a small percentage of people are non-programmers using programs with these necessary calculations done within the program(s).

For the following computer programs as written here, the user must be use a "plain" (ie. basic, using the ASCII text codes only in the source or origin text file) or unformatted (ie., not a [formatted] word processor) input and output text editor and a C language compiler. A compiler is a computer program that will transform or translates the input text, "source code", computer language program to its equivalent or effective "low level" machine-electronic numeric codes for the computers electronic circuitry. In short, the compiler program creates the executable (working, run time) program for the computer.

In program below, many extra notes about C programming and its syntax are given for clarification, however, a new computer programmer is invited to obtain some programming books about the subject of learning computer programming. This book here focuses upon writing some mathematical source code programs with the C Language syntax of computer programming, and specifically for user-made math functions often found on a scientific calculator or computer. These functions are built into the ANSI C language, but its of mathematical importance to show them here.
Here is the format or syntax of a programmers comment or note within programs in the C language: /* comment(s) */

```
/*****************************************************************************************************************************
SINE.C

Two examples of  functions to calculate the trigonometric sine of a
degrees angle value(s) input from the user.  The function implements
the SIN X series.

This function can be improved upon for use with more modern
computers that allow simplicity obtaining greater precision.

Comments are enclosed within these delimiters (bounding, set apart):
   Comments are placed between or inside the forward slash and the star
   character and then the comment is terminated by the  star character
   and a forward slash character.  Comments are not processed by the
   computer, but are notes for the writer or reader of the program to
   review later to help understand the computer program, parts or
   functions within it.  Comment syntax can also be placed around
   program code to omit it, such as for temporary test code.

This text or readable, "high-level" computer code or program requires
a C-compiler to convert it to a "low-level" executable or usable electronic
machine or computer code program consisting of only binary numbers
where each 0 or 1 bit will be represented by a high or low voltage level
to be used by the specific computer processor circuitry within the
computer.

The specific C programming language compiler used was the free (ANSI
compatible) TinyC Compiler which can be obtained or downloaded to your
computer from some websites.  The basic format of the command to compile
a c programming language file (ie. with a input filename like:  myfilename.c)
```

```
is:  tcc myfilename.c  , and this will create or output an .EXE (executable,
"runable" or "operational" computer program file with the file name of:
myfilename.exe    For example, the command:  tcc sine.c  will produce the
sine.exe file.

(c) J.A. 2014
*******************************************************************************************************************************************/

#include "stdio.h"       /* : Required header file, for the inclusion of some prewritten standardized input/output code,
                              such as the printf().   */
#include "math.h"      /* required for use of the prewritten pow() function used in the program */

double SINE(double angle);          /* : forward (prototyping) declaring a (C language, computer) function */
/*    A function, indicated by ( ), is code that can be written once and used over and over when needed,
      and  can be placed into any other program where it can be used or adapted-modified.  Some computer
       programmers concentrate on just writing small and specific computer functions, rather than full programs.

      "double"  essentially means a  double-precision floating point (non-integer) data type.  Here,
       the return value, and also the formal argument or "parameter" that is identified as "angle"
       are declared to have data types of double as opposed to for example an integer data type
       which has fewer bytes of memory assigned to hold its value.
       This function implements the SIN X series.
       The angle argument is in degrees measurement.
       The function's return value (data it sends back or "outputs" to the calling function/program) is
       the trigonometric sine of the angle.
       Accuracy of the return value is 9 digits for any angle between 0 to 90 degrees, almost 10
       digits for lower angles
*/


double SINE2(double angle);         /* : the function prototype of SINE2, and  this helps the programmer and compiler */
      /* This is an alternate method to calculate the sine of an angle.  This method uses
          a loop to calculate each term to add to the sum.  */

/*------------------------------------------------------------------------------------------------------------------------------------------------*/
void main(void)                        /*  an example program to use the SINE() function. */
{                                            /*  main() , or the "main function" is the formalized start of
                                             any C program, and you must include it   */
double angle;                         /* : declaring a variable for use, it is identified as "angle"
                                             and its data type (what kind of values angle will have) is
                                             double.  It is initially set or "initialized" to 45.0.  Technically,
                                             a data type also determines the amount of computer
                                             memory (bytes) to set aside to hold the value(s) of the
                                             corresponding variable.   Basically, a declared variable and
                                             its corresponding name or identifier resolves to be a pointer
                                             or address to a memory location that holds a value or data. */

for(;;){                         /* Defines a loop to be reused or redo the following program code or steps over and over.  */
            fflush(stdin);      /* Optional, clears out any possible garbage values from the keyboard (the standard input)
                                   memory buffer input line, so as to prepare for a  new text (keyboard) input from the user. */
            angle=0.0;
            printf("\n\nEnter the degrees angle (<=90%c) to find the sine value of, or 0 to exit:  ",248); /* 248 is the deg. sym.*/
            scanf("\n%lf",&angle);
```

```
          if(angle<=0.0){ printf("\a");  /* a beep sound */ exit(0);  };
          printf("\nThe sine of %.16g degrees is %.9g  and  %.9g",angle,SINE(angle),SINE2(angle));
        };    /* End of the defined loop, now go back to the start of the loop and let the user enter another value */

/*  This computer code, with a use of the printf() function, above will display the result (the angle and the sine of that
     angle) on the screen.  To "print" (ie., send) the std. output to a printer, disk or memory file, use the fprintf()  function.
     %g is a display or output text format specifier that indicates how the corresponding function argument is
     to be displayed on the computer screen.
     g means to display the corresponding data argument as a floating point value.  16 and 11 are the
     max. number of decimal digits to display, and these values can be changed to be more or less by the
     programmer.  When SINE(angle) is encountered (ie. the SINE function is called), the program code
     to be then run or processed will then take place at this SINE function (below), until a return (to the
     previous or calling code position) statement or other code is encountered.  */

 return;      /*  Essentially ends the computer program.  Sends the (computer processing) control back to the
                "calling program" that called this SINE.EXE program.  Often, the calling program is the computer
                 operating system or program.  */
};
/*-------------------------------------------------------------------------------------------------------------------------------------------------------*/
/*   SINE FUNCTION  ,  SINE()

      This computer function returns a result that is a double float sine value of the double float angle argument.  The angle
      must be a degrees angle.  This function uses the sin x series, in hard coded form,  to calculate sin x.

      (c)  J.A.  2014
---------------------------------------------------------------------------------------------------------------------------------------------------------*/
double SINE(double angle)
{
double z=(angle * 3.14159265358979323846 26)/180.0;       /* converts the input degrees angle to its
                                                             corresponding radian angle */

/*  Using this series of terms calculation for Sin z,  the lower the value of z, the input angle, will result in greater
     accuracy,  so it is best to keep the input angle less than 180 , and generally  90° or less for the best accuracy.
     This is mainly due to only using a few or limited number of terms of an infinite series of terms which expresses
     the true result. */

/* The above expression declares the identifier (or variable) called z
    and initializes (sets) it to an equivalent radian angle value of the degrees
    angle value that was sent to this function.  This radian angle value is what
    the sine series below works with. 16 digits of values, such as (pi) shown,
    are usually accepted with many computers and/or programming languages.
     *  =  the "star" text character,  is the syntax for the multiplication symbol for
    man computer languages. If your computer or electronic machine is capable,
    here is Pi to 31 decimal places:  pi=3.1415926535897932384626433832795  */

double sine;

/* Lets progressively add each next term of the sin series to the running total called sine. */
/* It is also possible to make a loop to create and add each new term.     */

sine = z;
sine = sine -  pow(z,3.0)/6.0;          /* for more of a homemade style, can use, for ex., z*z*z = z^3  for = pow(z,3.0) */
sine = sine + pow(z,5.0)/120.0;
```

```
sine = sine -  pow(z,7.0)/5040.0;
sine = sine + pow(z,9.0)/362880.0;
sine = sine -  pow(z,11.0)/39916800.0;
sine = sine + pow(z,13.0)/6227020800.0;
sine = sine -  pow(z,15.0)/1307674368000.0;

return sine;       /* Return or send this value back to the statement that called or used this function.  The call
                      or function call to invoke or use this function will now be set equal to this value.   */
};/*-----------------------------------------------------------------------------------------------------------------------------------------------------*/
double SINE2(double angle)
{
/*    Returns a double float sine value of the double float angle argument.  The angle must be a degrees angle.
      This function uses a loop to evaluate each term of the sin x series. 8 terms will give 9 correct or
      accurate (accuracy to) decimal places or positions.  You can increase the number of terms */

double z=(angle * 3.141592653589793238)/180.0;
double sine=0.0;
double numerator=z;
double denominator=1.0;
double  factorial=1.0;

int  termnumber=1;
int n=1;
int maxterms=9;     /* set this to the number of terms you want to consider and evaluate in the series */
double term=0.0;
double sum=0.0;

int s=0;     /* a toggle, for the positive and negative sign of the term */


for(;;){       /* a loop to calculate each term of the series and add it to the sum */

             term = numerator / denominator;         /* printf("\nterm=%.16g",term);     */

             if((s%2)==0){  sine=sine + term;   };    /* if s, or the term number is even */
             if((s%2)==1){  sine=sine - term;    };   /* if s, or the term number is odd */
             s=s+1;                                         /* printf("\n%.16g",sine);    */

             termnumber=termnumber+1;  if(termnumber==maxterms){  break; };

             numerator=numerator*z*z;
             factorial=factorial *(n+1)*(n+2);  n=n+2;     /* printf("\n***%.10g",factorial);   ,  factorials, test ok */
             denominator = factorial;
        };

return sine;
};
/*-------------------------------------------------------------------------------------------------------------------------------------------*/
```

To make the function such as that above more easy to read and comprehend, rather than use many multiplications of the same value by itself, this book has a program to demonstrate a function called integerpower.c that can calculate an integer power of any value.

The trigonometric and other tables in the appendix section of this book were created with the aid of a computer program. Before mechanical or electro-mechanical and-or electronic computers with fast and easy calculations, the only method to create theses tables was manually by "hand" (pen, hand written [scribed], paper, and mind-calculation). Each entry in the table had to be calculated, often by many people working together so as to decrease the effort and time needed, and to check or validate each others results.

Mentioned above was using the fprintf(), "print to file" function to place data into a file. The file must be opened for writing, and it should be closed formally when done writing to the file. To print a string text characters to a file the %s format specifier is generally used. If your program has calculated some numerical values, you can use a function that converts these numbers to text strings first. When text is sent to and saved in a file, to display that text, you will usually name that file so as to have a **.**txt filename extension. Rather than use C's standardized, built in functions to convert numbers to strings, or write homemade functions to do similar, you can use the "print to string or string print" function of **sprintf()**. The alternative to the sprintf() is to say, convert each number to a string using a special function,and then using the strcat() function to catenate or join each string. Using sprintf() makes building strings easy to display on the screen and-or send to a file. The arguments to the sprintf() are very similar to the printf(). The format for sprintf() is:

```
sprintf(char *string, string_with _any_format_specifiers_or_tags , data_arguments_corresponding_to_each_format_tag);
```

Example Computer Program To Directly Make A Text File With Data

```
/*---------------------------------------------------------------------------------------------------------
 MakeSineTable.c    Make a Sine Table.c
                                    Shows the very basics of making a table in a text file.
                                   (c) JPA 2024    ,  Author Of The Mathization ebook
---------------------------------------------------------------------------------------------------------*/
#include "stdio.h"       /*   for printf()     */
#include "stdlib.h"      /*   for exit()       */
#include "string.h"     /*   for strlen()    */
#include "math.h"       /*   for sin()        */
#include "conio.h"     /*   for getch()    */
/*-------------------------------------------------------------------------------------------------------*/
void main(void)
{
double degreesangle=0.0;      /*  for degrees angles  */
double radianangle=0.0;        /*  for equivalent radian angle  */
double sine=0.0;
long W = 0L;
long  F = 0L;
FILE * fp;
char  datastring[81];
int L=0;
int  e=0;

/*-----------------------------------------------------*/

fp=fopen("sine_table.txt","wt");   /* opens the file and-or erases an existing one */
if(fp==NULL){  printf("\n\a*** FILE ERROR OPENING FILE * * * , Press A Key");  getch();  exit(0);  };

e=fwrite("TABLE OF ANGLE AND CORRESPONDING SINE VALUES \n",1,48,fp);
if(e==0){  printf("\n\a* * * FILE WRITE ERROR * * *  , Press A Key");  getch();  exit(0);  };
```

```
for(;;){
          radianangle = degreesangle * (3.141592653589793238 / 180.0);   /* convert the degrees angle to its equivalent
                                                                                                                         radian angle for sin() */
         sine =  sin(radianangle);         /* test ok:     printf("\n %f  ",sine);   printf("%.15g  ",sine);    */

         sprintf(datastring,"Angle = %.2f   ,   Sine = %.15g  \n",degreesangle,sine);  /* build or print to a string with sprintf() */

         L=strlen(datastring);
         e=fwrite(datastring,1,L,fp);
         if(e==0){  printf("\n\a* * * FILE WRITE ERROR * * *  , Press A Key");  getch();  exit(0);  };

        degreesangle=degreesangle + 1.0;
        if(degreesangle>90){  fclose(fp);  break; };
       };

return;
};
/*-----------------------------------------------------------------------------------------------------------*/
```

A COMPUTER PROGRAM TO CALCULATE ARCSINE

```
/*******************************************************************************************************************************
ARCSINE.C

A function to calculate the trigonometric arcsine, or angle value of the input sine value.
The function implements the basic  ARCSIN X series only, and may be improved upon.

(c) J.A. 2018

*********************************************************************************************************************************/

#include "stdio.h"       /* : Required header file, for the inclusion of some prewritten standardized input/output code,
                             such as the printf().   */


double ARCSINE(double sine);         /* : declaring a (C language, computer) function */
/*    Here,  the return value is the corresponding degrees angle value of the input sine value..
      This function implements the basic ARCSIN X series.  */

/*--------------------------------------------------------------------------------------------------------------------------------*/
void main(void)
{
long double sine;

for(;;){                          /* a loop to reuse or redo the following program code over and over  */
          fflush(stdin);        /* clear out any potential garbage from the keyboard (the standard input)  memory
                                     buffer (temporary memory storage) of the user input text line */
         printf("\n\nEnter the corresponding sine value (up to 1.0)  of the  angle to find");
         printf("\n the corresponding angle, or 0 to exit:  ");
         scanf("\n%lf",&sine);
         if((sine<=0.0) || (sine>1.0)){ printf("*** Input Error ***   \a");  /*  <-- the ASCII bell or  beep sound */   exit(0);   };
         printf("\nThe corresponding angle of sin %.16g  is %.10g degrees",sine,ARCSINE(sine));
                                                                             /* Used .8g as a way to provided some rounding.  */
        };    /* end of the loop, now go back to the start of the loop and let the user enter another value */

return;
};
/*-------------------------------------------------------------------------------------------------------------------------*/
/* ------------------------------   ARCSINE FUNCTION  ----------------------------

     Returns a double float angle value of the double float sine argument.
     This function uses the arcsin x series to calculate arcsin x.   It is possible to
     calculate arcsin x in various other ways to improve accuracy, especially for
     sine values greater than about 0.65 which corresponds to an angle of about
     40 degrees.  For example, for improved accuracy for sin values > 0.5, you can
     convert that value to its corresponding cos value, and then use the arccos x
     series or a trigonometric identity, for example:

    To find arcsin (sin ¢) = arcsin  0.6  , we see that sin ¢ = 0.6, and we can convert this to the angles corresponding cos
    value:

    cos ¢  =  √ 1 - sin^2 ¢    =  √ 1 - 0.6^2    =  √ 1 - 0.36    =  √ 0.64    =   0.8
```

$\sin (x/2) = + - \sqrt{(1 - \cos x)/2}$: sine of half the angle trigonometric identity

$\sin (¢/2) = + - \sqrt{(1 - 0.8)/2} = \sqrt{0.2/2} = \sqrt{0.1} = 0.316227766$

arcsin 0.316227766 = 18.43494882° : multiplying this half-angle by 2 to find the actual angle

(18.43494882°)(2) = 36.86989765°

For the given SINE() function below, which can be improved upon, for increased accuracy, as mentioned above:

At sin 0.1 , the returned value is: 5.739170477 degrees
A calculator returns: 5.739170477 degrees

At sin 0.5 , the returned value is: 30 degrees .
A calculator returns: 30 degrees.

At higher values (>0.5) and higher, the ARCSINE() should be improved upon, as indicated above.

At sin 0.6 , the returned value is: 36.869893 degrees
A calculator returns: 36.86989765

At sin 0.7 , the returned value is: 44.426808 degrees
A calculator returns: 44.427004 degrees

(c) J.A. 2018

```
------------------------------------------------------------------------------------------------------------------------------------------*/
long double ARCSINE(double sine)
{
long  double x=sine;
long  double angle;      /* the formula gives the corresponding radian angle value, this will
                            then be converted  to its corresponding degrees angle value */
```

/* ******** Here is the arcsin x series, and further ahead is a simplified version:

$$\text{ARCSIN } X = X + \frac{(1)X^3}{(2)(3)} + \frac{(1)(3)X^5}{(2)(4)(5)} + \frac{(1)(3)(5)X^7}{(2)(4)(6)(7)} +$$

$$+ \frac{(1)(3)(5)(7)X^9}{(2)(4)(6)(8)(9)} + \frac{(1)(3)(5)(7)(9)X^{11}}{(2)(4)(6)(8)(10)(11)} + \frac{(1)(3)(5)(7)(9)(11)X^{13}}{(2)(4)(6)(8)(10)(12)(13)} +$$

$$+ \frac{(1)(3)(5)(7)(9)(11)(13)X^{15}}{(2)(4)(6)(8)(10)(12)(14)(15)} + \frac{(1)(3)(5)(7)(9)(11)(13)(15)X^{17}}{(2)(4)(6)(8)(10)(12)(14)(16)(17)}$$

$$+ \frac{(1)(3)(5)(7)(9)(11)(13)(15)(17)X^{19}}{(2)(4)(6)(8)(10)(12)(14)(16)(18)(19)} + \frac{(1)(3)(5)(7)(9)(11)(13)(15)(17)(19)X^{21}}{(2)(4)(6)(8)(10)(12)(14)(16)(18)(20)(21)} + \ . \ . \ .$$

$$\text{ARCSIN } X = X + \frac{X^3}{6} + \frac{3X^5}{40} + \frac{15X^7}{336} + \frac{105X^9}{3{,}456} + \frac{945X^{11}}{42{,}240} + \frac{10{,}395X^{13}}{599{,}040} + \frac{135{,}135\ X^{15}}{9{,}676{,}800} +$$

$$+ \frac{2027025X^{17}}{175{,}472{,}640} + \frac{34459425X^{19}}{3{,}530{,}096{,}640} + \frac{654729075X^{21}}{(3{,}715{,}891{,}200)(21)} + \frac{654729075(21)X^{23}}{(2)(4)(6)(8)(10)(12)(14)(16)(18)(20)(22)(23)} + \ . \ . \ .$$

```
******** */

/* Let's progressively add each next term of the arcsin series to the running total called angle. */

angle =  x;
angle = angle + (x*x*x) / 6.0;  /*  You could rather use the pow() function here, for ex.: angle = angle + power(x,3.0) / 6.0; */
angle = angle + 3.0 * (x*x*x*x*x) / 40.0;
angle = angle + 15.0 * (x*x*x*x*x*x*x) / 336.0;
angle = angle + 105.0 * (x*x*x*x*x*x*x*x*x) / 3456.0;
angle = angle + 945.0 * (x*x*x*x*x*x*x*x*x*x*x) / 42240.0;
angle = angle + 10395.0 * (x*x*x*x*x*x*x*x*x*x*x*x*x) / 599040.0;
angle = angle + 135135.0 * (x*x*x*x*x*x*x*x*x*x*x*x*x*x*x) / 9676800.0;
angle = angle + 2027025 * (x*x*x*x*x*x*x*x*x*x*x*x*x*x*x*x*x) / 175472640.0;
angle = angle + 34459425 * (x*x*x*x*x*x*x*x*x*x*x*x*x*x*x*x*x*x*x) / 3530096640.0;
angle = angle + (654729075 * (x*x*x*x*x*x*x*x*x*x*x*x*x*x*x*x*x*x*x*x*x)) / (3715891200.0 * 21.0);
angle = angle + (654729075 * 21 * (x*x*x*x*x*x*x*x*x*x*x*x*x*x*x*x*x*x*x*x*x*x*x)) / (3715891200.0 * 22.0 * 23.0);
/* At this last term above, the formula is converging very slowly and there is only a small change in value for more terms. */

/* Let's convert this radian angle to its corresponding  degrees angle.  */
angle = angle * (180.0 / 3.14159265358979323846264338327950);   /* :just using 9 to 15 decimal places of (pi) should */
                                                                                                      be sufficient for nearly all applications */
return angle;
};
/*-----------------------------------------------------------------------------------------------------------------------------------------*/
```

A COMPUTER PROGRAM TO CALCULATE AN ANGLES THREE TRIGONOMETRIC VALUES

```
/********************************************************************************************************************
trigvalues.c

A function to calculate the three main  trigonometric  values (sine, cosine, and tangent)
of a degrees angle value.  The function mainly implements a simplified version of the  TAN X
continued fraction, and the accuracy is very good at about 9 decimal places for angles from 0
through 90 degrees.

(c) J.A. 2018       : If you wrote a program or article, you own the copyright to it, and even if it is never published,
                      however, you may need some other verification that you indeed wrote it at such via a third party and-or
                      a dated email.  A verification is not as good as a formal copyright, but it is much better than nothing.
********************************************************************************************************************/
#include "stdio.h"
#include "math.h"

double trigvalue(double angle, int function);
/*    The angle argument is in degrees measurement.
      For the argument called function, use 1 for sine, 2 for cosine, 3 for tangent.
      The function's return value is the trigonometric  value of the angle.
      Accuracy is about 9 decimal places from 0 to 90 degrees.  */

/* here is p  ("Pi") to 31 decimal places:  pi = 3.1415926535897932384626433832795    */
/*                                          pi/2 = 1.5707963267948966                 */
/*----------------------------------------------------------------------------------------------------------------*/
void main(void)
{
double angle=0.0;

for(;;){                         /* main loop */
          fflush(stdin);        /* clear out any potential garbage from the keyboard (the standard input)  memory
                                   buffer input line */
          angle=0.0;

          printf("\n\nEnter the degrees angle to find the trigonometric values of, or 0 to exit:  ");
          scanf("\n%lf",&angle);
          if((angle<=0.0) || (angle>90.0) ){ printf("\aUse 0 to 90. Press A Key");  /* a beep sound */ getch(); exit(0);  };

          printf("\nThe sine value of %.6g degrees is %.9g",angle , trigvalue(angle, 1));
          printf("\nThe cosine value of %.6g degrees is %.9g",angle , trigvalue(angle, 2));
          printf("\nThe tangent value of %.6g degrees is %.9g",angle , trigvalue(angle, 3));

        };    /* end of the loop, now go back to the start of the loop and let the user enter another value */

return;
};
/*-------------------------------------------------------------------------------------------------------------*/
```

/* This continued fraction to calculate tan x is credited to **Johann Lambert**:

$$\cfrac{X}{1 - \cfrac{X^2}{3 - \cfrac{X^2}{5 - \cfrac{X^2}{7 - \cfrac{X^2}{9 - \cfrac{X^2}{11}}}}}}$$

. . . (and so on)

Ex. of the above simplified to one fraction, as used below in this program:

$$\text{TAN } X = \frac{10395X - 1260X^3 + 21X^5}{10395 - 4725X^2 + 210X^4 - X^6}$$: A FORMULA FOR EVALUATING TAN ¢

For angles greater than 45 degrees, for improved accuracy with the limited number of terms used:

$$\text{TAN } X = \frac{1}{\text{TAN } (p/2 - X)} = \frac{1}{\text{TAN } (1.570796327 - X)}$$: X is a radian angle

Some trigonometric identities to calculate the other trigonometric values:

$$\sin \text{¢} = \frac{\tan \text{¢}}{\sqrt{1 + \tan^2 \text{¢}}}$$

$$\cos \text{¢} = \frac{1}{\sqrt{1 + \tan^2 \text{¢}}}$$

Another way to improve accuracy is to implement a short table and the trigonometric identity of:

$$\text{TAN } (\text{¢}1 + \text{¢}2) = \frac{\text{TAN } \text{¢}1 + \text{TAN } \text{¢}2}{1 - (\text{TAN } \text{¢}1)(\text{TAN } \text{¢}2)}$$: TANGENT OF THE SUM OF TWO ANGLES FORMULA

TAN 1° = TAN 0.017453292519943 = 0.0174550649282
TAN 2° = TAN 0.034906585039887 = 0.0349207694917
TAN 3° = TAN 0.05235987755983 = 0.0524077792830
TAN 4 ° = TAN 0.069813170079773 = 0.0699268119435
TAN 5° = TAN 0.087266462599716 = 0.0874886635259
TAN 6° = TAN 0.10471975511966 = 0.1051042352657
TAN 7° = TAN 0.1221730476396 = 0.1227845609029
TAN 8° = TAN 0.13962634015955 = 0.1405408347024
TAN 9° = TAN 0.15707963267949 = 0.1583844403245

TAN 10° = TAN 0.1745329251994 = 0.1763269807085
TAN 20° = TAN 0.3490658503989 = 0.3639702342662
TAN 30° = TAN 0.5235987755983 = 0.5773502691896
TAN 40° = TAN 0.6981317007977 = 0.8390996311773
TAN 50° = TAN 0.8726646259972 = 1.191753592594
TAN 60° = TAN 1.047197551197 = 1.732050807569
TAN 70° = TAN 1.221730476396 = 2.747477419455
TAN 80° = TAN 1.396263401595 = 5.671281819618 : the tan of angles greater than 80° grow very large
*/

```
/*-------------------------------------------------------------------------------------------------------------------------------------------*/
double trigvalue(double angle, int function)
/*  Use angles 0 to 90 degrees.  set function=1 for sine, 2 for cosine, 3 for tangent */
{
double z=(angle * 3.141592653589793238)/180.0;    /* convert input angle to its corresponding radian angle */

double tangent=0.0;
double sine=0.0;
double cosine=0.0;
int a=0;
double numerator;
double denominator;

if(angle>45.0){ a=1;   z=(3.14159265358979323846 / 2.0) - z;  };      /* used for improved accuracy using the limited
                                                                           number of  terms with the higher angle value */
/* Calculate the tangent value of the angle:  */

numerator  =  (10395*z)  -  (1260*z*z*z)  +  (21*z*z*z*z*z);
denominator  =  10395  -  (4725*z*z)  +  210 *(z*z*z*z)  -  (z*z*z*z*z*z);
tangent = numerator / denominator;

/* Some basic value checking:  */
if(tangent <= 0.000000001){ tangent = 0.000000001;  };
if(tangent >= 5.6e+14){ tangent = 5.6e+14; };

if(a==1){ tangent = 1.0/tangent; };

if(function==1){     /* return sine value */
                    sine = tangent / sqrt(1+tangent * tangent);   /* : here,  using C's own square root math function */
                    return sine;
                  };

 if(function==2){     /* return cosine value */
                     cosine = 1.0 / sqrt(1 + tangent * tangent);
                     return cosine;
                  };

if(function==3){     /* return tangent value */
                    return tangent;
                 };

};
/*-------------------------------------------------------------------------------------------------------------------------------------------*/
```

A COMPUTER PROGRAM TO CALCULATE THE TANGENT VALUE OF AN ANGLE

```
/*************************************************************************************************************************
tangent.c

A function to calculate the trigonometric tangent value of a degrees angle
value.  The function implements the TAN X continued fraction.

(c) J.A. 2018

*************************************************************************************************************************/
#include "stdio.h"

double TANGENT(double angle);
/*      This function implements the TAN X continued fraction.
        The angle argument is in degrees measurement.
        The function's return value  is the trigonometric tangent value of the angle.
        Here is"Pi" to 31 decimal places:  pi=3.1415926535897932384626433832795
*/
/*------------------------------------------------------------------------------------------------------------------------------------------------*/
void main(void)
{
double angle=0.0;

for(;;){                          /* a loop to reuse or redo the following program code over and over  */
          fflush(stdin);      /* clear out any potential garbage from the keyboard (the standard input)  memory
                                    buffer input line */
          angle=0.0;
          printf("\n\nEnter the degrees angle (<=90 °)  to find the tangent value of, or 0 to exit:  ");
          scanf("\n%lf",&angle);
          if((angle<=0.0) || (angle>90.0) ){ printf("\a");  /* a beep sound */ exit(0);  };
          printf("\nThe tangent of %.16g degrees is %.10g",angle,TANGENT(angle));

        };    /* end of the loop, now go back to the start of the loop and let the user enter another value */

return;
};
 /*-----------------------------------------------------------------------------------------------------------------------------------------------*/
double TANGENT(double angle)
{
/*-----  NOTES

/*      This function implements the TAN X continued fraction.
        This function can be adapted to be used for the similar
        type of continued fraction for ARCTAN X.
        The angle argument is in degrees measurement.
        The function's return value  is the trigonometric tangent value of the angle.
        Here is "Pi" to 31 decimal places:  pi=3.1415926535897932384626433832795
        The smaller the input angle, the greater the accuracy for a given number of
        loops in this function.

      , (c) J.A. 2018
```

A TANGENT FUNCTION

This homemade C-language function uses this tan x continued fraction to calculate tan x.
x is the radian angle equivalent of a degrees angle.

This continued fraction is credited to Johann Lambert:

$$\cfrac{X}{1 - \cfrac{X^2}{3 - \cfrac{X^2}{5 - \cfrac{X^2}{7 - \cfrac{X^2}{9 - \cfrac{X^2}{11}}}}}}$$

. . . (and so on)

```
 -------NOTES  */

double z=(angle * 3.141592653589793238) / 180.0;    /* Convert  the input angle to its corresponding radian angle. */
double N;                                           /* This program uses variable z in place of X in the continued fraction shown. */
double tangent=0.0;

N=17.0;             /* set N to an odd value that is the "lowest denominator" to evaluate in the continued fraction.
                       When N=11, expect 3 decimal places of accuracy at 89 degrees, and 9 decimal places
                       of accuracy for 1 degree.
                       When N=17, expect 9 decimal places of accuracy at 89 degrees.  */

tangent=N;    /* setting tangent to an initial value, here N, the "lowest denominator"  */

 for(;;){      /* a loop to process the continued fraction from the bottom (the "lowest denominator") to the top (numerator) */
             tangent=(double)((z*z) / tangent);
             N=N-2.0;
             tangent = N - tangent;
             if(N==1){  tangent = z / tangent;  break; };
     };

return tangent;
};
/*-------------------------------------------------------------------------------------------------------------------------------------------*/
```

A COMPUTER PROGRAM TO CALCULATE AN INTEGER POWER OF A NUMBER

```
/*-----------------------------------------------------------------------------------------------------------------------------------
   integerpower.c    integer power.c

    Demonstrates a homemade function, integerpower(),  to calculate an integer indicated (as an exponent)
    power value (ie. the Number (N)) of any base value.   power = N = base^exponent

   (c) JA  2018
-----------------------------------------------------------------------------------------------------------------------------------*/
#include "stdio.h"

double integerpower(double basevalue, int exponent);
/* This function calculates the indicated (with the exponent)  integer power of the basevalue.  */

/*---------------------------------------------------------------------------------------------------------------------------------*/
void main(void)
{                                                 /* a sample-test program for the integerpower() function */
double base;
int exponent;
double power;

for(;;){
              fflush(stdin);
              printf("\n\nEnter the base of the indicated power or 0 to exit: "); scanf("\n%lf",&base);
              if(base==0.0){ exit(0); };
              printf("Enter the integer exponent of the indicated power: "); scanf("\n%d",&exponent);
              power=integerpower(base,exponent);
              printf("\n%g^%d = %.14g",base,exponent,power);
            };

printf("\n\nPress A Key."); getch();

return;
}
/*---------------------------------------------------------------------------------------------------------------------------------*/
double integerpower(double basevalue, int exponent)
{    /* This function calculates the indicated (with the exponent)  integer power of the basevalue.  */

double power=1.0;
int s;      /* for the sign of the exponent, in case the user entered a negative exponent  */

if(exponent<0){ s=1;  exponent = - exponent;  };    /* first calculate the power value for a positive exponent,
                                                                                    then take the reciprocal of that power value */
for(;exponent>0;exponent--){
          power=power * basevalue;
     };

if(s==1){  power = 1.0/power;  };

return power;
};
/*---------------------------------------------------------------------------------------------------------------------------------*/
```

A COMPUTER PROGRAM TO CALCULATE A ROOT (OR POWER) OF A NUMBER

```
/* -----------------------------------------------------------------------------------------------------------------------------------
    rootN.c    Calculates the nth root of a radicand value using a successive approximation.
               This program can also calculate a power of a number by using an indicated root (x) of
               of (1/x).  The same function was modified to create a power function.
               The functions created in this program use the standard C language pow() function,
               however a homemade pow() can be created and used instead.  A computer program
               called power.c, that is shown further ahead, will also calculate a power a number, and
               it uses the series sum method, and without using any of the standard C-programming
               language math functions to do so.

               Successive approximation will "zero-in" to the correct result, that is, each new or
               successive, intermediate or temporary result will have a greater numerical precision,
               hence more correct (decimal) digits to the right of the decimal point.

    (c) 2018 JPA
-------------------------------------------------------------------------------------------------------------------------------------*/
#include "stdio.h"
#include "math.h"   /* good to include when working with floats and doubles, etc.  It is
                        also needed for this program to work correctly for the C's standard
                        pow() that is used in this program. */

double  rootN(double index, double radicand);
 /* returns the index root value of the radicand
   Ex.    10 =  rootN(2,100);

This function, and specifically its arguments, can also be modified  to find powers:

  To find the second power of 10 = 10^x  = 10^2, the index of the radicand will be
  the reciprocal of this indicated power, here (1/x)=(1/2)=0.5.   100 = rootN(10,0.5);

   To calculate powers of (e), use 2.718281828459045235... as the radicand.  You
   can also make a specific  function(s) to solve for powers of (e).

The rootN() modified to more easily calculate powers is powerN();    */

double  powerN(double base, double exponent);

/* For both of these functions, generally, about 9 loops will usually give 9 or sometimes more
   decimal places of accuracy.  Both of these functions use the C-language pow(), but you can
   create or write your own pow() using the series for (e) and other mathematical steps.  */
/*-------------------------------------------------------------------------------------------------------------------------------*/
void main(void)
{
double  radicandvalue=0.0;
double  indexvalue=0.0;
double  rootvalue=0.0;

double Base1=0.0;
double Exponent1=0.0;
double powervalue=0.0;
```

```
printf("\nPROGRAM TO CALCULATE THE Nth ROOT or Nth POWER OF A VALUE");

for(;;){

    fflush(stdin);   /* optional, to clear any unused input  */

    printf("\n\nEnter the value that you want to find a root of, or 0 to exit program. \n: ");
    scanf("%lf",&radicandvalue);
    if(radicandvalue<=0.0){ /* such as a negative value */ printf("\ninput error \a"); /* a beep sound */  exit(0); }

    printf("\nEnter the indicated root to be taken. Ex. 2,   2.5,  4, etc.\n: ");
    scanf("%lf",&indexvalue);
    if(indexvalue<=0.0){ /* such as a negative value */ printf("\ninput error \a"); /* a beep sound */  exit(0); }

    rootvalue=rootN(indexvalue,radicandvalue);
    printf("\nThe %g root of %g is: %.15g",indexvalue, radicandvalue, rootvalue);

    fflush(stdin);   /* optional, to clear any unused input  */

    printf("\n\nEnter the base value that you want to find a power of, or 0 to exit program. ");
    printf("\nFor powers of (e) use: e=2.71828182845904523536 as the base.\n:");
    scanf("%lf",&Base1);
    if((Base1)<=0.0){  printf("\ninput error \a");  exit(0); }

    printf("\nEnter the exponent.\n: ");
    scanf("%lf",&Exponent1);

    powervalue = powerN(Base1,Exponent1);
    printf("\nThe %g power of %g is: %.15g",Exponent1, Base1, powervalue);

    printf("\n\nPress A Key");  getch();
    printf("\n______________________________________________________________________");

  }; /* end of for loop */

return;
};
/*------------------------------------------------------------------------------------------------------------------------------*/
double  rootN(double index, double radicand)
{
 /* This function returns the index root value of the radicand. */

double  root=0.0;
double  Rn=1.0;      /* initial guess, ex. for the 30th root of 10 to calculate faster, fewer loops */
double  Rn1=0.0;
double  temp1=0.0;
double  temp2=0.0;
double  diff=0.0;   /* difference */

if(radicand<1.0 && index<1.0 ){ Rn=0.000000000000001 ; };
/* some error prevention, or endless looping protection.
ex. such as for 0.5 root of 0.7 to calculate right  */
```

```
for(;;){

     Rn1 =   (   ((index-1) * Rn)  +   radicand/pow(Rn,(index-1.0))    ) / index;

          temp1=Rn;
          temp2=Rn1;
          if(temp1<0.0){  temp1=-temp1;  };
          if(temp2<0.0){  temp2=-temp2;  };
          diff=temp1-temp2;     if(diff<0.0){  diff=-diff;  };
          if(diff<=0.000000000000001){  break;  };

         Rn=Rn1;        /* for the next iteration of the loop (cycle, recycle) */
         /* printf("\n%g %g",Rn1,Rn);   : optional, to display intermediate results */
        };

return  Rn1;
}
/*------------------------------------------------------------------------------------------------------------------------------*/
double  powerN(double base, double exponent)
{
/* This function returns the x indicated power value of the base */
/* This function is based on the facts that:

              root^x  =  radicand  = number  =  base^x  = power

            , and that a root of a number can be expressed and calculated as a
              power of a number:

              x√ radicand  =  radicand^(1/x)

     This function implements a loop used to find a root of a number,
     and was modified to find the power of a number, mainly by taking
     the reciprocal of the indicated power    */

double  root=0.0;
double  Rn=1.0;        /* initial guess, ex. for the 30th root of 10 to calculate faster, fewer loops */
double  Rn1=0.0;
double  temp1=0.0;
double  temp2=0.0;
double  diff=0.0;       /* difference */

double index=0.0;
double radicand=0.0;

int t=0;    /* a toggle, set to 1 if the exponent is negative */

int loops=1;    /* this is generally not needed due to checking the difference between successive values,
                   but this is used in case there are certain numerical values that produce an infinite loop,
                   as was noticed with a base of 5.6 and a exponent of 3.4                                    */

if(exponent==0){ exponent=1; base=1;  };    /* this is for some error checking, prevention and correction, */
                                             /* in particular, if the user enters an exponent of 0.              */
```

```
if(exponent<0.0){ t=1;  exponent=-exponent;  };      /* make exponent positive, to initially calculate below as
                                                        a positive exponent */

index=1.0/exponent;    /* convert these two values for use in the unmodified root formula */
radicand=base;

if(radicand<1.0 && index<1.0 ){ Rn=0.000000000000001;  /* instead of Rn=1.0 as above */ };
/* This is some error prevention, such as for endless looping protection.
ex. Such as for 0.5 root of 0.7 to calculate right, because the result is actually less than 1   */

for(;;){

     Rn1 =   (   ((index-1) * Rn)  +   radicand/pow(Rn,(index-1.0))    ) / index;

        temp1=Rn;
        temp2=Rn1;
        if(temp1<0.0){  temp1=-temp1;  };     /* ie.,  if temp1 is negative in value  */
        if(temp2<0.0){  temp2=-temp2;  };
        diff=temp1-temp2;     if(diff<0.0){  diff=-diff;  };
        if(diff<=0.000000000000001){    /*  printf("\nloops = %d ",loops);   test ok */   break;  };
        /* printf("\ndiff = %.16g",diff);  test ok */

       Rn=Rn1;        /* for the next iteration of the loop (cycle, recycle) */
        /* printf("\n%g %g",Rn1,Rn);    test ok */
        loops=loops+1; if(loops==100){ break; }; /* endless loop protection for certain values */
       /*  printf("\nloop=%d",loops);  test ok */

      };

if(t==1){  Rn1=1.0/Rn1; };    /* if the exponent was negative, take the reciprocal value */

return  Rn1;
};
/*-----------------------------------------------------------------------------------------------------------------------------------*/
```

The above program used the standard C language built in power of a value function: pow()
If you want to make your own power function, here are the main steps:

1. Within a new program, create a power function. For ex: double power(base, exponent);
2. In this power(), calculate the natural logarithm of the given base, and multiply it to the given exponent of that base. This will be equal to the exponent or indicated power of (e). You can call the LN2() as shown within this book, within this power().
3. Convert the indicated power value to the power of (e). You can call the epower() as shown next within this book, within this power().
4. Within the power(), when done calculating the power value, return the calculated power value.

Ex. Find 5^2 , as derived previously in this book:

b^x = e^(a ln b)
5^2 = e^(2 ln 5)   =   (2.71828182845904523536...)^(2 ln 5)

The computer program, shown further ahead, called power.c utilizes the method just described.

A COMPUTER PROGRAM TO CALCULATE A POWER OF (e)

```
/* -----------------------------------------------------------------------------------------------------------------------------------------------
    epower.c

    A homemade function to calculate any indicated power value of (e).

    (c) JA 2018
-------------------------------------------------------------------------------------------------------------------------------------------------*/

#include "stdio.h"

double epower(double exponent);

/*-----------------------------------------------------------------------------------------------------------------------------------------------*/
void main(void)
{                                        /*  a sample test program to use the epower()   */
double exponent=0.0;
double power=0.0;

for(;;){                        /* a loop to reuse or redo the following program code over and over  */
        fflush(stdin);       /* clear out any potential garbage from the keyboard (the standard input)  memory
                                  buffer input line */

        exponent=0.0;
        printf("\n\nEnter the exponent of (e):  ");
        scanf("\n%lf",&exponent);

        power=epower(exponent);

        printf("\n(e) to the %.16g power is %.16g",exponent,power);

      };    /* end of the loop, now go back to the start of the loop and let the user enter another value */

return;
};
/*-----------------------------------------------------------------------------------------------------------------------------------------------*/
double epower(double exponent)
{
/* Notes---     Here is the power of (e) series:

                 _n=infinity
                 \
     e^x    =    \       _x^n_              :  0!  =  1   and   x^0 = 1                              : (credited to Euler)
                 /        n!
                 /_____
                  n=0

Expanding the terms of this series:

e^x  =  1  +  x^1  +  _x^2_  +  _x^3  +  _x^4  +  _x^5  +  _x^6  +  _x^7_  +  . . .  : a derivation of this is shown
                        2!        3!       4!       5!       6!       7!               in the appendix section.

Simplifying the factorials:
```

$$e^x = 1 + x^1 + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + \frac{x^6}{720} + \frac{x^7}{5,040} + \frac{x^8}{40,320} +$$

$$+ \frac{x^9}{362,880} + \frac{x^{10}}{3,628,880} \ . \ . \ .$$

To use an exponent less than or equal to 1 for the series for (e), e^x can be factored to smaller powers of (e):

Ex. e^5.2 = e^(5 + 0.2) = e^5 e^0.2 = e^(wholepart) e^(fractionalpart)

The integer power of (e) can be found by repeated multiplication of e=2.718281829...

Here is another method:

Ex. e^5.2 = e^10 e^0.52

The calculation below is set to return 16 decimal places of accuracy.

---------Notes */

```
double power=1.0;

double numerator=1.0;        /* the initial value for the first terms numerator */
double denominator=1.0;     /* the initial value of the first terms denominator */
double term=0.0;                /* the calculated value of each successive term in the series */
double sum=1.0;                /* the initial value of the first term */

double e=2.71828182845904523536;

int wholepart=0;                /* whole part of the entered exponent */
double  fractionalpart=0.0;   /* fractional decimal part of the entered exponent */
int N=1;

int t=0;  /* a toggle, set to 1 if the exponent is negative */

/*  printf("\n%.24g",exp(1.0)); getch();   /* c language's, power of (e) function , test ok,  2.7182818284590451 */

if(exponent<0.0){  exponent= -exponent;  t=1; };   /* if the exponent is negative, convert it to a positive exponent  */

wholepart=(int)exponent;
fractionalpart = exponent - wholepart;    /* this will be evaluated in the series */
numerator=fractionalpart;;
/* printf("\nwhole part = %d,  fractional part = %lf",wholepart,fractionalpart);     getch();  test ok */

for(;;){     /* a loop to find the value of each term and add it to the sum */
            term=numerator / denominator;
            if(term<=0.0000000000000001){   sum=sum+term; break; };
            sum=sum+term;

            numerator=numerator*fractionalpart;
            N=N+1;
            denominator=denominator*N;      /* to create the factorial in the denominator */
       };
```

```
/*  printf("\n%.16g",sum); getch();  test ok */

while(wholepart>0){      /* calculate:   e^(wholepart)  */
                                        power=power *e;
                                        wholepart=wholepart-1;
                                    };

power=power*sum;      /*  e^(wholepart)  e^(fractionalpart)  =  (power)(sum)  */

if(t==1){  power = 1/power;  };    /* if the exponent was negative */

return power;
};
/*-----------------------------------------------------------------------------------------------------------------------------------------------------------*/
```

A COMPUTER PROGRAM TO CALCULATE A NATURAL LOGARITHM

```
/* ----------------------------------------------------------------------------------------------------------------------------------------------
    LN.C    Natural Logarithm.c

    A computer program to calculate the natural logarithm of a number between 0 and <=2.  The program can
    be adapted  to build another natural logarithm program or function  to calculate the natural logarithm of
    higher values, and this is demonstrated below with the LN2() function which is based on the LN().

    In general, as the program is, it is very good for calculating the mantissa values of the logarithms
    on numbers that are less  than or equal to 1.

    (c) J.A. 2018
                                 _n=infinity
                                \
LN  N    =    LN (1+x)    =     \          (-1)^(n+1) .  _x^n_         :N<= 2 , x<=1        : A NATURAL LOGARITHM SERIES
                                /                          n             x=N-1                (Credited to Nicolaus Mercator)
                               /______
                                 n=1

Expanding some initial terms:

LN N  =  LN (1+x)  =   x^1    -   x^2    +   x^3    -   x^4    +  x^5    -   x^6   +  .  .  .          : N <= 2  ,  x <= 1
                        1          2          3          4         5          6                          x = (N-1)
-------------------------------------------------------------------------------------------------------------------------------------------------*/
#include "stdio.h"

double LN(double number);
double LN2(double number);
/*-----------------------------------------------------------------------------------------------------------------------------------------------*/
void main(void)
{
double number;

for(;;){                       /* a loop to reuse or redo the following program code over and over  */
        fflush(stdin);     /* clear out any potential garbage from the keyboard (the standard input)  memory
                              buffer input line */
        number=0.0;
        printf("\n\nEnter the number (between 0 and 2, and the lower, the better accuracy)  to");
        printf("\nfind the natural logarithm  of, or 0 to exit:  ");
        scanf("\n%lf",&number);
        if( (number<=0.0) || (number>=2.0) ){ printf("\a");  /* a beep sound */ exit(0);  };
        printf("\nThe natural logarithm  of %.16g is %.9g",number,LN(number));

        fflush(stdin);  number=0.0;
        printf("\n\nEnter the number greater than 0 to find the natural logarithm  of, or 0 to exit:  ");
        scanf("\n%lf",&number);
        if( (number<=0.0) ){ printf("\a");  /* a beep sound */ exit(0);  };
        printf("\nThe natural logarithm of %.16g is %.10g  and",number,LN2(number));
      };    /* end of the loop, now go back to the start of the loop and let the user enter another value */

return;
};
/*-----------------------------------------------------------------------------------------------------------------------------------------------*/
```

```
/*-----------------------------------------------------------------------------------------------------------------------------------------------------------*/
double LN(double number)
{
/* This function calculates the natural logarithm of a number less than 2.  It can be used to construct a program
    to calculate the natural logarithm of any value. */

double logarithm=0.0;

double numerator;
double denominator=1;

int termnumber=1;
int maxterms=45;     /* Set this to the number of terms you want to consider and evaluate in the series.  */

double term=0.0;
number=number-1;
numerator=number;

int s=0;            /* a toggle, for the positive and negative sign of the term */

for(;;){            /* a loop to calculate each term of the series and add it to the sum */

            term = numerator / denominator;

            if((s%2)==0){  logarithm=logarithm + term;   };    /* if s, or the term number is even */
            if((s%2)==1){  logarithm=logarithm - term;    };   /* if s, or the term number  is odd */
            s=s+1;

           /*  termnumber=termnumber+1;  if(termnumber==maxterms){  break; };                          */
            /* can also program the loop to terminate when the difference between two successive
               terms is a low value, for example: 0.000001, and this is shown and used right here:          */
              if(term<=0){ term = -term; };  if(term<=0.0000000000000001){    break; };

            numerator=numerator*number;
            denominator = denominator+1;
       };

return logarithm;
};
/*----------------------------------------------------------------------------------------------------------------------------------------------------------*/
 double LN2(double number)
{
/* ******This function calculates the natural logarithm of any positive valued  number.

    Here is an example of what takes place in this function,and it is described more in this books text.

     Ex.  Find or calculate:   ln 5

       ln 5  =  ln ((e^2) (0.676676416))
       ln 5  =  ln e^2  +  ln 0.676676416
       ln 5  =  2  +  (-0.390562087)
       ln 5  =  1.609437913
****** */
```

```
double N=number;

double logarithm=0.0;
double numerator;
double denominator=1;
int  termnumber=1;
int maxterms=34;      /* Set this to the number of terms you want to consider and evaluate in the series.
                                    A  value of 9 will give 9 correct decimal digits for ln 1.1, and less correct digits
                                    for higher values. For lesser numbers, and less than 1 in general, the accuracy
                                    is much  greater.  A value of 45 will give 3 correct decimal digits for ln 1.9             */
double term=0.0;

double e=2.71828182845904523536;
int x=0;   /* the indicated power of (e) */

for(;;){                   /* factor N, so as to get the factor of N that is a power of (e), and the factor of N that is
                                 a  mantissa value that is <=1  */
              if(N<=1.2){ break; };  /* 1.0 is adequate, but can try other values here, for ex., 1.2 gave good results */
              N=N/e;
              x=x+1;
        };

/* printf("\n-- %d -- %.9g",x,N); getch();  test ok  */

N=N-1;
numerator=N;    /* to  properly adjusted value of numerator = (x),  or (z),  for the series */

int s=0;            /* a toggle, for the positive and negative sign of the term */

for(;;){            /* a loop to calculate each term of the series and add it to the sum */

               term = numerator / denominator;

               if((s%2)==0){  logarithm=logarithm + term;   };    /* if s, or the term number is even */
               if((s%2)==1){  logarithm=logarithm - term;    };   /* if s, or the term number  is odd */
               s=s+1;

              /*  termnumber=termnumber+1;  if(termnumber==maxterms){  break; };                               */
               /* can also program the loop to terminate when the difference between two successive
                   terms is a low value, for example: 0.000001, and this is shown and used right here:           */
                  if(term<=0){ term = -term; };  if(term<=0.0000000000000001){    break; };

               numerator=numerator*N;
               denominator = denominator+1;
        };

         logarithm=x+logarithm;

return  logarithm;
};
/*------------------------------------------------------------------------------------------------------------------------------------------------------*/
```

A COMPUTER METHOD TO FIND A POWER OF A NUMBER

This method to find a power of a number utilizes the series of terms sum method, rather than successive approximation. This program and method does not rely upon any of the C-language pre-written standard math functions, and is completely homemade and may be modified if necessary. This program utilizes some of the homemade math functions shown previously in this book so as to create a power of a number function.

```
/* ----------------------------------------------------------------------------------------------------------------------------------------------------
    power.c

    A homemade function to calculate any power value.

    (c) JPA FEB 2018
-----------------------------------------------------------------------------------------------------------------------------------------------------*/
#include "stdio.h"
#include"math.h"

double power(double base, double exponent);
/* required for this function  to calculate any power value, are these two function: */
double LN2(double number);
double epower(double exponent);

/*-------------------------------------------------------------------------------------------------------------------------------------------------------*/
void main(void)
{
double base=0.0;
double exponent=0.0;
double logarithm=0.0;
double x=0.0;
double N=0.0;

/*  printf("\n%.16g",pow(0.32,0.32)); getch();  just test to compare to, result is 0.6944612105965807, requires math.h */

for(;;){                          /* a loop to reuse or redo the following program code over and over  */
        fflush(stdin);      /* clear out any potential garbage from the keyboard (the standard input)  memory
                                 buffer input line */

        base=0.0; exponent=0.0;
        printf("\n\nEnter the base of the indicated power:  ");
        scanf("\n%lf",&base);
        if( base<=0.0 ){ printf("\a");  /* a beep sound */ exit(0);  };
        printf("\nEnter the exponent of the base: ");
        scanf("\n%lf",&exponent);

        N=power(base,exponent);
        printf("\n%.16g",N);

      };    /* end of the loop, now go back to the start of the loop and let the user enter another value */

return;
};
/*--------------------------------------------------------------------------------------------------------------------------------------------------------*/
```

```
double LN2(double number)
{
/* This function calculates the natural logarithm of any positive valued  number.

    Here is an example of what takes place in this function,and it is described more in this books text.

     Ex.  Find or calculate:   ln 5

       ln 5  =  ln ((e^2) (0.676676416))
       ln 5  =  ln e^2  +  ln 0.676676416
       ln 5  =  2  +  (-0.390562087)
       ln 5  =  1.609437913
*/

double N=number;

double logarithm=0.0;
double numerator;
double denominator=1;
int termnumber=1;
int maxterms=34;      /* Set this to the number of terms you want to consider and evaluate in the series.  */

double term=0.0;

double e=2.71828182845904523536;
int x=0;   /* the indicated power of (e) */

for(;;){                        /* factor N, so as to get the factor of N that is a power of (e), and the factor of N that is
                                     a  mantissa value that is <=1  */
               if(N<=1.2){ break; };   /* 1.0 was originally used, and is ok,  but 1.2 seems to give better  results */
               N=N/e;
               x=x+1;
         };

/* printf("\n-- %d -- %.9g",x,N); getch();  test ok  */

N=N-1;
numerator=N;    /* to  properly adjusted value of numerator = (x),  or (z),  for the series */

int s=0;            /* a toggle, for the positive and negative sign of the term */

for(;;){            /* a loop to calculate each term of the series and add it to the sum */

                term = numerator / denominator;

                if((s%2)==0){  logarithm=logarithm + term;   };    /* if s, or the term number is even */
                if((s%2)==1){  logarithm=logarithm - term;    };   /* if s, or the term number  is odd */
                s=s+1;

               /*  termnumber=termnumber+1;  if(termnumber==maxterms){  break; };       */
                /* can also program the loop to terminate when the difference between two successive
                    terms is a low value, for example: 0.000001, and this is shown and used right here:           */
                   if(term<=0){ term = -term; };  if(term<=0.0000000000000001){    break; };
```

```
            numerator=numerator*N;
            denominator = denominator+1;
        };    /* end of the for loop */

        logarithm=x+logarithm;

return  logarithm;
};
/*------------------------------------------------------------------------------------------------------------------------------------------------------*/
double epower(double exponent)
{
/* Notes---    Here is the power of (e) series:

                    _n=8_
                   \
        e^x   =     \       _x^n_              :  0!  =  1   and   x^0 = 1
                    /        n!
                   /____
                    n=0

Expanding the terms of this series:

e^x  =  1  +  x^1   +   _x^2_   +   _x^3_  +   _x^4_   +    _x^5_   +   _x^6_   +    _x^7_   +   . . .  : a derivation of this is shown
                          2!          3!         4!          5!          6!           7!                  in the appendix section

Simplifying the factorials:


e^x  =  1  + x^1  +  _x^2_  + _x^3_  +  _x^4_  + _x^5_  +  _x^6_  + _ x^7_  +  __x^8__  +
                       2         6        24       120       720      5,040     40,320

        +   _ x^9__   +     _  x^10__   . . .
            362,880         3,628,880


To use an exponent less than or equal to 1 for the series for (e),   e^x can be factored to smaller powers of (e):

Ex.   e^5.2  =  e^5  e^0.2    =   e^(wholepart)  e^(fractionalpart)

The integer power of (e) can be found by repeated multiplication of e=2.718281829...

Here is another method:

Ex.   e^5.2  =  e^10 e^0.52

The calculation below is set to return 16 decimal places of accuracy.

---------Notes */


double power=1.0;
```

```
double numerator=1.0;         /* the initial value for the first terms numerator */
double denominator=1.0;     /* the initial value of the first terms denominator */
double term=0.0;                 /* the calculated value of each successive term in the series */
double sum=1.0;                 /* the initial value of the first term */
double P=0.0;                     /* the returned power of the number */

double e=2.71828182845904523536;

int wholepart=0;     /* whole part of the entered exponent */
double  fractionalpart=0.0;   /* fractional decimal part of the entered exponent */
int N=1;

int t=0;  /* a toggle, set to 1 if the exponent is negative */

/* printf("\n%.24g",exp(1.0)); getch();    c language's, power of (e) function , test ok,  2.7182818284590451 */
/* printf("\n*** %.16g",exponent);  getch();  test ok */

if(exponent<0.0){  exponent= -exponent;  t=1; };   /* if the exponent is negative, convert it to a positive exponent  */

wholepart=(int)exponent;
fractionalpart = exponent - wholepart;    /* this will be evaluated in the series */
numerator=fractionalpart;;

/*  printf("\nwhole part = %d,  fractional part = %.16g",wholepart,fractionalpart);     getch();   test ok */

for(;;){      /* a loop to find the value of each term and add it to the sum */
              term=numerator / denominator;
              if(term<=0.0000000000000001){   sum=sum+term; break; };
              sum=sum+term;

              numerator=numerator*fractionalpart;
              N=N+1;
              denominator=denominator*N;      /* to create the factorial in the denominator */
        };

/*  printf("\n%.16g",sum); getch();  test ok */

while(wholepart>0){       /* calculate e^(wholepart)  */
                              power=power *e;
                              wholepart=wholepart-1;
                            };

power=power*sum;       /*  e^(wholepart)  e^(fractionalpart)  =  (power)(sum)  */

if(t==1){  power = 1/power;  };    /* if the exponent was negative */

P=power;
return P;
};
/*------------------------------------------------------------------------------------------------------------------------------------------------------*/
```

```
/*------------------------------------------------------------------------------------------------------------------------------------------------------------------*/
double power(double base, double exponent)
{
double logarithm=0.0;
double x=0.0;
double N=0.0;

        logarithm=LN2(base);
        /* printf("\nThe logarithm of %lf is %.10lf",base,logarithm);  printf("\nPress A Key ");  getch();    test ok */

        x=exponent * logarithm;       /*  this calculates the indicated power = x =  exponent of (e)^x  */
      /*  printf("\nx = %.16g",x); getch();  test ok */

       N=epower(x);

return N;
};
/*--------------------------------------------------------------------------------------------------------------------------------------------------------------------*/
```

A COMPUTER METHOD TO SOLVE EQUATIONS

```
/*--------------------------------------------------------------------------------------------------------------------------------------------
EQSOLVE.C          Computerized Equation Solving
                    (c) J.A. 2014

A minimal basic format for solving any single variable (N) equation by numerical computation using successive
approximation.

Given an equation, start by placing it into the form of this example:

    N = f(N)

, that is, where N is a function of N itself, algebraically, this (N on the left side) cannot be solved for since it is also part of
the solution (on the right side), the way to overcome this is to make the N's ever-so slightly different in value.  We can
identify these two values as something like:  N1 and N2:

   N2 = f(N1)

Ex:

    x^2 + 2x = 15            : a "quadratic" equation, after solving for x
                               (not using the quadratic formula):

    x = sqrt(15 - 2x)          and for the computer solution method below, this is transformed to:

   N2 = sqrt(15 - 2N1)       : sqrt means square-root

                               Or by factoring the common factor x on the
                               left side of the original equation:

    x (x + 2) = 15
    x = 15/(x + 2)             and for the computer solution method below, this is transformed to:

   N2 =  15/(N1 + 2)

The concept is that N2 will be a better approximation to the actual true result value of the equation than N1 is.  N1 is then
set equal to the value of N2, and then the process (ie. equation) is repeated.  Caution must of course be taken in selecting
a value for the initial value of N1. For example, division by 0 will yield in an error.  You can even write the function to
accept an argument that is an initial guess.  Note,  this method below, as of written, will only produce one solution per
equation.  For the example above, the solution for one equation is +3.0 and -5.0 for the other.  Though they are different in
value, they both satisfy the original equation.  Hence, its always a good idea to graph the equation by setting it equal to
zero, and then setting it equal to y, and then plotting points and connecting them with a smooth (true, accurate, without
using straight line segments) curve.  The solutions (of  (x) in the equations expressed this way) will be the points where
the curve crosses the x-axis, where y=0.

The user/programmer can use the function shown next that is called mysqrt( ) as a basic format or example to follow for
their own functions.  The format can even be used to solve for constants.


--------------------------------------------------------------------------------------------------------------------------------------------*/
```

```
#include "stdio.h"


double mysqrt(double X);          /* A homemade square-root math function.  */
                                  /* Here, we let X represent the radicand.      */

/*------------------------------------------------------------------------------------------------------------------------------------------------------------*/
void main(void)
{
double radicand;                  /* doubles are 8 byte "floating" point data types in C */
double root;

radicand=2.0;
root = mysqrt(radicand);      /* call (transfer computer control or sends the program to) the
                                    mysqrt() function with the argument or variable identified as:
                                    radicand, and receive the "return(ed) value" (the numerical
                                    root) of the function and assign it to the variable identified
                                    as:  root.  The argument to the function can also be an
                                    explicit constant.  For ex.,   root  =  mysqrt(2.0)  */

printf("\nThe square-root of %.16g = %.16g",radicand,root);         /* display the result on the computer screen */

/*
   The output displayed on the screen will look something like:

   The square-root of 2.0 = 1.414213562...
*/

radicand=10.0;                /*  :Let's also find the square-root of 10.0, hence reusing the function already written. */
root = mysqrt(radicand);
printf("\nThe square-root of %.16g = %.16g",radicand,root);

return;
};
/*------------------------------------------------------------------------------------------------------------------------------------------------------------*/
```

```
/*-------------------------------------------------------------------------------------------------------------------------------------------*/
double mysqrt(double X)
{
double N1=1.0;            /* set to an initial guess for this function */
double N2;
double tempx1;
double tempx2;
double diff;               /* for the difference */

                           /* An infinite "for loop" (a program section that can
                              repeat endlessly until it's programmed to stop):   */
for(;;){

     /* - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - ----------- - - - - - - - - - - - - - - - - - - - - */
     /*                               Place Your Equation Here:                                  */

     N2=(N1 + X/N1)/2.0;           /* : a square root  successive approximation formula     */

    /* - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - -- - - - - - - */


    /*   The following code checks to see if the difference between
         N2 and N1 is as small as possible (for my computer), which
         will indicate they are near or practically equal, and the loop
         can cease since we found the true result.                          */


    tempx1=N1;     if(tempx1<0.0){  tempx1=-tempx1;  };           /*   :make positive if negative                   */
                                                                  /*  - is a negation operation, it will
                                                                      change the sign of the operand.
                                                                      Same as:  0 - (tempx1) = 0 - tempx1       */
    tempx2=N2;     if(tempx2<0.0){  tempx2=-tempx2;  };           /*   :make positive                              */
    diff=tempx1-tempx2;     if(diff<0.0){  diff=-diff;  };        /*   :get difference and ensure it's positive    */

    if(diff<=0.000000000000001){  break;  };                      /*   :Here,  the value of difference with respect
                                                                       to, or compared to, the value shown is the
                                                                       mathematical condition (logic or truth) that
                                                                       will determine if the loop should break
                                                                       stop or terminate) right here.   */

    N1=N2;         /* : Preparing for the next iteration (instance or program execution or running) of the loop.    */
   };              /* : THis closing bracket signals no more loop statements to process, so repeat loop          */


return N2;         /* Return computer control back to the program code (the next statement just beyond where
                      the function was called) with the value of N2.  N2 is the square root of X.  */
};
/*---------------------------------------------------------------------------------------------------------------------------------------------*/
```

A COMPUTER METHOD TO SOLVE DIFFICULT EQUATIONS

/*--
EQSOLVE2.C

This program demonstrates a way to solve a **single variable equation** (ie., an equation with just one variable within it, however there may be several instances of it in the equation), and it is very useful to solve difficult equations when there is no known or simple algebraic solution (ie., an equation to do so). This program contains an algorithm I made called a **Fast Algorithm To Solve Equations**. There is a alternative version of this algorithm placed in the **Extras And Late Entries** section at the end of this book that you may prefer. After making a study of this discussion and program, please then check the topic of "An improved algorithm to solve difficult equations", and which is similar and was based on this discussion, but with a slightly better algorithm. The program there with this algorithm is called there is called FESA.C

The relatively short C-language program given below solves for (x) in an example equation that is a multiple term exponential equation of the form:

N1^x + N2^x = N3 or even with more terms such as: RESULT = N1^x + N2^x + N3^x + ...

, and gives about 10 decimal places of accuracy of x. It is a difficult equation without an algebraic solution.

The example below solves the hard coded (set, but can be changed) and specific equation of:

5^x + 6^x = 61 : the solution is x = 2.0 , checking: 5^2 + 6^2 = 25 + 36 = 61

This method below does not use recursion. Its more of a fast "trial and error, precision method" so as to "zero in" (ie., find) the result quicker than simply incrementing it by only a very small amount after each loop. Requires only ten loops max. per correct digit position. Ex. For 9 places, instead of 1,000,000,000 loops for a net increment of 1 in (x) when incrementing by just 0.000,000,001, my process in the worst case requires only 100 loops, and on average (ie. digit 5 of 10 possible digits) of only 50 loops will be required to solve for (x). The program assumes that (x), the variable being solved for is positive. (x) is initially set to 0.0. Following the program are other code samples for other equations.
(c) J.A. 2014
--*/

```
#include "stdio.h"
#include "stdlib.h"
#include "math.h"
#include "float.h"
/*---------------------------------------------------------------------------------------------------------------------------------*/
void main(void)
{
double N1=5.0;              /* a value used in the given equation.  Can make this and other values a user input value. */
double N2=6.0;              /* a value used in the given equation  */
double RESULT=100.0;    /* right side of the given equation      */

double x=0.0;                /* to be found, this can set to a low positive value such as
                                    0.0000000001 if division by 0 error is to be avoided in the algorithm */
double TEMP=0.0;        /* ie. left side of equation */

int n2=0;                     /* outer loop counter variable */
```

```
int n=0;                        /* inner loop counter variable */

double a=1.0;                   /* (variable) increment to be added to current value of the variable (x) */

/* Below is the main algorithm for a fast method to solve an equation.  It is short with only a few lines of code statements,
   but  it is very powerful in terms of what it does.  It is just two loops, and one loop is an inner loop of the outer loop.  It is
   a "brute force" or "trial and error"  type of algorithm, but rather more of a "calculated or fast trial and error" algorithm.*/

/******************************  FAST ALGORITHM TO SOLVE EQUATIONS  *********************************************/

for(n2=0;n2<=15;n2++){     /* outer loop, get 15 digits of the result.  Can adjust this value for other computer systems. */

    for(n=0;n<=9;n++){                          /* inner loop, find a correct digit from 0 to 9 */
          TEMP=pow(N1,x) + pow(N2,x);    /* pow(N1,x) is a function for solving and returning:  N1^x  */
          /* printf("\nTemp = %.15g",TEMP);   optional, display the intermediate temporary result(s) */
          if(TEMP>RESULT){ x=x-a; break; };       /* halt, stop, break from the inner loop */
          x=x+a;   /* increment (x) by (a), so as to have a more accurate value of (x)  */
       };

   if( (TEMP - RESULT) <= 0.0000000000000001){ break; };   /*  break from the outer loop */
    /* if(TEMP==RESULT){ break; };  */       /* optional, break or halt the loop or repetition early if these are already equal */

    a = a/10.0;    /* (a) is essentially the weight of the next digit to find */
  };
/****************************************************************************************************************************/

printf("\nx = %.15g",x);             /* display the value of the independent variable (here, x) solved for */

printf("\n\nChecking the equation with the solved value for x:  ");
printf("\n5^x + 6^x = %.15g + %.15g = %.15g",pow(N1,x), pow(N2,x), pow(N1,x) + pow(N2,x));  /* should = RESULT */

printf("\n\nPress A Key To Exit");  fflush(stdin); getch();

exit(0);
};
/*----------------------------------------------------------------------------------------------------------------------------------*/
```

Here is the output with the hard coded values:

```
x = 2.28768427762655

Checking the equation with the solved value for x:

5^2.28768427762655 + 6^2.28768427762655 = 39.7212301043361 + 60.2787698956639 =100

Press A Key To Exit
```

```
/*--------------------------------------------------------------------------------------------------------------------------------*/
```

Here is code that can be substituted above for finding Square Roots.
For cube roots, the formula is almost identical:

```
   for(n=0;n<=9;n++){
        if((x*x)>N){ x=x-a; break; };       /* where N = root * root = radicand  */
        x=x+a;
```

```
        };

    if((x*x)==N){  break; };
    a=a/10.0;
   };
/*------------------------------------------------------------------------------------------------------------------------------------------------*/
```

This code below will work for cube roots, even when written in a form that cannot be solved with standard iteration:

For the cube root of 10 (or set 10 equal to a variable such as N):

```
for(n2=0;n2<=14;n2++){

    for(n=0;n<=9;n++){
        if(x>(10.0/(x*x))){ x=x-a; break; };              /* : Here N is "hard-coded" to 10.0,  */
        x=x+a;                                            /*   only for this example code.        */
       };

   if(x==(10.0/(x*x))){  break; };
   a=a/10.0;
  };
/*---------------------------------------------------------------------------------------------------------------------------------------------*/
```

Here is the loop code for solving the Cubic equation of: x^3 + x^2 + x^1 + C = 14

For this example here, C is arbitrarily set to 0, but should be coded into the program.

```
   x=2.0

   for(n2=0;n2<=14;n2++){

       for(n=0;n<=9;n++){
          if( (x*x*x + x*x + x)>14.0){ x=x-a; break; };              /* : 14 was "hard-coded" as a constant; best to use a variable */
          x=x+a;                                                     /*   so as to reuse the function for other values.              */
         };

       if((x*x*x + x*x + x )==14.0){  break; };
       a=a/10.0;
     };
/*--------------------------------------------------------------------------------------------------------------------------------------------*/
```

Extra notes on the above fast algorithm for solving a single variable equation:

If solving something such as: 100 = 10^x and using the above method to solve for (x), the indicated power, it will actually be solving for a logarithm value: x = log 100.   You can use any base and number as needed, perhaps (e)=2.718281828 as the base if you need to find a natural logarithms: logarithm = ln Number

If solving something such as: 0.5 = sin(x), it will actually be solving for: x = arcsin 0.5 , and note the value found will be a radian angle because it is natural, rather than the man-made values of degrees. The standard C language function for sin(x) considers the argument, here (x), as a radian angle.

For a dynamic equation or user input equation, such as a text string, after parsing it, perhaps the hard coded equation in this algorithm can be replaced by a hard coded equation such as::   result = term1 + term2 + ... term20 , and where each term is initially set to 0.0

A COMPUTER PROGRAM FOR A SIMPLE MULTI-FUNCTION CALCULATOR

```
/*-----------------------------------------------------------------------------------------------------------------------
calculator.c

    A scientific calculator program.  This program can be improved to
    implement any math function desired into this calculator program.
    Use the examples of the operations shown in this program to
    implement new operations into the calculator.  Look up C's
    prewritten math functions to use, such as in the math.h file.

   This type of calculator program is often called a "line calculator".
   There are various methods to input the desired expressions using
    the standard keyboard as the input mechanism, and the "syntax"
    or proper way to enter them for this program is perhaps the simplest.

    For some computers, you can run multiple instances of this program.

    (c) JA  2018
-------------------------------------------------------------------------------------------------------------------------*/
#include "stdio.h"
#include "math.h"
/*-----------------------------------------------------------------------------------------------------------------------*/
void main(void)
{
double operand1=0.0;   double operand2=0.0;
char command=0;        double result=0.0;   int validcommand=0;

start: ;                /* here, start is a label which is a point in a program which can be called with the goto command */
system("cls");      /* a "clear screen" command sent to the computers operating system-program */

printf("\nMATHIZATION BOOK  CALCULATOR PROGRAM");

printf("\n\nSymbol Operation     Example");
printf("\n");
printf("\n     +   add          5 + 2");
printf("\n     -   subtract     10 - 3");
printf("\n     * x multiply     4 * 5 or 4 x 5");
printf("\n     /   divide       9 / 3");
printf("\n     s   square-root  10 s s   for 2\\/ 10 ");
printf("\n     c   cube-root    8 c c    for 3\\/ 8  ");
printf("\n     p   power        5 p 2    for 5^2 ");
printf("\n     r   root         100 r 2  for 2\\/ 100 ");
printf("\n     e   power of e   2 e e    for e^2 ");
printf("\n     l   natural log  5 l l    for ln 5 ");
printf("\n     L   common log   10 L L   for Log 10 ");
printf("\n     S   Sine         45 S S   for SIN 45, ArcSin use: sinvalue S 1: 0.5 S 1 ");
printf("\n     C   Cosine       45 C C   for ArcCos use:  cosvalue C 1");
printf("\n     T   Tangent      45 T T   for ArcTan use:  tanvalue T 1");
printf("\n     a   angle conversion    degrees a 1 for radians, radians a 0 for degrees");
printf("\n     H   Pythagorean Theorem for hypotenuse (c), 4 H 3 , c = a H b , c = b H a");
```

```
printf("\n     Z   Pythagorean Theorem for side (a or b), 5 Z 4 , a = c Z b , b = c Z a");
printf("\n     h   program help  0 h 0");
printf("\n     q   Exit Program  0 q 0 ");

for(;;){                                             /* main loop, let user enter each expression to calculate */
        operand1=0.0;       /* clear out old values, set them to 0 */
        operand2=0.0;  command=0;   result=0.0;   validcommand=0;

        printf("\n\n> ");
        fflush(stdin);
        scanf("\n%lf %c %lf",&operand1,&command,&operand2);

       if(  (command == '+') || (command == '-') || (command == '*') || (command == '/') || (command == 's') ||
          (command == 'c') || (command == 'p') || (command == 'e') || (command == 'l') || (command == 'L') ||
          (command == 'S') || (command == 'C') || (command == 'T') || (command == 'a') || (command == 'H') ||
          (command == 'h') || (command == 'q') || (command=='x') || (command=='r')  || (command=='Z')  )
          {    validcommand=1;  };

       if(validcommand==0){  printf("\n\aERROR, UNKNOWN OPERATION OR COMMAND, ENTER> 0 h 0 for help");
                             continue;  };   /* the continue command will continue from the start of the loop */

        if(command=='+'){           /* add */
                             result = operand1 + operand2;
                          };

        if(command=='-'){           /* subtract */
                             result = operand1 - operand2;
                          };

        if( (command=='*') || (command=='x')){      /* multiply  */
                             result = operand1 * operand2;
                          };

        if(command=='/'){           /*divide */
                             result = operand1 / operand2;
                          };

        if(command=='s'){           /* square-root */
                             if(operand1<0.0){  printf("\n\aERROR, USE POSITIVE VALUES"); continue;  };
                             result =sqrt(operand1);    /* include math.h for this */
                          };

        if(command=='c'){         /* cube-roots  */
                             if(operand1<0.0){  printf("\n\aERROR, USE POSITIVE VALUES"); continue;  };
                             result = pow(operand1,0.33333333333333333333);
                          };

        if(command=='p'){         /* power */
                              result = pow(operand1, operand2);
                          };

        if(command=='p'){         /* root */
                              result = pow(operand1, 1.0/operand2);
                          };
```

```
if(command=='e'){          /* power of (e) */
                         result = exp(operand1);   };
if(command=='l'){         /* natural, base (e) logarithm */
                        if( (operand1 < 0.0)){
                                printf("\n\aERROR, USE A POSITIVE VALUE"); continue; };
                         result = log(operand1);     };   /* C uses log() for the e log base, and log10() for 10 base */

if(command=='L'){          /* common, base 10, logarithm */
                        if( (operand1 < 0.0)){  printf("\n\aERROR, USE A POSITIVE VALUE"); continue; };
                        result = log10(operand1);
                    };

if(command=='S'){           /* Sine */

                       if(operand2==1.0){                  /* for arcsine */

                        if( (operand1 < -1.0) || (operand1 > +1.0 )){
                                printf("\n\aERROR, USE A VALUE BETWEEN -1 AND +1"); continue; };

                                          result=asin(operand1);   /* convert this radian angle to its
                                                                       equivalent degree angle */
                                          result=result * (180.0/3.14159265358979323);
                                          goto displayresult;
                                          };

                        if( (operand1 <0.0) || (operand1 >180.0)){
                                printf("\n\aERROR, USE AN ANGLE BETWEEN 0 AND 180 DEGREES");
                                continue; };

                         operand1 = (operand1 * 3.141592653589793238)/180.0;
                                                      /* convert the degrees angle to its corresponding radian
                                                         angle for C-language sine function          */

                        result = sin(operand1);    /* include math.h for this */
                    };



if(command=='C'){           /* Cosine */

                       if(operand2==1.0){                  /* for arccosine */

                        if( (operand1 < -1.0) || (operand1 > +1.0 )){
                                printf("\n\aERROR, USE A VALUE BETWEEN -1 AND +1"); continue; };

                                          result=acos(operand1);   /* convert this radian angle to its
                                                                       equivalent degree angle */
                                          result=result * (180.0/3.14159265358979323);
                                          goto displayresult;
                                          };

                        if( (operand1 <0.0) || (operand1 >180.0)){
                                printf("\n\aERROR, USE AN ANGLE BETWEEN 0 AND 180 DEGREES");
```

```
                                continue; };

                        operand1 = (operand1 * 3.141592653589793238)/180.0;
                 /* convert the degrees angle to its  corresponding radian angle for  c-language sine function  */
                        result = cos(operand1);    /* include math.h for this */      };

if(command=='T'){           /* Tangent  */

                       if(operand2==1.0){                  /* for arctangent */
                                          result=atan(operand1);   /* convert this radian angle to its
                                                                         equivalent degree angle */
                                          result=result * (180.0/3.14159265358979323);
                                          goto displayresult;
                                          };

                        if( (operand1 <0.0) || (operand1 >180.0)){
                                printf("\n\aERROR, USE AN ANGLE BETWEEN 0 AND 180 DEGREES");
                                continue; };

                        operand1 = (operand1 * 3.141592653589793238)/180.0;
                                                      /* convert the degrees angle to its
                                                         corresponding radian angle for
                                                         c-language sine function          */

                        result = tan(operand1);    /* include math.h for this */
                     };

 if(command=='a'){            /* angle conversion  */

             if(operand2==1.0){    /* converting a degrees angle value to its corresponding radian angle value */
                                     result  = (operand1 * 3.141592653589793238)/180.0;
                                };

             if(operand2==0.0){   /* converting radian angle  value to its corresponding degrees  angle value */
                                      result= operand1  * (180.0/3.14159265358979323);
                               };
                     };


if(command=='H'){           /* Pythagorean, Right Triangle Theorem , can use letter P also */
                                                      /* Solves for the hypotenuse side of a right triangle */

                        result =sqrt( (operand1 * operand1) + (operand2 * operand2) );   /* ex. c = \/ a^2 + b^2  */
                     };

if(command=='Z'){           /* Pythagorean, Right Triangle Theorem  solve for the side of right triangle */
                                                      /*input format:  hypotenuse H side  */

                   if( (operand1 < operand2 )){
                       printf("\n\aERROR, HYPOTENUSE SIDE MUST BE GREATER THAN %.16lf",operand2);
                       continue; };

                   result =sqrt((operand1*operand1) - (operand2*operand2));  /* ex. b = √ c^2 - a^2  */
                     };
```

```
                if(command=='h'){   goto start;   };         /* program help  */

                if(command=='q'){    exit(0);  };         /* exit-quit the program */

/* ---------------------------------------------------------------------------------------------------------------------------------------------*/

displayresult: ;      /* DISPLAY THE RESULT OF ANY OF THE ABOVE OPERATIONS */

                 printf("\n  %.15g",result);
        };

return;
};
/*----------------------------------------------------------------------------------------------------------------------------------------------*/
```

SUMMING CONSECUTIVE TERMS OF AN ARITHMETIC SERIES

Given (N) consecutive terms of an arithmetic series, here is how to find the sum of those terms:

Algebraically, the terms of an arithmetic series can be expressed or noted as:

Term (N):	1	2	3	4	5	6	. . .
	F	F+d	F+2d	F+3d	F+4d	F+5d	. . .

Where (F) is the first considered term, term 1, and (d) is the constant difference between each terms value. Consider for example that the third term is the sum of the second term and the difference (d):

(term 3) = (previous term) + d = (term 2) + d = (term 1 + d) + d = (term 1) + 2d = F + 2d

Algebraically summing up the terms:

Term	Term Value	Total Sum Of Terms (A Running Or Continuous Sum Of This Term and All Previous Terms)	
1	F	1F	
2	F + 1d	2F + 1d	: since: (this term) + (previous sum) = (F + 1d) + 1F = 2F + 1d
3	F + 2d	3F + 3d	: since: (this term) + (previous sum) = (F + 2d) + (2F + 1d) = 3F + 3d
4	F + 3d	4F + 6d	
5	F + 4d	5F + 10d	
6	F + 5d	6F + 15d	

Notice in the sum expressions that the first term of each expression can be easily predicted to be NF, however the next term is awkward since the numerical coefficients (C) of (d) appear to be that of some unknown series. There is one "observable formula" for the value of C that will be shown ahead, and can you possibly guess it beforehand? Below is a method of how these constants can be calculated so that the sum of the first (N) terms of an arithmetic series can be evaluated as:

SUM = NF + Cd : SUM OF N CONSECUTIVE TERMS OF AN ARITHMETIC SERIES (*)
F = value of first term, N = number of terms, d = difference, C = numerical coefficient of d

For this analysis, let's say an arithmetic series has a difference (d) value of 2.0, and the first term is 3. We can express the sum of the first 3 terms as:

$$\begin{array}{rcl} S &=& 3 + 5 + 7 \\ +\ S &=& \underline{7 + 5 + 3} \\ 2S &=& 10 + 10 + 10 \end{array}$$

Algebraically adding the sum to itself, using the commutative law:
(Or by multiplying each side by 2, and then distributing it.)

Notice that each term on the right hand side is algebraically equivalent to (F1 + FN), where F1 is considered as the first term, and FN is considered the "last" term considered. Since there are (N) terms in question being added, and each is the same value, we can express this repeated addition with multiplication:

$2S = (F1 + FN)\,N$ solving for S:

$$S = \frac{(F1 + FN)\,N}{2}$$

: **SUM OF N CONSECUTIVE TERMS OF AN ARITHMETIC SERIES**
F1 is the first term value , FN is the last term value, and the common difference (d) between terms is not needed and-or known.

Since $FN = F1 + (N-1)d$ using algebraic mathematical substitution:

$$S = \frac{(F1 + F1 + (N-1)d)N}{2} = \frac{F1N + F1N + dN^2 - dN}{2}$$

$$S = \frac{2F1N + dN(N-1)}{2} = \frac{2F1N}{2} + \frac{dN(N-1)}{2}$$

$$S = F1N + \frac{dN(N-1)}{2}$$

: **SUM OF N CONSECUTIVE TERMS OF AN ARITHMETIC SERIES (based on the first term, and here, the last term is not needed and-or known)**

The first term in this expression above is equivalent to the first term of the expression for S shown previously: NF, (see the indicated * previously). Hence, the sum of the remaining term(s) must also be equivalent in value. That is:

$$Cd = \frac{dN(N-1)}{2}$$

Solving for C (the constant or numerical coefficient of (d)):
(ie., divide both sides by d)

$$\mathbf{Cn = \frac{N(N-1)}{2} = \frac{N^2 - N}{2}}$$

: **For Cn, n here is a subscript of C, that is equal to the term number N** We see that given a term number N, Cn is always a constant value associated with that term number and series, however since C changes in value from term to term, C in therefore not a constant in that aspect.

Checking, the constant of the 5th term should be 10 as indicated above:

$$C5 = \frac{5(5-1)}{2} = \frac{5(4)}{2} = \frac{20}{2} = 10$$

: checks

Checking, the constant of the 6th term should be 15 as indicated above:

$$C6 = \frac{6^2 - 6}{2} = \frac{36 - 6}{2} = \frac{30}{2} = 15$$

: checks

Here is an "observable formula" for C, the numerical coefficient of (d), however, it does not lead to a direct algebraic solution as shown above since it includes the previous value of C:

$Cn+1 = Cn + N$ Ex. $C6 = C5 + 5 = 10 + 5 = 15$

Here is a practical example of summing consecutive terms of an arithmetic series:

Ex. A checkerboard has a total of 8^2 = 64 squares on it. You are to place a (1) grain of salt on the first square, 2 grains on the second, 3 grains on the third, and so on. The grains of salt on each square will be equal to the square number. Each next square will have the number of grains as on the current square, plus one. How many grains of salt will you eventually place on the checkerboard?

This is an arithmetic progression type of problem. Clearly, the difference is 1, and there are 64 squares to consider, there is effectively 64 terms to mathematically consider. The first term is 1.

$$S = F1N + \frac{dN(N-1)}{2} = (1)(64) + \frac{(1)(64)(63)}{2} = 64 + 2016 = 2080 \text{ grains}$$

SUMMING CONSECUTIVE TERMS OF A GEOMETRIC SERIES

Given (N) consecutive terms, which can and will be considered as the first (N) terms of a geometric series below, what is the sum (S) of those terms?

Algebraically, the terms of a geometric series can be algebraically (mathematically, symbolically, generally) expressed as:

Term:	1	2	3	4	5	6	. . .	n
	F	Fr	Fr^2	Fr^3	Fr^4	Fr^5	. . .	Fr^(n-1)

Where (F) is the considered as the first term, and (r) is the constant ratio between consecutive terms. Each next term is found from: (term)(r) = next term. Mathematically , previous term = term = (next term) / r

An expression of algebraically summing up a total of (n) terms:

S = F + F^1r + Fr^2 + Fr^3 + Fr^4 + Fr^5 + . . . + Fr^(n-1)

Now, we can possibly derive a simple formula that is equivalent to the right hand side of this equation. Since each term has a factor of F in it, we can factor it from each term and get this equivalent expression:

S = F (1 + r + r^2 + r^3 + r^4 + r^5 + . . . + r^(n-1))

From the indicated factor to F as shown above within the grouping symbols, and considering just the first 4 terms of it, we can get:

1 + r + r^2 + r^3 factoring (r) out of the terms that contain it:
1 + r (1 + r + r^2)
1 + r (1 + r (1 + r))

The only observable possible formula is from the second term: For a total of (n-2) times, starting with (r) and add 1 to it, and then multiply this growing quantity by (r). This total value will be added to 1. The problem is that this is not a simple formula for the factor F as shown above, since it requires summing and multiplying almost the same number of terms as the series itself.

How about adding the series to itself, just like it was utilized during the derivation of the sum of an arithmetic series? This will not work here since the formula for the sum will simplify to the exact same series. However, rather than effectively multiply each side by 2 as in the sum of an arithmetic series, let's multiply each side of the equation by another value that is directly related to each term of the geometric series, specifically (r):

Let's try a geometric series where the first term is 4, r=2, and 3 terms are to be summed:

Eq. 1: S = 4 + 8 + 16 : r = 2, multiplying each side by (r) we get:

rS = 2(4 + 8 + 16) distributing:
rS = 8 + 16 + 32 which can be written or expressed as:

Eq. 2: rS = 32 + 8 + 16

Subtracting the first equation from the second we get:

rS - S = 32 + 8 + 16 - (4 + 8 + 16) = 32 + 8 + 16 -4 -8 -16 = 32 - 4 = 28

Notice that the only terms (always) left on the right hand side after the subtraction is the first term (F) of the original series and the last term (L) of the "multiplied" series which equals Lr. Noting this algebraically:

$rS - S = L - F$ using substitution for the value of the last term (L):
$rS - S = r^1 (Fr^{(n-1)}) - F$: $r^1 = r$, after distributing:
$rS - S = Fr^n - F$ factoring out common factors:
$S (r - 1) = F (r^n - 1)$ solving for S :

$S = \dfrac{F((r^n) - 1)}{(r - 1)}$: **SUM OF THE FIRST (n) TERMS OF A GEOMETRIC SERIES**
F is the term considered as the first term. (r) is the ratio between successive terms. (n) is the total number of terms.

Checking, the sum of the series shown (Eq. 1) should be 28:

$S = \dfrac{4(2^3 - 1)}{(2 - 1)} = \dfrac{4(8 - 1)}{1} = 4(7) = 28$: checks

If $(r < 1)$, the terms of r^n will approach to be a very small value of practically 0 for an infinite (or near infinite, large number for practicality) number (n) of terms, and the sum of this infinite number of terms will approach or converge to be a specific value of:

$S = \dfrac{F(r^n - 1)}{(r - 1)} = \dfrac{(-1)}{(-1)}\dfrac{F(r^n - 1)}{(r - 1)} = \dfrac{F(-1)(r^n - 1)}{(-1)(r - 1)} = \dfrac{F(1 - r^n)}{(1 - r)} = \dfrac{F(1 - 0)}{(1 - r)}$: as r^n approaches 0, $r^n \longrightarrow 0$

$S = \dfrac{F}{1 - r}$: Sum of terms when $r<1$ and the number of terms(n) approaches infinity

If the first term is F=1 we then have:

$$S = \frac{1}{1 - r}$$

Ex. If the first term is 1, and the ratio between each next or consecutive term is 0.5 (or 1/2):

$1 + 0.5 + 0.25 + \ldots$: in this example series, each next term is half of the previous. There is also an illustrative figure and computer program shown below for this type of series.
$1 + 1/2 + 1/4 + \ldots$

$S = \dfrac{1}{1 - 0.5} = \dfrac{1}{0.5} = 2$: note that the terms of this series are technically endless, yet this algebraic expression gives a definite or specific result.

Here's another observation of the above processes. The multiplying factor to (F) that was previously shown in the sum of (n) terms of a geometric series can be expressed as:

$\displaystyle\sum_{x=0}^{x=(n-1)} r^x$: simplest power series

Expanding the indicated series:

$$1 + r^1 + r^2 + r^3 + r^4 + r^5 + \ldots + r^{(n-1)}$$

Hence when the first term is F=1:

$$S = \frac{1}{1 - r} = 1 + r^1 + r^2 + r^3 + r^4 + r^5 + \ldots \qquad \text{:for this to converge} \quad -1 < r < +1$$

Or as:

$$1 + \sum_{x=1}^{x=(n-1)} r^x$$

Expanding the indicated series:

$$1 + (r + r^2 + r^3 + r^4 + r^5 + \ldots + r^n-1)$$

Equating the two sum formulas:

$$\frac{F ((r^n) - 1)}{(r - 1)} = F (1 + r + r^2 + r^3 + r^4 + r^5 + \ldots + r^{(n-1)})$$

By dividing both sides of this equation by F, or simply equating both of the factors to F, we get:

$$\frac{(r^n) - 1}{(r - 1)} = 1 + r^1 + r^2 + r^3 + r^4 + r^5 + \ldots + r^{(n-1)}$$

Transposing the (+1) term, and switching sides:

$$r^1 + r^2 + r^3 + r^4 + r^5 + \ldots + r^{(n-1)} = \frac{(r^n) - 1}{(r - 1)} - 1$$

If we add (r^n) to each side, we can get a formula (shown on the right side) for the sum of the first (n) integer powers of any value (such as r shown on the left side):

$$r^1 + r^2 + r^3 + r^4 + r^5 + \ldots + r^{(n-1)} + r^n = \frac{(r^n) -1}{(r - 1)} - 1 + r^n$$

Combing terms (ie., the fractions) on the right hand side:

$$r^1 + r^2 + r^3 + r^4 + r^5 + \ldots + r^n = \frac{r ((r^n) -1)}{(r - 1)}$$

: This is essentially the same expression previously derived, but here, r=F.

Another observation here is to note how a function such as on the right hand side of the equation above can be expressed as a sum of a specific series of terms.

Expressing this with summation notation, and letting the base = b = r :

$$\sum_{x=1}^{x=n} b^x = \frac{b((b^n)-1)}{(b-1)}$$

: **SUM OF FIRST (n) INTEGER POWERS OF A VALUE**
b = the base of the power = ratio (r) of the integer powers of it.
or= :

This sum can also be algebraically expressed as: $\mathbf{\dfrac{b^{(n+1)} - b}{b - 1}}$

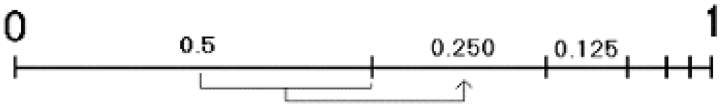
It may have not been apparent, but this is equivalent to the sum of the first (n) terms of a geometric series which has (F) and (r) set to the same value of (b).

Ex. If b=r=2 , and n=10:

$$2^1 + 2^2 + 2^3 + 2^4 + 2^5 + 2^6 + 2^7 + 2^8 + 2^9 + 2^{10} = 2046$$
$$2 + (2^1)2 + (2^2)2 + (2^3)2 + (2^4)2 + (2^5)2 + (2^6)2 + (2^7)2 + (2^8)2 + (2^9)2 = 2046$$
$$2 + 4 + 8 + 16 + 32 + 64 + 128 + 256 + 512 + 1024 = 2046$$

$$\frac{2(2^{10} - 1)}{(2 - 1)} = \frac{2(1024 - 1)}{1} = 2(1023) = 2046$$: the formula checks

Here is an illustrative example of a geometric series that is also a converging (to a specific value) sum of terms. It shows the remaining (fractional) values or terms after repeatedly dividing a value or length, here starting as a value or relative value of 1, by a constant value, here 2. This is the same as repeatedly multiplying the given value by the constant factor of r=0.5, (ie. (1/2) and resulting in "half" of a given value). The sum of all these remaining parts or quotients equals the original value, here 1 (or 100% of the initial value). In short and in a relative fractional or percentage manner, the sum of all the parts or fractions of the whole = 100% = 1. For starting values that are a multiple greater than 1, the terms and resulting sum will then be multiples as that shown for the series shown below that sums to 1: [FIG 214]



$$0.5 + 0.250 + 0.125 + \ldots = 1$$

$$1/2 + 1/4 + 1/8 + \ldots = 1$$

$$1/2^1 + 1/2^2 + 1/2^3 + \ldots = 1$$

$$1 = \sum_{n=0}^{n=\text{infinity}} \frac{0.5}{2^n} \quad \text{or} \quad \text{value} = \sum_{n=1}^{n=\text{infinity}} \frac{\text{value}}{2^n} = \sum_{n=1}^{n=\text{infinity}} \frac{\text{value } 2^{-n}}{1}$$

Note that above, 0.5 / 2^n = 1 / (2)(2^n) = 1 / 2^(n+1).

Note that the above could be expressed as: $x^1 + x^2 + x^3 + \ldots + x^n = 1$: where $x=0.5$ and $n \to$ approaches infinity.
And for just the first two terms: $x^1 + x^2 = x^2 + x = 1$: $x = GR = \sim 0.618$, Extra: $x^2 + x - 1 = 0$

By repeatedly dividing an object (ie., whole=100%) by two and making two pieces that are 50% of the original, it is the fastest way to create very small pieces of a certain size.

A COMPUTER PROGRAM SUMMING A CONVERGING GEOMETRIC SERIES

```
/* GeometricSeries.C     Displaying An Example Geometric Series With A Converging Sum Of Terms.c

   This is an example of repeatedly dividing a value by a value, and the resulting terms get less and less, and the
   sum of  those terms converge (gets closer and closer) to a specific value.  The classic example is repeatedly
   dividing 1 by 2, or multiplying 1 by 0.5, and the sum of all these (half, fractional, value) terms is 1.    Regardless
   of the denominator or (r), the sum is always 100% of the initial or starting value (numerator), and if 1 is used as
   the starting value, the sum is therefore 1.

   The sum of all the pieces or fractions (equivalent, or diminishing in value as for this series of terms) of a value
   is that value.  In general, the number of terms increase only slightly for a fixed denominator, or (r), as the
   numerator increases in value.  The higher the denominator, or the lower that the geometric ratio (r) is since
    (r)=(1/denominator),  the more terms required in the sum to the numerator or starting value since the fractional
    parts or successive terms will be smaller to continue working with.

    In the program below, if the input numerator is set to 1, and the input denominator is set to 2,  the series is the
    classic geometric series with (r) = 1/2 = 0.5 , and each fractional part and term is half of the previous.  This
    produces the terms, which always sum to the starting value, here 1, of:

      1/2  +  1/2^2  +  1/2^3  + . . .  =    0.5 + 0.250 + 0.125 + . . .  =    1      : sum of fractional parts = whole part or value

(c) JPA 2018
------------------------------------------------------------------------------------------------------------------------------------------------------*/
#include "stdio.h"
/*----------------------------------------------------------------------------------------------------------------------------------------------------*/
void main(void)       /* On a PC computer, you can usually redirect the screen text output to a file output, for example:  */
{                          /* "filename.exe" > "myfile.txt"    , >> for append ,  CTRL Z may be needed  to terminate the program.  */
                           /* "filename.exe" > prn    will redirect the screen  text output to a  line-text printer. */
double numerator=0.0;              /*  You can also use the mouse to mark and select the screen text to copy and paste.    */
double denominator=2.0;
double term=0.0;
double sum=0.0;

int n=1;  /* number of terms or loops processed */

fflush(stdin);  printf("\nInput the numerator, ex. 1.0:  ");       scanf("%lf",&numerator);
fflush(stdin); printf("\nInput the denominator > 1.0 so as to produce a converging series, ex. 2.0 , (r=1/denominator):  ");
scanf("%lf",&denominator);

for(;;){
           term=numerator/denominator;
           sum=sum+term;
           printf("\nterm %d = %.16lf , sum = %.16lf",n, term,sum);     /*  optional, display the value of each successive term,
                                                                                 %.16g uses exponential notation  */
           numerator=numerator-term;
           if(term<=0.0000000000000001){  break; };
           n=n+1;
        };

printf("\n\nterms=%d",n);  printf("\n\nPress A Key"); getch();

return;
};
```

```
/*----------------------------------------------------------------------------------------------------------------------------------------------------*/
```

Here is an example of the output of the above computer program:

```
Input the numerator, ex. 1.0:  1

Input the denominator, ex. 2.0 , (r=1/denominator):  2

term 1 = 0.5000000000000000 , sum = 0.5000000000000000
term 2 = 0.2500000000000000 , sum = 0.7500000000000000
term 3 = 0.1250000000000000 , sum = 0.8750000000000000
term 4 = 0.0625000000000000 , sum = 0.9375000000000000
term 5 = 0.0312500000000000 , sum = 0.9687500000000000
term 6 = 0.0156250000000000 , sum = 0.9843750000000000
term 7 = 0.0078125000000000 , sum = 0.9921875000000000
term 8 = 0.0039062500000000 , sum = 0.9960937500000000
term 9 = 0.0019531250000000 , sum = 0.9980468750000000
term 10 = 0.0009765625000000 , sum = 0.9990234375000000       : the sum is practically equal to 1.0 about here
term 11 = 0.0004882812500000 , sum = 0.9995117187500000
term 12 = 0.0002441406250000 , sum = 0.9997558593750000
term 13 = 0.0001220703125000 , sum = 0.9998779296875000
term 14 = 0.0000610351562500 , sum = 0.9999389648437500
term 15 = 0.0000305175781250 , sum = 0.9999694824218750
term 16 = 0.0000152587890625 , sum = 0.9999847412109375
term 17 = 0.0000076293945313 , sum = 0.9999923706054688
term 18 = 0.0000038146972656 , sum = 0.9999961853027344
term 19 = 0.0000019073486328 , sum = 0.9999980926513672
term 20 = 0.0000009536743164 , sum = 0.9999990463256836
term 21 = 0.0000004768371582 , sum = 0.9999995231628418
term 22 = 0.0000002384185791 , sum = 0.9999997615814209
term 23 = 0.0000001192092896 , sum = 0.9999998807907105
term 24 = 0.0000000596046448 , sum = 0.9999999403953552
term 25 = 0.0000000298023224 , sum = 0.9999999701976776
term 26 = 0.0000000149011612 , sum = 0.9999999850988388
term 27 = 0.0000000074505806 , sum = 0.9999999925494194
term 28 = 0.0000000037252903 , sum = 0.9999999962747097
term 29 = 0.0000000018626451 , sum = 0.9999999981373549
term 30 = 0.0000000009313226 , sum = 0.9999999990686774
term 31 = 0.0000000004656613 , sum = 0.9999999995343387
term 32 = 0.0000000002328306 , sum = 0.9999999997671694
term 33 = 0.0000000001164153 , sum = 0.9999999998835847
term 34 = 0.0000000000582077 , sum = 0.9999999999417923
term 35 = 0.0000000000291038 , sum = 0.9999999999708962
term 36 = 0.0000000000145519 , sum = 0.9999999999854481
term 37 = 0.0000000000072760 , sum = 0.9999999999927240
term 38 = 0.0000000000036380 , sum = 0.9999999999963620
term 39 = 0.0000000000018190 , sum = 0.9999999999981810
term 40 = 0.0000000000009095 , sum = 0.9999999999990905
term 41 = 0.0000000000004547 , sum = 0.9999999999995453
term 42 = 0.0000000000002274 , sum = 0.9999999999997726
term 43 = 0.0000000000001137 , sum = 0.9999999999998863
term 44 = 0.0000000000000568 , sum = 0.9999999999999432
term 45 = 0.0000000000000284 , sum = 0.9999999999999716
term 46 = 0.0000000000000142 , sum = 0.9999999999999858
```

```
term 47 = 0.0000000000000071 , sum = 0.9999999999999929
term 48 = 0.0000000000000036 , sum = 0.9999999999999965
term 49 = 0.0000000000000018 , sum = 0.9999999999999982
term 50 = 0.0000000000000009 , sum = 0.9999999999999991
term 51 = 0.0000000000000004 , sum = 0.9999999999999996
term 52 = 0.0000000000000002 , sum = 0.9999999999999998
term 53 = 0.0000000000000001 , sum = 0.9999999999999999
term 54 = 0.0000000000000001 , sum = 1.0000000000000000

terms=54

Press A Key
```

BELOW IS A SIMILAR PROGRAM SO AS TO HAVE A FURTHER EXAMPLE AND UNDERSTANDING OF HOW TO PROGRAM THESE TYPES OF PROGRAMS AND-OR FUNCTIONS.

```
/*  InfiniteSum.c   Infinite Sum, this is just an example of a function for successive
                    summing or approximation.  Of course, to have a definite
                    result, the terms must get smaller and smaller.  An infinite
                    number of terms is not used in this program, and the process
                    terminates to make the program practical and since the
                    computer system has a fixed maximum number of digits of
                    precision and for practical purposes, there usually isn't much of
                    of a need to proceed any further.

     (c) JPA  April 30, 2020
*/
/*----------------------------------------------------------------------------------------------------------*/

double infinitesum(double number, double divisor);       /* : C function prototype  */

/*----------------------------------------------------------------------------------------------------------*/
void main(void)
{
double n=0.0;
double divisor=0.0;
double result=0.0;
unsigned char ch=0;

for(;;){   /* an opt. infinite loop, so as to run the program easily many times */

   system("cls");
   printf("\nInput Starting Number: "); scanf("%lf",&n);
   printf("Input Divisor: "); scanf("%lf",&divisor);
   if(divisor <= 1.0){
                        printf("\n\aPlease use a divisor greater than 1, so the terms get");
                        printf("\nsmaller and this series converge to a specific value.");
                        printf("\nPress A Key"); getch();  continue;
                    };

   result=infinitesum(n,divisor);
   printf("\nResult = %lf",result);
   printf("\nThe change from %lf is %lf",n, result - n);  /* Interesting that the Change = result / divisor */
```

```
   printf("\nPress A Key, Use 0 to exit: "); ch=(unsigned char) getch();  if(ch=='0'){ exit(0); };
 };

return;
};
/*-----------------------------------------------------------------------------------------------------------------*/
double infinitesum(double number, double divisor)
{
double sum=number;
int loop=0;  /* not needed, but can be used for testing */
int term=1;

printf("\n%d   %lf",term,number);  term=term+1;

for(;;){
          number=number / divisor;
          if(number<=0.000001){ break; };
          printf("\n%d  %lf",term,number);
          sum=sum+number;

         /*  loop = loop + 1;  if(loop ==10){ break; };   for testing */
         term = term+1;
       };

return sum;
};
/*---------------------------------------------------------------------------------------------------------------*/
```

Here is an example of running the above program:

```
Input Starting Number:  100
Input Divisor:   4

1  100.000000
2  25.000000
3  6.250000
4  1.562500
5  0.390625
6  0.097656
7  0.024414
8  0.006104
9  0.001526
10  0.000381
11  0.000095
12  0.000024
13  0.000006
14  0.000001
Result = 133.333333
The change from 100.000000 is 33.333333
Press A Key, Use 0 to exit:
```

A DERIVATION AND UNDERSTANDING OF EPSILON (e)

This discussion is to provide some clarity of the constant (e) as found in many scientific, mathematical equations. If needed, please review the topic about Epsilon(e) previously shown in this book. The value of this constant is about: 2.718281828459045235. (e) is sometimes called Eulers number or (John) Napier's constant, epsilon or just plain (e) as seen in equations. (e) is an irrational constant like (pi), but (pi) can easily be described in a single sentence of the form: (pi) is the constant ratio of any, and all, circle's circumference and its diameter. (e) does not have a simple description, and hence its description is often omitted even though it is a very important value like (pi).

Values such as (e), (pi), and an angles trigonometric values are also called non-algebraic, or transcendental numbers, due to the fact that their values cannot be algebraically represented as a power (ex. n^x) value expressed with two rational numbers, that is, a rational base and rational exponent.

At the beginning of the year, say Jan. 1, you decide to deposit one (1) dollar (money) = $1 into a savings account at the First Simple Growth Bank in town. For this starting or "input" amount, the bank claims it will give you money that is one (1) dollar of interest (I) or "growth" at the end of a one year time period, hence doubling your money from 1 dollar to (1 + 1) dollars = 2 dollars. (starting input money + interest money) = $1 + $1 = $2. This is an increase of 100% from the initial starting or input value of $1, hence you can say that the banks interest rate (I) is 100% (or 1 = 1/1 in decimal).

Total = Original Deposit + Interest

$Mt = Mo + IMo$: Mo = original deposit of money. Since the interest rate (I) is 100%, and 100% = 1.0:

$Mt = \frac{Mo}{1} + \frac{(1)Mo}{(1)}$: Mt = total money in the account. Since the original deposit is $1, Mo = $1 = 1:

$2.00 = 1 + \frac{(1)(1)}{(1)(1)}$

You decide to split the total time period of 1 year into n intervals of time. If you let n=2 intervals, then at half the period or end of the first interval within that total time period, you should also receive half the interest since:

I% = (100% / n) = (100% / 2) = 50% or= 0.50 in strict decimal numeric form or= 1/2 in a proper decimal fraction form.

Half of the 1 year time period is: (total time period) / (n time periods) = (1 year) / n = (1 year) / 2 = (1 year)(0.5) = 0.5 years = 12months / 2 = 6 months

Half of the $1 dollar total interest for the year is: ($1 x 0.50) = $0.50 dollars or 50 cents. Your "net worth" or total money (Mt) at the bank at this point or time (6 months) in the total time period of 1 year should be $1.0 + $0.50 = $1.50. We are also considering that the bank allows this method of (partial) interest payment, and for this example, bank or system, we will assume it does. You decide to verify the above calculations using above formula:

$Mt = Mo + IMo$

$Mt = Mo + (50\% \text{ of } Mo) = Mo + 0.5Mo = \frac{Mo}{1} + \frac{(1)(Mo)}{(2)(1)}$

$Mt = \frac{Mo}{1} + \frac{Mo}{2}$ after substitution of values:

$1.50 = \frac{1}{1} + \frac{1}{2} = 1 + 0.5$: which verifies your claim

In general, if you split the interest total time period into N intervals, the (simple, non-compounded) interest you should receive per period is:

$\frac{100\% \text{ of the total interest}}{n} = \frac{1 \text{ (interest)}}{n}$: in decimal form

If n was 4 intervals:

time of each interval = (total time period) / (n time periods) = 1year/4 = 12months/4 = 3 months = 0.25 years

Since if n = 4 intervals, the (simple) interest to expect per interval period of 0.25 years or 3 months is also 0.25 of the total interest: (I):

$\frac{\text{Total Interest}}{n} = \frac{100\% \text{ I}}{n} = \frac{100\% \text{ I}}{4} = 25\%$ of the Interest, and expressing this mathematically:

$Mt = Mo + 0.25$ (Total Interest). Since: (Total interest) = 100% (Total interest) = 100% (1) = 1 = Mo :

$Mt = Mo + (25\% \text{ of } Mo) = Mo + 0.25Mo = \frac{Mo}{1} + \frac{(1)(Mo)}{(4)\ (1)} = \frac{Mo}{1} + \frac{Mo}{4}$

Notice the correlation to the number of time intervals chosen (n) and the denominator of the fractional value of the interest the bank will give you. If a time interval is a certain percentage of the total time period, the partial interest will be the same percentage of the total interest. Rewriting the formula, we get:

$Mt = Mo + IMo$: Here, initially considering the interest as: I = 100% = 1 , that is, as undivided.
For each and every interval of interest payment, I is now calculated to be:
I = 100%/n = 1/n :

$Mt = Mo + \frac{(1)\ Mo}{(n)}$ Factoring out Mo from each term:

$Mt = Mo + \frac{Mo}{n}$ or * $Mt = \frac{Mo}{1} \left(\frac{1}{1} + \frac{1}{n} \right)$: * (more will be said about this further ahead)

For this example where n=2, Mo=\$1, and It = \$1, at the end of the first interval of the 2 total intervals :

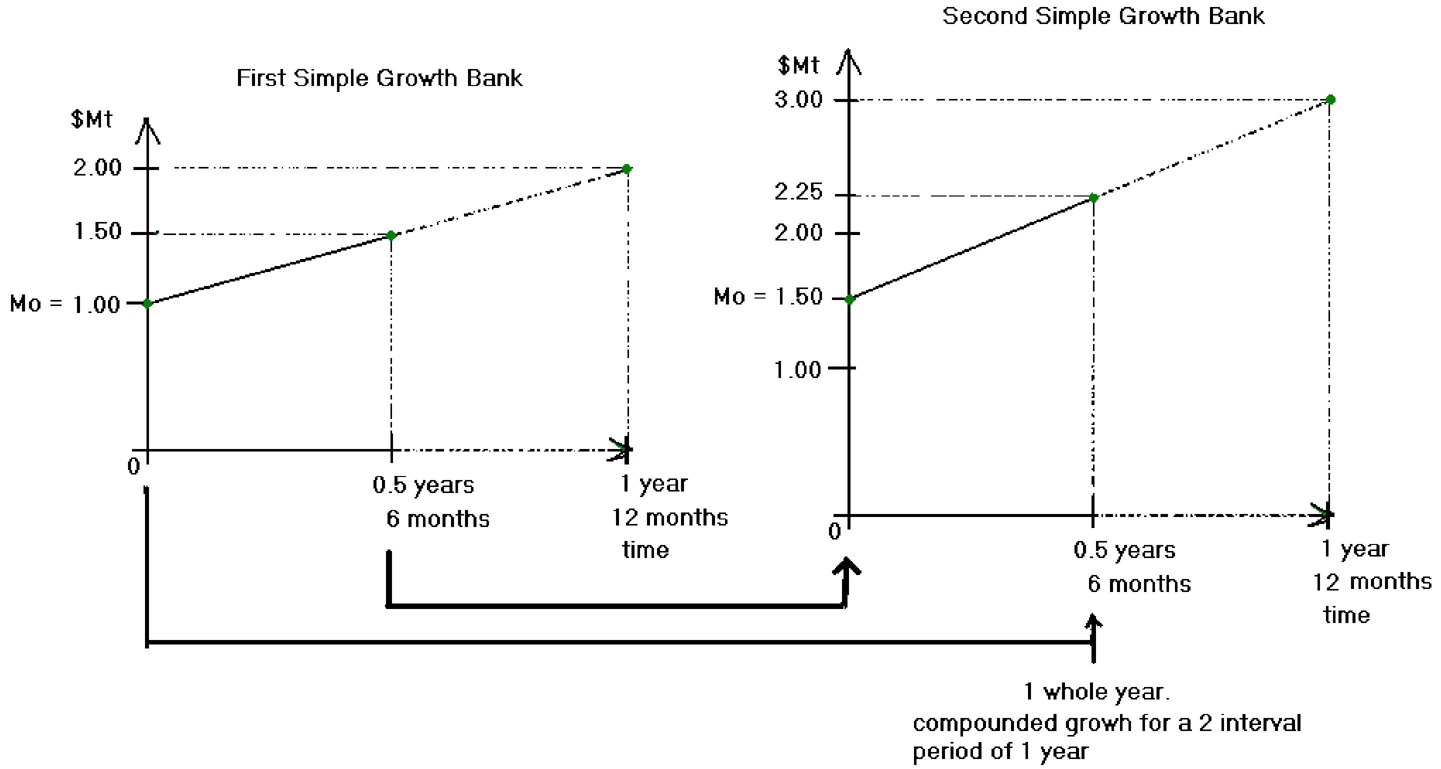
$Mt = \$1 + (0.50)\ \$1.0 = \$1 + \$0.50 = \$1.50$

If you then withdrawal this total money value of \$1.50, which can be expressed as just 1.50, at the end of the first interval (of a year divided into two intervals) and deposit it at a similar bank called: Another Simple Growth Bank, they will recognize your initial savings deposit as Mo = 1.50 instead of 1.00 as the First Simple Growth Bank did. Then at the end of the first interval (of a two intervals per year period, hence after 6 months) of one total time period (here, it's a year also), you should again receive 50% or half of the interest of \$1.50. Since I = 100%Mo, It =100%(1.5) = 1.5.

$In = \frac{It}{n} = \frac{It}{2} = \frac{1.5}{2} = (1.5)(0.5) = 0.75$

$Mt = \frac{Mo}{1} + \frac{Mo}{2} = \frac{1.5}{1} + \frac{1.5}{2} = 1.5 + 0.75 = 2.25$: By Dec. 31. Here at this bank, Mo = 1.50

This is certainly larger than the 2.00 you would of received if you kept your money in the First Simple Growth Bank after 1 year. The above "process" you have done is essentially known as compounded growth (due to interest being added to the (stepped, intervals or completely continuous with an infinite number of intervals) growing total value) which is growth based on, or with respect to, the current total or net growth value. The total growth in this example is growth based upon any and all previous growth (here, 1.50) of the total value in the bank which includes the initial deposit (Mo) and all the previous interest already added to the total value (Mt) in the bank. The value of Mt (or even the initial deposit) is then said to grow compoundedly after each interval of the total time period. [FIG 215]



1 whole year.
compounded growth for a 2 interval
period of 1 year

After each interval (n) of the total time period of growth, the new net value is calculated using the total value (Mt, or better yet, a subscripted variable like Mn which indicates which interval) or the entire growth from the previous interval (where the interest or growth was also added on to produce a new entire growth or growing value), and that the new value for this discussion about (e) is (as indicated above at *, as a multiplying factor to Mo):

* $\left(\frac{1}{1} + \frac{1}{n}\right)$ times larger than the previous interval value.

Algebraically:

$Mt = \frac{Mo}{1}\left(\frac{1}{1} + \frac{1}{n}\right)^1$: after the first interval of the time period

$Mt = \left[\frac{Mo}{1}\left(\frac{1}{1} + \frac{1}{n}\right)^1\right]\left(\frac{1}{1} + \frac{1}{n}\right)^1$: after the second interval, and simplifying this we have:

$Mt = Mo\left(\frac{1}{1} + \frac{1}{n}\right)^2$: after 2 intervals per period

At the end of a third interval (n=3), the total value of the growth would again be $\left(\frac{1}{1} + \frac{1}{n}\right)^1$ times larger:

$$Mt = \frac{Mo}{1}\left(\frac{1}{1} + \frac{1}{n}\right)^2 \left(\frac{1}{1} + \frac{1}{n}\right)^1 = Mo\left(\frac{1}{1} + \frac{1}{n}\right)^3$$: after 3 intervals of the total time or period

Clearly, an obvious pattern or correlation is that the value of the exponent of the expression for Mt is equal to the total number of intervals (n) in that total time period. Reflecting this correlation into the formula:

$$Mt = Mo\left(\frac{1}{1} + \frac{1}{n}\right)^n$$:Total Compounded Growth Formula of a (total time) period divided into n (compound growth) intervals.

We already know that for two (2) intervals of compounded growth that Mt = would be 2.5 times greater than the original or starting value of Mo. That is, mathematically:

$$\frac{Mt}{\text{The initial starting value of Mo}} = \frac{2.5}{1} = 2.5$$

$$Mt = Mo\,(2.5)$$

Increasing the intervals (n) of this compound growth process to see how many more times greater the total value (Mt) can be due to various compounded growths with an increasing number of intervals (n) :

$$\left(\frac{1}{1} + \frac{1}{100}\right)^{100} = 1.01^{1,000} = 2.70\underline{5}$$: after 100 intervals of compounded growth in the period

$$\left(\frac{1}{1} + \frac{1}{1000}\right)^{1,000} = 1.001^{1,000} = 2.716923932$$: after 1,000 intervals of compounded growth in the period

$$\left(\frac{1}{1} + \frac{1}{1,000,000}\right)^{1,000,000} = 1.000001^{1,000,000} = 2.718280469$$ This can also be expressed as:

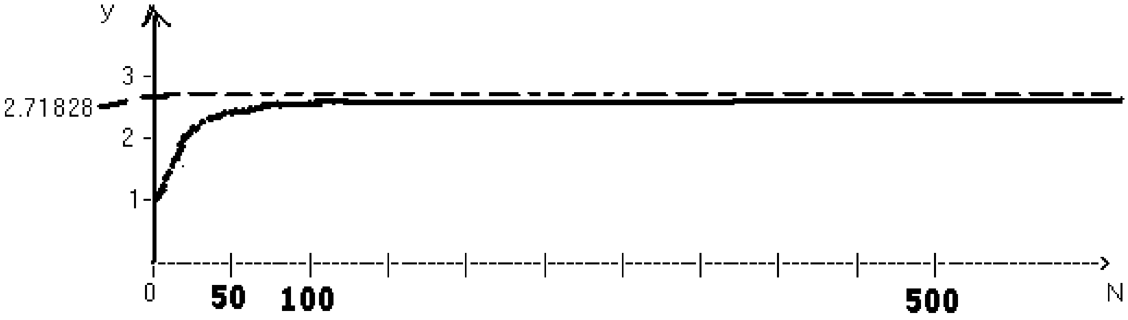
$$(1 + 0.000,000,1)^{1,000,000} = (1.000,000,1)^{1,000,000}$$: This clearly shows the base of the indicated power is only slightly greater than 1. A value of 1 raised to any indicated power greater than 1, including very high values, is only, and equal to 1. If 1 is increased by a very small , seemingly negligible, value, the result of that value raised to a very high value is significant. 1 will increase or grow by a factor slightly greater than 2.7

$$\left(\frac{1}{1} + \frac{1}{1,000,000,000}\right)^{1,000,000,000} = 1.000000001^{1,000,000,000} = 2.718281827$$

Rather than being some vary large value (as might be intuitively expected due to the large exponent) we see that this value is actually only approaching, converging to, or reaching a maximum limit (final value) of about 2.71828 as n approaches a (endlessly, infinitely) very high value. Here is why the value is not actually larger:

Given the power of: $y = \left(\frac{1}{1} + \frac{1}{n}\right)^x$

Note that when n approaches a value of infinity that the fraction (1/n) gets less and less, approaching a value of 0, and therefore, the base of this power actually approaches a value of 1. When n is approaching infinity and the indicated exponent x=n, the rate of change of this power value with respect to n is approaching a value of 0. That is, the slope of the curve is approaching 0, and it not rising much at all as n increases. In short, this is because any power of 1 is still 1 (ie. does not change in value), and any other higher power of a value only slightly larger than 1 will still be about the same power value. This all does seem a bit odd given a value of 1 with the smallest fractional portion possible, practically equal to 1.0, and raised to a very high power value is equal to 2.71828..., and is not so near to 1 or some very large value at all. The graph of the relationship of (y), and (n=x) somewhat resembles a logarithmic curve where slope of the curve eventually becomes very low, practically 0, and remains there for the remainder of the curve: [FIG 216]



This constant value (that the expression eventually becomes) is given the identifier symbol (e) or epsilon. This infinitely compounded growth concept is sometimes known as natural growth.

$$\frac{\text{Mt at the end of period}}{\text{Mo at start of period}} = (e)$$: ratio due to natural (infinitely compounded, with an infinite number of intervals) growth

Hence mathematically, given a value (Mo), it will be (e) times the original value at the end of one period of compounded growth where the number of growth intervals (n) approaches infinity. For example, if you deposit an initial value of $25.00 into a "Natural Growth" bank, this value will grow to (at the end of one period of total time, say 1 year):

Mt1 = (Mo)(e)
Mt1 = ($25.00)(2.71828)
Mt1 = $67.96

After two periods, your original value would grow to:

Mt2 = (Mt1)(e)
Mt2 = (Mo e)(e) = ($67.96)(2.71828) = $184.73
Mt2 = Mo e^2

Notice the pattern or correlation between the exponent of (e) and the period in question. Hence to calculate the total value at the end of 3 periods, you would simply make the exponent of the constant (e) equal to three (3):

Mt3 = (Mt2)(e) = Mo e^2 = Mo e^3

Hence, whenever you see an expression of the general form such as: ce^x , you will know that the value of (c), any constant value, will undergo (go through, be affected by) natural growth for (x) periods (and each single period has an infinite number of growth intervals) of time, and that the value that (c) will grow to is the value of this expression:

ct = ce^x :natural growth, ct = the total (t) value that the constant (c) will grow to.
Note, if you see e^x by itself with no multiplying constant (c), this actually means c=1 and e^x = 1e^x.
A value of 1 (ie. c=Mo=1) will undergo continuous compounded growth for (x) periods of time.

Ex. Let: $ct = n = e^x$ hence:

$n = 1e^x$: The constant of 1 will undergo continuous compounded growth for x periods.
Solving for x by taking the natural logarithm (ln) of both sides of the equation:

$\ln n = \ln 1e^x$
$\ln n = x \ln e$ since $\ln e = 1$ and switching sides:
$x = \ln n$ hence also, the expression $n = e^x$ could be written as: $n = e^x = e^{(\ln n)}$

Ex. $5 = 1e^x = 1e^{(\ln 5)} = e^{1.60948}$: for 1 to grow to a value of 5, 1.60948 periods are needed

Ex. $10 = 2e^x$ Here 2 will undergoing continuous compounded growth for x periods. Solve for x.
After dividing both sides by 2 for some simplification:

$5 = 1e^x = e^{(\ln 5)} = e^{1.60948}$: $x = \ln 5 = 1.60948$

In general, given: $n = ce^x$ dividing both sides by (c):

$n/c = e^x$ taking the logarithm of both sides:

$\ln (n/c) = \ln e^x$
$\ln (n/c) = x \ln e$: $\ln e = 1$

$x = \ln (n/c)$ also, according to the basic definition of a logarithm:

$e^x = \ln (n/c) = x$

Ex. Find x in: $37.6 = 1.6e^x$

$x = \ln (37.6 / 1.6) = \ln 23.5 = 3.157$

Ex. Find c in: $37.6 = ce^{3.157}$

$37.6 = ce^{3.157} = c\ 24.5$, $c = 37.6 / 24.5 = 1.534\underline{7}$

When (c) is not multiplied by a power of (e), but divided by a power of (e), the value of (c) will undergo natural decay:

$ct = \dfrac{c}{e^x} = ce^{-x}$: a common natural decay expression. Remember that $e^0 = 1$, ie.$x=0$, and $x<1$.
With ce^{+x} , there is a natural growth, even if $c<1$and is a fractional value.
With ce^{-x} , there is a natural decay

For example, when a heated object naturally cools down to any temperature, it's temperature will undergo natural decay and these facts will be included in a given formula relating the objects temperature with respect to time.

A derivation of the reciprocal of (e) is nearly identical to that for (e), except that instead of adding a value, a value is subtracted per interval of the period:

$e^{-1} = \left(\dfrac{1}{1} - \dfrac{1}{n}\right)^n = \left(\dfrac{n-1}{n}\right)^n = 0.367879441 = \dfrac{1}{e}$: as n approaches infinity

After one period of natural decay, a value will decay to about 0.368 = 36.8% of its original value, hence about (100% - 36.8%) = 63.2% of the original value is decayed or removed.

For calculating negative powers of (e), and as previously mentioned in this book:

$b^{-x} = 1 / b^x = (1^x / b^x) = \left(\frac{1}{b}\right)^x$: and since the base being considered here is (e):

$e^{-x} = 0.367879441^x$: an advantage of this method is that a reciprocal does not need to be taken

Since we have an algebraic-like expression for $e = e^1$, here is an expression for e^x:

In deriving the formula for (e), we used $x\%/n = 100\%/n = 1/n$, and essentially evaluated e^x where $x=1$, hence e^1. If (x) is some other value, then this changes the formula to:

$e^x = \left(1 + \frac{x}{n}\right)^n$: as n-->inf , (inf = infinity) , : Powers Of (e) , and:

This is also the same as the following which is perhaps more understandable at this point:

$e^x = \left[\left(1 + \frac{1}{n}\right)^n\right]^x$: as n-->inf , (inf = infinity) , Powers Of (e) , and:

You may view the topic of: DERIVATION OF THE SERIES FOR POWERS OF (e) to help understand this.

One other fact that was implied, but not discussed much above, is the fact that the time, time length or duration of each interval is mathematically inversely related to the number of intervals (n).

Expressing this in mathematical form:

time of each 1 interval = 100% total time / number of intervals = 1 / number of intervals or:

$t = 1 / n$: for a naturally compounded growth process, as n approaches infinity, (t) approaches 0.

If the total time is 100% = 1 for a relative analysis, and if this amount of time is equal to one complete period (P, or T) time, we can substitute this for the value of 1:

$t = \frac{P}{n}$ or= $\frac{T}{n}$: The higher the number of intervals (n), the "faster", "shorter", "quicker" or lower the time length of each interval. (t) and (n) are inversely related. T or P is the period or time length where the value undergoing natural growth will increase by a factor of (e) = 2.71828...

As an extra advanced consideration, if the number of intervals was considered as the frequency (ie. number of similar times or occurrences) of calculation or reckoning during the one complete period of time, and where new growth will be added on to the previous growths, then this has the standard form of:

t = time = 100%/ frequency = 1/ frequency : Each cycle or wave of a certain frequency has a a corresponding time value or duration needed to complete that cycle or wave. Frequency (f) and corresponding time (t) value are mathematical reciprocals of each other. $(f)(t) = 1$

PERT , The PERT formula

"Pert" or Pe^(rt) is a common formula for a bank that offers continuously compounded interest. It will be shown below how Euler's Number (e) comes into use at the bank and in other formulas elsewhere. Below, P = Principle (similar to Mo=initial money amount) is the starting amount that will be grown by the growth formula applied to it. Variable (I) = the interest rate.

Total Value = Starting Value + Interest

$Pt = P + PI$ Pt = total principle after interest is applied to it. Factoring P from each term:
$Pt = P (1 + I)$ If interest is to be calculated again after another period:
$Pt = Pt + (Pt)(I) = Pt (1 + I) = P(1 + I) (1 + I)$ Which can be expressed mathematically as:
$Pt = P (1 + I)^2$

If there are several intervals (n) in the period, the interest for each interval will be the total interest for that period divided by that number of intervals, and the exponent will also be equal to that number of intervals (of interest calculation) in the period:

$$Pt = P \left(1 + \frac{I}{n}\right)^n$$

Now in relation to Euler's Number (e) where the number of intervals approaches infinity for naturally compounded growth:

Letting interest rate = I = 100% = 1 = r , and n=x to express this as the classic power of a binomial expression for (e) which can be expressed with some grouping:

$$Pt = P \left(1 + \frac{1}{x}\right)^x$$ If the interest rate was not 1, but another value, this can be expressed as:

$$Pt = P \left(1 + \frac{r}{x}\right)^x$$ And this is equal to the following expression.

$$Pt = P \left(1 + \frac{1}{x}\right)^{xr}$$ We see that the rate (r) essentially becomes a power to (e). This is verified below at (*) and (**). This can be expressed as a power to a power:

$$Pt = P \left[\left(1 + \frac{1}{x}\right)^x\right]^r$$ As the number of intervals (x) in the period approaches infinity, this is equivalent to:

$Pt = P e^r$ If there are several periods (t) of time of this growth calculated, then the formula will include a power of (t) due to repeated multiplication, and which (t) could even be a fractional value.

$Pt = (P e^r)^t$ which can be expressed as:

$Pt = P (e^r)^t$:this is a power (of e) raised to another power, and this can be mathematically expressed as:

$Pt = Pe^{(rt)}$ **: the PERT formula.** P=starting principle amt., (e) is epsilon or Euler's Number, about 2.71828, r = interest or growth rate per time period of calculation , t = number of time periods being considered. For example. if the (infinitely compounded growth) time period is 1year = 12 months, and 6 months of time has elapsed: t = (6 months/12 months) = 0.5

A helpful way to understand this formula is to consider that when the interest or growth rate doubles, for example, that it will essentially have the same result as doubling the number of periods of (infinitely compounded) growth. The interest rate effectively becomes as like a multiplier value to the period of growth reckoning. Also consider:

* Given the basic growth formula:

$P\left(1 + \frac{I}{n}\right)^n$: I = interest rate (r), n = number of intervals of growth within the time period

When I=1, and n is very high, say n=1,000,000, we get:

$$P\,(1.000001)^{1,000,000} = 2.718280\underline{5} = e^1$$

If interest (I) (or the rate=r) doubles, from 1 to 2, and n is very high, this can be represented as:

$$P\left(1 + \frac{2}{1,000,000}\right)^{1,000,000}$$

$$P\,(1.000002)^{1,000,000} = P\ 7.3890413\ldots = P\,e^2$$

We notice that the power of e, is equal in value to the interest rate. A more generalized formula for all values of interest (r=rate) is then:

From: $P\,e^r$ or = $P\,(e^r)^1$, if the number of growth periods (t) is more than one (1), the generalized formula is:

$$P\,(e^r)^t = Pe^{(rt)}$$

As some extra verification, you can consider all of the following:

ce^1 represents the infinitely compounded growth of (c) for 1 period or time interval. $(ce^1)\,e^1 = ce^2$ represents the infinitely compounded growth of (c) for 2 periods or time intervals. Since the exponent of (e) corresponds to the time interval. After (x) time intervals, (c) will grow to a value of: ce^x. (c) can have any value including 1.

$$ce^x = ce^{(\text{time periods})}$$

If something, such as (c) was to grow (infinitely compounded) for 2 time periods, it is mathematically the same (equal to) as it growing at twice the rate, or twice as fast for just one time period or half of that total time of 2 time periods. That is, at half the time period specified, it essentially already grew by a factor of e^1.

$$c\,(e^x)^2 = c\,(e^x\,e^x) = ((ce^x)\,e^x)) = ce^{(x+x)} = ce^{(2x)}$$

$c\,(e^x)^2 = ce^{(2x)}$ and in general terms, for various rates and time:

$ce^{((\text{rate})(\text{time periods}))} = Ce^{rt}$ which is identical to the "Pert formula"

Also consider that mathematically, for example:

(e^x)^2 = e^(2x)

$$\left|\left[\left(\frac{1}{1}+\frac{1}{n}\right)^n\right]^x\right|^2 = \left[\left(\frac{1}{1}+\frac{1}{n}\right)^n\right]^{2x} = \left(\frac{1}{1}+\frac{1}{n}\right)^{2nx} = \left(\frac{1}{1}+\frac{2}{n}\right)^{nx}$$

as shown below:
(x can be 1 also)
When n--> infinity, that power of the binomial is defined as e=2.718...

e^2 = e^1 e^1 = (1/1 + 1/n)^n (1/1 + 1/n)^n = ((1/1 + 1/n)^n)^2 = (1/1 + 1/n)^2n = e^2 : when n is a very high value

If interest rate (I = r) doubles, say from 1 to 2 we would find this expression for the value of growth after (n) periods of time.

(1/1 + 2/n)^n when n is very high, this is equal to:

(1/1 + 1/n)^n (1/1 + 1/n)^n = e^2 = (1/1 + 1/n)^(n+n) = (1/1 + 1/n)^2n = ((1/1 + 1/n)^n)^2 = e^2

(**) We see a pattern that the indicated power; the exponent of e is equivalent in value to the interest rate. Also consider when n is a very high value:

e^2 = ((1/1 + 1/n)^n)^2 = ((1/1 + 1/n)^2)^n = [(1/1 + 1/n) (1/1 + 1/n)]^n

[(1/1 + 1/n) (1/1 + 1/n)]^n = [1^2 + 1(1/n) + 1(1/n) + (1/n)^2]^n

[1^2 + 1(1/n) + 1(1/n) + (1/n)^2]^n = [1 + 2(1/n) + (1/n^2)]^n : as n--> infinity, this last term approaches 0 very quickly, and is practically of no effect

e^2 = [1/1 + 2/n]^n , and if this is multiplied by [1/1 + 1/n]^n = e , the result is:

(e^2) (e^1) which can be expressed as:

[1/1 + 2/n]^n [1/1 + 1/n]^n = ([1/1 + 2/n] [1/1 + 1/n])^n = ([(1/1) + (1/n) + (2/n) + (2/n^2)])^n =

([(1/1) + (3/n)]^1)^n = [(1/1) + (3/n)]^n = e^3 as n---> infinity

A DERIVATION OF THE SERIES FOR POWERS OF (e)

The derivation of the series for powers of (e), or e^x, requires that you first know about the Binomial Theorem. Rather than perform repeated multiplication to "expand" a binomial (a two term expression) that is raised to an indicated power, a general formula has been developed and it is called the **Binomial Theorem**. The Binomial Theorem, as we know it today, was discovered in about the year 1665 by Sir Isaac Newton who also discovered many other properties, relationships and formulas, such as the basic formula for force: (force = mass x acceleration), and gravity (a force). The derivation of this formula begins with the following basic form of any binomial raised to an indicated power:

(a + b)^n

Since we are dealing with algebra, (a) and (b) can actually be or represent expressions, but for most practical cases, they just represent a simple numeric value. If the binomial is a difference of terms: (a - b)^n, you can perform this simple algebraic manipulation to place it into proper form, or just consider it in this form: (a + (-b))^n. The terms of the expansion of this binomial (a - b)^n are almost identical to those of a binomial with two positive terms except that now they will alternate in sign from positive to negative. Also note that regardless of the values (ie., signs) of (a) and (b) in the expression (a - b)^n, the result is always positive when (n) is even, and negative when the base is negative and (n) is odd. This can easily be verified by the rules for squaring or multiplication, for example:

(5 - 2)^2 = (3)^2 = 3^2 = (3)(3) = +9 : base is positive (here, +3), and n is even

(2 -5)^2 = (-5 + 2)^2 = (-3)^2 = (-3)(-3) = +9 : base is negative (here, -3), and n is even

Note that it is incorrect to simplify (-3)^2 above by removing the parenthesis before performing the squaring since:

-3^2 = - (3^2) = - (9) = -9 :since powers and roots have a higher precedence than summation

(5 - 2)^3 = (3)^3 = 3^3 = (3)(3)(3) = +9 : base is positive, and n is odd

(2 -5)^3 = (-5 + 2)^3 = (-3)^3 = (-3)(-3)(-3) = -9 : base is negative, and n is odd

Using repeated multiplication, expanding and simplifying some powers of the general algebraic form of a binomial:

(a + b)^1 = 1a^1 + 1b^1

(a + b)^2 = 1a^2 + 2ab + 1b^2 ex. 7^2 = (1 + 6)^2 = (1)^2 + 2(1)(6) + (6)^2 = 1 + 12 + 36 = 49

(a + b)^3 = 1a^3 + 3a^2b + 3ab^2 + 1b^3

(a + b)^4 = 1a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + 1b^4

Now, let's note the similarities or patterns in each expansion:

The total number of terms for the expansion will be one more than the indicated power of the binomial.
ex. (a + b)^2 , here the indicated power is with exponent n=2, so there will be (n+1) = (2+1) = 3 terms in the expanded form of that binomial.

The sum of the exponents of both the (a) and (b) variables in each and every term is equal to the indicated exponent (n) of the binomial.

If (n) is even, there will be (n+1) terms, hence an odd number of terms. The exponents of the (a) and (b) variables in the "middle term" will be equal to (n/2), and the sum of those exponents will be equal to (n).

The first term is: $a^n b^0 = a^n(1) = a^n$. For each next term, the exponent of (a) will be one less, and the exponent of (b) will increase by one. The last term is: $a^0 b^n = (1)b^n = b^n$.

The (numerical) coefficients of the (a) and (b) variables are symmetrical (ie. the same value) on each side of the middle of the sum of all the terms.

For the example above, the coefficients of the variables in each consecutive term are: 1, 4, 6, 4, 1, and note the similar value (here, 1 and 4) at both ends. Here is a formula for the numerical coefficients (c1, c2, c3, **. . .** cn+1) of those variables:

1. The first and last coefficient is 1.

2. The second coefficient is (n)

3. The next term's coefficient (ct+1) is found from it's previous term's coefficient (ct):

$$ct+1 = \frac{ct\ (\text{exponent of variable (a)})}{t}$$

: t = term number. All values on the right side correspond to term (t) which is essentially the previous term used to find the constant of the next term (t+1).

Any constants past the (n+1) term are 0, and therefore all those terms will have a value of 0. Here is a (non-standard) formula for these constants that does not rely upon the previous constant. This may be skipped over (to: *****) if its excessive to the reader, and since its non-standard:

Notice that for the first term of each expansion, the exponent of (a) is equal to (n). For the next term, the second term, the exponent of (a) is one less or (n - 1) and for the next term, the exponent of (a) is two less than (n) or (n-2), and so on:

$$c3 = \frac{c2\ (n-1)}{2} = \frac{\frac{n\ (n-1)}{1!}}{\frac{2}{1}} = \frac{n\ (n-1)}{(2)(1!)} = \frac{n\ (n-1)}{2!} \qquad : 2! = (t-1)! = (3-1)!$$

$$c4 = \frac{c3\ (n-2)}{3} = \frac{\frac{n(n-1)\ (n-2)}{2!}}{\frac{3}{1}} = \frac{n\ (n-1)\ (n-2)}{3(2!)} = \frac{n\ (n-1)\ (n-2)}{3!} \qquad : 3! = (t-1)! = (4-1)!$$

Summarizing the results:

$$c1 = \frac{1}{0!} = 1 \quad , \quad c2 = \frac{n}{1!} \quad , \quad c3 = \frac{n(n-1)}{2!} \quad , \quad c4 = \frac{n\ (n-1)(n-2)}{3!}$$

To be consistent with (t - 1)! , 0! is shown above and it is formally defined as: $0! = 1$.

The formula resulting from these expressions for the constants is:

1. Starting from the second term, each numerator has (t - 1) factors.

The first factor is (n).
Each next factor is 1 less:
The second factor is: $n - 1 = (n - 1)$
The third factor is: $(n - 1) - 1 = n - 1 - 1 = (n - 2)$, and so on.

In the last factor, the second term (the one being subtracted) is (t - 2).

2. Each denominator is (t - 1)!

* When these expressions for the constants are placed into the Binomial Theorem, the entire expression is formally known as the Binomial Formula:

(a + b)^n = c1a^n + c2a^(n-1) b^1 + c3a^(n-2) b^2 + . . . + c(n+1) b^n : **Binomial Formula**

Note that the first term that does not express the (b) variable since: c1a^n b^0 = c1a^n(1) = c1a^n
The last term does not express the (a) variable since a^0 = 1, however you can indicate (a^0, and b^0) it you want to be more formal in the ascending and descending powers of these variables.

Still, there is yet a pattern in these expressions for the constants, and the reader may skip this (to: *****) if it is excessive, and since its non-standard:

Each numerator essentially contains part of the most significant or leading portion of n!, and the denominator is (t-1)!. If the numerator is expressed as n!, some of the least significant or trailing portion of this factorial must be removed, and this can be done with division by that value that must be removed. Here is this formula:

$$ct = \frac{n!}{(t - 1)!\,(n - t + 1)!}$$

As a check, the sum of the indicated numerical coefficients (and not the value of the factorials themselves) of the two factorials in the denominator is always equal to (n).

* In the Binomial Formula shown above, when (n)<1, or a proper fraction, an infinite series of terms will be produced where the terms get smaller and smaller, and therefore, the sum of these terms (which get smaller and smaller in value) will converge or "zero-in" to a specific value.

The value of the exponent (n) need not be an integer, in fact, any value, such as a whole number (consisting of an integer or whole part, and a fractional part) for (n) can be used. Hence various roots and powers can then be found. Variables (a) and (b) can also be any value. Here are some of the results to expect using the Binomial Formula:

When (n) is an integer, the number of terms is fixed at (n+1).

When (n) is not an integer, or when (n) is negative (ie., the reciprocal of that power), the number of terms becomes infinite. Consider for example that a value raised to a negative power is the reciprocal value of the equivalent positive power. Still, the sum of the series of terms will converge to a specific value. As an illustration to why the expansion of this binomial is an infinite series of terms, imagine finding the square root of 2 with the Binomial Formula:

$\sqrt{2}$ = 2^(1/2) = (1 + 1)^(1/2) or (1 + 1)^0.5 : in this binomial, a=1, b=1, n=(1/2) = 0.5

The result to expect is the irrational value about 1.414213562**...**, and its' digits will not terminate or repeat at some point. To calculate any more digits for increased accuracy, you will need to do more calculations, hence there can never be a fixed number of terms for such a situation with "endless digits". Unlike when the exponent is an integer, here, the last coefficient (of the n+1 term) is not 1, and the exponent of (a) is not 0, so the process of adding more terms will continue.

Fewer terms are needed when one variable (a or b) is greater than 1 and when the other variable is less than 1. For example, instead of using (2 + 2)^1.5 for evaluating 4^1.5, you can use something like: (3.5 + 0.5)^1.5

When one of the variables, or terms, of the binomial is equal to 1, the Binomial Formula can be simplified or reduced to what is commonly called the **Binomial Series**. If (a) = 1, all of the powers of it are equal to one, and it can essentially be eliminated or removed as a factor in each term. Expressing this:

$$(1 + b)^n = c_1 + c_2b^1 + c_3b^2 + \ . \ . \ . + c_{n+1}b^n \qquad : \textbf{Binomial Series}$$

Even though the (a) variables were eliminated from the expansion, the same concepts and expressions for the constants are the same.

The results of using the Binomial Series are somewhat, but not exactly similar to that of the Binomial Formula:

Expect (n + 1) terms if (n) is an integer.

When (n) is not an integer, the expansion of the binomial becomes an infinite series of terms. The value of variable (b) must also be less than 1, otherwise, the terms will become larger and larger (since its' exponent is becoming larger and larger), and therefore, the series will not converge to a specific value. The lower (b) is, the fewer terms required for a specific number of (most significant, higher numeric weight, leading) accurate digits in the result.

The binomial series can be used for general mathematical calculations such as for taking powers or roots of values. Below, the binomial series is used to find a series for powers of (e):

The core or fundamental expression for (e) is a binomial raised to an indicated power (ie., indicated with an exponent). Since the first term (a) in the binomial is equal to 1, the binomial series can be utilized to "expand" it so as to find its value.

$$e = \left(\frac{1}{1} + \frac{1}{n}\right)^n \quad \text{or} = (1 + 1/n)^n$$

: (e) is defined where (n) is very high (infinite) in value. Raising each side to the (x) power:

$$(e^1)^x = \left[\left(\frac{1}{1} + \frac{1}{n}\right)^n\right]^x$$

using the power to a power rule on the right hand side:

$$e^x = 2.718281828^x = \left(\frac{1}{1} + \frac{1}{n}\right)^{(nx)}$$

To find any power of (e), this binomial can be expanded into a (power) series of terms by using the Binomial Series, where the exponent will be (nx), and (b) will be (1/n). Since (a)=1, it will essentially not be needed in this expansion since all powers of 1 have a value of 1, and 1 as a factor does not change any value of a term. Showing this algebraically, we will arrive at a simplified formula for finding powers of (e):

$$e^x = c_1 + c_2b^1 + c_3b^2 + c_4b^2 + \ . \ . \ . \qquad : e^x \text{ expressed as a power (of b) series}$$

$$e^x = c_1 + c_2\left(\frac{1}{n}\right)^1 + c_3\left(\frac{1}{n}\right)^2 + c_4\left(\frac{1}{n}\right)^3 + \ . \ . \ . \qquad \text{distributing the power to the num. and den.:}$$

$$e^x = c_1 + c_2\left(\frac{1^1}{n^1}\right) + c_3\left(\frac{1^2}{n^2}\right) + c_4\left(\frac{1^3}{n^3}\right) + \ . \ . \ . \qquad \text{simplifying:}$$

$$e^x = c_1 + \frac{c_2}{n^1} + \frac{c_3}{n^2} + \frac{c_4}{n^3} + \ . \ . \ .$$

$c_1 = 1$

$$c2 = \frac{nx}{1!}$$

$$c3 = \frac{nx\,(nx - 1)}{2!} = \frac{(nx)^2 - nx}{2!} = \frac{n^2x^2 - nx}{2!}$$

$$c4 = \frac{nx\,(nx - 1)\,(nx - 2)}{3!} = \frac{(nx)^3 - 3(nx)^2 + 2(nx)^1}{3!} = \frac{n^3x^3 - 3n^2x^2 + 2nx}{3!}$$

Now substituting the expressions of the constants:

$$e^x = 1 + \frac{(nx)(1)}{(1!)(n^1)} + \frac{(n^2x^2 - nx)(1)}{(2!)(n^2)} + \frac{(n^3x^3 - 3n^2x^2 + 2nx)(1)}{(3!)(n^3)} + \ . \ . \ .$$

$$e^x = 1 + \frac{(nx)(1)}{(1!)(n^1)} + \frac{(n^2x^2 - nx)(1)}{(n^2)(2!)} + \frac{(n^3x^3 - 3n^2x^2 + 2nx)(1)}{(n^3)(3!)} + \ . \ . \ .$$

$$e^x = 1 + \frac{x}{1!} + \left(\frac{x^2}{1} - \frac{x}{n}\right)\left(\frac{1}{2!}\right) + \left(\frac{x^3}{1} - \frac{3x^2}{n} + \frac{2x}{n^2}\right)\left(\frac{1}{3!}\right) + \ . \ . \ .$$

When (n), the number of intervals per period approaches infinity as for the definition of (e^1) and its corresponding value (about 2.718), all the fractions containing (n) in the denominator will rapidly approach a value of 0 and are therefore insignificant, "meaningless" or negligible in value, and those terms can be removed from the sum of terms:

$$e^x = 1 + \frac{x}{1} + \left(\frac{x^2}{1}\right)\left(\frac{1}{2!}\right) + \left(\frac{x^3}{1}\right)\left(\frac{1}{3!}\right) + \ . \ . \ .$$ simplifying:

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ . \ . \ .$$

: **SERIES FOR POWERS OF (e)**
(The derivative of (e) is also the same series)
This was found by the profound Swiss scientist
Leonhard Euler (1707-1783AD), and it is why (e),
"epsilon", is often called "Euler's Number".

When x = 1 we can find the value of e = e^1 = 2.71828 . . . :

$$e^1 = 1 + \frac{1}{1!} + \frac{1^2}{2!} + \frac{1^3}{3!} + \ . \ . \ .$$

$$e^1 = 1 + \frac{1}{1} + \frac{1}{2} + \frac{1}{6} + \ . \ . \ .$$: this is essentially 1+ (sum of reciprocals of the factorials)

$$e^1 = 1 + 1 + 0.5 + 0.16666666\underline{7} + \ . \ . \ .$$

: these four terms of the expansion sum to about 2.6$\underline{7}$
The specific value of each term will be less and less,
till the value of any new terms added in are nearly
meaningless since they will have a value close to 0.
The sum of all these terms will approach, "zero in", or
"converge" to have a certain or specific value, and here it is:
2.71828**...**

DERIVATIVE OF e^x

We can find the derivative of the function: e^x by taking the derivative of the expansion of e^x. To do this, take the derivative of each term and sum the derivatives. The result is amazing since the derivative of this function is actually the same function. That is, the derivative of e^x is e^x. Given (x), the derivative (dy/dx = de^x/dx) or slope of the curve at that point is equal to e^x = f(x) = (y) which is the corresponding coordinate of (x) on the curve. Obviously, the greater (x) is, the greater the value of e^x = slope at that point on the curve.

Since the first term is a constant, its derivative is 0.

The derivative of the second term is: $\frac{1\,x^{(1-1)}}{1!} = \frac{1x^0}{1!} = 1$

The derivative of the third term is: $\frac{2\,x^{(2-1)}}{2!} = \frac{2x^1}{(2)(1!)} = \frac{x^1}{1!}$

The derivative of the fourth term is: $\frac{3x^{(3-1)}}{3!} = \frac{3x^2}{(3)(2!)} = \frac{x^2}{2!}$ therefore:

$$\frac{dy}{dx} = \frac{d\,(e^x)}{x} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \;.\;.\;. = e^x$$

: **Derivative of e^x , series expression**
The derivative of e^x is e^x.

By knowing that the derivative of e^x is e^x, and that e^0 = 1, you can also write a Maclaurin series for e^x which will produce the same results as shown above for the series for powers of (e).

As another advanced verification to the derivative of e^x:

From: y = a^x : an exponential equation, and it's derivative is:

$$\frac{dy}{dx} = \frac{d(a^x)}{dx} = a^x \ln a$$

, if a=2.718281828... = (e), we have:

$$\frac{dy}{dx} = 2.718281828^x \ \ln 2.718281828$$

$$\frac{dy}{dx} = e^x \ \ln e = e^x \ (1) = e^x$$

We see that the derivative equals the original function. For other values of (a), lower or higher than 2.718281828 = (e), the derivative of a^x will not equal just the same function of a^x because it will also include the factor of (ln a). Since (e) is a valid value for (a), it gives a verification to the formula for the derivative of an exponential equation of the form: (a^x).

A similar formula for (e) is: e = lim n-->0 of: (1 + n)^(1/n)

Euler's Formula or Equation is: e^(ix) = cos x + i sin x : i is the imaginary unit or number of: sqrt(-1) = i

For example $\sqrt{-7} = \sqrt{(-1)(7)} = \sqrt{-1}\,\sqrt{7} = i\,\sqrt{7}$

In the series for (e), if you substitute (ix) for (x),
you can obtain this expression for Euler's Formula.

Besides the MOD operation, we have seen that only the trigonometric functions are periodic or cyclic (will repeat over and over with the same output, which can be seen on a graph of these functions). Euler's Formula above shows a relation of a standard algebraic equation and the periodic (an irrational) ones. It is derived from the combining of the series for (cos x) and (i sin x) term by term (which is very similar to combining of the series for (cos x) and (sin x)). Its two-dimensional graph has two curves on it, one for cos x and the other being an imaginary curve of (i sin x) that looks identical to sin x. In a three-dimensional graph, the curve looks like a spiral shape since the imaginary part (i sin x) causes a periodic rotation. When there is a rotation due to the imaginary part, the "curve or locus movement" is always perpendicular (at a right angle) to the current real (number) location.

Euler's Constant, or Euler's Number (e) has many strange properties associated with it, for example, of all the values of (x) for: $y = \sqrt[x]{x} = x^{(1/x)}$, y will have the maximum value when x = e. And that maximum value is about: 1.4446678610 . As x increases, the same value root of it will approach 1.0 in value as x approaches infinity. I have found that $(e)(\ln \sqrt[e]{e}) = (e)(\ln e^{1/e}) = 1$, and this 1 value also indicates that the two factors are reciprocals, hence the second factor equals (1/e) and-or e^{-1} , and that: $e^1 e^{-1} = e^{(1-1)} = e^0 = 1 = (e / e) = (e)(1 / e)$.

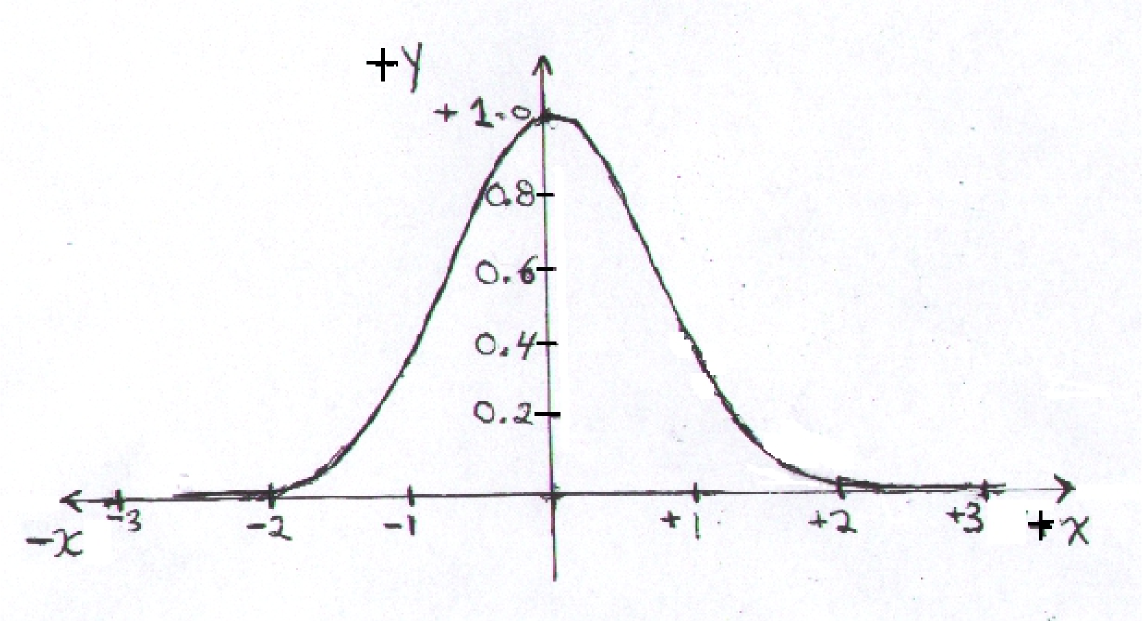
Another constant related to (e) is the "omega' constant" which is about 0.567143290. When (x) is this value, the equation below and other similar basic types of (e) and natural log equations are solved:

For example: $e^x = 1/x$: this equation is true when (x) equals the "omega constant" = ~ 0.567143290

If you want to draw the classic "bell (shaped) curve", try this:

$y = e^{-(x^2)} = \dfrac{1}{e^{(x^2)}}$: note, $e^{(x^2)} \neq (e^x)^2 = e^x e^x = e^{(x + x)} = e^{(2x)}$

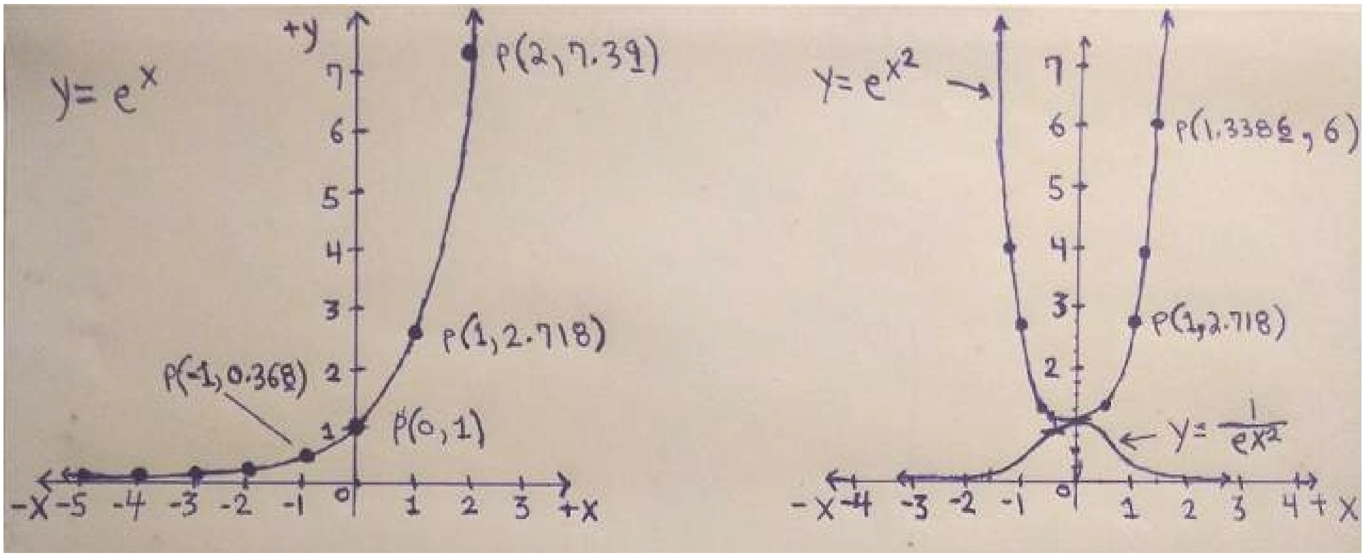
The maximum (y) value is 1 when x=0, the lowest absolute or signless value of x. This is due to the inverse mathematical relationship between a quotient (here, as y) and the divisor (here, as $e^{(x^2)}$). Points such as this are sometimes called the vertex of the curve. For this equation, the value of (y) approaches, but never is actually a value of 0. [FIG 217]



bell curve

In the above equation, variable (x) is squared, and the result besides a steeper curve shape, is that the values of the (y) and the equation are always positive in sign, regardless if (x) is positive or negative in value, and due to the rules for multiplying signed values . Here, the squaring of the variable creates the curve ("mirror") symmetry about the (y) axis.

Below are some examples of typical curves which involve (e) as a base of a power in an exponential equation, and the results are similar for values other than (e). In general, the greater (e) and-or (x) is, the steeper the slope of the curve.
[FIG 218]

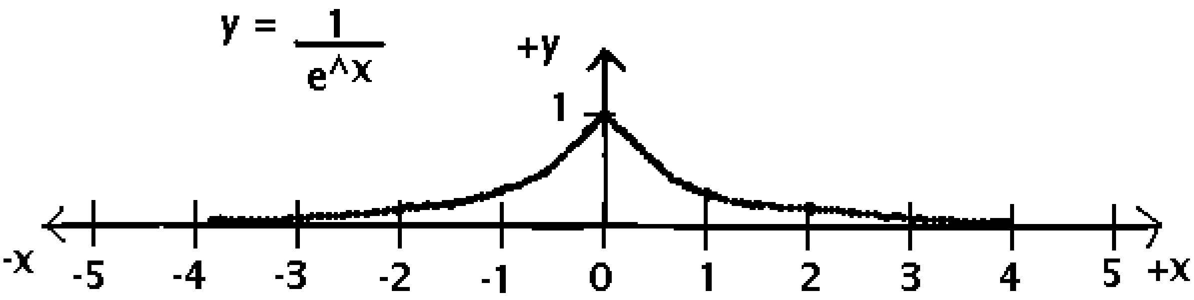


For curve above for y=e^x, notice that the values of (y) when x<0 (ie., negative in value) are reciprocal in value to the (y) values when (x) is positive, and that their product is therefore 1. For example: when x = 1, y = 2.718, and when x = -1, y = 0.368. The product of (2.718) and (0.368) = 1. (e^x) (1 / e^x) = 1 = (e^x) (e^-x) = e^(x-x) = e^0 = 1 . For this curve, when (x) is negative in value: y = e^(-x) = 1/e^x, and this curve is shown further below below.

On the right side is the bell curve of: y = 1/e^(x^2) , and the reciprocal of this equation and curve is: 1/y = 1 / e^(x^2) = e^(x^2), and this curve is also shown. Here, the corresponding values of (y) for the same value of (x) on each curve, are reciprocal in value.

The reciprocal curve, equation and values of: y = e^x is 1/e^x = e^-x , and this curve is shown below:

[FIG 219]



Note that: y = e^x + 1/e^x = e^x + e^-1x Is the catenary curve. This curve could be described as the natural sag or bend of a suspended (supported at its ends) line that is perpendicular to a field of attraction such as gravitational force.

For electronic circuits, the current and voltage within a capacitor (basically, 2 closely spaced metal plates acting like a battery or charge holder ("condenser", "accumulator"), or inductor (basically, a coil of wire that has the effect or temporarily impeding or resisting changes in charge or current) component will increase or decrease in value according to their respective and fairly similar equations which contain **Euler's constant (e)**. While at this topic, and briefly, a resistor component in series with a capacitor can reduce current to it and therefore increase the time it takes for that capacitor to fully charge up with current (electrons) and reach it's specific supplied or maximum possible voltage. Capacitors constructed to store more electric charges are said to have a greater "capacity" or ability to store charge, and will naturally take longer to charge. This is all helpful when designing and experimenting with electronic timer (ie., delay) and-or oscillator (periodic, cycled or repetitive signal pulses) circuits such as for radio [rf, radio frequency]. Radio waves are invisible waves of (electro-magnetic) energy at any frequency from 0 hertz to trillions of hertz or repeating cycles per second. Modern scientific studies have found that radio wave energy is composed of both an electric field and a magnetic field that is transmitted and received. These fields are transmitted together and having the same polarity, however they are at right angles to each other since this is how they were first created in the wire and-or transmission antenna. Many consider these fields as being composed of energy particles called photons, albeit many frequencies of it are not visible to the human eye, but they can be sensed by other electronic sensors and-or radio antennas. These sensors, including our eyes, convert this electro-magnetic energy into electrical energy (such as for a radio or solar-panel) and-or heat energy (such as for a green-house).

A DERIVATION OF THE NATURAL LOGARITHM SERIES

This derivation for the natural logarithm series stems from the facts that the derivative of e^x is itself, and from the derivative of the natural logarithm of (x) as will be explained here:

Given: $e^y = x$: taking the logarithm of both sides of this equation

$\ln e^y = \ln x$ by the log rules, this can be expressed as:

$y \ln e = \ln x$: since ln e = 1, this simplifies to:

$y = \ln x$ according to the basic logarithm definition, we then have:

$e^y = x$ taking the derivative (with respect to y) of both sides of this equation:

$\frac{d(e^y)}{dy} = \frac{dx}{dy}$: since the derivative of a power of (e) is equal to that same power of (e), for example, given: e^x , $d(e^x)/dx = e^x$. We now have:

$e^y = \frac{dx}{dy}$ since $e^y = x$, and $y = \ln x$ as shown above, and using substitution, we have:

$x = \frac{dx}{d(\ln x)}$ taking the reciprocal of both sides of this equation, and switching sides:

$\frac{d(\ln x)}{dx} = \frac{1}{x}$: **DERIVATIVE OF THE NATURAL LOGARITHM OF A NUMBER**

Note that the derivative of the natural logarithm of a number is equal to the reciprocal of that number. The higher the number (x), then the slope at that point on that log curve or graph approaches 0 which is the slope of a horizontal line. This is since given $y=\ln x$, and when (x) is high in value, changes in (x) will cause almost no effect or changes in the value of (y). Note also that while (1/x) is algebraic , (ln x) is not algebraic, but is transcendental in nature.

Now that we have the derivative of the natural logarithm of (x), we obviously cannot simply take the inverse (anti-) derivative of (1/x), which results to: (ln x), so as to find a usable expression (such as a series) for the natural logarithm of (x). Even if we tried taking the inverse derivative of $(1/x) = (x^{-1})$ using the method shown previously, it will also not produce a usable result since according to that method: increasing (adding) the exponent of (x), here -1, by 1 is 0, but then the division by this value of 0 is not mathematically allowed:

$\frac{1}{x} = x^{-1}$ taking the anti-derivative: $\frac{x^{(-1+1)}}{0} = \frac{x^0}{0} = \frac{1}{0}$: The quotient is undefined; has no specific value.

In order to find a usable expression for (ln x), we need more to work with, and this is an equivalent expression for (1/x). Perhaps then, there is a reciprocal series that is equivalent to (1/x) that we can use to find the inverse derivative of, and therefore have an expression for (ln x). As it turns out, the general reciprocal series is a bit more complex to work with than this "modified" reciprocal series shown below that we can use, but it will (only) evaluate the reciprocals of numbers (N) between 0 and 2:

$$\frac{1}{N} = \frac{1}{x+1} = \sum_{n=0}^{n=\infty} (-1)^n \, x^n = 1 - x^1 + x^2 - x^3 + \ . \ . \ .$$

: (modified) Reciprocal Series
: $0<N<2$
: $N = x + 1$
: $x = N - 1$
: $-1 < x < 1$

Unlike the general reciprocal series, there is no division required to evaluate the terms of this "modified" series. For this series to converge to a specific value, (x) must be less than 1, otherwise, the result would grow "without bounds" due to the increasing powers of it. More specifically (x) must be: $-1 < x < 1$. Remember, division by 0 is not allowed. The curve of this expression is actually the exact same curve as for the expression (1/x), the true reciprocal curve, except that it (all points) is effectively shifted leftward by 1 on the horizontal or (x) axis. Just the same, the curve of 1/(x+1) can be considered as being effectively shifted rightward by 1, or "one step ahead", with respect to the curve of 1/x. Consider this example data:

x	=	0	1	2	3	4	, . . .
y = 1/x	=	(undefined)	1.0	0.5	0.3333...	0.25	, . . .
y = 1/(x + 1)	=	1.0	0.5	0.3333...	0.25	0.1667...	, . . .

As an extra note, if N is greater than 2, you can still calculate its reciprocal using the above series by first factoring N (usually into a modified scientific notation form whose reciprocal(s) is already known). For example, to find the reciprocal of 4:

$$\frac{1}{4} = \frac{1}{10\,(0.4)} = \frac{1}{10} \cdot \frac{1}{0.4} = 0.10 \left(\frac{1}{0.4} \right)$$

: to divide a value by 10 or= multiply a value by 0.1, simply move its decimal point one position leftward.
$0.1 = (1)(0.1) = 1 / 10^1 = (1)\ 10^{-1}$

Then use the series to evaluate the reciprocal of 0.4, and then multiply it by 0.10. Here is the formula to determine the value of (x) for the terms of the series:

$N = 1 + x$
$0.4 = 1 + x$ solving for x:
$x = 0.4 - 1$
$x = -0.6$: use this value in the "modified" reciprocal series above

Don't forget that you can also use the log reciprocal formula:

$\ln N = -\ln 1/N$: for example $\ln 4 = -\ln 1/4 = -\ln 0.25$

The derivation of this reciprocal series uses the form of the binomial series shown previously:

$(1 + x)^n = c1 + c2x^1 + c3x^2 + \ . \ . \ . + cn{+}1\ x^n$: Binomial Series

To place $1/(x + 1)$ into a power of a binomial form, we will use its equivalent binomial form of: $(1 + x)^{-1}$

Letting n = (-1) and solving for the first four constants:

$c1 = \frac{1}{0!} = 1$, $c2 = \frac{n}{1!}$, $c3 = \frac{n(n-1)}{2!}$, $c4 = \frac{n(n-1)(n-2)}{3!}$

$c1 = 1$, $c2 = \frac{-1}{1}$, $c3 = \frac{-1(-1-1)}{2}$, $c4 = \frac{-1(-1-1)(-1-2)}{6}$

$c1 = 1$, $c2 = -1$, $c3 = 1$, $c4 = -1$

Substituting the constants:

$(1 + x)^{-1} = \frac{1}{x + 1} = 1 + (-1)(x^1) + (1)(x^2) + (-1)(x^3) + \ldots$ using distribution:

$(1 + x)^{-1} = \frac{1}{x + 1} = 1 - x^1 + x^2 - x^3 + \ldots$: checks , (modified) Reciprocal Series

Finding the inverse derivative of each term above to have the complete inverse (or anti-) derivative of this series, we have the following logarithm series which was developed by Nicolaus Mercator:

$$\ln(x + 1) = \sum_{n=1}^{n=\infty} (-1)^{(n+1)} \frac{x^n}{n} = +\frac{x^1}{1} - \frac{x^2}{2} + \frac{x^3}{3} - \ldots \qquad : x < 1$$

For this series to converge to a specific value, (x) must be less than 1. The terms will get smaller and smaller in absolute value. This series can evaluate the natural logarithm of values less than 2. This series is very important and-or useful, and a method has been developed as shown in this book of how to calculate the natural logarithm of numbers greater than 2.

For a simple formula for ln x , you can use the following, however without a computer or calculator, you will need to compute a power of (x) which for here, is actually a root of (x):

$\ln x = n(x^{(1/n)} - 1)$: as n-->inf , formula by Leonhard Euler
After simplifying, this is equal to:

$\ln x = nx^{(1/n)} - n$: as n-->infinity

A similar formula is:

$\ln x = \frac{x^n - 1}{n}$: as n --> 0

From the derivative of ln x, as shown previously, we mathematically have:

From: $y = \ln x$, $dy / dx = d(\ln x) / dx = 1/x$:

$\frac{d(\ln x)}{dx} = \frac{1}{x}$ from this equation, we mathematically have:

$d(\ln x) = \frac{1}{x}\,dx$ All the small parts or pieces of something considered as one whole thing will sum to that whole thing. This is called integrating. Each small part of (ln x) is noted as d(ln x), and all these will sum to the whole (ln x). Expressing the summation of all these small parts on both sides of the equation:

$$. \quad \int d(\ln x) = \int \frac{1}{x}\, dx$$ this results to:

$$\ln x = \int \frac{1}{x}\, dx$$

For an example of using the above, when the integral (of integrating or summing) is a definite integral, it is like an infinite area being defined, bounded or limited to a specific area or section that is below the curve (ie., between the curve of an equation and the x axis and where y=0). The result, the area in question, the bounded or definite area, is calculated as the difference of those two areas defined by upper and lower limits mentioned and-or indicated. With the above equation, the area beneath the curve of y = ln x, from the limits of x=1 to x=e =~ 2.71828 is:

$$\int_1^e \frac{1}{x}\, dx = \ln e - \ln 1 = 1 - 0 = 1$$

: The area beneath the natural logarithm of (x) curve between x=1 and x=e has a value of 1. This could also be said as the area between the curve and x-axis bound by the upper limit of (e) and lower limit of 1.

BASIC DEFINITIONS OF THE TYPES OF NUMBERS

Below are some of the various general types, categories, sub-categories, or classifications of numbers that you may sometimes see and-or read about, and it is would be helpful to at least know a basic or minimal definition of them. As you work more and more with numbers, you will obtain a natural, uncomplicated feel for what these numbers are.

natural number - Natural numbers are those used for basic counting and representing a quantity of whole (entire) things or units often found nature (plants, rocks, trees, stars). Ex. 1, 2, 3 , ...

counting number - Our man-made counting numbers are essentially the same as the natural or "natures numbers".

whole number - An integer value greater than or equal to 0, hence it is also positive in value. A whole number will also not have, include or indicate a fractional part (of 1). Ex. 0 , 3 , 7 , 1000 Whole numbers are used to enumerate complete or whole units, objects or entities. 0 is included since it numerically or symbolically represents an absence.

integer number - A signed counting number, hence can be positive or negative in sign. An integer also does not have, include or indicate a fractional part. Ex. -5 , 0 , +5 . The word "integer is rooted in the words: "wholeness", "complete", "integrity", "untouched" and probably; an "instance" or "next step" of. The word "integer" has some companion words such as: "increment" which means an "increase".

fractional number - A number that represents a value or quantity of something that is less than 1 or 1 whole entire unit or entity. A fraction of a thing is smaller piece or part of that entire or whole thing, hence it is not the whole thing. Its numerical value or representation of that part or quantity of the whole (considered as having a "relative" numerical value of 100% = 1), will therefore be less than 1.

Ex. 0.5 = 1/2 = 5/10, etc. , 0.5<1 , Ex. 0.1 = 1/10 = 2/20, etc. , 0.1 < 1

Although a fraction is always numerically less than, and only a part of the whole thing, if the whole thing is greater than 1, such as a group of things or units being considered, the actual or physical fractional part may then be numerically a value greater than 1. For example, if you have 100 bricks, a tenth (1/10 = 0.1, a fraction of 1 or the entirety) of those 100 bricks is 10 bricks, and this is more than 1 brick. These 10 bricks may also be said as being "ten percent" (10% = 0.1) of all the bricks. 10 bricks is still a fraction or part of all the 100 bricks considered as the 1 entirety, 100%, whole or all.

mixed number - A number containing a whole number part, and a fractional part or portion.
Symbolically: mixed number = whole number part + fractional part
or: mixed number = whole number part **.** fractional part
Ex. 5.2 , 5.2 = 5 + 0.2 , 5 is the whole number part, and 0.2 is the

fractional part that is less than 1.

positive numbers - Represent a value, quantity, or change of something that is greater than 0. Ex. positive 3 is written as +3 , + is the positive symbol

Increases (positive changes) are usually noted with a positive symbol. If the increase was 5, this would be noted as +5, and this will be combined or added to the reference value (0 is not indicated). If the reference value was +10, the resulting value is: +10 plus +5 = +15.

This positive number symbol (+) is also the same symbol for the mathematical addition, increment (ie., increase) or combine operation which is often called the "plus" operation and-or symbol.

negative numbers - Represent a value, quantity, or change of something that is less than 0. Ex.'s A negative temperature value such as -3°F, a debt (ie. a decrease, a loss), or a distance in the opposite (ie., by 180 degrees) direction than the positive, larger or increase direction.

negative 3 is written as: -3 , - is the negative symbol used to create or express a negative valued number or change which is for values less than 0 or some other reference value.

If you started with +7 and lost 3 (ie., a negative change, a decrease), the result is that: +7 and -3 combine or result to: +7 plus -3 = (+7) + (-3) = (+4) = +4

If you started with +0 and lost 3 (ie., a negative change, a decrease) the result is that: +0 and -3 combine or result to: (+0) plus (-3) = (+0) + (-3) = (-3) = -3

If you started with -3 and lost 1 (ie., a negative change, a decrease) the result is that: -3 and -1 combine or result to a bigger or larger negative number: (-3) plus (-1) = (-3) + (-1) = (-4) = -4 . Now consider the following:

If you started with -4 and gained or increased by 1 (ie., a positive change) the result is that: -4 and +1 combine or result to a lower negative number or toward the "(more) positive (values) direction".
(-4) and (+1) combined = (-4) plus (+1) = (-4) + (+1) = (-3) = -3

This negative number symbol (-) is also the same symbol for the mathematical subtraction, decrement (ie., decrease) operation and is often called the "minus" operation and-or symbol.

real number - A real number can be indicated (ex., a point) and-or located at a specific distance along a signed number line. A real number can be a rational or irrational number. In short, a real number is not an imaginary number.

rational number - A rational number is any number that can be expressed as the quotient or result of the division of two integer values. The fractional part of the result will be either 0, a number with fixed or certain number of decimal places, or a number which contains an "endless" repeating of the same decimal digits after a certain decimal position. All rational numbers are said to be "algebraic" (ie. can be produced (ie., a result) by a single variable polynomial equation).

Ex. 4/2 = 2.0 , 5 / 2 = 2.5 , 1/3 = 0.3333.... , 4/7 = 0.571428571428...

The word "(to) rationalize" basically means "to make (simple) sense or reason of".

irrational number - An irrational number cannot be expressed as the quotient or result of the division of two integer values. That is, it cannot be adequately expressed as a single fraction. The "expansion" or continuation of further decimal digits is endless and does not repeat (ie., have a constant pattern).

Ex. PI = 3.14159.... , e = 2.71828 .... , square-root 2 = 1.4142135.... , and the golden ratio = 0.618034.... are irrational.

The word "irrational" basically means: "does not make (simple) sense or reason", or is "beyond (simple or rational) reason and-or mathematical representation or expression".

Ex. (Pi) is the constant ratio of the Circumference of a circle to its Diameter: (Pi) = C / D = 3.14.... Even though (Pi) is a irrational number, it does not therefore mean that C and D are also irrational, and for example, it is very possible to have say a circumference equal to exactly 3, 5, 100 etc., a rational value. Here, mathematically: D = C / (Pi) , and D essentially becomes irrational in value since here it is being calculated by using an irrational value.

transcendental number - A transcendental number is a number that is "not algebraic", that is, it is not the root of a single variable polynomial equation with rational coefficients. A root is the solution of that variable in the polynomial equation that is set equal to 0. All transcendental numbers are also irrational (non-rational) numbers.
Ex. PI and Epsilon (e) are the best known transcendental numbers.
Note for example that squarer-root 2 is an irrational number, but it is not a transcendental number since the square-root of 2 is the "algebraic" solution of: $1x^2 - 2 = 0$.
The golden ratio (GR) is also irrational, but is not transcendental since it is the "algebraic" solution of: $1x^2 - 1x - 1 = 0$, and the solution here is found to be:
1/GR = 0.618033988749894 = 1.618004...
The equation shown can also be expressed as: $x^2 - x = 1$

The word "transcend" basically means "outside of" (such as standard mathematics such as basic algebra, or beyond "algebraic rationality"). "A number from out of the blue". "A number from out of nowhere". Transcendental numbers are real numbers. The trigonometric functions are considered as transcendental functions since the results take an infinite number of steps to complete, and therefore can't be represented or expressed as a ratio of two numbers or with a simple algebraic formula such as the angle or trigonometric value divided by some other value. In general, logarithm and exponential functions are to be considered as transcendental functions that generally result in a transcendental value, and especially due to how it may be calculated as a sum of an infinite series of terms that will eventually converge to a specific value after an infinite number of terms are utilized.

imaginary and complex numbers -

A complex number is a number that has both a real part and an imaginary part: complex number = real + imaginary
An imaginary number is a number that is imagined to have a numeric value, and is much like a theoretical concept. Whereas the negative of a number which can be symbolically expressed as: - (number) , and is like a reversal or 180° shift, opposite direction, or rotation of a (pos. or neg.) number, an imaginary number is preceded by the i or j symbol which represents the square-root of an effective 180° shift or rotation of a number, and which is therefore indicated as the square root of -1, and is often thought of as being a half, partial or 90° shift or rotation.

The imaginary part of a complex number is represented by the symbol of: j or i and is equal to the imaginary (ie., non-real) value of the square-root of -1.

Ex. $5 + 2i = 5 + 2(\sqrt{-1})$

Ex. $\sqrt{i}\ \sqrt{i} = \sqrt{-1}\ \sqrt{-1} = -1$: -1 is a real number

One possible way to possibly consider any real number is that they are all essentially found and-or already expressed in an equivalent digit form between 0 and 1 on the number line, hence all real numbers are effectively fractional values less than or equal to 1, and were effectively incremented by an integer value or multiplied by a power of 10. For example:

Given 0.5 , then 1.5, found between 1 and 2, is simply an increase of 1 integer step from 0.5 which is found between 0 and 1.

Given 0.314159265 , then 5.314159265 is simply an increase by 5 integer steps

Given 0.731 , then 7.31 , is simply 10 times higher

What can be said about some of the above examples is that it is essentially what also takes place when converting any value to scientific notation when the decimal point is shifted within a number.

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Common Constants, Formulas, Equivalents And Conversions Within And Between Measuring Systems

SOME COMMON CONSTANTS:

pi = 3.14159265358979323846... : the ratio value of: (circumference / diameter) of all circles. (pi) = C / D

e = 2.71828182845904523536... : epsilon, Euler's number, base (of measurement) for natural logarithms

GR = 0.618033988749894... : Golden Ratio (GR), a ratio of two specially related lengths or amounts.
1/GR = GR+1 = 1.618033988749894... , GR^2 = 1 - GR and GR^2 + GR = 1 ,
GR^2 = 0.38196601

Vc = 186282.4... miles/1s = 299,792,458 meters/1s =~ 300000 km/s :speed or velocity of light (c).
(in empty space, hence a vacuum)

Using (300 million) m/s is only about 0.0692% =~ 0.07 = 7 hundredths of 1 percent greater than the true value of speed of light, and is also ~= 186,000 miles/s = 186000 mi/s (3600s / 1h) = 669600000 =~ 670 million mi / hr

Vs = 1125 ft/s = ~ 343 meters/s =~ 767.27 miles/h : velocity or speed of sound through air, at about room temperature
0.343 km/s =~ 1235 km/h ~1100 ft/s or= ~ 350meters/s . Thunder sound called "thunder" will travel about 1 mile in 5 seconds.

g = 9.807m/s^2 = 32.1752ft/s^2 : Earth's constant gravitational acceleration of an object due to gravity which is the attraction of masses (material, substances, matter), and is measured as a force. This specific value is due to the specific mass of the Earth, and is generally not the same value on the Moon or other planets since their mass is more or less than that of the Earth.

From: weight = force = (mass)(acceleration) = ma = mg :

g = weight / mass = a = f / m : force = weight is measured with units of newtons (N),
: mass is measured with units of kilograms (kg)
: 1gram of mass = 1kg/ 1000 = 0.001kg of mass
(g) and (weight) are directly proportional to each other for a given mass.

Ex. Using a known mass of 1kg, and if it was weighed on a force scale to be 9.8N, and on some other planet or moon, it might weigh 1.6N, then (g) can be calculated as:

$$ge = a = \frac{f}{m} = \frac{9.8N}{1kg} = \frac{((9.8kg)/1m) / 1s^2)}{1kg} = \frac{9.807m}{1s^2} \quad : \text{for Earth}$$

$$gm = a = \frac{f}{m} = \frac{1.6N}{1kg} = \frac{((1.6kg)/1m) / 1s^2)}{1kg} = \frac{1.6m}{1s^2} \quad : \text{for Moon}$$

The ratio: gm / ge = $0.1631\underline{5}$ =~ (1/6). A mass on the Moon will weigh about 16% or one-sixth as that of the weight Earth. Since weight is directly proportional to (g), the ratio of the weight of an object on the Moon to that of the Earth also has the same ratio value:

gm / ge = weight on Moon / weight on Earth ~= 0.16 , and from this we have:
(weight on Moon) = (0.16) (weight on Earth) = (weight on Earth) / 6

Because of the above facts, only (1/6) the energy is needed to lift a weight on the Moon as compared to that of the Earth. A person with the energy and strength who can lift 30 pounds on Earth for a short time, can lift the equivalent of (30lbs)(6) = 180 pounds on the Moon for a short time.

Fg = G (M1M2) / r^2 = Force of gravity or attraction between two masses. (r) is the distance between the centers of the two masses (M). (r) has units of meters (m), and M=mass has units of kilograms.
F will have units of Newtons = N . Note that (kg)(kg) = kg^2 , and (m)(m) = m^2
G = universal gravitational acceleration constant =
G = ((6.674)10^-11) m^3 / (kg)(s^2) or= (6.674)(10^-11) N m^2 / kg^2

This basic knowledge and formula is mostly credited to Isaac Newton, and who once said that he standing on the shoulders [ie., the knowledge] of giants [previous others and their own accumulated contributions to science].

G is mathematically a constant of proportionality. This equation for Fg has the form of an "inverse square law", since Fg is inversely related to the square of (r) and therefore rapidly decreases as (r) increases.

Using F=Ma , M1=Me=mass of Earth, and M2=Mo is the mass of an object close to Earth, re = (re) = radius of the Earth, g can be calculated as: g = a = F/Mo = G Me / (re)^2
G can be calculated knowing an objects acceleration g =32ft per second, per second =32 (ft./s^2)
The **mass of the Earth** has been calculated at about 5.973 (10^24) kg = about 6 (10^24) kg.
The radius of the earth = re = 3959 mi =~ 4000 mi =~ 6437 km =~ 6.4 (10^6) m
The mass of the Sun is about 2 (10^30) kg

Using the values given above, a force of 1N will be the force due to gravity upon a mass of 1kg on the surface of the Earth which has a distance = (r) of 6.4 (10^6) m from the center of Earth and the center of its gravitational attraction force. r^2 = (6.4 (10^6))^2 m =~ 41 (10^12) m.
. Note, here, at this value of (r): Fg = G(M1 M2) / r^2 = Fe = weight = ma = mg , and where a=g=9.8 m/s^2 .

Since the meters and kilograms are human made units and value, the specific value of G is technically or mathematically, also a human made value, and so as to have units of Newtons.

The above equation: Fn = G (m1 m2) / r^2 resembles the standard equation for force which is:

force = (mass) (acceleration) and with a dividing variable, here (r^2), that considers the inverse-square type of relationship between (gravity) force and the distance (r) between the two masses or objects.

The more massive (ie., mass) an object, the more gravity potential it will have, however due to equal and opposite forces, the actual gravitational force applied to both masses will be the same value, and it could perhaps be called their mutual force or gravity. If one or both masses change in value, their value of gravity will be different, and it is directly related to how much mass is involved in that system.

When force = f = ma = mg, the force is being directly applied to both

two masses with direct contact, and when considering forces that are not directly applied, such as gravity, and the fact that its strength is inversely related to the distance between those two masses, (r^2) is used as a divisor, and it is not equal (t), but analogous to:
force = mg / r^2   or= gm / r^2

Also, as a reminder, a given force can more easily move a smaller mass, and-or do more work (force x distance) with it due to that the more massive object has more inertia and will require more energy to move it, and-or accelerate it to change its velocity. Also of note is that if one mass is less than 1 kilogram, there will be no effective magnification or amplification in the resulting force. This leads to the question of whether a more massive object has that much more gravity, and due to this equation, it can only be said that a more massive object has more gravity force potential, influence or possibility upon another smaller mass. Note also that:
G = Fn r^2 / (M1 M2)

A basic derivation of the gravitational force equation:

First, the gravitational force Fg between two objects or masses (M1 and M2) is the same value (regardless of their amount of mass) due to the concept of equal and opposite forces, and this could be expressed as:

Fg = F1 = F2   : here the Force(s) is the gravitation force

Fg = (M1 a1) = (M2 a2)

The equation is also written in such a way that if the values of M1 and M2 are exchanged, then Fg is still the same.

We know that the more massive an object is, the more gravitational force or attraction to another mass it will have, and its value of (a) will depend upon the size of that mass, and is directly related to that value. On Earth, the specific value of (a) is (g) = 9.81 m/s^2, and for another object or planet with a different mass, the local or surface value of (g) will be different. Since Fg = F1 = F2 is the same between both masses, and that F = ma, the resulting kinetic acceleration and velocity will be higher of the smaller mass than that of the larger mass.

We know that M1 is generally not equal to M2. Given a mass having a weight or force, if that mass increased by a factor, its corresponding weight (ie., an effect or force due to gravity [force]) will also increase by that same factor.

Something having twice the weight will make a dent twice as deep in the ground when dropped from the same height, and this is due to that the force upon reaching the ground was also increased by that same factor.
From :   F = weight = ma   ,   multiplying both sides by a factor (n):   nF = (n) weight = (n)ma

This implies that the Fg between a mass and another will be a multiple or relative (in numerical reference) or factor the mass value, and a multiple can be expressed as a product. Each mass will have its own local gravitational attraction (g) upon another mass. This force (Fg) is like an effective or resultant force due to an effective or resultant mass value so that Fg between them is the same value, hence of an equal and opposite value. The two masses essentially amplify the force of each other and can be expressed as a multiple or product of each other.

In F = ma , both (m) and (a) can amplify a (force=weight) when those values are amplified (increased by a factor of (n), and if (m) is constant, then (a) determines that force and-or weight.   Since (a) can amplify the force or weight, it is as if the given mass rather was amplified instead. The effective result for a two mass gravity system is that each mass is essentially amplifying each other so as to have an amplified, equal and opposite force (Fg). An equivalent force (Fg = Ma) upon each mass implies an effective or apparent amount of mass, and at the same acceleration (a=G) and its value was first mathematically determined by Henry Cavendish using a torsion (ie.,twisting) balance and various masses to cause a gravitational attraction force on the masses, and then a twisting of the balance by an angle that is determined by the amount of their gravitational attraction. Note that (g) does not equal (G). Technically, (g) is the value of the gravitational acceleration at the surface of the Earth, and that it actually gets less (ie., g is not a universal constant like G is) as the distance from the center of the Earth becomes farther. (g) is rather associated with the kinetic energy of a single object. In 1687, Isaac Newton first proposed the formula for the universal law of gravitation, but it took till 1798 when Cavendish gave the constant (G) a specific value and verified Newtons proposal.

If two masses have different values, the smaller mass will actually have a larger gravitational acceleration towards the larger mass, and this can be seen when you drop an object, and it moves toward the Earth due to Earth's gravitational constant (g), and that the Earth will barely moves toward that smaller mass, and this is due to Earths inertia, and that it will take much more force or energy to accelerate and move it. Acceleration is inversely related to mass:
$a = F / M$, and $v = at$.

pi^2 = 9.869604401089358 $\sqrt{\text{pi}}$ = 1.772453850905516 2pi = 6.283185307179586

1/pi = 1 / 3.141592654 = 1 / (C/D) = (D/C) = 0.3183098861837907

1/ (2 pi) = 0.1591549430918954

e^2 = 7.38905609893065 $\sqrt{e}$ = 1.648721270700128 2e = 5.43656365691809 1/e = 0.3678794411714423

e^e = $15.1542622414792\underline{6}$ e^3 = 20.08553692318766 $\sqrt[3]{e}$ = $1.395612425\underline{1}$

1^2 = 1 2^2 = 4 3^2 = 9 4^2 = 16 5^2 = 25 6^2 = 36 7^2 = 49 8^2 = 65 9^2 = 81 10^2 = 100

2^3 = 8 3^3 = 27 4^3 = 64 5^3 = 125 6^3 = 216 7^3 = 343 8^3 = 512 9^3 = 729 10^3 = 1000

$\sqrt{2}$ = 1.414213562373095 $\sqrt{3}$ = 1.732050807568877 $\sqrt{4}$ = 2 $\sqrt{5}$ = 2.23606797749979

$\sqrt{6}$ = 2.449489743 $\sqrt{7}$ = 2.645751311064591 $\sqrt{8}$ = $2.828427124746\underline{2}$ $\sqrt{9}$ = 3

$\sqrt{10}$ = 3.16227766016838 $\sqrt{100}$ = 10 $\sqrt{1000}$ = 31.6227766 $\sqrt{10000}$ = 100

$\sqrt{0.1}$ = 0.316227766 $\sqrt{0.5}$ = $\sqrt{5}$ $\sqrt{0.1}$ = (2.236067978) (0.316227766) = 0.707106781

$\sqrt{0.01}$ = $\sqrt{(0.1)(0.1)}$ = $\sqrt{0.1}$ $\sqrt{0.1}$ = (0.316227766) (0.316227766) = 0.1

$\sqrt[3]{2}$ = 1.259921049894873 $\sqrt[3]{3}$ = $1.442249570307\underline{4}$ $\sqrt[3]{4}$ = 1.587401052 $\sqrt[3]{5}$ = 1.709975946676697

$\sqrt[3]{6}$ = 1.817120593 $\sqrt[3]{7}$ = 1.912931182772389 $\sqrt[3]{8}$ = 2 $\sqrt[3]{9}$ = 2.080083823

$\sqrt[3]{10}$ = 2.154434690031884 $\sqrt[3]{100}$ = $4.64158883361\underline{3}$ $\sqrt[3]{1000}$ = 10 $\sqrt[3]{10000}$ = 21.5443469

The cube root of 1 million = the cube rot of 1(10^6) = 1 (10^(6/3)) = 1 (10^2) = 100 or 100 units

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METRIC PREFIXES: [For multiples or fractions of a unit or reference of measurement. Here are some common [prefixes, and this topic was previously mentioned in this book.]

kilo means: (1000) = "a thousand of"
centi means: (1/100) = "one-hundredth" or "one-hundredth of" the units used, indicated or expressed
milli means: (1/1000) = "a thousandth of"

ABBREVIATIONS:

in=inches, m=meters, cm=centi-meters, km=kilo-meters
ft=feet, g=grams, L=liters, lb=pounds, oz=ounces, fl=fluid (ie., a [liquid] volume)
s= seconds of time, hr=hours, m=minutes

FREQUENCY AND TIME:

Time=1/Frequency = Ts=1/Fhz. hz=hertz = vibrations, cycles or waves per second. Time is the total time of each and every similar cycle or wave, and it is mathematically inversely related to the frequency of that vibration, cycle or wave. The higher the frequency of a wave, the lower or shorter (ie., "faster" , "quicker") the time needed to complete that 1 wave.

Wavelength (W or Wl) is essentially the physical distance or length between similar or corresponding points (such as the peak [maximum] amplitude value, "peak to peak") on the wave (of vibration, energy cycle or pulses) during its cycle as it travels outward from the source. The wavelength of a wave will be different for energy waves that travels faster or slower such as the speed of light and the speed of sound in air. Wavelength can also be calculated using the frequency of the wave.

Since wavelength is a length or distance, and distance = (speed of movement)(time of movement) = (speed)(time):

electromagnetic **wavelength** = (speed of light)(Ts) = **(Vc)(Ts)** = (Vc)(1 / Fhz) = **Vc / Fhz**

As a helpful thought experiment about wavelength and the speed of light (photons) and-or electro-magnetic waves, consider if the physical wavelength of an electro-magnetic signal was 5cm of length, and then imagine that distance segment and-or wave structure then moving at the speed of light of about 186000 miles per hour which is another distance.

Another helpful thought experiment is to consider a frequency (f) of 1 cycle per time = 1 cpt = (usually) cycles per 1 second = cps.

If you had a frequency of 4cps, then each cycle is less or quicker in time, and here it is one-forth (1/4) that amount of time (t) of one second. It will be shown below that cps and spc (or cpt and tpc) are reciprocals in value:

frequency = cycles per time = c/t = cpt , and t = **time = time per cycle** = spc = time per cycle = t/c

If frequency = f = 4 cps = 4 cpt = 4 (c/t) = 4 hz , and t = time per cycle = seconds per cycle = tpc = 1 / cpt = 1 / cycles per time = 1 / frequency = 1 / f = 1 / 4cps = 1 / (4c / s) = 1s / 4c = 0.25 s/c

frequency = 1 / time or **cycles per time = 1 / (time per cycle)**

W = d = v t , if v = c = the speed of light, and time of each cycle is t = 1s , then, W = c (1s) = (d mi / 1s) (1s) = c miles. Usually for electromagnetic waves, the time of each cycle is much less than 1s, and therefore d is much less than c miles = 186000 miles. Ex. If the time of each cycle is 1ms then: W = d = v t = c (0.001) = 186 miles.

Ex. For typical AM radio wave with a 1 Mega-hertz-cycle , the signals wavelength and basic antenna length is:
(300,000,000 m/s) / (1Mhz 1/s) = (300(10^6) m/s) / (1 (10^6) 1/s) = 300m
This leads to a simple and commonly spoken formula for radio wave wavelength in meters:
(300 / frequency in Mhz) = wavelength in meters

Frequency (F) and Wavelength (W) are inversely related as shown in the formula above. Also consider these inverse proportions: Vc = (W1)(F1) = (W2)(F2) , Therefore mathematically: (W1/W2) = (F2/F1) , and this is an example of what is described as an "inverse ratio (of values and units)".

As a simple example, if a given frequency is multiplied by 2, the resulting wavelength is divided by 2.
For example, for a typical FM radio wave of 100Mhz, the wavelength is: 300m/100 = 3m
For a 1Ghz radio wave, 1Ghz = 1000Mhz, the wavelength is: 300m/1000 = 0.30m = 30cm

Compared to low frequency, long wavelength signals such as for AM radio, a high frequency, short wavelength FM radio transmitter and-or receiver antennas are much shorter in length, and may even be shorter by using a half or quarter wavelength antenna which are very practical, but they will reduce the transmission efficiency of the input electrical energy. You may research a common antenna called a **dipole** (ie., two pole, here 2 half-wavelength segments, unconnected, but on the same imaginary line, and where each segment will alternate in polarity with the alternating RF alternating signal applied to it.). A horizontal dipole antenna is more directional than a vertical single pole ("omni-directional") antenna, hence it can be used to communicate farther in the two opposite directions perpendicular to it.

Wavelength And Frequency Of Visible Light: 1Thz = 1 terahertz = 1(10^12)hz=cps = 1 trillion hertz = 1000Ghz
1nm = 1(10^-9)m = 1 billionth of a meter , and:

(**RGB**) =: **R**ed = 700nm , 428Thz , **G**reen = 530nm , 566Thz , **B**lue = 470nm , 638Thz = 638 (10^12)hz

White light appears as the brightest color, and it is due to its **primary colors**: RGB and where their frequencies are overlapping or combining and effectively simulating a higher frequency when they are all received and sensed at the same time. It could be said that white light is composed of the RGB (Red, Green, and Blue composite or component parts) frequencies. A prism of clear glass can physically separate or "extract" the colors or frequencies within white light since each frequency has a certain wavelength and "diffraction (bending, or turning) angle" when it is slowed by the glass medium. When any two of the R, G and or B colors of light are combined, a third, and apparently slightly brighter color is effectively created of which can even be combined with the third primary color. The resulting colors and-or "shades" (ie., intensity) of color depends on the intensities of all the colors and-or their frequencies involved.

Radio waves are transmitted, radiated outward, or broadcast from a metal antenna such as a straight length of wire. The (invisible) radio waves are emitted from the antenna much like how (higher frequency) visible light is emitted from a light bulb that contains (electric) energy. Radio waves can be considered as like invisible (ie., not visible to the human) photons containing energy. Consider how a heated metal, either by flame or electric current, will have a dark red to white glow of light emitted. Our eyes can only see a small portion of the electro-magnetic range or spectrum of frequencies. Though an antenna is nearly an open circuit, charges can still move in an antenna due to the electric forces of attraction and repulsion. Moving charges such as electrons in a wire will create both an electric and magnetic field. When charges, such as electrons, have gained kinetic energy, and eventually fall back into the orbit of an atom, they will release and-or radiate that excess energy as a photon, and here, particularly the invisible type containing both an electric and magnetic field. The frequency of the radio's oscillator circuit will determine the frequency of the (invisible) photons and-or radio or "energy wave" transmitted.

To receive radio-waves, another metal antenna is needed. The transmitted radio-wave energy strikes the receiver antenna and induces (causes) a signal (voltage and current) to be generated in that antenna wire, which is then processed by other electronic circuitry so as to demodulate (extract) the desired information (audio, video, data) from it.

The received signal may also be amplified so as to be more usable. The specific length of the transmission and reception antennas is based upon the wavelength of the radio-wave frequency in use, and so as to be most efficient. Using the proper length for the antenna is said as "tuning" the antenna for best efficiency. **AM is Amplitude Modulation** (a form of signal combining, here, so as to vary the strength or intensity) of the transmitted radio-wave energy transmitted is done by first applying and combining the (voice, audio, image) low frequency input information signal's varying electric voltage (ie., the signals amplitude) to the main transmitted radio frequency. Specifically, the amplitude (ie., AM) of the input information signal is added to the radio wave signal, and this step then changes the amplitude (ie., amount) of the radio wave signal. The main or fundamental transmitted radio-signal and frequency is sometimes called the carrier (of the information) frequency or wave. In a radio receiver, the audio or information signal is extracted or "demodulated" out from the received radio wave, and the carrier radio wave is discarded. **FM is Frequency Modulation** of the transmitted radio-wave by applying and combining the information signal's frequency to it so as to slightly alter the frequency of the main carrier wave, and then transmitting it. Since (electronic, RF, radio-wave energy, signals) noise can alter the amplitude of a received AM signal, FM radio is often used so as to have much more clear communication since amplitude noise is not much of a concern. Generally, FM radio communication is usually assigned or permitted to have less distance for communication than AM radio communication, and each FM radio station or "channel" requires a much wider (ie., more) "bandwidth" or frequency range than AM does. FM communication is generally more complicated in terms of circuitry than AM is. **Continuous Wave (CW)** communication is somewhat like AM radio communication, but CW uses only brief pules of an input audio "tone" signal such as a 1000hz sine wave for the information, and this is how (binary, two states for each bit or particle of information) Morse Code signals are transmitted, and for very long distance, and requires a relatively low voltage power battery supply and-or current. CW requires only a very narrow frequency bandwidth, say about +,- 2000 hz, near the assigned **carrier frequency** (ie., the main RF transmission frequency), and thereby reducing station crowding and signal interference from other stations. It also allows the input energy to be concentrated near the main transmission frequency of which both the transmitter and receiver are tuned to, hence the pulses of input energy are not wasted as unwanted transmitted frequencies and energy such as in a wider (frequency) bandwidth (or range) radio signal.

An LC (inductor and capacitor) tuner (technically a type of selective frequency filter) has a natural resonant or center frequency as mentioned in this book, and a natural bandwidth. Bandwidth is a frequency range. Max. Bandwidth = (upper frequency passed by the turner) - (lower frequency passed by the tuner) = Fu - Fl. Since there is an extensive small amplitude, frequency range of signals that exist near the upper and lower limits, usually the (best, usable, practical) "bandpass" range is defined as where the frequency amplitudes are reduced to 70.7% of the maximum value at the resonant frequency (Fr), and these frequencies are called the upper and lower (practical , signal) cut-off frequencies.

Quality factor of a tuner = Qf = R sqrt (Cf / Lh) = (Fr / Bandwidth). Since I is proportional to V, this limited bandwidth range represents about at 50% of the input power value of the signal: From: Pw = VI , Here: P = (0.707V)(0.707A) = 0.5 VA = 50% (VI) = 50% Pw

Considering the speed of light and other electromagnetic energy waves moving at 186000 miles per second = 299792 km/s =~ 300000 km/s and which is the speed of light through a vacuum such as in outer space:

Distances	**Approximate time needed**	
Sun to Earth = 93 million miles	500s = 8.333 min	: 93 million miles = 1 AU of distance , 1 AU / 8.333 min
Earth to Moon	~ 1.3s	: 240,000 miles away
1 Million miles	~ 5.38s	: 1 million miles =~ 0.0107527 AU (=~ 1/100 AU = 0.01 AU)
1 Billion miles	~ 1.5h	: 1 billion miles =~ 10.753 AU. (=~ 10 AU) Saturn is almost a billion miles away.
1 mile	5.3763441 micro-seconds	
1 km	3.335641 micro-seconds	
1 m	3.335646 nano-seconds	
1 foot	1.01825 nano-seconds	

In 1 second, light will travel about 186000 miles = 0.002AU per second , and:
In 1 minute, light will travel about: (186000miles/s)(60s) = 11.16 million miles = 0.12 AU per minute

In 1 hour, light will travel about 670 million miles = 7.2AU per hour.
In 24 hours = 1 day, light will travel about 16.08 billion miles =~ 16 billion miles = 172.8 AU =~ 173 AU per day
In 1 year =1 light-year =~ 365 days, light will travel about: 5.870 trillion miles= (365.25 days)(173 AU per day)=~ 63188 AU

TEMPERATURE:

The temperature (ie., heat, thermal energy) of a substance (matter, mass) is a measure of its atoms or molecules average movement (vibration, motion, moving, or "kinetic") and its corresponding energy level. When this type of motion does not increase in value, it is considered that the substance or mass did not absorb or acquire any more external energy. When a mass with a higher temperature is joined to one with a lower temperature, energy, via the atomic movements or vibrations, the energy of the mass with the higher temperature will be transferred to the mass with a lower temperature until the temperature of the two masses are the same value. The mass with the higher temperature will loose some energy, and the mass with the lower temperature will gain that same amount of energy.

When all atomic or molecular vibration or motion stops, this temperature is defined as "absolute zero" and is the reference of measurement for the Kelvin (K), or "absolute" [scientifically true or natural] temperature scale, and each degree of temperature increase in this scale is identical to that of the Celsius scale. Water freezes at about 273K = 0°C = 32°F, and boils at about 373K = 100°C = 212°F. Kelvin values are always positive in value. Though water freezes at 32 °F, it still has some trace amount of (atomic, and-or environmental) thermal energy within it, and can get as cold as absolute 0 in theory.

If an object or mass is now twice its initial starting temperature of reference value, it is thought to have gained the same value of its starting atomic or molecular thermal energy. The (real, solid) matter or mass (technically, the measure of matter) is storing that energy.

"Absolute Zero" where all thermal energy is depleted and atomic vibration or motion stops or "freezes" is defined as:
0°K = -459.67°F = -273.15°C. The Kelvin scale is credited to William Thomson who was also the 1st Baron of Kelvin (a river and area) of Britain, and he is more commonly known as "Lord Kelvin". He conceived this absolute or true temperature scale in about 1848 and is actually based on and-or an extension of the Celsius temperature scale.

°F = (9/5) °C + 32 : Degrees Fahrenheit , and mathematically:
°C = (5/9) (°F - 32) : Degrees Celsius
°K = °C + 273.15 = ((°F - 32)) / 1.8) + 273.15 :Degrees Kelvin
(9/5) = 1.8 =~ roughly 2 and (5/9) = (1/1.8) = 0.555.... =~ roughly 0.5 = (1/2) : for quick estimations

Fahrenheit temperature = F° = 1.8C° + 32 : a derivation of this formula is shown in this book , 1.8 = 9/5
Celsius temperature = C° = (F° - 32) / 1.8 = 0.555... (F ° - 32°) = K° - 273.15 , 0.555 ... = 5 / 9 = 1/ 1.8
Kelvin temperature = °C + 273.15 = ((°F - 32) / 1.8) + 273.15 : 273.15 =~ 273 =~ 270
0 °C = 32 °F = 273.15 °K : **water freezes to ice at here or if less, or ice melts to water if greater** , and
100 °C = 212 °F = 373.15 °K : **water boils in a pressure of 1 atm (14.7 psi) if at here or if more. Higher pressure essentially means harder to boil, and will take more energy for the steam or water vapor to be released and "gas off" or be released from the water.**

Some selected temperatures and their converted values:

-28.88°C = -20°F
-23.3°C = -10° F : 10 below zero or= 10 less than a 0°F reference temperature
-17.8°C = 0° F : 0°F was defined as the freezing point (value) temp. of a special "brine" fluid.
This method allows the Fahrenheit scale to have values below freezing at 32°F
water freezes: 0°C = 32°F : Water freezing point at standard atmospheric pressure of 14.7psi , 273.15 °K
Flowing water has some kinetic and thermal energy and wont freeze in air 32°F
Water in higher pressure will be more difficult to crystallize correctly to ice

"a chilly day": 10°C = 50°F : a slightly warm day can be perhaps from ~ 45°F to ~71°F
"room temperature": 22.2°C = 72°F : this is a human subjective, average comfortable living or "room" temperature
"a hot day" : 30°C = 86°F : The air temperature measured in the shade. ~87° or greater.
A high humidity level can often make the air temperature feel much warmer,
especially when the temperature is above 72°F.
water boils: 100 C° = 212°F = 373.15 °K

For temperatures above 200°F, the equivalent Celsius scale temperature is approximately half. Likewise, for temperatures above 100°C, the equivalent Fahrenheit temperature is approximately double.:

100°F = 37.8°C =~ 38 °C
125°F = 51.7°C
150°F = 65.5°C
175°F = 79.4°C
200°F = 93.3°C
225°F = 107°C
250°F = 121°C
275°F = 135°C
300°F = 149°C=about half
325°F = 163°C
350°F = 177°C
375°F = 190.5°C
400°F = 204°C
425°F = 218°C
450°F = 232°C
475°F = 246°C
500°F = 260°C
525°F = 274°C
550°F = 288°C
575°F = 302°C
600°F = 315.5°C
1000°F = 538°C
2000°F = 1093°C
3000°F = 1649°C
: **rounded, approximate values**
4000°F = 2204°C
5000°F = 2760°C
6000°F = 3316°C
7000°F = 3871°C
8000°F = 4427°C
9000°F = 4982°C
10000°F = 5538°C

As a quick "rule of thumb", notice above that at about 175°F and greater, the numeric part of the equivalent C° value is about half. Likewise, for temperatures greater than 200°F, or about 80°C, that the equivalent numeric part of the F° value is about double the Celsius equivalent. The Celsius equivalent is about half that of the Fahrenheit numeric value. It still needs to be stressed that a even few degrees Celsius is equivalent to many degrees of Fahrenheit. For values between 100°F and 200°F, the equivalent numeric part of the corresponding Celsius value is about 40% of the Fahrenheit numeric value. For values between 50°F and 100°F, the corresponding Celsius value is about 30% of the Fahrenheit numeric value.

Materials made of elements that are more dense can both absorb and-or conduct thermal (heat) energy better than that of less dense materials. A layer of metal can conduct heat well, whereas a layer of cardboard, (steady, non-flowing or non-moving) air, plastic or foam does not absorb and-or conduct heat well. The atoms of metals have many electrons and the atoms are also more "packed" into say a cubic centimeter of volume, and therefore making it a denser material than say air or water. This density of atoms increases the probability and-or efficiency of (mechanical, vibration) heat transfer from one atom to another. Air is not as dense as a metal and therefore it does not conduct heat very well as compared to that of a metal, and air is considered as both an electric and heat insulator. Denser materials with more atoms and electrons can store more thermal energy.
Although (non-metallic, non-molten) fluids, such as water, are not as dense as metals, they can still absorb and easily carry (transfer) heat energy from a source to a destination at relatively large distances away and speed of travel. An example would be a hot water heater which heats water to a higher temperature which then gets sent or directed through a pipe so as to be available at a distant location where it is useful such as for a washing and-or showering.

Given two different elements of say each being 1 cubic centimeter in volume, the denser element will have more mass (amount of matter, atoms) in that same volume or space. If those two materials are subjected to the same heat source, it will take the denser element longer to heat up to the temperature of that heat source because the denser element can store more thermal energy. Because the denser or "more mass" element can store a greater amount thermal energy, it will retain that thermal energy longer when sitting in the air, and will take longer to cool-down or reduce in temperature to the surrounding air temperature.

For reference, and as mentioned in this book, if one pound (weight, = 454g = 0.454kg) of water is heated and it increases by 1 degree of Fahrenheit temperature, it is said to have gained **1 BTU** (British Thermal [measurement] Unit) of (thermal) energy which is equivalent to about **1056 joules of energy**, roughly 1000J = 1kJ of energy. This value does not included any inefficiency or energy losses (ie., wasted energy) heating that material, here water, but rather is the thermal energy needed and-or gained by the water.

1 Newton (N) is the (constantly applied) force needed to accelerate 1 kilogram of mass by 1 meter per second, per second. 1N = (mass)(acceleration) = (1kg)(1 m/s^2). 1N =101g of force (ie., weight) in Earths gravity = 101**gf** =~ 100 gf = 100 **g**ram-**f**orce. 1kg-m = 1N of force = 0.225 lbs of force , mathematically: 1lb = $4.44\underline{4}$ N , 10N =~ 1kg of force in (g)

1**Joule** of energy and-or work is expended applying a constant force of 1 newton through 1 meter of distance.
1J = 1 (force)(distance) = Fd = 1 Nm = (ma)(d) = 1 (Newton)(meter) = (1kg) (m/s^2) (m) = 1 kg (m^2/s^2) =
1J = 1 Nm = 1 kg (m/s)^2

The joule (J) is a unit of energy, and its named in honor of **James Joule** (1818-1889), from England, and who studied heat energy, such as that produced and-or wasted in a resistance when an electric current (ie., electricity, electron flow) is passing through it. Since heat is a form of energy, heat and-or electrical energy has an equivalent mechanical (Ie., "dynamic", and due to physical forces of matter with kinetic energy) energy such as (force)(distance) = work , and which can produce the same amount of heat. Essentially, any form of energy, and which can be converted to one form or another, including work, can and should have the same units such as **joules (J)**.

Power (P) is the (average) rate of using energy during a time interval, and has units of Watts. Power is therefore not the total amount of energy transferred and-or used during a length of time. Power is how much (average) of total amount of energy was needed, transferred, and-or used during a period or amount of time.

Power = (total energy / time) = (joules / time) watts. : rate of using energy or= doing work **,** Also: 1W = (1V)(1A).

1 Joule (J) of energy used for 1 second of time can be defined as 1 watt of energy:

$1W = \frac{1J}{1s}$, mathematically: **1J = 1W-s = 1 watt-second**

If 1 watt of power is applied or used for 1 hour = 3600 seconds of time:

Total energy used = (power)(time) : derived from the power equation above

(1W)(1hour) = 1 w-h = $\left(\frac{1J}{s}\right)$ (3600 s) = 3600J of energy : 3600Joules = 1W-h = 1 Watt-hour

Note the difference of the meaning of the units mentioned above:

watts is the energy used per an amount of time, hence watts is a rate of using energy , and
watt-hours is the total amount of energy used : Wh = (watt-hours) = (watts)(hours) = (J/s)(3600s) = J / 3600

If 1 watt is applied , used or converted for 1 hour, this can be expressed as: 1 watt-hours = 1w-h = 3.412 **BTU**/hour = (1056 joules/BTU) (3.412 BTU) = 3603 joules/hour ~= 3600J/hour. Consider that if 1watt-second = 1w-s = 1J/s, and if the time length or duration of using this energy is 1 hour = 3600 seconds, then the amount of energy would be 3600 w-s= 3600 J/hour = 3.412 BTU/hour. Note that 3600 w-s is not equal to 3600 watts per second, but is rather the total amount of watts used during the 3600 seconds = 1 hour of time, hence a total of 3600 watts (total) per hour. If 3.142 BTU/hour = 1 watt-hours, then after multiplying both sides of this equation by 3600, we have: (3600)(1) watts-hours = (3600)(3.142 BTU/h) = 12283 BTU/hour. Also from the above information, 1BTU/hour = 0.293 watt-hours =~ 0.3 Wh.

For electricity: 1J = 1 w-s = for example =: (1V)(1A) (1s) or for example =: (10V)(0.1A)(1s) :Joules = watt-seconds

A **calorie** (thermal) energy unit is defined as the thermal energy gained within a gram = 1g of water when its temperature is increased by 1 degree Celsius. **1 calorie = 4.185J (joules) of energy**, and it is roughly equal to 4.2J For a given temperature, it is not too difficult to conceive that when the mass of an object is (n) times more than that of a lesser mass object, then it can hold that many (ie., n) times more total thermal (heat) energy. This book discusses calories, specifically for food, and which is sometimes and more properly called as "food calories" or "dietary calories" in more detail later in this book. If you heat, say about 30 grams of water in an insulated container with a thermometer in it, and by using an electric resistor (here, effectively wastes and-or converts electric energy to thermal or heat energy) submerged in it, it will take an amount of seconds. The average amount of energy to raise 1 gram of water by 1 degree can then be calculated as: Total Energy / grams = (V)(A)(s) Joules / (grams of water) , and the result should be about 4.185J / 1 gram = 1 calorie or= 1 scientific calorie. 1 food calorie is equal to 1000 scientific calories.

Freezing Sea Water To Remove The Salt, and other related topics:

Because of the salt (sodium chloride molecule, table salt) in ocean or sea water, salt water is more difficult to freeze, that is, it takes a slightly lower temperature for it to freeze at about 28.4°F =~ -2°C. Pure water takes 32°F = 0°C to freeze. This difference in the freezing point is about 3.6°F or 2°C less. This is due to the interference of the salt molecule with the water (H20 molecule) ice crystal being formed. A pure **crystal** of a material has only atoms or molecules of the same type and they are physically arranged in a consistent, repeating order or array. When salt water freezes, the ice will not have the salt in it, and this is one method, called freeze desalination,so as to **desalinate** salt water. It is generally incorrect to think that freezing water is the best way to remove impurities and-or make it safe to drink. If very cold ocean water can be placed in a pond in a cold climate, it will be able to freeze naturally and more rapidly since there is (hopefully) less wind and no ocean wave movement giving the water some kinetic energy and preventing it from freezing faster into a nice crystal form. Sea or ocean water is about 3.5% salt by weight, hence 1kg = 1000g of salt water will have about 3.5g of sodium chloride which is table salt. Ice (ie., frozen water) can be melted to a liquid water form by using various solar heating methods such as a warm greenhouse.

Before modern electric refrigerators were available, various methods that used ice, cool water, and-or cool earth were used to refrigerate items. When water freezes it will expand slightly, and this can put much force and-or pressure (force per unit area) upon a container or region, and may cause it to crack, and this is why it is good to fill cracks on building structures, roads and sidewalks with a water resistance material, such as tar (a high percent carbon mixture), before freezing temperatures and ice occur in the winter. Running or moving water can also enter cracks and cause sever land or dirt erosion and which is more difficult to repair.

If the salt water is very cold, it will only take a few degrees less to make it freeze. During heat desalination, it takes much energy to heat water to its evaporation temperature where the phase of water changes from liquid to gas. Direct solar thermal (heat) energy can be used to help pre-heat large amounts of salt water. This type of (pure) water is called "evaporated water" and it is sometimes available in some stores, and it is used for cleaning, car batteries, and wounds, but it is generally recommended to not drink it as is without mixing in some trace amounts of certain minerals to replace those not in it.

A somewhat related topic to desalination of water is the disinfection or sterilization of water, so as to prevent illness and disease. Using various screens and-or cloth, visible debris should be removed from water, however, this does not mean small germs have been removed. Heating filtered water hot for several minutes will help disinfect (ie., neutralize germs) water. Most sources have also mentioned about **5 drops of household, plain, fragrant free bleach per gallon of reasonably clear, filtered of visible particles source water**. Since a drop is standardized at 0.050mL = 0.00005L of volume, we have: (5 drops)(0.050mL/drop) = 0.25 mL = 0.00025L , and since household bleach has a density of about that of water, this equates to about 0.25g = a quarter of a gram-weight. For 10 gallons, this would equate to 2.5g For 100 gallons, this would equation to 25g (ie., about an ounce of bleach). Before drinking any portion of this water, allow some time (several hours if possible) to let the chlorine disinfect germs, and to then "gas off" (evaporate) and-or heat it to speed up this process. **A simple or quick rule is to use 1 teaspoon of household bleach per 5 gallons of water**.

Extra: Note that :

1 ounce of weight = 1 dry ounce = 1 oz = 28.3495 grams , 1 oz = 1 pound / 16 = (1/16) lb = 0.0625 lb
1 Imperial or British fluid ounce of water = 1 oz of weight
1 US fluid ounce of water = 29.5735296 grams = 1.043176441 oz of weight =~ 29.574 cc water = 29.574 mL of water =
=~ 30 grams of water = 30mL of water = 30 cc of water =~ 1 ounce of weight . 1 US fluid "food ounce" = 30 cc

DISTANCE OR LENGTH

We commonly use units of meters, centimeters, feet and inches for distance. An object moving can travel a distance and the formula for this distance is: distance = (speed) (time) = (velocity) (time) = v t. Speed or velocity is how fast something is moving. Mathematically: v = d / t , hence it is the rate of change in distance traveled or moved, and divided by the change in time (t = d / v) it took to move that distance. The units of measurement for velocity or speed is a combination of units, and is the distance units used divided by the time units used. For example: v = 10 m / 1 s.

There are various instruments to measure distance. For example: rulers, both English and Metric gauged linear scales, rope and wheeled rotation counter gauges for long distances, and English and Metric micrometer gauges for small distances and with a high numeric precision and-or fractions of a unit of measurement.

1 in. = 1" = 1 **inch** = (1/12) of 1 ft = 0.083333.. ft = **2.54 cm (exact) = 25.4mm** = 0.0254m :slightly greater than 2.5cm

The key to converting between metric and English distance units is to know this equivalent length value: 1 in = 2.54cm

1 in = 3 (average sized, from the center area of the barley seeds) barleycorn (ie., barley seed) lengths ,
2 in = 5.08cm =~ 5 cm = 50mm , 1 in / 10 = (1/10) in = 0.1 in = 2.54 cm/10 = 0.254cm = 2.54 mm.
1 in = 2.54 cm = 25.4 mm . 1 in / 16 = (1/16) in = 0.0625 in = 2.54 cm / 16 = 0.15875 cm = 1.5875 mm =~ 1.5 mm
1 mil = (1/1000)in = 0.001 inch = a thousandth of an inch = 2.54 cm/1000 = 0.00254 cm = 0.0254 mm = 25.4 um ,
1in = 1000mils , 1000 mil = 1 in = 2.54cm = 25.4 mm = 25400 um

1 meter = 1m = 100 cm = 1000 mm = **3.28084 feet** , =~ 3.281 feet =~ 3.3 ft , = 1.09361 yards =~ 1.1 yd
1m = 100cm = 1000 mm = 1,000,000 um = 1,000,000,000 nm = 1,000,000,000,000 pm
1m = 39.37 in = 3 feet + 3.37 inches = 3 feet + 3 inches + 5.92128 sixteenths of an inch =~ 3ft + 3in + 6 sixteenths of an inch =~ 1 yd + 3in. 1m = 100cm , 1in = 2.54cm , 100cm / (2.54cm/in) =~ 39.37 in =~ 3.281 ft

1 cm = since **1 in = 2.54 cm** , mathematically: 1 cm = 1 in / 2.54 = 0.393700787 in =~ **0.3937 in**
1 cm = **1 centimeter** = (1/100)m = 0.01m = 10mm = **0.3937 inches** :**~ 0.4 = 40% of an inch** , **10cm =~ 4in =~ 100mm**
1cm =~ (3.1496 / 8) in =~ (6.3 / 16) in , (3/8) in = 0.375 in which is a good approximation if you have an English ruler.
1cm = 1m/100 = (1/100)m = a hundredth of a meter = 0.01m = 0.010m = 10mm
1cm =~ 0.393701 inches, (in) =~ 0.4 in or 40% of an inch , ~ (6.2992 / 16) in =~ (6.3 / 16) in ~= 3 eighths of an inch
10 cm = 3.93700787 in =~ very close to 3.937 in = (3 in + (15/16) in) =~ 4 in

By repeatedly dividing an inch unit by two, or in half:

1.0 in = 4 quarters of an inch = 8 eighths of an in. = 16 sixteenths of an in. = 32 thirty-two-seconds of an in.
(1 in / 10) = 0.1 in = "one-tenth" of an inch , Ex. 3.1" = 3" + 0.1"
1.0 in = 1ft / 12 = 0.08333...ft = 2.54 cm exact = 0.0254 m

1/2 in = 1 half of an inch = (1/2) in = 1in/2 = 0.5in = (8/16)in = (2/4)in = (2.54cm/1 in)(0.5in) = 1.27 cm = 12.7 mm .
1/4 in = 1 quarter of an inch = 0.25in = (2.54cm/1 in)(0.25in) = 0.635cm =6.35mm = (1/2)in / 2 = (1in/2) / 2 = 1in / 4
= 1 fourth of an inch . 1/4 in = (1/2) in / 2 , hence = half, of a half of an inch
1/8 in = 1 eighth of an inch = 1in/8 = (1/8)" = 0.125in = (2.54cm / 1 in)(0.125in) = 0.3175cm = 3.175mm
(1/8)" = 0.125 in is slightly larger than (1/10)" = 0.100, (1/8)" = 0.125 in = 0.1 in + 0.02 in + 0.005 in
1/16 in = 1 sixteenth of an inch = 1in/16 = 0.0625 inches = 25.4mm / 16 = 1.5875 mm (about 1.5mm, or one and a one-half millimeters). Ex. 0.6875 in = (0.6875 in)(16 sixteenths / 1in) = 11 sixteenths. 1/16 = (1/2)(1/8)= (1/8)/ 2
1/32 in = 1 thirty-second of an inch = 1in/32 = 0.03125in = 25.4mm/32 = 0.79375mm =~ 0.8mm =
= slightly less than 1mm. (1/32) = (1/16) / 2
1/64 in = 1 sixty-fourth of an inch = 1in/64 = 0.015625 in = 0.396875mm = about 0.4mm =~ slightly less than half of 1mm. = (1/64) = (1/32) / 2 . 0.4 mm = 400 um

1/10 in = 0.1 in = (2.54cm / 1 in)(0.1in) = 0.254cm = (10mm/1cm)(0.254cm) = 2.54mm = 1"/8 - 1"/40 =~ 3/32 = 1.5/16
1/12 in = 0.0833 in = 0.2116cm =~ 2.1$\underline{2}$mm = 1.3333... / 16 =~ 5/64 =~ 2.5/32

3 in = 12in/4 = 1ft/4 = 0.25ft , 3 in = one quarter of 1 foot

4 in = 12in/3 = 1ft/3 = 0.333...ft = 10.16 cm = about 10 cm = 100mm , Note: 10cm length x 10cm width x 10cm height
= 1000cm^3 = a volume called a Liter (L). 10cm is the cube root of 1000 cm^3. 4 in = one third of a foot
6 in = 12in/2 = 1ft/2 = 0.5ft , 6 in = one half of a foot
8 in = (4in)(2) = (0.333...ft)(2) = 0.6ft

1 foot = 1 ft. , several of this unit are called feet and with the same abbreviation. 10 foot = 10 feet
1ft = 12in. , 12in (2.54 cm/in) = 30.48 cm =~ 30.5cm ~= 30 cm = 30.48cm x 10mm/cm = 304.8mm (roughly 300mm)
(1/10) ft = 0.1ft = 12 in / 10 = 1.2 in : "one-tenth" of a foot , Ex. 5.1" = 5" + 0.1" = five inches + a tenth of a foot
1ft = 0.3048 m = 30.48 cm = 304.8 mm =~ 305 mm
10 centimeters = 3.93701 inches , = ~ 4 inches
12 in =1 foot = 1ft = 0.3048 meters (m) , : average length of an adult male human foot, but later: 1/5280 of a mile (*).
1ft = 0.3048m , 5ft = 1.524m =~ 1.5m , 10ft = 3.048m =~ 3m , 100ft = 30.48m =~30m , 1000ft = 304.8m =~ 300m

1 mile = 1mi = 5280ft = 1609.34 meters = 1.60934 km : (*) a Roman mile was originally and arbitrarily just 5000ft which was the length of 1000 (ie., millennium or mil, for mile) complete cycles of average sized both (2) left and right steps or "paces" of an adult man out walking a distance for traveling. 1 stride = 2 paces. 1 step or pace = about 2.5 feet. At an average of 5 feet per stride, the length of 1000 strides is 5000 feet. The English mile is based on the Roman mile. The English or British mile is the multiple of 660 and 8 = 5280. 660ft is the length of 1 furlong commonly used for farming and property such as for an acre (an area of land). 1 furlong = (1/8) mile.
1 mile = 5280 ft (3 ft / 1 yd) = 1760 yards.

A speed or velocity of: 1mi/1hr = 5280ft / hr = 5280ft / 60min = 88ft / min = 88ft/60s = 1.46$\underline{7}$ft/s =~ 1.5 feet / s.
Or: 5280ft / 1hr = 5280ft / 3600s = 1.46$\underline{7}$ ft / 1s =~ 1.5ft / 1s = 18in / 1s

An average walking speed of an adult is about 2mi/hr =~ 3ft/s. A casual bicycle speed is about 6mi/hr on flat ground.
0.625 mi =~ 1 km

1 mile = 1.609 km =~ 1.6 km = 1609 meters = roughly one and a half kilometers = 1.5km
5 miles = 8.04672 km =~ 8 km
10 miles = 16.0934 km =~ 16 km : 10mi/hr = 14.66 ft/s =~ 15 ft/s
15 mi = ~ 24 km , 20 mi =~ 32 km , 25 mi =~ 40km , 30 mi =~ 48 km , 35 km =~ 56 km , 40 mi =~ 64 km
50 miles = 80.4671 km =~ 80 km , 60 miles = 96.561km =~ 100km
62.135mi = 100 km
100 miles = 160.934 km =~ 160 km , but closer to ~ 161 km

1 nautical mile = arc length distance of 1 arcminute on Earth = 1min of arc ¢ from N. pole to S. pole. = 1.852km = about
= 6076ft = 1.15078 miles. Nau is an old word prefix for things pertaining to boats and navigation. Each degree of angle contains 60 minutes of angle, and each minute of angle contains 60 seconds of angle:
360° = 360 arcdegrees = 360 (60) = 21600 arcminutes = 21600 (60) = 1296000 arcseconds
From the center point of the Earth: A 360° arc = an arc length of 1 circumference of the earth =~ 25000 miles
1° or 1 arcdegree on the surface of Earth corresponds to: ~ 25000 miles / 360° =~ 69.44 mi = ~ 70 mi
1 arcdegree = 360°/360 = 1° = 1 degree , 1 arcdegree = 60 **a**rc**m**inutes = (60 **am**)(60 **as**/am) = 3600 **a**rc**s**econds
From horizon to horizon = 360° / 2 = 180° = 21600 arcminutes / 2 = 10800 arcminutes = 648000 arcseconds
1 arcmin = 1arcdegree / 60 = 1°/60 = 0.016666...° = 0.01666$\underline{7}$ arcdegrees , 1 arcmin = 60 arcseconds
The Moon appears to us on Earth as being 0.5° wide = 0.5° / (0.01667°/1 arcmin) = 30 arcmin = 1800 arcseconds
The Moon's diameter is about 2159 miles , 1800 arcseconds / 2159 miles =~ 0.83$\underline{4}$ arcseconds / mile
Or: 0.5° / 2159 mi. = 0.000231$\underline{6}$° / mile
1 arcsecond = 1arcmin / 60 = 0.0166$\underline{7}$° / 60 = 0.00027777...° =~ 0.0002$\underline{8}$° arcdegrees

For what each degree on the Earth's circumference length corresponds to, we can set up an equivalent ratio or proportion equation: 360°/ 25kmi as = 1° / x mi , x mi = 69.44mi , (1°/60) = 0.01667° = 1 arcminute angle = 1''
1° is to 69.44 mi as = 0.01667° is to x mi., **1 arcmin is about 1.15 miles = (1/60) arcdegree**
Or: (25000mi / 360°) =69.4444 mi / degree = 69.4444mi / 60 arcminutes =~ 1.15741miles / arcminute

The arclength of 1 arcdegree on Earth is about: 25000mi / 360° = 69.444 miles
The arclength of 1 arcminute on Earth is about: (25000mi/360°)/60 = 1arcdegree/60 = (69.444 miles)/60 = 1.15741mi
The arclength of 1 arcsecond on Earth is about: 1.15741mi/60 =~ 0.01929mi = 101.852ft =~ 100ft =~ 30meters

DMS = **D.M.S** = An angle measuring system = arcdegrees . arcminutes . arcseconds = D° + M' + S" = arcangle or "angle of arc". This system turns the fractional parts of a degree into units - much like how a fractional part of an inch is called a sixteen or an eight of an inch, and so on.

1 knot = 1 naut = 1 nautical mile per hour = 1.852km / 1 hour = about 1.151 miles / 1 hour , hence a speed or rate
1 furlong = 1/8 mile = 0.125 of a mile = 660 feet = 10 chains. 1 furlong was also used to define the side length of 1 acre of area. A stadion or stadia is 1/8 of a Roman mile, hence 5000ft/8 = 625ft. A furlong and stadion (or stadium) are nearly identical. Furlong ("furrow length") is based on the word furlang which is the length of the furrow (trough) or planting length of a 10 acre field. A furlong is the length cattle could typically plough (plow) before needing rest.
1 rod = (1/40) furlong = 16.5ft . 1 chain = (1/10) furlong = 66 ft = 4rod

1 foot = 1 ft = 30.48 centimeters = 0.3048 meters = ~ 30 centimeters = ~ 0.3 meters
In relation to ancient Greece units: 1 pous =~ 12.13 in =~ 1 foot
1 yard = 1yd = 3ft = 36 inches = 0.9144 meters = 91.44cm , slightly less than a meter, about 90% of a meter = 90cm
1 pace = 30.0 inches = 2.5 feet = 2 ft + 6 in. The pace is now an outdated unit, and is mentioned in older texts.
A pace is the average distance from the heel of one foot to the heel of the other foot when "distance walking". Surely this depends on the physical condition of the person. The value shown here is for a single step, and a full step is simply twice this value. A pace is a way to measure distances, such as on trails, and without standard measuring devices - perhaps a string of known length. A pace is sometimes used described how fast a person or thing is moving, such as 100 paces per minute = distance / time = velocity. Using this method, the walker would not have to count their "steady or constantly paced" steps, and the distance traveled can be found by the time taken, and is mathematically: distance = (velocity)(time) = (known or typical steps / minute) (time in minutes).
1 stride = 2 full steps or 2 paces. The word "pace" is sometimes used as a word for "speed" and-or "ability".

1m = 1 meter = was defined as 1/10,000,000 = 1(10^-6) the distance from the Equator to the North-Pole, hence this assumes that this distance is 10,000,000 meters or 10,000 km =~ 6214 miles, and multiplying this value by 4 so as to have the total circumference of the Earth, we have Ce = 40000km = 24855 miles, and this is very close to the modern accepted value of about 24900 miles. From these values, the "other side, or halfway around the Earth" is 2(6214 miles) = 24855/2 = 12428 miles. Here the meter is defined by what we can see in nature, and is much like defining a length based on the average seed length of a common plant. To make the meter practical it can be placed in terms of other known units. If 10000 km = 6214 miles, then after dividing both sides of this equivalence by 10000, we have 1km = 0.6214miles =~ 3281 feet , and after dividing both sides by 1000, we have 1m = 3.281 ft.

1m = 3.28084 ft =~ 3.281ft = 100cm , dividing both sides by 100, we have: 1 cm = 0.03281ft
(0.03281ft)(12in/ft) = 0.39372in = 1cm , solving for 1in by dividing both sides by 0.39372,
we have ~=: 1in =~ 2.54 cm , 1in = (1ft / 12) = 0.08333333ft
1m = 39.375 in = 1.09375yd
1m = 3.281ft = 3ft + 3.372in = 3ft + 3in + 5.952 / 16 =~ 3 ft + 3in + 6(1/16) =~ 3ft + 3in + 1in/4 + 1in/8
=~ 3ft + 3in + 1in/4 + 2(1/16)
10m = 32.81ft
100m = 328.1ft =~ 330ft ,
1000m = 1km = 3281ft = 0.621402 mi

25m = 82.025ft =~ 82ft
50m = 164.05ft =~ 164ft
1m = (1/1000)km = 0.001km
1m = 100cm = 39.37 inches = 39in + 1(1"/4)in + 1.92 (1"/8) =~ 39.5in = ~40in = ~= 4 (10in) = 1.093yards
1m = 100cm = 1000mm = 1,000,000um : 1cm = 10mm and 1mm = 1000um , Extra: 1cm = 10000um

1 km = 1000meters = 0.621373 miles =~ 0.6 mi
1 km = nearly 5 furlongs ~= 5 (660ft) = 3300 ft . 5 furlongs = 1.00584km exactly defined
1 km = 3280.84 ft =~ 3281ft . 3280.84 ft / (5280 ft/mi) =~ 0.6214m =~ 62% of a mile , 1mi = 5280ft
1.609km =~ 1.61km =~ 1 mi
5 km = 3.107mi =~ 3 mi ,
8 km =~ 5 mi
10 km = 6.2137mi =~ 6 mi , 15km=~9km , 20 km =~12 mi , 25 km =~ 15 mi , 30 km = ~ 18 mi , 40 km =~ 24 mi
50 km = 31.069mi =~ 31 mi = roughly about 30 miles
100 km = 62.137mi =~ 62 mi = roughly about 60 miles

1mm = (1/1000) m = 1(10^-3)m = 0.001m = "one-thousandth" of a meter = "one millimeter"
A poppy seed is about 1mm in length and weighs about 0.0003g, hence there are about 3333 poppy seeds per gram.
1mm = (1/10) cm = 0.1cm = "one-tenth" of a centimeter =~ 0.03937in = ~ 0.04in =~ 0.63 sixteenths of an inch
Ex. 2.5 cm = 2cm + five-tenths of a (1) cm = 2cm + 5mm : (since a millimeter is a (one) tenth of a centimeter)
Ex. 1 in = 25.4mm = about 25mm , 2in = 50.8mm = about 50mm , 4in = 101.6mm = about 100mm
1um = 1 micrometer = (1/1,000,000)m = 1(10^-6)m = "1 micron" = "one millionth of a meter" = 0.001mm
1um = (1/1000) mm = (1mm / 1000) = 0.001mm = "one-thousandth of a millimeter" , 1 mm = 1000 um
1nm = 1 nano-meter = (1/1,000,000,000)m = 1(10^-9)m = "one-billionth of a meter" = "1 thousandths of 1 mico-meter"

1A = 1 angstrom = 1(10^-10)m = (1/10)nm = 0.1 nanometer = "one-tenth of a nanometer"

1 barleycorn (average barley seed length, from the center of the seed pod) = (1/3) inch = 0.3333 in
Barley is one of the ("cereal") grass plants (barley, wheat, oats, corn, rice) that produce edible seeds (or "corns") called grains. **Three average size barleycorn lengths from the center of the seed pod were used to defined an inch unit**. Surely, once this unit was defined, here naturally, wood and metal rulers of 1 inch were then rather used. Due to this natural definition, the length of the foot unit is based on the length of 1 inch, and 1 foot is defined as 12 inches, and then mathematically, 1 inch can then be "redefined" as (1/12) of a foot = 0.83333... ft.

The English shoe scale is based on the barleycorn = (1/3) in. For the length of the shoe, size 12 = 12 inches long, and each size lower is 1 barleycorn less in length, hence (1/3) of an inch less. Size 11= (12in - 0.33in) = 11.66 in. For some extra comfort, many will wear up to a size bigger (ie., longer) so as to have some extra toe ("wiggle room") space, and-or to consider a bigger size that may be usable for wide feet when wide-width shoes are unavailable.

1 cubit = 20.625" = 0.5236m (about 20" or half a meter) A cubit is an ancient distance or length measurement unit that was naturally defined as the length of an average man's arm from the elbow to the tip of the middle finger. Some cubit measurements vary from 18" (Bible) to 22", and 20in is an average = 50.8cm. **20.6 in = 1.71667 ft = 1 ft + 8.6 in = 52.324cm is the Egyptian Cubit; typically used**. This value is approximately half a meter = 0.5m. A cubit is an ancient measuring unit for length, and mentioned in the Torah, Old Testament book which is about 4000 years old at the current time of writing of this book. The Egyptian cubit was further divided into 28 (finger ("digit") sized widths) markings or steps. The cubit was often divided into steps of basic fractions (1/2, 1/3) of a cubit. A "palm" or "palm width" was "4 fingers wide". A cubit has 7 of these palm units, hence a palm unit is equal to a 1/7 fraction of a cubit unit. **As an example of expressing the difference and percentage between a measurement and a reference value:** If 20in is used as the cubit, then it is slightly less and in error of the 20.6in Egyptian Cubit by only: error = (value - reference value) = (20in - 20.6in) = -0.6in, and this error value as a percentage of the reference value is: (error value / reference value) = (-0.6in / 20.6 in) = -0.02913 = -2.913% =~ -3%. As a ratio of the two given measurements: (value / reference value) = (20 in / 20.6 in) = 0,97087 =~ 97%. That is, the average value is about 97% of the Egyptian reference value, and about: (1 - 0.97) = 0.03 = 100% - 97% = 3% percent less. Though this is a small percentage of error, it will become significant if may cubits are involved.

Mils. A mil is 1/1000 of an inch. A mil is sometimes used by a **machinist** when working with metal so as to be more precise than say a value with units of 32nds or 64ths of an inch. Since metric units are now common, the mil unit is not used much. The average human hair is 1 mil thick or wide. Mathematically, **1 in = 1000 mills**

1 mil = 1in / 1000 = 0.001 inch = a thousandth of an inch : note, 1" = 1in = 1 inch = 2.54cm = 25.4mm
1 mil = 0.001 in = 1 in / 1000 = 2.54 cm / 1000 = 0.00254 cm = 0.0254 mm = 25.4 um : um = micro-meters

1 mm = (1m / 1000) = 0.001m = 1000 um : mm = milli-meter
1 mm = (1cm / 10) = 0.1cm = "one-tenth of a centimeter" , 1 cm = 10 mm
1 mm = 0.03937000787 in =~ 0.03937 in =~ 0.04 in =~ 1.5 thirty-two-seconds of an inch = 1.5 (1 in/ 32) =
1.5 (0.03125 in) = 0.046875 in =~ 3 sixty-fourths of an inch = 3 (1 in / 64) = 3 (0.015625 in) = 0.046875 in
1 mm = 39.37007874 mils = about 39.37 mils =~ roughly 40 mils : derived from the above equivalence for mils
1 cm = 10mm = 10 (1mm) = 10 (39.37 mills) = 393.7 mils =~ 394 mils =~ roughly 400 mils
1 cm = 0.3937000787 in =~ 0.3937 in =~ 0.4 in , Note also that 1 cm =~ (3.15 / 8) in = 0.39375 and (3/8) in = 0.375 in
1 m = 100 cm = 100 (1 cm) = 100 (393.7 mils) =~ 39370 mils =~ roughly 40000 mils

1 inch = 1000 mils = 2.54cm (exact) = 25.4mm = 25400um : from this , the fractions of an inch can be found:
1/2 inch = (1/2)" = 0.5 in = 1.27cm = (1000mils / 2) = 500 mils = 12.7 mm
1/4 inch = (1/4)" = 0.25 in = 0.635cm = 250 mils : 1000mils / 4 = 6.35 mm
1/8 inch = (1/8)" = 0.125in = 0.3175cm = 3.175mm = 125 mils : = 1000mils / 8 =~ 3 mm
1/16 inch = (1/16)" = 0.0625 in = 0.15875cm = 1.5875mm = 62.5 mils =~ 1.5 mm
1/32 inch = (1/32)" = 0.3125 in = 0.079375cm = 0.79375mm = 793.75um = 31.25 mils =~ 0.8 mm
1/64 inch = (1/64)" = 0.015625 in = 0.0396875cm = 15.625 mils =~ 0.4 mm

1 inch = 1 in = 1" = 8 (1 inch / 8) = 8 (0.125 in) = 1 in
1 inch = 1 in = 1" = 16 (1 inch / 16) = 16 (0.0625 in) = 1 in
1 inch = 1 in = 1" = 32 (1 inch / 32) = 32 (0.03125 in) = 1 in
1 inch = 1 in = 1" = 64 (1 inch / 64) = 64 (0.015625 in) = 1 in

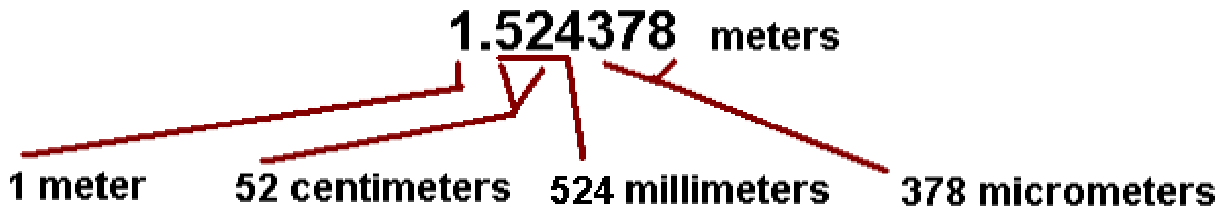
1 square-inch area = (1in)(1in) = 1in^2 = (1000mils)(1000mils) = (1000)(1000) mils^2 = 1,000,000 square mils
1 square mil = (1mil)^2 = (0.001 in)^2 = (0.001)^2 (in^2) = 0.000001 in^2 = a millionth of a square inch
= (0.0254mm)^2 = (0.0254^2)(1 mm)^2 = 0.00064516 mm^2
1 circular mil (CM) area has a diameter of 1 mil = 0.0254mm, and its area is: A =(pi)(r^2)= (3.14159265)(0.0254mm/2)^2 =
= 0.00050671mm^2 = (5.0671)(10^-4) mm^2

Wire diameter or "thickness" sizes may be expressed in mils (thousandths of an inch), or millimeters (mm, metric).

There are wheels, perhaps having a 2ft circumference, at the end of a rod which a user can push so as to measure long and-or awkward distances such as trails by the number of revolutions recorded on a counter, and then multiplied (ie., repeated addition) by the circumference of that wheel. Total Distance = (revs)(C distance)

1 fathom = 6 feet = 1.8288 m : a fathom is generally an old unit for water depth. Fathom is usually considered an old word for understanding, to make sence of.

How to comprehend a given value of meters. For example:



1m + 52cm + 4mm + 378um or:

1m + 524mm + 378um or :

1m + 52.4378cm

The decimal point in 1.524378 separates the number of whole meters from the fractional value of 1 meter:

number = whole meters + fractional value of one meter

number = meters + centimeters are hundredths of a meter + millimeters are thousandths of a meter +
+ micrometers are millions of a meter

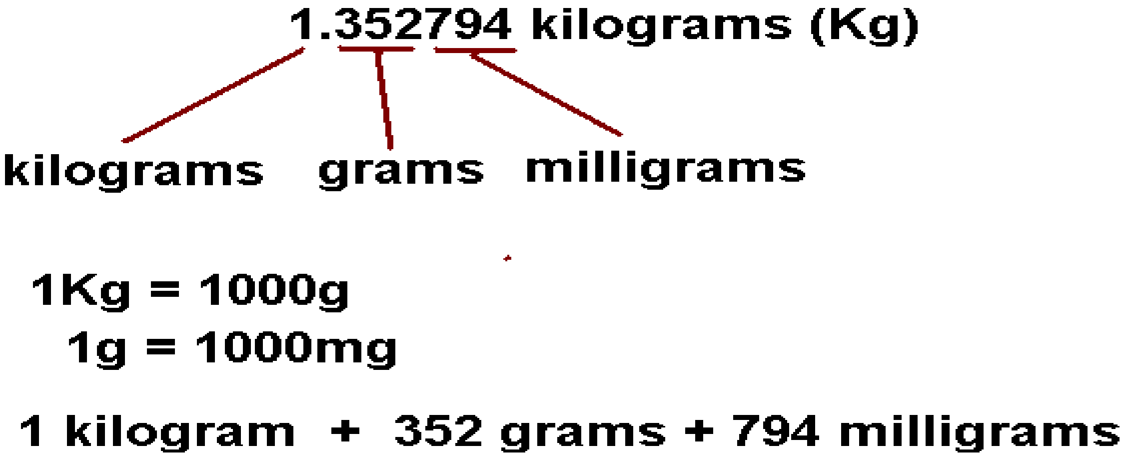
Ex. 52.4 cm = (52.4 cm)(1m / 100cm) = (0.542)(1m) = 0.524m

We see above that when we want to convert cm to meters, we have like units of (cm) in the numerator of one factor and (cm) in the denominator of another factors, and so as those units will cancel out and leave us with the units we want to have, and those are meters, and of which were placed in the numerator.

Ex. 52.4 cm = 52cm + 0.4cm = 52cm + 4mm
Ex. 1.542 m = (1.542m)(100cm/m) = 152.4 cm = 100cm + 52.4cm = 1m + 52cm + 0.4cm = 1m + 52cm + 4mm
Ex. 1.57347 km = 1 km + 573.47 m = 1 km + 573 m + 47 cm = 1 km + 573 m + 470 mm

For comparison: 1.43 ft = 1 ft + 0.43 ft = 1 ft + 0.4 ft + 0.03 ft = 1 ft + (4/10) ft + (3/100) ft : (1/10) ft = (12in/10)= 1.2 in

Here is an similar example of the above, but it is for how to comprehend a mass value with metric units such as kilogram (Kg) units. Note that 1 Kg = 1000g units.



Ex. 1.524378649 Kg = 1 kilogram + 524 grams + 378 milligrams + 649 micrograms =
= 1kg + 524g + 378mg + 649ug : here the value after the decimal point is less than 1 Kg
: micrograms may be expressed as **mcg** or **ug**

Ex. 0.003 Kg = 0 kilograms + 3 grams = 3g

Ex. 1.234 g = 1 gram + 234 milligrams = 1g + 234mg : here the values after the decimal point is less than 1 gram
Ex. 0.5g = 0 grams + 500 milligrams = 500mg
Ex. 0.001g = 0 grams + 1 milligram = 1mg

For a given amount of time, distance and velocity are directly related and proportional in value.

distance = (velocity)(time) = v t

From: d = v t , $\frac{d}{v} = t$, and if (v) changes by a factor of (n), then distance (d) will also change by that same factor of (n):

$$(n)d = (n)v\,t \quad , \quad \frac{d}{v} = \frac{d\,(n)}{v\,(n)} = t = \frac{d1}{v1} = \frac{d2}{v2}$$, and mathematically: $\frac{d1}{d2} = \frac{v1}{v2}$ and-or: $\frac{d2}{d1} = \frac{v2}{v1}$

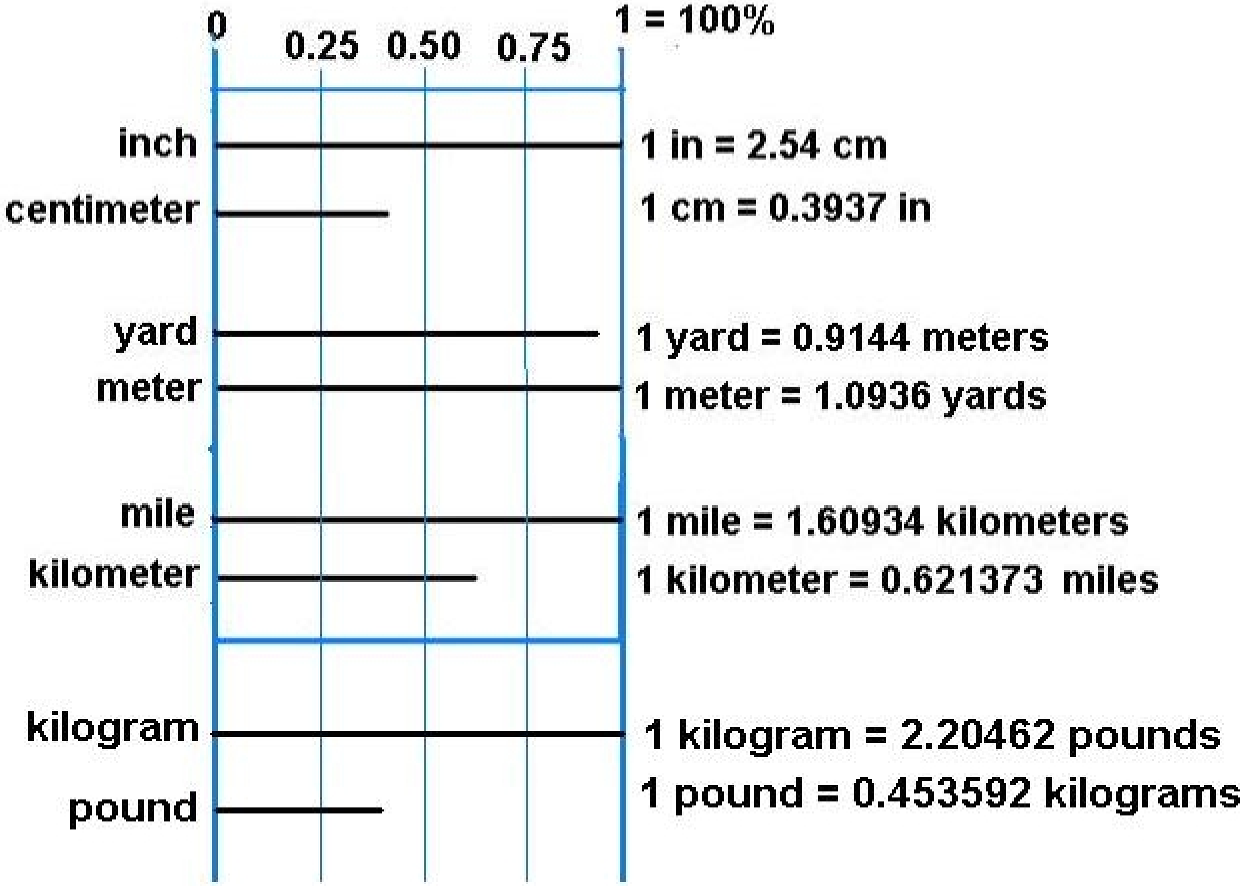
Note also that: d = vt is a linear or line equation of the form: y = mx , and $\frac{y}{x}$ = m = v = slope = a constant for a line.

Likewise, in a similar manner, if (v) is a constant value, (d) and (t) will be directly related and proportional in value:

d = v t , mathematically:
$\frac{d}{t} = v \quad , \quad (n)d = v\,(n)t \quad , \quad v = \frac{d}{t} = \frac{(n)d}{(n)t} = v = \frac{d1}{t1} = \frac{d2}{t2}$, and mathematically: $\frac{d1}{d2} = \frac{t1}{t2}$

Some Relative Sizes Of English And Metric Units

This figure is a quick and visual comparison of some length units. Also included is some mass and-or weight units The individual lengths are not drawn to actual size, but each set of two distances is rather drawn in relative or (fractional) percentage relationship to each other only. [FIG 219A]



SQUARE AREAS: (Some cubic units are also mentioned)

1 square inch = 1 in^2 = 1 in x 1 in = 2.54 cm x 2.54 cm = 6.4516 cm^2

1 square centimeter = 1 cm^2 = 1cm x 1 cm = 0.3937in x 0.3937in = 0.155 in^2
1 square centimeter = 1 cm^2 = 1 cm x 1 cm = 10 mm x 10 mm = 100 mm^2

1 square foot = 1 foot long by 1 foot wide = 1ft x 1 ft = (1)(1)(ft)(ft) = 1 ft^2 = 12in x 12in = 144 in^2 = 929.0304cm^2
1 square foot = (1ft)(1ft) = 1 ft^2 = (0.3048m)(0.3048m) ~= 0.092903 m^2 = roughly 0.1m or a tenth of a sq. mt.
1 square foot = 12in x 12in = 144 sq. in = 929.03 cm^2 =~ very roughly, ~ ~ 1000 cm^2 ~~ (1/10) m^2 ~~ 1m^2 / 10

1 cubic foot = 1 square foot x 1 ft. high = 1ft long by 1ft wide by 1ft high = 1ft x 1ft x 1ft = (1)(1)(1)(ft)(ft)(ft) = 1 ft^3
= 144 in^2 x 12 in = (base area of cube)(height of cube) = 1728 in^3 = 1728 cu. in

1 square yard = 3 ft x 3ft = 9 ft^2 = 36in x 36in = 1296 in^2 = 0.836127 square meters =~ 83.4% of 1m^2

1 square meter = 1m x 1 m = 1m^2 = 100cm x 100 cm = 10000 cm^2
1 square meter = 10000cm^2 = (10000cm^2)(0.155 in^2 / 1 cm^2) = 1550 in^2 = 1.196 square yards
1 square meter = 10.7639 square feet = 1550 square inches

1 square kilometer = (1000m)(1000m) = (3280.84ft)(3280.84ft) = 10763911.11 ft^2 = 0.386102 square miles
1 square kilometer = 1000m x 1000m = 1000000 m^2 = 1 million square meters

1 square mile = 5280ft x 5280ft = 27878400 square feet = (1.60934 km)(1.60934 km) = 2.589975 km^2 =~ 2.59 km^2
1 square mile = 2589998.11 square meters =~ 2.59 square kilometers
1 square mile = 640 acres , and if this is or was thought of as an actual square shape, it would be about 25.3^2 acres

1 acre, a unit of area, is defined as 660ft x 66ft = 1furlong x (1/10) furlong = 43560 square feet (ft^2) The square root of 43560 sq. ft. is about 208.7 feet, which is slightly over 200 feet long on the side of a square shaped area. An acre is therefore about 200ft x 200ft. = 40000 ft^2. An acre was defined as the average amount of land that an ox could till (plough, "plow") in a day, and the furlong was used for its dimensions, and the long length considers the plowing of a long (660ft) straight distance of 1 furlong along the typically narrow (66ft average) planting widths along rivers. The long furrow length meant fewer difficult turnings (reversing direction) of the oxen or horses. 1 square acre of land = 43560 square feet = 43460 ft^2 = (208.7 ft)^2 = (208.7ft / (3ft / yd))^2 = (69.567 yards)^2 =~ 4840 yards^2

1 acre = 63.612m x 63.612m = 4046.856 square meters = 0.004046866 square kilometers = 0.0015625 square miles

A square mile is 5280ft x 5280ft = 27,878,400 square feet , and 27,878,400 ft^2 / (43,560 ft^2 / 1acre) = 640 acres.
If each side of a given square area, say A1, is increased by the factor of (x), the total area increases by the factor of x^2 :
A2 = (xL)(xW) = x^2 LW = x^2 A1 , and A2/A1 = x^2

In the metric system, an area of: 10m x 10m = 100 square meters is called an "are" (based on the word for "area", the word "square" appears to be based on the word "are" or "area", and "square" then means "an e**q**ual **s**ided **are**a").

10m = 32.9701 feet ~= 11yds. 100 sq. m = (10m x 10m) = (10m)^2 = 100m^2 = (32.9701ft x 32.9701ft) =~ 1087 ft^2.
100 (= "hect") of these ares is called a hectare = (100 sq. meters / are) x (100 ares) = 10,000 square meters = (1/100)km^2 . 1km^2 = (1000m)(1000m) = 1 million square meters.
1 hectare = 2.47105 acres =~ 107639.104 ft^2 =~ 328^2 feet =~ 328 ft x 328 ft =~ (100m x100m)= 100 m^2 = 0.01 km^2

Extra: If something is said to be, for example: 0.75 square inches, this could be thought of as a square inch and then with a reduced dimension, such as its height being 0.75 inches long. If 0.75 square inches of total area was to have a square shape with equal side length dimensions, each side would be the square-root of this amount of area when it is considered as having a square shape with an area of 0.75 cu-in, and that value here is the square root of 0.75 in^2 which is equal to 0.86603 in. A square with sides = s = 0.86603 in will have an area of 0.75in^3.

In a square, L=H, hence its area is equal to A = L x H = L x L = L^2 , and the square-root of this area is L.

Surface Areas

The general formula for the area of a rectangle is : A = (length)(width) , if both dimensions increase by a factor of (n), the area will increase by (n^2): A2 = (n length)(n width) = n^2 (length)(width) = n^2 A1. If only one dimension increases by a factor of (n), then the area will increase by that same factor of (n); A2 = (n length)(width) = (n)(length)(width) = n A1

A cube has 6 square sides of equal sizes or areas. Each side is s^2, therefore, the surface are of a cube is:

Surface area of a cube = 6 (Area of 1 of the cube faces) = 6s^2 **: surface area of a cube**

For a rectangular prism, much like a stretched cube:

Surface area of a rectangular prism = Sum Of Area Of Its 6 Sides . There will be 3 pairs of 2 equal area sides.
Each area can be found from:
Any rectangular area = (length of side)(width of side) = (L)(W) sq. units

Arp = (areas of side pair 1) + (area of side pair 2) + (area of side pair 3)
Arp = 2(L1 W1) + 2(L2 W2) + 2(L3 W3) : terms with a common factor of 2:
Arp = 2 [(L1 W1) + (L2 W2) + (L3 W3)] :surface area of a rectangular prism

For the **surface area of a sphere**: A sphere structure can be created by rotating a circle or disk 180 degrees (half of a full rotation of 360 degrees) about its diameter line.

As = 4 (pi) r^2 = 4 (pi) (d/2)^2 = 4 (pi) d^2/4 = (pi) d^2 **: surface area of a sphere** = (pi) (area of a square)
r = radius , d = diameter. Also , **As = Cd or= CD**

For irregular and-or uncommon structures having flat and-or curved sides of various dimension, its total surface area can be found by the sum of each of those flat sides.

Smaller Objects, In Or Of A Larger Object, Will Have A Larger Total Surface Area

If you decrease the radius of a sphere, you obviously (by observing the formula for surface area of a sphere) will decrease its surface area. If you have a large amount of tiny spheres, say within another known amount of volume, such as that of a sphere or cube, their net or total surface area is greater than that of the largest possible sphere in that volume. A large value of surface area has some practical applications, particularly when trying to get maximum contact between substances and-or for more efficient processes. Consider a block or cube cut in half, there will now be more surface area.

The ratio of a spheres surface area to its volume is found using division of the two formulas, and is: (3/r). The smaller the radius (or diameter) of a sphere, the resulting value of (3/r) will increase, and this effectively means that the ratio of the surface area to volume, of a smaller sphere, is larger.

One way to help verify this is to imagine a unit cube or volume of 1 (unit^3)=cubic unit. If you divide its dimensions in half, each part, cube or smaller volume, will be only: (1/2)^3 = 0.5^3 = 0.125 unit^3 , but the total volume will remain the same. (1 unit^3 / 0.125 unit^3) = 8. There will be 2^3 = 8 smaller cubes if you divide a cubes dimensions in half. This is somewhat similar to dividing the radius of a sphere in half, but with a sphere, its surface area will decrease by 4, and its volume will decrease by 9. The total volume will be the sum of all the smaller segments of a certain volume. The total surface area will be the sum of all the smaller segments of a certain surface area. Dividing a volume up into smaller volumes will not result in a sum of those volumes being greater than the source volume.

The surface area of the cube with 1 unit, with 6 sides or surfaces, is: 6 (length x width) = 6 (1 x 1) = 6(1) = 6 (units^2).
The surface area of that cube divided up into 8 smaller cubes when its dimension are halved (divided by 2) is:

8 (surface area of one small cube) , and
8 ((6) (length x width)) = 8 ((6) (0.5 x 0.5)) = 8(6)(0.25) = 48 (0.25) = 12 square-units
Note the total number of (smaller) surfaces is 8 times more, and that the total surface area doubled.

For a thought experiment consider a circle, and all about that circle's circumference on the inside or outside of it are smaller half-circles side by side, and clearly, the (arc) length of each half-circle or semi-circle is longer than the portion of the larger circle's circumference it subtends or is on. This helps verify that the surface area is also larger for the smaller semi-circle(s) and theoretical smaller spheres. Consider a very large circle where the circumference is line-like for a small segment of it and the shortest distance is directly along its path, clearly an arc of a semi-circle placed on that segment is longer, and it has a larger surface area.

Some selected speed or velocity conversions:

1ft/second = (1ft)(3600) / (1s)(3600) = 3600 ft / hour = 0.681818 miles/hour = 1.097338 km/hr =~ 1.1 km / hr =!~ 1km/hr
1ft/second =~ 0.3048 meters/second =~ 30.5 cm/s = 305 mm/s
1 m/s = 1 meter / 1 second = 3.28084 ft / s = 60 m/min = 3600m/1hr = 3.6 km/hr

1 mile/hour = 5280 feet/hour = 1.609 km/hour = (1609 meters/h) / (3600s/h) = 0.4444 meters/s = (5280ft/h) / (3600s/h)) =
= 1.4667 ft/s **=~1.5 ft/s** = 88 ft/min
1 mile/hour = 1609.344 meters/hour =~ 1.609km/hour
1 mile/hour = 26.8224 meters/min = 0.44704 meters/second
10 miles/hour = 880 ft/min = 14.667 ft/s =~ 4.47 meters per second = 16.093km/hour =~ 15 ft/s roughly
60 miles / hour = 1 mile / minute = 88 ft/s =~ 26.8 meters/second = 96.48 km / hr = about 100 km/hr =1.608 km/min
100 miles/hour =~ 0.078 mi/s = 146.7 ft/s =~ 150 ft/s
1000 mi/h = 0.78 mi/s =~ 1500 ft/s

1 mile / s = 1.60934 km / s = 60 miles / 1 min = 3600 miles / 1 hr
1 meter/second =~ 3.281 ft/second = 3600 meters per hour = 3.6km/hour =~ 11812 ft/hour = 2.237 miles/hour
1 km/hour = 1000 m/hr = 16.667 m/min. = 0.2778 m/s = 0.621 miles/hour =~ 3281 ft/hour =~ 0.91154 ft/s **=~ 1 ft/s**
10 km/h =~ 6.21 mi / h =~ roughly 6 mi / h
96.48 km / h =~ 60 mi / h
100 km /h =~ 62.1 mi / h =~ roughly 62 miles / h

1 km/s = 1000 m/1s = 0.621371 mi/s = 3280.84 ft/s = 3600000 m / 1h = 3600 km / 1h 1 hr = 60 s

1mile / 1s = 5280ft/s = (1mile)(3600) / (1s)(3600) = 3600miles / hour = (3600miles/hr) (1.609km/mile) = 5792.4 km/h
5 miles / 1s = 5 (1mile / 1s) = 5 (5280ft/s) = 26400ft/s = 5 (3600miles / 1 hour) = 18000 miles / 1hr = 28962 km/h

For angular velocity = w, and (linear) revolution velocity, see the section below titled: ANGLES AND ROTATION.

MASS: (The amount of real and-or physical matter, material or substance.)

Every object and-or (physical, real) substance will have mass (ex. grams), weight (depending on the local gravity, ex. Newtons, pounds), and volume (ex. cc, liters, cups. Mass and Weight are proportional to each other. An amount of mass can be determined by its weight. Two substances that weigh the same will have the same amount of mass, but then this does not determine the specific types of matter that the mass is, such as what element it is. Different elements have different densities or amount of matter per unit volume. Given a unit volume of two elements, their densities (= mass/volume) will not be the same, but different. Due to this, their weights per unit volume will also be different. An element or type of mass can be determined by its density (= weight / volume). Mass is independent of the gravity placed upon it, that is, mass is a universal concept, whereas the weight of an amount of mass depends upon the local gravity field and-or force upon it and giving that mass an acceleration: **Weight = weight of a mass = Force created due to gravity = force = (mass)(acceleration) = ma , in gravity, a mass will experience acceleration (g), hence: Weight = force = mg , and mathematically, m = weight / g**

The word "mass" appears to be rooted in the ancient Greek word "maza" and Latin word "massa", with a general meaning of "to mash and-or collect into a larger pile". Various words and-or spellings such as "make", "maker" and "massage" are rooted in those words and meanings. "Mass" has a general meaning of a large amount of a substance or material, hence the word "massive". Scientist use the word "mass" to usually mean the amount of physical substance(s) or matter, and which can be determined or measured by its corresponding and proportional weight (ie., the force due to gravity upon that mass).

In general, if the volume of a substance changes by a factor of (n), its mass and-or weight will also change by that same factor of (n).

gram is a word based on the ancient Greek word "gramma" which means a small weight, and may be related to the word "gravity" which has the general meanings of "seriousness" or "grave", weight, and matter which means the physical material or substance which has weight. It could also be rooted in the concept of a "**gra**in or **gra**nule [small piece] of **m**atter".

A gram is technically a unit of mass and not weight. The weight of an object is often expressed as the corresponding gram units of mass, and this is due to that the mass of an object is universal, whereas an objects weight depends on the local gravity of the planet. Weight is a unit of force and not mass. Weight = force = ma = mg. Mass = m = f / a = force / g = weight / g . (g) is the local gravitational acceleration. On Earth, g = 9.71m/s^2 = 32.2ft/s^2

1g = 1 gram of matter. Matter is real, physical material, and mass is the measure of it and has units of grams. This matter could be composed any substance, element, atoms, molecules, etc. Though having a different number of atoms or molecules, A mass of 1 gram of any substance(s) still has the same number of **atomic mass units** (**amu** or= "daltons" [in reference to John Dalton who studied such things]) which are protons and-or neutrons. The specific number of amu so as to have a gram of mass or matter is defined as 1 mol of amu. 1 mol of amu is 6.18 (10^24) amu. 1g of mass = 1 mol of amu, such as protons. Since a hydrogen atom only has a proton, 1g of mass = 1 mol of hydrogen atoms. 1g of any substance will also have the same weight since the number of amu in each gram of that substance is the same. A "common weight scale" is calibrated so as to show the corresponding weight as being 1g. Weight and mass are directly related and proportional. If something has twice as much matter or mass, it will weigh twice as much and vice-versa. The mass or amount of matter or material of an object can be found by weighing it. See the topic of WEIGHT shown below. 1 amu is sometimes called a **nucleon** since it is part of the nucleus of an atom.

1g = 1 gram. This was initially defined as the weight of 1 cubic centimeter volume of water due to Earth's gravity force. Anything that weighed 1g therefore had the same about of substance or matter. 1 gram of cotton has the same weight and mass (ie., matter, substance) as that of 1 gram of iron. Using volumes or containers to measure a true amount of substance has various problems such as randomly sized air gaps between the substance, such as between apples, grain, cotton, fabric, etc., and it creates a doubt as to what amount of matter or substance is actually in that volume. Surely a solid cube of gold weighs less than a hollow cube or box of gold that is the same size and looks exactly like the same volume. The way to avoid this problem of actual substance s to rather know the mass and-or weight of the item in question rather than its volume.

1g = the (corresponding) weight of 1cc of water or water mass = 1cm^3 volume of water = 1mL volume of water.

1g is also the base unit of the amount of mass, matter or substance, and without any influence of gravity.
1g of mass has 1g of weight displayed on a common weight scale, and this allows finding an amount of mass by using a common weight scale calibrated to display the corresponding mass value of an object or substance.
1g of mass, matter, material of any substance or element will have the same weight. This is since it will have the same total number of fundamental mass particles or atomic mass units (amu) which are protons and neutrons.
1g of water is actually 1 gram of water molecules which are composed of hydrogen and oxygen atoms, therefore, a fraction of that gram of mass or material of water is hydrogen and another fraction of that gram is oxygen.
Since atoms of different elements have different mass, the weight of those atoms is different. An atom of hydrogen does not weigh the same as an atom of oxygen. A mol count of hydrogen atoms does not weigh the same as a mol count of oxygen atoms. Still, a gram of hydrogen atoms does weigh the same as a gram of oxygen atoms.
A water molecule is composed of 2 hydrogen atoms and 1 oxygen atom. Hydrogen has 1 amu per atom, and oxygen has 16 amu per atom. 2 hydrogen atoms will have a mass of 1amu + 1amu = 2amu. A water molecule therefore has 2amu + 16 amu = 18 amu per water molecule. Since 16 amu / 2 amu is 8 times more, the weight of the oxygen atoms in any amount of water will be 8 times more than the weight of the hydrogen atoms, yet each water molecule has 1 oxygen atom per 2 hydrogen atoms, and that there is not 8 times more atoms of oxygen in a water molecule. Mass does correspond directly to the number of (amu) particles, but not to atom or molecule sized particles which contain various amounts of amu or mass depending on the specific element or substance such as a molecule, and therefore these particles will have a different (total) mass. An oxygen atom has more amu per atom than a hydrogen atom, and therefore it has more mass and weight than a hydrogen atom.

1g = 0.035274 oz = ~ 3.5 hundredths of an ounce , 1 oz = 28.3495g , 1 **Troy ounce** = 31.1034768g, 1 Troy ounce = 1 toz = $1.097\underline{1}$ oz = 1.097 oz =~ $1.\underline{1}$ oz in the (outdated, 1400's) English Troy mass system: 1 troy ounce = 20 pennyweights, and 1 pennyweight = 24 grains of the avoirdupois system. 1 Troy pound = 12 troy ounces. Troyes is one of the French towns of which many merchants went to buy, sell and-or exchange goods. The British Imperial units system replaced the Troy system in 1824. Troy ounces is still used for some precious metals.

A "**food ounce**" in the U.S.A. is standardized as simply being 30g of mass, and when in agreed use.

1g = 1(10^-3)kg = 0.001kg : a gram is 1gram / 1000 = "a gram is one-thousandth of a kilogram"

1g of mass actually weighs = force = (mass)(acceleration) = (0.001Kg)(9.81m/s^2) = 0.00981Newtons

1 milligram = 1 mg = one-thousandth of a gram = 0.001g.
1 microgram = 1mcg = 0.000,001g = one-millionth of a gram = one-thousandth of a milligram = 1mmg = 1 ug = 1mcg

A slug is a unit of mass in the (older, previous) US Customary or US Common, and British measuring systems. This unit has become outdated and is nearly obsolete, but it may be useful to know about for any possible reference and-or conversion needed. The word "slug" is from same word used in reference to piece of bulk-metal or "bullion" yet to be processed into smaller metal items. A piece of metal such as a coin yet to be stamped with a design is sometimes called a "blank" or "slug".

1 slug = 1 slug of mass = a mass of 1 slug

From: force = F = (mass)(acceleration) = ma

$$m = \frac{F}{a}$$

: extra, note that for a given force, if the acceleration is high, it is because the mass is low, and this is an inverse mathematical relationship. When a force is applied to a mass, more massive things are more difficult to change their motion and-or accelerate due to their inertia.

1 slug = (1lb of weight) : a slug of mass defined as a "gravitational pound" (GPound) due to a pound of weight.
(_1ft__) mathezation:
(1s^2)

(1lb of weight) = (1 slug)(_1ft_) = F = m a
(1s^2)

1 slug of a mass [matter, material atoms, real substance] has a weight [a force] of 32.17405 lbs due to gravity (= ~ 32.2ft/s^2), and this is 32.2 times more than (1ft/s^2). 1 slug =~ 143.1173 N

weight = force = (mass)(acceleration)
32.174 lbs =~ (1 pound-mass)(32.2 ft/s^2) : a (pound-mass) , (ie., a pound of mass) is the mass that will weigh 1 lb on a (force=weight) scale on Earth's surface and gravity (ie., g=acceleration due to the force of Earth's gravity) value there, hence it is said a being or correspondingly having
1 pound-weight = 1lb of wt = the wt of 1 lb or simply as = 1 pound : wt = weight

F = ma = (1 pound-mass)(32.2ft/s^2) = (1 slug)(1 ft/s^2)

From: mass = force / acceleration = weight / acceleration , we have from the above equation:

(1 pound-mass) = 32.174lbs weight / (32.2 ft / 1s^2) , mathematically:
(1 pound-mass) = (1lb-weight) / (1ft. / 1s^2) , from this we have:
(1lb-weight) = force = ma = (1 pound-mass) (1ft / 1s^2)

1 slug = a mass having a weight of 32.174 lbs = 14.59390312 kg of mass , dividing each side by 14.59390312 , we find:
1kg = 1000g = 0.068521765 slugs , dividing each side by 1000, we find:
1g = 0.000068521765 slugs = 6.85218 (10^-5) slugs

Since 1slug of mass will have a weight of 32.174lbs, after dividing each side by 32.174 we find:
1 lb of weight (a force due to gravity acting upon a mass) corresponds to matter having a mass of 0.031081 slugs

1kg = 1000g = 0.0685217659 slugs, and in the force of gravity, this will weigh 2.20462lbs.
1g = 0.0000685218 slugs = 6.85218 (10^-5) slugs

Note: A foot-pound = ft-lb is a unit of work and-or energy:

Work = Energy = (force)(distance) = (distance)(force) = (1ft)(1lb) = 1 (ft-lb) , with the units rearranged so as not to be confused with torque as mentioned below.

Ex. The Work or Energy required to move a 1lb weight object a distance of 1 foot is 1ft-lb. Applying a constant force of 1lb through a distance of 1ft, is 1ft-lb of work=energy.

In numeric value, work = energy used or required: f = ma =1lb = (0.454kg)(9.81m/s^2) = = 4.454 N , 1ft=0.305m , E=W = fd = (4.454N)(0.305m) **= 1 ft-lb = 1.35582 Nm = 1.35582 Joules**

Note: A pound-foot = lb-ft is a unit of torque or "twisting or rotating force" placed upon and-or applied to the rotational or pivot point at the center, and which is like an amplified input force when that force is transferred via a lever-arm and-or radius..
Torque = (force)(lever-arm) = (force)(radius distance from center) =
Torque units are (lb force)(ft) = lbf-ft. **1 lbf-ft** , and which is sometimes simply called a
1 lb=ft =~ (0.453592 kgf) (0.3048 m) = **0.138255 kgf-m = 1.355818 Nm** = **1 ft-lb** (see

1ft-lb mentioned above, ie., 1 (ft-lb) can be used to cause a torque of 1lb-ft = 1 (lbf-ft)

WEIGHT: Weight is equivalent to the downward force of a mass due to the influence or force of gravity (g) of the Earth being applied to and pulling downward on that mass and giving it acceleration, and even when the object appears still, and not in motion, and this can be observed by the constant weight value indicated for object on a weight measuring scale.

Weight = a Force caused by gravity = (mass)(acceleration) = F = ma or= mg

Some units of weight are outdated, but are still needed for past reference and conversions.

In the S.I. or Metric System, weight = force due to gravity = (mass)(acceleration) = (kg)(9.81m/s^2) Many modern "(metric) weight scales" will automatically convert the weight of the mass to just its mass value. For example, an object that weighs 9.8N = (m)(a), has a mass of 1kg and will be displayed on the scale as 1kg. This type of scale is actually a (user friendly) "mass scale" since for many people would rather have the (universal) mass value,rather than its corresponding force or weight value. 1Newton of force or weight is defined as = 1N = (1kg)(1m/s^2) = ma = f . A mass might have much kinetic energy KE= mv^2 / 2, but if it does not push or collide with an object, no force (the application or transfer) will be applied to that object, and therefore, no energy will be transferred to it. If a force is applied to an object, the mass that caused it will decelerate: F = ma , a = F / m.

1N = 1 Newton of force and-or it's equivalent weight = 1N = (1kg)(1m/s^2) = (mass)(acceleration) = force = ma. The velocity of a 1kg mass or object will change or increase by (1m/s) and for each 1 second of time if this amount of force (ie., the application of energy, inducing or causing a transfer of energy) is still being applied to that object and increasing its kinetic energy. In the presence of Earth's gravity force which applies a higher acceleration of 9.8m/s^2 near the Earths surface altitude (or practically less than a few hundred miles high), the mass would be less:

1N = force or= weight = (mass)(acceleration) =~ (0.1kg)(9.81m/s^2) = (100g)(9.81m/s^2) ~= 0.981N
1N = 101.97g =~ 102g :or equivalent gram-weight (force) on Earth = **roughly 100 g = 0.1 kg = 0.220462 lbs**
1N = 3.5969452724 oz =~ 3.6oz =~ 102g
1N = **0.22481 lb** : noted as about or roughly 0.25lbs = a quarter of a pound, shown above.
9.81N =~ 10N =~ the force of 1kg in Earth's gravity (a=g) = weight of 1kg of mass = **1kg weight** = 1kgw = 2.205lbs

1 lb = 1 pound of force. 1 pound of force or simply "1 pound". An amount of mass that happens to weigh 1lb on a weight = force scale, could be said as being the weight of a mass of: 1lb-mass = "1 pound of mass" = 1 mass-lb, or a mass having a force of one pound in Earth's gravity force and acceleration of about: 32.2ft / s^2 = 9.81m/s^2 Weight and mass are proportional, and if one changes by a factor, the other will also change by the same factor.

1 lb = 1 pound = 1 lb-weight = 1 lb-force = 1lbf = 1 one pound of weight in the avoirdupois system of weight = 7000 average center grains of barley. A barley grain is a seed of barley and is therefore called a "barleycorn". The corresponding mass which would weigh or have a force or weight of 1 lb in the influence of Earth's gravity is called a pound-mass = lbm. The pound is used in the US Customary and (British) Imperial Measuring system, however, the metric or SI (System International, or International System) measuring system use kilograms for mass, Newtons for force, and meters for distance are now widely used. The English word and unit of weight called a "pound" is based on the Latin word "poundus" which is a unit weight. A phrase sometimes still heard is "pound it down" or "pound it in", which basically mean to force or press something down or in. So that people do not have to have barley seeds and to count them all, equivalent or reference weights of 1 pound and other weights are available and are commonly used, such as for calibrating (checking, adjusting) weight scales so as to be accurate. The (previous in history, ancient) Roman libra (L) or pound (lb) was equivalent to about 0.75 pounds today.

1 lb = **16 oz. of weight or force** = **corresponds to a mass of**: 0.45359237kg = 453.59237 grams = about **454 grams** in the influence of Earth's constant gravity force causing an acceleration of (a=g=9.8m/s^2) and causing a force = weight = (mass)(acceleration) = ma = mg. 1lb =~ 45% of a kilogram, or vaguely as 0.5kg = half a kilogram

Since 453.59237 grams = 453.59237 cc of water, taking its cube root we have a side length of: 7.683432 cm. = 3.025 in =~ 3in + 1/32 in =~ **3 in** , and as a cube, this side length would then make:
(3.025 in)^3 = 27.68064063 in^3. 1 lb of water =~ 454 grams of water = 0.454L of water = 454cc = 454 mL vol

1 lb = 4.44822 Newtons. In other words: 1 pound of weight (a force) =~ 4.448 newtons of force =~ 4.54N
force = (mass)(acceleration) = f = ma =~ (0.454 kg)(9.81m/s^2) =~ 4.45N

1 lb = 1 pound (a weight) is defined as the weight of 16 fluid (a volume) ounces of water. 16 fluid ounces is defined as the amount of volume called a pint. 1 pint = 2 cups. 1 cup = 8 fl.oz of volume of any substance and regardless of its weight because the volume concept does not consider weight, and-or is a different concept than the weight concept. 1 lb of another substance other than water is still defined as 16 weight oz of weight of that substance.

In the United States customary system, mass can also have units of pounds, and it is more properly said as pound-mass (lb-m) as opposed to pound-force (lb-f). 1 pound of mass is defined as a mass which weighs (id., a force) 1 pound in the force of Earth's gravity. The usage of pounds (weight) is more common than using slugs (mass), particularly due to many scales are calibrated in pouinds of weight rather than slugs of mass.

Ton = "tun" = An English or short-ton=2000lbs. A long-ton = a metric ton = 2240lbs = 1.12 ton = 1000kg
A force that is said to be 1 million pounds of force can lift a maximum weight of 1 million pounds = 500 tons.
A ton of weight (ie., force) = (2000lbs)(4.448N / lb) = 8896 Newtons of force. 1 (10^6) lbs = 4.448 (10^6)N.
The word ton is derived from a certain container called a tun, of which when filled, it had 2000lbs of weight of a liquid substance, and may also be derived out of the word s**ton**e which was a certain unit of weight.

1kg = 1000g = 2.204622622 lbs =~ 2.2lbs = weight of 1L of water = weight of 1000cc of water.
1g = 1(10^-3)kg = 0.001kg = weight of 1 cc = 1mL of water , 1lb = 0.45359237 lbs

1kg = 35.27392 oz

A metric tonne is 1000kg = 2204.62lbs = with a "long" or English ton equivalent of 1.10231 English ton

1 English stone = 14lbs = 6.3503 kg , a very old unit of measurement that is not used much anymore.

1g = 15.43235835 grains (average sized center barley grains, hence a natural, non-made made, unit of measurement)
1 **grain** = (1/7000) avoirdupois pound = 0.06479891g =~ 65 mg =~ 0.0022857 oz

1g of mass in the influence of Earth's gravitational force field will weigh (a force) of 0.0098067N =~ 0.01N.
100g = 0.981 N =~ 1N.
454g = 4.45374 N =~ 4.54N = 1lb of weight
1000g = 9.81 N =~ 10N

1 pennyweight = weight of 24 average, center located grains of barley =~ 1.5552g

1 dram = (1/16) of an ounce in the common **avoirdupois** pound (lb) system = 0.0625oz = 1.772g. A dram is a also unit of mass and volume (fluid-drams) in the **apothecaries** (older, pharmacy) system where 1 dram = (1/8) fl.oz. = 0.125fl. oz. **The avoirdupois (bulk goods and weights) system was the English and European system just before the metric system.** In ancient Greece, 1 obol = 0.1g , In ancient Rome, 1 unus or ounce = 27.41g, and 1 libra = 329g

1 oz = 1 weight and-or "dry ounce" (and **not** a "fluid, volume ounce size"), = a fraction of a pound (= lb) unit = (1/16)lb = 0.0625 lb = 16 drams = 437.5grains of barleycorn = **28.34952313 grams** ~= **28.35 g** in the common avoirdupois system, and 480 grains in the apothecaries and-or troy system = 31.103g = 1/12 lb. A US **"food ounce"** is often rounded and simplified to: 1 US fl-oz =~ 29.57 cc =~ 30cc = 30mL and it could be said that 1 US oz of weight is about equivalent to the weight of 30g of mass. **The US weight ounce and British or Imperial weight ounce are equivalent in value.** There is a slight difference between US and British or Imperial fluid volumes. **Many**

countries are now using the equivalent Metric system: grams, meters and liters in place of any of older systems and their units. The United States is currently (2025) using both the older US Customary System and the modern Metric system, and with a future where the metric system is used mostly. Nonetheless, you may still encounter older and-or outdated measuring system units, and will need to make a conversion to the corresponding number of metric units, etc.. The word "ounce" is based on the ancient Roman words "unus" and "uncia", meaning "once" or an "instance of", and of where the words "unit" and "inch" (= 1/12 foot), come from.

1 oz =~ $0.0283\underline{5}$ kg = $28.3\underline{5}$ g :of mass , **This is a key to remember for converting English and Metric mass values**
1 oz = $0.27801\underline{4}$ Newtons of weight and-or force. : of weight or force = (mass)(acceleration) = ma = mg
1 toz = 1 **troy ounce**, a standard made in the city of Troyes, France, and of which is located in the central area of north-eastern France. 1tz ~= $31.1034\underline{8}$ g 1tz =~ 1.097143 oz ~= 1.097oz ~= 1.1oz

1 carat (ct) = 1 "carat weight" = 200mg = 0.2g = 0.007055 oz , hence 1g = 5 carats , such as for the mass of precious, rare minerals, stones and metals, such as a diamond. This word is derived from the word carob seed used as a common reference weight. 1 oz = 28.3495g = (28.3495g) (5 carats/g) = $141.74\underline{8}$ carats

A karat = a fractional (here equal to 1/24) measure and unit for the percentage of pure gold in a metal alloy. 1 karat means that the substance is or contains (1/24)% pure gold. (1/24)% = $4.16\underline{7}$% , 14 karat is (14/24) = 50% pure gold , 24 karat is (24/24) = 1 = 1.0 = 100% pure gold. 1 karat means that roughly 4% of the total mass and-or weight of a substance will be pure gold. 10 karat will mean roughly (4%/carat)(10 carats) = 40% of the total mass and-or weight will be pure gold.

Some commonly available reference and-or calibration weights and lengths:

When you do not have a known reference or calibrated weight or mass, such as 1oz or 1g, you can use some common items which are known to have a certain weight, mass or volume. You can use multiples and-or combinations of the known weights or masses so as to obtain or sum to a larger reference weight and-or mass. Many of the items listed below will require a fine (weight to equivalent mass) scale which can measure, say a gram to a high precision, say 2 decimal places such as (1/100) = 0.01 of 1 gram, and if you want to verify its mass and-or weight. If the item(s) you are weighing need a container to hold them in, be sure to first measure the weight of this container so as it can be subtracted (ie. to "tare", remove) from the total weight measured or displayed. Some electronic scales have a tare button which will set the scale to a value of 0.0, and to be pressed or activated after the empty container is placed on it.

1 **U.S. penny coin** = 1 cent = (1/100) of 1 dollar (1 USD $1 = 100 cents), monetary value, and base unit of the USA monetary system. If this coin was made in 1981 or earlier, it is mostly copper and with a mass of 3.11 grams = (3.11g)(1oz / 28. 349523g) = 0.109702 oz.. For 1983 and later, 97.5% zinc, weighs 2.5 grams = 0.0881855oz. The mass of two modern "zinc pennies" will be 5 grams. Its diameter is 0.75 in. = 3/4 in. = 12 sixteenths of an inch = 6 eighths of an inch = 19.05mm = about 19mm. Rim thickness is defined at 1.52mm. 4 of these penny coins side by side in a line, will have a total length of 3 in. 16 of these would have a total length of 12in = 1ft. 2 modern pennies would have a mass of: 2.5g + 2.5g = 5g = the mass of 1 U.S. nickel coin.

1 **U.K. pence coin** = penny coin = 1p = 1 cent or (1/100) of 1 pound monetary value. 1p = 0.01 GBP (Great Britain Pound). From 1992 and later, the coin is made with copper coated steel. mass = 3.56g, diameter = 20.3mm , thickness = 1.65mm.

1 **Euro cent coin** = (1/100) or 1 cent of a European Union dollar = E0.01 or $0.01E. Since 1999, it is composed of steel with a copper coating. Its diameter is 16.25mm, its thickness is 1.67mm, and its mass = 2.30g

1 **Rupee coin of India** = stainless steel since 1992 , mass = 3.76g , diameter = 21.93mm , thickness = 1.45mm,

1 **U.S. nickel coin**, 25% nickel, 75% copper, weighs 5 grams = 0.17637oz. , width is 0.835in = 21.21mm , rim thickness is 1.95mm This data excludes same sized nickel coins minted from 1942-1945 so as to conserve copper.

1 **Canadian nickel coin**, year 2000+, weighs 3.95grams = about 0.139oz , diameter = 21.2mm , thickness = 1.76mm

1 **Austrailian nickel or 5 cent coin** = 75% copper, 25% nickel. mass = 2.83g, diameter = 19.41mm , thickness = 1.3mm

1 **U.S. dime coin**, nickel clad, pure copper interior with a nickel-copper plate, 91.67% total copper, weighs 2.268 grams Diameter is 0.705 in = 17.91mm =~ 18mm = ~ 0.7 in. A pre-1965 silver dime weighs 2.5g. Also check the U.S. quarter coin.

1 **Austrailian dime or 10 cent coin** = 74% copper, 25% nickel, mass = 5.65g, diameter = 23.6mm , thickness = 2mm

1 **U.S. quarter coin**, nickel clad plate, 91.67% copper, weighs 5.67 grams = 0.2 oz. A 1932 to 1964 silver quarter is 90% silver and 10% copper, weighs 6.25 grams = 0.2205 oz. Diameter = 0.955 in = 24.26mm =~ 1 in. The mass or weight of silver in this quarter dollar (q) coin is 90% of its total mass or weight: 6.25g (0.90) = 5.625g . In terms of troy ounces this is: 5.625g / (31.1035g / toz) = 0.1808 toz of silver in 1 quarter (q).
1 **oz** = 28.3495g , 1 **Troy ounce** = 1**toz** = 31.1034768g =~ 1.1 oz , 5.625 g =~ 0.1809 toz
(0.1809 toz / 1q.) (5q) = 0.9045 =~ 0.90 = 90% of 1 Toz. (0.1809 toz / 1q.) (6q) = 1.0854 Toz =~ 1 Toz of silver
(0.1809 toz / 1q)(4q) = 0.7236 toz = the amount of silver in "$1 dollar "**face**" or "coin currency or money" **value**.
Note that "**spot price**" is the current price of silver, and the "sale price" is usually a few percent higher due to that the sellers handling and profit-business charge which is usually called the "(sellers) **premium**" price and-or is a percentage of the price of the total silver purchased, and which is usually **about 5%** added to the price per ounce.

1 **Euro dollar coin** = E1 or $1E, As of 2002, diameter = 23.25mm, thickness = 2.33mm, mass = 7.5g , various metals

1 **Austrailian dollar coin** = 92% copper, 6% aluminum, 2% nickel , mass = 9g , diameter = 25mm , thickness = 2.8mm

1 **U.S. Dollar Bill** = 75% cotton and 25% linen [flax plant] = 6.14in =155.956mm long = about 156mm, and 2.61in=66.294mm wide. Its diagonal length is about 6.912in long = 175.56mm = about 7in. Its weight is designed to be 1g, but could vary by a few hundredths of a gram depending on the moisture in it. This 1g defined weight is for the paper being in an environment that is said to have about 50% or a global average air humidity.

Craft wire, steel metal, ~0.5mm thick, ~35mm long = 3.5cm long = ~ (22/16)in = 1in + (6/16)in = 1.375in , weighed 0.1g
Since 0.1g is to 1.375in, we can express this as: 0.1g/1.375in , Multiply num. and den. by 10, we have 1g / 13.75in
3.5mm of this wire mathematically corresponds to a mass of 0.01g = a hundredth of a gram = 10mg

1 **Human hair width**, on average is about 0.1mm = one-tenth of a millimeter = 100um thick. 10 hairs wide =~ 1mm. The thickness of a common solarcell is about 125 um, and they are also brittle, and therefore will crack or break if bent a few degrees. For reference, a common soda or soft drink can has a wall thickness of about 100 um = 0.1 mm.

Width of an average adult **thumb** finger at the widest part is about 2 cm = (2 cm)(1in / 2.54 cm) =~ 0.7874016 in =~ 0.8 in

The mass of an average a **hard, air-dried yellow popcorn (corn, maize) seed or kernel** is about 0.1614545g=~ 162mg, and its length has an average value of about 7mm = 0.7cm . Therefore: 1g / (0.1615g / 1 kernel) =~ 6.2 average kernels / 1 gram, and 6200 kernels per kilogram (kg). This mass per kernel can be found from the total mass of a total sample size, and dividing by the number of samples. The **digital scale** used for small weights and-or mass had a precision (ie., position, and not necessarily accuracy, but very close to the rated precision) of one-hundredth of a gram = 0.01g, however when weighing a single kernel which is known to vary slightly in size and therefore weight, it usually does not represent the average weight of a single kernel. For example, one single (1) kernel was measured by this scale as only about 0.12g =120mg, but another kernel could measure as perhaps between 0.1g and 0.2g. The average expected kernel mass value from a large sample such as 220 kernels with a mass of 35.52g used for this measurement and calculation is: (35.52g total / 220kernels) =~ 0.162g = 162mg / 1 kernel.

A **steel metal bottle cap** used for a crimp (folded, mechanical) seal, often on a soft drink bottle, is about 2.02g =~ 2g

A solid, clear **glass sphere or glass marble** 13.7 mm in diameter weighs about 5.25g. Weight is proportional to volume. If the weight increases by a factor of (n), the volume increased by that same factor of (n) or vice-versa.

1L volume in a **cube shape** having sides that are almost 4 inches in length. 1L = 1000cc ~ (4in)^3 = 64 in^3 = 64 cubic inches. 1L of water is 1000g of matter and weighs 2.2046 lbs. You can make a homemade 1L volume by weighing out 2.2046 pounds of plain water and marking its height. 453.6g ~= 454g of water is equal to 1 lb of water.

A common **aluminum metal, soda, soft drink soda can** has a basic shape of a cylinder, and has a volume of 12 fluid ounces = 355mL = 355 milliliters = 0.355L, and is 2.13 in. = 5.4102 cm in diameter across the top rim. The main body cylinder has a diameter of 2.6 in = 6.604cm, and the total height of the can is 4.83in. = 12.27cm. The empty can has a mass of about 14g =~ 0.493835 dry ounces =~ half an ounce = 0.5oz. 32 of these cans would weigh approximately 1 pound-weight = 0.453592kg ~= 454g. Pound-weight = pound-force = "pound".

12 full of liquid soft drink cans weighs approximately: (12 cans x 12 fl.oz/can) ~= 144fl.oz =~ 144oz / (16oz / lb) = 9 lbs. The total weight of a "**12 pack of soda**" including the weight of the cans is therefore 9 pounds plus 6oz for the aluminum cans, and a total of: 9lbs + 6oz/16oz = 9lbs + 0.375lbs = 9.375lbs at minimum if considered pure water.

A (very near, metric) standard of **1s and-or 1m** can also be verified by a pendulum as discussed in this book. A pendulum that can define and has a swing of 1 second, is naturally called a "seconds pendulum".
1m = 100cm = 3.281 ft = 3 feet + 3.37 inches = 3 ft + (3.37in x 16 segments/in) =
3 ft + 53.92 sixteenths of an inch ~= 3 ft + 54 sixteenths of an inch =~ 3ft + 3in + 6 sixteenths of an inch =
3 ft + 3 in + (1/4) in + (1/8) in

[FIG 220]

The scale shown has units of millimeters. The coins, from left to right are: quarter, dime nickel, and penny. As an additional reference, a common AA size battery is typically a total of 50 mm = 5 cm long, and 14 mm = 1.4 cm wide. 1 cm = 10 mm. The image above has been magnified, and should not be considered as the actual physical size. The 4 reference coins shown in the image have a total width of about 81.5mm = 3.201in The combined width of the penny and the nickel is 40.25mm =~ 40mm = 4cm. A strip of paper 4cm long (L) can be folded in half so as to have 2cm = (L/2) or half-length, and that can be folded in half so as to have a 1cm = (L/2)/2 = L/4 or quarter-length. This 1cm length of paper can be divided into two 5mm= (L/4)/2 = L/8 or one-eighth-length. For a 1/3 length of a line or one-third of any length= L/3, one method to find this length is to cut a strip of thin paper that is the total length in question, and first bend it into an S curve shape, and then a Z shape, and then flatten this structure so as to have three equal lengths, and with each being (L/3).

1 fluid ounce volume of water weighs nearly 1dry or solid ounce of weight, and is about 1.040843 ounces.
1 U.S. fluid ounce volume of water weighs 29.57353 grams = 29.57353 cc of water ~= 30 "food, fluid, grams" = 30cc
1 fluid ounce of volume of water is about 29.57353 mL of water = 1.80469 in^3 =~ (1.2175 in)^3 : ~ 30mL
1 British or "Imperial" fluid ounce = about 28.41mL , and weighs 1 oz = 1/16 of a pound
1 cup volume of water = 8 fluid oz, volume, of water =~ 8 dry or solid ounces, weighs 236.588 grams = 236.588cc of water
2 cups volume of water = 16 fluid oz, volume of water =~ 16 dry or solid ounces = 1 poiund = 1 lb
1 cc volume of water weighs 0.033814 dry or solid ounces = 1 gram of mass
1 liter volume of water = 1000cc of water = 1000g = and weighs about 2.2 lbs = 33.814 dry oz of weight =~ 2.113375 lbs
1 U.S. gallon volume of water = 16 cups volume of water =128 fluid ounces volume of water weighs 8.326744 pounds

The size of a cube having a volume of 30 cc = 30 cm^3 =~ (3.1072 cm)^3 is roughly 3 cm

A much less common today, U.S.A. "customary measuring system" used the slug as the unit of mass, and a pound as the unit of force (such as weight). 1 slug = 14.5939 kg. A mass of 1 slug has a weight, due to the force due to gravity, of 32.2 pounds. 1 pound of constantly applied force can accelerate a mass of 1 slug by 1 ft/s^2. From: force = weight = (mass)(acceleration) , 32.2 pounds = (1 slug)(32.2ft/s^2). Hence 1 slug of mass of any substance(s) will weigh 32.2 pounds. In this system, which often uses Imperial (an older British system, before the metric system) units, the units of pressure is = force / area = pounds / ft^2 , and-or pounds / in^2. Torque in this system is torque = (force)(distance) = (force)(radius) = pound-inch = lb-in or in-lb. Here, the radius or "lever-arm" value is essentially a "force multiplier" value. Another common torque unit is the foot-pound or pound-feet.

Various units having pounds:

Modern science generally uses the metric system with kilograms (mass), Newtons (force), seconds (time) and meters (distance). Pound units are sometimes still used as in the United States of America and Great Britain. Conversions between old and modern (such as the metric system) measuring systems is often necessary.

Using the word "pound" for corresponding mass and weight seems like a good thing since mass and weight correspond to each other and are mathematically proportional to each other, however, it can be confusing to some people who are left in wonder if the topic is about mass or force. Weight is the force due to a mass when the constant force of gravity (causing acceleration = a = g) is applied to it. On another planet where (g) is not

the same, then a pound of force, and a pound of mass will actually (physically) have different values than those on Earth. Using mass or matter with units of kilograms is more practical and universal, and is the same physical value and measurement regardless of the value of (g) on that planet.

pound-mass = (lbm) = a measurement of mass (amount of real matter, material or substance) via its weight and giving it units of pounds. Note that a pound of mass will weight differently if the gravity is different than Earths gravitational force of acceleration (a = g).

pound-force = (lbf) = a measurement of force and giving it units of pounds. In the force of gravity (g), 1 pound-mass, or a mass of 1 pound has a weight (= force in gravity) of 1 pound-force or 1 pound-weight. Just the same, if given 1 pound-force, the corresponding mass to cause that force in gravity is 1 pound-mass.

foot-pound = (ft-lbf) = a unit of work = energy. Joules (J) is the unit of work = energy in the SI system.
Work = (force)(distance) = (pound-force)(feet) with units of "foot-pounds"

pound-foot = (lbf-ft) = "pound-feet") or= foot-pound = a unit of torque (twisting force) applied about a point.
Here: Torque = (force)(lever-arm distance) = (force)(radius) = (pound-force)(feet)

If a reverse manner, it could be said that the moment of this torque is that it can provide pound-force at a distance of feet from the rotation point.

For reference: 1L volume = 1000cc = 1000g mass of water = 1 kg of water and weighs 9.81N = 2.20462 lbs =~ 10N .
The dimensions of 1L cube are nearly 4in per side, hence 1L has a volume of about (4in)^3 = 64 in^3 = 64 cubic inches.

Some mathematical relationships of pounds and Newtons:

For mass: 1 pound-mass = 0.453592kg = 453.492g =~ 454g , after dividing both sides by 0.453592 :
1kg = 2.2046244 pound-mass = 2.2046244 pound-weight
1kg was defined in terms of distance and water as the mass of: 10cm^3 volume of water = 1000 cc

Force = (mass) (acceleration) , when (a)=(g = 9.81m/s^2 = 32.2ft/s^2), Force = Weight = force due to gravity.

A force of 1N that is constantly applied to 1kg of mass or mater, will keep increasing its velocity or speed by an acceleration of: a = F / m = N / m = (kg m/s^2) / kg = 1 m/s^2 or= 1m/s for each second of acceleration.

Force = weight = ma = (1kg)(9.81 m/s^2) = 1 Newtons = 1N = x pounds of weight

In the US Customary System: Force = (mass)(acceleration) , if (a)=(g = 32.17405m/s^2),
F = (slugs)(32.2 ft/s^2) = weight with units of pound-weight

Since 1m = 3.280804 ft , after dividing both sides by 3.280804: 1ft = 0.3048 m

If the acceleration is: $a = \frac{32.2\text{ ft}}{1\text{s}} = \frac{9.81\text{m}}{1\text{s}}$: = g , Here, this is the acceleration of a object due to the constantly applied force of gravity applied to it..

F = N = (m)(a) , and due to gravity, (a)=(g) and: F = N = weight = ma = mg

9.80665N = (1kg)(9.80665m/s^2) = (2.2046244 pounds-mass) = 2.2046244 pound-force = "pounds"
= weight in Earth's gravity force of: 32.2ft/s^2

Dividing both sides by 9.80665 , we find **1N =~ $0.22480\underline{9}$ pounds** of weight = pound-force or= weight-force

Force = 1 Newton = ma = $0.2248\underline{1}$ pounds ~= one-quarter of a pound =~ 0.25lb such as weight, solving for pounds by dividing each side by $0.2248\underline{1}$:

Force = **1 pound ~= $4.4482216\underline{2}$ Newtons** : a measure of force , 1 lb = 16 wt-oz =~ 454g of force = 454 gf
1 wt-oz =~ 28.375 gf , 1 gf =~ 0.0353 wt-oz

Work = (force)(distance) = (Newtons)(meters) = N-m Joules of energy , and since: 1m = $3.2808\underline{4}$ ft
0.3048m = 1ft:

Work = (force)(distance) = (($0.22480\underline{9}$) (pounds)) N x (($3.28\underline{1}$) (feet)) m Joules , when N=1, and m=1:

Work = (Newtons)(meters) = 1Nm ~= ($0.737562\underline{2}$)(pounds)(feet) = $0.737562\underline{2}$ lb-ft J = energy

Work = **1N-m = $0.737\underline{6}$ lb-ft** Joules of energy , dividing both sides by $0.737\underline{6}$:
: this can be thought of as lifting $0.737\underline{6}$ lb a distance of 1 ft,
or lifting 1 lb a distance of $0.737\underline{6}$ ft.:

Work = **1 lb-ft = $1.3558\underline{2}$ N-m** J : 1 pound-foot =~ $1.357\underline{6}$ Newton-meters

Finding mass and weight from each other. You can find the mass of an object by dividing its weight by (g).

From: force = (mass)(acceleration) = weight , we mathematically have:
mass = (force / acceleration) = weight / g : mass is essentially the weight value with the gravity factor and multiplier removed, and here its divided out

Force has units of Newtons (N). Weight (a force) is actually a measure of an object or mass in a gravitational field such from Earth. Gravity is constantly pulling (attracting) on the object with mass and causing it to accelerate. This will cause the object to have a force, and this force is called the weight of the object. Weight, being a force, is also measured with units of Newtons.

Mass (amount matter, and not its force) and weight (a force) are directly related and proportional to each other having a constant of proportionality of (g):

(g) = (weight / mass) = (force / mass) = (mass)(acceleration) / mass = (acceleration),

Mass and weight can be calculated from each other if the local force of gravity is known which causes an attraction and acceleration of a nearby mass. Technically, given two masses, their gravitational attraction or pull on the other mass is the same value for both, and this concept is called "equal an opposite (direction) forces".

A 1kg mass will have a downward force or weight equal to: (1kg)(9.8m/s^2) = 9.8N or weight.
Therefore, to find the weight or downward force corresponding to this mass with Kg units, simply multiply it by 9.8

The corresponding mass associated with 9.8N of weight is from: mass = (weight / acceleration) = (weight / g) = weight / (9.8m/s^2) = 9.8N / (9.8m/1s^2) = 1N / (m/s^2) = 1kg amount of matter. Therefore, to find the mass of a weight, simply divide the weight by 9.8 and assign the result as having kilogram units, and this is what most "weight scales" are actually calibrated to do, and the result displayed is actually the objects mass (in grams or kilograms units), rather than its weight (Newtons) force due to Earths constant gravity force. Mass is "universal" and a constant value, where as (g) of Earth is not, and it is just a local (specific location, or localized) value in the universe. The local value of (g) on each planet will cause an object to weigh (a force) different value.

1g weighs = (mass)(acceleration) = (0.001Kg)(9.81 m/s^2) = 0.00981N : a common scale is calibrated (ie., adjusted) to display the corresponding mass, and will display 1g.

Note that gravity is a force of attraction of two masses. The amount of force is determined by the sizes of the masses and their distance from each other since the "gravitational field strength" and its effect (of attracting other mass) is less as the distance between those two objects with mass increases. When a force is constantly applied to a mass, it will therefore cause that mass to not only be attracted, but to also accelerate and get faster and faster in speed as it nears the other object. As two objects get closer, their local (due to their masses) gravitational, "attractive" or "pulling" field strength or "force of gravity" is also stronger.

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Mass in the presence of the constantly applied force of gravity near Earth's surface will then have an acceleration of (a) = g = 9.80665 m/s^2. =~ 9.81 m/s^2. Weight is a value of Force.

F = (m)(g) = the local, Earth "Weight" of that mass , therefore, Weight = (m)(g) , and mass = $m = \frac{\text{Weight}}{g}$

$F = (m)(g) = \frac{(\text{Weight})}{g}(g) = \text{Weight}$

[This space for edits.]

VOLUME: A measure of 3-dimensional (length, width, height) size or space, such as of containers and-or a size or an amount of a substance such as liquid or solids. A volume of something is essentially how much "room" something occupies within a container and-or will need.

1 fluid oz (a liquid, a volume) of water weighs 1.040843 oz =~ 1 dry oz for most practical purposes, therefore 16 fl. oz of water weighs about 16 dry oz = 16 oz = 1 pound = 1 lb : a dry ounce is essentially a non fluid, solid matter weight and not its volume, such as fluid ounces. In general, only 1 fluid volume ounce (1 fl. oz) of water equals 1 dry ounce (1 oz) weight ounce, and other elements or substances (such as mercury or oil) will weigh more or less per fluid ounce. In general, fluid ounces are a volume (3 dimensional space and-or shape having some length, width, and height) and not a weight. **1 fl. oz = 1 foz = 29.57353 mL of volume = 0.02957353 L =~ 29.576 cc**

The word "fluid" or "liquid" is not always indicated to distinguish it from a weight oz, and of which only 1 fluid oz of water actually weighs 1 dry oz. If the substance is not water, then this equal value (volume and weight) correspondence is not true, and should not be considered or used. When volume is considered, it is best to consider and-or indicate "oz" explicitly as "fluid oz", "fl. oz", "volume oz", "fluid volume oz". The Liter (L) is the common base unit for volume in the metric system. **1 fl. oz. = 1 volume oz :fluid ounces are not equal to weight ounces unless it is liquid water**

1L = 1000cm^3 = 1000cc = 10x10x10 cubic centimeters, (fluid) volume. 1 deciliter = (1/10)L = 0.1L =100cm^3 = 100mL
1in = 2.54 cm, therefore 1cm = 0.3937in., 10cm = 3.937in ~= 4in , 1L= (3.937in)^3 = 61.02334 in^3 = 61.02334 cu-in
A gallon volume has 231 cubic inches of volume. 1gal / 1L = 231 in^3 / 61.02334 in^3 = 3.7853447 , therefore:
1gal = 3.7853747 L and after dividing both sides by 3.7853747 , we have: 1L = 0.264170303 gal

1L = (1000 cm^3)(1000 mm^3/cm^3) = 1,000,000 mm^3
1L volume of water = 1000 cc water, and therefore its mass is: 10g x 10g x 10g = 1000 grams = 1 kg mass of water.
One-thousandth of a liter = 1L/1000 = 0.001L = 1mL = 1000cc/1000 = 1 cubic centimeter (which weighs 1 gram)
By filling a container with water, you can find the volume of it by the corresponding volume of water or the mass of it.
For example, if a contain was filled to some level by 400 grams of water, and knowing that there is 1g water / 1cc:
Setting up a proportional or equivalent fraction equation: 1g water / 1cc = 400g water / Xcc , Xcc = 400cc
1L = 1000 cm^3 = 1000 mL
1 liter = 1L = 33.814023 fluid ounces = 2.1133764 pints = ~ 2 pints = 1 quart = 1 qt.
The weight of 1L water at it maximum density at 4°C weighs 1 Kg = 35.274 dry oz = about 32 oz or slightly over 2lb =~ the weight of 1 quarter (ie., 1 quart = 2 pints) gallon of water
1 liter = 1/1000 of the volume of a cubic meter. (1m)^3 = 1m^3 = 1m x 1m x 1m =100 cm x 100 cm x 100 cm =
= (100cm)^3 = 1000000 cm^3
1 liter = 0.264172 gallons = roughly or slightly more than a quarter of a gallon.
1 gallon = 1 gal = 128 fl oz = 128 foz. and 1L / 1gal = 33.814 foz / 128 foz = 0.264172
1 liter =~ 1 quart (quart or quarter =1/4 of 1 gallon) ~= 2 pints ~= 4 cups =~ 32 fluid or volume ounces
1 liter = 1.05669 quarts (a quarter, one-fourth) of a US gallon. 1 quart = 0.94635 liters
1 liter = 61.0237 cubic inches ,and the cube root of 61.0237 in^3 gives a side length of about 3.9372 in = 10cm , or ~4in
1L =~ (4 in)^3 = 4^3 in^3 = 4^3 cu.-in. 3.9372 in = 3 in + nearly 15/16
1 liter = 4.22675 cups , =~ roughly 4.25 cups = 4 cups + 1/4 cup , and since 1 cup of water = 8 fluid ounces of water,
1L = (4.22675cups)(8oz/cup) = 33.814 floz
1L =~ 0.3058 chous units in ancient Greece =~ 10.3 fl. oz

1 cubic foot 1 cu. ft = 1 ft^3 = (12 in)^3 = 1728 in^3 = 28.3168667 Liters =~ 28L + 317 mL

1 fl oz = a volume defined as the volume corresponding to 1/16 of a pound of water, but is then used for any liquid
1 US fl oz = 1L/33.814 = 0.029573549 L = ~ **29.5735296mL = ~ 30mL** =~ 30cc which is how it is defined in the U.S.A. food industry as a "food, fl. oz". 1fl oz in the S.I. or British "Imperial" system is: 28.41306mL = 0.02841306L .
1 fl oz =~ 29.574 cm^3 = a cube having sides of the cube root of this value, and which is about: 3.1cm in length
1 fl oz = 591.471 drops =~ 600 drops and: (600 drops)(0.05 mL) = 30 mL = 30 cc of volume
1 fl oz = 1.8046875 in^3 , A cube of this volume will have a side length of ~ 1.2175 in =~ 1in + (7/32)in
Solving for 1 in^3 = 1 cubic inch by dividing both sides by 1.8046875:

1 cubic-inch of volume = 1 in^3 = 0.554112554 fl oz = slightly over half a fluid ounce of volume

1 dL = 1 deciliter = a tenth of 1L = 0.1L = 0.10L = 0.100L = 10 cL = 100 mL : the dL unit is not used much

1 cL = 1 centiliter = a hundredth of 1L = 0.01L = 0.010L = 10 mL : the cL unit is not used as much as the mL unit

1 mL = 1 milliliter = a thousandth of 1L = (1/1000) L = 0.001 L = 0.033814 fl oz = **1cc** = 1g of weight if the substance is water.

1ml = 1cc = 20 drops : here, the drop unit is scientifically defined or a standardized drop such as from a "eye-dropper"

3785.41178 mL = 128 fl.oz = **1 US gallon = 8.34 lbs of weight if water =~ 3.785 kg if water**

1000 mL = 33.8140227 US fl.oz. = 1L = 1 kg if water : =~ 34 US floz

946.352946 mL **= 32 fl.oz. = 1 quart**
750 mL = 25.360517 fl.oz.
500 mL = 16.907 fl.oz.

473.176473 mL = **16 fl.oz.** = **1 pint** = **1 lb of weight if water**
354.882 mL = **12 fl.oz = 1 typical can of soda drink**

300 mL = 10.1442 fl.oz.
295.735296 mL = 10 fl.oz.
250 mL = 8.45351 fl.oz.

236.588237 mL = 8 fl.oz. = **1 cup of volume = (1/2) lb = 0.5 lb if water**

200 mL = 6.7628 fl.oz.
147.867648 mL = 5 fl.oz.
100 mL = 3.3814 fl.oz.
88.7205887 mL **= 3 fl.oz**
75 mL = 2.53605 fl.oz.
50 mL = 1.6907 fl.oz.
30 mL = 1.01442068 fl.oz. =~ 1 fl.oz. = 30g if water

29.5735296 mL = 1 fl.oz. = 1 oz of solid or dry weight if liquid water =~ 30cc = 30g if water : = ~ 30mL

25 mL = 0.845350568 fl.oz.
20 mL = 0.676280454 fl.oz.
15 mL = 0.5072103431 fl.oz. = 1 tablespoon of volume = 3 teaspoons of volume
14.7867648 mL =~ 15 mL = **0.5 fl.oz.**
10 mL = 0.33814 fl.oz. = 10 cc = 10 cm^3 = 10g if water
5 mL = 0.169070114 fl.oz. = 1 teaspoon of volume : 1 tsp is scientifically defined as slightly less than 5 mL
1 mL = 0.0338140227 fl.oz. = **1cc of volume = 1 cm^3 of volume = 1g if water**

1 cc = a cubic centimeter = 1 cm^3, and if in a cube shaped volume, it will have a length, width and height length of 1 cm
1 cc = 1 cubic centimeter = 1 cm^3 = **1 mL** = one-thousandth of a liter = (1/1000) Liter = 0.001L , 1cc of water will weigh 1 gram on an (equivalent mass) scale. 1 cm^3 = 1cm of length x 1cm of width x 1cm of height = 1cc = 1 cm^3 = 10mm x 10 mm x 10 mm = 1000 mm^3 . **1cc = 1mL = 20 drops**
1 cc = 0.0610237 in^3 , = and taking the cube root of this we have it being equal to: (0.393700692 in)^3 =~ (0.4 in)^3
1 cc = 0.033814 (US) fluid ounces =~ ((6.4/16) in)^3

1 drop = a volume = 1mL / 20 = (1/20) mL = 0.05 mL = "five, one-hundredths of a milliliter" = 1/20 cc = 0.05 cc = 0.05 cm^3 in volume. If this volume was a cube shape, it would then have a side length of the cube root of 0.05 cm^3, and that equals 0.3684 cm = 36.84mm = 0.145in = slightly over (1/8) in = 0.125 in. The volume of 20 drops would be equal to: (20)(0.05mL) = 1mL = 1cc. Since 1mL = (1/1000)L = 1 cc of water is 1g, 1 drop of water weighs = (1/20)g = 0.050g = 50mg = 0.0016907fl oz =~ 0.0017 fl vol. oz. Volume sphere, such as a drop = Vs = (4/3)(pi)(r^3) , r = cube-root of: ((3/4)Vs / (pi)) = cube-root of (0.2387324 Vs) = cube-root of ((0.2387324)(0.05 cm3)) =~ 0.22854cm = 0.09 in =~ 0.1 in =~ (1/10) in. =~ 2.3mm = (1.449 /16) in

1 drop = 1 mL / 20 = (1/20) mL = 0.050mL = 50uL , hence there are 20 drops / 1 mL = **20 drops / 1 cc**
1L = 1000 cc = 1000 mL , and the ratio of : 1L / 1 drop = 1000 mL / 0.05mL = 20000 ,
hence mathematically, there are **20000 drops / 1 L**

1 drop = 0.05 mL = 5 (10^-2) mL = 5 (10^-2)(10^-3) L = 5 (10^-5) L = 0.00005 L = 50 uL = 50 mico-Liters
1 drop of water = 0.05 mL = 0.05 mg = 5 (10^-2) mg = 5 (10^-2) (10^-3) Kg = 5 (10^-5) Kg = 0.00005 Kg
1 drop / Liter = 0.05 mL / L = 0.05 mL / 1000 mL = 0.00005L / L , if water: = 0.00005 Kg / L

1 pinch = a non-exact measurement of volume typically related to food. It is considered as an amount of a substance (ex. salt, sugar, spice) that can be picked up, pinched or contained between a persons fingers - typically the thumb and index finger. 1 pinch = 1/16 teaspoon. to 1/8 of a teaspoon. 1 pinch = 0.3125 mL to 0.625 mL

1 cup = 8 fluid (volume) oz = 14.4375 cubic inches (in^3) =~ (2.435 in)^3 cubic inches = (6.1849 cc)^3 = 236.59 cc
1 cup fluid volume is commonly considered as weighing 8 dry oz of weight if water (only), but is technically 8.3214oz of weight in the United States, U.S. customary measurement system = 236.59 grams of weight since 1 U.S. customary oz is 28.34952 grams =~ 28.35g
1 cup = 1/16 of a US gallon = 8 fl. oz = 0.236558 liters =~ 0.25L =~ about a quarter of a liter. 1 cup = 1/4 gal
2 cups = 1 pint = 16 fl. oz.
4 cups = 1 quart = 32 fl. oz = 2 pints = a **quart** is a **quart**er or 1/4 of a gallon =~ about 1 liter

1 pint = 1/8 gallon = 2 cups = 16fl. oz , and if it contains water substance, that water weighs: 1.04375 lbs = 473.437g ~= 1 pound = 1 lb = 473.437 cc = slightly less than half a liter = 1000cc/2 = 500cc . 2 pints can be thought of as roughly 1 Liter. 1 pint is defined as the (fluid) volume of 1 pound = 16 dry ounces, weight of water. Also by this definition, 1 fl. ounce = (1/16) of a (U.S. customary) pint.

1 quart = 2 pints = 4 cups = 0.94635 liters = 946.35 mL = 946.35 cc : a quart is sometimes approximated as ~ 1L
1 **quart** = 1/4 gallon = 0.25 gal = "a **quart**er of a gallon" = 32 fl. oz.
1 gallon = 128 fluid ounce. By this definition, 1fl. oz = (1/128) of a (American, U.S. customary) gallon
1 gallon = 4 quarts = 8 pints = 16 cups = 128 fl. oz = 3.785412 liters = ~ 3.785L =~ 3785ml =~ 3785cc = 3785g if water

1 gallon of water weighs 8.3454 pounds = 3.785412 kg = 3785.41 grams of weight : for water substance, mass only
1 US gallon has 128 fl. oz of volume and-or 231 cubic inches, and this is the American ("customary") gallon. A fl. oz is then (1/128) of 1 gallon. This makes (128 fl. oz / 231 in^3) = 0.5541126 fl. oz per 1 in^3, and it could be said as being roughly or about 0.5fl.oz = half a fluid ounce per cubic inch. Mathematically , 1 fl. oz = 1.804687351 in^3 , and if this volume had a cube shape, its side would be the cube-root of this value, and this is: ~ 1.2175 in = =~ 1in + 7 (1in / 32) = 1 in + (7/32) in. The dimensions of 1 gal would be the cube-root of 231in^3 = = ~ 6.1358 in = 6in + (1/8)in + (1/64) , and this is roughly equal to a cube of **~ 6 in.** per dimension (ie., L,W,H).

The **British** ("**Imperial**", standardized) gallon of which the American gallon is roughly based upon, has 277.4194 cubic inches, and is based on the volume of the weight of 10 pounds (weight) of water. 10 pounds of weight = 160 oz = 4.5359237kg = 4535.9237g. Since the substance is water, it was called a fluid ounce. 1 gal = 160 foz. Hence 1 fl oz = (1/160) gal = (4535.9237g / 160) = 28.349523g which corresponds to ~ 28.35 cc =~ 28.35 mL . This value seems to have been updated or redefined later as being slightly more as: 28.41306 mL
The physical dimensions of a British fluid ounce of volume is: (160 foz / 277.4194 in^3) = 0.576744 foz / in^3.
The British fl. oz. is also defined as (1/20) of a British pint, and there are 8 British pints in a British gallon, and

1 British pint = (160 fl oz / 8) = 20 fl. oz. A British quart or "quarter-gallon" = 2 pints = (160 fl. oz) / 4 = 40 fl oz.

1 British pint = 2 cups, hence 1 cup = 20fl. oz / 2 = 10 fl. oz = (160oz / 16) = 1gal / 16 = (1/16) gallon , hence 16 cups per gallon. Note that 10 lbs = (10)(16 oz) = 160 oz, and 1 Imperial gal = 160 fl. oz. , and therefore, 1 fl. oz weighs 1 oz of dry or solid weight. There are: 16 oz / 1 lb and 16 fl. oz / 1 lb , also, 16 oz / 16 fl .oz and 1 oz weight / 1 fluid oz water

1 British gallon = 1.20095 US gallons , 1 US gal = 0.832674133 British gal

As of the year 2022, the American gallon still has common use, however the use of the British gallon unit has faded more due to the more popular metric system with liters (L) as its base or reference unit of or for volume measurement. Nonetheless, the values and conversions between measuring systems is a problem to the average person, and therefore, there is still a significant effort for the world to adopt and use the metric system.

1 UK or Imperial **bushel** has the volume of 10 Imperial or British gallons of liquid water. An Imperial gallon is the volume of 10 pounds of water. 10 Imperial gallons of water is equivalent to about 9.61 US gallons. Today, a bushel of a product is associated with the weight of a (dry, non liquid) specific product that would normally be contained in that bushel sized volume. 1 bushel = 8 dry gallons (UK) = 32 quarts = 36.37L = 8.257 US gallons. 1 bushel = 9.3092 dry gallons (US) = 37.237 quarts (US) = 35.2393L. A bushel sized basket is often used for gathering and-or measuring produce such as fruits and vegetables from farms. A bushel is a unit of volume measurement, and therefore, it does not have a specific weight due to that various products, crops or foods to put in it will weigh differently due to their sizes, shapes, quantity or count, amount of water, etc, however for a single product or food, there will be a typical or average weight per bushel to expect. Crop pickers often use low-cost, lite-weight wooden baskets of various fractions of a bushel so as to gather crops. In good planting and growing conditions, an acre of **wheat** (ie., wheat grass, a tall grass, and with sprouts and seeds being edible) will yield (ie., produce) about 45 bushels of whole-wheat or even more for some wheat varieties, and where each bushel weighs abut 60 pounds. An acre can therefore produce about (45 bushels)(60 pounds / bushel) = 2700 pounds of ground whole-wheat (seed) flour (fine particles). 1 pound of whole-wheat can produce about 1 pound of whole-wheat (ground) flour, and about 0.65 pounds of white-flour.

1 Peck = a unit of volume for non liquid products, and is equal to 1/8 of a bushel, hence it is a volume of 4 quarts. 1 peck is equivalent to the volume of 2.3273 gallons = 9.3092 quarts (US) = 8.80977L. A bushel is equal to 8 pecks. The peck unit is not used much as of the writing of this book in the year 2022.

1 Barrel = US customary, such as for a variety of products, is 42 gallons =159L of fluid volume for the wooden (often with oak wood) ones created first, and then 55 gallons = "200L" (actually 208.2L) steel "drums" (oil) barrels which came later in 1905. Aluminum and plastic barrels, containers, "kegs" or "casks" of various volume sizes and shapes are also used and it depends on the product placed within it. Barrels are used to hold and protect goods during storage and shipment. The shape of a barrel often resembles that of a cylinder, and often with a height that is 1.5 times greater than its diameter. A US 55 gallon barrel has a 22.5 in. diameter and a 33.5 in height.

1 Cord = Usually a volume of cut and dried wood, and with pieces of it to be routinely placed in a stone and-or metal fireplace to heat a home. 1 cord is typically in these dimensions: 24 ft. long, by 4ft high, by 1.333 ft = 16 in deep. Multiplying these dimensional values together yields a volume of: 128 cubic feet = 128 ft^3. A solid cube of this size will have a side dimension of the cube root of this value, and which is: 5.04 ft =~ 5 ft. Since pieces of uneven wood will have spaces, this 5 ft value will average higher at perhaps 5.75 ft. Due to all the gaps (spaces) between each piece of stacked wood, it would be ideal to know the weight value of a cord of wood of a certain type of tree when dried sufficiently, and rather than the volume value. This is a large weight; several hundreds of pounds, and it is possible to construct a weight scale out of a floating container with a calibrated weight and-or depth scale on its side that corresponds to different weights within it.

1 Dozen = not a volume, but a (linear) count of 12 items. A **gross** (meaning large, huge, many) is twelve dozen, hence a dozen of a dozen, or a dozen dozen, and the number of items in a gross unit is (12)(dozen) = 12(12units) =

144 units or items. Food purchases from a store are sometimes often called groceries, and the seller is often called a grocer (ie. a large or gross amount wholesaler) . The words "gross" and "groceries" may be rooted in the words "grow" and "growth". A more generic and common word for edible groceries is food.

For fluids, such as water, a volume (ie. space or 3 dimensional size, and not a weight) measurement:

1 fluid or volume ("food") ounce = 30ml = 0.030L , a food (weight) ounce is also considered as 30g of weight = 30ml water
1 cup = 8 fluid oz (ounces) = 0.240L = 240ml = 240cc : = 240g = 1/2lb = 8 dry oz. of weight if the substance is water.
1L = 4.1667 cups : this considers fl. oz as "food ounces" with 30mL per ounce, and (8)(30mL) = 240mL per 8oz cup = 0.240L , otherwise, 1 cup = 0.23659 Liters and 1 Liter = 4.22675 cups

Fluid (fl) ounces, fluid or liquid measurement is actually a volume or space (as in size) measurement, however 1 fl. oz of water actually does nearly weigh, and is considered about 1 oz of weight or "dry weight" (non volume, but rather weight due to gravity). Consider that a cup full of plastic foam particles will weigh less than a cup of water, and that a cup of lead metal will weigh more than a cup of water. All these substances of matter can occupy the same volume, but these same volumes of the substances have different weights, and this is due to that their density (amount of mass per unit volume) each, has different value. Fluid or liquid essentially means having no discernible space between the particles, and can flow and fill any shaped volume. Fluid essentially means a flowing liquid or substance. or liquid or substance that can flow.

Here is a list of the common fractions of a cup, based on using: 1/8 cup = 30cc = 1fl.oz, = 30 gram "food ounce"

3/4 cup = 6 fl.oz = 180ml = 180cc , 2/3 cup = 5.33 fl.oz = 160ml = 160cc
1/2 cup = 4 oz fl. = 120ml = 120cc , 1/3 cup = 2.67 fl.oz = 80ml = 80cc
1/4 cup = 2 fl.oz = 60ml = 60 cc
1/8 cup = 1 fl.oz = 30ml = 30cc volume = 30g of weight if the substance being measured within is water only.
1/8 cup = 2 tablespoons (tbs) = 6 teaspoons (tsp) = 1 fluid ounce (oz) volume = 1 ounce of weight if water only

To find the basic equivalent amount of ml or cc given a volume of fluid ounces, multiply the fluid ounces by 30cc.

1 teaspoon of (fluid) volume = 5mL = 5cc (typical), substance weight could vary widely depending on the substance properties. Note that some "home teaspoons" may vary in the volume that they can hold, and may not be 5mL. **If possible, get some calibrated measuring (volume) spoons and scoops, so as to ensure better accuracy.**
1 teaspoon = 5ml / (0.05 ml / drop) = 100 scientifically defined drops
1 level teaspoon of water is 4.929cc ~ 4.93 grams weight = about 5cc = 5 grams of water weight = 1/6 oz = 0.1667 oz
1 teaspoon of water actually weighs ~0.17337 oz when not rounded to 1/6 "food ounces" = 30g / 6 = 5 grams
6 teaspoons =~ 1 US fluid ounce of volume
1 level teaspoon of water is equal to 1.33... fluid drams, which today is a unit of measurement that is not used much.
3 level teaspoons of water will equal the volume of about a 1 level tablespoon of water
1 level tablespoon of water is about 3 level teaspoons of volume ~ 15mL = 15cc = 15 grams if the substance is water
1 tablespoon volume of water weighs 0.52011 oz = about 1/2 fl oz, hence 2 tablespoons = 6 teaspoons is about 1 fl. oz
1 cubic centimeter = 1cc = 1 cm^3 = 1mL = 1milli-Liter = (1/1000) L = 0.001L = 0.0610237 cubic inches
1 cubic centimeter of water weighs 1 gram = 0.0352717 oz = 0.002204481 lbs
1 cubic inch = 1 cu. in^3 = (2.54 cm)^3 = 16.3871 cubic centimeters = 16.3871 mL , and would be 16.3871 grams of weight if the substance is water
1 cubic inch of water weighs about (8.345404lbs/gallon) / (231cubic inches/gallon) = 0.0361lbs/in^3 = 0.578oz/in^3 = 16.386 grams/in^3. A cubic inch of water weighs a little over half an ounce of weight , as shown next:
1 cubic inch = 0.554111786 US fl. oz of volume : slightly more than half a US fluid ounce
1 cubic inch of ice (frozen, solid water, a water crystal) weighs about: (0.578oz)(density of ice) = (0.578 oz)(0.9167) = 0.5298526 oz =~=~ 15.0210462g. Frozen water or ice, such as an icecube or iceberg) will float near the water surface.
1 cubic foot of water weighs about 62.427118 pounds, and is about 7.481 gal. Since iron is 7.874 times denser than water, a cubic foot of iron weighs 7.874 times more, hence: (62.43lbs/f^3)(7.874) = 491.57lbs/f^3
1 cubic foot = 1ft^3 = (12in)(12in)(12in) = (12in)^3 = 1728 in^3 = (1728 in^3)(16.3871cc / in^3) = 28316.9 cc = 28.317L
1 cubic foot of water would weigh: (1g/cc)(28316.9cc) = 28316.9 cc = 28.317L = 28317g if water = 28.317kg ,

and (28316.9g) / (453.6 g/ lb) = 62.437 lbs
1 cubic yard of water weighs: (62.4272lbs / 1 ft^3.) x (3ft. x 3ft. x 3ft) = 62.4272lbs x 27 = 1685.53lbs
1 cubic meter = (100cm)(100cm)(100cm) = (1,000,000)cm^3 , if the substance is water which is 1000cc / 1kg
= 1000cm^3 / 1 kg, then 1m^3 = 1,000,000cm^3 / (1000cm^3 / 1 kg) = 1000kg = 1000L of water
1m^3 of water = 1000kg of water = 1 metric ton = 2205lbs of weight if water = slightly over 1 ton of weight

Items that are "irregular shaped" and difficult to measure or calculate the volume of can be submerged into a liquid such as water to determine the change (due to the "**displacement**", "water displacement", moved to the side, essentially **the change in volume** caused by the submerged object) in volume of that liquid which is equivalent to the volume of the item. If the item is of a single and consistent (solid) material such as a flat or round piece of metal, it can be weighed, and then its volume can be calculated by the knowing weight per volume, or volume per weight of that material or substance:

Density of an element = d = (mass / volume) = (weight **/** g) **/** volume = (weight)(volume) **/** g
mass = (density)(volume) = (force **/** g) = (weight **/** g) :g = Earth's gravitational acceleration = 9.81m/s^2
weight = (density)(volume)(g) = (mass)(g) = ma = force
volume = (mass **/** density) = (weight **/** g) **/** (density) = (weight) **/** (density)(g)
g = weight / mass **=** force / mass = gravitational acceleration (due to gravitational force on Earth's surface)

Since ice (frozen, solidified water) floats on liquid water, ice has a lower density than liquid water. The density of ice is about 92% that of liquid water (1g/1cc): **Density of ice = 0.9167g/1cc** For a given volume, the weight of ice will also be 91.167% that of liquid water. Ice will float near the surface of water, but just barely float since it is not too much lighter than liquid water. Ex. Only the tip of an iceberg is above water.

Some Selected Volume And Weight Values (approximate values)

1L = 1 liter, of water substance weighs 1kg ~= 2.2046 lbs , 1L of volume = 1000 cubic centimeters = 1000 cc
1L = 0.264 U.S. Gallons = 0.0353 cubic feet = 61.0237 cu. in. , **1L of water is defined to weigh 1kg at 4°C = 39.2°F**
1 cc = 1mL water weighs 0.002205 lbs = 0.035oz = 1 gram of mass at 4°C and maximum water density (mass/volume).
1 fluid ounce of volume = 29.57cc. 1 fl-oz of water weighs 0.0651985 lb = 29.57g weighs 29.57g
1 U.S. Customary Gallon ~= 128 fluid-volume ounces = 231 in^3 / (1728 in^3 / 1 ft^3) = 0.133681 ft^3 ~= 3.7854 liters
1 US Customary Gallon is typically considered as weighing (128 fl. oz / (~16 fl. oz / lb)) = 8 pounds (actually 8.34 lbs wt)
1 Imperial Gallon (British) ~= 4.546 liters =~ 1.20093 US gal =~ 1.2 US gal
1lb of water is 16 fluid-volume ounces = 453.5924 cc of water = 0.4535924 L ~= 0.454 kg of mass if water

1 cubic inch = 1 cu = 1 in^3 = (2.54 cm)^3 = 16.387064 cc = 16.387064 mL = 0.0163872 L , Dividing both sides by 16.387064, we find:
1 cubic centimeter = 1 cm^3 = 1 cc = 0.061023744 cubic inches. 1 cm =~ 0.3937 in =~ 0.4 in = "4 tenths of 1 inch"
1 cc = 1 cm^3 = 1 mL volume =~ 0.033814 US fl. oz
1 US fl. oz = 29.5735296 mL = 29.5735296 cc = 0.0295735296 L . If water, it will weigh 29.574 g. 1 US fl. oz. of water weighs 1.04317556 wt. oz , therefore, 1 wt. oz of water = 0.95861142 fl. oz. =~ 0.96 fl. oz.
1 cu. in water weighs 0.03613 lbs = 16.3871 grams = the weight of 16.387064 cc of water indicated above
1 lb of water =~ 27.6778 in^3 of water, and taking the cube-root of this, we have a cube with a side of : **3.0249 in =~ 3 in**

Extra: If something is said to be, for example: 0.75 cubic inches, this could be thought of as a cubic inch and then with a reduced dimension, such as its height, of being 0.75 inches long. If 0.75 cubic inches of total volume was to have a cube shape with equal side length dimensions, each side would be the cube root of this amount of volume when it is considered as having as a cube shape or volume of 0.75 cu-in, and that value is: 0.90856 , and (0.90856 in)^3 = (0.90856)^3 (1 in)^3 = (0.75)(1 in^3) = 0.75 in^3

In a cube, L=W=H, hence its volume is equal to V = L x L x L = L^3 , and the cube root of this volume is L

1 cu. in. = 0.554111786 US fl. oz. volume =~ half a fluid ounce = 0.5 fl. oz
1 US fl oz = 1.80469 in^3 , taking the cube root of this we have a cube with a side length of: 1.217496 in =~ 1" + (3.5/16)"

1 cu. ft = 1 cubic foot = 1ft^3 = 1ft x 1ft x 1ft = 12 in x 12 in x 12 in = 12^3 in^3 = 1728 in^3
1 cu. ft = 1 ft^3 = 1 ft x 1ft x 1 ft = 30.48 cm x 30.48 cm x 30.48 cm = 30.48^3 cm^3 = 28316.85 cm^3
1 cu. ft. = 30.48^3 cm^3 = 0.3048^3 m^3 = 0.028316846 m^3 , from this we have:
1 cu. m = 1 m^3 = 35.31466672 ft^3 = roughly: (3ft x 3 ft x 3 ft) = 27 ft^3 = 1 cu. yd = 1 yd^3 + 8 ft^3 = 1yd^3 + (2ft)^3
1 cubic inch = 1 cu. ft / 1728 = 0.000578703 ft^3
1 cu. ft. = 7.481 gallons = 28.317 liters = 1728 cu. in. = 28316.8466 cm^3
1 cubic yard = 1cu-yd = 1cyd = 1yd^3 = 1yd x 1yd x 1yd = (3ft L)(3ft W)(3ft H) = 27 ft^3 = 27 cubic feet
1 cu. m. = 1 cubic meter = (1m)^3 = (1m)(1m)(1m) = (100cm)^3 = ((10)(10cm))^3 = (10^3)(1000 ccm) = 1000 Liters =
= 35.31455 ft^3 , and if this volume was a cube, it would have a side length of: ~ 3.280835 ft = 3ft + (9/32) in

Substance	**Weight / U.S. Gallon**	**Substance**	**Weight / cubic yard = 27 ft^3**	**Substance**	**Weight / cu. ft**
alcohol	6 lbs	dry sand	2600 lbs = 96 lbs / cubic foot	oils	55 lbs
gasoline	6.2 lbs	concrete	4000 lbs = 150 lbs / cubic foot	water	62.4272 lbs ~ 28.342kg
oils	7.7 lbs	water	1686 lbs = 765.234kg	steel	490 lbs =~ 500 lbs
water	8.34 lbs = 18.37 kg equivalent mass value		, 1L of water "weighs" 1 kg	lead	708 lbs
liquid paint	11 lbs (typical)		on a weight to mass scale	gold	1206 lbs

For any same volume chosen and filled with a material, such as for example: a cubic foot of gold, and a cubic foot of steel, the ratio of the weight of gold to the weight of steel is: (1206lbs/cu.ft) / (490lbs/cu.ft) = 2.46 It could be said that gold is 2.46 times heavier than steel, or roughly 2.5 times heavier than steel. It will be also shown below that gold likewise has about 2.46 times the mass (the measure of real, physical substance) as that of steel, and that is why gold is also heavier. For reference: 1 gallon = 3785.312 cc = 3785.312 L of volume , 1 lb = 454.592g = 0.454492 L = 454.492 cc if water , and 1 cu. ft = 1 ft^3 = 28316.8466 cc = 28.3168466 L of volume

The density of gold is 19.28 g/cc, and the density of steel is 8g/cc. The ratio of the density of gold to that of the density of steel is:is about : (19.28g/cc / 8g/cc) = 2.41 . This is about the same ratio of their corresponding weights, hence **weight and density are directly related and proportional**. **Since weight is directly related and proportional to mass, mass is also directly related and proportional to density. Given the same volume of two elements or substances, the ratio of their weights, and the ratio of their masses are the same.** For the above example, the radio of the density of gold and steel is 2.41, and the ratio of their corresponding weights is:
(1206 lbs/cu.ft) / (490 lbs/cu.ft) =~ 2.461 , If 500 lbs/cu.ft was used for steel, the result is: 2.41

Gold has 19.28g/cc , Dividing both num. and den. by 19.28 so as to make an equivalent fraction, we find that
gold has 1g/0.05187cc or using the reciprocal of this we find: 0.05187cc / 1g = ~ 0.052cc / 1g

Steel has 8g/cc. , Dividing both num. and den. by 8 so as to make an equivalent fraction, we find that steel
has 1g / 0.125cc or by inverting (ie., reciprocal) this expression, we have: 0.125cc / 1g

Since steel has a lower density than gold, therefore, steel has less mass and weight per same volume than gold does, and it will take more volume of steel so as to have an equivalent mass and-or weight of gold such as 1 gram, or 1 ounce of weight, and therefore: **density and volume are inversely related. Note that 1 gram of steel has the same mass as that of 1 gram of gold, aluminum, or any other element. A gram is not related to the density, volume or size of a mass, but it is actually a specific amount of mass (ie., real substance, matter or material) for any and all elements. A gram of any element will also weigh the same, and for the common weight-to-mass or weight-to-grams calibrated scales, a gram of any element will be said as "weighing" 1 gram.**

Given a certain volume of a specific substance or element, if the volume of it changes by a factor of (n), both its mass and weight will also change by that same factor of (n), but its (natural, at standard air pressure) density will still remain the same value. It is of note that gasses can be easily compressed by using a force and-or pressure so as to increase the density of it, such as to make some liquid oxygen from a large volume of oxygen gas.

Alcohol, the natural form. from say wood, is **methanol or methyl-alcohol** is C1H3OH = CH3OH. C1H3 is methyl, and OH = O1H1 is an extra hydroxy or hydrogen-oxygen molecule. This type of alcohol is **poisonous,** and some is sometimes made during the fermentation process of making edible grain or food alcohol that is called **ethyl alcohol or ethanol** that is produced by yeast, and the methanol must be removed before it is consumed, and this is usually done by the manufacturer. The USA law permits a maximum of 7 grams per liter, and the EU permits a maximum of 10 grams of methanol per liter of pure, 100% ethyl-alcohol.

Methanol is sometimes called "wood alcohol". Its molecular weight or "molar mass" is about 32 grams per mol-atoms. A wood alcohol molecule has 32amu. Each carbon atom has 12amu, each oxygen atom has 16amu, and each hydrogen atom has 1 amu. The density (mass/volume) of methyl alcohol is 0.792g/cm^3 = 0.792g/mL = 792g / L = = 0.792kg / L = 792kg / m^3 . This being < 1g/ml, will float on liquid water since water has a greater density of 1g/mL. 792g =~ 27.937 wtoz = 1.746 lbs

Water has a molar-mass of 18.01528 g/mol since each water molecule = H20 contains 2 hydrogen atoms and 1 oxygen atom, and that each hydrogen atom contains 1amu, and each oxygen atom contains 16amu. The total sum of amu in a water molecule is: 2amu + 16amu = 18amu. Note that for the metric system, the (mass) density of water was defined as 1g/cm^3 = 1000 g / 1000 cm^3 = 1kg / L.. 1g = 1L/1000 = 0.001L = 1mL. 1 gram of mass in the influence of Earth's gravitational acceleration force will weigh (a measure of force) nearly a hundredth of a Newton: F = ma = (0.001kg) x (9.8067m/s^2) = 0.0098067N =~ 0.01N. 100g of mass will weigh about 1N. 1kg of mass will weigh about 10N.

Although alcohol weighs less per unit of volume (a gallon size in the example above), an alcohol molecule weighs nearly twice as much than that of a water molecule, and this is due mostly to the carbon in alcohol. A larger, heavier (more massive) molecule will occupy more space in a given volume, and this results in having less molecules in a given volume. Less molecules for a given volume is indicative of that substance weighing less per unit volume. Note that less molecules of any single element will always result in less weight.

Some extra mathematical considerations about area and volume:

Area of a square = square area = $As = s^2 = d^2 = (d)(d)$: $s = d$ = side distance of a square
$V = s^3 = d^3 = (d)(d)(d) = (d^2)d = A d$, Ex: (base Area)(height distance) : V = volume of a square
cube root of volume = $s = d$
$V / A = d^3 / d^2 = d$

If the dimensions of an area increased by a factor of (n), the area will increase by $(n)^2$:

$A2 = (n \text{ length})(n \text{ width}) = n^2 (\text{length})(\text{width}) = n^2 A1$

If only one dimension of an area increased by a factor of (n), the area will increase by that same factor of (n):

$A2 = (n \text{ length})(\text{width}) = (n) (\text{length})(\text{width}) = n A1$

If the **area** = A increases by a factor of (n), and becomes A1, the side distance (s=d) increased by the factor of the square root of (n):

$A2 = nA1 = n(d^2) = n(d d) = \sqrt{n} d \sqrt{n} d = (\sqrt{n} d)^2 = (\text{side})^2$: here, let: $A1 = d^2$

If d increases by a factor of (n), A increase by a factor of (n^2):

$A2 = (d2)^2 = (n d1)^2 = n^2 d1^2 = n^2 A1$, and $A2 / A1 = n^2$: here, let: $A2 = (d2)^2$ and $A1 = (d1^2)$

If the side (s=d) of a square area increases by 1, A will increase by (2d + 1), and this value is not a constant, but depends on the given, initial value of (d). :

$A1 = d^2 = (d)(d)$: here $d = d1$ in reference to A1 , $d2 = (d + 1)$: technically, d^2 is a quadratic equation, b=0, c=0
$A2 = (d2)^2 = (d+1)^2 = (d+1)(d+1) = d^2 + d + d + 1 = d^2 + 2d + 1$: a quadratic equation in variable (d)

The difference in these two areas is how much the area will increase:

$(A2 - A1) = d^2 + 2d + 1 - (d^2) = d^2 + 2d + 1 - d^2 =$
$(A2 - A1) = 2d + 1$: a linear equation , mathematically:

$A2 = A1 + \text{increase} = d^2 + \text{increase}$
$A2 = (d2)^2 = A1 + (2d + 1) = A1 + 2d + 1 =$
$A2 = (d2)^2 = (d+1)^2 = d^2 + 2d + 1$: A2 with respect to A1 when (d) increases by 1 , a quadratic equation

---

If the dimensions of a volume increased by a factor of (n), the volume will increase by $(n)^3$:

$V2 = (n \text{ length})(n \text{ width})(n \text{ height}) = n^3 (\text{length})(\text{width})(\text{height}) = n^3 V1$

If only one dimension of a volume increased by a factor of (n), the volume will increase by that same factor of (n):

$V2 = (n \text{ length})(\text{width})(\text{height}) = (n) (\text{length})(\text{width})(\text{height}) = n V1$

If the **volume** increases by a factor of (n), the side distance (d=s) of that volume increased or increases by the factor of the cube root of (n):

nV = n (d^3) = n (d d d) = 3√n 3√n 3√n (d d d) = (3√n)^3 d^3 = (3√n d)^3

If (d) increases by a factor of (n), V increase by a factor of (n^3):

V2 = (n V1) = (d2)^3 = (n d)^3 = n^3 d^3 = n^3 V1 : V2 / V1 = n^3

If the side (s=d) of a cube volume increases by 1, V will increase by (3d^2 + 3d + 1). First,

V1 = (d^3) = (d)(d)(d) : here d = d1 in reference to V1
V2 = (d2)^3 = (d+1)^3 = (d+1)(d+1)(d+1) = (d+1)^2 (d+1) = A2 (d+1) =
= (d^2 + 2d + 1) (d +1) = d^3 + d^2 + 2d^2 + 2d + d + 1 =
V2 = d^3 + 3d^2 + 3d + 1 : V2 with respect to V1 when d increases by 1 , a cubic equation

The difference in these two volumes is how much the volume will increase:
(V2 - V1) = d^3 + 3d^2 + 3d + 1 - (d^3) = d^3 + 3d^2 + 3d + 1 - d^3 =
(V2 - V1) = 3d^2 + 3d + 1 : a quadratic equation , mathematically:

V2 = V1 + increase
V2 = V1 + (3d^2 + 3d + 1) = V1 + 3d^2 + 3d + 1 =
V2 = (d2)^3 = (d+1)^3 = d^3 + 3d^2 + 3d + 1 : a cubic equation

Here is a somewhat or related note about the mass, gravity, and volume of an object. If the volume of a substance or material doubles, the mass of that substance or material doubles. If the length (such as a diameter or radius doubles) of an object doubles, its area (A = l x w) doubles: A = l x w. If l or w doubles, the total area is now: 2A = (2l x w) = 2(l x w), but if the object was a sphere or cube, its volume increases by the cube of this factor (here 2) value, and this is equal to a factor of: (factor)^3 = 2^3 = 8 applied to the initial volume.

Vcube = l x w x h = l x l x l = l^3 . : In a cube, l=w=h. If the side length of a cube is doubled:
(2l) x (2l) x (2l) = (2l)^3 = (2^3)(l^3) = 8(Vcube) : the volume is 8 times more, and volume is said to change or increase rapidly as the linear dimensions change

From this, if a volume is 8 times more, its dimensions are the cube root of 8, hence 2 times (twice) that of the reference mass volume. In general: linear dimension increase = cube root of volume increase factor.

If only one dimension of a volume doubled, then the volume will only double:

Ex: If v1 = (l)(w)(h) , and if h doubles: v2 = (l)(w)(2h) = 2 ((l)(w)(h))= 2(v1) = 2v1

As a simple thought example, if you had a cylinder filled with a liquid, and if the length of that cylinder doubled, the total volume is doubled. This is also the same total volume of two of the original cylinders. If the mass of an object increases by (n), its weight, volume and gravity will increase by (n).

Ex. A volume 1 liter of water = 1L has a mass of 1kg of water substance, and will weigh 9.81N.

If this volume is doubled:

A volume of 2L of water will have a total mass of 2kg, and will weight 19.62N.

In the **Extras And Late Entries** section of this book, and in the topic: Some of the commonly used units and conversions for volume measurements, there are some volume and conversions mentioned of which were mentioned previously and some possible new understandings for the readers.

TIME

Time is often called the fourth-dimension. Physical changes, motion and-or processes are not instantaneous and are said to need an amount time or duration to complete. An object can't be located at two positions simultaneously or at the same instant or time value, and if it were so, it would make two objects from one object. If timed stopped, there would be no further processes or changes possible, and it would be as things were essentially frozen or halted in position. An object in motion, fast, slow or temporarily stopped will have three standard location or position coordinates of: p(X, Y, and Z), and it will also have a fourth or time coordinate (T), hence the coordinates of a point or position could be expressed as:
p(X , Y, Z , T)

The time dimension or simply Time is a measure of and between past (already happened, "time ago"), present (now, currently), and-or future (to happen later) events (happenings, realities, other time points or reference times). Time elapsed is much like a total length duration and number of (accepted) time units of measurement. In general, time is often how long a certain event or process takes, hence it is also the difference between the ending time value and the starting time value. The basic units for time measurement are usually seconds (s), which is a fraction (1/60) of a minute, which is a fraction (1/60) of an hour, which is a fraction (1/24) of a day, which is a fraction (1/365) of 1 year of standard "(24 hour) clock time". The Earth rotates (spins) on its (imaginary) polar axis of rotation, once a day unit of time which is equal to 24 hour units of time. The Earth revolves around the Sun in one year (=365 days) unit of time. "Current time" is the present or "now" time and-or date. Time on Earth is in reference to these natural periodic (regularly repeating, and usually having the same time duration between and-or during these events, cyclic), events, and clocks and calendars are therefore also made so as to be periodic.

Things that happened in the past are things that happened (became true, real, reality, existed) at a previous, lesser or earlier time. Things that happen in the future are things that have not happened yet and will happen at a later or increased amount of time past the current time. The "present", or present time, is the "current" or "now" time, and of which things are happening now, existing at this "current" (or more correctly for time as: "currant") instant in time. The "cur" portion of similar words that mean "to flow or mean" is derived from the ancient word "ker" or "kers", and of which is the basic of "car", "chariot", "occur", and "chair".

In the simplest wording, a time value is a measure of past, present and future events using an accepted unit of time such as the second. If the present is now, this can be considered as the time of reference for all other times, and it will therefore have a value of 0. The past or a previous time will have a negative signed time value, and a future time will have a positive sign. Time is a gauge of when an event happened, or will happen, and-or the time duration of that event. The time on an accurate clock is the "now" or current time, and is in reference to an accepted reference time and that is usually the Greenwich Mean Time (GMT) as mentioned in this book.

Now,
A basic "timeline" of events: **Past <**.......................**Present**,....................> **Future**
. . . -5 , -4 , -3 , -2 , -1 , **0** , 1 , 2 , 3 , 4 , 5 **. . .** : time units
Already Happened Reference To Happen (s) , (min) , (hr) ,
of time measurements (day) , (yr)

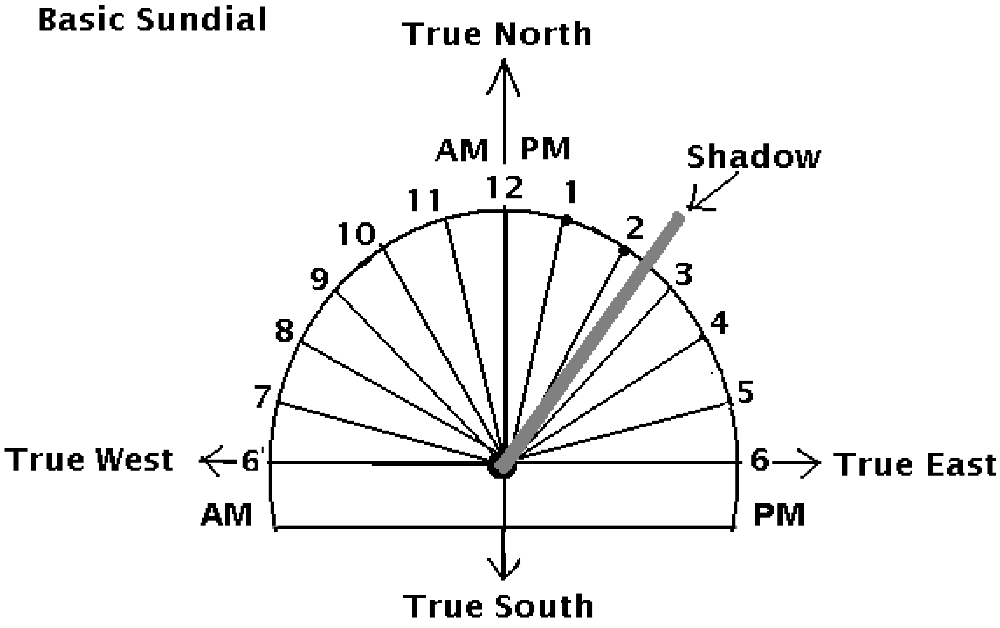
Ex. An event will happen 3 time units from now, and be 1 time unit long.

The event will start at (now + 3) = (0 + 3) = 3 , and end at:
(start time) + (duration time) = (start time) + (change in time) = (ending time) = (event time) + 1 = (3 + 1) = 4

If a previous (past, before) event started 7 units before the event at 3 units, the start time of that past event in reference to the current time of t=0 is:

(event time) + (change in time) = 3 + (-7) = 3 -7 = -4 = 4 time units ago from the current reference time of 0

The Earth revolves or orbits around the Sun at about 1 degree of arc of a circular orbit per day. Every day, it also appears that the Sun revolves or orbits the Earth, and this is due to that the Earth is rotating (from the westward to eastward direction at its surface) on its (true, geographic North to South) polar axis. As the apparent angle between the ground (horizontal) and the Sun changes throughout the day, it will create a shadow from an object such as a vertical rod or stick, and this shadow will likewise change throughout the day as the Sun appears to move from its current position. Since the Sun appears to change position by 15° per hour, the corresponding shadow of the rod will likewise appear to change by 15° per hour, and a basic sunclock, solar-clock or commonly called a **sundial** can be created from this concept. This vertical rod is perpendicular (ie., 90°) to the local, flat surface level or horizontal sundial scale where the numbered hour lines are placed. From where the center of the vertical rod or stick is, draw radius or "radial" (like the spokes of a wheel) lines extending from it that are 15° apart about it, and these lines will be used as hour lines of the sundial of which the current hour is indicated by the shadow of the rod. This has described a basic sundial and is very simple, practical and useful for a large geographical area, particularly several hundred miles radius of your geographical location, such as your latitude and longitude on Earth, and since the apparent Sun angle will be about the same value in this region or area. A basic sundial can also keep a standardized time for a group of people that are also using the same basic sundial design. The time at all geographical points along a (north to south) longitude line of the Earth is the same. For the initial calibration of this basic sundial, the hour lines or planar dial or scale of this sundial is placed on level or horizontal ground. Then, the 0 or 12pm hour line of this sundial must be aligned with the true north and not the (unstable in location) magnetic-north direction line of the Earth **unless agreed upon** to use, such as during the daylight when the stars cannot be seen. For a person in the Norther hemisphere, true North is usually considered in the horizontal direction to the North Star which is also known as Polaris. People living south of the solar plane are blocked from viewing the north star by Earth's local horizon surface and-or land being in the way of view. **The angle to the star Polaris is equal to the latitude of the observers location**, and this angle can be used in the construction of some types of sundials. The 6pm hour line will then be automatically aligned in the true south direction of the Earth. The sundial will only function during the local 12 daytime or daylight hours of the day; hence from 6AM to 6PM, with noon (12) between these two time values, and if the local sunlight is not blocked by clouds, hills or mountains. Again, for the sundial as shown below, **the rod at the center must be vertical with respect to the horizontal and-or local horizon**, and the numbers and lines are placed horizontally about this rod and on the ground or a stand. [FIG 221]



The shadow is indicating the time is about 2:30pm

On the spring (~ March 21) and fall (~ September 21) equinox day of the year, the Sun will appear to "rise" at the distant horizon in the Eastern direction, and the shadow of the vertical pole on the sundial will indicate 6AM in the morning. If you were at the equator, the Sun will appear to travel exactly westward to eastward, and will be directly overhead and the highest in the sky for the day at noon (12pm) on the two **equinox** ("equal day and night") days of the year. At local noon (12PM), the shadow created will have its smallest length depending on the sundials latitude. Low latitudes will have the shortest shadow seen. At solar or Sun noon, the Sun is directly in line with your local longitude, and the shadow of the pole will be the shortest length throughout the entire day, and it will point to the true north direction if you live in the northern hemisphere of Earth. At the equinox days of the year, the equator is in line with the solar (ie. Sun) plane of Earth's orbit around the Sun, and the day time will be 12 hours long, and the night time will be equivalent at 12 hours long. Considering up to an hour before sunrise, and-or up to an hour after sunset, there is often enough light to see.

Another type of sundial has its pole tilted at an angle equal to your local latitude. The top edge of this structure is in line to the star Polaris. This discussion in this book about sundials and the displayed time is for an ideal sundial, such as if placed on Earth where the solar plane is currently at, and informally always considered at the equator, and where the hour lines of the sundial are separated by 15°, but there are more precise equations for these hour line positions and which depend on your latitude from the equator. The more your latitude, the greater your tilt away from the solar plane and-or Sun, and the longer the resulting shadow created. There is much literature, books, clubs or "societies" devoted to sundials and clocks. For simplicity, a vertical stick in the ground, and 15° hour lines is acceptable so as to have a common time within the same region or area on Earth.

If the weather is cloudy, and-or at nighttime, a water-clock or sand-clock can be used. It is even very possible to tell the time by the position of the stars throughout the night as they (apparently) revolve about the north star (Polaris). After each hour of time, the start will appear to have revolved about Polaris by an angle of 15°.

The current or present numeric time of day is actually the duration from the reference or start of the day considered as 0 hours, 0 minutes, and 0 seconds which is at midnight (typically indicated as "12am" [but is actually 0am] on a typical time measurement device or clock) where the Sun is directly on the opposite side of the Earth. At you local noon (12pm or actually 0 pm), the Sun will appear to be at the highest angle (with respect to your horizontal or ground level at your location) in the sky for the current day of the year. Because of the apparent tilt of the Earth throughout the year, the apparent Sun angle angle in the sky will slowly change (~ +-23.5°) in a cyclic, periodic or repeating manner. The maximum height and-or apparent angle value of the Sun throughout the year depends on your latitude angle or position from the reference 0° Equator line.

The Earth spins (rotates) on or about it's (north to south, central polar) axis in 1 day (or 24 hours) of time. Each hour, the Earth rotates on it's axis about 15 degrees per hour. This gives us the illusion that the Sun, Moon and stars are going around the Earth at about 15 degrees per hour. (24 hours) x (15 degrees/hour) = 360 degrees, a full rotation.

The Earth will go around or orbit the sun after 365 days of time = 1 year. (A 360° circle or rotation is based on this value.) More precisely, the Earth orbits the Sun in about 365.256 days = ~ 365.25 = 365 and 1/4 of a day. 1/4 of a day is 24h/4 = 6h. After 4 years this unaccounted for 1/4 of a day becomes a full day, and a day (called "leap (year) day") is added into the year as February 29 to account for this unaccounted for 1/4 day each year. In short, it takes slightly longer than 365 days for Earth to go around the Sun, and "New Years" day is celebrated 6 hours earlier than when Earth actually completes the full orbit around the Sun, and an adjustment must be, and is eventually made as February 29, an extra day placed in the year. 365.256 is the true sidereal or ("about the sun") year time.

Each day of time is divided so as as having 24 hours of time. 1 hour = 1 day / 24 = (1/24) day = about 0.0417 day = about 4% of a day of time. Each hour has 60 minutes of time. hence 1 minute is (1/60) of 1 hour. The word "minute" means a small part or fraction. Each minute has 60 seconds of time, hence each 1 second is (1/60) of 1 minute. The word "second" essentially means a "second minute" part, which means a minute part of another minute part, and this is essentially a fraction of a fraction of an hour: A second is a fraction of a minute which is a fraction of an hour.

From 60s = 1m , mathematically:

1 second = (1 minute) / 60 = (1 hour / 60) / 60 = (1/3600) hour = 1s = $0.000277\underline{8}$ hr

From 60m = 1h , after dividing both sides by 60: 1m = (1/60)h = $0.01666\underline{7}$ h

1 Day = 24 hours = 1440 minutes = 86400 seconds. 1 Year = 365.25 Days = 31,557,600 seconds

24 hr / 1 day = 1 hr / (1 / 24) day = 1 hr / $0.041666\underline{7}$ day , that is 1 hr = $0.041666\underline{7}$ day =~ 4.2% of a day

1 day / 86400 s , dividing num. and den. by 86400: = 0.000011574 day / 1 s , that is 1s = 0.000011574 day

Years past present and of the future are often expressed in reference to the yearly calendar or date system based on the year of the birth of the famous spiritual teacher and social influencer named **Jesus** (the English transliteration of the Jewish (ie., ethnic race) name of **Yeshua** , and of which many have transliterated it as **Jesus,** and the Muslim faith calls him **Isa**). Jesus from Nazareth, Israel, and who is also called by many as the Messiah, God's Prophet, Son Of Man (Mankind), Christ and the Light to the world - much like a new beginning offered to mankind from the God of the universe. Jesus and many of the unique events surrounding him are believed to have been prophesied (ie., spoken and then written, predicted by other prophets) many years before his birth. That year is considered as the modern starting or reference year of 1 or "1 A.D." (in the Latin language: "A.D." = Anno Domini = "In the year of the Lord", invented in 525AD in eastern Europe). He lived in the city of Nazareth of Israel after being born in the city of Bethlehem in Israel about 2,025 years ago from the current year (about 2025A.D) of this book writing. Years before this are indicated as: year-value BC, and BC means "(in the year) Before Christ". Ex. 1 BC is in the first year or 1 year previous or before 1 AD, the "first year" of this date system. There is not a defined year of 0, and since 0 or "nothing" was apparently an odd concept at those times. 2 BC means in the second year or roughly 2 years before 1 AD. 10,000 BC is roughly 10,000 years before 1 AD. The "first century" is technically the first one-hundred years after 1AD, hence it consists of the years of and between 1AD and 99AD. Things which happened in the "second century" happened between the years 100AD and 199AD. 1900 or 1900AD is roughly 1900 years since or after 1AD. The year 2020AD is considered as a year in the 21st century which started at year 2000. Some may consider AD as meaning "after death, of Jesus", but this is technically incorrect, and many have estimated Jesus left the Earth between 30AD and 36AD. Related to the abbreviation and meaning of AD is CE which means "in the **C**ommon **E**ra" (ie., of the common or current times). BCE means "before the Common Era" = BC also. Who made the CE dating system? It was **Dionysius Exiguus,** a Romanium Monk, who introduced it in 525AD, but it took a few centuries for it to become commonly used. He calculated that 1 AD as being 753 years after Rome was founded, and that he was then in the CE year of 525AD, and that year then being 1278 years after Rome was founded. These values have since been accepted worldwide as being fair.

A Year or 365 days is typically divided into 12 month (named after the word Moon) units of time, with each month being (1/12) of a year which is about (365/12) =~ 30 days of time duration or length, and this amount roughly corresponding to the length of one Moon cycle or orbit around Earth. Each month has 4 units of time called a week = (1/4) of a month unit. 1 year = 52 weeks. Each week has a 7 day cycle with each new day called: Sunday (Day of the Sun, Sun-day), Monday (Moon-day), Tuesday (Mars-day), Wednesday (Woden's-day or Mercury-day), Thursday (Thor's-day or Thunder-day, Jupiter's-day), Friday (Frigga's day, Freya's day or Venus-day), Saturday (Saturn-day). These names are generally of ancient Roman and-or northern-European origin, often in reference to some god that was once considered.

MONTH OF THE YEAR	**DAYS IN THE MONTH**	**DESCRIPTON OF THE MONTH**
1. January	31	[named after Janus, a Roman god for times and changes, February is from "Februa" = purification]
2. February	28	[a "**leap-year**", ex. 2020, is a time correction every 4 years, and February will be 29 days long]
3. March	31	[1st month in Roman calendar, named after planet ("wandering star") Mars, and a Roman god of War]
4. April	30	[means "second"]
5. May	31	[named after Maia, a Roman fertility goddess, and possibly means major and older]
6. June	30	[named after Juno, a Roman queen god, and possibly means "junior" as meaning younger and youth]
7. July	31	[named after Julius Caesar, a famous Roman leader, born ~100BC, died 44BC, about 56 years old]
8. August	31	[named after Augustus Caesar, first Roman Emperor, born 63BC, died 14AD, son of Julius Caesar]
9. September	30	[name derived from "seven", since it is the seventh Roman month from the start of March]
10. October	31	[name derived from "eight"]
11. November	30	[name derived from "nine"]
12. December	31	[name derived from "ten" or "deca" , "decimal" means pertaining to 10]

The etymology or word origin of the words "Moon" and "month" are generally based upon each other. A lune is an ancient word for a crescent (ie., crest, having a peak) shape, such as that made by two circles that are not aligned. The Moon is sometimes called as Luna. Things pertaining to the Moon are sometimes noted using the word "lunar". The word "lute" for a string musical instrument, often having a half-round-like shape, seems to be based on the word "lune" and "flute".

Typical or common time indicating, notation or expressing formats:

Date (within the Year)
month / day / year :in U.S.A
day / month / year :in Europe, etc.

Time (within the day)
hours **:** minutes **:** seconds
[Standard or Civilian Time uses two 12 hour periods: 1 to 12 for hours.
Military Time uses numbers 0 to 23 for indicating the hour,]

"Time duration", "time length" or "time change or= the change in time" is the difference between two time points or measurements. If a process started at 10s, and ended at 30s, the time duration or length of that process is:

time duration = ending time - starting time and for this example:
20s = 30s - 10s

Typically, on many "count up" (duration) timers or manually, the starting time is considered as 0s.

If a new process is said to take twice (2) as "long" as another, its time length, duration, or simply time, will be twice as much or "twice as long":

The new process, length, or duration time can be expressed with multiplication as:

old duration x (time or duration factor) = new duration , or simply:
old duration x factor = new duration or:
(old time) x (factor) = new time which can be expressed as:
new time = (factor)(old time) : a linear equation , for ex:
new process time = 2 (old time of process) , for example:
10 minutes = 2 (5 minutes)

Ex. If a new process is said to take only half (1/2 = 0.5) as long as another, its time length, duration, or simply time, will be half as long, length or duration as of the reference time value given, and is said to be twice (2) as fast, hence it will be a less or shorter time duration. The new process, length, or duration time will essentially be expressed as a division of the old duration time. The time of the new faster or quicker process is now less than, and is only a fraction of the time duration or length as that of the old process:

new time = (factor that is a fraction or multiple of the old time duration) time or:
new time = (factor)(old time) and for the given example:

new time = (1/2) (old time) = 0.5 old time which can be expressed as:

$$\text{new time} = \left(\frac{1}{2}\right)\frac{\text{old time}}{1}$$ which can be expressed as:

$$\text{new time} = \frac{\text{old time}}{2}$$

Here, the new lesser time value of that is a factor of (1/2 = 0.5) of the old time can be considered effectively as a 200% quicker or faster in the speed of getting things done, hence "twice as fast" = "in half the time". Consider this analysis:

From the concepts of: distance = (speed) (time) = 1 complete distance

$$(\text{1 completed process}) = 1 = (\text{effective speed of process}) \times (\text{total time of process})$$

Since their product is 1, these factor values are reciprocals of each other:

$$(\text{effective speed of the process}) = \frac{1}{(\text{total time of process})}$$

$$(\text{total time of process}) = \frac{1}{(\text{effective speed of the process})}$$, for the example:

If the total time of the process is halved, or divided by 2, and which is the same as multiplying it by a factor of (1/2 = 0.5):

$$(\text{effective speed of the process}) = \frac{1}{(0.5)\,(\text{total time of process})} = \left(\frac{1}{0.5}\right)\left(\frac{1}{(\text{total time of process})}\right) = 2\left(\frac{1}{(\text{total time of process})}\right)$$

The effective speed or rate of the process so as to have a 1 completed process is shown to be doubled (ie., a factor or 2 more) when the time is halved. For both sides of the above equation to be in balance, the (effective speed of the process) must also be multiplied or increased by a factor 2.

Ex. If the time duration, or simply the time, of an event or process is said to be 1.5 times faster:

$$\text{new duration} = \frac{\text{old duration}}{1.5} = (0.667)\,(\text{old duration})$$

: hence only 66.7% of the old time duration which means it's now a less time (duration), hence a faster or quicker process in terms of time length or duration. Consider:

(time of process) (effective or relative speed of the process) = 1 process done
and:

$$(\text{old total time of the process}) = \frac{1}{(\text{effective or relative speed of the process})}$$, for the example:

$$(\text{new total time of the process}) = \frac{1}{1.5\,(\text{effective speed of the process})} = (0.667)\,\frac{1}{(\text{effective speed of the process})}$$

For both sides of the above equation to be in balance, the (effective speed of the process) must also be multiplied or increased by a factor (1/1.5) = 0.667 . This can be expressed as:

(new time of process) = 0.667 (old time of process)

In short, a process is said to be (x) times faster or quicker, the amount of time needed for that process is a fraction of (1/x) that amount of time. If a process is said to be (x) times longer, the amount of time needed for that process will a factor of (x) times more.

For a **mechanical clock** that uses physical force to move (ie., rotate) the gears so as to rotate the attached time indicators (ie. "hands", or pointers), the passage of time is indicated on a circular disk surface. For the hour hand, pointer or indicator, each quarter hour = 60minutes/4 = 15minutes, corresponds to a 90° of rotation. For the minutes hand or indicator (or "pointer"), and-or the seconds hand or indicator, each 15 minutes or 15 seconds, respectively, of time passage corresponds to 90° of rotation of the corresponding hand or indicator. This amount is one-quarter or 1/4 of a full circle and-or rotation (ie., turning). 15 is 1/4 of 60. 60/15 = 4. 15min/60min = 0.25 = 1/4 , 15sec/60sec = 0.25 = 1/4. and 360° / 4 = (360°)(1/4) = (360°)(0.25) = 90°. Each 1 second and-or minute corresponds to: 360°/60 = 6° Each hour of time corresponds to 30 degrees of pointer rotation: 12hours / 1 rotation = 12 hours / 360° = 1 hour / 30°. If the hour hand corresponded to 24 hours (a full day) instead of 12 hours (a half of a day), each hour would then correspond to 15 degrees of rotation per hour. Extra: The rotation of the Earth on or about its polar axis, and-or the apparent movement of the stars across the sky, is also 15° / 1 hour.

> A mechanical clock is much like a mechanical counter with a numeric base of 60. When the seconds hand rotates once after 60 seconds of time has elapsed, the minute hand is incremented or increased by 1. When the minutes hand rotates once after 60 minutes of time has elapsed, the hours hand is incremented by 1. When the hours hand rotates 24 times, a day of time has passed.   When 60 units of seconds and-or minutes have been expressed on the clock, the values "rollover", "back to" to 0, and to "start over", advancing and-or incrementing again.

In the summer months, the northern hemisphere is effectively tilted toward the Sun during the day, and it will receive more daylight time, and shorter night time. During this same time of year, the southern hemisphere is effectively tilted away or back from the Sun during the day, and it will receive less daylight time, and will be in its winter season.

ANGLES AND ROTATION:

Degrees and radians are units of angles and-or rotation. Degrees are man made practical units made before the concepts of radians was known, whereas radians are units derived from the natural and true concepts of circles and rotation. Degree units are still commonly used everywhere for practical, non-scientific uses.

1 degree is a fraction of a full rotation that is 360 degrees about a point. 1° / 360° = 0.002777...
1° = (1 part of a full rotation / total parts of a full rotation) = (1 part / 360 parts) = 0.002777... parts of a rotation
1 degree = 0.0174532925199433... radians

1 radian = 57.29577951308231... degrees : *

> From C = 2(pi)r =~ **6.2832** r , hence:
> The radius can go about the circumference about 6.2832 times: C / r = 2(pi) =~ 6.2832
> Total angle in degrees about a point or circle = 360° : man made
> Total angle in radians about a point or circle = 2(pi) =~ **6.2832**...
>
> $$\frac{360^\circ}{(6.2832 = 2(\text{pi}))\text{rads}} =\sim \frac{57.3^\circ}{1\text{ rad}} \quad : *$$

A **steradian** is a "solid radian angle" shape that is naturally based, and is a radian angle of 1 rad rotated about its center axis, like a (3-D) cone. Energy or "(energy) rays - straight, line like outward travel" transmitted from an ideal (spherica-like) infinitely small point source can be thought to have this type of cone shape as it travels away from that location and outward and into a larger circular area.

360 degrees = 2(pi) radians = 6.283185307... rads = 1 revolution or rotation = 1 rev

In some astronomy and navigation angle measurements: 1° Degree =or has 60' Minutes , and 1 Minute =or has 60" Seconds. DMS is the Degrees, Minutes, and Seconds, system. Here, a minute ('), or arcminute, is an (arc) angle that is a fraction (1/60) of a degree = $0.0166\underline{7}$° A second ("), or arcsecond, is an angle that is a fraction (1/60) of 1 minute = (1/3600) of a degree = $0.000277\underline{8}$° The value of 60 was chosen for use since it evenly divides 360° up into smaller angles. Arcsecond and arcminute are not associated with time values such as seconds of time, but rather specific angles values. 1 arcdegree, or "1 degree of arc", corresponds to 3600 arcseconds.

Ex: 5° 10' 26" = 5 degrees, 10 minutes, and 26 seconds in the DMS (Degrees, Minutes, Seconds) angular notation system.

The known visual or apparent angular diameter or "width" of the Moon is about 0.5° and this corresponds to 30 arc-minutes of angle. 0.5 degrees / (0.01667 degrees per arc-minute) = 30 arc-minutes. 1 degree of arc. = 60 minutes of arc. Since angle = arctan (tan ¢) = arctan (opp/adj) = arctan (2159miles wide/ 240000miles away) =
~= arctan (0.009) = $0.51\underline{5}$°

From: velocity = distance / time, we have: angular velocity = w = angle / time = ¢ / time = ¢ / s

We must use the "natural value" of radians for the angle in these types of equations, rather than the man made degrees units of which was decided that there should be arbitrarily and non-scientific 360° degrees, angles or parts per circle, and probably since it nicely approximates the (days / year) known in those ancient times.

For the amount of rotation (rather than distance) per second, we have rotational velocity or rotational speed:

From: linear velocity or speed = $\frac{\text{distance}}{\text{time}}$, we have rotational velocity or speed = $\frac{\text{amount of rotation}}{\text{time}}$

angular velocity = angle / time = w = radians / s = rads / s :it is very possible to measure it with revolutions or degrees per second:

1 revolution / s = 1 rev / s = 1 rotation / s = 1 rot / s = (2)(pi) rads / s =~ 6.28 rads / s = 360° / s

If there is a multiple of revolutions (1 rev = 360° = 2pi rads) or revs per second:
w = (radians / s) = ((revs)(2)(pi) radians) / second : angular velocity in radians per second

Rotational speeds:

revolutions per minute = rpm = rev per min = revolutions / minute = (rps)(60)
revolutions per second = rps = rev per sec = revolutions / second = (rpm / 60)

w = ((2)(pi) (rps)) radians / second = (2)(pi)(rpm/60) radians / s or= (pi)(rpm/30) radians / second : w = angular velocity

1 rps = 1 cycle per second = 1 cps =1 hertz = 1 hz = 360° / second = 1 rev / s = (2)(pi) radians / s =~ 6.28 radians / s

hz = cycles per second = c/s = cps = (2)(pi) radians / s : frequency = F is a-the rate of cycles = cps

1/ frequency = time of each cycle, wave, revolution or rotation in seconds , hence:

1 / Fhz = 1 / (cycles / second) = seconds / cycle = s / cycle = time / cycle = Period of cycle = T = P

F = cycles / s = (2)(pi) radians / s = the frequency expressed in terms of revolutions per second or cycles per second, and radians per second = the angular velocity = w

rps = revolutions per second = revs / s = rot / s
rpm = revolutions per minute = revs / min. = revs / 60s = rot / min

1 rps = 1 rev / 1s , and after multiplying num. and den. by 60:
1 rps = 60 rev / 60s = 60 rev / min = **60 rpm** , multiplying both sides by N:
N rps = N 60 rpm

rpm / 60 = rps , Ex. 120 rpm = (120 rpm / 60) = 2 rps , 600 rpm = 600 rpm / 60 = 10 rps

For the linear velocity of a point on a circle:

For a circle, circumference = C **= 2(pi) r =~ 6.28 r** : circumference (C) is a distance , (r) = radius of the circle

From : speed = distance / time = v = d / t , for the speed of an object rotating about the center point of a circle, the distance traveled for each rotation or orbit is equal to c:

$$v = \frac{d}{t} = \frac{C}{t}$$

: the velocity when the distance traveled = 1C = 1 revolution.
: here, v **=** equivalent rotational velocity at point (r) from the center, rather than angular velocity = w , and:

Total distance = (C)(revolutions) = (C)(revs) = 2(pi)(r) (revs) = (6.28 r)(revolutions)

angular velocity = w = $\frac{\text{(amount of rotation)}}{\text{time}} = \frac{\text{¢}}{t}$, for each revolution: ¢ = 6.28 radians , **w =** $\frac{6.28}{t}$ using substitution:

$$v = \frac{d}{t} = \frac{C}{t} = Cf = \frac{6.28\,r}{t} = w\,r$$

: linear velocity of a point on a circle or in a circular orbit, and in terms of the angular or rotational velocity. Also:

$$d = vt = C f t = C \text{ (revolutions / second) (seconds)}$$

Ex. A vehicle wheel has a circumference of C = 2m, and the vehicle is moving at 7m/s, what is the frequency of the rotating wheel? From v = C f , we have: **f = v / C** = (7 m/s) / (2m) = 3.5 (1/s) = 3.5 / s = 3.5 hz = 3.5 cycles / s =
= 3.5 revs / s = 3.5 hz

$$w = \frac{v}{r} = \frac{\text{radians}}{s}$$ **: rotational or angular velocity in terms of linear velocity and the radius of rotation. v = w r , r = v / w**

Converting between degree and radian angular velocities. This essentially resolves to converting between degrees and radians, and since the time value will be the same, such as 1 seconds.

$$w = \frac{n \text{ radians}}{s} = \frac{\frac{n\, 360^\circ}{(2)(\pi)}}{s} = \frac{n\ 57.29578^\circ}{s} = \frac{\text{deg}}{s}$$

$$w = \frac{n \text{ degrees}}{s} = \frac{\frac{n^\circ \text{ rads}}{57.29578^\circ}}{s} = \frac{\text{rads}}{s}$$

distance = d = (speed) (time) **= v t** = (w r)(t) = **wrt** : a point at distance (r) from the center of rotation will travel this (equivalent linear) distance in the given amount of time

Ex. An automobile (ie.,a "car") is traveling at a velocity or speed of 20 miles per hour = v = d / t = 20 mi / 1hr.
Since: 1 mile = 5280 feet and 1 hour = 3600 seconds , for the distance traveled in 1 second:

$$\frac{20 \text{ mi}}{1 \text{ hr}} = \frac{(20\text{mi})(5280/\text{ft/mi})}{(1\text{hr})(3600\text{s/hr})} = \frac{105600 \text{ ft}}{3600\text{s}} = \frac{29.33 \text{ ft}}{1\text{s}}$$, to find how many revolutions per second = rev/s = rps :

From: distance = (velocity)((time) , velocity = speed = distance / time = (d / t)

An angular velocity or rotational velocity of: w = (1 revolution / time) = (6.28 radians / time) = (6.28 / time) ,
and this corresponds to a linear velocity of: v = (circumference / time) = (6.28 r / time) : r = radius

If the wheels of that car have a radius of 1ft: C = 6.28(r) = (6.28)(1ft) = 6.28 ft
1 rev corresponds to a distance of 6.28 ft, and x rev corresponds to 29.33 ft? , using a proportion equation:

$$\frac{1 \text{ rev}}{C} = \frac{1 \text{ rev}}{6.28 \text{ ft}} = \frac{x \text{ rev}}{29.33 \text{ ft}}$$, x rev = (29.33 ft) (1 rev) / 6.28 ft = 4.67 rev , And this corresponds to 4.67 rev / s = 4.67 rps since the distance was expressed per second.

$$\frac{4.67 \text{ rev}}{s} = \frac{(60)\,(4.67 \text{ rev})}{(60)\quad(1\text{s})} = \frac{280.2 \text{ rev}}{60\text{s}} = \frac{280.2 \text{ rev}}{1\text{min}} =\sim 280 \text{ rpm} = 280 \text{ cycles / min}$$

From the above concepts, we can find that 1 mi/hr = $1.466\underline{7}$ ft/s =~ 1.5 ft/s , hence 10 mi/hr is about 14.6ft/s =~ 15 ft/s, and 100 mi/hr is about 147 ft/s =~ 150 ft/s. The average adult human walks at about 1 mi/30min = 2 mi/hr ~= 2.93 ft/s =~ 3 ft/s.

A special note about the actual units of frequency and time in terms of cycles or repetitions:

Frequency and time are mathematically inversely related, and are mathematical reciprocals of each other. In some other words, this could be stated that the number of cycles, repetitions or rotations are inversely related to the time. As the frequency or amount cycles increases, the duration of each cycle decreases:

$1 = \text{(frequency)(time)}$,

$1 = F\,t$, and mathematically:

$F = \frac{1}{t}$ and $t = \frac{1}{F}$: reciprocals, and showing that the units are also reciprocals:

$\text{frequency} = F = \frac{\text{cycles}}{1\text{s}} = c/s$: c/s = hertz

$\text{time} = t = P = T = \text{period of cycle} = \frac{\text{seconds}}{\text{cycle}} = s/c$

If (t) = T = P decreases by (n), the corresponding frequency will increases by that same factor of (n).
If (t) increases by (n), the corresponding frequency will decrease by that same factor of (n).

$$1 = F\,t = 1\,Ft = \frac{(n)}{(n)}\,Ft = (n\,F)\left(\frac{t}{(n)}\right) = \left(\frac{F}{(n)}\right)(nt)$$

Ex. If t = 1s , **F** = 1 / t = (cycles / t) = cycles / 1 s = **cps** : cps = "cycles per second" = hertz = **Hz**

Ex. If F = 1000 hz = 1000 cps , t = seconds / cycle(s) = s / F = 1s / 1000hz = 0.001s = "one milli-second"
F = cps = **c/s** , 1 / F = 1 cps = (1 / (c/s)) = **s / c = t**

Ex. If n = (1/10) = 0.01 = F = 1000hz / n = 1000hz / 10 = 100 hz = 100 cps , t = 1 / F = 1s / 100 hz = 0.01s
We see that if the frequency decreases by 10, the time or period of each cycle increases by 10 which indicates a longer wavelength, and which is actually 10 times longer. If the frequency increases by 10, the time of each cycle or period decreases by 10, and which indicates a shorter wavelength, and which is actually 10 times shorter for this example.

In short, the time or duration of each cycle or period is inversely related to frequency.
Wavelength is inversely related to frequency, hence it is also directly related to the cycle or period time.

Extra: What is "red-shift" in terms of a frequency(s)?

This topic is an advanced topic, and you may skip over it. If you were to walk along and rub a stick against evenly spaced wooden fence boards, and at a constant rate (ex., boards per foot of distance and-or boards per unit of speed and-or time), a certain frequency of striking those boards will be obtained and usually heard. If you were to increase your velocity to a higher value, and-or the fence and its boards were also moving towards you (ie., here for this example, parallel to you), the frequency of striking the boards will increase. This concept is basically called a "shift in frequency", and if the fence or say a distant galaxy was moving away from you as you struck the boards, then the frequency of that would then decrease because it would take a longer time to strike the boards of the fence. A "shift" (ie., change) in frequency is when the light from a galaxy moving away from the observer, and the expected frequency will become shifted into the longer wavelengths, as if the waveform was being stretched longer, and which have a lower frequency. In this instance, it would be called a "**red-shift**". A "blue-shift" or increase in the expected frequency is rare in astronomy, and the Andromeda galaxy is said as having a "blue-shift" since it is coming closer to our Milky-Way galaxy and-or vice-versa. Perhaps in a few million years these two galaxies will collide, however since the space between any two stars is a few **l**ight-**y**ears (here, about **4 ly**) of distance on average, and that all the known stars are relatively very small as compared to this distance, the galaxies will mostly "pass through" each other. The stars in each galaxy will be only be minimally affected by each others gravity and there will be relatively few collisions of stars and-or their planets.

[This space for book edits.]

SOME COMMON MATH FORMULAS

The following formulas and discussions are presented so as the reader may have at least a basic, acceptable or minimal awareness, knowledge and understanding of some of the typical scientific concepts they may possibly encounter at some point in various fields of study and-or in everyday life. For most people, this would be enough to have available and to consider, but these topics when fully explored could easily fill volumes of other books that are more needed for those who have focused on these specific topics or fields of study, experimentation, and work.

distance = (speed) x (time) :The most common equation of these three variables. We also see that to know distance, we need to first know what speed = velocity is. Mathematically:

speed = distance / time :after the division is performed, speed is essentially the rate of an amount of distance units traveled with respect to and during one unit of time . Speed is how fast or how steep, intense or rate" (ie. a rate called "speed"), and its value is defined as how much distance has, is will changed with respect to the change in time. After division and-or simplification of the fraction, this equation essentially reduces to how much distance changed or how much an object traveled in 1 unit of time. Speed (s) is formally called **velocity** (v) in science studies.

Ex. 100miles / 2 hours = 50 miles / hour
Ex. 200 miles / 4 hours = 50 miles / hour , hence the same distance and time rate as the last example. Mathematically:

time = distance / speed : here, time can be mathematically described and-or equal to as the ratio of the change in distance with respect to the amount of speed

Examples of the (mankind made) units of measurement for distance are units of length such as: feet (ft), miles (mi), kilometers (km), etc.

Examples of the units of measurement for time are: seconds (s), minutes (min.), hours (hr), months (mo), years (yr), etc.

Two times values can even be expressed as a ratio, and for example: "days per year" = days/year = = 365.25 days /1 year, and the reverse ratio of this is: 1 year / 365.25 days. Both of these fractions include 1 year unit. Dividing the numerator and denominator values of the first fraction by 365.25, we have these equivalent fractions which include 1 day unit and the equivalent amount of year units: 1 day / 0.00273785 yr = "1 day is to 0.00273785 years" and-or "0.00273785 years is [equivalent] to, or corresponds to 1 day".

Examples of the units of measurement for speed or velocity, and which is a combined unit of distance divided by a unit of time such as: (miles/hour) = "miles per hour" , (feet/second) = "feet per second".

When speed is not an average value, but for a specific point in time during the travel, it is often called the (current or instantaneous) velocity. Distance is a length value.
Units, such as for example: speed = meters/second, or miles per second, are sometimes called "derived units" that are composed of other basic units of measurement.
velocity is another scientific word for speed.

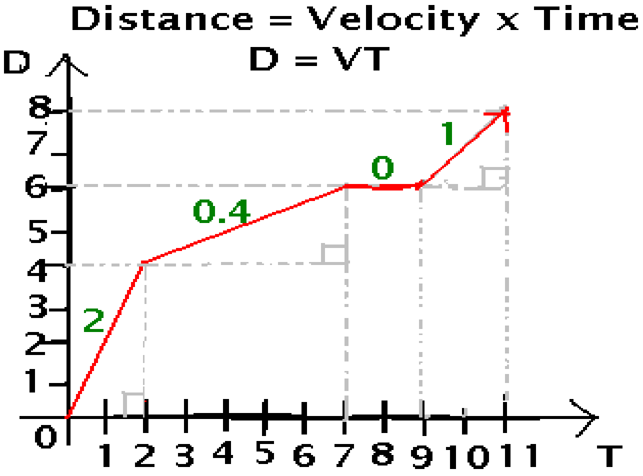
velocity = change in distance / change in time , and if both values started at 0m and 0s, then this formula reduces to the same formula as the speed formula:

velocity = distance / time , and this becomes the same formula for speed

velocity or speed is a ratio or rate of two other values. It is the rate of change in distance with respect to a change in time.

If speed or rate is constant, the distance equation: D = VTs is a linear or line equation of the form: y = mx , or more formally as: y = mx + b where (m) is the rate of change of (y) with respect to (x). (m) is also called the "slope" or steepness value of the line. (b) would be an initial start point or distance already covered when (t) is considered 0.

Below is an example graphing of the distance equation of Distance with respect or in reference to Time. The graph helps visualize the situation and the understanding of it. The velocity or speed is indicated in green next to the current total distance traveled. [FIG 225]



velocity = (change in distance) / (change in time) : the slope of the red line indicates (v)
Ex. From T=2 to T=7. velocity = (6-4) / (7-2) = 2 / 5 = 0.4

First, the equation of D = VT is a linear or line equation of the form y = mx. Clearly in the above graph, the line is not continuous from the start, and the line now becomes line segments that essentially "curves" or changes direction and-or steepness since the velocity of the object at and during those time values has changed.

Ex. This example also shows a deeper understanding of the distance equation, along with a forward example of **acceleration** which is discussed shortly thereafter.

If an object travels at distance of 1 meter after 1 second of time, it is said that the speed or velocity (v) is 1 meter per second of time. v = 1 m/s. The total distance traveled after 10 seconds is:

$$d = v\,t = \left(\frac{1m}{1s}\right)\left(\frac{10s}{1}\right) = 10m \qquad : v = d / t$$

It the total time of travel was 20s, t=20s , the total distance traveled is:

$$d = v\,t = \left(\frac{1m}{1s}\right)\left(\frac{20s}{1}\right) = 20m$$

If an objected is constantly increasing (ie., changing) its speed or velocity by 1 meter of distance per second after each second of time, it is said to have an **acceleration** of (a) = (change in velocity / time) equal to (1 meter per second) more, after each second of time, hence:

$$a = \frac{(\text{change in velocity})}{1s} = \frac{1m/s}{1s} \quad \text{and this is mathematically equal to:} \quad a = \frac{1m}{s^2} = \frac{1m}{(1s)^2} = \frac{1m}{(s)(s)} = (1m / s) / s$$

We also see that the (change in velocity) mathematically equals:

(change in velocity) = (acceleration) (time) = at , and if the initial velocity v1was 0m/s:

(v2 m/s - 0 m/s) = v2 = **v = a t**

If the initial velocity is 0m/s, after 10s, the (accelerating or increasing) velocity will have a value of:

$v = a\,t = \frac{1m}{1s^2}\,\frac{(10s)}{(1)} = \frac{10m}{s}$: here, at v1=0m/s, $a = \frac{v}{t}$

For simplicity of some of these expressions, let the delta symbol of: Δ represent the change in a value:

(change in a value) = (difference in the values) = (value2 - value1) = Δ value

$a = \frac{\text{(change in velocity)}}{\text{(change in time)}} = \frac{(v2 - v1)}{(t2 - t1)} = \frac{\Delta v}{\Delta t}$, If v1 = 0, and t1 = 0, then: $a = \frac{v2}{t2} = \frac{v}{t}$

Δ v = (v2 - v1) , therefore: v2 = v1 + Δv , Note, if there was no change in velocity: Δ v =0, there was no acceleration: a=0, and vice-versa

Δ d = (d2 - d1) , therefore: d2 = d1 + Δd , if d1 = 0 , d2 = Δ d

Δ t = (t2 - t1) , therefore: t2 = t1 + Δt , if t1=0 , t2 = Δ t

Since: d = v t

Δ d = Δ (v t) , to have a change in distance, a change in velocity and-or time is needed:

Δ d = Δ v Δ t = a Δ t Δ t = a (Δ t)^2 : to have a change in (v), a change in (a) is needed, and of which is caused by a change in applied force.

If a velocity, say v2, is 2m/s faster than its previous value of v1, the change in velocity is: Δ v1 = 2m/s: new velocity = old velocity + change in velocity = v2 = v1 + Δ v1 = v1 + 2m/s. If the time of traveling at this new velocity is 3 seconds, the increase in distance that corresponds to, or is due to this change in velocity is:

Δ d = Δ v Δ t

Δ d = (2m/s) (3s) = $\frac{6\text{ m s}}{1s}$ = 6m , also, and as an example and check:

Δ d = a (Δ t)^2 , First, a = Δ v / Δ t = (2m/s) / (3s) = 0.667 m/s^2

Δ d = a (Δ t)^2 = (0.667m/s^2) (3s)^2 = (0.667m/s) (9s^2) = 6m : checks

Some useful mathematical considerations:

d2 = d1 + Δ d1 , therefore: Δd1 = (d2 - d1) = (Δ v1 Δ t1)

d2 = (v1 t1) + (Δ v1 Δ t1) , since v2 = v1 + Δ v1 , Δ v1 = v2 - v1 :

d2 = (v1 t1) + (v2 - v1) (t2 - t1) , for the above example:

d2 = (v1 t1) + (2m/s) (3s)

d2 = (v1 t1) + 6m

dt = total distance = sum of each distance segment, portion or instance of the total distance traveled

dt = d1 + d2 + . . . dn

Each distance, length segment or portion of the total distance has a corresponding velocity and time of travel value associated with it, and needs to be considered in the calculation:

dt = (v1 t1) + (v2 t2) + . . . + (vn tn)

A change in velocity or speed per unit of time is the basic definition of **acceleration.** When velocity changes or increases, it is due to an acceleration that has happened. When velocity decreases, it is due to a deceleration that has happened.

Ex. If more fuel or energy is applied to the engine of a vehicle, more power and force can eventually be applied to the wheels of the vehicle, and it can then accelerate (ie., change, increase) its speed. The vehicle will "go faster". If the fuel is reduced and-or the brakes are applied to the vehicle, the vehicle will decelerate (ie., change, decrease) and its speed will decline. The vehicle will "go slower".

For speed or velocity changes to happen, some force must act upon the object so as to speed it up or slow it down, and-or change its direction. Where the line in the above graph changes direction and-or slope is where there is where a change in velocity happened, hence an acceleration or deceleration has happened. On this graph, the speed changes are abrupt (happen quickly in a short amount of time, such as due to a short pulse or application of force) rather than a more gentle increase and curve. Generally accelerations or decelerations to reach a new speed value happen over a reasonable amount of time.

Ex. If the change in velocity was from 30 feet per second to 40 feet per second, the change in velocity was (40fps - 30fps) = +10fps, and if the change in time for this to happen was 20 seconds, the acceleration could be describe as :

$$\textbf{acceleration} = \frac{\textbf{(change in speed)}}{\textbf{(change in time)}} = \frac{\Delta v}{\Delta t}$$

: acceleration is the rate of change of speed with respect to a change in time

acceleration = +10fps / 20s = +0.5 feet / 1s = +0.5 fps.

Whenever the change in speed is to a lower value, a deceleration has happened and the change in speed will have a negative sign, and therefore the acceleration value would have a negative sign which essentially indicates a negative acceleration = opposite of an acceleration = a deceleration.

Acceleration or deceleration is a measurement for how the speed or velocity of an object changes over or with respect to time (t) due to a force acting upon that object. It is the rate of the change in speed per unit of time. If an object is going very fast but maintains its speed, no forces acted upon that object to change its speed, and therefore, it did not accelerate to a higher or lower speed (velocity). It had 0, or no acceleration.

acceleration = (change in speed) / (change in time) , or after simplifying the fractions:
acceleration = (change in speed) / (1 time unit)

The formula for acceleration is actually be expressed as a rate of another rate that is also with respect to the same variable that is time (T), and because of the definition of acceleration, we can express acceleration in relation to just distance and time, and with this formula for acceleration, it will include a squared variable of time (T)..

Since **speed** or velocity = (D / T), and using substitution into the above formula for acceleration:

acceleration = (change in velocity) / (change in time) , and this reduces to:
= (change in speed) / (unit time) = (change in (D / T)) / T =
= (change in distance) / T^2 = D / T^2

Ex. 3m/s^2 = An acceleration of "3 meters per second squared" = "3 meters per second, per second" = (3m/s) / s . A change in velocity of (3 meters per second) for each second of time considered.

Ex.: If the acceleration was noted or calculated at: "five meters per second squared" = = 5 (m / s^2) , this could be thought of as the objects speed or velocity increased by 5 meters per second of time for (per) each second it was accelerating or increasing in speed. If an objects initial velocity was 100 meters per second of time, and the time of accelerating was 10 seconds, the objects new or total velocity would be:

First note that mathematically from equation for acceleration:

(change in velocity) = (acceleration) (change in time) = AT : a multiplication or product

total velocity = original velocity + change in velocity : all values with respect to time
And:

Vt = Vo + (acceleration)(time) , and:

Vt = 100 m/s + (5 m/1s^2)(10s)
Vt = 100 m/s + 50 m/s combining like terms:
Vt = 150 m/s

Also of note as shown above is:

(acceleration)(time) = (a speed value) when the change in time was from an initial time value considered as 0 seconds, and therefore, the change in time is equal to just the total time value that the acceleration was applied.

A helpful summary of velocity and acceleration:

velocity (v) is the rate of change in distance with respect to, or in reference to the change in time, or in other words it is the time rate of change in distance. v = d / t = Δ distance / Δ time . At a certain point, value or instant in time where the change is very small or "infinitesimally small", velocity is the derivative (ie., like a slope value) of distance with respect to time.

Acceleration (a) is the rate of change in velocity with respect to, or in reference to the change in time, or in other words it is the time rate of change in velocity. a = v / t = Δ velocity / Δ time = Δ d / Δ t / Δ t = Δ d / (Δ t)^2 . At a certain point, value or instant in time where the change is very small or "infinitesimally small", acceleration is the derivative (ie., like a slope value) of velocity with respect to time.

[This space for edits.]

Gravitational Acceleration , The acceleration of an object due to the constant force of gravity.

When the force of Earth's gravity (attraction) is constantly applied to a falling object, it will cause that object to not just change speed once to some new value, but will cause the object to keep increasing in speed since the force is always or constantly applied to it, and is therefore not just a temporary event or pulse of applying the force during a short or quick amount of time. **Force** can be defined as the application or transfer of energy from one thing to another, and the longer a force is applied to an object, the more energy it can gain or store. With the application of force to an object, the object can accelerate and gain speed. The speed an object is part of the total energy that the object is storing and which can be applied and given to some other object. A change in speed requires an acceleration and which requires a force.

Galileo make a public demonstration that two objects of different weight (and mass) rolled down a declining plank or dropped from a tall height, will both reach the end or ground at the same time for each method and-or from other heights. Before this, it was "naturally common and natural" to think that a heavier object would reach the ground first. Galileo demonstrated that there must be some force upon all and various weighted objects, and it must be a constant value. This type of motion due to gravity (ie., gravitational force, the force of gravity and-or mass) alone, is sometimes called "free-fall" or "free-fall motion" since there was no input energy or motor. Galileo found that the distance covered per unit time is mathematically related to the square of the time value. The objects were accelerating (increasing in velocity or speed) and at a constant rate which must be due to a constant force applied to them. **Isaac Newton** later called this force "gravity" since it is in the downward direction, and he later wrote the very useful equation for any applied force in about 1687. Later in this book, there is a article called "**A Note On Newton's Force And Gravity Equations**" which mentions how Newton most likely developed this formula for force:

(force) = (mass) (acceleration) or= **f = ma**. : with units of: $\frac{\text{(kilgrams)(meters)}}{(\ \text{s}^2\ )}$ = **(kg-m / s^2)** = **N**ewtons
(kg) (m per sec.) (per sec)**.**
"per sec" = "for each second"

If you drop an object from a low height onto the ground, it will make a small dent in the ground, but if you increase the height and drop it again, the dent will be deeper. This indicates there was a larger force applied to the ground. Since the object has more time to fall, it gains more and more speed due to the constantly applied force of gravity. A change in speed or velocity is called an acceleration. A constantly applied force will cause an acceleration. The larger the mass and-or weight of the object, the larger or deeper the dent will be. We find that the falling or moving object will apply a force to the ground, and that amount of force is directly related to the mass of the object and the speed it is traveling. When the object impacts (strikes, collides with) the ground, its speed will decrease, and this is called a deceleration. If the ground has a sponge, spring or soft dirt on it, the time to decrease the objects speed of 0 is now longer, and the deceleration (or "negative acceleration") = (- a = change in velocity / change in time) will not be a high such as for a direct impact, but will be a lower value, and this will cause the speed or velocity of the object to slow down just before impact and produce a smaller dent in the ground. It could be said that the object lost some of its energy (ie., ability, force ability) to make a dent. We find that force is now rather, or more "technically", directly related to mass and acceleration since the velocity is constantly changing. Acceleration (or deceleration) considers how much of the objects velocity was actually used (or lost) during the application of the force or transfer of energy such as an impact. Acceleration (a) is the rate of change in velocity: (a) = (v2 -v1) / time. If the change in velocity is small, the acceleration or deceleration will be small, and the applied or impact force and dent will be less. If the change in velocity is high, such as a what happens during a quick (a low value of time) impact, the acceleration or deceleration is very high and the impact force and dent will be larger. The transfer of energy (energy / time = Joules / s = power , watts) is at a higher rate for quicker impacts where the change in velocity is higher, hence the acceleration is higher. When an object slows down, it will loose some of its force or dent ability, and this is technically due to that it lost some of its (kinetic, moving) available energy. A box that is pushed across the floor will slowly loose its kinetic energy due to (surface) friction (ie., a force to overcome) with that floor surface, and its mass of the box will remain the same, but its remaining kinetic energy and remaining velocity value will decrease to 0 because of the (drag) force of friction and transfer of energy (mostly as heat, and some a sound) causing a deceleration of that object. In general, the heavier the box or weight to move across a surface the more the friction force. Smooth, hard and flat surfaces have the low friction.

A derivation in this book shows that: $F = ma = \frac{Energy}{distance}$ or $= \frac{J}{d}$: Energy = Work = Fd : units of Joules = J
Here, d = distance = meters

If a given amount of energy is depleted in a short distance (and time), such as for a strong impact, the value of force will be high. Also from this equation: $a = J / md$ which indicates acceleration is related to the amount of energy applied to a mass. A higher amount of energy can apply a higher amount of force. A higher amount of force can apply more energy. Also from this itself, we have: Energy = J = mad = Fd = Work .
Shown in this book in this book is that: Kinetic Energy = mv^2 / 2 Joules = Fd Joules = mad Joules

A force, such as gravity, that is constantly applied to an object will cause that object to have an acceleration or change in speed and-or direction. An acceleration due to gravity is called gravitational acceleration and has a constant value since the mass of Earth and its gravitational force is constant. Earth's gravitational acceleration is often given the symbol of (g). **(g) = 9.80665 m/s^2 =~ 9.8$\underline{1}$ m/s^2**. Expressing this in the general force equation: f = ma = mg.

The weight of an object is the mass of that object being influence by the force of gravity, causing it to accelerate and effectively creating a force equivalent to the weight of that object: **weight = f = ma = mg**
With this equation, the mass of the object can be found, and the gravitational acceleration:

m = f / a = f / g = weight / g and **a = g = f / m = weight / mass : the higher the acceleration, due to a higher force applied to the mass, the higher the weight and-or the (force per mass).**

Earth's gravity (attraction) force is due to Earth's large amount of mass (material), and it effectively extends outward from its center and causes an ("gravitational" [force]) induced acceleration of: g = 32.$\underline{2}$ft/s^2 = 9.81m/s^2 upon any sized mass or object in a vacuum (no air or other matter, and zero internal pressure), but is nearly the same value for objects in an air medium when the object is smooth and has low air resistance (ie., air collision, "drag" or "drag force" in the opposite direction, reducing its kinetic energy) due to it colliding with air and slowing it down by an effective application of a force in the opposite direction. Another way to consider this is that the object essentially looses some of its energy as it compresses the air beneath it and causing the air to have some pressure or force in the upward direction, and the object will have difficulty gaining any more kinetic energy due to gravity.

velocity = initial velocity + change in velocity
v = vo + v1
v = vo + (g)(t) : see (*) below for a derivation of velocity = (g)(t)
v = vo + (9.8m/s^2)(t) : time (t) will also have units of seconds (s).

(*) When the initial velocity of the object is 0m/s, as when simply released by a persons hand at a height above the ground, and the time is considered as 0s:

acceleration = (change in speed) / (change in time)
a = (final velocity - 0) / (final time - 0) = velocity / time = v / t , therefore, mathematically:
v = at , since we are dealing with Earth's gravitational acceleration (g), this equation can be expressed as:

Ex. If the initial velocity is 0m/s, and an object accelerates at: a = 1m/s , the velocity of the object after 10s is:

$$v = \text{initial velocity} + \text{change in velocity} = 0 + \text{change in velocity} = a\,t = \left(\frac{1m}{s^2}\right)(10s) = \frac{10\ m}{s}$$

v = at = gt : the units of t = time are usually seconds (s), unless otherwise noted.
v = (g)(t) = (9.8m/s^2)(t) = 9.8m / s :this has the form of: velocity = distance / time , from: distance = (speed)(time)

Here is an algebraic example of acceleration that uses the gravitational acceleration (g) constant. Shown is how to find the distance an object has traveled after it has been dropped from a height and after an amount of time has elapsed:

a = acceleration = change in speed / time = $(S1 - S0) / t$ mathematically:

$at = (S1 - S0)$: S0 is the starting speed of the object, hence its is equal to 0 at t=0. S1 is the speed of the object after at a certain time value. If S0=0, the equation can be reduced to:

$v = at = gt = S1$ = current or instantaneous velocity or speed of the object after a certain time has elapsed

distance = (velocity)(time) = $\dfrac{(g\,t)\,t}{(2)} = \dfrac{gt^2}{2}$: $g = 9.8\underline{1}\ \text{m}/1\text{s}^2$, t = time in seconds (s).
: this denominator of 2 is basically due to using the average velocity throughout the distance and where the starting velocity was 0m/s rather than a constant or an average value throughout that distance. In other words, the distance would of been effectively and incorrectly doubled, and therefore must be halved.

These equations for a (unaided, from rest) free falling body are for relatively short distances or heights since air resistance or "drag" (**d**e**g**r**a**din**g** and deceleration force, in the opposite direction) will keep increasing on the object as the more speed it has. The shape of the object, such as a hin flat plate of some material (such as cardboard) can cause significant drag to where this equation can't be used, but perhaps a unique equation that uses some average values measured for a certain object to get only an average result. Drag (force) can be reduced if the cross sectional area of the objected is reduced. A thin, pointed and (directional) stabilized vertical rod shape will have less drag than a wide, flat shape. As a surface moves through the air, it both collides with and compresses the air directly in front of it, and like a spring, it will then have a force in the opposite direction. This force will therefore reduce the speed and-or kinetic energy of the object. The maximum speed or velocity of a certain shaped object is called the objects "terminal velocity". At terminal velocity, motion still continues, but acceleration ceases since the gravitational force is essentially eliminated due to the drag force having the same value in the opposite direction. The object will still remain in constant motion due to its kinetic energy, and keep moving or falling in its (downward) direction due to the force of gravity still affecting the motion of the object.

It is of note that the force due to gravity is similar to that of a magnetic field and electric field where the strength of the field (g=a) and or gravity exponentially gets weaker as the distance from the source increases, and is in fact, related to the inverse square law. In a pseudo equation: (force2 at distance2) = force1 / (distance1^2 from force1)

Some other related equations to consider:

$f = ma = m\,\dfrac{(\text{change in } v)}{(\text{change in } t)}$, If v0 = 0, and t0=0: $\mathbf{f} = ma = \dfrac{mv}{t} = \dfrac{m\,(d/t)}{t} = \mathbf{\dfrac{md}{t^2}}$

$\mathbf{d = vt = \dfrac{f\,t^2}{m}}$: the distance an object will move after a constantly applied force will apply and transfer (kinetic) energy to it and accelerate it and increase its velocity. If the force is applied in outer space, the objects distance will continue to increase at the same velocity, and without any more force applied. Since d and m are mathematically inversely related, if m gets smaller, the distance the object can be moved given the same value of force. From the above equation, we also have many other equations to explore:

$\mathbf{\dfrac{d}{t^2} = \dfrac{f}{m} = a}$ **:** since $f = ma$, $a = f / m$

$\mathbf{d = v\,t = (a\,t)\,t = a\,t^2 = (f / m)\,t^2}$ as shown above : $t^2 = d / a$, therefore: $t = \text{sqrt}(d / a)$

$\mathbf{t^2 = \dfrac{dm}{f}}$ **, and:**

$$t = \sqrt{\frac{dm}{f}} = \sqrt{\frac{d}{a}} = \frac{d}{v} \quad \text{and} \quad t^2 = \frac{d^2}{v^2} = \frac{dm}{f} \text{ , hence } f = \frac{mv^2}{d} \text{ , hence } \frac{v^2}{d} = a = \frac{d}{t^2}$$

Mathematically: v^2 t^2 = d^2
Taking the square root of both sides: d = v t

For consideration: E = W = f d , f = E / d
a = f / m E / dm , E / f = dm / m = d = v t
E / f = KE / f = mv^2 / 2 f = d = v t

force = (mass) x (acceleration) Force is the application of energy to an object. Energy used up, transferred or applied is called and measured a work or "work done". For the units of measurement of force:

F = (mass)(acceleration) , having units of: (kilograms)(meters/s^2) = Newtons = N
1N of constantly applied force could be thought of as that needed to accelerate 1 kilogram at 1meter per second, for each second that force is applied. If you know an objects acceleration during a certain amount of time, you can find its velocity at a certain point or instant in that time using: v = at

work = a force applied for a distance, its units of measurement are equal to that of the units of energy which is Joules (J) since this is how much total energy was used up, applied and-or transferred.

work = (force)(distance) = (mass)(acceleration)(distance) =
= (ma)(Meters) Having units of: (kilograms)(meters/s^2)(meters) =
= (kilograms)(meters^2) / (s^2) = kg (m/s)^2 = kg (v^2) joules

The rate of using energy or doing work is called power (P), and its units are watts (W):

power = **Pw = work / time = energy / time**.
For the units of power: **W = watts = joules / second**

For example, consider that a battery may have a lot of available energy that was slowly accumulated over some time value, and stored within it, and if only a small amount of it is used during each unit of time, the power value and-or energy usage rate will be a low value. For example: 5 watts means 5 watts of power used per second = 5 joules of energy used or drawn from that energy supply (ie., an electric battery) per second = 5W = 5J/s.

Due to an applied force, an object will then change speed which is called **acceleration**. (accelerate = change from one speed to another) and-or direction. If the object is caused to go faster, it has gained (**kinetic=movement**) energy which is an ability to do work such as applying a force to something else. It is possible to define energy as stored force. For example if you compress a spring with your hand, it will store the force used to compress it, and you can feel an equal and opposite (directional) force or pressure applied to your hand as the spring tries to release it's new stored and-or **potential energy** (ie., potential or ability to do work). Another example is if you stretch something elastic such as an elastic cord or rubber-band, there will be both a stretching force outward, and an compression force inward of equal value.

As another example, a rechargeable battery can be charged by the force of falling water at a distant hydroelectric generating station. The battery with stored energy is also said to have "(electric, charge) potential (potent, future ability, available to be applied) energy". Since water has a much higher density than air, moving or flowing water at the same velocity as that of air will contain or have much more moving matter, and the total kinetic energy and-or or power per second (ie. watts) will be much greater. Water is about 816 times denser than air. The density of water is typically: 1g/cm^3, and the density of air at sea-level is typically 0.001225g/cm^3. Density of water / density of air = (1g/cm^3) / (0.001225g/cm^3) =~ 816. In general, a hydro-electric generator will produce more power or electricity than a wind-generator when the velocity of the water and wind are the same. A high dam can be used so as to increase the velocity, energy and resulting force of the water striking the turbine of the electricity generator and producing a torque (rotation force) so as to rotate it and the rotor of the generator connected to it.

Force can be thought of as like a **pressure**, influence, ability, cause or driver upon something else. An object in outer-space will keep it's speed and direction if there is no forces (like a collision and-or friction with another object(s), etc.) causing it to loose, or gain (kinetic) energy and cause it to accelerate and change it's speed and-or direction. A "push" (away), or a "pull" (toward) is applying a force and in a certain direction so as to give a motion or movement to an object so as it will go (move, or to move faster) in that direction. An object can also be compressed if forces are applied to it from opposite (ie., 180°) directions, much like a clamp. An example is a compressed spring which is now storing the applied forces in the springs newly bent, twisted or tensioned atomic and-or crystalline structures, and the spring now has (stored) potential energy that can be transferred and applied as a force to something when needed.

The basic formula for force as indicated previously above, is: f=ma , and this formula is credited to **Issac Newton** at around the year 1667AD.

The units of force are the units of mass times the units of acceleration, hence (kilograms)(m/s^2) and are usually called Newtons (N). f = (m a) Newtons

A force of: 1N = $0.2248\underline{1}$ lbs =~ 0.225lbs of weight : roughly, a quarter of a pound
A force of: 1lb = 4.4482 N =~ 4.45N of force : roughly, four and a half Newtons

Also, by the original equation: acceleration = force / mass
: = "the force applied per unit of mass" : per = "for each"
and:
mass = force / acceleration

Since: force = (mass)(acceleration) = f = ma

a = f / m , If a constant force of 1N is applied to a mass of 1kg for 1 second, it will accelerate or change its velocity or speed by 1m/s^2 = 1 meter per second, per second
a = f / m = (ma) / m = 1N / m = (1kg)(m/s^2) / 1 kg = 1m/s^2

Since: force = (mass)(acceleration) = f = ma, 1kg of mass will weigh (ie. have a force): weight = f = ma = (1kg)(9.8 m/s^2) = 9.8N, or roughly 10N, on Earth where g = 9.8m/s^2. (g) on the Moon is smaller since the Moon has less mass than the Earth, and therefore it will have a less gravitational force on objects or masses. The **mass of the Moon** is about (7.35)(10^22)kg and this is about (1/8) that of the Earth, but it is also condensed into a smaller sphere or radius than the Earth, and its gravity is not actually (1/8 = 0.125) of that of the Earth, but a little higher at (1/6 = $0.166\underline{7}$). (g or gm) on the surface of the Moon (m) is about (1/6) that of the Earth (e), therefore:

(g)moon =~ ge/6 = (9.81m/s^2)/6 = 1.635m/s^2. A 1kg mass on the surface of the Moon will weigh: f=ma = (1kg)(1.635m/s^2) = 1.635N. A 1kg mass on the surface of the Earth will weight six times more at 9.8N. (g)mars = 3.72 m/s^2 which is about 37.9% as that of Earth. Lower gravity also means a lower amount of rocket engine force needed and therefore, a lower amount of rocket fuel needed to overcome the local gravity.

The weight units may commonly (though incorrectly, since weight and mass are different concepts) be given the units of mass (here, kilograms) when the weight value is understood to be a force (Newtons) or weight value. A 1kg mass may be said as weighing 1.633kg on the Moon, and as weighing 9.8kg on the Earth due to the effects of gravitational acceleration (g), and that weight, a force = f = ma = mg. Still, a force of 1N is the same on both the Earth and the Moon. Basically, weight is like an effective or enhanced force due to gravity acting upon the object or mass. A mass and its value will not change, and is "universal", and will be the same value anyplace in the universe.

Extra: M earth / M moon =~ 6 (10^24) **/** 7.35 (10^22) =~ 0.81 (10^2) =~ **81** times more, but only about 6 times more gravity than the Moon. For a general comparison, consider 1 cube, and if its sides (L,W,H) doubled (2), the cube becomes 8 times as much mass and-or volume, hence an exponential and-or cubic (here, n^3) increase when the linear dimensions increase. Given the Earth is 4 times wider than the moon, 4^3 = 64 for starters, however, the Earth also has a much larger metal core with a high mass, and hence why its actually closer to 81 times as more. Using Vsphere =~ 4.1889...(r^3), the ratio of the volume of Earth to the volume of the Moon can be shown to be about **50** times more.

A force applied to an object (mass) will continue to give it acceleration until that force is released or removed to 0 applied force. The amount of force that was applied to the object can therefore be indirectly measured by the effects of it after its application to the mass and the change in velocity or acceleration that the mass had. The measure of the force applied is directly related to both the mass and acceleration of it. f=ma. That is, acceleration of the mass is rather directly related to the force applied and indirectly related to the mass: a = f / m. It is also more difficult to accelerate (or decelerate) a larger mass, and to give it more kinetic energy. As can be seen in the formula, mass is indirectly related to acceleration and vice-versa. Given a fixed and constant value of force applied to a mass, if the mass is larger, the acceleration is then smaller.

We found that the value of acceleration is inversely related to mass, and that the smaller the mass given a certain force, the greater the acceleration of that mass due to the applied force upon it. In other words, a force will have more effect on a smaller mass. Larger masses are said to have more **inertia** or "resistance" to change in acceleration and hence more of a (natural, property of the mass alone) resistance to a change in speed (ie. motion). It could also be said that larger masses resist change in their total energy, hence it will therefore take a longer time for them to gain or loose their stored energy, and that they have a greater ability to do work as compared to smaller mass. A simple illustrative analogy example would be a large or massive volume or pot of hot water, and a small cup of hot water. The large pot of hot water has a greater heat capacity or "heat-momentum" and will take a longer time to initially gain, such as heat up, and then release its acquired energy into the surroundings and cool off. Another illustrative example of inertia would be like trying to change (increase or decrease) the same depth of water in a container with a large surface area of water as compared to that of a small container with a small surface area of water. In short, more massive or larger things have a greater capacity to store energy, but also have a greater resistance or "slowness" to a change in some value or physical property such as gaining or loosing energy, such as temperature and-or speed (ie. kinetic energy). A larger battery is said to have more "energy capacity" or power ability than a smaller battery of similar construction.

For a "thought example", consider an object traveling in a direction in space at some value of speed, and a faster moving object comes straight behind it and headed in the same direction. The faster moving object will now effectively have some resistance to its free motion in space when it collides with the slower moving object. The faster moving object will essentially cause the slower moving object to increase its speed due to the force of the collision. The faster moving object essentially transfers some if its (kinetic) energy to the slower moving object. The faster moving object will be reduced in speed and (kinetic) energy, and the slower moving object will increase in speed and (kinetic) energy. The total (kinetic) energy, before and after, of this system of two objects is still the same since energy cannot be created or destroyed, and this is a part of the "conservation of energy" concept.

Pressure is a measure of force applied to a unit of area. : It is essentially a measure of the distribution and-or concentration of force, or the force concentration in a given area.

pressure = force / area = $P = F / A$: therfore, force = F = (pressure)(area)

For force, the units for the measurement of mass are kilograms in the metric system. It is still common to hear of "pounds per square inch" (psi) in the older English system.

Ex. If a weight (F) of 100 lbs is applied to an area that is 10 square inches, the pressure applied can also be expressed as: 100lbs / 10 in^2 = 10lbs / 1 in^2 = 10 psi

An object that is sharpened to a point, such as a nail, pin, knife or needle, will have a high pressure or force per unit area available at the small tip area for the given applied (input) force value, and is then able to apply that concentrated (output) force to penetrate surfaces more easily. Essentially, all the applied input force will be concentrated at the point or tip which has a very small area.

Since Pressure = force / area, if the area decreases, the pressure or "pressure force" increases. This is a good example of an inverse physical and mathematical relationship.

Daniel Bernoulli's (1700-1782), from Switzerland, created **Bernoulli's Principle** which he formulated in 1738, and it mathematically states the pressure and speed relationships of fluids and gases, such as moving through a pipe of different diameters and-or cross-sectional area. The basics of his understanding and equation shows that there is an inverse relationship to the speed of a flowing or moving liquid and its pressure, such as within pipes of different diameter. What is happening is that as the speed increases, the effective pressure or force upon the inside surface of the pipe and within the substance itself decreases. Essentially the force or pressure is now being applied at less of an angle, or less directly (ie., perpendicularly) to the wall of the pipe and is now more directed towards the forward motion of the substance in front of it. This helps to push it and give it kinetic energy. As speed increases, more of the input energy is converted to to kinetic energy of the substance rather than static pressure (ie., substance compression or stored potential energy). For a substance to even move in a pipe, there must be a net or combined pressure difference place upon it, and it will naturally move by force to or into a region of lower pressure and at an increased velocity due to the net pressure difference upon it. If you had a tall container of water and put a small hole or tube in the bottom region of it, the pressure (when >14.7psi) in and of the water will cause a stream of water to move out of that hole and into the lower pressure region (here normal atmospheric pressure of 14.7psi). The net or effective pressure upon the water is the difference in the pressure in the water where the hole is, and the pressure of the atmosphere outside that container. Ex. pressure difference = change in pressure = = (20psi at the bottom - 14.7psi external pressure) = 5.3psi. and this amount of effective, difference and-or resulting pressure will determine the velocity of that stream of water.

As for Bernoulli's principle, this is actually the principle happening when faster moving, lower pressure air is over the top surface of an **airplane wing** (see FIG 246), and it effectively reduces the normal downward pressure or force of air on that wing surface (area). Therefore, there is effectively a higher air pressure or force on the opposite side or beneath that wing. The difference in pressure or force leaves an net upward force giving the airplane a upward or vertical "**lift**" force so that it can go higher and-or remain at some height and resist falling due to the constant force of gravity upon the airplane. If the upward "lift" force is greater than the weight (downward force due to gravity) of the plane, the plane can go higher. When the two forces are in balance, the plane will be flying in a "level" or horizontal direction. An airplane can only fly so high due to the thinning atmosphere at higher altitudes, and that there will then be only a tiny pressure difference upon both sides of the wing and only a small to no lift force possible to overcome the weight of the plane and the force of gravity. By increasing the speed of the airplane, the pressure difference on each side of the wing will increase, and even the force of the air striking the bottom wing surface can also help increase the pressure difference upon both sides of the wing so as to lift the airplane further upwards. Even if the pressure difference (Pd) between the upper and lower surface of a larger wing is the same, the effective lift force (here as F) will then be larger since from: $P = Pd = F / A$, we have: $F = Pd\ A$.

If a material can be compressed due to an applied force or pressure, its density will increase. Water, that is effectively a condensed gas due to its atomic structure of hydrogen and oxygen, is said to be nearly incompressible. Air can be compressed to a high compression ratio (pressure / volume), such as often done in a combustion engine for automobiles and other machinery. It is possible to compress air that it will become dense enough to be liquefied. An engine converts one form of energy (ex. gasoline, oil) into mechanical power such as for a car or machine, and is often called a "**motor**" which is that which provides **motional** energy, power and-or motion (ie., movement).

Momentum = **Mass x Velocity = mv** , also then: velocity = momentum / mass

Momentum is a measure of an objects ability to keep moving at the same velocity and direction, hence it is technically a vector quantity having both a magnitude and direction. It is also much **like** a measure of the total (moving or kinetic) potential energy or force available due both the object's mass and its speed. **Momentum can be thought of as a measure of an objects motion** at a given moment in time, and rather than its kinetic energy in Joules. A small particle having a low mass going at high speed can have the same value of momentum as a very large mass going at a slow speed. Also, consider that there is a law of motion stating that an object in motion will continue or keep that speed and direction until some outside force acts upon it, and this could be described as part of the concept of momentum. Momentum is rather a measure of its ability to keep its motion, and the more mass and-or velocity an object has, the more energy needed to alter its direction and-or speed, hence it has more momentum or ability to keep its motion, including if it was initially at rest where it has the same mass, but 0 velocity or speed.

If a first object with more momentum strikes a second object, it applies a force to that second object, and vice versa, since that second object is also then applying some force to that first object, and this is an example of the concept of equal and opposite forces. Some of the energy or momentum of that first object will be transferred to and gained in that second object with less momentum. Basically, what is happening is that the object with more momentum (and has more "potential influence" upon another object) is using up, loosing, transferring, or applying some of its KE energy and momentum to a second object which gains that amount of transferred energy or momentum. In particular, since when the masses remain the same, an energy or momentum increase is indicated by a velocity increase in that second object. The first objects mass will still be the same, but it will have a reduced velocity. The total sum of the KE energy or momentum of both masses will remain the same after a collision of two masses or objects.

The more the mass or "massive" an object, the more inertia (resistance to changes in velocity which is a resistance to acceleration) it will have, and therefore, the more difficult (ie., more energy needed) to change its speed (ie., motion).

One easy to understand example of momentum is when water is flowing in a pipe, and a water valve is shut quickly. Before it is shut, that flowing (moving) mass of water had speed. The water is said to have momentum (ie., a certain movement and ability to keep moving). When the valve (such as connected to a control knob to control the water flow into a sink and-or tub) is shut off quickly, the kinetic energy available or stored in the water then needs a place to go and be applied elsewhere, and the water pressure in the pipe and upon the side walls of the pipe will quickly rise, and it will cause the pipe to rattle and-or "knock" like someone hit it with a hammer and to make it move and produce quick pulse of sound. This effect is commonly known as the "water hammer".

It is said that for every force there is an equal and opposite force. The downward force of an object near the Earths surface is actually equal to the weight of the object. As the object sits on the ground or table, the object will be stationary and not moving, and this is because of the surface tension of the Earth prevents that mass from going further downward, even though the force of gravity is constantly applied to it.

A lower mass object has less inertia or resistance to an applied force, and its speed is therefore more affected by the (same) value of the force of attraction of two masses. It is simply easier to move a lesser amount mass; less force and-or energy is required. It could be said that a certain value of force applied per unit of mass is greater when the mass is less. Acceleration is mathematically equal to force per unit of mass and is therefore essentially a measure of the concentration and effectiveness of the applied force, much like how pressure is the force applied per unit area. A concentrated or more local (ie., in a close and-or small .area) force will be more effective at doing local work. The Earth with its much higher mass value will have a much greater inertia (ie., resistance to external forces, and therefore less changes in motion) value than the smaller object. The Earth will still move (ie., be attracted) toward the lower mass object, but the acceleration, change in velocity and distance will be very small and unnoticeable:

force = (mass)(acceleration) , solving for acceleration:

acceleration = force / mass : the lower the (constant) mass affected by the (constant) force, the greater its (constant) acceleration value. Higher acceleration values mean a greater achieved velocity and distance traveled during an amount of time. Mathematically, when the force is a constant value, (mass) and (acceleration) are reciprocal in value.

A momentary pulse or application of force upon an object will give it either pressure and-or a movement having a direction and velocity. If that force is constantly applied to the object it will accelerate and gain more motion or speed. When an objects speed is increased, its distance covered per unit of time will increase: An objects current speed is directly related to current and all other past accelerations due to applied forces, and distance = (speed of travel or motion) (time of travel or motion) = st = vt.

The equation for force = ma , is very similar to that of momentum = mv . A mass with a higher velocity has the capability of applying a higher force. Now consider two masses traveling in the same direction at the same high velocity, and it will still take a change in velocity (ie., an acceleration) for one of the masses to apply even a small force to the other mass and their same velocity (slow or fast) is irrelevant to the amount of force applied due to their "relative motion", therefore force is rather related to acceleration (or deceleration), rather than velocity. Something that accelerates quickly to a speed is initially due to a high force causing a high acceleration. If the speed of an object decreases quickly, such as when it hits the ground, it has a high value of deceleration, and this is due to an equal and opposite force of the Earth upon it. Earth, an object having a high mass and therefore momentum and then being more difficult or resistance to change. If the Earth was able to move due to the impact, some of the kinetic energy of the object would be transferred to Earths kinetic energy or movement or velocity and the dent of the impact would not be as much, indicating that the force of the impact was less and the deceleration was less.

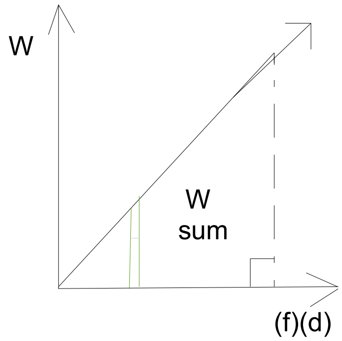
Kinetic Energy = (KE) = $\frac{mv^2}{2}$: Kinetic energy is the (potential) energy an object (mass) has due to its amount of mass and its motion. If the speed (velocity) is doubled, KE increases by a factor of 4. This also means that the input energy must also increase by 4 so as to double the speed.

m = mass of the object, with units of kilograms, v = velocity of the object, with units of $(M/s)^2$ = (meters per second) squared. The units for energy, such as kinetic energy, is joules (J):

1J = 1Nm = newton-meters = (force)(distance) = (ma)(distance) = work = energy

Friction and-or collisions with other objects will cause an object to waste, transfer or loose some of its kinetic energy. It will slow down unless more force is applied to it. Since it will take energy to apply a force, force can be described as the application of energy.

Here is a verification to the kinetic energy formula shown above: [FIG 226]



Work = (force)(distance). A small bit of work is done when there is a small bit of distance the force was applied. The (infinite) sum of all these small bits, slices or rectangles = (force)(very small change in distance) of work equals the total work done, and that can be visualized by the area beneath the curve above, and clearly it is equal to the area of a right triangle having sides of (f) and (d). One theoretical slice or rectangle is shown between the green colored lines. Though this rectangle is slightly longer on one side, when the rectangles gets very small due to the change in distance being very small, this difference also gets very small and negligible or insignificant to the (much larger) total sum.

$$\text{Area} = W = \mathbf{KE} = \frac{(f)(d)}{2} = \frac{(ma)(vt)}{2} = \frac{\left(m\frac{v}{t}\right)(v\,t)}{(2/1)} = \mathbf{\frac{mv^2}{2}}\ \textbf{Joules}$$

If the velocity of an object increases by a factor of (n), its KE increases by a factor of n^2:

$$\frac{m(nv)^2}{2} = \frac{mn^2v^2}{2} = n^2\,\frac{mv^2}{2} = n^2\,(KE)$$

This also indicates that for example, to increase velocity by (n) will require (n^2) times more input energy. For example, to double (2) the speed of an object will require 4 times the input energy. In space, with no friction losses, once a desired speed or velocity has been reached, the energy applied to create a thrust force upon the object is no longer needed, and the object will continue to travel at that faster speed. The object will also have 4 times the kinetic energy as it did previously. To slow the speed of an object will require a force in the opposite direction to its forward travel. A force in the opposite direction will reduce its velocity and available kinetic energy. The object will decelerate (a negative acceleration) as long as the force is applied. The energy and-or force required to stop a slower moving object having less kinetic energy will become lower.

current = **voltage / resistance** Therefore: resistance = voltage / current and voltage = (current) x (resistance) This is commonly known as **Ohm's Law** for electricity since he recognized and then formulated these physical and mathematical relationships into this simple, but very useful formula used in the study and measurement of electricity.

In honor of George Ohm, an **ohm** is the basic unit of resistance. **Resistance** is any device (such as a resistor) or material that will resist (impede, interfere) and therefore reduce the total flow and-or amount of current. Current is a description of "flowing" or moving electrons (microscopic, particles of an atom) of which have momentum, and therefore energy to do work. Electrons have a very tiny mass and their motion and-or direction will be influence by an (attractive or repulsive, polarity) electric field (ie., an electric or "electromotive" force (emf) field, or field of force) which induces (causes, applies) a force upon them, and they will gain momentum. This is very similar to how a magnet with its magnetic field can influence, force and move iron particles that are within that magnetic field (of influence).

A resistor that reduces current flow from within it (which also affects the current flow of the entire circuit or wires to and from that resistor)can basically be thought of as blocking some of that current from passing through it by reducing or removing the momentum and energy of some of the electrons. This can be simply considered as removing some of the energy and-or power of the current. An analogy to this would be like a valve (flow restrictor or adjuster) or some blockage being in a pipe (directs and allows the transfer or conducts the flow of the water fluid) which then reduces the flow and amount of that water to and from that location.

When a resistor removes some of the energy from the flowing electrons, there will be an associated power loss, wasted and-or unavailable energy and-or power to the rest of the circuit. Usually, the energy removed from the current is transformed or converted to thermal or heat energy. Resistors can get quite hot an they need to be designed large enough to withstand that amount of (thermal) energy or power they gained and without being damaged (ie., melted, fire). Resistors are also often designed so as to radiate their heat (thermal energy) into the surrounding air so as that resistor will reduce and-or maintain its temperature to a safe level.

In general, elements that have a lower resistance, due to having more "free", movable, less bound electrons, have both higher electrical (electron) conductivity and higher thermal (heat) transfer or conduction ability. Solid metals such as silver, gold and copper have high electrical and thermal (heat) conductivity as compared or in reference to most other metals. **Graphite** which is a specific form of carbon, and having 1 unbound, relatively free and-or easy move (via an electrical force and-or energy) electron in its crystal structure, and therefore it has good electrical and thermal conduction ability. Diamonds are another form of (crystallized, dense) carbon, are poor conductors of electricity since the crystal structure does not have any free electrons, but are one of the best thermal (heat, energy) conductors.

Aluminum is a good conductor of both electric and heat energy, and is much more economical as compared to the better ability of silver. Although several time more expensive than aluminum, copper with its lower resistance value per foot and-or cross-sectional area, is perhaps the most commonly used metals for conducting electricity (ie., transferring power using electricity (ie. electrons transferring electric power)) and for creating magnetic fields in the wire coils of electric motors, and which the magnetic force is then used to give a (kinetic energy) movement or motion to the rotor (rotating) part of that motor and something such as an electric fan or wheels that are connected to it.

The units of current (the flow of electric charges or particles, such as electrons of atoms which have gained some kinetic energy, such as from a battery, to move and do work) are Coulombs of charge per second or= **Amperes** (A or sometimes (a)). A **Coulomb (C)** of charge (ex. electrons) is a specific amount **(6.28 x 10^18) of charged particles** such as electrons. An amount of 1 Coulomb of charge flowing (like a current) or passing by a point in 1 second is called an (1) ampere in honor of **Andre Ampere** ("amp" = A) who made significant advances in the study of electricity, specifically electro-magnetism, and charge movement or flow. Ampere also invented the (electric to magnetic, or "electromagnetic" movement which is essentially a type of [non-360 degree rotating] electric motor or movement) ammeter (or "amp meter") to measure the amount of current flowing through a wire. This device with its internal coil of wire has similar properties to the later developed (current-to-magnetic [electro-magnetic] movement) device called a loudspeaker which transforms or converts electrical signals or pulses into mechanical audio signals or pulses.

Electro-magnetism , (1820)

Hans Oersted (1777-1851), from Denmark, is credited to initially discovering electromagnetism in 1820 when he noticed a wire with electricity going through it would cause a (magnetic, directional) compass to change its direction away from the magnetic north direction of the Earth. This indicated that electricity in the wire was causing some magnetism in the wire. By using a compass about the wire, it can be deduced by observation that the magnetic field is circular about the wire.

In simple terms, magnetism and-or a magnetic field of a magnet is much like a gravitational field, but a magnetic field is much stronger locally within a relatively short distance, and that a magnetic field has two poles like an electric field, and can be thought of as a starting pole and a ending pole for the (imaginary) lines of force. Studies of the atoms of a magnet have shown that a directional alignment of atoms and their electrons spinning about the nucleus will create a net magnetic field in that direction. Due to the unpaired and-or unbounded valence electrons with the neighboring atoms in the crystal structure of **iron**, iron is well suited for both being a magnet and-or influenced by magnet.

A **permanent magnet** requires no electricity to create a magnetic field. When a piece of iron was made to be hot and in an external magnetic field, and then cooled, the alignment of atoms and their electrons remained fixed or permanent. Passing current through a wire will cause the electrons to be aligned and flowing in the same direction, and this will cause a net magnetic field, and here which is at a right angle to the electric field created by the flowing electrons. A magnet has a **north magnetic pole** and a **south magnetic pole**, and the magnetic forces at those poles will attract the magnetic poles of opposite polarity, and repel the same polarity. A magnet will also attract ferrous (having iron metal) materials in it by inducing that material to become magnetized and then having magnetic poles.

The electro-magnet , (1830)

In 1830, **Johann Schweigger** (1779-1857), from Germany, found that if the wire was placed into a coil shape, that the magnetic field created when a voltage (and resulting current) was applied would be concentrated, and therefore stronger. A device he made is an (electric) galvanometer, but unlike just sensing static electric charges and-or fields, his device could measure the flow of charges or electric current. An electro-magnet is sometimes called a **solenoid**, especially when it is used to cause a movement of a nearby piece of metal.

Magnetic induction or the creation of electricity by using a magnet and wire , (1830) , the creation or generation of electricity without a voltaic battery.

In 1830**, Michael Faraday** discovered magneto-electricity, whereas a magnet **(**either a non-electric permanent-magnet or an electro-magnet made from a coil of wire with a electric current [flow of electric charge going through it]) can induce a voltage and current in a nearby wire and especially a coil of wire where the effect is essentially larger (ie., more produced, generated or induced voltage and-or current) and more noticeable and practical. This effect only occurred when there was a relative motion between the magnet and-or coil of wire. More technically, it is rather a changing magnetic field strength, and which can be due to the relative motion and-or position (ie., distance to or from) between the wire and the magnet. If the relative motion stops, the magnetic field strength is a constant value at the wire, this field is not then changing position so as to not apply a force to the electrons and give them some kinetic energy to move or flow, and the induce current will likewise stop.

In 1821, Faraday invented a primitive device that created a consistent and circular motion due to the applied electricity, hence he created a **primitive type of electric motor**, and it inspired others to experiment with electricity and-or to create motors.

Ampere soon took great interest in Oersted's discovery of electric current creating a magnetic field in a wire, and basically he noticed that the larger the battery voltage, the greater the magnetic field produced, and this indicates that the amount (ie., current value) of electricity in the wire was now larger. Surely more electricity (ie., modernly called current or current flow) in a wire would also make the wire a higher temperature and we know this today as due to the resistance in the wire and causing a power loss in it, and we know this modernly that the kinetic energy of the electrons is converted to heat energy in a resistance (analogous to "electric friction") to the flow of current or electrons.

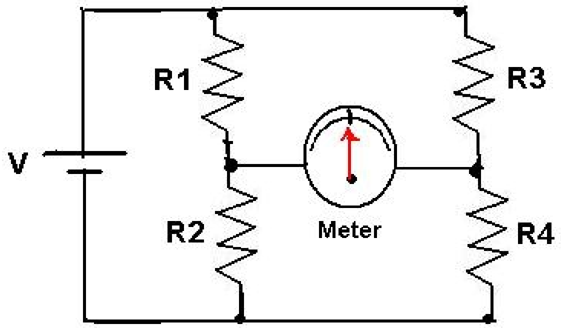
Oersted's Law is that electricity will cause a magnetic field around a wire, and this is electro-magnetism. **Faradays' Law** of electro-magnetic (or more correctly "magneto-electricity") induction is essentially the inverse of Oersted's Law.

In, 1827, **George Ohm** (1789-1854), from Germany, expressed the mathematical relationships of the electrical and-or physical relationships between voltage (V), current (A or I), and resistance (R), and created his famous equation for electricity called **Ohm's Law**, and which is presented in this book. This law relates how those variables and values affect each other, and it can be used to solve for an unknown value when given the two other values.

In 1834 **Heinrich Lenz** (1804 - 1865), from Russia , noticed that the current induced in a wire by a (changing in strength) magnetic field will flow in a direction so as to create a magnetic field of opposite polarity and-or direction, and this will then affect, oppose and-or or repel that which caused it, therefore also including motion. This is called **Lenz's Law**.

Charles Wheatstone is given most of the credit to an electric circuit for a device that can measure resistances, however an ammeter can also be used to determine a resistance when considering Ohm's Law and using: R = V / I . The circuit is called a **Wheatstone Bridge circuit** in 1843**.** See the figure below. This circuit was first initially invented by **Samuel Christie** in 1833, and it is essentially two voltage dividors (R1 and R2, R3 and R4) circuits in parallel with (R1/R2) = (R3/R4) equal in value when (voltage) "balanced" (ie., no voltage difference and-or remaining voltage potential), and with one resistor replaced with a sensor or to be inparallel to one. Any resistance "unbalance" will cause a voltage unbalance or difference and a current to then flow through the central bridge between these two voltage dividors. This current can be measured or put through a resistor and sensed as a voltage. It was also useful for telegraph communication such as for using uncoded (non-Morse coded with "dot" and "dash" signals) letter signals to be indicated on a (letter) pointer (or indicator needle) and readout display (visual) system. The basic circuit bridge circuit to measure a resistance did it by balancing the galvanometer readout pointer by using a resistance with an equivalent and known value also placed into the circuit. The equivalent and known resistance may also be a variable resistor (or "rheostat" [adjustable and-or stationary]) or resistance that is generally given credit to **Moritz Jacobi** (1801-1874) from Russo-Germany (Prussia) and who also invented **one of the first useful (electro-magnetic) electric motors** in 1834. If a third terminal is used between the end terminals of a variable resistor, it is called a **potentiometer** (ie. adjustable the voltage potential, or voltage adjustment device) or (variable) voltage divider that divides a voltage into two (same or different) values, and of which either or both can be used for some purpose. [FIG 226B]

A basic, Wheatstone Bridge Circuit.



A diode bridge circuit is similar in construction, however its function is very different than a difference meter. Both circuits mentioned are often drawn as a diamond shaped construction where the two upper circuit elements, here R1 and R3, are shown as connected to a common point, and R2 and R4 are shown as connected to a common point. The meter's pointer or indicator in the figure is showing it is centered or "balanced", hence there is no current flowing through it in either direction, and-or no voltage difference.

The units for voltage (ie., emf = electromotive force) are called volts (v). Electromotive force (such as for repulsion or attraction of charges [charge particles, usually electrons]) or "electric pressure" is what will cause charges to move from their position and "flow" as an electric flow or current. When two objects are charged (have an unbalance of charges on and-or in them, that is, having an excess of either a positive or negative charge) to opposite electric or charge polarity (a concept much like the two opposite polarities of magnets, and which attract or repel other magnets) they will attract or repel each other due to the static (stationary, unmoving, accumulated electromotive) charges and their corresponding forces. Electrons are considered as having a negative charge and surrounding electric field around them. Protons are considered as having a positive charge and surrounding electric field around them. These fields are naturally greater in strength at closer or shorter distances to those particles. Electrons and protons will attract each other. Objects with a net negative charge and objects with a net positive charge will attract each other. Objects having the same electric or charge polarity will repel each other.

The strength of an electric (force, or force inducing) field is called its electric field strength, and it varies in value depending on how much charge (Q) there is and how far (r) from the source it is measured. The basic formula for this is: **E = Q / r^2**. Electric field strength (E) at a certain point or location in it can be measured as, or by the (attractive or repulsive) force upon an electrically charged particle at that point or location in that field. Here are some related equations, and the reader may skip over them, and review them when ready and-or needed:

$$E = \text{Electric field strength} = \frac{\text{force}}{\text{charge}} = \frac{F}{Q}\text{ , with units of: }\frac{\text{Newtons}}{\text{Coulomb}} = \frac{N}{C}$$

Mathematically: F = EQ : note, the amount of charge of a single electron is:

(1.602564)(10^-19)C , since 1electron =
1 particles and charge out of
(6.24)(10^18) particles and charges = 1C

1 charge / 1C = 1 / (6.24)(10^18 charges =

1 charge = 1 electron / (6.24)(10^18) charges =
1 charge = charge on 1 electron = 1.602564 (10^-19) C

Or in brief: 1C of charged particles or electrons = (6.24)(10^18) charges or electrons, and if each side is divided by the factor of (6.24)(10^18), we find the charge of 1 single single electron to be: 1.602564 (10^-19) C

Since force is the application of energy upon something else, work is essentially a measure of using or applying energy for a physical distance or length, Therefore, work can also be measured with the same units used for energy, and those are in Joules. Note that since energy from a source is needed to do something, after that energy is used up with its application or doing work, that amount of energy has been removed from the source and-or system and transferred elsewhere during the work.

W = Work = (force)(distance) = Fn Dm , with units of Newtons-Meters = Joules

Mathematically: F = EQ and W = (EQ) Dm = FnDm and E

For negative charged particle such as an electron, the **electric field lines** are said to go **inward** to it from an external positive charge particle and-or quantity of where its field extends **outward** from it, and no matter how distance and weak its field is, it will have some influence upon another charged particle. Consider that very distant objects such as planets still have a weak influence of gravity upon each other, and this can be observed over a long duration of time and observations and noticing a slight change in a planets expected orbit.

Voltage (V) is technically defined as equal to the amount of work or energy required to move or force charge across or between two points. Since work is essentially equal to an amount of energy, voltage is also an amount of energy and is often called as "voltage-potential energy" or simply a voltage potential (ie., ability to move current so as to transfer energy (via electrons) and apply power or work). A higher voltage has a greater ability or potential to induce and move electric current. Consider this form of Ohm's Law where: (current) = (voltage / resistance) = (I) = V / R. Current is directly related to the voltage inducing, forcing or generating it. If the voltage increases by a factor of (n), current will increase by the same factor of (n).

Work is a measure of using or applying a constant force (= mass x acceleration) through a distance, and it could be said as being how much total energy was required and used, and the units of work therefore contain the same units for energy, which are Joules.

Voltage = Work / Charge or = Joules / Coulomb : "energy per coulomb of charge"
V = J / C

From the above equations, we can mathematically express an electric field strength using voltage and the distance between two points in an electric field (E):

V = J / C = W / Q = FD / Q = (EQ) D / Q = ED mathematically:

E = Electric field strength = Fn / Qc = Vv / Dm = , "volts per meter" , and:
E = FnD^2 / Q , and:

W = work = Fn Dm = Vv Qc = energy needed to do that work, and or transferred

Considering Ohm's Law: I = V / R, a voltage of 1 volt will cause (force and move) a current of 1 Amp (=1 Coulomb of charge per second) through a 1ohm resistance or resistor. If this resistance is halved to a lower value of 0.5 ohms, the same voltage can then force a greater current of: I = V / R = 1.0v / 0.5ohms = 2 amps through it. If a resistance was doubled, it will take twice the amount of voltage to force the same amount of current (here, 1 Amp) through it. 2v / 2 ohms = 1A.

Vvolts = Iamps x Rohms = V = I x R : typical expression for Ohm's (Electrical) Law

Ex. A resistor of 100 ohms has a voltage of 10 volts applied across it, calculate the amt. of current flowing through that resistor: I = V / R = 10v / 100ohms = 0.010A = 10mA

Today, there are some digital multi-meters (DMM) available that are inexpensive, and are very useful measuring and test tools for electricity and electronics. In particular, they can measure voltages (V volts), current (A amps), and resistance (R ohms). Values that can't be directly measured can be calculated from the other two known values using Ohm's Law: I = V / R or V = IR or R = V / I

CAUTION: Electric power can cause fires, damage and-or injuries to people. Consider necessary safety precautions. For example, when working with electric devices, first unplug them from their power supply, household outlet and-or remove the battery or other power sources for that device. Be cautious around capacitors that may still contain a static charge and its potential energy, and therefore, electric power which can shock you, and-or cause a fire if the charages in the capacitor drain rapidly due to a low resistance across its leads or wires.

Power = a measure of the amount of energy being generated, delivered and-or used over a time period, hence it is the rate of using energy (Joules) during a time period, and is essentially calculated like an average value:,

Power = energy / time : with units called watts : energy or= work has units of joules
Ex: Joules per second = Joules / second = watts, : **watts** is a derived unit composed of two other units since it is a rate. Also Joules = (Watts)(Time) = Ws

In terms of electricity: **Pwatts = Pw = V x I** : the derivation of this equation will be shown ahead in the book
Pw = (V) (I) = (J/C) (C/s) = (Joules/Coulomb)(Coulombs/seconds) = J / s

Even though a power value may be relatively a low value or rate, energy can also be accumulated over time, and a common example is charging a battery to store electric charges. Once a battery is charged it can deliver a high level of power for relatively brief periods of time as compared to the time needed to charge that battery, and then battery must be recharged before further use of it. This is typically what happens with many solar-power energy systems, and then the collected, accumulated and stored energy can be used to power some high power tools and other devices if needed. Any excess energy generated by the solar panels of a solar-power energy system would also be lost is not used and-or saved, such as in a battery.

1 horsepower = 1hp is defined as nearly 746 watts = ((550 (ft)(lbs) of force) / second, 1Kw = 1000w = 1.34hp
Ex. If an engine is said as having a 1hp output and can turn a generator that is 90% efficient at producing electricity, then it is capable of generating (1hp)(0.90) = (746w)(0.90) = 671.4w

James Watt (1736-1819), from the United Kingdom (UK, England, Great Britain) calculated the amount of work, and the equivalent energy needed and done by the average working horse. The (hp) was commonly used as a unit of power that is used for comparison of the power ability of various machines that eventually replaced horse and water-wheel power. The most common unit of power as of the year 2020 is called the **watt**. James Watt also invented an improved version of the steam (hot, pressurized water gas) engine in 1876 which helped the industrial (ie.,manufacturing) revolution which reduced time and costs, and improved the availability of many things till this day although most steam engines have now been replaced by high power electric motors. Watt's steam engine considered the previous steam and (first practical) piston (a cylinder with a disk in it which pressure can move that disk and then the connected rods attached to it) engine developed in 1712 by **Thomas Newcomen** (1664-1729), from England. Newcomen based his design on the first practical steam pump (engine) which was made by **Thomas Savery** (1650-1715), from England, in 1698 which was made to pump water upward from flooding mines. Newcomen's use of a piston powered by steam is based on the study and ideas of **Denis Papin** (1647-1713), from France, in 1690. Papin invented the first type of pressure cooker called a "Steam Digester" in 1679 and which was a heated container that had a high pressure release valve to prevent a very internal high pressure from damaging the container. This valve had a piston and (tub)e cylinder design. Due to the internal pressure increasing due to the steam having a high kinetic energy, the boiling point of the liquid, such as water, increases, and this super-heated liquid and steam holds a much greater thermal and kinetic energy, and which softens and-or breaks down the substances put in the container. Papin knew **Christiaan Huygen**s and **Robert Boyle** and some other scientists, and after contemplating about his vibrating piston, he came up with the possibility of a steam powered engine which utilized a piston held in position by a force (here a weight) which could be set as needed. He also invented the first leg powered paddle-wheel boat, and this has a type of mechanical propeller (transfers input force and-or power to the water, and which then due to an equal and opposite force, the water pushes and-or propels [e. moves, causes motion] the boat in the opposite direction - here forward.. The input power, the propulsion system for this boat to travel was human leg power, although the concept of the paddle wheel was known and used since antiquity. A paddle is basically a flat wooden or metal board attached to a larger wheel consisting of several paddles about it, and which the wheel will rotate due to the input power, A paddle can be thought of as the flat end of an oar. Papin's ideas were significant steps for further advances in many fields of study including steam powered machinery, the paddle-wheel steamboat and eventually the internal combustion engine.

The density of water at just below boiling temperature; 100°C = 212°F, is about 0.96 g/cm^3. The density of steam at just above boiling temperature is much less dense at about 0.0006 g/cm3, and it is a highly kinetic or energized gas. If 1 gram of water (which has the dimensions of 1cm^3) was converted to steam, it would theoretically occupy a space or volume that is about (0.96 g/cm) / (0.0006 g/cm) = 1600 times greater than 1 cm^3.
1600 cm^3 = a volume of 1.6 Liters

The most commonly known seam engine are those used for steam powered trains ("locomotives"), and some electric power generators. The basics of operation is that liquid water is heated to its gaseous steam phase, and which is then placed into a cylinder (tube shape) so as to force a piston away from it, and of which is attached to a rod and other mechanical parts ("linkages") so as to transfer power to the train wheels or generator. This steam, still at a reasonably high kinetic energy and pressure, will then be put into a large tank to cool off and reduce in pressure, and of which creates a vacuum which restores the piston to its original position. The water is recycled for use. Several one-way (ie., one direction flow) valves are also used to facilitate this process which is not unlike a modern combustion engine.

1 foot-pound = 1 ft.lb = the energy needed to move a weight or apply a force of 1 pound through a distance of 1 foot. **1ft.lb = about 1.35582 joules of energy and-or work**. 1ft.lb / 1s or = 1lb ft / 1s = about 1.35582 watts This concept and unit is of a general form of:
Energy = Work = (force) (distance) = (distance) (force) here, but see the torque units here:

For torque, 1 pound-foot = 1lb.ft = about 1.3558 Nm , Nm = Newton meters , hence the units of a force value . This is a pound of force applied at a (perpendicular) distance of 1 foot from the pivot, axis, center or turning point, and is essentially the length of the radius and-or lever-arm where the force was applied at.

Ex. For tightening or un-tightening bolts to a certain recommended and safe value.

As mentioned above, Pw = (V)(I), and here are some other mathematical equivalences based on Ohm's law: :

Pw = V x Ia = (I x R) x I = I^2 R = V x (V/R) = V^2 / R

Ex. 0.5 amps of current is flowing through a 100 ohm resistor: Pw = (0.5^2)(100) = 25w

Ex. If a 100 ohm resistor has a maximum power rating of 1W, what is the maximum continuous (ie. dc = direct current) current that can be allowed to flow through it when the resistor is being electrically (electronically) operated at it's maximum power handling rating?

From: Pw = I^2 R , Ia = sqrt (Pw/ Ro) = sqrt (1 / 100) = sqrt (0.01) = 0.1A
And:

If this amt. of current was flowing through that resistor, the voltage applied across it would be: Vr = Ir Rr = 0.1a x 100ohms = 10 volts, but this is not necessarily the maximum voltage that can be across it, for higher voltages can be across that resistor if less current passes through it so as the actual power applied to that resistor can still be equal to or less than the maximum power rating of that resistor.
And:

$$\mathbf{Pw} = \frac{\text{energy}}{\text{time}} = \frac{J}{s} = \left(\frac{J}{s}\right)\left(\frac{Qc}{Qc}\right) = \left(\frac{J}{Qc}\right)\left(\frac{Qc}{s}\right) = \mathbf{V\ I\ \ watts}\quad , \quad \mathbf{Energy = Work = Joules = (Pw)\ (time)}$$

$$Pw = \frac{\text{energy}}{\text{time}} = \frac{J}{s} = \frac{\text{work}}{s} = \frac{\text{(force)(distance)}}{s} = \frac{\text{(mass)(acceleration)(distance)}}{s} = \frac{Nm}{s}\ \text{, therefore:}$$

J = Nm = (Newtons)(meters) or= Newton-meters

Since P = (voltage)(current) , if given a certain amount of power, and if the current is reduced to a very small value, the total (kinetic, motion) energy and power delivered by those charges would correspondingly be a very small value. If the voltage (ie., emf = electromotive force) applied to that smaller charge is increased, such as by the same factor that the current was decreased, then the same amount of power could then be delivered. Consider this numeric verification example :
Pw = 120w = J/s = (voltage)(current) = (120v)(1A) = (120v)(n) (1A / n) = (120,000V)(0.001A) :n=1000

Ex. A certain battery having a voltage of 3v is powering a 3v white light LED which is drawing a current load of 25mA. How many joules of energy is it using each second, and-or draining from the battery each second? How many watts of power is it using each second?

From: V = Je / Qc , we have: J = V Qc , From A = Qc / s , we have: Qc = A s , and therefore:

J = V A s , = (3v)(0.025A)(1s) = **0.075 J** : this is the energy used or drained for each second of time.

P = J / s = 0.075 J / 1s = **0.075W** = V A = (3v)(0.025A) , and the amount of energy used for (1 hour = 3600s) is therefore 3600 times more:

J = V A s = (Pw)(s) = (3v)(0.025A)(3600s) = 270 J = 270 W-s (ie., watt-seconds)

Extra: Since 1 hour = 3600s , 1s = 1h / 3600 = 0.000277$\underline{8}$ h
270 W-s = 270 (0.002778 h) = 0.075 wH (ie., watt-hours)

Summarizing the above equations in the above example and-or to consider so as to solve for a variable:

V = Je / Qc : voltage

E = J = VQ = V A s = P s = P t : energy

A = Qc / s , Q = As : current

Pw = V I = E / s = J / s : power

s = Qc / A = (total charge) / (time rate of using or draining that charge) : time
s = J / P = Ej / Pw = (energy) / (time rate of using or draining that energy) : time

1J = 1W-s , 1 Wh = (1W-s)(1h) = (1W-s)(3600) = 3600W-s = 3600 J : total sum of energy used during a period of time.

A Reminder: A common rotating electricity generator converts mechanical input energy and-or power to electrical output energy and-or power, and a motor is the opposite of this. A solar cell converts light energy into electrical energy. An LED converts electrical energy into light energy.

MASS AND WEIGHT HAVE DIFFERENT MEANINGS, BUT ARE DIRECTLY RELATED

Mass is the physical amount of matter (the real, solid material, and not having any extra or empty space other than that between atomic particles), such as the quantity of atoms and other particles of a given substance such as a certain elemental metal (gold, iron, etc) or object, and weight (of a mass) is the force that a mass under the influence of gravity (a constant force, and which will cause a constant acceleration) will exert upon or apply to an object beneath it or holding it up, including a scale. **Mass and weight are directly related and-or mathematically, directly related and proportional in value, and so if one changes by a factor of (n), the other will change by that same factor of (n).**

This weight (force) of a mass will depend upon the specific force of gravity (ie., an attracting force) or other forces acting upon that mass. The force of gravity varies among planets, moons, etc., due to that they have different total amounts of mass which determines their effective (average) or apparent gravity to attract other masses. The more mass an object has, the more gravity it creates around itself. On Earth, the constantly applied force of gravity upon or influencing any basic mass or object that is at low altitudes near the surface of the Earth, will cause it to accelerate at about 9.81m/s^2 =~ 32.17ft/s^2 = (g) towards the center of the Earth. (g) was previously mentioned in this book, and it is a value of acceleration induced by the force of gravity, and is therefore commonly known as the gravitational acceleration (constant for Earth). Force = weight = mass x acceleration.

A weight measuring scale constantly showing the objects weight is a verification that weight (a measure of the total effective force) and acceleration, due to that constantly applied force, does not cease even when the object is at rest, such as it resting (not apparently moving) on the scale measuring its weight, and that gravity (a force which causes acceleration when constantly applied) is still being applied to objects even at rest on the surface of the Earth.

Objects at rest on the surface of the Earth do not move and have any further downward motion toward the center of the Earth because of the equivalent supporting upward force and surface tension (a force) of the Earth's surface, or a supporting table (of which is also an object at rest on the Earth's surface). The result is that the (mathematically positive and negative signed) directional forces balance or cancel each other out, resulting in 0 total or net force applied to the object and it remains motionless, but still, that does not actually stop the constant force of gravity from constantly being applied to the object and having a weight.

Even as you lay, sit or stand still, you can still feel or react to the (downward) force of gravity being applied to you. From force = (mass)(acceleration), a 1kg of mass (matter, material, substance only) will weigh (1kg)(9.8m/s^2) = 9.8 Newtons as a measure of force. From the force formula, mass and weight (a force) are numerically proportional, so for example, if mass doubles, the weight of it will also double, and in general it will increase or decrease by the same factor, and this is necessary to create equivalent valued fractions or portions (ie., ratios) having the same value. The constant of proportionality or ratio of weight and mass is (g):

$$\text{Weight} = F = (\text{mass})(\text{acceleration}) = ma = mg \quad , \quad \text{therefore:} \quad \frac{\text{Weight}}{\text{mass}} = g = a$$

Note that the air pressure (typically 14.7psi at Earth's surface), much like a low density fluid, is not considered since it is applied to all sides of the scale and-or object, and hence does not have a net downward weight.

IMPORTANT: Most scales are actually (and perhaps unknowingly) calibrated to measure mass rather than the weight of that mass. The scale essentially divides the weight in Newtons (N) by the gravitational acceleration constant and displays the result value which is the mass of the object and not its weight. People will usually call this the weight value, but it is technically incorrect. It is better to display mass since it is a universal or constant value associated with the constant amount of matter, whereas weight depends on local force of gravity, such as the gravity value of a planet such as Earth. This constantly applied force of gravity will cause an object to accelerate, therefore giving it energy which can be used to apply a force such as its weight. Weight = F = ma.

weight = force = (mass)(acceleration) : the units of force are Newtons (N) and Units of mass is Kg, and units of acceleration is m/s^2

Many formulas use the mass value of an object rather than its weight value. If given an objects weight value, to convert it to its corresponding mass value, then divide it by the force of gravity (g) = 9.81m/s^2 = 32.2ft/s^2 since from F = weight = ma = mg , mass = weight / g :

mass = weight / acceleration = N / g : the common units of mass is the kilogram (kg)
N is Newtons, the units for forces.
g is Earths gravitational acceleration constant of ~= 9.8 m/s^2

Since 1 gram = 0.001Kg = 1(10^-3)Kg , the weight of 1 gram of matter is:

weight = force = (mass)(acceleration) = (0.001Kg)(9.81m/s^2) = **0.00981N** : **weight of a gram of mass**

Most common scales will not display 0.00981N for when a mass of 1g is placed upon it, but the scale is calibrated to display the corresponding mass value of that weight sensed or influencing the scale. The scale will display 1g. This weight scale therefore is a mass scale. Since a kilogram is 1000 times more mass than 1 gram, its weight will correspondingly be 1000 times more than the weight of 1 gram:

weight of a Kg of mass = (weight of 1 gram)(1000) = (0.00981N)(1000) = **9.81N** : **weight of a Kg**

Density = Mass / Volume : Density is a measure of the amount of mass or substance (material such as the atoms, or atomic particles) per, or in a, three dimensional area of space or volume. Density can also be thought of a s the concentration or amount of matter within a given volume or space. A denser substance can be said as being more concentrated.

The units of mass are grams. If the units of volume are cubic centimeters, as commonly used in density tables. The units of density are derived units composed of the units for mass and volume, and for here, they are: (g/cm^3) = "grams per cubic centimeter".

If the mass of an object is unknown, it can be found from the objects weight since:
weight = force = (mass)(acceleration) = ma = mg ,
and m = mass = weight / g = (weight kg) / (9.8 m/s^2)
Most common "weight scales" display the equivalent mass of that weight, such as grams (g).

The amount of material or substance of water in 1cc volume is defined as the reference and unit mass of 1 gram.

1 gram of mass of water per 1 cubic centimeter (cc = cm^3) volume (space) is defined as has a density of 1g/cc = 1g/1mL at 4°C. This is also the max. density of water. This is the common reference (of measurement) and or unit of mass (ie.,the mass unit) when it comes to comparing densities of other elements and materials.

1 gram of water is also indicated as 1 gram of weight on most scales of which are calibrated so as actually express or display its corresponding mass values. This value is then more correctly called a "gram-weight" or "mass-weight" value. The weight or force of 1 gram of mass is called 1 gram-weight or 1 gram-weight unit, and commonly (and incorrectly) as 1 gram of weight. The readier is reminded that the units of mass are grams, and the units of weight or force are Newtons, therefore, the mass and weight of something are two different concepts and quantities.

An element that is twice as dense as another will have twice the mass or atomic substance per 1cc of volume of it, and therefore it will weigh twice as much.

A less dense liquid or liquefied material will float upon a more dense liquid or melt such as a hot molten (liquefied) metal that sinks lower due to its increased density, weight and therefore pressure upon any material below it.

An object made of a denser element, but still having the same weight as an object made of an element not as dense, will have (or "occupy", "take up", "fill") a volume that is less. For example, one ounce of gold will have a smaller volume than an ounce of aluminum, and this is because gold has a higher density or amount of material (atoms) per unit of volume than that of aluminum.

From: density = p = mass / volume = m / V , mathematically: **m = pV**

Force = F = ma = pVa , if the acceleration is Earth's gravity acceleration = (g) =9.81m/s^2 :

F = ma = Weight = pVg = (density of the substance) (Volume of the substance) (g) = **PA**

Weight = F = ma = (mass)(acceleration) Newtons , mathematically:

mass = F / a = Weight / a

ex: mass = 10N / g = 10N / (9.81 m/s^2) = (10 kg-m/s^2) / (9.81 m/s^2) =
= 10 (1000g) / (9.81) = 10000g / (9.81) = 1019.3$\underline{7}$g = 1.01937 kg

$$\mathbf{p = \frac{mass}{V} = \frac{Weight}{g\,V}}$$

: if something weighs more per unit, or the same volume, it has more mass and has a higher density (p). , (g) is a constant. Volume, if unknown, can be measured, and one way is to immerse the object into water and measure the amount or volume of water that it displaces. To see the largest change in the level of water, you can use a container that has a smaller cross sectional area. To calculate volume if the object has one material or element and is a solid structure : **V = Weight / pg**

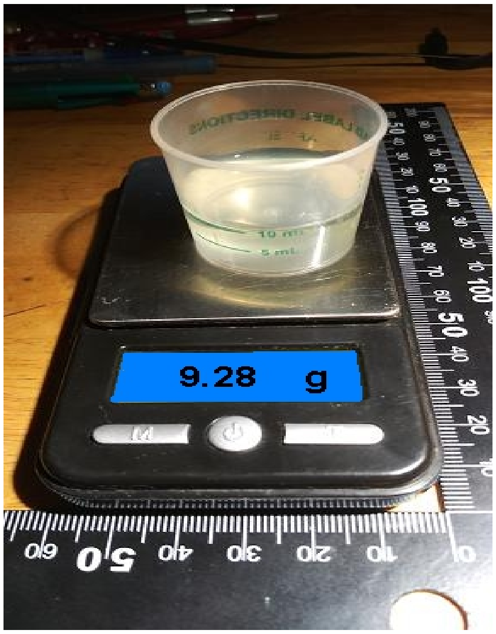
Ex. For the weight of 1000cc of volume of water. Note, 1000cc = 1000g of water.
F = Weight = pVg = (1g / 1cc)(1000cc)(9.8m/s^2) = (1000g)(9.8m/s^2) = (1kg)(9.8m/s^2) =
9.8 kg-m / s^2 = 9.8 (1(kg-m/s^2)) = 9.8N

Ex. In a small measuring cup, 10cc of volume of a food oil was "weighed", and its equivalent mass was displayed as 9.28g after the mass (2.71g) of the measuring container was subtracted or "tared". Note, 10cc = 10mL of volume = 10 (1/1000)L = 10 (0.001)L = 0.01L

The density of this food oil is = p = mass / V = $\frac{9.28g}{10cc}$, after dividing both the num. and den. by 10, creating an equivalent fraction:

p = 0.928g / 1cc : and this corresponds to the known density of most food oils. Water has a density of 1g / 1cc, hence the density of this oil is less than that of water, and this oil will float on water. Because oil has less density than water, it has less mass and weight per same volume as water. Dividing both num. and den. by 0.928 , and inverting the fraction, we find the volume of 1 gram of this oil substance: volume per gram = 1.0776 cc / 1 gram

Because the oil is less dense than water per volume, it has less mass in the same volume or space, and it will take more volume of oil so as to have an equivalent amount of mass as 1 gram of water. Here is an image of the density test: [FIG 227]



The length ruler is shown as a size reference, and each graduation or step is a millimeter.

If the substance in the measuring cup was 10mL = 10cc of water, it would have a mass of 10g since each 1cc of water has a mass of 1g. This is actually a good way to first calibrate a measuring cup and-or determine if its volume is gauged or marked correctly before measuring the volume of substances which have the same volume but different mass and its corresponding weight. To calibrate a weighing scale, you can purchase or use an object with a known calibrated mass and-or weight. The (average) mass of some objects such as coins is given in this book. A 1983 or later U.S. copper coated, zinc penny coin has a mass of 2.5g. Four of these coins weighs about 4 (2.5g) = 10g.

A List Of Some Common Elements, Substances, And Their Density

As mentioned previously, the density of a substance is formally defined as the amount of substance or mass per unit volume, hence it is the amount of mass per unit volume. **Density = mass / volume.** Usually, for this measurement and-or calculation, density is expressed as grams per 1 cubic centimeter.. 1 cubic centimeter of volume = 1cm x 1cm x 1cm = 1^3 cm^3 = 1 cm^3 = 1 cc = 1mL = 1 (1L / 1000) = 1(0.001)L = 0.001L = 1(10^-3)L : L = liters, a unit of volume

1 cc of water is defined as being the mass unit of 1 gram, and the 1 "gram of weight" or "mass-weight" concept where on a weight scale, 1 gram of water is commonly said to also weigh 1 gram, and without mathematically considering the gravitational force (g) acting on that mass and giving it force or weight. In short, the weight of that 1 gram of mass of water when in the effect of gravitation, does affect a weight (or force) scale but it is usually calibrated so as to express the weight (a force) of 1 gram of water as 1 weight-gram which is often commonly called as its weight or grams of weight, but is actually its (or any other substance being "weighed" on that scale) expressed mass value. Most scales in common use are therefore a "mass-weight" or simply a mass scale where a mass is first determined by its corresponding weight (a force) which is then converted to its corresponding mass value and displayed. This may all seem a bit odd at first, but it also allows the mass of a substance to be found by using an inexpensive weight scale, and without the need for a mathematical calculation by the user such as:

From: force = weight = mass x acceleration = ma, mass = F / a = F / g

Under the force of gravity a mass of 1kg will weigh (a force) about: f=ma = (0.001kg)(9.8m/s^2) = 980655 Newtons, however, a common scale is calibrated to indicate or express this the weight as being its corresponding amount of mass of 1kg which is commonly or simply said being "1kg of weight" which actually means "1kg of mass-weight" or "the corresponding mass of the weight".

1 cc of volume is equivalent to (1/1000) Liter = 1mL = 0.001L of volume.

1cc volume of water is defined as weighing "1 gram-weight" or "1 weight-gram", or more commonly (and incorrectly) as just "1 gram", but then this value by itself is actually 1 gram of mass since a gram unit is a unit of mass (substance) and not a unit of weight (a force) which has units of Newtons. 1 gram = 0.001kg of mass will actually weigh 0.0098N when being induced (being forced, pulled) by the force of Earths gravity which has a value of (g) = 9.81m/s^2 due to the mass and gravity of the Earth.

A "relative density" is how many times that the mass and-or weight of a substance is greater than that of 1cc = = 1mL = 1g of liquid water at 4°C, which is defined as the 1 gram of weight standard even though a gram is a unit of mass and not of the force (with units of Newtons) induced by gravity. This density is actually defined at the temperature where water is most dense but not hard yet like ice (crystallized or frozen water) which is actually slightly less dense per unit volume than of liquid water since the frozen water ("ice") actually expands slightly in size as it is freezing into its (atomically, atoms) arranged crystal structure. As water freezes into ice, the resulting force of this growing and expanding solid crystal structure can cause various types of natural erosion and damage such as cracks to man-made structures.

If ice is placed in water which is slightly more dense, that ice will rise upward from pressure beneath and around it, and the cooler and denser water will move (ie., sink) below it, effectively pushing the ice upward. The ice will remain floating on the denser liquid water, but it will not rise upward any further since air is less dense than ice. Denser objects have a larger weight and force per unit volume and-or area, hence they will exert a larger pressure (force per unit area) on any material placed below it and this effectively causes it to move into (ie., displace, sink - go downward) that material in the downward direction of the gravitational force attracting it. Just the same, cooler water will sink below warmer water, and warmer, less dense water will rise upward and away from the denser cooler water, and this is similar to a hot air balloon rising upward to a less dense region in the air.

Note also that there is no consistent linear relationship to an elements density and its melting point (temperature, "heat of fusion" where the atomic bonds are greatly weakened and-or become solid after cooling). For example, lead is considered highly dense and is therefore heavy, but unlike gold, lead has a fairly low melting point or temperature where it turns from a solid state, phase, structure or form, to a liquid form where the atomic bonds necessary for crystallization are much weaker. The atomic bonds are weaker due to the heat energy absorbed and this increase in kinetic energy (ie., atomic movement) prevents crystallization, solidifying or bonding of the metals atoms. For comparison, gases have very weak or no atomic bonding.

To convert grams per cubic centimeter to grams to cubic inch (ci = in.^3):

From: 1 cubic inch = 1 cu. in^3 = 16.3871 cubic centimeters = 16.3871cc , we can multiply a given number of grams in the grams-per-cubic centimeter value by 16.3871 so as to find the number of grams per cubic inch (ci = in^3).

Water, being 1g/cc will be (1g)(16.3871) = 16.3871g/ci .

Steel, being 8g/cc will be: (8g)(16.3871) = about 131.1g/ci , and since mass per unit volume is 8g and is (8g/1g) = 8 times more than that of water, that amount of mass will be both 8 times more and also weigh 8 times more than that of an equivalent volume (here, 1gram) of water.

To convert grams to pounds: Since 453.592g = 1 lb = 16 oz, divide the number of grams by 453.592 to convert to pounds. From this we can mathematically find that:

To convert grams to ounces: Since 28.35g = 1 oz, divide the number of grams by 28.35 to convert to ounces.

For when 30g = 1 "food ounce", then simply divide the number of grams by 30 to convert to "food ounces".

If a density is noted as 8 grams per cc of volume: $\frac{8g}{1\ cc}$

The reciprocal of this is: $\frac{1cc}{8g}$

Dividing both the numerator and denominator by 8, we can find the unit ratio of the amount of volume per 1 gram unit:

$$\frac{\frac{1cc}{8}}{\frac{8g}{8}} = \frac{0.125cc}{1g} \qquad \text{or:} \quad 1g / 0.125cc = \text{"one gram per 1/8 cc"}$$

Here is a list of some common elements and their density at room temperature and standard atmospheric or "air" pressure, which is 14.7lbs per square inch = 14lb / 1in^2 at sea level. This "1 standard atmospheric pressure = (1 atm)" value is reduced when the elevation, height or altitude is higher than at sea level. The air pressure per unit area is reduced since the amount or the mass of air ("air column", volume) in the atmosphere above that location is less and therefore weighs (a force) less and exerts less of pressure (force/unit area) upon things below it. As the altitude increases, the amount of air gets more "thinner", or more "sparse" per unit volume (ie., density) and is not as dense per unit volume and exerts less pressure on objects located there. The air at lower elevations has a greater pressure because of the weight (ie., force) and pressure of the "column" or amount of air pushing downward upon it. Because of this, the air at lower elevations is denser and-or compressed more.

In the following (partial) list or table of the more common elements, g/cc is "grams per cubic centimeter", and although (g) is a unit of mass (material, substance) and not weight (a force on that mass), here (g) is used as rather a "weight-gram" or "grams of weight" per cubic centimeter. The main reason for this method is that corresponding mass of a substance can be found from the weight of that object.

For a specific substance or element, mass and weight are proportional. For example, 2 pounds of iron will have twice as much mass as the 1 pound of iron. If the weight of a substance doubles, the mass of that substance will also double and vice-versa. For both measurements, the ratio of the mass per weight, or weight per mass, is the same value. An element that is more dense will have more weight per unit of volume.

Given g/cc or density of an element at standard temperature and pressure, we can find g/L by multiplying num. and den. by 1000 since 1000cc = 1000mL = 1L . Ex. if an elements density is 0.25g/cc 0.25g/mL, its density in grams per liter is: 0.25g/1mL (1000/1000) = 250 g/L. Also 0.25g/1cc = 1g/4cc after dividing num. and den. by 0.25

Since the volume of 1 cubic-meter = 1m^3 = 1000L = (100cm)^3 = 1000000cm^3 , for the mass of a liquid or gas at standard temperature and pressure (STP) per cubic meter, and which is equal to a density = mass / volume , such as for water which is 1g/1cc is: (1g/1cc) (1000 cc/L) = 1000g / L , After multiplying both num. and den. by 1000 we have: 1000g/1L (1000/1000) = 1000000g/1000L = 1Mg / 1000L = 1Mg/1m^3 = 1000kg / 1m^3

1 cubic foot of volume = 1 ft^3 = 1728 in^3 = 28.317 L
1 cubic inch of volume = 1 in^3 =~ 16.387$\underline{1}$ cc = 0.0163871 L
1 cubic centimeter of volume = 1cm^3 = 1 cc = 1L / 1000 = 0.001L = "a thousandth of a liter" =~ 0.06102374 in^3
1L of volume = 1000cc = 1000 cm^3 = 61.0234 in^3 = 0.035314$\underline{5}$ ft^3 = 0.264 gal.-US
1 gallon of volume = 1 gal =~ 3.78541178 L =~ 3785.4 cc

A list of many common elements, some substances (compounds of two or more elements) and their densities is given below. They are also in order from lower density to higher density.

ELEMENT AND ITS DENSITY (In grams of mass per 1 cubic centimeter of volume = g/cm^3)
(g/cm^3 = g/cc = g/mL = a volumetric density or amt. of mass or matter, measurement value)

This list also includes a few substances that are compounds of two or more elements.

hydrogen = 0.0000838 g/cc , liquid hydrogen is 0.07g/cc = 70g/L at 1atm (14.7psi) , temp is -423°F = - 253°C
helium = 0.0001785 g/cc , liquid helium is 0.125 g/cc at 1 atm (= 14.7psi) , temp is -452°F = -269°C
methane = 0.00072 g/cc , a compound of carbon and hydrogen, CH4, methane is flammable (can burn, a low level or slow combustion) Liquid methane is 0.423 g/cc = 423g/L = Liquid or Liquefied Natural Gas (LNG). A mixture of liquid methane and liquid oxygen can make a rocket fuel. A methane filled balloon will rise upward in the air.
water steam [hot, water vapor or gas] =~ 0.0006 g/cc , at just above the boiling temperature of water
nitrogen = 0.0012506 g/cc , liquid nitrogen is 0.804 g/cc at 1 atm (14.7psi) , temp. is -320°F = -196°C
oxygen = 0.001429 g/cc , liquid oxygen is 1.141 g/cc at 1 atm (14.7psi) , temp. is -297°F = -183°C
Liquid oxygen being (1.141/ 0.001429) - nearly 800 times denser than gaseous oxygen, will fill a volume nearly 800 times larger when it turns back into a less dense gas. An **oxide** is a molecule composed of an element and oxygen. For ex.: zinc oxide (ZnO), and iron oxide (Fe2O3).
chlorine = 0.002898 g/cc = 0.002898 g/mL or ~ 2.9g/L when both the 1g weight and the 1 cc volume are multiplied by 1000 so as to find the equivalent weight of the substance in 1L of volume.
air ~0.00124 g/cc = ~1.24g/L : the density of the mix of gasses in the air at sea level. See the note below.
: 1m^3 = 1000L, and the volume of air per cubic meter is: 1.225 kg / m^3

The standard air mixture of gasses that we breath is about: 78% nitrogen, 21% oxygen, 1% argon, and a small amount is carbon dioxide (CO2) but this value for (CO2) could dangerously increase to many times more without proper air ventilation in a room as the oxygen level gets depleted (reduced, used up, converted to carbon dioxide) from breathing. Plants use carbon dioxide to produce sugars with the aid (energy) from sunlight in a process called photosynthesis. Carbon dioxide is denser, and therefore heavier than common air. Air also contains traces of some other elements. Liquefied **air** is normally 0.870 g/cc at 1 atm (14.7psi) and has a temperature of about -200°C = -392°F =~ -400°F. This can be done by what is called the **Linde process** which condenses (increases its density up to about 700 times. 0.870 / 0.00124 ~ 702) the gas, but also heats it, and this will then be passed through a heat exchanger or radiator system to cool and liquefy it to its high density. It must be kept at pressure or it will boil (ie., evaporate into the air at standard pressure and temperature), such as seen as a vapor, cloud or mist in the air. Note that its density increased by 700 times, but its volume decreases by that same value of 700.

lithium 0.53 [a lightweight, soft metal at room temperature, a small amount is used in rechargeable batteries]
acetone 0.785 [a simple keytone - carbon and oxygen molecule , very volatile (evaporates easily) and flammable, typically used as a solvent for making plastics , is **miscible** (mixes well, completely) in water]
alcohol [pure, non-poison ethanol 0.789 , isopropyl 0.786 "rubbing" alcohol, poisonous typically used for antiseptics]
[Methanol is ethanol that is "denatured" with poisonous methanol, typically used for fuels and solvents]
[Alcohol boils at about 173°F = 78.33 °C , Methanol boils at about 148 °F = 64.4 °C]
potassium 0.862 [a soft, and very low radioactive (radiates or emits energy and-or particles) metal, many people are low in potassium and it is found in some foods and-or some salt-substitutes].
food oils 0.91 to 0.93 is typical [ex. olive oil 0.917, corn oil 0.93, will **float** on water since oil is less dense or heavy. Denser (m/V) objects also weigh more per unit of volume; (W/V), hence have more force of gravity upon them.]
sodium 0.971 [a soft metal, melting point of only 208°F, it is highly reactive with water, Table-salt contains sodium]
water (solid, ice) 0.9167 [ice is cold, crystallized water, and is less dense than liquid water because the crystal solid expanded into a larger volume, hence some of that material, here ice, is outside that reference volume, here 1cc = 1 cubic centimeter = 1 cm^3]. Water freezes at 0°C
water (liquid) 1.0 [comparative reference density, defined at 4°C, slightly above freezing, and max. water density. Water, H2O1 = H2O, as a liquid, is essentially many condensed (ie., now denser, more of a solid) molecules of hydrogen and oxygen gas. Sea-Water is about 1.03g/1 cc . Liquid water can be

thought of as condensed or concentrated water vapor (humidity and steam), and ice can be thought of as condensed or concentrated liquid water. When water vapor looses its thermal energy, it can condense and form into snow (fine ice crystals, at high altitude, where its very cold) and-or rain drops when the temperature is not water freezing (at or below 32°F = 0°C)] **Though water feels and is much softer than hard ice, water is actually more dense than ice in terms of the amount of mass per volume = mass density = mass/volume.**

amber 1.1 typical or average [hard or petrified ancient tree sap, usually a variety of pine tree, and usually having an amber (orange-brown) color, it does not have an atomic crystal structure, but is amorphous. Though slightly denser than pure water, it will float in brine (ie., high salt concentration) water]

milk 1.3 [typical, for whole milk containing some fats (oils)]

PVC 1.38 typical [Polyvinyl Chloride, a plastic, often used for electrical and water pipes. Made from petroleum (ie., crude oil). PVC pipes and material can be softened (at about 200°F) with hot air such as in an oven or hot-fan blower, and then bent into shape before cooling. **HDPE** is High Density Polyethylene, a relatively soft plastic made from petroleum, and can be recycled and-or melted (about 350°F) in an oven and made into products, including bulk plastic material available for other smaller projects. Being softer, the density of HDPT is lower than that of PVC, and is only about 0.95 g/cm^3 and will float on water which has a density of 1g/cm^3 = 1g/mL]

honey 1.42 typical

chalk 1.5 typical [calcium carbonate , ex., a construction material , **Do not breath in dust of any substance.**]

calcium 1.54 [a soft metal, used in some alloys (mixes) of metals, melting point of 1548°F , a bio-nutrient]

table sugar 1.59 [a food sugar, can be produced to have a crystalline, solid form]

magnesium 1.74 [a lightweight brittle metal, used to strengthen some alloys, highly reactive (burns) in water, can also be found in Epsom Salt (magnesium sulfate=MgSO4 which contains about 450mg of Mg per 5g, and plants so as to do photosynthesis. many people are magnesium deficient which can cause some health problems; various nuts and seeds like pumpkin and sunflower seeds are high in magnesium, and seeds can be ground and added into other foods, such as soups, as flavorings].

phosphorus 1.82 white, 2.34 red, 2.69 for black [white, highly reactive, a **non-metal**, soft, waxy, used in fertilizers] Phosphates are a certain molecular form of phosphorous that our body uses for a variety of functions and structures such as bones of which also have calcium. It is said that too much phosphorus can reduce calcium, bone mass and strength and cause dangerous higher levels in the blood and body via accumulations or deposits. Calcium can also reduce phosphorus absorption. Phosphorus is part of the **ATP** molecule for energy in our body. The RDA of phosphorous for the body is about 700 mg = 0.7g for an averaged size adult male. White and yellow phosphorous is toxic and can ignite in the air. People with kidney problems and-or disease may have to severely reduce their phosphorus foods intake (consumption, eating) due to the kidneys have problems excreting it from your body,and then it can build up in the body and cause a myriad or range of problems such as calcium deposits in the body. Animal products tend to have a high level of bio-available phosphorus. Many soda drinks have phosphorus acid added into their ingredients mix, and is actually a (subconsciously addictive?) source of phosphorus. Please try to be wary (ie., aware) or knowing of nutrition needs and in foods.

beryllium 1.85 [a rare brittle metal, a nickel or copper-beryllium alloy is used to improve heat and electric transfer]

carbon dioxide 1.96 ["dry ice" is cooled (frozen) and solid (condensed) carbon dioxide, with a density of 1.5 typical]

opal 2.1 typical [a mineral-like silicate (ie., silicon-dioxide) and water mixture, 8% water typ., colorful for gems]

sulfur 2.07 [a **non-metal** element, it is also a necessary nutrient for life, such as for making amino acids and-or proteins]

table or food salt 2.165 [a compound, NaCl, composed of 40% sodium ions with a net positive charge, and 60% chloride atoms (chlorine ions with a net negative charge, gained an electron from the sodium) by weight. These ions electrically attract and bond together to form each molecule. This is a ratio of (40% / 60%) = (0.4 / 0.6) = (0.2 /0.3) = (2 parts sodium to 3 parts chloride). That's also a total of 5 parts, and (2/5) = 0.4 = 40%, and (3/5) = 0.6 = 60%. Sea water which has a typical density of about 1.03g/cc, and is about 3.5% salt, hence it is slightly more dense and heavier than pure water. 1cc of table salt weighs 2.165g. 5g = 1 teaspoon of table salt has:
(5g)(40% sodium) 5000mg (0.40) sodium = 2000mg = 2g of sodium. 5g -2g sodium = 3g chloride

Note that granulated table salt crystals have much air space between the dense, hard crystals, and it is then less dense being at about 1200mg / 1cc = 1.2g / cc. 1.2g / 2.165g = about 0.55, hence the density is now only about 55%, nearly half the density, as it is in solid, granulated form. 5g of granulated table salt will require about:5g / (1.2g / 1cc) = 4.2 cc.

Salt is one of the many essential mineral nutrients for the human body, and requires more than a trace amount. The FDA (The United States, Food and Drug Administration) recommends that an adult male diet have less than 2300 mg = 2.3g of sodium per day. To get, consume or intake 2g = 2000mg amount of sodium from food salt = sodium chloride, will require about 5g = 5000mg of table salt which has the volume of about (1/2) of a level 5cc volume teaspoon. People having a lower body weight and-or a sedentary lifestyle will usually require a lesser amount than this. The RDA of essential chloride is 3000mg = 3g.
A persons body salt level is usually measured via a blood sample. Having too low of salt in the body can cause **hypomatremia**, and will cause dizziness and cramping, and in severe cases, shock (very low oxygen and-or blood flow) and death. A minimal amount of total salt per day has a recommendation of 500mg =0.5g . An estimated amount of salt lost (excreted by urine and-or sweat) per minute by working hard, jogging, or running is about 15mg per minute on average, hence these people need to consider replacing this lost salt and water by eating some more. When the body has or will get a low level of sodium in it, the body will try to maintain it, and not excrete it in the urine. When the body has an excess of sodium, it will try to excrete it through the urine, and will cause the person to feel thirsty so as to drink more fluids so as to excrete it in the urine and lower the level of sodium **concentration** [**(**amount of substance / reference value**)** = **(**sometimes called the particle or substance or particle density / reference value**)**] in the body. A concentration value in a substance or mixture might be stated as a percentage, for example: 35g per 1kg = 35g / 1000g = 0.035 = a 3.5% (of) sodium concentration (per unit volume and-or any amount of the substance . A concentration might be expressed as parts per thousand = ppt , or parts per million = parts of substance per million parts of the solution, "potion" or mixture = ppm

Salt is vital for the body and is used in many internal and-or chemical processes. Salt of any amount will cause a corresponding rise in blood pressure due to the fact that the body will retain more water, and this will cause a higher pressure upon your blood vessels since the heart will need to use more force to move a larger volume of blood fluid through the body. Normal and-or average blood pressure is usually noted as: 120 / 80 ("120 over 80"). Drinking water can help reduce a high amount of salt in the body. To much salt in the blood is medically called **hypernatremia**. The normal amount of sodium in human blood is about 140 **m**illi-**m**ol **Eq**uivalents per liter, or units of: (mmol Eq / L) or= (mEq / L) = about **3.22mg/ml = 3.22g/L**, and note that 1ml = 1cm^3 = 1cc, Adult human blood has a density of about 1.057 g/cc, and the amount of blood in a human is about 7.5% of the total mass, hence also about 7.5% of their corresponding weight. Infants will have a higher blood mass of about 9.5% An average adult person has about 1.36 gal = 5.11L =~ 10.5 pints of blood in their body, and can donate up to about 1 pint = 0.473L (nearly half a liter = 0.5L) of blood, but may then feel dizzy and need to drink water, and it helps if it contains some electrolytes such as a small amount of table salt. A derivation of 140 mmEq is shown in the **Extras And Late Entries** section of this book.

To much sodium can cause high blood pressure, a stroke, a heart attack, kidney disease, etc.

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Studies have now shown that **potassium** levels in the blood also controls blood pressure. Many people are lacking in the amount of potassium and other nutrients (elements [minerals] and vitamins [ie., vital minerals in the form of other created molecules and-or compounds such as vitamin A, B, C, D, E, etc.]) in their blood. Potassium helps regulate fluid volume in the body, it is an important electrolyte like salt is, and is used for body processes. The recommended intake of potassium for an adult male is currently 3400mg / 1 day = 3.4g/day. Please check that your table salt is supplemented with iodine. If you are on a salt restricted diet then you might not then receive the (typically minimal) RDA of **iodine**.

limestone 2.2 average value
graphite 2.25 typical , a specific physical form or instance of a carbon element
carbon 2.267 [a **non-metal** element that is a vital part of living in all hydrocarbon (or hydro-carbon) based life forms, it is rooted in the word for charcoal (burnt, charred wood). Carbon-12 = C12 is a nuclide of carbon and is used for the reference atomic mass unit. A **carbide** is a hard crystal compound made from carbon and a metal such as iron, and is often used as a coating for some high wear items such as cutting and abrasive tools. Graphite and graphene are forms of carbon that can **conduct electricity** (ie., electric current, and-or energy) due to their atomic crystal structure where three carbon atoms are joined leaving a free electron per carbon atom which can carry its charge and kinetic energy.]
silicon 2.33 [a brittle semi-conductor metal, often used in diodes, transistors and solar-electric generating panels. Silicon can be extracted from quartz and-or sand which is basically silicon-dioxide. Silicon has a melting point of about 2500°F, and silicon carbide, 3.21g/cm^3, has a melting point of almost 5000°F]
boron 2.34 [a very hard, liteweight, brittle, generally a non-metal, low electrical conductance, and which is often used for "doping" (adding trace [tiny] amounts) to semi-conductors to improve conduction]
window glass 2.5 typical [heat tempered glass is stronger, but brittle, "safety glass" usually has a plastic layer]
marble 2.65 typical [a hard crystalline stone, usually from metamorphic limestone from calcium carbonate deposits]
aluminum 2.698 [will eventually react with air to produce a hard aluminum oxide, oxygen "rusting", surface coating. The melting point of aluminum is about 1220°F]. Aluminum is named after the mineral ore called alum. Aluminum was first discovered by Hans Christian Oersted, of Denmark in 1825.]
marble 2.5 average [a dense form of limestone, more corrosion resistant than limestone]
granite 2.7 typical [a hard stone, mostly quartz 2.65 which is silicon dioxide [like glass], but also has feldspar=6.0]
diorite 2.9 typical [a hard volcanic rock, often with black and white spotted colors]

antimony 3.0 [a mettaloid, brittle, used mostly for alloys with other metals]
barium 3.5 [a soft, reactive and toxic metal, used for some vacuum tube parts and semi-conductors]
diamond 3.51 typical [its density is not very high, but it is a very hard crystal, brittle (non-malleable or non-bendable), and is a slightly denser form of carbon (graphite)]

titanium 4.54 [a low density, lightweight, but high strength metal, relatively brittle and can crack rather than bend]
iodine 4.93 [a **non-metal**, poisonous, but a necessary nutrient in small ("trace") amounts to make vital or essential thyroid hormone which helps regulate body metabolism. Iodine is scarce and not generally needed by land plants, but ocean fish and plants, such as seaweed and kelp, do absorb dissolved iodine and concentrate it. Because of not eating seafood, inland people often need to be supplemented (such as in table salt) with iodine. Seawater has about 50 parts per billion (ppb) that is iodine. This is equal to 50 ug per liter of seawater. Evaporating the water out of **sea water** is one way to extract the minerals in seawater. 1L of seawater will contain "sea salt" (sodium-chloride, a compound of elements) weighing at 10 grams of sodium and 20 grams of chloride, plus other lesser quantity of dissolved minerals (elements) such as magnesium chloride, sulfate, calcium chloride, calcium, potassium, sulfate, bromine, iodine, etc. **In the United States, the (min.) RDA of iodine is 150 ug/day. Many people are unhealthily deficient in iodine and some other minerals. A well known and trusted form of iodine is called Lugol's Iodine (a potassium iodide mix) that is intended as a nutritional supplement.**

Note that evaporating seawater (especially when warm) may actually "gas-off" the (volatile, fragile) iodine into the air. Some sea plants, such as seaweed and kelp absorb iodine and have a much higher, less volatile, concentration of it. A pre-made mixture of alcohol and iodine is used to fight germs on some surface (non deep) skin wounds such as scratches, otherwise seek formal medical attention and treatment.

germanium 5.3 [a semi-metal or metalloid, brittle, similar to silicon and used for semi-conductor electronics]

basalt 5.5 typ. [a dark to black lava (= melted, volcanic) rock]
radium 5.5 [a highly radioactive and dangerous element, many more times naturally radioactive than uranium]
gallium 5.9 [a soft metal, brittle when cold, liquid at about 85°F, used for some semi-conductors and mirrors]

vanadium 6.11 [a rare, soft and ductile metal, its oxide resists corrosion, and is used in high strength alloys]
zirconium 6.52 [a hard metal, malleable, high strength, non-reactive]

neodymium 7.0 [a "rare-Earth" metal, soft, reactive, used in some high magnetic field strength magnets and lasers]
zinc 7.134 [a metal with a melting point of 787°F, often used in alloys, and for an additive in other products]
chromium 7.15 ["chrome", hard, non-reactive, used for a stainless, protective and shiny plating for other metals]
tin 7.29 [a metal with a low melting point of 750°F that can be reduced further if mixed with bismuth and-or zinc]
manganese 7.3 [a hard and brittle metal , used in some alloys for magnets , glass making, a chemical reaction catalyst ; a catalyst is a substance that helps to improve a chemical reaction, hence it speeds the chemical reaction process and reduces the total time and-or energy needed for a complete chemical reaction. An analogy would be that a catalyst is the spark which causes a combustion. Ex. To help remove potentially dangerous fumes or gas particles from the byproducts or exhaust of combustion engines, a metal such as platinum, palladium and-or rhodium is used to effectively filter it via a chemical reaction of this catalyst metal, and where the hot gas molecules of the exhaust are split apart into less dangerous gasses and water vapor. This device is often called a "catalytic converter".]
iron (steel is iron with some carbon) [7.87 for low carbon mild hardness steel, 8.0 for grade "316 stainless steel"] [Sometimes called ferrum (Fe), ferrous or ferrite, iron melting point is 2700°F, steel melting point is 2500°F, its name may mean "metal from the hot fire or furnace", "fire metal" or "furnace metal". Iron oxide is surface rust due to air-oxygen contact, Iron is an abundant element on Earth. **Harry Brearley** from the United Kingdom is credited to making the first practical stainless steel in 1913, and of which does not rust if made correctly. Most places that prepare and-or serve food to the public are required to have a stainless steel sink and food preparation table among some other things such as a license, being clean, possible refrigeration, etc.

The body uses a small amount of iron, as atoms, to carry oxygen through the body. Iron supplements may help some people lacking iron in the blood, but a blood test is generally needed to know if a person is indeed lacking in iron. **Do not take extra and-or unneeded iron supplements thinking it will enhance your general health since a high level is toxic, and the same could also be said for most other supplements.** Most of the iron atoms in a person is mostly found as part of the hemoglobin protein in red blood cells. Low iron levels is also a serious health issue and cause a variety of symptoms, and the main one is breathlessness. Eating nutritious foods is good to prevent and-or treat many health issues within the body, and of which includes the physical mind processes.

Iron-pyrite ($FeS2$ = 1 iron atom to every 2 sulfur atoms) or iron-sulfide is iron with traces of sulfur in it, and it occurs naturally, atom by atom in the presence of flowing water containing the supply of atoms, sometimes as a beautiful crystal form. It has a yellow color as that of shiny brass and is somewhat rustproof. The density of iron-pyrite is less than that of plain iron or steel, and is about 5gm/cm^3. Pyrite is a semiconductor with an electrical (for conduction) "bandgap" of about 0.95eV and could be used as a part in a "crystal" (ie., sold-state, radio).] Forged steel is compressed or "hammered" steel when it is red hot, and is stronger due to the increased size of the metal crystal structures. Many tools and machine parts are made from forged steel. Some iron artifacts have been dated to about 2000 BC, but before about 1200BC, smelted iron was generally not available to most people, and hammered iron from meteorites was sometimes used in a very limited basis. Bronze (an alloy of copper and tin was in much use by the time iron products became available. The first iron production and use is usually thought to be in south-eastern Europe, and India. Every piece of iron and-or steel was surely valuable at those times and until to the "industrial revolution" beginning at about 1800AD where many products began to be "mass (ie. massively, many) produced" and became much cheaper and available due to advances in science, technology and transportation. The ore form of iron is very abundant in the Earth.

brass 8.6 typical [an alloy of varying proportions of mostly copper, and with zinc, and is golden-yellow in color]
cadmium 8.65 [a soft and malleable metal, toxic, used in some rechargeable batteries]
bronze 8.5 to 8.9 typical [a hard alloy of mostly copper, and with tin, typically has a reddish-brown color]
[brass is much less of conductor of electricity than copper, and bronze is even worse than brass]
[bronze is harder than copper, and it melts at a lower temperature due to the tin additive]

cobalt 8.9 [used in some blue pigments, alloys, magnets, the B12 vitamin molecule contains 1 cobalt atom]
nickel 8.91 [a hard, malleable, an nonreactive or non-corrosive metal, **a potential toxic health risk from its dust**. **Avoid any dust from any metal or substance.** The melting point of nickel is about 2650°F]
copper 8.96 [an orange-red colored metal, has very good thermal and electrical conduction. The melting point of copper is 1984°F]
bismuth 9.81 [a dense, heavy metal with a low melting point of 520°F, an alternative to using toxic lead metal, when melted, it will contract, and like ice, it will expand when cooled to a solid, bismuth is not toxic]

molybdenum 10.3 [used in some stainless steels and other alloys]
silver 10.5 [a metal with excellent heat and electrical conductivity, but higher in cost than copper. silver reflects the most light for any metal, and is ideal for mirrors such as for reflecting telescopes. The melting point of silver is about 1760°F]

lead 11.342 [a dense metal, but relatively soft and malleable (ex. by hammer) metal because of weak crystalline atomic (atom) bonding. It is **toxic or poisonous**, and has a low melting point of 621°F]

mercury 13.53 [a dense liquid at room temperature, but highly soft being a liquid, **highly toxic or poisonous**]
tantalum 16.7 [a non-reactive or "inert" metal, used in some alloys, has a high melting point]
uranium 18.95 [a slightly radioactive metal element when stable, toxic, with a long half-life of decay, it is also malleable where its shape can be changed by hammering an pressing without it cracking]

tungsten 19.25 [a hard and brittle metal, has the highest meting point of 6192°F, and highest tensile strength - which is basically a stretching or tension strength before breaking failure, such as from the max. suspended load weight (as a pulling or stretching force) applied to the end(s) of the length of a certain standardized reference cross sectional area of the material that is held in a fixed in position. Tungsten has almost the same density as gold. The high melting point of tungsten is due to having a strong crystal structure - similar to carbon, having 6 shared and-or many bonded electrons.]
gold 19.28 [very dense and heavy, but relatively soft and malleable due to a weak crystalline structure bonds, gold has a shiny, yellow color which does not oxidize (ie., loose an electron(s) and become "gold-oxide") and rust or tarnish (discolor) in the air, etc. Gold has a melting point of about 1950°F]
plutonium 19.82 [a radioactive metal with a long half-life of decay, particles of it are toxic, such as if inhaled]

platinum 21.46 [a dense or heavy, malleable metal, high melting point of 3220°F, very non-reactive or "unreactive" to other chemicals, and is therefore superior to stainless steel which also does not corrode or rust easily]
iridium 22.6 [rare, very dense - heavy and hard, but brittle and will break or fracture by relatively low amounts of force, high melting point and corrosion resistant metal. The heaviest of metals are believed to be mostly at the core of Earth, and with only a trace amount in its crust (ie., surface), but higher concentrations can be found in some meteorite impact sites, and this indicates that the meteorite was probably from the core of a large asteroid or planet.. About twice as dense as lead.]

Most metals have a gray to silvery type of color except copper and gold which have a unique color due to its atomic (atom) and crystallized structure. Metals conduct electric current fairly well, and each metal element will have a unique resistance to the flow of electricity. Gold has a very low resistance to the flow of current as compared to that of steel, however gold and silver are very expensive and are then impractical to use because of that, and copper then becomes the preferred electricity conductor for most applications.

PHASES OR PHYSICAL STATES OF MATTER

The 3 common "states" or "phases" of matter are: solids, liquids, and gases. Given a substance such as an element or molecules, it can change to a different state, usually depending upon its temperature and-or state of energy. A "4th" state of matter is called **plasma**. "Plasma" is an old word that means something that is soft, and the word "plastic" is also a based on the word "plasma". Plasma is created when electrons are removed from many atoms of the substance. The plasma will then contain many positively charged ions and negatively charged electrons. Plasma's often give off some light, especially when a current is passing through or within it and giving it energy to do so. When an energized (kinetic energy) electron falls into orbit, it will release its gained energy as a photon of light. A plasma is like a very hot gas(s), and due to the high kinetic energy and-or velocity of the particles. Stars, such as the Sun, are composed of hot plasma consisting of gases of different elements - mostly hydrogen and helium. **Helium** is created by fused (fusion, joined together) hydrogen atoms in the very hot (high kinetic energy) and high gas pressures (ie., force / area) deep inside the Sun. A helium atom has 2 protons, 2 neutrons, and 2 electrons. Helium was discovered by **Pierre Janssen**, of France, in 1868 while observing helium's spectrum line in the light from the Sun, and this gas was later named after the Sun (Helios in Greek). Helium gas can be found on Earth, but it is often in relatively small quantities mixed with natural gas (**methane**, CH_4, classified as the simplest hydrocarbon, here,a molecule with 1 carbon atom in the center and then surrounded by 4 hydrogen atoms). Hydrogen has 1 proton and 1 neutron, and when two hydrogen atoms are joined together, they create helium which has 2 protons and 2 neutrons. The excess kinetic energy of the two hydrogen atoms will be released as heat (infrared or infra-red light) and light energy of that Star.

The three phases of water are commonly called: ice = frozen or crystallized water, liquid water, steam (water vapor or gas). With each phase of water, the molecules have a different arrangement.

The **phase or state of matter** mostly depends upon the temperature applied to that matter (ie., element), but may also depend upon the pressure applied to it such as to change the state of the matter. The states or phases of matter are usually then closely related to the density of that matter. Water has a rare property that in its solid form or ice, its density is less than that of liquid water form, and this is due to that water will slightly expand in size or volume at the molecular level when it crystallizes to ice.

Consider water, if it is heated to be hot (high) in temperature, it will boil (disassociate, "boil off", "come off", "loosen", "vaporize", "break away from", "release") and turn into to steam (a gas) particles. If steam is cooled, it looses energy and condenses together into tiny drops of which is a liquid state. If a drop of water is cooled, it will condense even further and freeze into a solid (ie., ice). Water is also a special case of where when it is frozen, it is actually not less in size, but slightly larger in volume. The atoms in the ice or solid water have a low kinetic energy and they have a greater ability to join, bond and remain near each other and form the (atomic, molecular))crystal structure of ice. The atoms in a gas have a higher kinetic energy. Gases have a low density, while solids have a high density. The **air** that we breath is a gas at standard temperature and pressure, and is mostly a mixture of **nitrogen (78%) and oxygen (21%)**. Ammonia (NH_3) is a gas that is a nitrogen and hydrogen compound, and it is used in fertilizers and cleaning products. Some ammonia is produced when protein is processed in the body. Ammonia has a "sharp", strong pungent smell and is found in urea and "smelling salts" that are sometimes used to awaken people after fainting and-or make them more alert.

Plasma is a state of matter that is a very hot gas and is high in kinetic energy, and its atoms have broken apart into protons (positively charged) and electrons (negatively charged). An example is a gas light such as a neon ("new gas", a natural trace element found in air) or fluorescent light which is basically a glass tube filled with a gas that gets **ionized** and turned into a plasma by applying electricity through it. Fluorescent lamps are coated on the inside by a material that will fluoresce, which means when struck by invisible high energy light radiation, it will radiate visible light radiation.

One definition of a gas is that a small unpressurized amount of it will fill or expand outward into any (empty) container of which it is placed into. When a gas spreads out into a larger volume, the density of that gas will decrease since there is now less matter (here, gas) per unit volume. A gas spreads out because the atoms or molecules are not joined together as like that of a liquid or solid, and may repel each other electrically, and-or have thermal, kinetic energy.

One definition of a liquid is that is will fill any volume it is placed into, but due to its density and the gravity (force) of Earth, it will only fill the lower (ie. in height) volume of that container. Unlike for the gas, the volume and density of the liquid will generally remain the same when pressurized. A liquefied element or molecules is more dense than the gas(s) that compose it. A liquid can be defied as the state of a substance when it is between its solid and gas form. A liquid is usually less dense than a solid.

Matter that is less dense is more compressible. Gases are more compressible than liquids. Liquids, though only slightly compressible, are more compressible than solids. In brief, gases can be converted to a denser or liquid state by using the Linde-cycle or process invented by **Carl Linde** (1842-1934) from Germany in 1895. In this process the gas is compressed and this increases the pressure of it and effectively increase its density. A side-effect is that the temperature of the gas will also rise when it is compressed. The hot, compressed gas is then let to cool off (release much thermal and-or kinetic energy and then it reduces in temperature (getting colder, more dense) and pressure) by thermal conduction (ie., transfer) through the wall of the smaller volume holding it, and-or through a heat exchanger, and then allowed to then expand and reduce in pressure, and the cycle is repeated until the gas is liquefied. This is also basically how a modern food refrigerator works. Liquid oxygen, nitrogen, liquefied natural gas (LNG, methane), and liquefied propane gas (LPG) are common examples of liquefied gases. Once liquefied and cold, if warmer air then comes into contact with liquid oxygen or nitrogen, some of that cooler liquid at the surface and boundary of the warmer air will then increase in temperature boil off into the warmer, higher energy air and make a visual steam or "fog" appearance of the less dense gas.

We see that the phases or states of matter (or atoms) are mostly due to the influence of external physical conditions such as temperature and pressure applied to that matter and its initial state.

A solid metal, usually with a hard crystalline atomic or molecular structure, can become soft when heated to a hot temperature, and before it is melted to a liquid state or status. The high heat will weaken its hard crystalline structure. When a metal is soft, it is more pliable (bendable, malleable), and more "plastic-like", and can be more easily **forged** or pressed or cut into a shape using a high amount of force and-pressure, such as with a hammer. This is usually how coins are made more easily, and also to have some design or inscription placed on them by another carved piece of harder metal. A person called a "black smith" can make many types of metal (usually of iron or steel) items by hammering and-or cutting the hot metal piece against a much larger piece of metal called an anvil. The metal is then allowed to cool naturally and-or is dipped ("quenched", heat extinguished) into water or an oil so as to harden it more quickly and to give the metal some specific property such as increased hardness. To anneal a piece of metal is heat it hot and then slowly cool it so as to help remove internal stress (ie.,hard areas) and to make it slightly softer to work upon.

The **melting point** of a metal or solid is the temperature at which it changes physical state from a solid to a liquid. When a metal is liquefied, it will increase in volume (ie., thermal expansion) and become slightly less dense , but when it cools to be a solid form again, it will decrease in volume (ie., thermal contraction). A metal can also boil and evaporate up into the air when its pressure becomes greater than that of the air or gas pressure. Adding in some other metals and-or impurities can affect the melting point of that alloy of metals since it alters the crystal structure of the resulting metal, and generally weakens the electrostatic bonds of the atoms and it will melt at a lower temperature. For example, Rose Metal is an alloy of bismuth, lead and tin which has a low melting point of about 205 °F A metal mixture or alloy is said to be **eutectic** ("melts easily and-or evenly into a homogeneous [ie., as one] mixture") when its melting point for the entire mass is lower than the lowest melting point (m.p. or mp) of any one metal in the mixture. Various types of (electronic, and plumbing) welding solder are eutectic alloys (mixes) of metals, and some do not even contain lead metal, hence making it safer, but at a higher monetary cost such as if it contains relatively high cost silver metal. Some other eutectic alloys are: Wood's metal (~ , 50% bismuth, 25% lead, 15% tin, 10% cadmium) with a melting point of 158 °F, Field's metal (~ , 50% indium, 33% bismuth, 17% tin) with a melting point of 144 °F, and Galinstan (~ , 68% gallium, 22% indium, 10% tin, with a m.p. of -2°F) of which can melt in your hand much easier than pure gallium metal (m.p. of 86°F) can. Galinstan can replace toxic mercury in many applications such as in thermometers. Note that most metals and-or elements can be toxic, depending on the element, amount, and use of it. Inhaled dust particles is a danger of any substance, so wear a breathing mask (ie., a breathing, debris filter), gloves and eye protection. A blood test can determine if someones blood contains high values of a metal(s) and-or other substances.

The amount of energy needed to change a specific element from a solid state to a liquid or melted state depends on the amount of mass of it and the type of element since each element has a various atomic bond strengths. This amount of energy to change between a liquid (near the solid state) and solid (near the liquid state) is called the **energy or heat of fusion (Hf rating or ratio)** and has units of: joules / kilogram or joules / gram. Fusion Energy = (mass)(Hf). For example, given a big mass or block of ice that is at the minimal freezing temperature, it will require a particular amount of energy to melt just one gram of it so as to change to and be in the phase or state of liquid water, and theoretically still be at the same temperature, and this specific amount of energy is called the heat of fusion of water. The heat of fusion or "latent heat" amount of energy for water is about 79.7 cal / gram = practically about 80cal / gram = 334 J / gram at standard atmospheric pressure. In a reverse manner, if liquid water is frozen, about 80 cal of heat / gram of water needs to be released as the kinetic energy of water is changed to another state, and here, the energy is released as heat (thermal energy). The amount of energy, the heat or energy of vaporization, Hv, to change from a liquid state to a gas state is not generally the same value as the heat of fusion. Hv of water is more than other elements and is about 2245 J / gram to change from liquid to water to ("steam, vapor") gas. This relatively high amount of energy is due the large number of hydrogen bonds in all the water molecules in a gram of water. Steam, and particularly when concentrated and-or having pressure (ie., force) due to its kinetic energy, is said to contain high thermal and-or kinetic energy and is therefore useful for (mechanical, input force operated [ie., as the supply force, "fuel" or energy] and-or output) machinery such as a steam engine for moving trains.

An example of the many different alloys of two or more metals:

Nichrome (NiCr) is a mixture ("alloy", joining, combination) of nickel and chromium, and might have a small amount of manganese and-or iron. Nichrome has a relatively high resistance per unit of length due to it having a relatively low number of free electrons. This wire is often used for various types of high temperature (up to about 2000 °F) heating elements such as for coffee makers, toasters and soldering irons that get hot and-or create warm air for heating rooms. This ability make nichrome wire valuable in terms of common usage. Nichrome dust and-or fumes from being machined do have some toxicity. Nichrome wire is also called "resistance wire" or "heating wire". The resistance of nichrome wire, and like other metals, depends on the alloy ratio used, length and cross sectional area, and as a basic reference, 22 gauge nichrome-60 (contains 60% nickel ,and 40% chromium) wire has about 1 ohm of resistance per foot of length. and 32 gauge has about 10 ohms of resistance per foot of length. Unlike other metals, the resistance of nichrome wire does not change much per degree change in temperature, hence it is said as having a low thermal coefficient. This is good if your making a calibrated, high power (and heat ability to then dissipate)) resistor from the wire and are expecting various amounts of current (and power loss) through it . The retail (ie., for the consumer) cost of 22 gauge nichrome-60 wire is averages about 15 USD per 100 feet of length. Generally, when making a nichrome resistor components for electronic devices other than for heating purposes, it is wrapped around and-or encased in a hard ceramic type of non-flammable and non-melting material to electrically insulate it. The higher the amount of power applied to the wire of a given length, the hotter it will get. If the wire is made shorter, it will bet hotter for a given amount of power, and when the wire is made longer, it will get cooler for a given amount of power. For general electronic device where the current and-or power drops are relatively low, the maximum power rating of nichrome wire, say at its melting temperature, is generally not exceeded. For an air heater it is better to have a longer length of "heating element" wire, such as nichrome wire, so that the total surface area of the wire is larger, and so that there is then more contact with the cooler air and the thermal heat transfer and-or or exchange to the air is more efficient and-or faster.

CAUTION: For high power, household devices, and of which often have a low resistance such as for an air heater, it is recommended to **not use an extension cord** (ie., essentially, a two wire cable to transfer AC power) with it. This rule is due to the voltage divide concept with two low resistance values, here the device resistance and the cord resistance, and that the cord will therefore have a significant amount of power loss in it which will raise its temperature, and it could melt its insulation and-or cause injury and-or fire.

When a metal is put in a more pliable state.

When a metal is made pliable (ie., somewhat like plastic, bendable), its state could be described as being between the state of a solid and the state of it being liquid. It is generally obvious that a thin metal is more pliable than a thicker piece of that same metal. A common example of deliberately placing a metal in this pliable state is with blacksmithing, and this is briefly discussed next.

A **blacksmith** can take a piece of pre-made steel from a steel factory or scrap (unused product remainder, "left over", unused, used and-or to be recycled) metal, and cut and hammer (ie., "forge") it to produce a product. It is of note that thin metals are often bent and-or shaped without first heating them when using the powerful bending and-or shaping machines which have a high amount of force, accuracy and quality. Heating them will make them more pliable for these machines to work with, and particularly for bending and-or shaping thicker pieces.

Before blacksmiths and machinist works with the metal such as steel, they will soften or "**anneal**" the steel by heating it hot and then allowed to slowly cool so that it is slightly softer and less brittle to work with, and this allows it to be more easily shaped and is less prone to breaking in the process. Once the steel has been worked into the desired shape, it is usually "**heat-treated**" so as to have a better, more uniform and stronger (atomic, atom) crystal structure that is more resistance to wear since it is harder. This process is usually done by heating the steel to a high temperature and then quickly quenching or dipping it in water or oil. Making the steel harder will also make it more brittle, therefore, this process may also be followed by what is called "**tempering**" the steel so as to help reduce its brittleness slightly and yet still maintain a fairly hard steel, and this process is to heat it up to a lower temperature, and then let the steel cool slowly.

LIST OF THE KNOWN ELEMENTS

1 Element , 2 Symbol , 3 Atomic Number , 4 Standard Atomic Weight (g / mol) , 5 Density = p = (g/cc)

3: (# of proton mass units per atom)
(2)(Atomic Number)=Atomic Mass
= Atomic Mass Units = AMU
for the element atom

4: ("molar mass")
(Total sum of mass of
1 mol of atoms)
(typical, average)

5: (Mass of 1 cc of volume)
(g/cc = g/cm^3 = g/mL)
Weight is proportional
to mass, volume, density
and gravity. Mass is
constant: m = p V = F / a

1	2	3	4	5	
Actinium	Ac	89	227	10.07	: radioactive , a bright colored, soft metal, has a faint blue glow
Aluminum	Al	13	26.98	2.698	: a lite-weight metal, often used for soda cans, etc.
Americium	Am	95	243	12	: man-made, radioactive, often used in smoke detector alarms-
Antimony	Sb	51	121.76	6.68	-having less than 0.5ug of Am-241, half-life of 450 yrs
Argon	Ar	18	39.95	0.001784	:a gas, at STP (Standard [reference] Temp. and Pressure)-
Arsenic	As	33	74.92	5.72	-STP = 0°C = 32°F and 1 **Atm**osphere = 14.7psi
Astatine	At	85	210	6.35	: a non-metal
Barium	Ba	56	137.33	3.59	: a soft metal, used in the medical and electronics fields
Berkelium	Bk	97	247	14.78	
Beryllium	Be	4	9.012	1.85	:a lite-weight, hard brittle metal, for alloys, crystals called emeralds
Bismuth	Bi	83	209	9.81	: melts at 520 °F , slightly toxic - avoid inhalation of its dust
Bohrium	Bh	107	270	26.5	: man-made, synthetic, has a short half-life
Boron	B	5	10.81	2.34	: has a high melt point at 3770 °F; has a strong crystal structure
Bromine	Br	35	79.9	3.12	: a non-metal
Cadmium	Cd	48	112.41	8.65	: a soft metal, a carcinogen to lung and body, **toxic** build up
Californium	Cf	98	251	15.1	: a synthetic element, highly radioactive, decays to curium
Cesium	Cs	55	132.9	1.88	: some isotopes are radioactive, melting point is ~ 83 °F
Carbon	C	6	12.011	2.267	: a non-metal,but the graphite forms can conduct electricity
Calcium	Ca	20	40.08	1.54	:a soft metal, reactive with water, a nutrient, used in alloys
Cerium	Ce	58	140.12	6.7	: a moderately soft metal, named after the proto-planet Ceres
Chlorine	Cl	17	35.45	0.0029	: a gas at STP
Chromium	Cr	24	52	7.15	: stainless steel has about 10% or more chromium
Cobalt	Co	27	58.93	8.9	
Copernicium	Cn	112	285	23.7	
Copper	Cu	29	63.55	8.96	: greatly used for electricity wires, coils, electromagnets, pipes-
Curium	Cm	96	247	13.52	: synthetic, highly radioactive - copper melts at 1948 °F
Darmstadtium	Ds	110	281	34	
Dubnium	Db	105	268	29	
Dysprosium	Dy	66	162.5	8.54	: can be used to increase max. temperature of NdFeB magnets
Einsteinium	Es	99	252	8.84	
Erbium	Er	68	167.26	9.07	
Europium	Eu	63	152	5.24	
Fermium	Fm	100	257	9	: man-made = synthetic, highly radioactive , decays to californium
Flerovium	Fl	114	289	14	
Fluorine	F	9	19	1.7	:a gas at STP, very high reactivity, used to help melt metals, toxic
Francium	Fr	87	223	1.87	
Gadolinium	Gd	64	157.25	7.9	a rare metal, used to improve MRI scans, may cause kidney issues
Gallium	Ga	31	69.72	5.9	: a metallic liquid at just 85.58 °F = 29.76°C , for alloys and -
Germanium	Ge	32	72.64	5.3	: used in semiconductors - semiconductors
Gold	Au	79	197	19.28	: a precious metal, malleable , non-reactive, melts at 1943 °F
Hafnium	Hf	72	178.5	13.1	:a rare metal , for special alloys , melting point =~ 4050°F

Hassium	Hs	108	269	41	
Helium	He	2	4.0026	0.0001785	a gas at STP
Holmium	Ho	67	164.93	8.8	a rare earth metal, reactive , used in lasers, MRI machines
Hydrogen	H	1	1.00784	0.00008988	a hydrogen atom has 1amu, 1mol of hydrogen is ~1 gram- -hydrogen is a gas at STP
Indium	In	49	114.82	7.31	
Iodine	I	53	126.91	4.93	a non-metal , a vital nutrient in trace amounts, often put in salt
Iron	Fe	26	55.85	7.87	melts at 2800 °F
Iridium	Ir	47	192.22	22.6	rare, high corrosion resistance, melts at about 4430°F
Krypton	Kr	36	83.9	0.00373	a gas at STP
Lanthanum	La	57	138.91	6.15	a soft metal, reactive in air and water, various uses
Lawrencium	Lr	103	266	16	
Lead	Pb	82	207.2	11.342	melts at 328°F , **toxic** , moderately soft, very malleable
Lithium	Li	3	6.94	0.534	the least dense metal, soft , reactive with water , floats in H2O
Livermorium	Lv	116	293	12.9	
Lutetium	Lu	71	175	9.84	rare, a hard metal
Magnesium	Mg	12	24.31	1.74	an esential nutrient to humans and for photosynthesis in plants
Manganese	Mn	25	54.94	7.3	hard, but brittle, used in alloys, an essential nutrient
Meitnerium	Mt	109	278	28	man-made, synthetic , has a short half-life
Mendelevium	Md	101	258	10.3	
Mercury	Hg	80	200.6	13.53	a liquid at about -38°F = - 38.8°C , toxic - need a license
Molybdenum	Mo	42	96	10.3	a hard metal, an essential nutrient, used in alloys
Moscovium	Mc	115	289	13.5	
Neodymium	Nd	60	144.24	7.0	used in a mix of metals (NdFeB) so as to make strong magnets
Neon	Ne	10	20.18	0.0009	a gas at STP
Neptunium	Np	93	237	20.4	
Nickel	Ni	28	58.7	8.91	used in some magnets, coins and stainless steel
Nihonium	Nh	113	286	17	
Niobium	Nb	41	92.9	8.57	used for some alloys, and special magnets for MRI scanners
Nitrogen	N	7	14.007	0.0012506	a gas at STP
Nobelium	No	102	259	9.9	synthetic, highly radioactive, short half-life , decays to fermium
Oganesson	Og	118	294	5	
Osmium	Os	76	190.23	22.5	rare, the most dense metal known, 22.5 times denser than water
Oxygen	O	8	16.0	0.001429	a gas at STP
Palladium	Pd	46	106.42	12.02	a reare, prescious metal
Phosphorous	P	15	30.974	1.82	a non-metal , white phosphorous is toxic and can ignite , glows
Platinum	Pt	78	195.08	19.46	a rare, prescious metal, used to reduce smog from vehicles
Polonium	Po	84	209	9.35	extremely radioactive, danger
Potassium	K	19	39.1	0.862	a soft metal, reactive to water, a nutrient in small amounts
Plutonium	Pu	94	244	19.82	extremely radioactive , danger
Praseodymium	Pr	59	140.91	6.7	
Promethium	Pm	61	145	7.26	
Protactinium	Pa	91	231.04	15.39	
Radium	Ra	88	226	5.5	"ray" , highly radioactive, from decayed uranium , danger
Radon	Rn	86	222	0.0097	a gas at STP , from decayed radium , radioactive , danger
Rhenium	Re	75	186.21	21	rare, very dense, high melting point of 5767 °F
Rhodium	Rh	45	102.91	12.4	rare, very expensive, high heat tolerable, corrosion resistant
Roentgenium	Rg	111	282	28.7	very dense
Rubidium	Rb	37	85.47	1.53	
Ruthenium	Ru	44	101.07	12.4	a very rare metal, hard, corrosion resistant , used in electronics
Rutherfordium	Rf	104	267	17	man-made, synthetic , has a short half-life , decays to nobelium
Samarium	Sm	62	150.36	7.53	a rare metal, used in high temp. samarium-cobalt magnets
Scandium	Sc	21	44.96	3	a rare-earth element
Seaborgium	Sg	106	269	35	

Selenium , Se , 34 , 78.97 , 4.8 : a non-metal, is one of the trace, essential nutritional minerals
Silver , Ag , 47 , 107.87 , 10.5 : a precious metal , melts at 1762 °F
Silicon , Si , 14 , 28.085 , 2.33 : used in semiconductors and solar-cells , melts at 2577 °F
Sodium , Na , 11 , 23 , 0.971 : a soft metal, reactive to water, part of the food Salt NaCl molecule
Strontium , Sr , 38 , 87.62 , 2.64 : radioactive
Sulfur , S , 16 , 32.065 , 2.067 :a non-metal (ie., does not conduct normal electricity, not reflective)
Tantalum , Ta , 73 , 181 , 16.7 : dense, used in high capacitance capacitors and alloys, mp=5468F
Technetium , Tc , 43 , 97 , 11.5
Tellurium , Te , 52 , 127.6 , 6.28 : a rare metal, brittle, increases conduction with temp. and light
Tennessine , Ts , 117 , 294 , 7.2
Terbium , Tb , 65 , 158.93 , 8.25 : relatively soft, used to hehlp make green phosphors, electronics
Thallium , Tl , 81 , 204.38 , 11.85
Thorium , Th , 90 , 232.04 , 11.71 : has potential for safer nuclear fueled , electric power plants
Thulium , Tm , 69 , 168.93 , 9.32
Tin , Sn , 50 , 118.7 , 7.29 : commonly used in alloys of metals, melts at 500 °F
Titanium , Ti , 22 , 47.87 , 4.54 : has a low density, but strong
Tungsten , W , 74 , 183.84 , 19.25 : very dense , has a very high melting point of 6192 °F
Uranium , U , 92 , 238.03 , 18.98 : a nuclear fuel for the generation of electricity via fission and heat
Vanadium , V , 23 , 50.94 , 6.11 : a hard metal, used in alloys
Xenon , Xe , 54 , 131.3 , 0.0055 **:** a gas at **STP** (**S**tandard [reference]**T**emperature and **P**ressure)
Ytterbium , Yb , 70 , 173.05 , 6.95
Yttrium , Y , 39 , 88.9 , 4.47
Zinc , Zn , 30 , 65.38 , 7.134 : melts at 787 °F , used in some batteries
Zirconium , Zr , 40 , 91.224 , 6.52 : cubic zirconia is a crystal of zirconium dioxide (ie., has oxygen), and it may look like carbon diamond gem crystal, but with more rainbow colors or prism effect output of the incoming light.

The above values for atomic mass considers the typical amounts of natural isotopes in typical samples of the element, and are then essentially an average atomic mass, and where a small fraction of that mass is due to the mass of the low percentage of isotopes (ie., slightly changed atom structure) in the sample of the element.

Carbon has the highest melting point (mp) temperature of any element. The melting point of carbon is 6420 °F, and this is even higher than that of tungsten. This is due to its atomic structure and bonds. Graphite (a crystalline form of carbon) electrodes can be used in high current electricity and heating applications. Regular Carbon electrodes have a shorter time of use than graphite, but are also less expensive.

As of the year 2024, the 23 elements (or 20% of all the known elements) with atomic numbers from 95 through 118 are man-made and-or synthetic, and since they have not yet been found naturally. These elements are radioactive to some degree, and the higher the atomic number, the shorter the half-life time, hence they decay (release radiation) and-or change to another and more stable element more rapidly, hence radiate more radiation. A few other elements having atomic numbers 84 to 94, except 90 and 92 and including 43 and 61, are very rare and as only trace elements naturally, and they are rather also produced synthetically in particle accelerators ("atom colliders" or "atom smashers") and nuclear reactors by using an element with a lower atomic number and adding some protons or neutrons to its nucleus. It does this by giving a high kinetic energy to an atomic particle(s) which will be aimed at and strike the target material or element. Americium is a synthetic element that is used in many smoke detectors so as to help save lives and property, but there is other ways to detect some smoke such as by using a light sensor. Americium is made from a certain isotope of plutonium and has a much lower half-life. Americium decays to neptunium.

Hydrogen is considered the first and fundamental element, and that all other elements came from it later by applying huge forces to hydrogen atoms and further atoms of other elements created. The "Big-Bang" central explosion (still difficult to prove as of 2024, and some say a higher intelligent power could create it) which probably started the expanding universe. It is now thought that neutrons are another state or construction of a proton which has 2 up quarks and 1 down quark particles. Helium contains 2 protons, 2 neutrons,and 2 electrons. A neutron has 1 up-quark and 2 down-quark particles.

Sometimes when a radioactive or unstable element with extra neutrons emits radiation and decays, a proton and electron is created from a neutron. From observations, it is understood now that the mass of a neutron is slightly more than the mass of a proton, and therefore the atomic number of an atom of a element may have a small fractional value added to it. It could be thought that the mass of a neutron is equal to the mass of a proton and electron, and-or that a neutron is composed of the both mass of the proton and electron, and that the proton and electron charges either do not exist in the neutron and will arrive only when a neutron breaks apart and-or decays, and-or that the electric charges balance and cancel each other when in the neutral charge neutron.

Of the 118 known elements as of the year 2025, about 80% of them are **metals**. Of the remaining 20% of elements, they are classified as **metalloids** (about 5% of the elements, typically elements of which semiconductors are made from after mixing traces of other elements in them), and **nonmetals** (about 15% of the elements).

The 7 common **metalloids** are: antimony, arsenic, astatine, boron, germanium, silicon and tellurium

The 11 typical **nonmetal** elements that are gases are at STP (Standard temperature and pressure conditions): hydrogen, helium, oxygen, fluorine, neon, nitrogen, chlorine, argon, zenon, radon, krypton

The 6 typical **nonmetal** elements that are solids at STP are: carbon, phosphorus, sulfur, selenium, bromine (liquid at room temperature), iodine

The 6 **noble gases** at STP are: argon, helium, krypton, neon, radon (dangerous for long exposure), and xenon Nobel gases are also called inert gases, and of which are nonreactive (ie., stable) and non-bonding with other elements so as to make molecules. Their outer electron shell is full, hence it does not contain a loosely bound electron(s), or an electron gap or hole for other atoms and their electrons to bond with. These gasses are usually nonflammable, colorless, odorless, and have a relatively low boiling point temperature.

It is interesting to note that at standard temperature and presssure, the first two elements: hydrogen and helium are gasses, and already **the third element: lithium is already a solid and not a gas**. This is so even though llithium is a low density solid and metal, but it is still much higher than that of a gas. If the atoms of an element can join together into a crystal structure, they will create a solid.

How a star such as the Sun releases heat and-or light energy.

A star contains condensed hydrogen gas where each hydrogen atom has 1 proton and 1 electron. This hydrogen is from large hydrogen gas cloud in space, and of which will naturally condencse together naturally via the gravity of its atoms. This condensed hydrogen gas can then ignite due being subjected to intense pressure at the center of the Sun, and this then becomes a self-sustaining chain reaction of the **fusion** or joining of hydrogen atoms together, and into helium atoms. During this process, excess mass and energy is converted to energy (usually light, photon energy) and is radiated during that process where essentially two or more atoms becomes one atom. This fusion process in a star will convert hydrogen atoms into helium atoms. In particular, 4 hydrogen atoms consisting of a total of 4 protons and 4 electrons, will eventually be fused together, one at at time, so as to form 1 helium atom having 2 protons, 2 neutrons, and 2 electrons. Here are the common proton-proton chain of fusion steps and results within a star:

1 Hydrogen atom fused with 1 hydrogen atom becomes 1 deuterium atom having 2 protons.
Deuteriuim is an (heavier, more mass) isotope of hydrogen, and is sometimes called as hydrogen-2.

2. 1 deuterium atom fused with 1 hydrogen atom becomes 1 heliium-3 atom having 2 protons and 1 neutron.

3. 1 helium-3 atom fused with 1 helium-3 atom becomes 1 helium-4 atom having 2 protons and 2 neutrons,
The 2 extra protons will become free protons after this process. Helium-4 has slilghtly less mass than the hydrogen atoms used to make it. Helium-4 is a isotope of helium, and it is stable.

Common Particles Within Atoms (Atomic Particles)

Fundamentally, each element has a unique and the same number of protons, neutrons and electrons in each of its (similar) atoms. These basic parts of an atoms are what give the atom its mass (a measure of the amount of physical matter). Protons and neutrons are called atomic mass units, and are generally thought of as the main particles of mass in an atom. Note that the number of **amu** (or u, **atomic mass units**) per mole of an element in basic atomic theory is:

amu per mole of atoms of an element = (number of amu per atom of the element)(number of particles in a mol unit)
amu per mole of atoms of an element = (2)(atomic number)(Avogadro's number)

The number of amu per gram of atoms of an element in basic atomic theory is:

amu per gram of an element = (amu per mole of atoms of the element)(number of mol of atoms per gram)

The number of atoms per gram of an element in basic atomic theory is:

atoms per gram of an element = (amu per gram of the atoms of an element) **/** (amu per atom of the element)
or (amu per gram of the atoms of an element / 2) **/** (number of protons in each atom)

: For hydrogen, atoms per gram = (amu per gram of hydrogen), and not divided by 2 since hydrogen does not have a complementary neutron for each proton. Also hydrogen has 1 amu per atom, and which is 1 proton.

The number of atoms of an element, per mol of amu is: (Avogadro's number) / (amu per atom of the element)

Note also that 1 mol of amu is defined as having a mass of 1g, hence the value above is also equal to:
number of atoms of an element per gram = (Avogadro's number) / (amu per atom of the element).

It is believed, and expressed here in simplicity, that the "Big-Bang" (ie., like a highly energetic "supernova" or explosion of a star) origin of the universe contained all the initial particles (electrons, protons and neutrons) of the atoms and mater, and that the first atom created shortly afterwords was the hydrogen atom. It would later take the gravitational forces of a large hot energized star to create (ie. due to a "fusion" of particles) most of the other "heavier (in weight)", more massive elements from the hydrogen gas and some other low density atoms or elements such as helium. To create all the other elements that are more dense than iron, a star will need to undergo a supernova (ie., explode, creating tremendous pressures in regions of and near it) and-or collide with a relatively small (about the mass of the Sun), very dense neutron star that is the remnant of some "collapsed star" such as from a previous supernova, and has an abundance of neutron atomic particles that do not have an electric charge. In general, and statistically, the greater the density of an atom or element, the less abundance of it and the less stable it is, and can "decay" into, change or convert to a lighter (less dense, less weight), less massive element (same atoms) and having greater stability. After the (probable) big-bang, by the force of gravity (ie., force of attraction of matter), matter condensed into stars, orbiting planets, moons, asteroids, comets, and galaxies of stars held by their own gravity. The majority of the mass of the Earth and its life are sometimes said as being made of "**stardust**". In general, given a hot molten planet, the more heavier, denser elements will sink to the core region.

The metals called **alkali metals** are generally found in nature only as an oxide (molecule contains an oxygen atom) form since the react with other substances such as water and easily loose a valence electron and become positive charged ions, here with a net charge of (+1), and are sometimes called cations (cathode or cathodic ions). Sodium, potassium, lithium, rubidium and caesium (aka: cesium) are alkali metals, often called the "lithium members, family or group" of metals in the periodic table of elements. These metals are soft and very reactive, and will oxidize in the air and react highly with water and-or moisture. Sodium, potassium, and a trace amount lithium are essential or vital nutrients for the human body. Alkali metals area also aid to be base (on the pH scale, > 7.0) metals and will form hydroxide ions (OH-) when dissolved in water. Calcium metal is a soft metal, and is also an essential or vital nutrient, but is not in the same group of metals as those just mentioned, and is in the "alkaline earth metal" group along with metals such as magnesium and barium, and these are also often found in nature in its oxide (molecule contains an oxygen atom) and compound forms such as limestone, and having lost two valence electrons, and hence having a net positive charge of (+2). About

4% of the Earth's crust in weight is calcium metal, hence it is farily common. **Hardening of the arteries** is a serious health issue and is caused by **calcium** buildup on the walls of blood vessels; seek doctor and-or nutritional guidance.

Notice that most elements except the gases and a few other elements are metals. Carbon, sulfur, phosphorous and iodine are the most common non-metalic elements. Some elements such as the ones known as semi-conductors are sometimes called **metalloids**. The most common metalloids or semi-metals are silicon, germanium, boron and antimony,, tellurium, arsenic Metalloids have properties of both metals and non-metals, hence they are often semiconductors which do not conduct electricity very good. Metalloids are brittle, and this can be easily seen while handling a thin and brittle, glass-like silicon solarcell.

The density (d = m / V) of metals is generally or universally a constant and-or average value with not much deviation from it, however the density of gasses greatly depends on the heat and-or pressure applied to them, such as by an atmosphere and-or a heat source such as it being deep in the ground and-or a volcano chamber. We see that the density of gasses can vary widely by the physical conditions applied to it, and so their density is usually expressed as its density at a **standard temperature and pressure (STP)**.

(The reader may skip over this discussion if it is too complex right now and-or may view it at another time.)

An example of comparing density (p) and atomic mass (g/mol) values of two elements:

	P	(g/mol)	p
Aluminum , Al	13	26.982	2.698
Iron , Fe	26	55.845	7.87

: P = number of protons ,

Iron has twice the number of atomic mass particles in each atom as compared to aluminum:

Calculated here using just the given number of protons for each atom:

$$\frac{\text{\# of protons in an atom of iron}}{\text{\# of protons in an atom of aluminum}} = \frac{26}{13} = 2$$

: iron has twice the mass , or divide the amu in each element atom

Now comparing the corresponding densities, (g/cc):

$$\frac{\text{density of iron}}{\text{density of aluminum}} = \frac{7.87 \text{ g/cc}}{2.698 \text{ g/cc}} \sim= 2.91\underline{7} =\sim 3.0$$

: about three times as much per unit volume. A cubic-centimeter of iron will weigh about 3 times more than a cubic-centimeter of aluminum. Also note the equivalent fractions to make for density: 1g / volume = 1g / x cc. Ex. Iron has 7.87g/1 cc = 1g / 0.127 cc

If the mass or amount of a substance or material of something such as a particular element increases by a factor (n), then the weight of that something will also increase by the same factor of (n), and vice-versa. If the mass and-or weight of something increases by a factor (n), then the volume of that something will increase by that same factor of (n), however, the linear physical dimensions (L, W, H, such as for a cube shape) after that increase will only change by the cube-root of (n). For example, given 1 cube (ex, L=1 unit, W=1 unit, H=1unit) that has a mass of 1 gram, if you then make a larger cube of that same substance, say L=2 units, W=2 units, H=2 units, then this cube is: 2 x 2 x 2 cubic units = 2^3 cu. units = 8 cu. units. At t 1 gram/1 cu. unit, this is then 8 grams total mass. The cube root of 8 unit^3 is 2 units^1 = the linear dimensions of it.

Geometrically, mass and weight are one dimensional, linear or a "**scalar**" concept of a unit, and a volume unit is a three dimensional. Consider a unit cube with each side length (L, W and H) of 1 and having a volume of 1 cubic unit. If the length of the side of a cube doubles from 1 to 2, a linear change, the corresponding volume increases from 1 to 4 cubic units, hence the corresponding volume changes are not linear or proportional. If the linear (dimensions, mass, weight) change is (n), the corresponding volume change is (n^2), hence the change is **exponential** (not linear, as in a certain factor or constant) increase.

A doubling of atomic mass units (AMU) does not mean there will be a doubling of the size (width, volume) of a theoretical spherical nucleus of an atom. The sphere-like or shell-like atomic arrangement of the atomic mass units within each atom and the orbiting electrons, and of the atom alignments and-or crystal structure and its atom alignment and spacing of a particular element will affect the amount of substance (ie., mass) in a given volume, and this will then be reflected into the volume and density value of that particular element. **Density of an element is not proportional to the atomic number of that element, but it is usually related to its value. Higher atomic number elements generally have a higher density.**

Consider having 100 atomic mass units such as protons. If you were to arrange them into a side by side square plane (2 dimensional) shape, there would be $\sqrt{100}$ = 10 atomic mass units x 10 atomic mass units. 100 atomic mass units side by side would be 100 units long, but the area would only have 10 mass units per side or dimension. In short, the density and-or volume of an element is not proportional to the number of atomic mass units of an element. Consider "heavier", "more dense" elements with more atomic mass units per atom will then have a smaller volume per gram of mass.

If you were to arrange 100 atomic mass units into a side by side cube shape (3 dimensions), there would be $\sqrt[3]{100}$ =~ $4.64\underline{2}$ atomic mass units for each unit of length, width and height. To complicate matters, various sized atoms having various amounts of mass (amu = atomic mass units which are protons and-or neutrons), and various atomic arrangements and-or crystallization will also affect the volume of a mass, and therefore the amount of atoms needed to fill a given amount of volume such as a 1cc. Generally, entire atoms are worked with since the particles of it are essentially locked into position and cannot be normally isolated and that their mass is too small to be isolated, counted and of practical use. Even the mass of even an atom is too small to be isolated, measured, counted and be of practical use, and a huge number of atoms called a mol (or "mole", an old word for mass) of particles is used as the reference unit for the number of atoms to be worked with in a practical and measurable (by weight) manner.

If you had 10 atomic mass units and increased it by a factor or 10 or "10 times" to 100 atomic mass units, the corresponding weight would increase by 10, but the corresponding volume would only increase about 4.64 times as shown above. 4.64 is considerably less than 10. The volume in this example would have 4.64 units per side or dimension.

Atoms with more **atomic particles** (neutrons and protons) or atomic mass units (amu) have more atomic mass units per atom. Their atomic mass per atom is higher, or in other words, their atomic mass unit density per atom is higher. Atoms with more atomic mass units will correspondingly and proportionally weigh more. Bigger atoms with a larger diameter and volume will require more volume for a given number of atoms, and therefore mathematically, for a given volume, there will be fewer larger atoms per volume than the number of smaller atoms that have fewer atomic particles (amu). As indicated above, volumes for the number of (amu) particles and-or atoms are not proportional. For a given weight, there will be also be fewer larger mass, diameter or volume atoms than the number of smaller mass, diameter or volume atoms, but still, the total number of particles (amu) due to the larger mass (amu) atoms will be the same amount as that of the lower mass (amu) atoms for a given weight. Weight is therefore a better indicator, comparator, or "scale" of determining an amount of mass, material or substance than volume is.

On a scale, if two substances have the same weight, their amu count or total mass is the same, and regardless of the various types of atoms (ie. elements) which have different amu amounts.

As a thought example:

Substance or Element	amu per atom	atomic weight per atom
A	5	5 units of amu weight or "5 weight units"
B	10	10 units of amu weight or "10 weight units" Here, since mass is twice as much, its weight is twice as much. That is, if mass doubles, weight doubles.

Given 1 atom of substance B, it will take 2 atoms of substance A so as to have the same amount of total (amu) mass and weight.

Weight is a property that depends on both the amount mass and the strength of the local gravity influencing it with a force and giving it that weight value. A mass value is universal, however weight values are not. For example, the weight (a force) of an object will be less on the Moon than it is on Earth, and this needs to be considered for various kinds of science research.

A gram was originally defined as a unit of weight, and specifically, it was defined as the weight of 1cc of water at 4°C. If some other substance weighed the same amount, then it has the same mass in terms of the number of amu, but not in terms of the number of atoms which for different substances or elements have a different number of amu per atoms.

For a given weight value, say an ounce of weight, when the amu per atom increase, such as for another element, there will be fewer atoms needed for the same weight.

The more amu particles, substance, or material in a unit volume, the more mass in that volume and the denser that material is said to be. density = mass / volume . As indicated before, given atoms with a larger mass, there will be fewer of them for a certain weight. This is an inverse relationship. For a given number of atoms, larger mass atoms will occupy a larger volume than that of lower mass atoms. More lower mass atoms can be put into a certain volume than larger mass atoms.

For learning more about matter and mass, please see the following discussion called: More About Matter, Mass And Mol,

Fundamentally, for pure samples of an element have the same number of protons and electrons, and without considering the possibility of a small percent of atoms being isotopes. An **isotope** is a **nuclide** (an atom of an element that has a different number of protons and-or neutrons. "ide" is a word suffix with a general meaning of a combination of things) with the same number of protons and a different number of neutrons than the average, common or fundamental atom of the same element. For an example, "atomic" or fundamental hydrogen with one proton is sometimes called **protium**, and hydrogen with 1 neutron (created from a hydrogen proton, during fusion) added is an isotope of hydrogen called **deuterium.** Deuterium is used in "heavy water" that cools and prevents hot nuclear fuel rods from melting, and it also absorbs radiation. Hydrogen (one proton) with 2 neutrons added is called **tritium** which is usually man-man = artificially made = synthetic, often from colliding lithium atoms witih neutrons, and of which is ("low energy" beta) radioactive with a half life of 12.3 years. Tritiuim decays (or "reduces or converts to" after releasing radiation) to **helium-3** which is stable, and which has 2 protons and 1 neutron. Helium-3 is very valuable as a future energy fuel, and there is a large amount of it on the Moon's surface. The radiation from tritium can also be used to cause a phosphor coating to glow (ie., gain energy and will then emit that energy as visible light) and can then be used in indication lamps for low-light and-or night-time usage. The cost of tritium is about $30 USD per milligram as of the year 2024.

Atomic Mass (matter) = (2)(Atomic Number) = (atm) of each atom of that element
= (2)(number of protons)
= (number of protons) + (number of neutrons) = atomic mass units = amu

When the probable or actual number of neutrons are considered, the atomic weight or mass of an atom is, at minimum, twice the atomic (proton) number plus some small percentage of that atomic number due to a high probability of some isotopes (have extra neutron(s)) of that element that may be present in the sample. This small increase generally increases as the atomic number increases. Though an atomic (proton) number of an element is constant, the more accurate atomic mass is an average or expected value that can be considered. The mass of an electron is very negligible and is generally not considered for the atomic mass (or weight) of any atom. The mass of an electron is sometimes noted as about (1/2000) or more modernly and precisely as about (1/1836) that of 1 proton or 1 neutron. The sum of the mass of all the electrons in an atom is still negligible and does not even come close to the mass of one proton or neutron. With this reasoning, 2kg = 2000g of mass will have about 1g of mass due to electrons.

Ex. The most common or pure form of carbon has an atomic (proton) number of 6 has an atomic mass of 12 and is identified as C12 ("carbon-twelve") and has a fundamental atomic mass of 12amu (atomic mass units = protons + neutrons). When considering a small percent of those carbon atoms in the sample being probable isotopes, such as C14 ("carbon-14") which has 2 extra neutrons and which increases the total number of atomic mass units in that atom, and the "weighted" or average mass is slightly higher than 12, is considered as 12.0107amu. Though this is only a small percent increase, but when considering many millions of probable isotope atoms in the sample of the element, this small percentage may need to be considered. Certainly, a pure (all similar atoms) sample of C14, having slightly more mass than C12, weighs slightly more than C12 of the same volume.

Ex. Iron atomic (proton) number is 26, and its atomic mass is therefore (2)(26 amu)=52 amu, however due to isotopes often being present in most (practical) samples, its atomic mass or "molar mass" (g/mol) is listed as 55.845. The concept of mol ("mole", unit of the amount of matter) is given a robust discussion ahead of this topic. Taking the reciprocal of this value we can find (mol/g), and it is: 1 / (55.845g/1mol) = 0.017906706 mol per gram. If 1 mol of atoms is to 6.023 (10^23) atoms, then 0.017906706 mol is to (x) atoms, after solving for (x), x atoms = (mol)(atoms / mol) = (0.017906706)(6.023)(10^23) atoms = 0.10785 (10^23) atoms , in 1 gram of iron. or: 55.845g = 1 mol atoms = (6.023)(10^23) atoms , after dividing both sides by 55.845: 1g iron = 1.0785 (10^22) atoms.

For stable, (electrically) balanced or neutral atoms, the number of electrons (-, neg. charge) equals the number of protons (+, charge). Neutrons have approximately the same mass as that of the protons, but neutrons do not have an electric charge to consider for electric and electro-chemical reactions.

Natural isotopes of elements that have a very long half-life and are said to be stable and naturally emit none or very low radiation. Unstable or radioactive isotopes (or a specific "nuclide" form of an atom with a certain number of neutrons or protons) decay into other atoms. Isotopes can also be artificially ("man-made", not naturally, "synthetically") produced by using the high energy and speed of atomic particle collisions such as in particle colliders.

Ex. C12 is "carbon twelve" which is the fundamental form and atomic mass of carbon. C14 is a (radioactive) isotope of carbon or C12. It has two more neutrons, and is made naturally in relatively small quantities (about 1 in a trillion atoms of carbon) due to cosmic radiation, and can be absorbed (ie., breathed, etc) by living things until they no longer are alive. C14 has a half life of about 5700 years, and can be used to help determine the age of the material which contains it. C14 has two extra neutrons as compared to C12. Much of the weight of living plant matter and trees is due to the presence of water, but the solid matter remaining is mostly carbon due to carbon-dioxide gas from the air that the plant matter "breathed" as part of their **photosynthesis** process to produce sugar from sunlight and carbon-dioxide. The photons from the sunlight will split apart the **carbon-dioxide (CO2)** molecule into carbon and oxygen atoms, and produce a sugar fuel for the plant. The water in living matter gives it substance for growth, nutrient and process fluidity and mobility (or "transport"), and is also used to as part of its circulation (ie., transportation, distribution) of nutrients and-or waste removal, and temperature control. All the **life** that we know of is said to be mostly hydrocarbon (a water and carbon molecule and-or compound) based with other various atoms (minerals, elements) used to create other molecules such as for vitamins (nutrition) that the body uses for certain necessary processes for its sustenance and energy, growth and repair. Natural substances that compose and-or promote living organisms are called **organic**. C14 radioactively decays naturally (back) into N14 = nitrogen14 and which is the most common and stable form of nitrogen. Plants absorb carbon dioxide from the atmosphere and convert it into sugar, carbon and oxygen. A dry piece of wood has a high carbon content of about 50% and about 40% oxygen by mass or weight, and the remaining 10% is trace minerals. About 1% of a dry piece of wood is converted to wood ash after it is burned (combustion). The dry ash contains forms of about 50% calcium, 25% potassium and 7% magnesium, and reasonable trace-levels of sodium, phosphorous, silicon, and iron. The type of the plants or trees, and-or its soil and environment will help determine what minerals it contains. If the heat was high during the combustion of the wood, then some elements may be turned into gas and essentially "vaporized" into the atmosphere if not collected, and this is what happens to most of the carbon mass. It is usually converted into carbon dioxide (CO2) gas, and-or is converted to poisonous **carbon monoxide** (CO1 = CO) gas and soot (mostly carbon particles) if it was weakly (**incomplete combustion** such as due to a low amount of oxygen) burned (combusted with oxygen). Soot, a fine particle, carbon byproduct of combustion can dangerously clog or block a chimney and-or reduce its performance of getting the fumes outside into the air, and away from your lungs to breath. It is part of the visible "smoke". Carbon monoxide is flammable. It can also be a health and fire hazard. Carbon monoxide (CO1 = CO) with one oxygen atom per molecule is very similar to **carbon dioxide** (CO2) which has two oxygen atoms in its molecule.

Carbon monoxide (CO) can attach itself (connect, bind) to hemoglobin in the blood which carries oxygen to the body, and therefore, it can deprive the body of oxygen it needs, and this can cause sickness and-or can be fatal. People have fallen asleep and later died when the concentration of CO in the blood increased. Carbon dioxide is also produced by people exhaling, and this can be a health problem if there is not a sufficient amount (% per volume) of oxygen available in the local area, room, etc, and some symptoms are dizziness and concentration issues. If a person has carbon monoxide or carbon dioxide poisoning, remove them from the local environment, and-or ventilate the area so as to breath in less CO or CO2. Seek medical attention, dial 911 if in the United States, and one treatment is to breath in pure oxygen for an amount of prescribed time. Carbon dioxide can also lower the pH in the blood and body, and therefore it can cause an acidity issue.

Plants enjoy carbon dioxide, and release (ie., essentially exhale) oxygen. Carbon monoxide and carbon dioxide are odorless colorless. Some sensitive people can smell a high level of carbon dioxide as an acid-like smell, but not carbon monoxide. Even very low concentrations of carbon monoxide can cause health issues. CO is sometimes used in the chemical industry to make products. Dry-ice is the frozen, solidified form or state of matter of carbon dioxide gas. There are electrical devices and-or alarms (much like a smoke and-or fire alarm) that are available which can sense the level of carbon monoxide or carbon dioxide in the local environment.

> The density of carbon dioxide is: 0.00198 g/cc , and the density of carbon monoxide is: 0.00145 g/cc
> Since air has a typical density of about 0.00124 g/cc, = 1.24g/L both carbon dioxide and carbon monoxide are heavier than air gas (mostly nitrogen and oxygen) and will tend to be at a lower level in a room with no fan circulating and mixing the gasses in that room.

A **candle**, when ignited (ignition, "started") by another heat source, and becomes lit (ie., has a "flame", self ignition, chain or continuous reaction process, once it is started) is a form of combustion of the carbon, hydrogen and oxygen in the wax. This chain reaction continues until the fuel, here wax, a hydrocarbon, is used up or gone. Heat and light energy is released. The heat energy is due to the energy stored in the oil or hydrocarbon molecules that get separated during the combustion process, and the light is due to electrons that have gained kinetic energy due to the combustion process, and then those electrons convert and give off (ie., radiate) that energy as light energy when they join with an atom that needs an electron. A byproduct of the combustion or burning of a hydrocarbon (ie., a water and carbon molecule) in the air (usually containing O2 molecules = a molecule of two oxygen atoms bonded or joined together) is a small amount of water steam (H20 as its gas state), and carbon dioxide (CO2).

The international system (SI or S.I.) unit of mass or matter is the mole (mol) unit. The number of atomic mass units (protons and neutrons) in a hydrogen atom is 1amu. 1 mol of hydrogen atoms or protons will weigh about 1 gram. The number of atomic mass units (protons and neutrons) in a carbon atom is: 12amu. 1 mol of carbon atoms will weigh about 12 grams since it has 12 times more atomic mass units or particles than that of hydrogen which only has 1 proton for its nucleus. It can be said that there are 12 grams of weight / mol of carbon atoms. The atomic mass or molar mass (matter) number or rating of C12 in grams per mol = **grams / mol** = 12 grams / mol or 1 mol / 12 g . Dividing both the num. and den. by the atomic number we can find the number of mols per gram, or matter per gram weight = **mols / gram,** which is essentially the reciprocal of the atomic weight: (1/12) mol / 1g = 0.0833 mol of carbon atoms / 1g. The larger the atom, the more heavier and more massive it is having more atomic mass units or particles of mass, and therefore a lower number of atoms needed to make 1 gram of matter.

The (weight, not mass) density of carbon is: 2.26g / 1 cc . If 1 mol of carbon is to 12 grams, how many mols of carbon are in just 2.26g of it? First, 1 mol carbon atoms / 12 g = (1/12)mol carbon atoms / 1g : after dividing both num. and den. by 12 so as to create an equivalent fraction. The result is:
0.083333 mol / g , then: (2.26g)(0.083333 mol / g) = 0.18833 mol In general:

mols / gram = 1 / (grams per mol) = 1 / atomic weight rating **: g / mol = atomic weight rating**
atomic weight = molecular weight

x mol = (grams of an element) (mols / gram)

If a formula said to use 0.2217 mol of carbon, how much would that amount of carbon weigh? Since 1 mol of carbon is to 12 grams, and creating a proportion type of equation:

$$\frac{\text{1 mol carbon atoms}}{\text{12 grams of carbon atoms}} = \frac{\text{0.18833 mol carbon atoms}}{\text{x grams of carbon atoms}} \qquad \text{mathematically:}$$

$$\text{x grams of carbon atoms} = \frac{\text{(0.18833 mol carbon atoms)(12 grams carbon atoms)}}{\text{(1 mol carbon atoms)}} \approx 2.26 \text{ grams}$$

In general:

x grams of an element = (mols of the element) (grams / mol) : converting mols of an element to grams

Due to that a substance can be compacted and-or compressed to a smaller volume, and that also increases its density for that unit volume, such as with a powdered form of a substance such as carbon dust or flour, it is generally not recommended to determine the weight of something by using a volume of that something. Depending on how much it is compacted or not, 1cc of carbon powder may not weigh exactly 2.66 grams. It is usually best to always use a substance's known weight so as to measure the weight or mass of a substance, rather than use the volume of it. If known measurements were previously made and if you do use a consistent substance and volume of it in a repetitive manner to measure mass or weight, then it is possible to get practical, but not necessarily precise or scientific results with that method of using a volume of a substance.

Since: density = (g/cc) and molecular weight = (g/mol) , multiplying the left equation by the reciprocal of the corresponding sides of the right equation, or= dividing the corresponding sides:

$$\frac{\text{density}}{\text{molecular weight}} = \frac{(g / cc)}{(g / mol)} = (g / cc)(mol / g) = (mol / cc)$$

: **mol per cubic centimeter of an element**

mol = (cc)(density) / (molecular weight)

$d = m / V$, $m = d V$

Electrons and their orbit about the nucleus of an atom

In 1913, **Neils Bohr** conceived the basic model of an atom that we commonly know of and accept. The basic model is that electrons orbit the nucleus which is composed of neutrons and protons. Since larger, more dense and-or heavier atoms have many electrons, some of those electrons will then be in less crowded, more distant orbits where there is not too much of negative charge from other electrons to repel them out of orbit. These orbits having the same radius distance from the nucleus are called (orbital or electron) **shells**, and they are spaced apart at precise distances from the nucleus and each other. These shells can be considered as like the surface of a sphere, and the farther from the center, the surface area increases and then giving much space for electrons to coexist in that same orbit.

The negatively charged electron orbits the positively charged nucleus which contains protons, and is due to the attraction of those charges rather than the very negligible amount of gravity. The minimum distance for an electron to orbit from the nucleus is called the **Bohr radius,** and therefore it is also the closest shell. This distance was found with the aid of Planck's constant (h). Max Planck is discussed in this book in a few areas. Each other shell is at a integer multiple of this distance. This Bohr radius distance is due to the wavelength or distance that this closest electron orbits about the nucleus. For an electron to break free from its orbit and become a free electron, some energy (kinetic and-or electric force) must be applied to it.

The orbital shells are given a letter and-or a number such as: K=1 , L=2 , M=3 , N=4 , and so on. Each shell has a maximum number of electrons, and that shell is then said as being fully filled. The first shell can have a maximum of 2 electrons since there is a relatively low amount of "room" and-or surface area for those electrons from not interfering with each other electrically (ie., here, the repelling charge and its corresponding force to nearby other charges).

The surface area of a sphere is As = 4 (pi) r^2 , if the radius =r doubles to a value of (2r), then (2r)^2 = 4r , hence the surface area will double when the radius doubles. The maximum number of electrons in the second shell will be 4 times more, hence (2 electrons) (4) = 8 electrons maximum in the second shell. If the radius increases by one more unit to (3r), then (3r)^2 = 9r, hence the surface area will increase by 9. (2 electrons)(9) = 18 electrons maximum in the third shell. Due to this calculation process and observing the results, the maximum number of electrons in a shell is:

maximum number of electrons in a electron orbit shell = (2 electrons)(shell number)^2

In the standard Periodic Table of elements, each next row downwards shows elements that have one more electron shell.

Electrons in the shell that is the farthest distance from the nucleus are called valence electrons, and they are said to reside in the valence shell. These electrons are relatively loosely bound to the nucleus since the electric force greatly reduces when the distance increases, and in an inverse square way. In a pseudo formula:
(electric field strength2 at r2) = (electric field strength1 at r1) / (r2)^2

The electrons in a primary shell n may be concentrated in what is called a **subshell** of a shell. Subshells are assigned the letters and-or number: s=1 , p=2 , d=3 , f=4 , and g=5. The maximum number of electrons in these shells is:
2 for s , 6 for p , 10 for d , 14 for f , and 18 for g. We can notice that each new shell can have 4 more electrons.
To then identify the second subshell of shell 3 , this notation would be used: 3p , but another possible notation to consider might be: (shell number, subshell number) , and then 3p = (3 , 2) or= (3-2) or= (s3,ss2)

Subshells are also associated with the energy level of those electrons, and a subshell does not need to be filled in order to begin having electrons in the next shell. Subshell s or 1, is for lower energy, specifically lower angular momentum (p=mv), lower velocity electrons which then reside slightly closer to the nucleus. The orbits of several electrons in some subshells can compose a unique shape, and when they are elliptical orbits. Subshells are usually filled with electrons starting at the first subshell (s), and then proceeding to be filled into higher numbered subshells..

This book will generally not further discuss electron shells or orbits because it is beyond the scope of this book, however, some people might appreciate a basic mention of them as given here. If interested further, you can also research the Aufbau principle for electron orbits, shells, etc.

Some other helpful formulas to remember for finding some needed values are :

weight density = weight / volume : (weight-density / g) = mass density , g=a=9.81 m/s^2
mass density = mass / volume : mass or matter density , mass = force / acceleration = weight / g
mol density = atomic mass rating of an element = g / mol or mol / g : "atomic density" or "mol density"

Elements that are less dense (less mass and-or weight per volume) are said to be "lite elements", and their atom size is very small due to having a low number of atomic mass units (amu). To make a gram (weight) of this element will require many more atoms than that when using heavier atoms which have more atomic mass units per atom.

The identity of a (pure) element can be found by weighing a known volume size or amount of it and comparing it to what is listed in a table of elements. Mass density = mass / volume = mass / cc and Weight density = weight / cc

To help identify a metallic element, if a metal contains iron or is pure nickel, a magnet will stick to it. Usually stainless steel which does contain iron will not attract a magnet, but there is a certain type that will. Color and-or hardness (or softness) are good metal identifiers. When a metal is heated in a flame or other mechanical process such as a grinder, it can emit a certain color which can identify it. Fireworks are colorful displays that often use metals to produce the various colors. Some metals are toxic, so it is best to handle metals and any dust, fumes and-or smoke with caution, and do not breath in any of that. Use proper ventilation, eye and breathing protection. **Spectrometry** is a field of study to determine substances in a sample by using and observing (with a spectrometer instrument) various frequencies (ie., seen as colors, for example, a green color when copper is heated hot, such as for a "flame test", "(light) emission" spectrometry to identify an element) of light emitted when a substance is heated and **emits** certain colors and-or frequencies of light. This light is produced when the electrons gain kinetic energy and become free electrons, and then will radiate that excess energy as a frequency of light when it "falls" back into an orbit about the atom. The electron is then said to be in its ground state, hence 0 excess energy than normal. Some ther materials and the colors seen: sodium = yellow , iron = yellow-orange , calcium = red-orange, potassium = violet. When light such as starlight passes through a gas, that gas will **absorb** a particular light frequency and is an indication of what that gas is. The word-prefix of "spect" is related to vision, and a rainbow of colors is sometimes called the (visible) spectrum of colors, and an event to see is sometimes called a spectacle.

For the number of mol particles, such as atoms of a gas, in a volume of a gas = molar-volume

It was previously noted that 1 mol of gas atoms at standard temperature and pressure (STP = 0°C =32°F, and 14.7psi =
= 1 atm = 1 bar = ~100000Pa) will naturally (no additional temperature and-or pressure) occupy or be in the volume of 22.4L of that gas. This concept is also for a gas of a particular element and not for a mixture of gases. 1 atm = 1 **atm**osphere unit of air pressure = 1 bar. **Bar** is an old unit for weight, and then for pressure, and which for air, its pressure became called as **bar**ometric pressure. "Bar" comes from the old Greek and Hebrew word "baros" meaning weight, burden and-or a load, and of which brought about the word "bear" as in "to bear of hold a weight" and is the same as the English word (letter) spelling for the bear animal. A similar sounding word of "bare" has a general meaning of "without".

1 atm = 14.7 psi =~ 6.668 kg / in^2 = 6.668 kg / 6.4526 cm^2 = 1.0335421 kg / cm^2 =~ **1 kg / cm^2**

24.4L =~ 5.92 gallons =~ 1367 in^3 = a cube shape of about: (11.1 in.)^3 which is nearly a cubic foot. When 24.4L of gas is put into a container, and at STP conditions, the inside of that container walls and the internal gas will have 1 atm =14.7 psi of pressure. As you can imagine, if you try to fit any volume of gas into a smaller container, both the density and pressure of that gas will increase.

Ex. The density of helium = 0.0001785 g/cc. Since the volume of a liter is defined as 1000cc, and after multiplying both the numerator and denominator of this value by 1000, we have: helium = 0.1785 g / 1L = 1g/5.602241L. This is for grams of atoms of helium. Note that a helium atom has 2 protons and 2 neutrons, and therefore has 4 atomic mass unit per atom. The molar mass of hydrogen atoms is listed as about 4 grams of mass / mol. Setting up a proportion type of equation:

If: it is known that there is a density of: $\frac{0.1785\ g}{1L}$ then: $= \frac{4g}{XL}$ with: X =~ 22.4 L
1L = 1000 cc = 1000mL
: Equivalent ratios and-or proportions

And:

$$\frac{0.1785\ g}{1L} = \frac{Xg}{22.4L} \quad : \text{with } Xg = 4g$$

Extra: As the temperature of a gas increases, it will have more kinetic energy and will occupy (if able to expand outwards) a larger volume and will get less condensed, more dispersed, and have less matter, less weight, and less density (of matter) per unit volume. It will therefore take a larger volume of that gas and its new current state (status, conditions, temperature, pressure) so as to have 1 mole atoms of it. A common example of a gas trying to occupy a larger volume is in the cylinder of a combustion (gas, petrol) engine when the fuel ("gas") and air (ie., oxygen) mixture combusts which is a quick release of stored chemical (bonded atoms) energy. Combustion is a chemical reaction causing a high pressurized expansion or "explosion" due to excited hot gas atoms that have a high kinetic energy. This energy will create a force that will greatly increase the internal pressure in the cylinder (ie., a strong metal, combustion and containment chamber) space, and the hot gases will push downward on the (movable) metal piston within the cylinder inside the engine. This energy and-or force applied to the piston will eventually be applied to the wheels of the vehicle or to some other machine so as to provide a rotational force (ie., a torque) such as the wheels pushing against the road surface so as to produce a forward movement of the vehicle. **Combustion**, essentially a form of burning or "consuming", is a chemical reaction of the (carbon) fuel and (oxygen, or "oxidizer" substance) air mixture. Combustion can be described as (rapid, fast) oxidation where oxygen or some other element removes electron from another element and-or substance. During combustion, heat (ie., atoms with kinetic energy) and its resulting pressure (ie., force) will form. If the fuel is a **hydrocarbon (ie., a hydrogen and carbon molecule)** combusting in the air (ie., with oxygen), there will also be a substances formed such as hydrogen, carbon dioxide (a carbon and oxygen molecule) and water (hydrogen and oxygen) in the (initial) form of hot steam (ie., water gas). To start ("ignite", initiate) the chemical reaction of combustion, a source of high heat is needed, and an example of this is when the air temperature inside a piston goes to a high temperature value when it is being compressed and has a high pressure. This initial compression, to essentially "start" (begin the cycle of combustions) the self-running, chain-reaction of compressions and combustions was used with the first type of automobiles manufactured. Here, a person would start a car (automobile) by manually (physically) turning a crank lever with some force so as to move (or "turn") and compress a piston(s) in the engine so as to begin an initial combustion inside that engine of the vehicle. Later, by 1911, geared, high torque, mechanical power and-or force, electric "starter motors" began to be used to "turn-over" (move, cycle) the engines pistons and start a vehicle.

Combustion (ie., oxidation) can also happen slowly ("slow combustion") such as with aging wood or paper after a long time, and it will get darker and have a higher percentage of carbon as the water evaporates. Combustion can happen very quickly as during the explosive combustion with "fireworks" displays. The rusting on iron is due to oxidation. Rust particles are iron-oxide (a molecule of iron and oxygen, and with the aid of water) and it usually has a brown-red or dark-orange color. A candle or a fire with burning sticks is a form of combustion, and the smoke is a steamy mix of hydrocarbon (mostly water and carbon) with some other trace gases and minerals. It is even possible to collect this "fire gas" substance(s) so as to produce a high energy liquid fuel, and which may be called something like "wood-gas". Charcoal is wood that has been already been "cooked" or processed so as to have a higher carbon content, and then it is used as a portable and popular fuel to cook food without much smoke, however, in the process of making charcoal, much potential energy of that wood slow combustion is lost into the surrounding air, and charcoal is then a questionable item in terms of being energy wasteful. The price of charcoal purchased at stores is relatively high as compared to unprocessed dry wood, and therefore it is only purchased occasionally by most people who use it. Oxygen is needed for combustion to occur, and sometimes some substances are added to a fuel to increase its combustion efficiency, and these are called oxygenates, and are said as increasing the "octane" of a fuel.

UV Light

Also of note is that sunlight (especially high frequency, high energy ultraviolet light) can i**onize** [when enough energy is applied to cause electrons to leave their orbit] things and actually promote rusting. **Ultraviolet (UV) light** is generally dangerous to many materials and eyesight. UV light is composed of high frequency, high energy wave of photons. Since ultraviolet light was mentioned, here is a further note about it:

UVA is a (relatively) lower frequency ultraviolet (ie., if it could be seen as an approximate color) light. UVB is medium frequency ultraviolet light, and UVC is high frequency, ultraviolet light. UV-A will pass through regular plate glass, and block most of the higher in frequency, UV-B and UV-C radiation. Laminated glass with the plastic between two layers will block UVA light. Amber and blue colored containers block much of UV light, and is often used for medicine or "pill" containers. Welders wear eye shields made of (usually) dark green glass which blocks both infrared (ie., low frequency light, heat rays) and (**ionization** [can give much (kinetic) energy to an electron(s), and so as to remove them from an atom and-or molecule, and this also then creates positive charged ions], damaging) UV (high frequency) light such as UV-C. These electrons can then travel, and are called "free electrons" meaning it is unbound to an atom. Sometimes molecules of these ions will form due to UV light UVA and UVB can cause sunburn, and typically in just an hour of exposure to direct sunlight and-or sunlight reflected off of bright snow. Having clothing and a hat on, a tan and-or **sunscreen** ("sunblock") lotion helps prevent sunburn and-or UV exposure. Eye protection such as UV rated sunglasses can block much UV light, and sunglasses also reduce the typical bright light, and helping your eyes to maintain normal light sensitivity.

Some electric lamps produce some UV light, intended for specific use (such as sterilization of nearby surfaces). UV light can also help make safer drinking water. Some modern glues can be "cured" (ie., dry faster, crystalize) when exposed to UV light, and modern dentistry and some craftsmen often use such glues and UV light.

Prolonged exposure to UV light can cause skin cancer (melanoma) because of then having a higher chance of ionization and resulting damage to cells and DNA mutations that were not eliminated from the body but rather caused cancer cell growth. Cancer can spread to other parts of a persons body and become even more dangerous. Sunlight in a limited and-or well regulated sense is generally known to help promote the body to create vitamin-D3. Under legal supervision and prescription by a doctor, there are light-therapy, UV light treatments and-or lamps for treating some skin diseases, and the patient must know when and how to use them, such as how close and for how long without causing unintended UV damage and-or other health issues. **Pets, guests and family members need to be aware of a potentially dangerous UV lamp, and to never turn it on.** Vitamin-D3 can be recommended by a doctor, and is often available "over the counter" (ie., without a medical prescription required). If you are sensitive to the sunlight or not, it is recommended to wear a wide brim hat, or at least a "baseball" type of hat. It is also recommended to wear bright colored clothing in the summer so as to reflect a good percentage of the light to limit your UV light exposure and to prevent sunburn. Sunglasses are a good option to consider for any time of the year. **Inform others,** such as your children, about prolonged sun exposure and its dangers. For example, for people who have not come outside routinley for at least a few minutes at a time during the springtime for a slight protective tan to prepare for summer, and then easily get sunburnt in the bright and intense summer sunlight and its energy. It is even possible to get a suntan and-or sunburnt in the winter when the conditions are just right, such as a bright sun reflecting off of the snow around midday (noon, 12pm) time. Light reflecting off of water or other bright surface can also contribute to a suntan and-or sunburn.

In space near the upper atmosphere of Earth, about 10% of the Sun's radiated energy is UV light and-or energy, and with **3%** of it reaching the surface of the Earth at the Equator at noon and after the other 7% was already absorbed by the atmosphere. The higher the altitude, the density of the atmosphere decreases, and the greater amount of Solar radiation received there. Of that 3% mentioned, 95% of it is UVA, and 5% is UVB. At the surface of the Earth, about 45% of the received energy is visible light, and with abut 52% of it being infrared light (ie. heat energy). A small amount of microwaves and x-rays from the Sun will reach the surface of the Earth.

Mohs Scale = The Mohs Scale Is a type of a "molecular hardness" (resistance to visible scratching) scale invented in 1822 by **Friedrich Mohs**, (1773-1839), from Germany, who studied rocks and geologic landforms. The values could be thought of as a measure of the amount of force needed to scratch a sample piece, however, the scale is not exactly linear, and is more of a slight exponential-like scale or mathematical relationship of the amount of force needed to scratch the element. **Note that hardness and density are not necessarily directly related,** for example, lead is very dense and heavy, but is also soft, melts easy and has a low Mohs Scale rating. For an element to be visibly scratched (technically, not a repeated cut or plowed groove), it must be done by another element with a higher Mohs Scale rating. Generally, elements and materials that have stronger atomic bonds will have a higher Mohs rating, and it could be said that those elements are more crystallized at the atomic (atom) level. The Mohs scale was developed to determine what element a rock and-or material may be composed of.

When a Mohs Scale value for an element is greater than 1, the amount of force needed to scratch it is roughly proportional to the square or cube of that value. Here are some typical Mohs Scale values:

Potassium = 0.4 , Sodium = 0.5 , Lithium = 0.6 : low hardness , **soft metals**

Talc ("soapstone", which is similar to plaster ("paste, layer", a pre-word for (bendable) "plastic"), and talc is a mineral compound of magnesium, silicon and oxygen. gypsum = Calcium-sulfate , chalk = calcium-carbonate = **1.0**

Lead , Gallium , Calcium , Tin = 1.5 : lead is quite dense, but has low hardness (ie. "soft") , lead melts at 621°F , gallium melts at 85.6°F , tin melts at 450°F

Plastic = 1.75 typ , Alabaster (crystalline, calcium-sulfate) = 1.5

Alabaster = a denser, crystalline calcium-carbonate stone = 2.0 , Amber (a hard tree sap) = 2.25 typ
Bismuth = 2.25 has a relatively low melting point of about 521°F , Sulfur = 2.0 typ. (= typical)
Zinc = Magnesium = Silver = Gold = Table-Salt , 2.5 : gold is quite dense (mass/volume) and therefore relatively heavy, but it is only as hard as zinc , hence relatively soft. zinc melts at 420°F , magnesium melts at 1202°F , silver melts at 1763°F , gold melts at 1946°F
Aluminum = 2.75 , aluminum melts at 1220°F
Copper = 2.85 av. , copper melts at 1985°F

Calcite (calcium carbonate) = Coral = Antimony = 3.0 , Platinum = 3.5 , Brass = Pearl = Limestone = 3.5 typ. Bronze=3.7 typ. , Mother Of Pearl (Abalone , animal shell from the ocean) = 3.7 typ.

Iron = Nickel = 4.5 , Platinum = 4.25 av. , Marble = 4.0 typ. [essentially a dense limestone]

Apatite = 5.0 : often a green to blue colored stone containing a compound of phosphorous
Steel = 5.25 av. , Glass = bone = 5 typ. , Opal = 5.7 typical, somewhat brittle (will often have a chonchoidal (curved) fracture, and not cleaved along a flat plane) if bumped. Opal has an internal iridescence or reflective shine of colors when light is reflected from it. , Manganese = Cobalt = 5.0
Turquoise (or Turquois) = 5.5 av. which is a copper-aluminum-phosphorous-oxygen (oxide) mineral
Stainless Steel = 5.7 typical - note that tempering the steel to alter its crystal structure will change its hardness, the more harder, the more brittle usually.

Pyrex glass = 6 typ (borosilicate glass, thermally annealed, withstands high heat, low thermal expansion, but can still shatter due to a high temperature differences - thermal stress).
Titanium , Feldspar = 6.0 (a crystal mixture of elements: potassium, calcium, sodium, aluminum, silicon) = berryllium (has crystals of aquamarine and emerald) = 6 , hematite = 6 typ.,an iron oxide , Manganese = Germanium = Uranium = 6.
Silicon = Iridium = 6.5 typ.

Ceramic = 7.0 , a man made stone from mostly clay which is mostly silicon dioxide, mixed with water, shaped, dried and fused into a solid using high heat. Porcelain (a more dense ceramic) ~ 8
Quartz (a silicate crystal mineral of silicon dioxide) = garnet = diorite = granite = flint = vanadium = 7 , Hardened Steel = Tungsten = Amethyst = Emerald 7.5

Topaz = 8 , a silicate (silicon and oxygen molecule) mineral with aluminum and fluorine, a gemstone
Chromium = 8.75 typ

Corundum (is aluminum oxide, abrasives, crystals with some trace elements such as chromium oxide are called sapphire and ruby, can be made artificially with high heat from current) = Tungsten Carbide = 9
Moissanite = a natural silicon-carbide mineral, a diamond-like gem substitute, can be man made = 9.25
Boron = 9.3 is quite dense and hard, but brittle , Silicon Carbide (carborundum, silicon and carbon) = 9.5

Diamond = 10 : very hard form of graphite carbon = 0.5 , but diamonds are also very brittle due to the crystalline atomic bond structure, and will shatter or cleave rather than bend or absorb-store applied mechanical energy. 10 is the highest value in the Mohs scale.

Chromium is the hardest metal known. Corundum is about 2 times harder than quartz. Boron is about two 2 times harder than corundum. Diamond is about 3 times harder than corundum and about 2 times harder than boron. Cubic-zirconia is the (diamond-like) crystal form of zirconium (a metal) dioxide (being part oxygen, here specifically 2 oxygen atoms in the molecule) and has a Mohs hardness rating of 8.5

Besides using the Mohs scale, there are some other newer methods and scales created to more accurately measure and represent the physical properties of elements or materials such as the **Rockwell scale** which is a tensile strength test that uses indentation (such as an indent from a moving or dropped weight of specific dimensions) which also indicates a materials hardness by the depth of the dent. Harder materials will simply have a smaller and lesser amount indentation. Softer material will have a larger and-or deeper indentation. The **Vickers scale** and **Brinell scale** are somewhat similar to the Rockwell scale, and also uses an indentation (depth of a dent) method.

The Rockwell scale and the initial indentation machine was invented by Hugh and Stanley Rockwell from the U.S.A. in 1914. Though this entire subject is beyond the scope or range of this book, and is rather useful when and where needed such as at some factory or for material production, here are some examples of the Rockwell scales that can be used to test a materials hardness:

For the Rockwell A hardness scale = HRA, a diamond tipped cone, and with an applied force of 60kg is used.

For harder materials, the Rockwell B hardness scale = HRB, a sphere tip having a 1/16 inch diameter, and with an applied force of 100kg is used.

For more harder materials, the Rockwell C hardness scale = HRC, a diamond tipped cone, and with an applied force of 150kg is used.

The Rockwell Hardness of a material is then expressed as: HR = (N - hd) , where d is the depth in millimeters. For the Rockwell A scale: h = 500 and N = 100, For the Rockwell B scale: h = 500 and N = 130. For the Rockwell C scale:
h= 500 and N = 100. This HR equation has the form of:

> Rockwell Hardness value = HR = (Arbitrary Initial Standard Reference Value) - (multiplier)(indentation depth in mm). The less the depth of the indentation, the higher the Rockwell Hardness value. The recommended scale to use depends on the specific material(s) being tested.

A scale for wood hardness:

Gabriel Janka (1864-1932), from Austria, made a test and resulting **Janka measurement scale** to determine the relative surface hardness, compression and-or general strength of various species (ie., types) of wood. The amount of natural density, drying (water reduction), grain direction(s), and age ("seasoning", dryness, "carbonization") will also effect this value. The test is standardized for wood having a moisture content of 12%. The Janka measurement is done using a dent test using a 11.28 mm = 0.01128 cm (= slightly greater than 1 cm) =~ (7/16)in =~ nearly a (0.5 in) steel ball and the amount of force or more specifically, the Torque (= Force x distance from rotation point) to place half of it into that wood. The circular area (pi x r^2) at the diameter of this sphere and-or produced dent is **1 cm^2**. First, **lbf = pound-force =** (equivalent, weight) **pounds of force,** and here its also for a certain amount of defined area, hence a pressure (P = F/A) value.

Ex. Buloke (Bull-Oak) = 3700 lbf to 5000 lbf , Ebony = 3500 lbf , Bamboo = 3500 lbf or more typ. , Hickory = 1800 lbf , Apple = 1700 lbf , Hard-Maple = 1448 lbf , Oak - 1350 lbf , Ash - 1330 lbf , Walnut = 1000 lbf typ. , Cherry = 970 lbf , Paper Birch = 900 lbf , Sycamore = 750 lbf , Yellow Pine = 600 lbf , Poplar = 540 lbf , Pine (soft, Fir) = 500 lbf typical , Basswood - 430 lbf , (common) Aspen = 360 lbf typ. , Cedar = 350 lbf , Balsa = 70 lbf.

The price of wood often depends on both its rarity and-or hardness. Woods such as ebony and rosewood are endangered trees, and it is advised to seek alternative woods and-or materials such as carbon fiber or hard maple. Trees that have fallen naturally are generally sought after as a fair resource, and so as to obtain its wood for practical use before it decays naturally and is then much of valued resource.

If wood dries out further, and then having a lower moisture content per volume, and a higher carbon content per volume, it will tend to then have a higher Janka value for the same type of wood. This wood has bwecome denser.

Some of the following are mostly new book discussions, and it includes some brief, additional and-or alternate discussions of previous and-or further book discussions. Definitions and some general history of the commonly recognized discoverers and inventors are mentioned, and so as to help the reader have a further understanding of a concept and-or formula.

Torque = (**Force Applied) x (Lever-Arm Distance)** : this is not exactly the same as Work = force x distance, and as described below, torque is essentially the resulting or effective amount of force output after a change or transformation (ie. here, an effective amplification, or growth) of a given applied or input force. With the Work equation, the force is not amplified as it is with the torque concept and equation.

Torque is a measure of the resulting and effective rotating, turning or "twisting" **force** (ie., a force that will cause a rotation about a center point), and if that force is constantly applied, it will cause something to accelerate about a fixed point. An example is a long wrench tool being pushed or pulled so as to loosen or tighten a screw or a nut on a bolt. Another example is a rod, beam or lever which has a pivot (turning or rotation) point or "fulcrum" (support, post) set at some distance along it, of which the rod, beam or lever is able to rotate on or about, including up and down vertically. Levers are used as simple (mechanical) machines to have a "mechanical advantage (gain, increase, amplification)" so as a small force or weight can lift objects that have a heavier weight, although requiring more time and-or distance to do so. Torque may sometimes be mentioned as "moment" or "moment force". Often, once a bolt and-or nut is "broken free" or "loosened" from applying its ("clamping") pressure to mechanically join things together, the torque needed to completely remove it will be much less than that used to tighten it.

Lever-arm is a term associated with torque forces, and is the distance from the point a force is applied on the lever, beam or bar, to the point of pivot or rotation of that lever. A regular lever, or compound pulley, as mentioned in the book are well known as force multiplier, but taking longer to do so. With the concept of torque, the lever material rather transfers the input force to the center point of rotation, and applies the output force there. The Lever-arm distance is essentially like a radius or distance value from the pivot point as if that point was the center point of a circle. Since force is (mass x acceleration), we see in the torque equation that the lever-arm distance is essentially a multiplier to the force applied, and technically, this is because the lever-arm (or radius) distance essentially increases or multiplies the acceleration. Given a constant value of applied force, if you increase the lever-arm distance, then the output torque (twisting force) will increase by the same factor the lever-arm was increased. The equation shown above assumes that the applied force is perpendicular (at a right angle, or 90° to it) to the lever arm, otherwise, when the force is not applied in a perpendicular manner, the applied force is actually less, and the more general equation to use for such conditions is:

Torque = (**F**orce) (**L**ever-arm) (Sin ¢) : where the angle is the angle between the lever and the applied force. When the applied force is perpendicular (90°), Sin angle is at it's maximum possible value of Sin 90°= 1, and all the force is being applied to produce torque and the equation simplifies to the previous one shown: **T** = **F L**

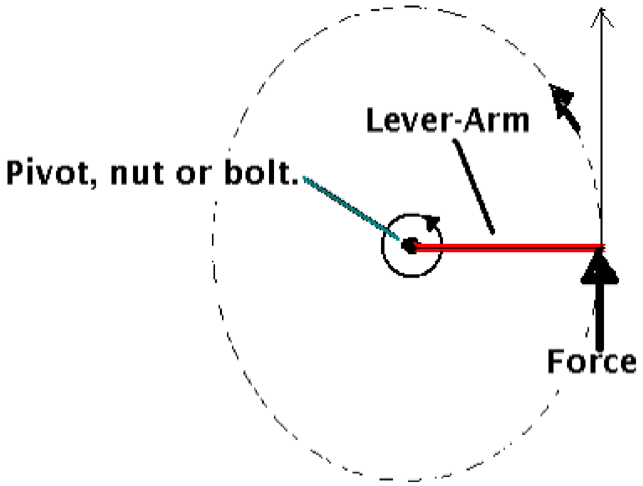
[SEE FIG 228 below]

Increasing lever-arm increases or amplifies torque (force) because the acceleration of a point (where the force is applied) on the lever is greater when it is farther from the (center) point of rotation, and that the velocity and distance traveled about the point (center of rotation) is greater per same amt. of time. Force (f = ma) is directly related to acceleration, and since acceleration is a change in speed (velocity) per change in time, and speed (velocity) is a change in distance per change in time, therefore, acceleration is directly related to the distance traversed per unit time. If acceleration increases, or is desired to increase, force must be increased. Acceleration will essentially become a multiplying factor to the mass when force is being considered since:

f = ma = force = mass x acceleration. Using substitution we see how these factors affect the torque equation:

Torque = (Force)(Lever-Arm) = (mass)(acceleration)(Lever-Arm) with units of **Newton-meters** = Nm

A longer lever-arm (such as that of a long wrench tool, ratchet, or machine lever) increases the acceleration which increases the output or torque force. If the lever arm or radius distance increases by a factor of (n), the torque will also increase by that same factor of (n). It is incorrect to think that a common, long screwdriver will provide torque. [FIG 228]



Ex. Here, all these conditions will apply the same torque or "twisting force" of 100 ft-lbs:

100lbs of force at 1ft = 50lbs of force at 2ft = 10lbs at 10ft = 1lb at 100ft.

A note about the units to use or express for torque:

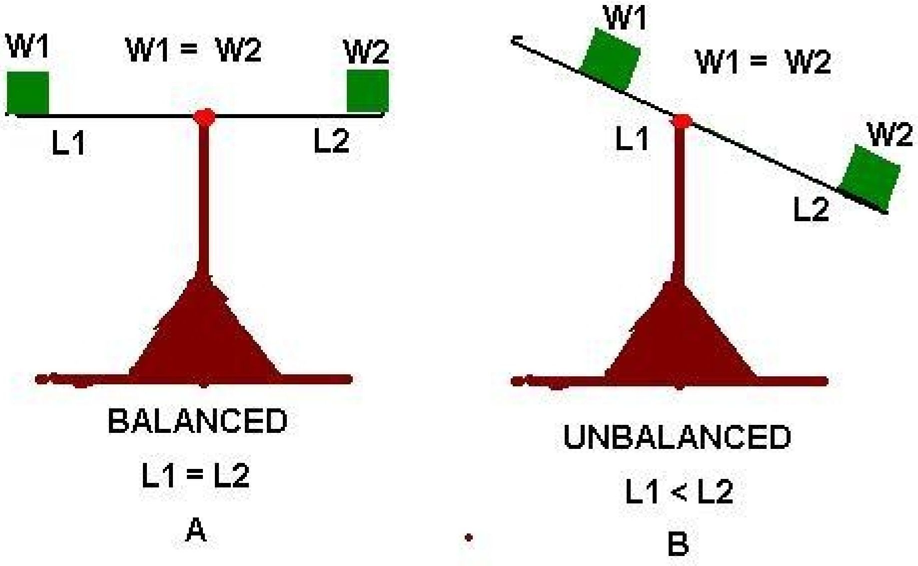
More correctly, 100 ft.-lbs of torque should be expressed as: 100 lbs-ft. = 100 pound-foot. That is, the force should be expressed first in the units of torque. The expression or units of: Fd = (force)(distance) is actually an amount of work or energy in Joules. This value does of (Fd) does equal (Nm) which is considered the standardized unit for torque, but don't consider a torque value equal to a joule value of energy or work. Note that (torque)(distance) would rather be equal to: (force)(lever-arm)(distance) and this is clearly a different value than work or energy. For the units of torque as (Nm), the value of m=distance in meters) is not the arc distance moved, but is rather the distance the force was applied from the center of rotation, hence the lever-arm distance. You may still sometimes see torque expressed inunits of ft-lbs = (foot)(pounds), hence the distance expressed first. 1ft-lb = 12 in-lbs

From the Torque equation, we have: $\text{Force} = \frac{(\text{Torque})\ \text{Nm}}{(\text{Lever-Arm})\text{m}}$: with units of Newtons = N

$\text{Lever-Arm} = \frac{(\text{Torque})\ \text{Nm}}{(\text{Force})\text{N}}$: with units of meters = m

An example of where torque is used is for a stated, recommended (for safety) "tightening" or "tightness" value for a nut or bolt. Once tightened, this but or bolt will compress two or more piece of material together like a strong clamp applying force and-or pressure (force/area) on each side of it, and it will give that structure much rigidity and-or stability at that location. Note that a nut may have a specified and-or rated amount of torque to apply to it for safety and legal reasons, and so as to not damage that nut and-or bolt by excessive force, and also to prevent injury of the user. There are various torque-wrenches manufactured and are available in the tool sections of some stores. They are often a type of ratchet wrench which accepts a socket adapter of the size that will fit the dimensions of the nut being tightened. The specific torque value needed can be set on these wrenches, and a click-sound will be heard and-or indicated when the nut has been tightened enough.

Some considerations about torque by using the concepts of a balance scale that is used to measure mass and-or weight of objects. The balance mechanism is essentially two levers of the same length on both sides of a pivot point. A calibrated reverence weight and-or mass may be utilized for this measurement depending on the specific balance scale used. [FIG 228B]



For both part A and part B of the above figure, the weight values (W1 and W2) have the same value, therefore, W1 = W2. As mentioned, weight is a force in the downward direction due to the force of gravity constantly attracting and accelerating a mass. Weight = Force = (mass)(acceleration) = m a .

In part A of the figure, since the lever arm distances (L1 and L2) of the weights are both the same length from the pivot point, this system is in balance and is not moving, and specifically here, not rotating due to gravity pulling upon the weights. The "balance beam" of which weights are upon will be horizontal and-or level with the ground. The torque = T = (Force)(Lever-Arm distance) = FL or rotation force produced by each side is the same value, and the result of these opposite signed torque values is zero net torque as if no weight or force was applied to either or both side. Another value is that the support structure is holding up or supporting the sum of the two weights and beam.

In part B of the figure, the weights are still the same value, but one is moved closer to the pivot point, and has a lesser value for its lever arm. Although the weights are the same value, L1 is now less than L2, and this system is now unbalanced and is rotating. It is now as if W2 is greater than W1, but again, they are still the same value. What is actually less is the torque or rotation force produced by W1 and its corresponding lever arm distance. The torque of W2 has not changed, but it is now greater than the torque produced by W1 and its lever arm. The lesser that L1 is, the less the resulting torque. An actual balance scale for measuring weights and-or mass will usually have some type of springs or physical stops so as to prevent too much rotation and possible damage to the scale.

Torque or torque force is the rotation force around the pivot point. For this unbalanced system shown above, the result is that W1 and W2, and their supporting beam structure will be vertical, and where W2 at the bottom due to its greater torque force. The net or total torque upon the pivot point is the difference between the two torques applied:
Tt = (T2 - T1).

When given any two weights of any value, their torques can be balanced (ie., produce and equal torque value) or set equal by adjusting their corresponding lever arm lengths. In general, for this to happen, the larger weight will need to have a shorter lever arm distance.

Archimedes discovered the **Lever Law** that states: **(M1 L1) = (M2 L2)** , and for the lever to be balanced, both sides are equal, and-or that mathematically: (M1 / M2) = (L2 / L1) , and these are reverse ratios. Note that due to gravity, the mass (M) results in a force or weight of (Ma = Mg), and that torque = (F L). For the lever to balance and-or be set to no net torque: (T1 = T2). Archimedes did knot know Newtons specific equation (F = ma) for force at that time.

When using a wrench upon a nut or bolt - essentially being the pivot point or axis of rotation, and if there is no second weight or force involved, all the input energy, force and torque (T= F L = m a L = m (v / t) L) created is applied about the pivot point, nut or bolt. As can be seen in the formula, the greater L is, and even if the input force is the same, the greater the (effective linear) velocity will be about the center point. This also effectively gives a higher value of acceleration. Note that the angular velocity (= ¢ / t) is the same at every point along the lever arm distance. Technically, for an object to change direction, there must be a force applied, but in the case of orbits, the force applied is gravitational force (= ma = mg) pulling upon the spacecraft, and the result is a circular orbit.

Another way to explain the force increase upon a nut or bolt due to a long wrench applying torque is:

Energy In = Energy Out : and-or: Work In = Work Out
F in D in = F out D out : Energy = Work = (Force) (Distance) with units of Nm = Joules

Given a long wrench, the radius distance from the center will be long, and the distance a point on its far end will travel is a much further effective linear distance about that center point for any given angle of rotation. Mathematically:

$\text{Fout} = \dfrac{\text{Fin D in}}{\text{D out}}$: since D out is a relatively low value, and less than D in, F out will be larger than F in.

Also: Power out = Power in : or more correctly: Work in = Work out , or Energy in = Energy out , : joules
F2 d2 = F1 d1 : F1 = input force , F2 = output Force = amplified input force
$(n\ F1) \dfrac{d2}{n} = F1\ d1$: showing that F2 is a factor of (n) times more than F1, and actually (n) is the lever arm distance (L). (d2 / n) = d1 , hence: d2 = n (d1)

Weight is a resulting force due to the constantly applied force of gravity (causing g = a = an acceleration in velocity = 9.81m/s^2 = 32.2 ft/s^2 on Earth) upon a mass. This force of gravity gives the mass an acceleration of (g) and increases the energy of that mass. Essentially, mass and gravity are amplifying each other, and are expressed as multiplying factors of each other, and the resulting force or weight is:

F = weight = (mass)(acceleration) = (m)(a) = (mass)(g) : g = a = 9.81 m/s^2 = 32.2 ft/s^2 Note that:

1lb of weight is 1lb of force = about 4.4482N of force =~ 4.45N of force =~ 0.45359kg of force =~ 454g of force
1 kg of mass will have a weight of 9.80665 newtons of force = 9.80665N of force ~= 9.81N of force = 2.2046 lbs
1 lb = 0.45359 kg = 453g . 1N =~ 0.22481 lbs = roughly a quarter of a pound , 1N =~ 102g =~ 0.1kg

Here are the derivations of the above conversions:

$\text{F newtons} = (\text{mass})(\text{acceleration}) = (m)(a) = (\text{kg}) \left(\dfrac{m}{(s^2)}\right) = \dfrac{\text{kg-m}}{s^2}$: N = newtons = force units

1N = (m)(a) = (1kg) (1m / 1s^2) = 1 kg-m/s^2

When the acceleration (a) = (g) = 9.80665 m/s^2 , such as for weight force:
9.80665N = (m)(a) = (1kg)(9.80665 m/s^2) = 9.80665 kg-m/s^2 = 2.2046 lbs of force

1kg of mass will weigh 9.80665N when in the gravitational field ("force field" or "field of force") of Earth.

If the weight of a mass or object is 1lb = 16 dry-ounces, the corresponding mass in kilogram units is 0.45359 kg:
4.448$\underline{2}$N = (m)(a) = (0.45359 kg)(9.80665 m/s^2) = 4.448$\underline{2}$ kg-m/s^2 = 1lb of force

100lbs =~ (4.45N / lb)(100 lbs) = 445N , 1000 lbs = 4445 N , 1 ton = 2000 lbs = 8890 N

1ft = 0.3048m ~= 0.3m.
1lb-ft =~ 1.355817Nm :derived from: (1lb)(1ft) =~ (4.44822N)(0.3048m)=~ 1.355817Nm , therefore mathematically:
1Nm =~ 0.7375624 ft-lb

Though kilograms are often thought of as a weight value, it is actually an amount of mass, and the corresponding weight or force of an amount of kilograms of mass is shown above as:

F = weight = (m)(a) = (kg)(m/s^2).

The reason for thinking a kilogram is a weight value is that for example, a 1cc = 1 gram of water is said to have a "weight" of 1gram or gram-weight. Most scales will actually display the corresponding mass value and it is commonly said or spoken as being a weight value, and which is a loose way of saying it was a measured value from a device (scale) that initially senses the weight of the mass. Most weight scales are calibrated to show an equivalent amount of mass of the object. Therefore, if a scale indicates 1 gram, you will know that the mass of that object is equivalent to 1 gram of mass. As mentioned several times in this book, weight and mass are mathematically proportional to each other, if one variable increases or decreases by a factor (n), the other will also increase or decrease by that same factor. A mass value of 1 gram of any type of atom (ie., element), object, or substance will also have the same weight, and this is how **the total mass of an object can be determined, and that is by measuring its corresponding total weight.** If you know the volume of the object, then its density = (mass / volume), and then you can find what (if consistent) element it may be from an element density list. (weight / volume) = (force / volume) =(ma / V) = (mg / V) could be defined as a "weight density" of an object.

Since letting mass density = d = (m / V) , and letting: weight density = (mg / V) , we can see that if we divide the weight density of an object by (g = 9.81 m/s^2) , we will then have its corresponding mass density = (m / V) and this is very useful if you do not know the volume of an object.

Energy can be defined as the ability to do something such as work. Therefore, a measure of energy is a measure of this ability to do work. The units of energy are Joules (=J). A force (ie., influence) can be used to apply and transfers energy from one object to another. For a force to apply and transfer energy, there must first be energy to create that force. Energy can be accumulated, concentrated and stored, such as for example: in batteries, fuels, and our muscles.

Work can be measured as equal to the total amount of energy used up and-or transferred, and therefore, its units are also the same as that for energy, and that is Joules. In term of forces (the application of energy) and distance:
Work = (force)(distance) with units of the energy needed and-or used, and that is **Joules (J)**.

Power is a measurement of the rate of using energy or doing work, hence power is a "work-rate", and therefore it is equal to the total amount of energy used per unit of time.

power = energy / time = work / time : with units of joules/second = J/s , and is a rate value; the rate of, and-or using energy

The units of power are **joules per second** which is also called watts (w). 1 **Watt** is 1 Joule of energy used and-or transferred per second of time.

1watt = 1w = 1J/s.

From the above formula, we can derive that 1 Joule of energy is equivalent to 1 watt of energy used or applied for or during 1 second of time.

J = w-s. : 1 Joule of energy is equal to 1 ws = "one watt-second" of energy =
1 ws = an energy rate of 1 watt = 1J/s applied for a length of time equal to 1 second of time.
Ex. 5ws = 5 Joules , could be from an energy rate of 1 watt = 1J/s applied for a time length of 5 seconds, or 5 watts = 5 J/s applied for a time length of 1 second.

Another form of this equation is: **Joules = energy = (power)(time)** **: total joules or energy**

For electricity or electronics, 1 watt of energy usage is equivalent to 1J/1second which is equivalent to 1 volt = 1V of electrical pressure or force (emf = electro-magnetic force due to charges repelling or attracting) causing and moving 1 ampere (= 1A) of charge (current, electrons) past a point in a circuit. The equation for watts then becomes:

Power = Watts = (energy/second) = (joules / second) = (voltage)(current) = V I , and this is derived in this book.

A joule of energy is also equivalent to the energy needed to apply 1 Newton of force for or through 1 meter of distance:

1J = energy = work = (force)(distance) = (Newtons)(meters) = N-m = 1 w-s

1J = N-m = (force)(distance) = (mass)(acceleration)(m) =
= Mam = M(m/s^2)m = M (m^2/s^2) = M(m/s)^2 = Mv^2

Mv^2 is nearly the same equation of kinetic energy = KE = Mv^2 / 2 = joules / 2 , and one way to consider this value be half, is that it is due to equal and opposites forces which limit the energy transfer by half.

Extra: Also of note: J = (force)(distance) = F v t = M a v t

Horsepower (hp) is an old and large unit of power = (energy / time) with units of watts. Power is essentially the average amount of energy or work used during a time period or amount of time.

1 horsepower (hp) is 746 watts. 1 hp = 746w = 746 J/s = 178.3 calories/s.
$1hp = 748w = \frac{746\ J}{s} = \frac{746\ N\text{-}m}{s}$

Work is a measure of applying a constant force (F) or energy to something (object) over a length (distance) of movement. The units of work are the same as the units of the measurement of energy, and that is Joules (J). Since work is measured with units of energy, work is also a measure of the total amount of energy used and-or transferred to something(s) else:

Work (W) = (Force) x (Distance) :Joules (J) of energy , mathematically:

Note that here, distance is the distance of which the constant force was applied and the time needed to do so. Also, note that mathematically. and expressed like an average ratio or rate:

Force = Work / Distance = energy applied per unit of distance.

$\text{Force} = \frac{\text{work}}{\text{distance}} = \frac{\text{energy}}{\text{distance}} = \frac{J}{m} = N$: mathematically, force is the rate of energy used per unit distance, or the rate of applying energy over a distance, and for this example it is how much energy was used or applied for each meter. For an object to accelerate, the force must be constantly applied, and therefore, the energy used and-or needed must be constant.

Work = (force)(distance) = (Newtons)(meters) = Nm Joules of energy. , Mathematically:
Work = (mass)(acceleration)(distance) Joules
Work = (mass)(acceleration)(speed)(time) = mavt Joules : acceleration = a = change in v / change in t,

a = change in v / change in t , and if the initial values were: t=0, and v=0 , a = v / t = (d/t) / t = d / t^2

Since d = (speed)(time) = vt, and the initial velocity was 0, then that speed or velocity was not always applied during that amount of time (t), and the value of v= (d / t) is not then actually correct to use for (v). What value of (v) should then be used? The average value of the two velocities can be used: v av = (V final - v Initial) / 2 , and this assumes there was a steady increase in the velocity during that amount of time due to the (constant) acceleration during that time:

$$\text{Work} = (m)(a)\frac{(v)}{(2)}(t) =$$

$$\text{Work} = (m)\frac{(v)}{(t)}\frac{(v)}{(2)}(t)\ \text{Joules} = \frac{mv^2}{2}\ \text{Joules} =$$

Which is a measure of the total energy needed to do that work and-or the gain in kinetic energy (KE) of an object in motion due an applied force and-or work upon it. mass (m) has units of kg, and velocity (v) has units of m/s.

An object just moving in space at a fixed velocity and without accelerating (a=0), then the factors of acceleration and time are not influencing factors of its ("stored", "potential") equivalent energy value, and therefore are not even considered as factors in the equation. The object has energy due to just its mass and velocity. Because the object has mass and velocity, the object has momentum which is set as being equal to this value. Momentum p = mv = (mass)(velocity), and it is defined as a measure of an objects ability to keep moving and-or its resistance to changes in velocity and-or direction. The momentum of an object is directly related to the kinetic energy (KE) of that object. In short, it takes more energy or force to influence (change [increase or decrease]) the speed, and-or direction of a more massive object. Since v = d / t, momentum can also be describe as a measure of a mass through a distance during an amount of time: mv = (mass) (distance) / t

Ex. A mass of 1 kg which has a velocity of 3m/s^2 has $\text{KE} = \frac{(1\text{kg})(3\text{m/s})^2}{2} = \frac{(1\text{kg})\ (9\ \text{m}^2/\text{s}^2)}{2} = 4.5$ Joules

If this energy was to be used to apply a force to a second object, and due to equal and opposite forces being applied to the first and from the second object, only half of that energy is actually transferred to the second object, and the amount of energy transferred is: KE / 2 Joules. It is as if half the energy was not transferred due to the resistance or inertia of the second object. Another explanation of this formula is that when an object1 initially collides with object2, object1 will begin to decelerate, and its velocity and energy will decrease. Object1 will start to loose energy, and since object1 is applying a force = (m)(a) to object2, object2 will gain energy, increase its speed and-or possibly be compressed. This formula is also mathematically derived in this book.

As a "thought concept or experiment", if object1 somehow transferred all of its energy to object2, then surely that amount of energy increase would be equal to the KE of object1. Since an amount of KE is related to the square of the velocity, if the velocity changes by a factor of n, then the KE increase (added) of object2 will also be a factor of n^2. This also implies that to increase the velocity of an object by a factor of (n), then n^2 more Joules of KE energy must be applied to that object. Likewise, if the mass of an object increases by a factor of (n), then the amount of its potential energy increases by that same factor if it is still moving at the same velocity.

If you were applying a force (f=ma) pushing a box across the room on the floor, surely the box must experience an acceleration so as to have a velocity and move. At the same time, the force of friction or drag between the box and the floor causes the box to loose energy, decelerate and essentially loose any acceleration and energy it had. If the same pushing force and drag force is constantly applied and the velocity of the box appears to be the same, this is due to very brief, nearly instantaneous, periodic acceleration and deceleration of the box. It will be as if only half the acceleration, or average accelerate is effectively being used.

Assuming no energy losses during applying energy so as to produce a force upon an object through a distance, which is the work, the energy used and transferred to an object is equal to the increase in that objects kinetic energy.

Our muscles store energy and it can be used to apply force so as to do work and-or apply energy to another object, and due to the "equal and opposite force" concept, some of the total amount of energy needed to move an object will essentially be "used up" or lost in our muscles as heat energy and wont be transferred to the object as a gain in its momentum. The energy gained by the object is equal to: work = (force)(distance) joules, but this equation does not consider any losses and the total amount of energy actually required by the person or a machine, and this value of work is rather the energy gained by the object and-or the minimal amount of energy required by the person or machine. Essentially, only half the total energy involved will be applied to the object.

If a force is applied to an object for a certain distance and-or time in space, that object will gain energy in the form of momentum. The force and-the (limited) amount of energy used will actually move it an infinite distance and also at the same speed if there is no other forces applied to it. For the work formula , work = (force)(distance) joules, the force and distance values for this equation are only for when the force and energy was initially applied to the object so as it to give it that supply of energy.

If a meteor has a speed of 15000 mph, it is calculated that it will take about 25 seconds for it to go 100 miles of distance, and which is about the thickness of Earth's atmosphere. If the meteor does not come in (ie., travels, moves, enters) at a direct (perpendicular to Earth's surface, a 0° angle) course, that length will be longer than 100 miles and the time needed to travel that distance will be longer, say 1 minute on average. Once the meteorite enters Earths atmosphere it, will collide with the air and compress it and loose much momentum and energy due to the "friction" and "drag" resistance of the air. Much of the energy will be converted to heat energy, and the meteorite may "burn up" and become many small bits of dust and gas. This can be seen, and it is often called a "shooting-star". Its speed will decrease further, and therefore it will take even more time to reach the surface of the Earth. Since the surface area of the Earth is about 75% water, a meteor will reach land (solid surface, non-water) about 25% of the time on average, and generally away from populated areas.

Ex. Since KE = mv^2 / 2, if the speed of the wind (which has mass due to the air gases) doubles, its instantaneous kinetic energy for an instant of time will increase by 4 if it is considered as a single mass or object:

KE = mv^2 / 2 = 0.5 mv^2 , and if the velocity of a thin layer or mass of air or wind doubles:

KE = m(2v)^2 / 2 = m4v^2 / 2 = 2mv^2 and this is 4 times more KE, but more needs to be considered since we need to consider the KE in a volume of air that will pass a point and-or be available to the wind-blades of the wind electric generator every second.

Kinetic Energy (KE) of the wind is being used to power a wind- turbine so as to make electricity or power some mechanical equipment such as a mill or grinder, and the result would be 8 times the power if the wind speed doubles. An instant of air mass can be considered as a thin, two dimensional layer of air, such as the bottom layer of a cube, however over time such as 1 second, this thin layer of air becomes a volume of air which has three dimensions, and the total KE of the air during that time would be the sum of all these layers of KE during.

Consider: power = energy / time = KE / time = joules / second

The thin layer or area of the air can be considered as an instance, bit or "differential" of a larger volume, and noted as dV. To find that volume, we can take the anti-derivative of it; and here in this example: dV = 2mv^2 (which is the energy in each thin layer of air and-or instantaneous or bit of volume of that air per second, and when the speed had only doubled) and get the total or sum of all those thin areas or bits of volume so as to have the full volume (V) of air, and the full KE value of that volume of air during 1unit of time (here it is 1 second).

Consider: Volume = a^3, and that the derivative of it is dV / da = 3a^2 = an area (A) equal to a thin layer (ie., an area) of, and instantaneous bit of that volume. From this it can also be inferred that a higher dimension is equivalent to and-or composed of an infinite number of bits or layers of the next lesser dimension.
Volume = a^3 = (aaa) = (a)(a^2) = (a)(Area). As a side note, Volume is mathematically equivalent to Area times the square root of that Area. Ex. If Area = (2)(2) = 4, Volume = (Area)(Area^0.5) = Area^1.5 = 4^1.5 = 8 =
= 2^3 = (4)(2) = 8.

Also note mathematically: Volume / a = Area, and that as mentioned above or taking an infinitesimally small change, called a differential (d), in each variable, we have:

d Volume / da = dV / da = d Area = same ratio value of Area = Volume / a , or:
From: Volume / a = Area , we have: Volume = (Area)(a) , and after taking the differential of each side:
d Volume = d ((Area)(a)) and mathematically:
dV = d ((A)(a)) = d A da

The sum of each calculated, real bit and-or segment of volume = Sum of (dV) = equals the whole volume (V).

For our equation above of 2mv^2 , and if we take the anti-derivative (ie. summing up all these derivative, bit, component or instantaneous layers of area, which equals dV for each bit or layer), we have:
2mv^(2 + 1) / 2 = mv^3, and if speed doubles, KE will increase by the cube of that factor value, and here it is the cube of 2, and it is: 2^3 = 8 times more KE per second, or power than the KE at the half the speed or velocity.

What is the mass of air passing a point in 1 second. The density = (mass / volume) of air is noted at about:
Ma = 0.001225 g/cc. Multiplying both the numerator and denominator by 1000, we have:

Ma = (0.001225 g / 1 cc)(1000 . 1000) = 1.225 g/1000 cc = 1.225 g / L .
A cubic meter = 1m^3 = 100cc x 100cc x 100 cc = 1,000,000 cc , after dividing by 1000cc / L:
A cubic meter = m^3 = 1000L .
A cubic meter of air has a mass of: (mass / 1 L)(1000 L / 1 m^3) = (1.225 g / L)(1000L / 1 m^3) = 1225g / m^3 = 1.225 kg / m^3 .

If the windspeed was 1 m/s, then (1.225 kg of air / m^3) will pass by a point per second.
This could also be expressed as a mass rate per time of:
(mass / volume) / time = (mass)(time) / (volume): 1.225kg / 1m^3 / 1s = 1.225kg-s / m^3 , or = (density / time)

Setting up a proportion to find the volume of air that will pass a point per second in the windspeed is 10 m/s:

$$\frac{\text{speed}}{\text{density}} = \frac{1\text{ m/s}}{1.225\text{ kg / m\^{}3}} \quad \text{as} = \frac{10\text{ m/s}}{\text{X kg/m\^{}3}} \quad \text{we have:} \quad \text{X kg/m\^{}3} = (1.225\text{ kg/m\^{}3})(10\text{m/s}) / 1(\text{m/s}) = (1.225\text{kg/m\^{}3})(10) = 12.25\text{ kg/m\^{}3}$$

Power, which is the rate of using energy which is: energy / time can also be expressed as work done per unit time: Pw = Work / time , hence power is a measure of the rate of doing work and-or the using energy per unit of time. The derived or combined units for power is called watts (w).

Pw = (energy / time) = (joules / time) = (work / t) = (force)(distance) / (time) = (Nm / t) = (J / t) = (V C / t) = V A watts

In terms of torque: power (Pw) = (torque) x (rotation speed) : rotational speed = w = angular velocity = radians / s
Pw = (torque)(w)
Or: w = 1 revolution / s = 1 rev / s = (2)(pi) rads / s
w = (2)(pi)(revs) radians / second .

rpm = rev per min = revolutions / minute = (rps)(60)
rps = rev per sec = revolutions / second = rpm / 60
w = (2)(pi) (rps) = (2)(pi) (rpm/60) = (pi)(rpm/30)

This type of equation above is often found in the study of wheel and gear systems, and is derived below. A discussion of basic gear systems is given in this book. The units of force are usually called Newtons, the units of energy is Joules, and the units of power are Newton-meters per second = Nm/s = Joules/second. We see in this equation of power, that torque and speed are mathematically inversely related for a given (fixed, constant, limited) amount of power, and such as for the input and output power of gears or wheels. Note, that even though they are inversely related, they are generally not mathematical reciprocal in value to each other. It is also important to note that in a gear system, that the corresponding gains (multiplying factors) in torque and speed are actually mathematical reciprocals. The equation may look unique in terms of the variables used, but it is actually related to the previous equation shown. Torque is related to force, and speed is related to distance and time.

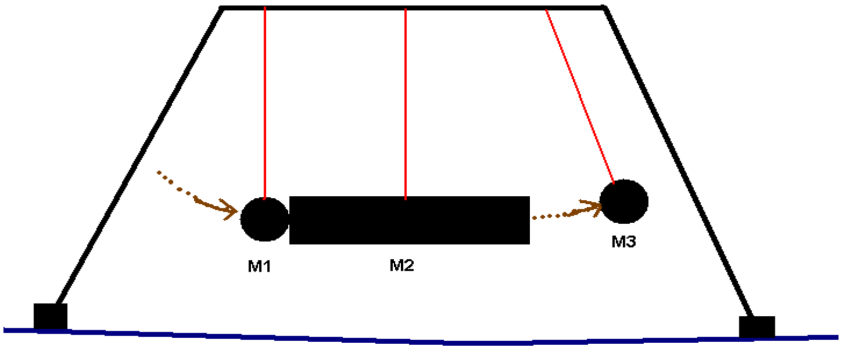
For the above equation: As derived in the **angles and rotation** section this book:

linear velocity = v = angular velocity / radius = w / r
distance = v t = (w r) t

Power = energy / time = work / time = (force) (distance) / time = ((torque) / r) ((wr)t) / t simplifying:
Power = (Torque)(w) joules

A BASIC ANALYSIS OF COLLISIONS OF MASS AND THE TRANSFER OF ENERGY

Below is a mechanical system of 3 masses at rest without any initial motion and having energy: M1, M2, and M3. M1 and M3 have the same value of mass, and M2 has many more times the mass of either M1or M3. [FIG 229]



This device shown in the above figure is very similar to a device called "Newton's Cradle", "Newton's Balls" or "Newton's Pendulums" and where it has only solid metal balls (ie., spheres), and with each having the same mass. That device is also credited to Abbe Marriot, and Christiaan Huygens to study the collisions or impacts of masses in about the mid 1600's. Today, Newton receives most of the credit because of his famous equations for motion such as force = (mass)(acceleration) = f = ma. Fundamentally, this device is also an image of how mechanical (ie., physical and-or motional) forces are transferred even through a single piece of material, atom by atom, from the input to the output at a high speed.

When M1 is swung and let go, it will collide into M2. The energy of M1 will be transferred to M2 via the applied force of M1 due to its mass and velocity, and which is its **momentum** (p or P = mv). With this system shown in the figure, there will be a small energy loss for each collision, and due to a transfer of some energy into sound and (KE) movement losses due to friction.

The since the mass of M2 is much larger than M1, M2 will not accelerate and then move much due to that its mass is larger and therefore has larger inertia or resistance to change. Then M2 will transfer nearly all of its newly gained energy to M3 by applying a force to it. After M3 reaches its maximum height, it will swing back in the reverse direction, and a cycle of these collisions will keep happening until the loss of energy due to the small amounts of friction and sound deplete the input energy to 0 Joules.

Energy in = Energy out : conservation of energy , theoretically 0 losses, full transfer of energy
KEm1 = KEm2 = KEm3 : In reality, there is a small amount of friction losses for each cycle.

Since: $KE = \frac{mv^2}{2}$ for a given, constant or limited amount of kinetic energy (KE), the relationship of (m) and (v^2) is an inverse relationship. When the mass (m) increases, its velocity will be less. When the mass (m) decreases or is less, its velocity will be more. Note that this equation has a quadratic equation form with a squared independent variable.

If all the kinetic energy (KE) of a mass such as M1 is transferred to a larger mass at rest such as M2, the velocity or movement of M2 will be less, but its amount of KE will be still the same. When M2 applies force to M3, it will transfer its KE to M3. Since M3 is a smaller mass, its KE will still be the same, but its velocity or movement will be more:

Momentum in = Momentum out : conservation of momentum (p, here using P), theoretically 0 losses,
Pm1 = Pm2 = Pm3 full transfer of momentum. Whereas (KE) is a quadratic type of
(M1)(V1) = (M2)(V2) = (M3)(V3) relationship to (v), momentum is a linear relationship to (v).

Work in = Work out
Work due to M1 on M2 = Work due to M2 on M3
(force)(distance) = (force)(distance)

When M1 strikes M2, M1 will apply a pulse of force to M2. M1 will loose energy and decelerate, and this is also due to the concept of equal and opposite forces. The inertia of the large mass of M2 will be like a resistance in the path of M1, and so the force applied to M2 is as if it was "reflected" (by compression and expansion of the atoms) back to and upon M1, especially when the velocity of M2 is very low, such as 0, and it will take some time for it to accelerate to a velocity. Since this "reflected force" is in the opposite direction and upon M1, M1 will decelerate and loose velocity and momentum (ie. energy). The momentum and energy that M1 transferred or "lost" will be gained by M2.

An "**elastic collision**" or impact is when much of the applied energy or momentum is "reflected" back to the object(s) of the same mass, and their KE will be, or nearly be, the same as before the collision as if none was lost and-or transferred to something else. In short, their momentums will be exchanged, and there is no loss of momentum and-or energy.

Forces cause motion, but forces also often causes a very high speed compression (force, pressure or "shock") wave of matter (ie., atoms) that transmit the energy from one atom to the next via force, much like how sound is a compressed wave or vibration of air that transmits the pulse of energy, atom by atom through the air.

When M1 collides or impacts with M2, the force and available amount of energy is quickly transmitted in what is described as a pulse of energy and-or an impact force. Here, quickly transmitted means a very short, almost imperceptible, amount of time to do so. The change in velocity with respect to time is high, therefore, the acceleration (here, in the form of a deceleration) is also very high during that short amount of time.

$$a = \frac{\text{change in velocity}}{\text{change in time}} = \frac{\text{Vtraveling} - 0}{\text{a fraction of a second}}$$

: at impact the velocity is 0, especially if the object that got impacted did not move much

Since: F = ma , and if (a) is a very large value, the impulse (a quick impact) force will be a very high value, but only briefly in time, and over a very short distance. Because the force is high, and force is the application of energy, the energy "release" or transfer (joules/second) will be a very high rate. We see that a quick acceleration and-or deceleration can effectively increase or amplify the force created to a very high value for a short amount or pulse of time. This high force value can cause metal to bend in a high speed collision, and especially mores so when the objects are traveling toward each other and with each having their own kinetic energy (KE=mv^2 / 2) and momentum (p=mv) which will increase the resulting decelerations and (impulse, quick-impact) forces during the collision. As a thought concept, when the objects area headed toward each other at the same velocity, if the velocity of one object was then twice as much, its KE increases by 4.

If an object collides with another object that is already moving, such as in the same direction and slower so as there is actually a collision, then only some of its KE will be converted to force upon colliding with that object, therefore, it will keep the energy that was not transferred to the object. The energy transferred to the object will increase its KE (joules) and momentum (movement, p = mv).

Since: Work = (force)(distance) = W = (F)(d) = (ma)(d) , we can mathematically derive:

$$F = ma = \frac{W}{d} = \frac{\text{Joules}}{d}$$

: force expressed as a rate of using energy per unit of distance,
and:
Mathematically: a = F / m = J / md and:
mad = Joules and: Fd = work = Joules and: Fvt = Joules and:
Fv = Joules / t = energy / time = Power in watts and
F = Joules / (vt) and: momentum = p = mv = m (d / t) = md / t

The change in momentum (p) of an object just before collision and after collision where its momentum is 0 because all of its energy has been transferred or released to something else is:

(change in momentum) = (change in p1) = (m1)(v2) - (m1)(v1) = (m1)(0) - (m1)(v1) = - (m1)(v2)

The negative sign indicates the change in momentum (p , a change in movement) was a decrease.

From: a = (change in velocity) / (change in time) = $\Delta V / \Delta t$: extra: (change in v) = a (change in time)

$$F = ma = m\frac{\Delta v}{\Delta t} = \frac{\Delta p}{\Delta t} = \frac{\text{(change in momentum)}}{\text{(change in time)}}$$

: force can be measured as a rate of change in momentum with respect to a change in time. The units could also be expressed as energy / distance = = joules / distance .

$F = \Delta p / \Delta t$, also , $\Delta v = F \Delta t / m = a \Delta t$ and $F = ma = m v / t = p / t$ and $F t = mv = p$

If there is no change in momentum or motion such as a change in velocity, then no force has been applied and v is the same.

The greater the change in momentum and-or the shorter the time of that change, the greater the resulting brief pulse or impact force applied. During a collision, velocity and momentum will change since p = mv.
Also: v = p / m = momentum / mass

$\Delta p = F \Delta t$: (change in momentum) = (force applied) (change in time) = a measure of an **impulse**
(the application energy by a force)

The longer the time that a force is applied to an object , the greater the change (increase or decrease) in its momentum. The greater the force applied to an object, the greater the change in its momentum during a given amount of time. Δp may sometime be symbolized with letter J, but it does not mean Joules of energy.

Force can be described as the application of energy to an object and-or which changes the momentum of an object.

p = momentum = mv with units of: (kg)(m/s)

If one Newton of force is applied for 1 second of time, it is called a Newton-Second = N-s and is the unit of impulse (I), and which is a measure of the force applied and the time it is applied.

Impulse = (Force)(change in time) = (change in momentum) = (mass)(change in velocity) = (Newtons)(seconds) = N-s = (m)(a)(s) = (kg-m / s^2)(s) = (kg-m / s) = mv = momentum. From this we can also have: mv = Ft and F = mv / t = momentum / t = m a

Impulse = I = F t = m a t = m v

If energy is used up by a person trying to apply a force to something very massive and it does not move (ie., change its momentum = mv), and since none of that input energy was transferred to that massive object, there was no actual, apparent or useful work=energy done in the terms of transferring energy to that object. Surely, a force was used and energy was applied, but none of it was apparently transferred, and-or saved (ie., gained) by the massive object, and the energy used up by the person is lost and-or wasted as other forms of energy. In the equation: Work = (force)(distance), if the distance is small or 0, then the work is small or 0 regardless of how much energy (Joules) was used. Work is rather an amount of transferred energy to something else. This is one reason why W = (F t)Joules is not used for the transfer of energy, but rather W = (Fd) Joules is used. Using: (Work)(time) = (force)(distance)(time) = Fdt = (Joules / time)(time) = (power)(time) = Joules , would be the total energy input, and including any losses to the (useful, desired) momentum of an object moving a distance.

A moving object such as an automobile or bicycle have wheels which will experience friction with the road. This can be thought of as a type of collision between the wheel and the roadway. Due to this the moving object will continually loose a certain amount of its kinetic energy each second, hence a certain amount of power each second, and the moving object velocity will continually be reduced unless more kinetic energy is given to the object by a force. If the energy input is equal to the friction losses, the objects speed will remain constant. If the energy input is greater than the friction losses, then the object will accelerate and continually increase in velocity. The effective input force given to the object and-or its mass is:
(effective input force) = (input force) - (friction force) , $F = ma$, $a = F / m$ = (effective input force) / (mass of object)
The effective energy input is: (effective energy input) = (energy applied) - (energy losses due to friction).
As the object slows, the graph of its velocity and KE with respect to time will decline downward just like a downward parabola, and have a high negative slope value. This is so since kinetic energy = $mv^2 / 2$ has the squared variable in it and it is therefore a quadratic and-or parabolic equation.

The mathematical relationship between kinetic energy and momentum:

First: Momentum is a measure of the total movement, motional and-or ability of an object to either keep moving or stopped. $p = mv = vm$ is a linear equation

Kinetic energy is a measure of the stored energy of a moving object, and which therefore has an equivalent amount of potential or stored energy. Its units like other forms of energy are joules. $KE = mv^2 / 2$ is a quadratic equation, and that KE greatly increases when v increases. It will also, for example, take 4 times more KE and-or applied force so as to double an objects velocity.

A falling object will increase in motional (ie., momentum, p) and kinetic energy (KE) due to gravity accelerating it.

$KE = \frac{mv^2}{2}$ and **p = momentum** = mv , we have $v = p/m$ and by squaring both sides: $v^2 = p^2/m^2$, we have:

$KE = m v^2 / 2 = m (p^2 / m^2) / 2 = p^2 / 2m$ = (momentum)^2 / 2 (mass) , solving for p = momentum:

$p^2 = 2 m KE$, taking the square root of both sides:

$p = \sqrt{2 m KE} = 1.414 \sqrt{m KE}$

Also, from $KE = (mv^2) / 2 = (mv)(v) / 2 = p v / 2$

Before moving on to another topic, lets see how the values of mass and velocity will vary if the kinetic energy (KE) remains (nearly, and practically) a constant value such as in the Newtons Cradle, (KE) device.

From: $KE = \frac{mv^2}{2}$ we have: $m = \frac{2(KE)}{v^2}$ and $v = \sqrt{\frac{2 (KE)}{m}}$

If we set (KE) equal to 1=100% of some value, we can create a relative analysis so as to see how the variables related to (KE) will change in a relative way, such as by what factor, and so as to have a more intuitive approach to how (KE) will change when one of these variables changes.

$1 = \frac{mv^2}{2}$, since we are rather interested in changes, * we can consider this analysis without the denominator of 2, and we see that these are factors of the product of 1, and are then mathematical reciprocals of each other:

$m = \frac{1}{v^2}$ * , if one factor increases or changes by a factor or (n), the other changes by the factor of (1/n) which is the the inverse of that factor. Ex.:

$$1 = \frac{(n)}{(n)}\,\frac{mv^2}{1} = (nm)\,\frac{(v^2)}{n} = \frac{m}{n}\,\frac{(n)(v^2)}{(1)}$$

It could be said that the relative changes to each factor are also reciprocal in value, and expressing this as another equation:

1 = (relative change in m)(relative change in v^2) mathematically:

$$\text{(relative change in m)} = \frac{1}{\text{(relative change in } v^2\text{)}}$$

: relative change in v = $(v2 / v1)$ = result v. / starting v.

$$\text{(relative change in } v^2\text{)} = \frac{1}{\text{(relative change in m)}}$$

: relative change in m = $(m2 / m1)$ = result m. / starting m.

$$\text{(relative change in m)} = \frac{m2}{m1} = \frac{1}{\left(\frac{v2}{v1}\right)^2} = \frac{1}{\text{(relative change in v)}^2}$$

$$\text{(relative change in v)} = \frac{v2}{v1} = \sqrt{\frac{1}{\left(\frac{m2}{m1}\right)}} = \sqrt{\frac{1}{\text{(relative change in m)}}} = \frac{1}{\text{sqrt (relative change in m)}}$$

Ex. If the KE of a mass1 of 1kg is being transferred to a mass2 of 10kg, how will the velocity of mass1 change?

(relative change in mass) = (mass2 / mass1) = $(10 / 1) = 10$

(relative change in velocity) = $\sqrt{1 / \text{(relative change in m)}} = \sqrt{1 / 10} = \sqrt{0.1} \approx 0.316$

Its velocity, whatever it may be, will change by a factor of 0.316 . That is $(v2 / v1) = 0.316$
For example, if its velocity = $v1 = 5$m/s , $v2 = (0.316)(v1) = (0.316)(5\text{m/s}) = 1.58$m/s

$KE = mv^2 / 2 = (1)(5)^2 / 2 = (10)(1.58)^2 / 2 = 12.5$ joules : When (KE) in = (KE) out , due to no energy losses, and a complete energy transfer and conservation of energy.

The discussion about Bernoulli's equation in this book has a similar analysis to the above.

Momentum is conserved for two objects that collide

total momentum before a collision = total momentum after a collision

total momentum = momentum of object1 + momentum of object2 = $(m1\ v1) + (m2\ v2)$, mathematically:

$v1 = [\ (\text{total momentum}) - (\text{momentum of object2})\ ]\ /\ m1$
$v2 = [\ (\text{total momentum}) - (\text{momentum of object1})\ ]\ /\ m2$

If after a collision where object2 collided with object1, and if object2 stopped moving, all of the energy and-or momentum of object 2 was transferred to object1, and the momentum of object1 will then be the total momentum of the system (here, of the two objects). This conservation of momentum (ie., the total momentum being conserved and not reduced or lost) is similar to the conservation of energy concept.

How much (kinetic) energy is transferred during a collision depends on the amount of force and the distance (and time, since to go a distance takes time) it was applied.

energy = work = (force) (distance) , Joules
energy = work = (mass)(acceleration) (distance) , Joules

Note that (KE) is a second-order or quadratic equation, and momentum is a linear equation. When velocity changes by a factor of (n), the (KE) is significantly changes by the square of that factor , and the momentum (p) changes in a linear manner by the same factor:

$KE1 = mv^2 / 2$ and $KE2 = m\,(nv)^2 / 2 = (m\,n^2\,v^2 / 2) = n^2\,(mv^2 / 2) = K1\,n^2$

$p1 = mv1$ and $p2 = m\,(nv1) = n\,mv1 = n\,p1$

From: $KE = mv^2 / 2$ and momentum $= p = mv$ and $m = p / v$ and $v = p / m$

$$\mathbf{KE} = p\,v^2 / 2v = \mathbf{pv / 2} = p\,p / 2m = \mathbf{p^2 / 2m} \quad \text{and} \quad \mathbf{p = 2\,KE / v = \sqrt{2m\,KE}}$$

Sometimes an **impact force** is described in terms of the (impact) energy in joules = (force in newtons)(distance in meters), rather than Newtons of force. For example: "the object impacted with a force of 1000 joules", and this rather means: "the object impacted with 1000 joules of energy (ie., KE = kinetic energy).

Energy = KE = Work = (Force)(distance) = Fd joules = $ma\,d = \frac{mvd}{t^2}$, and mathematically:
: in an impact (a) is a deceleration for the object that looses energy

Force = F = Energy / distance = Joules / distance = $KE / d = \frac{mv^2}{2\,d} = \frac{m\,v\,v}{2\,d} = \frac{mv}{2t}$: $d = v\,t$, $t = d / v$
$1 / t = v / d$

If the given energy is transferred in a shorter distance and-or shorter time, the impact force will be higher. An object with a greater mass and or velocity will have a greater amount of kinetic energy, and will produce a higher impact force.

One way to actually measure a force is by using a force gauge which can indicate the highest force encountered during an impact.

If an object is dropped from a height from above ground level, and due to the concept of equal and opposite forces, there will also be a force placed upon that object during the impact, and the value of 2 in the denominator of the impact force expresses that the (KE) energy available will apply and-or create (ie., induce) an equal amount of force to both the object and the ground it lands upon. The ground may obtain a dent in it, and the object may break apart. In other words, the energy got divided, some used to make the dent in the ground, and some used to break apart the object.

Some extra equation for consideration:

From: KE = mv^2 / 2 , we have: m = 2 KE / v^2

From: F = m a , we have: F = 2 KE a / v^2 = 2 KE g / v^2

KE = mv^2 / 2 = F v^2 / 2g

When KE = GPE , KE = GPE = mgh = mv^2 / 2

v^2 = 2 g h and $v = \sqrt{2 g h}$:

Such as for a falling object just before it reaches the ground . Notice that mass is not a factor in this equation, hence various masses will have both the same velocity and time of impact when dropped from the same height (ie., distance). Due to the constant force of gravity accelerating the mass, the linear equation: distance = height = (velocity) (time) is not valid due to that the velocity is not a constant value due to the force of gravity constantly applied to the object and accelerating the velocity of the object which starts from v=0 units of distance/second and continuously increases.

Calories = An indicated calorie number or level is basically the total energy available, and in these modern times, it is usually in reference to food(s) substances only, and for the specific food(s) in question since the total calorie value will vary for each different food substance or mixture and the amount of it being considered. This energy can be used by the body to do things, such as the fuel for moving muscles so as to move or do things which will require some energy (ie., fuel) to do. Any of the available calorie energy that is not used immediately by the body is either stored (perhaps as some fat, muscle or other tissue for a future use by the body) and-or eliminated from the body. 1 calorie of energy is formally defined (ie., its calculation and measurement) as the amount of energy needed to raise the temperature of 1 gram of water to 1°C higher than a given temperature. The English unit conversion of this is: 1 gram of water = 0.038814 fluid ounces of volume , and 1 °C increase = 1.8 °F increase or change. 1 calorie is equivalent to about: **1 cal = 4.187 J =~ 4.2 joules** of energy. Note, that to heat a gram of a certain substance or element other than water, it may take more or less than 1 calorie, and the specific value is called the thermal ability or thermal capacity of a substance. Also this equation assumes that all of the input energy to try to increase the thermal energy of the water or substance will increase of the thermal energy of the water or substance by that input amount, and this is generally not the case due to the specific situation and its inefficiencies of the energy transfer. For example, with a flame heating a pot on a stove, some of that energy is wasted as hot air around the side of the pot and it will then not heat any water in the pot. **1J = 0.239 cal**

In the cells of our body and other life forms are smaller structures called organelles that make and-or do things. One organelle type is called **Mitochondria**, and these produce **ATP** (**A**denosine **T**ri-**P**hosphate, a nitrogen, hydrocarbon sugar and phosphate molecule) from chemical reactions from the foods we eat, and this ATP molecule with stored chemical energy and-or ability provides chemical energy for the body to make and-or do things. In short, it takes energy from food to create other forms of energy that our body needs. You can tell by it's name, phosphate (**phosphorous**) is part of this ATP molecule. A typical cola soda may contain about 30mg to 55mg of phosphorous in the form of phosphoric acid, and too much soda will usually decrease the vital calcium level in the body and interfere with some other essential minerals. People with kidney disease will have trouble removing any unnecessary phosphorous from the body. Because white phosphorous can radiate a green light when in contact with warm air, its name is based on the words "photon bearer". White or yellow phosphorous can cause caustic burns and ignite in air, and is the toxic form of phosphorous. Red phosphorous is made by heating white phosphorous to about 500°F in an airless (ie.,no oxygen) container, and is not toxic, but it is flammable and is used in the production of fire matches.

The average human adult at rest, say sitting and-or lite activities, will use about 100 watts of energy = 100 Joules of energy per second for powering all their body processes (blood flow, digestion, breathing, etc.) of which much is eventually converted to heat and keeping the body's internal temperature at a comfortable 98.6°F = 37°C, and of which some of this heat energy is constantly being lost to the cooler air that is in contact with the skin and the body then needs to keep reheating itself. 100 W = 100 Joules of energy / 1 s =~ 23.88 cal / s =~ 24 cal / s , and this would total to 2063232 cal / day =~ 2000 kcal = 2Mcal = 8640000 J / day . It is often, though technically incorrect in terms of units, commonly said that adults should consume a scientifically estimated, **"2000 cal / day diet"** so as to replace this (estimated, averaged) amount of energy used up and converted to mostly heat energy. For a person doing moderate exercising and-or doing general work, this 100W = (100 Joules of energy used / 1s) of power value can increase to say 300 W of power, and these people will then require three times more food, calories and its corresponding energy so as to replace this used up (ie.., converted to heat, etc) energy and to sustain and power their work ability. Many animals have a similar or slightly higher body temperature than that of humans.

A calorie in terms of food: A calorie value or level is not an indication of a food's nutritional (nutrients= vitamins [elemental compounds] and minerals [elements]) content or value. A food may be high in calories, but low in nutrition. Many diets include some physical exercise and limit the number of calories you should eat, and then you will therefore need to get your proper nutrition (vitamins and minerals, and with at least the minimal daily amounts recommended) in that food you do eat, and assuming your body actually did absorbed all the available nutrition in what you did eat. Some people eat well to get their nutrition, but have digestion problems, and so that nutrition available may not actually be getting absorbed into them, and they should therefore consider deliberate supplementation (ie., a vitamin pill(s)) after consulting with their doctor. For people having some difficultly loosing weight, it is still better to at least not gain any more weight, and which is a very positive start of controlling their weight. Having food is comforting and life maintaining, however it can actually become detrimental to our health if it contains inadequate, low nutrition, and-or if we are already overweight and not controlling our weight gain. Try to always stay aware and consider these matters every time you shop for and-or eat foods. Too much sugar, oils (fried foods, chips) dairy (cow milk, cheese, sour creme, etc.), salt, and bread (wheat) products have been a problem for many people. Vegetables, some fruit, and a general variety of foods are usually recommended. Some foods are also more enjoyable, depending on how they were prepared and cooked. For example, finely shredded raw carrots are easy to eat, are somewhat sweet and make a wonderful addition along with some other available shredded vegetables into ramen ("quick-noodles" [cook quickly]) or a "veggie sandwich" with a small amount of your favorite salad dressing.

Ex. A certain bag of potato chips that has a total of 252g indicates that 1 serving for a person is 28g = 1oz, and that it has 150 calories. How many grams and calories would 3 servings be? 3 servings would be (3 servings) (28g/serving) = 84g. 150 calories is to 28g as is x calories is to 84g. 150 cal/28g = x cal/84g. After solving for x, we have x= (3)(150 cal) = 450 cal. A 252 gram bag of these chips will have: 252g / (28g/1 serving) = 9 defined or standard servings. This bag will also have at total of: (252g)(150 cal / 28g) = 1350 cal.

Examples of calories in common foods:

Food	**Calories (averaged over a variety) / gram**	**: 1 gram = 1g = 0.035274 oz (weight ounce, wtoz)** **1 oz =~ 28.3495g =~ (30g food 1 oz.)** **1g = weight of 1cc volume of liquid water** **1 fluid oz. water = 29.573555 grams = 30g food oz**
Water	**0**	**1 fl. oz. Water =~ 29.57 cc volume**
Food Salt = Table Salt =	0	
Food Oils	9	= about (9 cal/g) (30g/oz) = 270 cal. / oz
White Sugar	3.8	, about 112 cal./oz
Instant Dried Potatoes	3.5	, about 100 cal./oz dry : can rehydrated with plain water, hence 0 added calories
Popcorn	3.8	, about 110 cal. / oz
Rice	3.55	, about 100 cal. / oz dry
Peanuts	5.63	, about 160 cal. / oz : about 50% of the mass and-orweight of a peanut is due to its oil
Peanut Butter	5.85	, about 166 cal. / oz
Meats	13	, (13 = 4 calories protein + 9 calories fats) = about (400 cal./oz) = 6400 cal/16oz = 6400 cal/lb
Breads	2.75	: (2.75cal)(30g/oz) = 82.5 cal / oz = 1320 cal / 16 oz = 1320 cal / 1 lb
Pasta (noodles)	3.55	=~ 100 cal./oz = 1600 cal / 16oz = 1600 cal / lb
Vegetables	0.5	
An Olive	1.3	:a green olive weighs about 4 grams. About 15% on average of the weight or mass of a green olive is oil. A back, ripe olive can have about 22% oil on average.
Cola Soda	~ 0.38	, (mostly from white or cane sugar) , 12 fl.oz can =~ 355cc =~ 360g , having ~ 140 cal / .can
Vodca , 100%	7.5	, (about 210 cal./oz = 210 cal./28g = 7.5 cal./g)
Average Beer	~ 0.43	, (a 12 floz can will have about: (12 floz)(12.3 cal/oz) = 148 cal. / 1 can of beer
Chicken Egg	1.3	, 65 cal./ medium size chicken egg =~ 50g , hence : 65 cal. / 50g =~ 1.3 cal./g
Mozzarella Cheese	2.71	, 76 cal. / 1 oz = 76 cal. / 28g : from cow milk , slightly aged , mild taste
American Cheese	3.68	, 103 cal. / 1 oz = 103 cal. / 28g : Usually a blend of mild cheddar, Colby, and may have other common cheeses , made from milk

Specific Heat Capacity is the amount of energy rating to raise the temperature of 1 Kg of a substance by 1 °C, and when no losses are considered. Water has a specific heat capacity of 4186 J/Kg/°C = 4.186 J/g/°C , and it will have this much more thermal energy when no losses are considered.

Mass-Energy Equivalence : E = Mc^2

First, **energy does not have a specific form, material or shape, and is rather an ability, "the potential" or "power" to do things**. Materials can store and-or release energy such as to apply it and release it to other materials that will absorb it. Energy is measured in standard units called Joules (J) or Joules of energy.

There is a concept or theory that since we can use matter or "mass" to store and-or release some energy, in a reverse manner, that energy can be condensed into some mass, hence effectively as being a "matter battery" or "battery of matter" which can contain energy. A good example is a fuel to release its energy and make heat, light or motion (such as in a vehicles gas combustion engine) so as to have and give useful energy to another mass so as to do other things. There is a famous equation for the mathematical relationship of mass and energy, and it was developed by the German physicist **Albert Einstein** (1879-1955) in about the year 1905:

E = M c^2 : E = energy in joules , M = mass in kilograms , c = speed of light in meters/second =~ 3(10^8) m/s
c^2 ~= (3(10^8))^2 m/s =~ 9 (10^16) m^2/s^2

This formula indicates that even a small mass, such as 1 gram, has a very large energy capacity, storage, potential or ability within it. The energy is stored as the **strong nuclear force** that is much like the stored energy of a compressed spring. This force holds the quark particles together that compose each proton and neutron. This force will also keep the protons from electrically (ie., charge force) repelling each other in the nucleus. The strong nuclear force is much greater than the electric force, but it only acts strongly at the nucleus area of the atom, hence for a short distance only. According to the formula, the density of the mass, or the type of element(s) the mass is composed of, does not have an effect on the resulting energy equivalence. From the formula, since (c) is a constant, if M is increases by a factor of (n), then E increases by that same factor of (n), and vice-versa. (c) is a constant of proportionality in the mathematical relationship between (of) M and E, and specifically, its value is how many times E is numerically always greater than M. (E / M) = c^2. The ratio of the value of E to M is always the same constant of (c). The reciprocal of c^2 is how many times M is numerically less than E.

Another consideration is that the smaller the mass, the less energy needed to make it go at a certain or given speed, and here, the speed of light. It takes little energy for an atomic particle, which has a very small mass and momentum, so as to be highly accelerated and have a high velocity such as the speed of light. E / M = c^2 , and if M is changed or amplified by a factor, E is also changed by that same factor so as to have an equivalent fraction with the same ratio value, here c^2.

According to this formula and theory, 1 gram of mass which equals 0.001kg can (theoretically) be converted to 9(10^13)J = 90 (10^12) J = 90 trillion joules = 90 Tera-Joules = 90 TJ . This is enough energy for a 1 MW = 1 mega-watt = 1 million watts = (1) (10^6)J / s = 1 MJ / s power station to supply full (100%) and continuous power for 2**.**854 years:

From: watts = joules / s = J / s , we have: time = joules / W = (energy available) / (rate of using the energy)
90 TJ / (1MJ / s) = 90 (10^12) **/** 1 (10^6) / s = 90 (10^6) s = 90,000,000 s = 90 million seconds of time
Since 1 year = 31,536,000 s : 90,000,000s **/** 31536000s/1yr = 2**.**854 years

Einstein's equation shown above is actually very similar to the (older, and more useful) kinetic energy equation: **KE = mv^2 / 2** , and it could be thought that the if the matter itself also turned into pure energy, then this KE value would be twice or double as it is expressed in Einstein's equation: mv^2 = mc^2. Even though Einstein's mass-energy equation contains the speed of light variable (c), his equation actually considers the mass when it is at rest and does not have any extra or kinetic energy associated with it and which would increase its total energy. His equation rather considers the energy as potential or stored energy due to the matter only.

It was previously shown in this book that force = mv^2 / d , hence: fd = mv^2 = work = energy , joules
If v = c , the speed of light , we have energy = mc^2

Again , this is the kinetic energy formula without the 2 in the denominator. What can we make of this? Consider when you lift an object a height to give it an amount of (gravitational) potential energy, your muscles have used and converted an equivalent amount of work and-or energy but have lost it as heat energy, hence this is a reason why the common KE formula equals mv^2 / 2 or half of the total energy used in the system as indicated by the 2 in the denominator. This is much like the concept of equal and opposite force, and perhaps could be considered equal and opposite work = energy , joules.

When atoms are made, tremendous kinetic energy and force is needed to force together, and essentially fuse together, the protons, neutrons and electrons of an atom. Even each neutron and proton are now understood (after research beginning 1964) to be composed of three quark (with mass a mas of 1/3 of a proton or neutron) particles each, and with gluons holding the quarks together with what is called the "strong atomic or nuclear force". Don't confuse this force with magnetic or electric force. This energy used to force particles so as to be together as an atom, is stored in that atom as strong atomic or nuclear forces and this could be thought of as being like a compressed spring with stored energy, and if the spring is cut, the stored energy is released. The more particles in a particular type (ie., element) of atom, the more total energy need to make it, and the more stored, potential energy in it.

A helpful theory of what causes gravity. The field of gravity from a mass, unlike that of electric and magnetic fields, does not seem to curve around to itself an-or have two poles of attraction and-or repulsion. If mass is associated with energy, a larger mass has a larger amount of energy, and therefore a smaller mass has a less amount of energy. The force of gravity seems to be proportional to the amount of mass, hence therefore proportional to the energy within that mass. Surely, it takes more energy to split and-or create atoms having more mass particles (atomic mass units = protons, neutrons, etc.) because they took more energy to create and they can therefore store more energy, much like atom sized springs compressed together at the nucleus and storing the energy it took to do so, and particularly for protons which have the same (positive) electric charge and repel each other. Gravity could then be said as due to that energy (or energy and mass) attracts other energy (or energy and mass), and is commonly said as (just) the matter, and not the conceptual energy it has that attracts each other. With this theory of gravity, and object with more mass will have a higher "gravitational [potential] energy" potential and-or ability to attract other mass. In short, gravity is then a form of stored and-or accumulated energy, much like how the strength of magnetism and electricity are.

The concept of **relativity** takes into effect of how the observer (apparently) sees, experiences, and-or understands (or assumes) things. For example, if you were in some type of vehicle, such as a car going 50 miles and hour, or even going 1 million miles an hour, you would generally not even notice how fast that vehicle is going. With respect, reference, relative, or in relationship to you, it would seem or appears as that you and the vehicle and all the objects in it are going 0 miles per hour or not moving in reference to you and each other. This all happens because you and the vehicle are both going the same speed and direction, and there isn't any difference in speed between you and the spaceship your in. To a distant stationary observer on a planet, they would indeed see the spaceship vehicle with you in it traveling quite fast with respect or relative to them and their own corresponding position (location), speed and direction. It is often said that speed or velocity is "relative", or "relative to the observer (and their own speed)". Due to this fact, is not to difficult to then image that distance and time are also relative to the observer. Now consider the possibility that gravity is also relative to the observer, surely, for two astronauts or objects falling at the same rate inside a spacecraft orbiting about the Earth, they will sense no downward force of gravity upon them, and it is as if g=0m/s^2 , but it is the relative or apparent value to them. It will still require energy to make a force and apply it (the energy) so as move an object in "weightless" space, but once it is applied, the object will keep moving unless another force acts upon it. Extra: While on this subject, due to the mass value of an object, it will still have an inertia (ie., effective resistance to the change in motion, and-or that an amount of time may be needed to transfer the applied or input energy) value, and its value is relative to the amount of mass. If an orbiting spacecraft were to reduce speed and-or stop in its orbit, it will begin to feel the effects of gravity and it will be pulled closer to the Earth by not having enough velocity to maintain the altitude of orbit, and the spacecraft will reduce in altitude, such as needed for landing the spacecraft.

Some conversions for energy, work and power

BTU = **B**ritish **T**hermal **U**nit. If the temperature of 1 pound (the weight of 16 fl or volume ounce of water) of water is increased by 1 degree Fahrenheit, it is gained 1 BTU of (thermal) energy. 1lb = 1 pound =~ 453.592374g = = ~ 454g =! 0.454kg. 1°F = 0.555°C , 1°C = 1.8°F . 1 BTU = 0.293 watts, therefore 1 watt = 3.413 BTU

Calorie = A (thermal, energy) calorie or cal is a unit of (thermal energy) measurement when the temperature of 1 gram (= 1cc) of water increases by 1 degree Celsius or Kelvin. 1 cal =~ 4.184 Joules of energy.

Foot-pound = A unit for the amount of energy to lift or move a weight of 1 pound upward by 1 foot of distance. In terms of torque or twisting force, 1ft-lb is the (perpendicular applied) force of 1 pound on a lever arm that is 1 foot long. If the force is still 1 pound, but the lever arm is now 2 feet long, then the torque or twisting force is 2 ft-lbs, and this is also equivalent to a weight of 2lbs on a lever arm of 1 ft. Torque = (F)(L).

1 Joule of energy = about 0.000948 BTU = 0.239 calories = 1 watt-seconds = 1Nm = = 1 Newton-meters , (ie., the energy = work of 1 Newton of constantly applied force to an object through 1 meter of distance). 1BTU =~ $1054.8\underline{5}$J =~ 1000J = (~ 1000 kJ) = 0.293w

Note that force = (mass)(acceleration)
1N = (1kg)(m/2^2) = the force that can accelerate 1kg of mass 1 meter per second/per second that it is applied.

1 Joule in terms of electricity is the amount of energy needed or used to force or move a total of 1 ampere or 1 coulomb of charges / 1 second of current through a resistance of 1 ohm for a duration of 1 second. `V = J / C , therefore: J = VC = fd = work = energy. The voltage drop across and-or through this resistance is IR = (1A)(1ohm) = 1V. If the resistance was then 10 times more at 10 ohms, and the current was still 1 ampere, the voltage drop across and-or through this resistance is IR = (1A)(10ohms) = 10V and the energy needed, used up (or wasted as heat) would be 10 times more in that 1 second of time, and that would be 10 Joules. An equation to find the amount of electrical energy is:

From: **V = J / C and 1A = 1C/s** we have: **J = VC = VAs** , Ex., as described above: J = (10v)(1A)(1s) = 10
Note that VA = power = P watts , and 1J = 1W-s
VAs = Ws = (Power)(time) = (J/s)(s) = J = Joules

Pw = $\frac{\text{joules}}{\text{time}}$ = (voltage) (current) = $\frac{(J)\ I}{(C)} = \frac{J\ I}{I\ s} = \frac{J}{s}$: 1W = 1J/s
: From: Ia = 1C/1s , 1C = (1a)(1s) = "1 amp-second"

joules = J = (Pw)(t) = (V)(I)(t) : also by Ohm's Law: V = (I)(R) and I = V/R ,

joules = (I^2)(R)(t) = (V^2 / R)(t)

Work = (force)(distance) joules = (mass)(acceleration)(distance) joules = Energy to do that work

The relationship between these energy units:

1 BTU = about $1055.05\underline{6}$ joules = 252 calories . : Consider: 1 BTU = $\frac{454g}{1°F} = \frac{Xg}{1°C}$ and $\frac{454g}{1.8°C} = \frac{Xg}{1°C}$ and:

(1°C / 1.8° C) 454g = Xg = 0.555(454g) = 252.2g

Or by considering a (mass)(temperature) product:

(454g)(1°F) = (454g)(0.5555C°) or:
(454g)(1.8C°) = (Xg)(1°C) and:

454g = (Xg) (1°C / 1.8°C) = Xg (0.5555) , solving for Xg:
Xg = 252.2g

1BTU is = 454g raised 1F° = 252g raised 1°C = 252 calories = 1055.05$\underline{6}$ joules , mathematically:
1 joule = 0.0009478$\underline{2}$ BTU =~ roughly 0.001 BTU =~ a thousandth of a BTU
1 calorie = 4.184 Joules = about 0.003966 BTU =~ 0.004BTU =~ 4 thousandths of a BTU

1 watt (W) of power (ie., energy used up or converted to another over form time = an energy rate) = 1 Joule per second = = 1 J/s . From this we have: 1W = 1J/s , therefore: 1J = 1 (W)(s) = 1 W-s = 1 watt of power applied for 1 second

1 watt = 1J/s , if we multiply both sides by 1000, we have 1kW = 1000 J/s
Hence, mathematically: (1kW)(1s) = 1000J , and if we multiply each side by 1hr = 3600s, we have:
(1kW)(3600s) = (1000J/s)(3600s) = (power)(time) = **PwTs = Total Energy**
1 kWh = 3600000 J = 3.6 (10^6) J = 3.6MJ = 3.6 mega-joules

1 horse power = 1HP = 550 foot-pounds = 746 watts = 745J/s = 0.746Kw. 1Kw = 1000w = 1.34$\underline{1}$hp

Energy = Work = (force)(distance) = 1 ft-lb or 1 lb-ft = about 1.3558 Nm = 1.3558 joules of energy

Other related considerations:

Power = "strength applied or needed" = $\frac{\text{Energy}}{\text{time}}$ = $\frac{\text{Work}}{\text{time}}$ = when the fractions are reduced = $\frac{\text{Joules}}{\text{second}}$ = J/s : like an average

Total Power used or needed = $\frac{\text{Total Energy}}{\text{Total Time}}$

Power = $\frac{\text{Energy}}{\text{time}}$ = Joules / time = Work / time = (force)(distance) / time = (force)(velocity) : d=vt , v=d/t , t=d/v

Force = $\frac{\text{Power}}{\text{velocity}}$ = (mass)(acceleration) , velocity = $\frac{\text{Power}}{\text{Force}}$ = $\frac{\text{Joules / s}}{\text{Force}}$ = $\frac{\text{Joules}}{(m)(a)(s)}$

time = $\frac{\text{Total Energy Used}}{\text{Energy Use or Rate}}$ = $\frac{\text{Joules}}{\text{Power}}$: = Ts = time to deliver, acquire or use an amount of power

velocity = distance / time = meters / second

acceleration = change in velocity / time = (change in d/t) = (meters / second) / second = meters / second^2 = m/s^2

Force = (mass)(acceleration) = (mass)$\frac{(\text{meters})}{(\ s^2\ )}$, mass = $\frac{\text{Force}}{\text{acceleration}}$ = $\frac{\text{Force}}{m/s^2}$ = $\frac{\text{Force } s^2}{\text{distance}}$

Force = $\frac{(\text{mass})(\text{distance})}{s^2}$ = (mass)$\left(\frac{d}{(s)(s)}\right)$ = $\frac{(\text{mass})(v)}{s}$: (Force)(s) = (m)(v) = momentum, or change in momentum

An object having (n) times more mass will take (n) times the force and-or energy to move it the same given distance, for example lifting a mass and-or weight a height. For a given amount of mass, if the force and-or energy applied is reduced by (n), the acceleration is reduced by (n), and the time needed to move it a given distance will increase by (n).
If the mass increases by (n), Energy applied and-or stored / unit mass is still the same ratio.

Work = Energy = (Force)(distance) = lb-ft = Nm = (mass)(acceleration)(distance) = (m)(a)(v)(t) , Joules

(**n**) Work = (**n**) Energy = (**n**) lb-ft = (**n**) Nm = (**n**) (force) (distance) , Joules

Ways that thermal or heat energy can be transferred from one object or place to another.

In general, the density of a substance is a major contributor to its ability to conduct heat. Metals have a high thermal energy conduction, liquids are not as good, and gasses are poor thermal energy conductors. Materials having much air space or "gaps" in them are often used for insulating (ie., protect, separate, make of no effect) objects from both higher and lower temperatures by greatly reducing the thermal conduction around an object. The study of heat effects in materials, and the spread and-or transfer of thermal energy is called **thermo-dynamics**.

1. **Conduction** = Direct contact, such as from the atoms of a hot object in contact with the atoms of a cooler object. The cooler object will increase in temperature, while the hot object will therefore decrease in temperature. Friction or mechanical resistance is an example where some kinetic energy is converted into thermal energy, friction is usually a sign of wasted energy, an energy loss. Fluids and substance blended or rated at high speed in a container can actually increase in temperature due to friction.

2. **Convection** = To convey or transfer of heat energy by a medium such as a liquid or gas. This transfer can be assisted by pumps and fans. The warm air from an electric heater is an example. Hot. low density and weight air rises upward like a flowing current, and heavier weight and density cool air falls downward (ie. sinking) like a "draft" (flow) of current. This warm air can then transfer its heat to another object via conduction or direct contact.

3. **Radiation** = Transmission through space. An example is solar (sun) heat (thermal) radiation (energy transmission). Thermal radiation is actually a form of light which itself is a form of electromagnetic radiation.. Invisible radio waves are another example of electromagnetic radiation and is often called "RF" or Radio Frequency energy. The energy in this radiation can then heat up objects. A solar air heater can convert solar radiation into warm air, heat convection after the heat is first conducted to the air at the collectors surface.

"Saving heat" energy is preventing heated objects from cooling down and wasting the energy and money it took to heat it up initially. **Insulation** (ie., a separation from) is the general method to prevent thermal energy loss, and insulation basically slows down and-or prevents thermal transfer, that is, it does not easily absorb and-or transfer heat energy.

Friction is a direct contact, (slight) collision or (slight) impact between two objects and-or surfaces and will cause heat to be created due to the forces applied to the atoms in those surfaces. Friction will also cause some of the (kinetic = motion) energy of a moving object to be transferred to that which is causing the friction, and the energy (KE, motional) of that moving object will be reduced, and therefore, its speed will reduce. An example is when you push a box on a floor, it may move fast with a high kinetic energy at first, but friction will reduce or decay (ie., slowly loose) all of its KE to 0 joules. Force and-or friction can even cause the direction of an object to change, and this could be considered as a deflection in motion. Friction can cause an object to turn in its direction of contact as can be seen by slowing (perhaps by the friction of a piece of wood against it, the left wheel or motion of vehicle and it will want to turn somewhat leftward as the right side of the vehicle tries to continue forward. If the left side movement was to stop completely, the right side would pivot or rotate about the left side. A rudder (a thin, flat, vertical and steerable piece of wood or metal) in the water beneath the end of a boat can be used to steer that boat by slowing one side of that boat. The collision between the rudder and the water will also cause the boat to loose some kinetic energy and its velocity will also slow. That energy will be transferred to the water.

For an example of direct contact or conduction of thermal energy:

A piece of ice such as an "ice-cube" is a solid piece of water. Is crystallized water (H20), a crystal of water, or water crystal, and with atoms bonded, held or locked together. For this to happen, the liquid state or condition of that water must be changed to a solid state, and this is done by cooling that water to its freezing or solidifying temperature of 32°F = 0°C or less. When the water cools, its molecules loose thermal or kinetic energy, and can no longer stay loosely moving about due to the energy they had at a higher temperature.
If a piece of ice is placed into a liquid that has a temperature higher than the freezing temperature of water, the frozen

molecules in the ice will start to gain kinetic energy and move about. The ice will melt or dissolve to be liquid water. Since the liquid of which the ice was placed in has used up or lost some of its thermal energy to melt the ice and increases the thermal energy in the ice. The temperature of that liquid that the ice was place in will decrease or cool, especially if it was a small volume and-or mass, and there will be less of a temperature difference between the liquid and the ice, and the rate or speed of the amount or volume of the ice melting will decrease. To make a cool drink or substance to a lower temperature, ice can be placed into the liquid, and the melt water of the ice will become part of that liquid solution, and dilute it. Some of the dense cool water will also sink into the liquid, and this is a form of convection which transfers (here, cooler) thermal energy.

Usually, an object that is dense, having many particles, is more able to more gain kinetic or thermal energy and can then store much thermal or heat energy. That object is said to have a high **thermal mass**. Thermal mass can be considered as: energy / mass . Ex. joules / gram.

Just like a large mass that is difficult to move, a large mass will take longer to move and-or heat up, but when it does, it is difficult to reduce since it contains a lot of energy and-or motion. It could be considered as like a "thermal momentum" or "thermal inertial" of that substance. A flywheel is a large rotating mass to store kinetic (motional) energy, and its energy value is directly determined by the amount of mass and its rotational speed. A small solid mass cannot store as much thermal energy as a larger mass of the same material, and the smaller mass has a lower total thermal mass or energy in it, and it will "cool off" or reduce (or gain) in temperature much more quickly than a large mass of the same material. A larger surface area for the same given mass will also increase the rate at which an object can change in temperature due to having more ability to conduct heat into the air (or other material) due to the larger contact area between those substances..

A **thermal capacity, heat capacity** or **specific heat** rating of a mass or object is the amount of energy needed to raise its temperature by 1 degree, usually in Celsius temperature. The word "specific" in specific heat, is in reference to the specific, unique substance and-or element being considered, which has a specific associated thermal capacity and rating.

$$\text{Heat or Thermal Capacity rating of a mass or object} = \frac{\text{energy needed per mass}}{\text{degree change in temperature}} = \frac{\text{Joules}}{1\text{C}^\circ\ 1\text{g}} = \text{J} \ / \ \text{g}.1\text{C}^\circ$$

Ex. Water has a high heat capacity of about (4.2 J / C°) per gram of mass , hence (4.2J / g) / 1°C = (4.2J / g-°C). It could also be said that water has 4.2J of energy per gram for each degree rise in Celsius temperature. It will also take 4.2 joules of energy to raise the temperature of 1 gram of water by 1 degree Celsius, and if there was no energy losses doing so. If there are (4.2J/C) / (1 gram), and if we multiply the numerator and denominator of this fraction by 1000, we have there are (4200J/C) of energy per kilogram of water. Water has a density of 1 gram / cc. This value of 4.187J =~ 4.2J is also defined as the **calorie** energy unit.

Ex. Copper, which is much denser than water has a low heat capacity of about 0.385 J / C° per gram of mass. Copper has a density of nearly 9 grams/ cc, and this is 9 times more than the density of water, and the ratio of their heat capacities is that water is about 11 times more, and-or copper is 11 times less.

Compared to other substances, water is fairly difficult to heat or change in temperature due to its strong atomic bonds, and that its density is fairly low at 1g/cc, and as compared to the much higher densities of metals, and yet it therefore retains heat very well, and is a fluid which can transport thermal energy in a practical manner through a pipe, etc. To much thermal energy applied to water will cause it to change its physical state to a gas called steam or water vapor when it boils. The higher the temperature of water, the lower its density.

If the water being heated is (N) grams, it will take (N) times more energy to raise its temperature by 1°C :

$$\text{Total energy needed for 1°C in temperature in N grams of water} = \text{Ng} \ \frac{(4.2\text{J})}{\text{g}} = \text{Ng } (4.2) \text{ Joules}$$

The above value is for a 1°C change in temperature, and if the change in temperature is Tc times higher, it will take Tc times more energy to do so. This is so since the energy needed is the same for each degree change, step or increase in temperature, and the total energy is the sum of energy needed per degree change. This sum can be expressed as a multiplication by the number of degrees change in temperature = Tc.

Total energy needed for a change in temperature of Tc of N grams of water $= Tc\ Ng \left(\frac{Joules / g}{1\ C°} \right)$ Joules =

= Tc Ng (4.2) Joules for water

4.2 Joules is also called a **calorie** of thermal energy. 1 calorie will raise the temperature of 1 gram of water by 1 degree Celsius. 1 calorie = 4.2 joules = 3.96567 **BTU** (British Thermal Units) ~= 4 BTU. A BTU is the amount of energy needed to raise 1 pound (about 16 fl. oz = 454 cc) of water by 1 degree Fahrenheit. 1lb = 0.454kg = 454g. 1J =~ 0.0009478 BTU , and 1 BTU =~ 1055.075 J : 1 BTU can be roughly considered as equal to about 1000J. It will take about 4220J =~ 4 BTU of energy to raise the temperature of 1kg = 1000g = 1000cc of water by 1 degree Celsius. 4BTU will also raise the temperature of 1 pound of water by 4F° , or 4 pounds of water by 1°F.

This above example also assumes all of the energy used to heat the water was converted to thermal energy of that water, and with no wasted energy or energy losses. The equation presented was theoretical, and there will be some loss of energy. For an object to retain its thermal energy, temperature or heat value, it must be insulated from its surroundings by something that does not remove or conduct heat or thermal energy away from it. This is the case as for thermal energy storage, where that energy is stored until it is needed. A Thermos (tm, R) or other similar device is a double walled glass container that is insulated by a **vacuum** between each wall, and can keep and-or cook foods if a process known as "thermal (retention) cooking". Here, hot water and the item to bee cooked such as noodles is placed within the container, and all that is needed is time to cook and soften the noodles rather than more energy. Likewise, in theory, a well insulated house would not have to use any more energy to be heated to a comfortable temperature once it was warmed up. If there are no energy losses in a container, the substance within it can be heated by using just a small amount of power, but it may take considerable time depending on its mass and-or volume: From: Energy input = Power = Joules / time = P = J / t, we mathematically have: J = Pt, and for heating purposes, this must be thermal or heat energy.

A vacuum is the best thermal insulator due to the lack of direct material contact, but it will allow radiation such as infrared thermal energy (ie., a form of light that we can not see) to pass through it. Due to this, a Thermos (tm, R) type of container is often coated on its outer surface with a reflective material, and this will help maintain the internal temperature at a stable higher temperature, and from reducing due to that type of thermal energy loss. The thermal energy loss per second is a very small amount of joules, and this can be replaced by a small amount of joules of thermal energy directly applied to the substance - perhaps by a small, low-power electric heater. Just the same, a thermal insulator can also keep things within it at a stable cooler temperature and from absorbing external thermal energy.

How much total thermal energy does a mass of water have? It depends on the amount of mass, its temperature, and its energy storage capacity per gram. From the above discussion, we know that for each degree change in Celsius temperature, a mass of water will require and then acquire 4.2 Joules of energy per gram. Let the temperature value (To) of an object be its change in temperature from the "absolute zero" (zero thermal energy) and where there is no (0) kinetic and-or thermal energy of any use in that mass or object:

Consider: $\frac{\text{Total thermal energy}}{\text{Total mass}} = \frac{\text{Joules}}{1\ g}$: equivalent ratios or fractions, and after the division of just the numeric parts of the values. From this we have:

Total thermal energy = (mass) (energy / gram of mass) = Ng (joules / g) = joules

We will now consider a given temperature of the mass:

Total thermal energy in a object or mass of N grams = To (Ng) $\left(\frac{\text{Joules / g}}{1\ \text{C}^\circ}\right)$ = Joules :for **To**, see the note below

K = C° + 273.15 : K = kelvin temperature , C = Celsius temperature
C° = K - 273.15 F = Fahrenheit temperature , C = (5/9) (F° - 32) , F = 1.8C + 32

At absolute 0 = K = 0 = -273.15° C = -459.67° F : temperature, and-or thermal heat, start at 0°K, and not 0°C.
Change or difference in temperature Celsius above absolute 0 = Temperature - (-273.15°) = Tc + 273.15°

Let **To** in the above equation be the total temperature change or difference above absolute 0: To = Tc + 273.15°
Tc is the current temperature of the substance. Both values have units of Celsius degrees.

Ex. If the mass is 1g of hot, near boiling liquid water, and its temperature is 212°F, = 100°C, how much thermal energy (Et) does it have?

K = 100°C + 273.15 = 373.15 : ie, equivalent Celsius degrees above absolute 0

Et = (Ng)(change in temperature)(thermal energy per gram per degree Kelvin)
Et = (1g)(373.15°C)(4.186J/gC) = 1562J

Element or Substance	**Density (g / cc) At STP**	**Heat Capacity Rating (J / C° / g) = (J / g C°)**	**: (essentially a "thermal mass" rating) Typical or Average Values**
Water	1.0	4.186	: slow or difficult to heat , but hot steam has great KE energy,
Lithium	0.534	3.6	
Paraffin Wax	0.87	2.48	: a wax made from petroleum (ie.,crude oil)
Water Steam	0.006	2.0	: at 100°C
Water Ice	0.917	2.1	: at 32°F = 0°C , and 1.4 for -100°F = -73°C
Alcohol	0.79	2.2	
Sodium	0.97	1.2	
Magnesium	1.74	1.05	
Air	0.0013	1.012	: =~ 1.0
PVC	1.38	1.67	: Polyvinyl-Chloride
oils	0.93	1.75	: averaged, for food oils, petroleum is 2.1
clay	1.5	0.91	: for the denser clays
Aluminum	2.7	0.9	
Salt	2.16	0.88	: sodium-chloride = "table salt" = "food salt"
Concrete	2.3	0.88	
Glass	2.89	0.82	
Granite	2.7	0.79	
Carbon	2.25	0.71	
Steel	8.05	0.48	: doesn't store too much thermal energy, but is easy to heat
Copper	8.96	0.39	
Silver	10.49	0.23	: very easy to heat, its atoms have a high thermal transfer in it
Gold	19.3	0.13	
Lead	11.35	0.128	: easy to heat and melt (at 621°F) to a liquid

1g mass of water = 1cc volume = 1mL volume or water that has gained 4.186 Joules of energy will have increased 1°C higher in temperature. Likewise, it could then be said that if 1cc = 1mL of water was now 1°C higher in temperature, then it has gained 4.186J of energy. As seen above, metals with high density, dense crystal-like atomic (ie., atom) structure, can be heated relatively easily due to their thermal transfer from atom to atom. The above could be also stated as:
(4.186J / 1°C) / 1g = (4.186 ((J / g) / C°))

From this information, if 1L = 1000cc = 1000mL = 1000g of water was 1°C higher in temperature, that water has gained: (1000cc)(4.186 J / 1°C) = 4186 J of energy.

If 1L of water changed by 10°C higher, that water has gained: First, 1L = 1000cc = 1000mL = 1000g if water: (1000g)((4.186 J / g) / 1°C)(10°C) = (4186 J)(10) = 41860J of energy.

For water: $\frac{4.186\text{J}}{1°\text{C}}$ is to , then $\frac{1\text{J}}{0.2389°\text{C}}$ is to , : after dividing both the num. and den. by 4.186

If the temperature of an amount of water decreased by 1°C, then it has lost 4.186J / 1cc = 4.186J / 1g if water.
If 1L = 1000cc = 1000g if water decreased in temperature by 1°c then it has lost: (1000g)(4.186J/g) = 4168J of energy

A **thermal conductivity** rating of a material is how well (ie. how fast) heat or thermal energy can be transferred or flow through it over a time period. It is the rate of heat or energy transfer within itself. A thermal conductivity rating is essentially based on the amount of energy needed per unit time which is power or watts, and so as to change the temperature within the substance at a distance of 1 unit away and by 1 degree increase in temperature.

Some metals have a very high **thermal conductivity rating**. Silver, copper and aluminum have a high thermal conductivity rating, and they also conduct electricity very well and therefore have a low electrical resistance to the flow of electricity (electrons, current). Air and-or insulation foam have a low thermal conductivity and is used help keep things (objects, a volume) remaining at a certain temperature, and to also minimize wasted heat and-or energy. A vacuum has no thermal conductivity and is the best temperature or heat insulator. Here are some approximate thermal conductivity ratings (W / m k°) of some materials, and they can be considered as considered relative values. For example, silver has a rated value of 410, and it therefore transfers heat 410 times faster than that of glass which has a rating of 1. Although water can store much thermal energy, it is not a good thermal conductor of thermal energy within itself. Hot water can rather be transferred to where it is needed via pipes such as to deliver thermal energy. Hot steam (ie., water vapor substance, gas), rather than its temperature can do much work due to its high (gas) pressure and-or kinetic energy.

air=0.024 , pine wood=0.15 , water=0.6 , glass=1 , concrete=1.6 , ice=1.58 , lead=35, steel=48 , iron=80 , aluminum=210 , gold=310 , copper=390 , silver=410 , diamond (dense, crystallized carbon) =1500 av.

Note that lead is very dense, and yet has a very low thermal conductivity.

thermal conductivity = (thermal or heat power transferred) / (degree change / unit area) / unit distance

The formula may also be expressed as:

thermal conductivity = (joules / time) / 1 C° / m : 1C° change in temperature = 1°K change in temperature

thermal conductivity = $\frac{\text{(watts)}}{1\text{C}°\ \text{m}}$: the reference distance may be for example, 1cm, hence over that distance, a reference volume that the heat must pass through is 1cm^3 = 1cc. The reference distance is usually expressed with units of centimeters or meters. If centimeters, the resulting digits are the same, but are a thousand times less.

Ex. How much energy will be needed to heat a fluid ounce of "tap water" from a pipe to boiling temperature?

A fluid ounce volume of water = about 30cc of volume, and has a mass of about 30 grams
The average temperature of tap-water is about: 55°F =~ 13°C.
When water begins to boil, it has a temperature of: 212°F =~ 100°C

From this data, the water will require a temperature increase or change of: 100°C - 13°C = 87°C

From: Total energy needed for 1°C in temperature in N grams of water = $Ng \frac{(4.2J)}{g}$ = N(4.2) Joules

Energy needed = (mass)(J /g C°) (change in temperature) : (J/g) / C° = J / gC° , or simply:
Energy needed = (mass) (energy per gram / degree) (number of degrees)

Energy needed = $30g \frac{(4.2J)}{g} \frac{(87C°)}{1C°}$ = 10962 J = 10.962kJ =~ 11kJ : (number of degrees) can also be noted as the (change in temp.).

This could be considered as that it takes about 11 kJ / 1oz or= 11 kJ / 30g, to heat tap water to a boiling temperature. Hence, for 2 ounces, or twice as much water, it will take: (2 oz / 1)(11kJ / 1oz) = 22k J or twice as much energy.

Power = $\frac{\text{energy}}{\text{time}}$ watts : power is the rate of using energy, mathematically, Energy = (Power)(time) J

If that water requires 2 minutes to boil: time = 2 minutes = (60s/min)(2min/1) = 120s, the rate of the input power needed and applied to the water is on average:

$$P = \frac{10962J}{120s} = \frac{91.35\ J}{1s} = 91.35 \text{ watts}$$

For the above example and values, regardless of any possible inefficiency or energy losses and that value during this change of temperature process, when the mass of water has reached a certain temperature, that mass has gained 11kJ of energy, and in the form of thermal heat (kinetic motion of the molecules and-or atoms).

If the temperature of a mass increases by (n), the total thermal energy within that mass will increase by (n). If the amount of mass with a given temperature increases by (n), the total thermal energy within that mass increases by (n).

If given two different metals with the same mass, say 1g, and if they are placed in an oven with a certain air temperature inside it, those two metals will absorb thermal energy via (heated air) convection until they reach the same temperature of the air in that oven. If the metals have a different density and associated thermal capacity rating, then the amount of thermal energy that each will then have or store in its atoms, will be a different value. Metals with a higher thermal capacity can store more thermal energy per gram. Metals with a higher thermal conductivity will heat up or increase in temperature at a quicker or increased rate with respect to time than a metal with a lower thermal conductivity. Silver will heat up much faster than steel will, and this is because silver has a higher thermal conductivity rating and this is determined by its density, atom and crystal structure. Note also that silver has a lower thermal capacity than that of steel. In general, it could be considered that silver rather conducts thermal energy better than it stores it.

Extra: **John William Draper**, (1811 - 1882), from England, discovered in 1847 what he named the **Draper Point** temperature at which all solid matter starts to produce (ie. make, "glow", "infrared" thermal radiation) some visible, faint red light at about 977 °F = 525 °C =~ 798 °K. This is roughly about 1000°F. This color of relatively low frequency of light in the visible "spectrum" (frequency range or bandwidth of frequencies) has a frequency of about 83 (10^12) hz = 83 trillion hertz = 83 Thz. If this temperature is placed into the Stefan-Boltzman equation as shown in this book, the result is about 22000 watts of (rf, light, thermal radiation) energy per square meter of surface area. Some examples of where thermal radiation is needed is for cooling via heat-sinks ("draws or absorbs heat from a nearby object, and radiates it away), and cooling in shaded regions of a spacecraft to prevent overheating.

When heat retention is needed, then the loss of energy and-or heat via thermal radiation is to be avoided. If you had an object or dwelling that requires (x) joules of energy per second (hence a power value = x J/s = x watts) to maintain a certain temperature, then surely, that object or dwelling is loosing and-or radiating (x) joules of energy per second. A well constructed, perfectly insulated (ie., via a vacuum, etc) object or structure will maintain or slowly (J/t) and (°T/t) reduce in temperature.

A SIMPLIFIED THERMAL ENERGY TRANSFER EXAMPLE

In this example we will be given two different masses or objects of the same metal, say iron or copper, hence they will have the same density. If it helps, you can consider these pieces of metal as being cube shaped. We are to find the resultant or final temperature ("the thermal equilibrium temperature") of the combined masses after the two metals are joined together at a side, and considering no external heat energy losses.

Helpful considerations for when the masses are joined together at their surfaces:

1. If given an evenly heated mass or object of the same element and-or density, it has a total amount of thermal energy associated with it. Any portion of that amount of mass has the same portion of the total thermal energy of the total amount of mass. For example, 10% of a mass will contain 10% of the total thermal energy of the total mass. Each different sized portions or samples will still have the same temperature, but the mass and temperature products ($m\ °T$) will be different, hence having a different total thermal energy.

2. Heat or thermal energy will transfer from the hotter mass to the cooler mass. There must be a temperature difference for heat to be transferred. A temperature difference can be considered as the resulting "heat pressure" which will cause a transfer of thermal energy. Hotter substances or elements have atoms and-or molecules that have an increase in (thermal) energy, and they are then moving and-or vibrating faster and will have a higher kinetic (ie., movement) energy, and which can be transferred to other atoms and-or molecules during a collision, and thereby loosing some of its kinetic energy and then getting cooler as the object it collided with gains (thermal and-or kinetic) energy. Remember that when and object collides with another other, each is effectively colliding with the other and will affect each other in some way.

3. The hotter mass will get cooler, and the cooler mass will get hotter, and the amount of temperature change for each depends on the amount of mass for each. If the two masses have the same temperature before being joined, the larger amount of mass tends to then have a slightly lower temperature, and a smaller amount of mass tends to change more and quicker in temperature. Over time, the rate of the change in the temperatures of the masses with respect to time will decrease, and then those two objects will eventually be at the same temperature or "equilibrium temperature". The equilibrium temperature will have a value that is between the two temperatures, and will be closer in value to the temperature of the largest mass.

4. The more area of surface contact of the two masses, the faster that the heat can transfer or conduct from one to the other. An example is with the construction of a "heatt-sink" and-or radiator, such as to cool a motor or electronic part, or to heat a room quickly. A heat sink will absorb heat from a device, and that heat can be transferred, radiated to, and-or absorbed by a cooler air mass. This will cause the heated air to become less dense and it will then rise upward and allow more (cooler) air to then come into contact with the heat sink which may sometimes be called a (heat) radiator.

5. When the masses or objects become the same temperature, there is no temperature difference or imbalance, and therefore, there is no "thermal (energy) pressure" to cause a transfer of thermal energy. Their temperature are at equilibrium or balance.

6. The total amount of thermal energy is the sum of the thermal energy in each mass. If there are no losses due to air and or infra-red thermal radiation, this amount of energy will be the same value after the two masses are joined together.

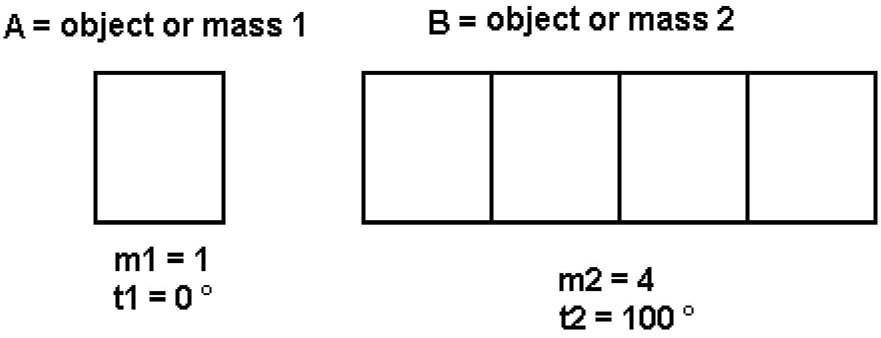
Et = E1 + E2 : starting and ending total amount of thermal energy
Mt = M1 + M2 : total mass

7. At thermal equilibrium the temperatures of the joined objects are the same. The amount of energy (ie., Joules) in each object depends upon its mass. A larger mass at a give temperature will have more thermal energy associated with it than a smaller mass at that same temperature. Given a certain temperature, if the mass is twice as much, its thermal energy is twice as much, hence mass and energy are directly proportional when the temperature is constant. The ratio of the energy in each mass will equal the ratio of their corresponding masses when they are the same temperature. E1 / E2 = M1 / M2 or= V1 / V2 = since volume is directly proportional to mass.

8. A larger mass will take longer to heat up to a certain temperature, and will then have more total thermal energy, and this value could be described as its "thermal mass" and-or ability. A larger mass will take longer to heat up to a certain temperature, and this could be described as "thermal inertia" or the slowness or resistance to a temperature change.

9. The amount of Joules in an object may be difficult to calculate by some, and so another practical way that energy can be measured is what can be called the "mass-temperature" product or thermal-mass product. The more mass, the more thermal energy at a certain temperature. The more the temperature of a certain mass, the more thermal energy it has.

   For this analysis, we will let: E = energy in joules= thermal-mass product = (mass)(temperature) , [FIG 230]



When m1 and m2 are combined together at their flat edge area, what will the resulting or equilibrium temperature of their total mass be if there are no losses in thermal energy? .

The total energy of this system is the energy of mass1 plus the energy of mass2:

Et = Em1 + Em2
Et = (m1)(t1) + (m2)(t2) = (1)(0) + (4)(100) = 0 + 400 = 400 : at the both start and end at equilibrium
: (m)(t) = "mass-temperature product" and can use (v)(t) = "volume-temperature product"

The total mass of this system is:
Mt = m1 + m2 = 1 + 4 = 5

The total amount of energy at equilibrium temperature (Te):

Et = (Mt)(Te) or can use: **Et = (Vt)(Te)** :if you have measured volumes (V), such as water fluid.
400 = (5)(Te) , solving for Te:
Te = Et / Mt = 400 / 5 = 80 considering each mass having this new temperature, their energy is:

Em1 = (m1)(t1) = (1)(80) = 80
Em2 = (m2)(t2) = (4)(80) = 320

As a check, the ratio of these two masses is:

m1 / m2 = 1 / 4 = 0.25

Mass2 had 4 times the mass as that of mass1, and at equilibrium, mass2 will have 4 times the energy as that of mass1. The ratio of their corresponding thermal energy is:

E1 / E2 = 80 / 320 = 1 / 4 = 0.25

The author has tested the above formula using 1 container of cool water at 63°F, mixed with 4 containers of hot water at 114°F, and the measured temperature was about 103°F, and the calculated temperature, using the above method, was about 103.8°F The slight error was probably due to both a slight measurement error and some minimal heat losses in the large combining and mixing container, and of which probably absorbed some thermal energy.

Solving for the temperature needed for a volume and-or mass so as to have a certain equilibrium temperature of two volumes (V) and-or masses :

Ex. If V1 = 4, T1 = 100°F, and V2 = 1 , what is T2 if the equilibrium temperature (Te) after mixing these two volumes of water is to be 120°F ?

Et = Vt Te = (V1 + V2) 120°F = (4 + 1) 120° = (5) 120° = 600° : total thermal energy

Et = Vt Te = E1 + E2 = (V1)(T1) + (V2)(T2) , solving for T2:

$$T2 = \frac{Et - (V1\ T1)}{V2}$$

$$T2 = \frac{600° - (\ (4)(100°)\ )}{1} = \frac{600° - 400°}{1} = 200°F$$: this temperature is close to water boiling at 212°F

How is the specific heat rating of an element determined? In brief, it is done by performing an experiment, often by immersing the metal in question into a volume of water and noting all the changes, and then making a calculation. The mass and temperature of both the water and metal are also measured before this takes place. A loss in temperature and-or (thermal) energy of one is a gain in temperature and-or (thermal) energy of the other.

Let **Hs = Specific Heat of an element = (J / gC°)**

When an object is heated by a certain number of degrees higher, it gains thermal energy or heat, and therefore, there was a change in energy:

(change in thermal Energy) = (mass)(change in Temperature)(Hs) = (m)(Hs) ,* and note that mathematically:

$$\frac{\text{(change in thermal Energy)}}{\text{(change in Temperature)}} = \frac{(E2 - E1)}{(T2 - T1)} = (m)(Hs)$$:**for a given object and element, (m) and (Hs) are constants**

The thermal energy and temperature of an object or mass are linear and-or proportional, and for a certain mass, they will always have the same ratio value:

$\frac{E1}{T1} = \frac{E2}{T2} = \text{ratio}$, * Mathematically:

$Hs = \frac{\text{(change in thermal Energy)}}{\text{(change in Temperature) (mass)}}$, with units of : J / g °C , and the loss of thermal energy in the water is equal to the change, gain or increase, of thermal energy in the metal

Thermal Expansion

When a metal is heated, it will cause that area of the metal to expand due to the excited or energized (vibrating) atoms and the forces created when they expand, and this is particularly seen by heating a long thin metals strip in its middle, and its cooler ends will effectively bent in an arc shape toward the source of heat.

A bi-metallic strip is created when two pieces of different metal are welded together along their length. Each metal has a different amount of thermal expansion due to the heat applied, and the rated value of the expansion is called its thermal expansion coefficient value (CTE). Due to the heat applied, this will cause the ends of the metal with a higher thermal expansion value to bend or arc towards the other metal. Obviously, a device than can bend due to heat can be used as the temperature sensor for various temperature measuring devices. Lengths, widths, areas, and volumes are affected by thermal expansion. Much like water freezing to ice, the forces involved in thermal expansion can be high, and devices and structures can crack and then be a potential functional and-or safety problem.

When an object such as a strip of metal is heated uniformly, it will expand by a certain percentage uniformly and then it will be longer as long as its temperature remains the same. When the object cools, it will contract and eventually be the size it was before it was heated.

Coefficient Of Thermal Expansion (CTE) = $\frac{\text{(percent increase in any length of the material)}}{\text{(1 degree increase in temperature)}}$

A Short Note On The Thermal Affects of The Air Temperature On The Human Body

Our body temperature is 98.6°F on average, and this is considered a healthy body temperature, however it does not necessarily indicate any medical condition or problems. We actually feel most comfortable on average when the air temperature is near to 72°F and which is called "room temperature". We feel comfortable at that temperature and it is most likely due to that any excess internal body thermal (heat) energy created due to internal functions and also to keep us warm in cooler weather is more easily transferred to the air, and our bodies do not have to create any more internal heat (if possible) to keep us warm. Clothing helps insulate us and keeps us warm in cooler weather, and several layers of shirts, a hat, socks, shoes and-or slippers will help keep a person stay warm. As for pants, the author has found that "blue-jeans" (ie., dungarees, durable cotton material) are a poor choice to wear in cool or cold weather. Loose fitting "sweat pants" (ie., a thicker soft cotton) are then a better choice to wear, and two layers of pants can be worn if the temperature is very cold and windy outside. A winter or cold weather coat is also needed in cool or cold weather conditions. A "hoodie" coat is acceptable, but only if the person has layers of shirts on, hence a oversized coat may be necessary for a person to feel comfortable. The author has found that an indoor, dwelling temperature as low as 60°F = 15.5°C is livable and comfortable if people dress properly, and this will also help conserve energy and money needed for it. Keeping active also helps keeps us warm in cooler weather. .

When the air temperature gets lesser than say 72°F or "room temperature", the heat from our body is more easily transferred to cooler (lower temperature) air since the difference is growing, and it is then more easier to transfer thermal energy to it. At cooler air temperatures, our skin in contact with the air will reduce in temperature or "cool down" or "cool off" (ie., reduce in value) as the heat from it is more easily transferred to the cooler air and the it slightly increases its temperature before it rises upward into the more cooler air since warmer air is slightly less dense. Cooler skin will then absorb heat from the body, and this process will eventually cool the body to a lower temperature if there is no thermal or heat energy created by the body to keep it at a constant temperature.

If the air temperature is greater than 72°F, then the higher it is, the more difficult it is for the body to transfer energy to the warmer air since the temperature difference is less, and this is why it feels very warm or hot at say, beginning at about 85°F. Today as of about the year 2010, an electric air-conditioner to reduce a rooms air temperature and-or humidity is no longer an expensive luxury, but nearly an inexpensive necessity since it is often warmer indoors than outdoors during the summer months, and that is no way to live - stay cool, comfortable and be healthy indoors. Being hot often raises people "tempers" and discomfort, and things become more challenging to complete. During hot temperature, people can find shade to stay cooler, drink cool liquids, and-or wear a wide hat and- sunscreen ("sunblock") lotion if necessary.

More About Temperature And Altitude , How Does Air Temperature Change With A Change In Altitude

The troposphere of the Earth is from sea level (0 feet high) to about 30000 feet high, and the air temperature in this zone will decrease at a fairly steady or linear rate as the altitude increases at a fairly steady or linear rate. As altitude increases, the air gets less dense and-or has less mass and thermal or kinetic energy. The air near the Earths surface is denser, and therefore warmer. Some of this warmer air temperate, especially near the ground, and up to about 30000 feet is due to the air being in contact wit the relatively warm surface of the Earth and then rising upward, and also due to warm air temperature mixing and distribution due to the clouds and wind .

For an average value about the Earth, the air temperature will decrease at a rate of about (-0.003566 °F) per foot higher, and-or increase at a rate of about (+0.003566 °F) per foot decrease in elevation. To make an estimated temperature calculation at a certain height above a location, you will need to know the locations current air temperature:

$$\frac{-0.003566\ °F}{1\text{ ft higher}} = \frac{-3.566\ °F}{1000\text{ ft higher}} = \frac{-3.566\ °F}{0.3048\text{ km higher}} =\sim \frac{11.7\ °C}{1\text{ km}}$$

: 1 km = 1000m = 3280.8 ft = 0.621364 mi
1 mile = 5280 ft = 1.609364 km
1m =~ 3.28 ft , 1 ft =~ 0.304878m

Some other equivalent fractions to consider:

- 1°F / 278.86 ft higher = -1°C / 85.5 m higher = 0.0117 °C / 1m higher

°F Air Temperature Above = Air Temperature °F At Your Location - [0.003566 °F (change in feet higher)]
°C Air Temperature Above = Air Temperature °C At Your Location - [0.0117 °C (change in meters higher)]

The above equations is also from and having a linear equation format of: y = mx + b:
Air temperature at your current altitude =
Air temperature above your current altitude + (change in temperature / distance)(change in distance) =
(change in temperature / distance) (change in distance) + air temperature above your current altitude

For some reference: 32 °F = 0°C and 0°F = -17.7778
A change of 1°F corresponds to a change of 0.55555°C
A change of 1°C corresponds to a change of 1.8 °F

The Stefan-Boltzmann Law : **Ludwig Boltzmann**, (1844-1906), from Austria, and who studied energy loss and **entropy** which basically means the eventual disorder of a system, and particularly when studying gas molecules in a container that will eventually collide and go into random directions and velocities, hence a disorder.

This law states that the energy radiated or emitted per unit of surface area of a piece of matter per unit of time is related to the fourth power of its temperature. The simplified general formula is: **Energy = c T^4** , and where **c** is the Stefan-Boltzmann constant =~ 5.6704 (10^-8) W / m^2 K^4 . **T** is the absolute temperature, hence in degrees Kelvin. If T of the matter has doubled, the internal and radiated energy will increase by 2^4 = 16, and this also implies that the object has also gained this much more thermal energy. This formula is for an ideal **black-body** (or **blackbody)** that absorbs all radiation or energy, and that it radiates it very well once it is at the temperature of its environment, and this is called black-body radiation. This black-body radiation factor is equal to 1 = 100%. A "grey-body" rather absorbes and emitts radiation of a llimited range of frequencies. A conceptual "white-body" will reflect all radiation and not rise in temperature. **The full Stefan-Boltzmann Law equation is: Power = c (Area) T^4 , hence P / A = c T^4**

A **heat-sink** is often a piece of aluminum metal with a large surface area in a small volume of space or size, and it is used to protect sensitive electronics by absorbing its heat and then releasing or radiating this excessive heat energy into the surrounding cooler air, and therefore, this system is very similar to an electric air heater.

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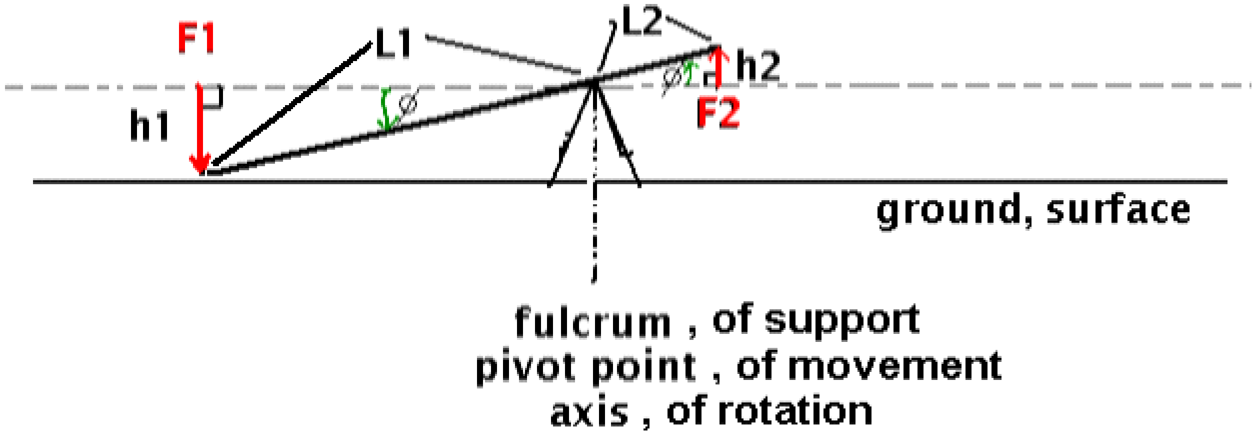
A LEVER ANALYSIS

A lever system can provide a gain in force, but requires more input distance and-or time needed so as to do something such as lifting a heavy weight. A lever is known as a "simple machine" and has a "mechanical advantage" of some sort, and here it is an amplification or "gain" in force. [FIG 231]



Though a lever arm does provide a gain in force, it does not provide a gain in the total amount of energy needed, and the total amount of energy needed to move an object a certain height will be the same as that without using the lever. The red arrows in the image rather show the relative height length moved at the ends (ie., input and output) of the lever and not the relative amount of force. Still, the arrows do show the direction of the input and output forces. F1, the input force is lower than F2, the output force, when a lever is to have a mechanical (ie., force) advantage or gain.

Total Energy Out = Total Energy In	and since power is the energy used per time:
Total Power In = Total Power Out	The input Power is lower with a mechanical (force) gain, but it takes longer, and therefore, the total input power will be the same as the total output power.

The lever machine or system allows the user a lower rate of using or inputting energy or force so as to eventually lift the object, but then the time needed to lift that object a certain height using that low rate of energy input or power will be longer so as to effectively apply or accumulate the necessary total amount of energy needed to lift that object a certain height. Since there is a lower rate of input energy per unit of time needed, the corresponding force needed to apply that input energy is lower, hence its easier for the user.

W out = (force out)(distance out) = W in = (force in)(distance in) : since (force out) > (force in) due to the increase in output force. For equation balance (distance in) > (distance out) Here the distances are the heights moved.

For these two values to be the same, if one factor is multiplied by a certain value (n), the other non-corresponding factor on the other side of the equation must be multiplied by the reciprocal (1/n) of that same value, which is the same as dividing it by that value (n). In short, for a given or constant value such as the value of the above equation, if one factor of it increases, the other factor of it must decrease.

For this analysis of a mechanical lever system, we can, for practical purposes, analyze it with basic right-triangle analysis even though the system actually rotates about a pivot (ie. rotation) point in a circular manner. Levers are mostly used to help lift a heavier, more weighted (ie., a force) object by using a smaller input force for a longer period of time. This almost seems impossible, but it becomes very possible and real when the input leverarm distance (L1) to the fulcrum (pivot or rotation point) is greater than that of the output leverarm distance (L2). L1 is the distance from the fulcrum that the input force (F1) is applied. L2 is the distance from the fulcrum to where that the output force (F2) is applied to the object. The twisting or turning input and output forces or torques will be analyzed using simple direct, linear (non-twisting or rotating) forces. How can a smaller input weight or force lift a larger output weight or force? This will be mathematically shown now.

First consider that using weights (a force) of the same value and distance on each side of the fulcrum point of the lever, that the lever would be in balance and not moving downward or upward (ie., rotating). The effective output torque would equal the input torque, or in simple words, the effective output force would equal the input force:
Tout = Tin and-or Fout = Fin : we will also use: F2 = Fout and F1 = Fin

If the weights or forces are different values, then their distances along the leverarms can be adjusted so as the system is kept in balance and not moving or rotating due to an unbalance of forces and resulting in some net force being applied to and then moving (here, rotating) the lever.

From the concepts of work (W), a measurement of work equals a constant force applied through a distance so as to do that work. For this lever system and analysis, we will let the distance (D) moved equal the height (h) moved:

W = (force)(distance) = (F)(D) = (mass)(acceleration)(distance) = mad : = energy , joules

Win = Wout this is equivalent to: energy in = energy out and power in = power out less any losses

$(F1)(h1) = (F2)(h2)$ from this we have the reverse ratios of: $\frac{F2}{F1} = \frac{h1}{h2}$ mathematically:

$F2 = (F1)\frac{(h1)}{(h2)}$:clearly as h1 increases, the output force F2 increases. The input force (F1) amplification value or factor is equal to the ratio of: (h1/h2). To increase (h1), the length of the leverarm (L1) on that side of the fulcrum must be increased. The longer (h1) and (L1) is, the larger the output force (F2).

In the figure shown, the angles on each side of the fulcrum are equivalent vertical angles, and the lines (h) dropped to the lever create similar right-triangles. The lever arms (L) are effectively a (hypotenuse) side of these right-triangles created. Corresponding sides of right-triangles are proportional (same portions or fractional value of their respective triangles) when one is essentially just a similar triangle or magnified construction of the other:

$\frac{h1}{L1} = \frac{h2}{L2}$ mathematically:

$\frac{L2}{L1} = \frac{h2}{h1}$ or $\frac{L1}{L2} = \frac{h1}{h2} = \frac{F2}{F1}$

We see that the height ratio is equal to the corresponding leverarm ratio which can then be mathematically substituted in the above equation for F2:

$F2 = F1\frac{(L1)}{(L2)}$: * The gain in input force (F1) is equal to the ratio of the input and output lever (L) arms. The greater L1 is and-or the lower L2 is, the greater the input force amplification factor. From this we have the familiar torque (T) equation(s):

$(F2)(L2) = (F1)(L1)$, which can be expressed as a ratio of torque or force values:

$\frac{F2}{F1} = \frac{L1}{L2}$: these are also "reverse ratios" since corresponding values are not in the same numerator and denominator. Consider also for example, the less the output lever arm (L2), the greater the output force (F2)

Here is a related note about actual levers and their lever arms:

torque (force) multiplier = tm = (reverse ratio of lever arm lengths or distances) , (see * above)
torque (force) multiplier = tm = (lever arm length of the input force) / (lever arm length of the output force)

Note that if using a lever arm system to lift something up a height, the height multiplier (h1/h2)=hm or increase is inversely related to the torque or force multiplier (F2/F1) = fm. height multiplier = hm = (1/fm). For example, if the force multiplier is 10, the output height multiplier will only be: hm = 1/fm = 1/10 = 0.10 = a tenth of the force multiplier. If the input height was 10 inches, the output height is: (10 inches)(hm) = 10"(0.10) = 1 inch

Some other helpful relevant equations are similar to those mentioned:

From: work = (force)(distance) = energy : input energy = output work , and with standard units of joules
: input work = output work

W1 = F1D1 = W2 = F2D2 : W1 = Win and W2 = Wout , mathematically:

$F = \frac{W}{D}$, and for a given amount of work or energy, if the distance decreases, the corresponding force value will be higher since these values are mathematically inversely related.

$D = \frac{W}{F}$, and for a given amount of work or energy, if the force increases, the distance to do that work and-or use-up its corresponding amount of energy (Joules) will be less.

Notice that the time it takes a point at the end of L1 to go the distance of h1, is the same amount of time it takes a point on the end of L2 to go the distance of h2:

Since: distance = (speed)(time)

D = VTs

$Ts = \frac{D}{V}$

If the time (Ts) value is the same or a constant, then if a distance (ie., h) is less, such as the output height or distance, then (vertical, linear) velocity will be less on that side of the lever. If the input distance or height is larger, the (vertical, linear) velocity will be more on that side of the lever.

In terms of angles moved per unit time, since the angles are the same value due to being vertical angles, both sides of the lever will rotate at the same rate or angular velocity = degrees or angle per unit time = (¢ / time).

For the lever system, the input and output time of the (input and output) applied forces or resulting amount of work is the same, but the specific time value or amount of time to get the result of lifting the weight a certain height is now longer, but "easier" (less force).

If there is a weight at the end of a lever and you are trying to raise it a height, the torque (ie. an "amplified force") due to that weight must also be considered, and this could be considered as the effective weight that we are trying to move a height. A basic equation for this is: Effective weight = Torque = (Force)(Leverarm) = (Weight)(Leverarm)
To raise that effective weight to a height, we must now provide a torque that is greater than the torque due to that weight. The torque needed to be applied to the opposite side of the lever so as to raise that effective weight will be at least:
Force = (Torque) / (Leverarm) and-or (Leverarm) = (Torque) / (Force)

If a weight is too difficult to lift to a height such as to be used for a lever, many smaller objects can be used such as rocks, bricks, dirt, etc. The gear analysis shown previously in the book is closely related to this lever analysis, and vice versa.

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More About Matter, Mass, And Mol

Chemistry is the study and measurement of matter (ie., chemicals composed of atoms and molecules) and its interactions with other matter and-or energy such as heat, light and-or electricity. The word "chemistry" is rooted in a description of Egypt as "Khemia" which is based on the fertile "kem" (dark or black) soil material called "khem or "kemet" found about the large and fertile Nile river delta near the Mediterranean Ocean. The word "chemical" is based in the word of "khem". khem basically means substance(s), and chemistry basically means the study of substance (ie. matter). From a modern point of view, the ancient Egyptians had much knowledge about chemistry or matter and which then inspired the world in that field of science. **Matter** is a real physical material or substance. The word and meaning of "matter" is based on the words of "mother" and "source". The word "material" is based on the word "matter". **Mass** is a measure of the amount of any type of substance or matter. The word "mass" is a form of the word "material" and which is being combined or assimilated (based on the word "similar") together so as to have a quantity of it. A **mol** is a unit for a defined quantity of similar objects or particles of fundamental units of matter such as for example protons, atoms, or molecules.

We can divide an objects weight (a force) by (g) to find its mass.

From: force = (mass)(acceleration) = weight = (mass)(g)

:g is the amount of local gravitational acceleration caused by the constantly applied gravitational force upon a mass, therefore producing the corresponding force or weight of that mass. (g) at Earth's surface is ~= 9.81 m/s^2.

Force or weight has units of Newtons = (kg m/s^2) = the result of an acceleration applied to a mass

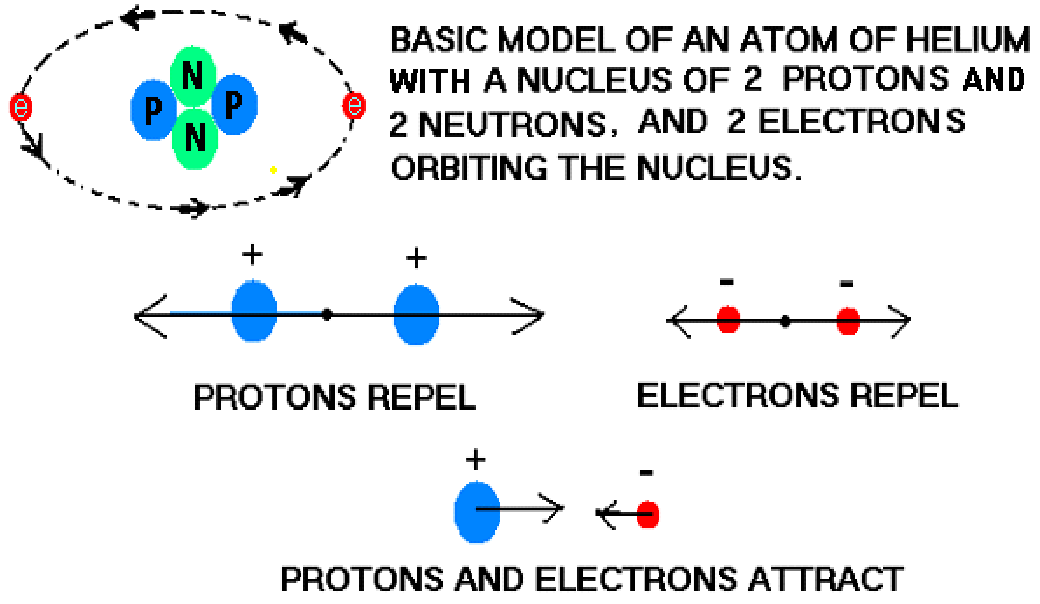
$$\text{mass} = \frac{\text{force}}{\text{acceleration}} = \frac{\text{force}}{\text{g}} = \frac{\text{weight in Newtons}}{9.81 \text{ m/s\^{}2}}$$

, with units of kilograms

Most household scales are calibrated so as to automatically do the above calculation for mass using its weight, and then to display the result in grams or kilograms. Most scales are technically "mass scales" and not "weight or force scales".

Atoms are commonly understood as the building blocks of matter (ie., "mass"), and are the smallest part of a unique material or substance called an element (of matter). An atoms fundamental structure is composed of particles of material (ie,. matter that has a measurement of mass) called protons and neutrons located at the center or nucleus of an atom, and electrons which orbit that nucleus of the atom. The number of protons in an atom is formally and usually equal to the number of neutrons. The number of electrons is equal to the number of protons. Each element (ex., oxygen, iron, gold) has a unique atom or atomic structure because it has a different number of particles than that of an atom of another element. The unique structure of the atoms of a particular element gives that element its unique physical properties such as mass and its corresponding weight, density, color, hardness, bending or twisting strength, melting temperature, electrical conductivity, light energy sensitivity, etc. [FIG 232]

Please note that chemistry experiments can be very toxic and dangerous. Please get an education first, and-or follow the precise rules made by those who have came previously.



The above figure shows a basic representation of the main (atomic) particles of an atom, and their electric forces. The electric force of attraction or repulsion is a much stronger force than gravity at the same distance. For example, static electricity can cause (lightweight) things to be lifted upward, hence against the force of gravity.

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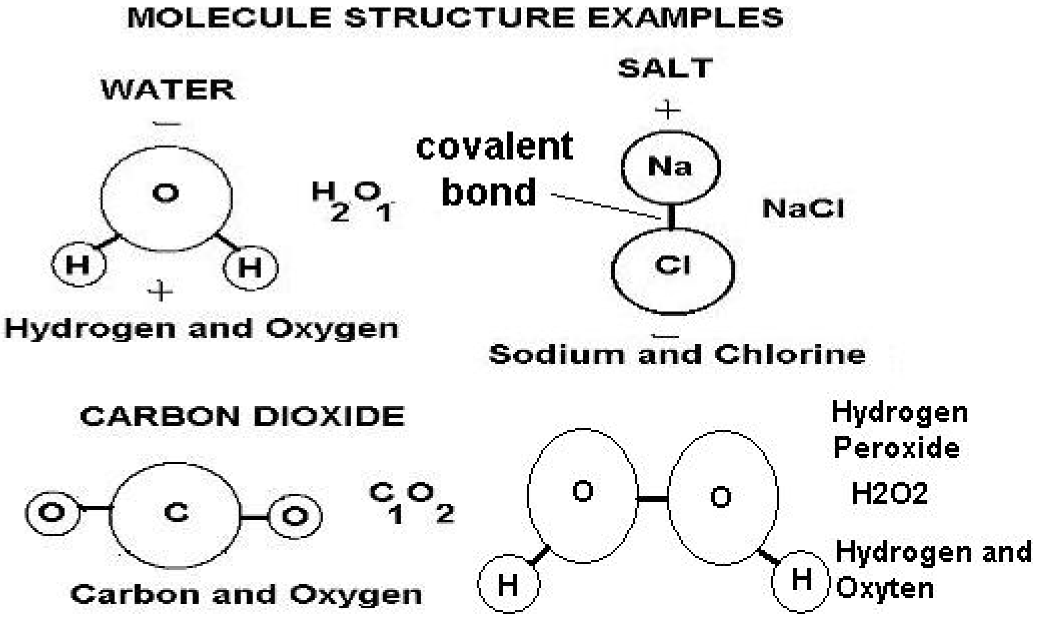
Each element is composed of particular atom structure that has a specific amount of (sub-atom or sub-atomic) particles (such as protons, neutrons, and electrons) of mass, and therefore, each atom of a particular element will have a particular amount of total mass to it. The more particles in an atom, the more mass it has and will be "heavier" (more) in weight. If the proton and-or neutron particles in an atom increase by a factor of (n), the total mass and-or weight will also increases by that same factor of (n). When not expressing the unique mass of an atom, a value of mass (matter) generally does not indicate what specific element(s) of matter it is in reference to. For example, 1 gram of copper, and 1 gram of gold both have the same mass of 1 gram, but each different atom of an element has a different number of atomic particles per 1 gram. 1 atomic particle, such as a proton or neutron, is said to have a mass of 1 atomic mass unit (amu). The mass of an electron is so small, it is considered negligible and is usually not considered for most of the basic and useful calculations involving mass.

If we want a specific number of mass particles (usually atoms, but could be its particles of atomic mass units = amu), say from different elements, it is nearly impossible to count them since they are to small to even see, but we can use the corresponding weight of their combined (ie., multiples of) mass (each having a very small weight) to effectively count, know, or closely estimate the number of particles in that mass. Since each element has a different number of atomic mass units (each amu has the same mass, and therefore each amu has the same weight) for each atom, the number of atoms of an element to have a given weight will be different, but the total sum of mass, regardless of what element(s) considered, is still the same for a given weight.

The number of particles (atomic mass units, or atoms) used by chemistry to make a substance is usually in reference to units called mol, and is a specific and large quantity of particles, and which will be discussed shortly. The weight of a mol of particles (usually atoms which contain or are composed of the atomic mass units) of each different element has a corresponding and unique weight, and this information can then used in a equivalent fraction or proportion equation so as to find the number of particles or mol units in question when given the total weight, or to find the total weight when given the number of mol units of those particles.

A **molecule** is two or more different element atoms bonded (joined, usually by sharing electrons) together into one unique unit or structure. The word "molecule" is based on and has the meaning of "massive [mass]" and "small" ["cule", a old word suffix meaning "small"], hence the word "molecule" means a small piece of an object or mass. More modernly and formally (ie., scientifically, standardized), a molecule is the smallest indivisible part of a **compound** (ie., "combined", several, joined as one) of two or more elements joined together. An example of a molecule is a water molecule (H2O) that is made of 2 atoms of hydrogen (H), and 1 oxygen (O) atom joined together into a new substance of matter. Liquid water, a compound substance is therfore composed of two or more water molecules. A **mixture** is two or more substances (elements and-or compounds) mixed together. Note that a mol (a specific numerical amount) and a molecule (a structure of two or more atoms) are two different concepts. [FIG 233]



Also indicated in the above molecules is the "electric polarity" (if its substantial) of the molecule, and this is useful for some chemical and-or electo-chemical reactions. The main common factor of chemistry and electricity is the electron. For a simple example of how this can be is that in chemistry, an electron is shared by atoms when they combine together to form molecules, and in electricity, a moving electron is used to carry energy through a circuit.

In chemistry science, when substances and-or chemical processes and reactions are being made or considered, sometimes a specific (from an existing formula or recipe) amount or mol of the substance(s) are to be used. Having and using a specific number of mol of particles (usually atoms) ensures having enough particles of that material or substance to make another substance, and to reduce waste and prevent making a chemically wrong substance and-or mixture with too much or too less of something. A chemist will easily weigh (rather than count the mass) each substance so as to have a particular number or mol of particles (atoms, electrons, protons, molecules) of it needed for the chemical process, etc.

Some fields of science and work that may include and-or require some chemistry knowledge are healthcare, pharmacy (medicines), agriculture (farming), nutrition, and research. In about the first half of the 1900's, the agricultural scientist, chemist, and teacher **George Washington Carver** (1864-1943), from America, made many significant discoveries in the field of agriculture, chemistry, and new products, with many being plant based, and so as to benefit both industry and mankind. Many of his new products were created from the peanut plant and of which is also said to be a "complete protein(s)" in terms of nutrition. Carver created and promoted the concept of **plant rotation** by periodically planting a different crop(s) in the same location (ie., farm, garden) so as to help replenish (ie., fertilize, replace) surface soil nutrients, particularly the nitrogen compounds created by bacteria on plant roots so as to have "nitrogen fixing" and nitrogen for the plant to grow. Soil nutrients can be depleted and-or lost by plant "takeup" or absorption of nutrients into its roots so as to grow bigger and-or higher. Rain runoff (removal, erosion) from the top layer of soil will also deplete valuable nutrients for land based plants. For a grown food to be nutritional, the soil it was grown in must first be nutritional. Carver, being an African-American, also promoted racial harmony, and his scientific knowledge and expertise was desired by many other scientists and farmers, and of which some new scientific discoveries are still being based upon today.

A unit for counting nearly invisible particles, atoms or molecules by weight and-or mass:

A **mol** (based on the concept and word of "molecule", and sometimes pronounced as "mole", and sometimes referenced as "molar") is a standardized (accepted) unit of measurement (reference, for comparison to) that is essentially just a certain number of objects, much like how the dozen (doz) unit equals a group or count of 12 and is a defined as a specific and accepted (standardized) number of objects, and yet is used and understood as a (single, 1) unit of measurement or reference. For example 5 dozen doughnuts = (5 doz) (12 doughnuts / 1 doz) = 60 doughnuts.

How many things are in a mol unit?

A mol or "mole" (from the word "molecule" meaning small, minute) is simply defined as: **1 mol = (6.02214076)(10^23) particles**. This strictly numeric or unit-less value was found by experiments, measurements, and calculations of charged (electric, and which protons have an atomic mass and weight) particles, and some years after it was proposed. See Coulomb's law or equation about the basic formula for the amount of electric force due to particles with electric charge. This numeric constant and particle count called a **mol** unit is also called **Avogadro's Number (N_A)** or constant since **Amedeo Avogadro** (1776-1856), from Italy, first proposed (ie., a theory) a fundamental particle of matter count (linear, scalar, proportional) relationship to mass (ie., amount of matter) and weight in about the year 1811. It must be mentioned here in this book that it was already known since antiquity that there is a direct or linear relationship between volume and weight of a particular substance. For example, a cubic inch of lead weighs more than a cubic inch of iron, and at twice the volume, these specific cubes of volume of an element would then weigh twice as much, however, the question eventually became: "Is their a fundamental particle count then the same also?". The answer we know of today, after Avogadro's proposal, is that different elements have atoms that have a unique amount of mass, and therefore, given the same weight of two elements, there is a different number of atom particles and with each type of element atom having a different mass. Still, if the weights are the same value, the total mass (due to the amu = atomic mass units or particles) will be the same, but their atom sized particle count per unit of weight and-or volume will be different. Given a mol of atoms of two different elements, a mol of atoms of one element will weigh more and have more total mass than that of a mol of atoms the other element. If an atom has more mass (due to more amu), a mol of those atoms will weight more.

Avogadro proposed (now known as **Avogadro's Gas Law**) that the same volume (such as in an inflated balloon or rigid container) of different gasses (which have different natural densities) at the same temperature and pressure will probably have (on average) the same number of (randomly moving, and smallest of) particles (or what was also called then as "molecules") of matter which will keep each balloon inflated at the same size (ie., volume), and due to it having the same number of particles (with the same average thermal and kinetic energy level) applying pressure to its inside surface. This also indicates that an atom or particle of different elements (here, only gas elements were used) probably have a different mass in their natural, unpressurized state, and therefore, probably have a different size, structure or construction (ie., today, now known as atoms, and with their specific number of smaller parts now understood to be atomic mass units (amu) that are protons and neutrons).

Today, we also know that the kinetic energy of a particle of mass is: KE = (mv^2) / 2, and that temperature can change the kinetic energy of a particle and vice-versa. In the early 1800's, when **electrolysis** was used in water, the volumes of the two gases created was twice as much for hydrogen gas as for the oxygen gas, and hence the chemical formula for a water molecule became H2+O1 = H2O. Water (technically a compound [ie. substance, of many molecules] of or joining of two elements, here into water molecules) was then found to be not a single element, but is composed of two elements which must be joined together to make a particles (here, a molecule) of it. Avogadro did not yet know the actual number, or even an estimate of this particle count per unit volume, mass or weight of the gas, and it took until 1909 when **Jean Perrin** (1870-1942), a scientist from France, made experiments and gave a good estimate of it, and it has been refined in value after many years of research. The numerical constant of Avogadro's number is named in honor of Avogadro. In relationship to mass: **1 mol of protons (atomic mass units) is defined as having a mass of 1 gram**. For example, since each hydrogen atom has just 1 proton, **1 mol of hydrogen atoms has a mass of 1 gram.** The molar mass or mol per gram = mol / gram , of hydrogen atoms is therefore: 1 mol atoms per gram = 1 mol per gram. At the **s**tandard (reference) **t**emperature and **p**ressure (**STP =** 68°F =~ 72°F = 273.15K, and 14.7psi), **1 mol of a gas atoms, and which all have a low density (mass/volume) and which are all greatly spread out and-or colliding due to their containment and movement, will occupy a volume of 22.4 L = 5.917454 gallons =~ 6 gallons of volume.** Note that a gas having some kinetic and-or thermal energy will naturally expand outward and fill a container it is placed in. As it expands to fill the container, its density will decrease if it was initially compressed (ie., pressurized and having a higher density), but its total mass in that entire container and volume will remain the same value.

Note that a mol amount of atoms does not have a general or specific defined weight, and since atoms of different elements have a different structure or atomic (amu) particles in each atom. For example, a mol of iron atoms will weigh more than a mol of hydrogen atoms. This is also true for different gasses of different elements and-or compounds.

The atomic structure of a carbon-12 atom was found to contain 6 protons and 6 neutrons, hence it has 12 atomic mass units per atom = 12 amu / atom. It will be shown in a calculation below that 1g of carbon-12 has 1 mol of amu, and-or that 12g of carbon-12 is 1 mol of atoms = 1 mol-atoms. 1g of any element will have 1 mol of amu, but not atoms, except for hydrogen which has only 1 proton in its nucleus.

Note, as mentioned above, each atom of carbon-12 also has 12 smaller particles of matter (atomic mass units = amu = protons and neutrons) with mass and with each having 1 atomic mass unit (amu). The atomic or molar mass (mass of a mol of it) of carbon-12 atoms is about 12 grams of weight per mol of atoms or= 12 grams per mol-atoms:

12g carbon-12 = 1 mol of carbon-12 atoms : or= 1 mol-atoms of C-12 , and since each C-12 atom has 12 amu:
12g carbon-12 = 1 mol of carbon-12 atoms = (12 amu/atom)(1 mol atoms) = 12 mol-amu : = 12 mol of atomic mass units

If 12g of carbon-12 is to 12 mol-amu, then mathematically:

1g of weight (ie.,1g of mass being displayed on a common "corresponding weight to mass scale") of carbon-12 atoms is to 1 mol-amu = 1 gram of mass = 1 gram unit of mass or= 1 mass-gram

As a reminder, most "weight scales" are actually calibrated to display the objects corresponding mass value associated to its measured corresponding force=weight=Newtons value. A common "weight scale" is more correctly a "mass scale" whose value was automatically calculated (in the scale) from the corresponding weight of that mass.

If another element(s) or substance(s), other than carbon-12, has a weight of 1 gram, then it will have more or less than the number (1 mol) of atoms of carbon-12 in 1 gram of mass, but it will still have the same amount of mass and weight as that of 1 gram of carbon-12 = the mass and weight of 1 gram of atomic mass units (ie., protons)

If two substance, and regardless of their size or volume, have the same total mass, the weight of each substance is the same, and vice-versa.

1g of mass (matter, substance) placed on a common "weight scale" which is actually a "corresponding mass of that weight scale" will be displayed as 1 gram.

1 gram of mass = 1 gram of amu = mass of 1 mol of amu = mass of 1 mol-amu

The actual weight or force of a gram or other mass value can be found from:
F = weight = (m)(a) = (m)(g) newtons : with units of kg m/s^2. Note, 1g = 0001kg , and g = 9.8m/s^2

Ex. How many particles, objects, cells, atoms, protons, molecules, items or things is 5 mol?

Since: 1 mol = (6.022)(10^23) things if we multiply both sides by N, we have:
N mol = N (6.022)(10^23) things using substitution of the actual value in question, here 5:

5 mol = 5 (6.022)(10^23) things
5 mol = 30.11 (10^23) things = 3.011 (10^1)(10^23) things = 3.011 (10^24) things or items

Total Items = (N mol) (items per mol unit) :Total Items or Total Quantity Of Items In N mol Units

If Given: 3.011 (10^24) things, how many mol units of things is this?

Since: N mol = N ((6.022)(10^23)) things using substitution of the actual value:
N mol = N ((6.022)(10^23)) things = 3.011(10^24) things

We can solve for N by dividing both sides by the number of items in each mol unit:

N mol = 3.011 (10^24) / (6.022(10^23)) = 0.5 (10^(24 -23)) = 0.5(10^1) = 5 , Hence a formula is:

N mol = (number of items) / (number of items in 1 mol unit) : N mol Units Of A Given Quantity Of Items

If the number of particles (atomic mass units = amu, protons, neutrons) in an atom of a particular element is greater than that of another atom of a different element, the mass of that atom is greater, and the number of atoms per 1 gram (ie., 1 gram of mass) will then be less because it will take less atoms to make that 1 gram total amount of mass or matter, but still, the number of particles in 1 mol unit of particles (such as atoms) is always the same value of (6.022)(10^23) regardless of the specific element atom and its specific amount of atomic mass units (amu) and density. mol is a unit-less unit of measurement and is a count, quantity or number only, and is not a unit of mass. A unit of mass is the atomic mass unit = amu which is defined as the mass of 1 proton or 1 neutron. The question now is, how many amu of matter are in 1 gram of matter? The answer is **1 mol of amu of matter = 1 gram of matter.** 1 mol of hydrogen atoms, with each atom having just 1 proton for its mass, it will have a total mass of 1 gram and is said to weigh (on an equivalent or corresponding mass scale, "weight-grams" or "weight to corresponding grams" scale) 1 gram.

For any substances or elements having 1 gram of mass or matter, they will have the same total number of atomic mass units (amu) since these units are what actually determines a mass value and-or its corresponding weight.

Carbon12 (C12) is a common element with an atom containing 6 proton and 6 neutron mass particles (ie., atomic mass units or particles, amu) with each particle considered as having about the same mass, size (volume) and weight. The 6 electrons of C12 are not considered in the particle count for mass because their mass is negligible, practically 0. Therefore, a carbon12 atom has a total of 12 atomic mass units (amu), and is said to have an (summed total) atomic mass of 12amu. Each 1 particle in a carbon12 atom has a mass of (1 / total number of particles in that atom) = (1/12) the total mass of that atom. Note that the atomic mass of hydrogen is only 1amu since it is composed of just 1 proton, and does not have a neutron. A Carbon12 atom therefore has 12 times more mass (real material, matter) than the atom or element of hydrogen. For the same given number of mol of hydrogen and carbon12 atoms, carbon12 will always have 12 times more mass and correspondingly 12 times more weight than that of the hydrogen. 1 gram of hydrogen and 1 gram of carbon12 both have the same mass, and will therefore weigh the same value since they both have the same number of amu in that 1 gram of mass. Since 1 gram of matter has 1 mol of amu, 1g of hydrogen and 1g of carbon12 will both consist of 1 mol of atomic mass units, but not 1 mol of atoms. Since an atom of carbon12 has 12 times more mass than a hydrogen atom, a mol of carbon12 atoms will weigh 12 times more than a mol of hydrogen atoms. Any number or mol of carbon12 atoms will always have 12 times more weight than the same number or mol of hydrogen atoms.

In terms of mass per atom, or "mass density per atom", or "density or amount of mass per atom":

carbon12 has: 12 amu per atom = 12 amu / atom : mass per carbon12 atom
hydrogen has: 1 amu per atom = 1 amu / atom : mass per hydrogen atom

Since the mass of a carbon12 atom is greater than that of a hydrogen atom, It could be said that the "atomic weight" of a carbon12 atom is also greater than that of a hydrogen atom.

Because larger atoms have more volume (ie., physical space), there is fewer of them in a certain mass value and-or volume value, and-or or per unit volume. The atoms / volume density is lower for larger sized atoms with a greater number of amu or mass. Therefore, the (mol of atoms) / volume of those elements is also lower.

A mol of a "heavier element" (ie., atoms have more mass and or weight / atom = amu / atom) will have more mass, and therefore will weigh more than that of a mol of a "lighter element". A mol of carbon12 atoms will weigh 12 times more than a mol of hydrogen atoms since mass is directly related and proportional to mass.

hydrogen has 1g / mol of atoms , and since carbon12 atoms have 12 times more amu or mass:
carbon12 has a mass of 12g / mol of atoms = 1g / (1/12) mol of atoms = 1g / 0.083333 mol of atoms.
The reciprocal of this is: 0.083333 mol of carbon 12 atoms / gram of mass.
In short, **more massive, larger or "heavier" elements, have less atoms / gram, or less mol-atoms / gram , but also have more grams of mass / mol of atoms.**

The "atomic weight" listed on a table of elements, such as the common Periodic Table Of Elements, is actually the atomic mass units (amu) of each atom. 1 fundamental particle of matter has is 1 unit of mass = 1 atomic mass unit = 1amu. 1amu has a corresponding fundamental unit of weight (whose value depends on the locations gravity in the universe) which could be called an (fundamental) atomic weight unit. Therefore, in any location in a universe, the "atomic weight units" = "atomic mass units", or stated as: "atomic weight" = " atomic mass units".
The volume of 1cc = 1 cubic centimeter = (1/1000)Liter = 1mL = 1cm^3 of water at 4°, (ie., at the max. density of water, having the most mass of water per volume), is defined as having a mass of 1 gram. If an amount of material weighs the same as the weight of 1cc of water, and having any volume, it is said that it also contains the same amount of mass as that of 1 cc of water, and that amount of mass is 1 gram. In short, **if substances have the same weight, they have the same total amount of mass** since these two values always directly correspond to each other, and it is said that they are linear or proportional to each other. A 1cc volume of a denser substance and-or element will have more mass / volume and therefore have more weight / volume. For example, 1cc of iron has more mass than 1cc of water, and will weigh more by the same ratio or factor of two masses.

1 mol of atoms (ie., "mol-atoms" sized particle units) of carbon12 atoms will have (6.022)(10^23) atoms of carbon12. It actually takes 12 grams (mass) of carbon12 so as to have that many, 1 mol, of atoms. Since each gram of carbon12 element is composed of atoms with 12 amu particles each, the number of atoms in that gram is 12 times less than the number of its atomic particles (protons and neutrons) in its total mass such as 1 gram. This is why it was necessary to have 12 grams of the carbon12 element - so as to have 1 mol of complete carbon12 atoms. The **molar mass** (or mol-mass) of an element is the number of grams of it needed to have 1 mol of atom particles. The molar mass of carbon12 is 12 grams per mol = 12g / mol, or expressed as: 1 mol per 12 grams = 1mol / 12g. For other elements, their molar mass value will be a different amount of grams, yet the mol of atoms is always the same value.

1 atomic mass unit = 1amu, is the mass of a proton or neutron which are the two fundamental parts of the nucleus of an atom, and which has most of the mass of the atom. 1 mol of atomic mass particles is defined as having a mass of 1 gram. Since carbon-12 = C12 has 12 amu per atom, 1 amu can be defined as (1/12) the mass of a C12 atom. If 1 gram of C12 has 1 mol of amu, and to have 1 mol of entire or complete C12 atoms, it will take 12 times the amount of that mass. It will take 12 grams of C12 so as to have 1 mol of C12 atoms.

For C12, there is 12g / 1mole of atoms , dividing numerator and denominator by 12: we have:
1g / ((1/12)mol) atoms = 1g / ((6.022)(10^23) / 12) atoms = 1g / (0.502 x 10^23) atoms , and therefore, there is (5.02 x 10^22) C12 atoms in 1 gram of C12. We can multiply this value by so as 6 to find the total number of either protons, neutrons, or electrons in 1 gram of C12.

Ex. If 12g of C12 atoms is equal to 1 mol of C12 atoms, what is the mass of one-thousandth (0.001) of a mol of C12 atoms? [As a reminder, most modern scales are calibrated to display the equivalent mass value of a given weight].

Setting up a proportion type of equation:

(1mol-atoms of C12) / (12g of C12) = (0.001mol-atoms of C12) / (Xg of C12) , after solving for Xg of C12:
X = (0.001) 12g = 12g / 1000 = 0.012g = 12 (10^-3)g or= 12(0.001)g = 12 mg of C12 material

Ex. If 3g of C12 is added into a mixture, how many mol of C12 atoms did you add into that mixture?

First note that given 12g/mol, it can be also said there is: 1mol/12g. = 0.0833mol / 1g or 1g / 0.08333mol.
1g of C12 = 0.08333mol , and multiplying both sides by 3, we get: 3g = 0.25mol. This is also the result after multiplying both the numerator and denominator by 3 in the last previously expressed fraction above.

Or by setting up a proportion type of equation:

(1mol of C12) / (12g of C12) = (Xmol of C12) / (3g of C12) after solving for X:
X = 0.25mol of atoms of C12

Since 1 mol atoms of C12 is (6.022)(10^23) atoms, and that there is 6 protons in each atom, if we multiply a given mol-atoms value by 6, we can find the number of protons (or neutrons, or electrons) in 1 mol-atoms of C12:

(X mol-atoms)(Y particles per atom) = (X mol-atom)(Y) particles = X 6.022 (10^23) Y particles = XY 6.022 (10^23)
(1 mol-atoms)(6 particles / atom) = (1)(6.022)(10^23)(6 protons) = 36.132 (10^23) protons in a mol of c12 atoms

As the number of atomic mass particles in the atom structure of different elements is increased, the total mass of a 1 mole-atoms quantity of that element will also increase by the same factor, and therefore, the weight will also increase proportionately (ie., by the same factor) to this mass increase. Since the mass of an electron is negligible, the mass of an atom is considered due to its protons and neutrons only.

Extra: The neutrino

Although beyond the scope of this practical book, but still worth mentioning here just for the awareness of it, is that the smallest particle of matter found (as of of the year 2025) is called a **neutrino.** Its mass is considered a million times less than that of an electron, and it is then difficult to measure. A neutrino has a great ability to travel through all other matter without colliding into larger particles of matter, and it is sometimes called the "ghost particle". Occasionally, a neutrino will collide with a nucleus of an atom, and the more neutrinos there are, it will greatly increase that chance. Scientist **Juan Collar** from the University of Chicago, USA, has even proposed a radio-like communication system using neutrinos. This could send the particle signals though a mountain range, deep into the ocean or through the Earth to the other side. Neutrinos are thought to be composed of just 1 whole particle with no sub-particles or composition particles, and have no electric charge, and have no magnetic property, charge or (force) field, hence it is considered as a "neutral particle", and its name reflects this.

There are three neutrino types, and with one of these types being associated with one other atomic particle such as an electron, and that this specific neutrino is sometimes called an "electron neutrino". The subatomic particles were usually declared as real after many years of experiments, observations and producing similar results with the aid of particle accelerators and-or particle colliders (ie., "atom smashers"). It was also found that most particles have an anti-particle, even a neutrino, and which will cause both particles to go out of existence or disappear when they are joined together, and this process will emit some photons (ie., light, RF energy, radiation) during this process. Note that a neutrino does not orbit the electron or any other atomic particle, and that neutrinos are rather decay products of atomic interactions (ie., fission or fusion) and-or the nucleus decay of an unstable atom such as after emitting an neutron. Besides the electron neutrino, the two other types of neutrinos are the muon neutrino, and the tau neutrino. It is said that the universe if then always filled with an huge, uncountable number of neutrinos, and of which even pass through our bodies and Earth without much issue. Our nearby Sun, a star, emits the most percentage of neutrinos that encounter Earth. A neutrino can be affected by the (short range) weak nuclear force, and gravity just like any other mass.

In 1930, **Wolfgang Pauli** was first to propose a particles that would later be called a neutrino particles in 1932 by **Enrico Fermi**. The first neutrino was visually noticed in 1970 by using a "bubble chamber", of which is used to help observe particles paths, collisions, and other particles created. As of the year 2025, neutrinos are still being heavily studied. If interested further, please research atomic particle physics and the current, standard model of particles.

The **Periodic Table Of Elements** has much numerical data (information, knowledge, (periodic, repeating) trends or physical similarities, patterns and groupings) about each element, and a value called the **atomic number** is used to help identify a particular element and is equal to the number of protons in the atom of that element. If the atom is neutral or balanced in electric charge, the atomic number or proton number is equal to the number of electrons in that atom. The **atomic mass** (or atomic mass units, amu) of a specific element is the number of protons and neutrons in one atom of it. The total mass of the electrons in an atom is considered as negligible or so small as to be of no practical consideration for the total mass and weight of an atom. The **atomic mass number** of a specific element, if considered as a number of grams of that element that will then be equal to the required (mol, molar) mass of that element necessary to have 1 mol of atoms of that element. For example, carbon, or more specifically carbon12, has an atomic mass number of 12 since it has 12 amu per atom. It will take 12g of carbon (ie., carbon atoms) so as to have 1 mol of particles of carbon atoms, or 1 mol-atoms of carbon. In 1869, the first formal and modern periodic table or chart was made in Russia by a chemist named **Dmitri Mendeleev** and it included the 56 known elements at that time, and with the probable properties of some unknown elements yet to be found being proposed (ie., extrapolated, deduced) by the data from the known elements. The periodic table also organized the elements into similar (electron) electro-chemical properties. Modern chemists and other scientists often study and know the periodic table fairly well so as to be more knowledgeable and proficient in their field of science (ie., exploration and study).

When the number of grams in a grams per mol-atoms measurement increases, such as having an element with a higher atomic mass, we find that the number of mol-atoms is actually less per gram for the "heavier" (more massive in mass, and therefore more weight) elements. The result is that the total volume (such as the number of cc of volume) of that material or element will be less per gram of mass, and-or that 1 cubic volume full of that material weighs more than that of an element with a lower atomic mass number for each atom. Consider atoms as having a sphere shape, and that atoms with more amu per atom have a larger diameter and volume for each atom, and therefore fewer of them will actually fill a set volume such as a cubic centimeter.

Ex. The atomic mass of iron is about 55.85. Iron atoms have more mass than C12 atoms because iron atoms have many more atomic mass units (amu's) of protons and neutrons in each atom than that of C12. If there is about 55.85 grams of iron / 1 mol of iron atoms, and dividing both numerator and denominator by 55.85, we have 1 gram of iron per (1/55.85) mol. of atoms. Since (1/55.85) is less than (1/12), we see that the number of particles (here, atoms) is less per gram for "heavier" (more particles or amu per atom), more mass(ive), and therefore more weight elements. One result of all this is that the volume of that material or element needed per gram of a more massive element is less, and-or that the weight per 1 cc volume is more than that of another "lighter", less massive element such as C12 with only 12g/1mol = 1g/(1/12) mol. The g/cc, or grams per unit volume of iron is greater than that for carbon12.

Ex. How many grams is 100 billion atoms of C12. 100 billion atoms = 100(1)(10^9) = (1)(10^2)(10^9) = = (1)(10^11) atoms. Using the known data for C12, it has 12 amu per atom, and its atomic mass number is 12, and will therefore require 12 grams of it so as to have 1 mol of C12 atoms:

$$\frac{\text{12g}}{\text{1mol}} = \frac{\text{1g}}{\text{0.083333333 mol}} \quad \text{or} = \frac{\text{1g}}{\text{(6.022)(10^23) / 12) atoms}} = \frac{\text{1g}}{\text{(5.018)(10^22) atoms}} \quad \text{: for C12 only}$$

Expressing a proportion equivalence or equation:

$$\frac{\text{Xgrams}}{\text{(1)(10^11) atoms}} = \frac{\text{1g}}{\text{(5.018)(10^22) atoms}}$$, after solving for X: Xgrams = 1g / (5.018)(10^11) and

X grams =~ (0.1993)(10^-11) grams =~ 0.2(10^-11) grams = 2 (10^-1)(10^-11)g = 2 (10^-12)g = 0.000,000,000,002g
= 2 picograms

From this we can also find that 1pg corresponds to 50 billion atoms of C12.

Ex. It was found previously that 1 gram of carbon12 element had about (5.02)(10^22) atoms. We can solve for the mass of just 1 carbon12 atom if we divide 1 gram of mass by (5.02)(10^22).

___1 gram of carbon12 element___ or: _ (5.02)(10^22) atoms of carbon12__
is to: (5.02)(10^22) atoms of carbon12 is to: 1 gram of carbon12 element

Dividing the numeric values in the numerator and denominator, we find:
1 atom of carbon12 element has a mass of about (1.993)(10^-23)g =~ (2)(10^-23)g ~= 2(10^-26)Kg

Since C12 has 12amu, the mass of one C12 atom can also be found by multiplying the mass of 1 amu by 12. **The mass of 1 amu or proton = 1.6726219 (10^-27) kilograms = 1.6726219 (10^-24)g** , and this will be derived here:

Since a carbon12 atom has 12 atomic mass units (6 protons and 6 neutrons), we can divide the mass of 1 carbon atom by 12 so as to find the mass of 1atomic mass unit or particle:

2 (10^-23)g / 12 =~ 1.672621898 (10^-24)g :Mass of 1 atomic mass unit or particle such as a proton or neutron. (mass of 1 electron) < (proton mass / 1000). MASS OF A PROTON. The diameter of a proton is about 0.833 femtometers = 0.833fm = 0.833 (10^-15) meters. The reciprocal of this value is how many protons could be set or lined up side by side in 1 meter. If you divide that value by 100, you can find the number of protons lined up in a centimeter. If you then cube (take the 3rd power) of that value, you can find the number of protons in a cubic centimeter.

Since 1 gram unit is one-thousandth of a kilogram unit, 1g = 0.001kilogram, we can express this result in units of kilograms:

1.673 (10^-24)g =~ 1.673 (10^-24) (0.001)kg = 1.673 (10^-24) (1)(10^-3)kg = (1.673)(10^-27)kg

Ex. An estimated number of cells in the average 150 lb adult male human body is 30 trillion human cells = (30)(10^12) cells. That is about 200 billion cells per pound of weight. Some scientists have calculated the number of bacteria and fungi cells in the human body to be about the same number value as that of human cells, however a human cell is much bigger, and therefore it has much more mass than most bacteria and fungi cells. On average, a bacteria mass is only 0.02 or 2% of the mass of a human cell, hence about 2% of the mass and-or weight of a human is due to bacteria. The average diameter of a red blood cell is 7.2 um. The average length of a bacteria is 5 um.

If each cell was to be given 1 atom of carbon12 element, the total mass of that amount of carbon12 element needed is:
(mass of 1 atom of carbon12 element) (30 trillion) = (2)(10^-23)g (30)(10^12) = 60 (10^-11)g =
6 (10^-10)g = six-hundred trillionths of a gram = 0.6 billionths of a gram = 0.000,000,000,6g =
= 600 pico-grams = 0.6nano-grams . **Extra:** (150 lb / 30 trillion cells) = (5 lb / trillion cells) =
(1 lb / 0.2 trillion cells) = (1 lb / 200 billion cells)

Ex. How many mol of atoms is 30 trillion atoms? Expressing this as an equivalent proportions type of equation:

1 mol is to (6.022)(10^23) atoms as is = x mol is to 3 (10^13) atoms
1mol / ((6.022)(10^23)) = x mol / ((3)(10^13) after solving for x:
x = 0.4982 (10^-10) mol = 4.982 (10^-11) mol =~ 50 (10^-12) mol = 50 pmol : 50 pico-mol

The atomic number (ie., protons per atom) of element carbon12 is 6, and the atomic number of element iron is 26. Carbon12 has an atomic mass (ie., amu per atom) of about twice (due to an equal number of neutrons) its atomic number = (2)(6amu) = 12amu.

Iron has an atomic number (ie., amu) of 26, and has an atomic mass of about (2)(26amu) = 52amu per atom. The ratio of the atomic mass of iron to carbon is about: 52amu/12amu = 4.3333... An iron atom has about 4.33 times more mass than a carbon12 atom, and for a given mass or volume, iron will weight about 4.33 times more per unit volume than that of carbon12.

> 1 mol of iron, and 1 mole of carbon12 both have the same number of particles, such as atoms, but the total mass of each atom and-or 1 mol-atom count of the elements is different. 1 mol of carbon12 atoms has a mass of 12grams. Since the number of atomic particles in an iron atom is 4.333 times more than that of carbon12, 1 mol of iron atoms has 4.333 times the mass:
>
> 1 mol of carbon12 atoms = 12g of mass and 1 mol of iron atoms = 4.333(12g) = 52 grams of mass
>
> This example above verifies that when the numeric part of the atomic mass of an element is considered as having units of grams, it is equal to the mass of 1 mol count of atoms of that element.

Pure water is a compound (ie., a mixture or combination) containing molecules (ie., bonded, joined or combined atoms) made of hydrogen and oxygen atoms. The molar mass of water is 18.01528 g/mol and is often noted as simply 18g/mol. Drinking or nutritional water, such as spring water, contains vital and-or healthy amounts of some trace (small amounts) elements. Pure water is often used and-or required for chemical reactions so as to avoid contamination and chemical interactions by other elements. The common chemical formula and (symbolic) equation for pure water is:

2 hydrogen atoms bonded to 1 oxygen atom --> 1 water molecule : --> means "results to" or "equals", and in symbolic form:
$H_2 + O_1$ or= H2 + O and is often simply expressed as: H2O : chemical formula for water

A water molecule can be considered as the smallest particle of the substance or compound known as water, and so as to still be considered as being water. In 1 water molecule, often expressed as H2O1 or= H_2O with the 1 understood, there are 2 hydrogen atoms, and 1 oxygen atom joined (ie., a compounded) together at their electrons so as to make 1 water molecule. The ratio of the number of 2 hydrogen to 1 oxygen atoms in a water molecule is therefore 2. To make any amount or mass of water molecules, such as 1 mol of water molecules, it will therefore take twice as many hydrogen atoms than the number of oxygen atoms. 1 water molecule contains 2 hydogen atoms and 1 oxygen atom. 1 mol of hydrogen atoms is 1 grams of mass or matter. 1 mole of oxygen atoms is 16 grams of matter. .A mol count of water molecules or H2O = H2O1 will contain 2 mol of hydrogen atoms and 1 mol of oxygen atoms. It could be said that 2 parts of 1 mol atoms of water will be hydrogen atoms, and 1 part of that same 1 mol of water will be oxygen atoms. It could also be said that 2 mols of hydrogen atoms and 1 mol of oxygen atoms are needed to make 1 mol of water molecules. Also, 1 mol of water molecules contains 3 mol of atoms since each molecule contains 3 atoms. The volume size of 2 mol of hydrogen gas at standard temperature and pressure (STP) will be: 2 mol (volume of gas/ mol) = 2(22.4L) = 44.8L

Since the atomic mass of a hydrogen atom is 1, since it has 1 amu, then 1 gram of hydrogen is 1 mol of hydrogen atoms. 2 mols of hydrogen atoms is therefore 2 grams of it. Since the atomic mass (amu(s)) of an oxygen atom is 16, it has 16 times the atomic mass as that of a hydrogen atom, and therefore, there is 16 grams of oxygen atoms in 1 mol of oxygen atoms. Therefore, in 1 mol count of water or H20 atoms, there is 2 grams of hydrogen atoms + 16 grams of oxygen atoms. Therefore, **1 mol of water molecules has a mass of about 2 grams + 16 grams = 18 grams**. This can be measured and-or weighed out on a (weight to corresponding mass) scale. In terms of volume, 18g of water has a volume of 18cc = 18mL of water. This is slightly more than a 15mL tablespoon of water - about 1 tablespoon and 1 teaspooon. The mol/mass or "molar mass" of water molecules is 18 grams/1mol. It could also be said that a gram (or any other mass, or weight) of water has 2/18 = 0.1111... = about 11% of it being hydrogen atom mass, and 16/18 = 0.8888... about 89% of it being oxygen atom mass - practically 90%. This is so, even though the number of hydrogen atoms in a water molecule is greater at twice as much than the number of oxygen atoms, but the (atomic particle(s), amu) mass of a hydrogen atom is much less than that of an oxygen atom.

Note that the ratio of the atomic mass (ie., amu) of oxygen to the atomic mass of hydrogen is 16/1, and so if there are twice as many hydrogen atoms in a water molecule, the ratio of the atomic mass of oxygen to the atomic mass of hydrogen in water or a water molecule is: (atoms)(oxygen amu) / (atoms)(hydrogen amu) = (1 x 16) amu / (2 x 1) amu = 16/2 = 8. It could be said that the mass of water or a water molecule unit or particle is due to that there is 16 atomic mass units or particles of oxygen and 2 atomic mass units or particles hydrogen for each molecule of water. Therefore, any mass or amount of water has (16/2) = 8 times more of its mass (and-or weight) due to oxygen than its mass due to hydrogen.

Technically, a gram or any other specific amount of any matter such as atoms and-or molecules of an element or elements will have the same number of atomic mass units (amu) and will also weigh the same on a scale. A mol or mol count of different atoms and-or molecules will generally not have the same total mass (ie., grams) due to the different and total amounts of atomic mass units within those atoms or molecules. For example 1 atom of hydrogen has, contains or is composed of just 1 amu, and 1 atom of oxygen has, contains or is composed of 16 amu. 1 mol of hydrogen atoms will have (1 amu / atom)(1 mol atoms) = 1 mol amu. 1 mol of oxygen will have (16 amu / atom)(1 mol atoms) = 16 mol amu. Since 1 mol amu of hydrogen is defined as having a mass of 1g, 16 mols amu, here from oxygen, will have 16 times more mass, hence a mass of 16 grams.

A water molecule has a mass of 18 atomic mass particles or units. 16 are from oxygen, and 2 are from hydrogen.

Water molecule = H2 O1 = (2 hydrogen atoms) + (1oxygen atom)
= (1amu/atom + 1 amu/atom) + (16 amu/atom) combining the amu's:
= 18amu / water-molecule = mass of a water molecule and:

$$\frac{\text{waters atomic mass due to atomic particles or mass units from oxygen}}{\text{waters atomic mass due to atomic particles or mass units from hydrogen}} = \frac{\text{16 amu (1 atoms)}}{\text{1 amu (2 atoms)}} = \frac{\text{16 amu}}{\text{2 amu}} = 8$$

Therefore, mathematically:

waters atomic mass due to oxygen = 8 (waters atomic mass due to hydrogen) : 8 times more mass particles
waters atomic mass due to hydrogen = (waters atomic mass due to oxygen) / 8 : 8 times less mass particles

16 amu oxygen / 18 amu total = 0.8888... oxygen, or 2 amu hydrogen / 18 amu total = 0.1111... hydrogen

Since 0.1111.. of waters mass is due to hydrogen , (1 - 0.1111...) = 0.8888... of the mass of water is due to the other substance(s) in that water molecule, and here it is just oxygen. It could be said that about 88% of the mass or weight of water is due to the mass or weight of the (condensed, liquefied) oxygen (gas) in it, and that about 11% of the mass or weight of water is due to the (condensed, liquefied) hydrogen (gas) in it.

0.8888.. oxygen / 0.1111 hydrogen... = 8
or= 16amu O / 2 amu H = 8

: a water molecules internal amu particle ratio, and **not** the ratio of the number of atoms (here 3) or their ratios in a water molecule. In short, the mass of an atom from different elements is not the same value, and that an equivalent mass of different elements will have a different number of atoms. Both of these facts are due to the different number of atomic mass units (amu) or particles (protons, neutrons) in atoms of different elements.

Mathematically:

(0.1111...)(8) = 0.8888... or~ =: (11.1%)(8) = 88.8%

When a recipe or formula states to use a certain amount of a substance2 that is a certain percentage of another substance1's weight, rather than its volume, it helps ensure that the mass (the true quantity of material or matter) of substance2 is also the same percentage. Stating and using a certain volumes of a material is problematic for others since a volume (physical size) of material can change due to the physical structure of the material used and-or if it had moisture (water) in it, and-or if it was compacted (compressed) or not. A compressed substance (such as wheat flour) will weight more in a certain volume, than when not compressed. Using the weight in a recipe ensures the same mass or amount of substance. Mass and weight are directly proportional to each other. For various studies and measurements of weight and mass, it is beneficial to obtain an inexpensive and accurate digital weight scale that can measure small objects, say from a fraction of a gram and up to100g or 500g, and possibly another scale for heavier weight objects.

Some formulas to have available for chemistry studies, and of which variables can be mathematically derived from:

(total mass of an amt. of an element) = (molar mass of the element)(mols of that element) = (g/mol)(mol) = grams

(total number of particles) = (mols)(Avogadro's Number) : Avogadro's Number = Na = NA ~= 6.02214$\underline{1}$(10^23)

Using substitution for mols into the above equation:

(total number of particles) $= \left(\dfrac{\text{total mass}}{\text{molar mass of an element}}\right)$ **(Avogaddro's Number) :here, the total number of atom particles since molar mass is defined for atom particles**

$= \left(\dfrac{\text{g}}{\text{g/mol}}\right)$ **(NA) = (mol) (NA) as shown above (grams/mol-atoms)**

For electricity studies:

1 mol of electrons has, and is 1 mol of (negative) electric charges, and this amount of charge is called a (Michael) Faraday unit of electric charge, and this is somewhat of an outdated unit for an amount of electric charge, and rather the Coulomb unit for electric charges is now used instead. The Faraday (F) unit was mostly used in electro-chemistry science such as for electrolysis. Note also that the units of capacitance are called Farads and is still widely used today.

F = charge per mol of electrons ~= 96500 coulombs per mol of electrons = 96500C / mol : Faraday's constant
F = 96500C per 6.022 (10^23) electrons , mathematically:

$\dfrac{\text{Coulombs}}{\text{mol of electrons}} = \dfrac{96500\text{C}}{6.022\ (10\text{^}23)\ \text{electrons}}$ * = 1.602457655 (10^-19) C : charge per electron

1 electron charge = 1e = 1.602457655 (10^-19) C , also mathematically:

(1 mol of electrons)(charge of each electron) = Coulombs (C) of charge of a mol of electrons = Faraday constant

* This fraction, when num. and den. are divided by 96500, can be reduced to about: 1C / (6.24(10^18)) electrons. There are that many electrons per Coulomb (C) of charge, and-or that many electrons flow past a point in a circuit when the current is 1A = 1C / s = 1 Coulomb of electric charge (Q) per second = 1 Coulomb of electrons per second

A current of 1A = 1 coulomb of charge / 1s = 1c / s. If a current of 1A flows for 1 hour of time = 3600s, how many coulombs of charged have flowed past a point? 1c / 1s = Xc / 3600s. After solving for x,we find x=3600c

How long will a current or charge flow of 1A of electrons take to total 96500C = 1Faraday = 1 mol past a point? From 1A = 1C / 1s, we can write a proportion or equivalent fraction equation of: 1C / 1s = 96500C / Xs, after solving for Xs we find it equal to 96500s. If we divide this value by 1h=3600s, we have 96500s / (3600s/1h) = 26.81 hours. If the amount of

coulombs of charge was 5 times less, or the amount of current was five times more at 5A = 5Qc/1s , that time of 26.81 hours would be 5 times less at: 26.81h / 5 = 5.36h . 1 minute = 60s and 1 minute as a fraction of an hour of is:
1min / 1h = 60s / (3600s/1h) = 0.0166$\underline{7}$h , or simply: 60s / 3600s = 0.0166$\underline{7}$ =~ 1.67% and:
1 min / 0.01667h = x min / 0.36h , after solving for xmin , we find xmin = 21.6min , hence 5.36h = 5h and 21.7 min

To split a mole of water molecules = 18g of water into hydrogen gas and oxygen gas, will require at least a total current of 2 mol of electrons since each water molecule requires 2 electrons to be removed or unbonded and pass through the external battery circuit. 1 mol is to 96500C as= 2 mol is to Xc: 1mol / 96500C = 2mol / Xc , solving for Xc we find: Xc = (2)(96500C) = 193000C of charge. Note that if the water gets hot, some its volume be lost as steam, and in practice, more than 18g of water would actually be used - perhaps 500 to 1000 grams would be used, but only a maximum of 18g of that total volume of water would be converted to hydrogen and oxygen if only 2 mol of electrons flowed in this electrolysis circuit, and regardless of the time taken.

According to Einstein's famous equation: E=mc^2, the nucleus of atoms contains a very high amount of stored potential energy, and likewise it took at least that much energy to create that nucleus. The more protons a nucleus has, the greater the nuclear energy required to make that nucleus, and the greater the stored potential energy in that nucleus. In terms of molecules of two or more atoms, they take a much smaller amount of energy to create since the atoms of the molecules share electrons that are relatively at a large distance from the nucleus, and are bound by the electric force rather than the strong nuclear force such as found at the nucleus and its particles. To split a mol (6.022 x 10^23) of water molecules into its hydrogen and oxygen parts will require about 237000 Joules = 237kJ, plus about 20% more lost, wasted or transferred as heat energy during the process. Water molecules can be split apart by using electrolysis, and where a voltage is applied to it. This will create an electric circuit and current through that water, and which will replace electrons when they combine with positively charged ions (atoms that are unbalanced or non-neutral in terms of electric charges).

A mole of water has a mass of about 18g. To start removing electrons from this molecule so as to split it apart will require a minimum voltage of about 1.23v potential difference across the water if it is warm, and slightly more voltage if the water is cool. Since the atoms of each water molecule share 2 electrons, also called its (molecular, atom) bonding electrons, it will take 2 times the total energy to free them all from their molecular bonds than it does just to free one electron from its molecular bond. When a single electron is influenced by the energy of a 1V (ie., 1J / 1C of electrons) potential difference, it is said to have gained 1 electron-volt of 1eV of energy. For water to split into hydrogen and oxygen atoms, it will require at least 1.23v potential difference because it will take about 1.23ev of energy to break an electron from its water molecular bond, and 2 electrons will then require twice as much or about 2.46ev of energy.

1V = (1J / 1C) = (1J / (6.24)(10^18) electrons) , and after dividing the numerator by the denominator, we have:

1.602 (10^-19) Joules / 1 electron = **1ev** **,** To split each water molecule of water will require:

2.46 ev / water molecule = (2.46 ev / water molecule) (1.602 (10^-19)J / 1 ev) = 3.94$\underline{1}$ (10^-19) J / water molecule

(3.941 (10^-19)J / water molecule) (1 mol of water molecules) =
(3.941 (10^-19)J / water molecule) (6.022 (10^23) water molecules) =

237327 J =~ 237 kJ to split 1 mol of water molecules = 18g of water molecules : 1 tablespoon = 15mL = 15cc = 15g of water. 1 teaspoon= 5cc = 5g

From **V = J / C** , we have: **J = VC** , and from : **1A = 1C / 1s** , we have: **C = As** , and using substitution:
J = VAs = **(Pw)(Ts)** , and if J=237000 , V = 1.23 , and A = 5 how long will it take, hence solve for s = seconds of time:

s = J / VA = 237000J / (1.23v)(5A) = 237000J / 6.15 watts = 3853$\underline{7}$s = 10.7h

The ratio of hydrogen atoms to oxygen atoms in a water molecule is: 2 hydrogen atoms / 1 oxygen atom = 2. The corresponding amu's of the numerator and denominator is: 2 amu / 16 amu = 8 . The amu due to oxygen is 8 times more than that of hydrogen. When a water molecule splits into its component hydrogen and oxygen gas atoms, there will be twice as much hydrogen atoms produced than oxygen atoms, but the mass and corresponding weight of those oxygen atoms is 8 times more than that of the hydrogen atoms. In water molecules: hydrogen mass / oxygen mass = 1 / 8 , or oxygen mass / hydrogen mass = 8 / 1 or 8 to 1 = 8:1. 88.8..% of 18g or 1 mole of water = 16g of oxygen , and 11.1..% of 18g or 1 mole of water = 2g hydrogen.

Of an interesting note is that when the oxygen and hydrogen gas atoms having kinetic energy then combine to form (denser) liquid water, they will loose some of that kinetic energy. Likewise when liquid water turns into (cooler and denser) ice, it must loose some of its thermal and-or kinetic energy to do so.

During the **electrolysis of water**, it is generally considered as being the result of electrostatic forces at the two metal or graphite (ie., a form of carbon) electrodes or terminals in the water, and of which an external voltage is applied. Warm water will also supply some thermal energy to help the process of splitting water molecules apart, and possibly the electrostatic forces from a nearby electron passing through the water will also help the process of splitting water molecules apart. At the positive electrode, also called the anode, nearby water molecules are electrostatically split into positive charged hydrogen ions (ie., protons) and oxygen (a gas) atoms. The 2 electrons, 1 from each hydrogen atom, will be attracted to the positive electrode and travel through the circuit and through the battery and be repelled to the negative or cathode electrode. These electrons will then be attracted to the positive charged hydrogen ions and will create a complete hydrogen atom which is a gas. Since pure water does not easily conduct electrons, an electrolyte such as salt and-or an acid is usually added to the water to help conduct the electrons through it so as to complete the electric circuit. When table-salt dissolves in water, the salt molecules are split apart by the water molecules into both positive sodium ions and negative chlorine ions. These ions will help move electrons through the water, and so as to conduct electricity. The acid can also put many positive charged hydrogen ions (ie., protons) into the water, and which will aid the electron conduction through it. It is also possible to rather add a base into the water. Note that a table-salt (a sodium and chlorine molecule) electrolyte will release some dangerous chlorine (ie., like "bleach") gas when it undergoes electrolysis in the water, and you may wish to do this outside a dwelling (home, building, etc). Electrolysis is sometimes used to remove rust off of a metal.

In a galvanic cell, chemical reactions create and-or release energy and create free electrons and an electric potential (ie., voltage = J/C), hence creating a cell or battery. In electrolysis, the electric potential or energy of a battery is used as the energy to cause and power an electro-chemical reaction.

In electrolysis, oxidation, loss or removal of an electron(s) from the electrolyte happens at the anode electrode. These electrons are attracted to the positive (+) side of the battery and will be sent to the cathode terminal after being given some energy and repelled by the battery. Near the anode, reduction or gaining of an electron takes place.

Electroplating a metal can be done by a form of electrolysis. Electroplating is where one metal is coated with a thin layer of atoms (or ions) of another type of elementary metal. The metal where the plating metal or metal coating is deposited is said as being plated and-or electroplated. Here, the cathode or negative electrode is the metal to be coated or plated by the other (anode, positive electrode or terminal) metal and-or metal particles (fine and-or dissolved) suspended in the electrolyte solution. At the anode and-or electrolyte solution near it, positively charged ions of it will be created and they will travel via electrostatic forces through the electrolyte and will combine with the electrons on the cathode metal surface, hence placing an entire atom of the anode metal onto its surface. Electrons from the anode metal will attracted to the positive side of the applied battery (ie., electric potential, electric force) and will pass through it to the cathode, and so as to have a complete electrical circuit for the flowing or moving free electrons. The metal to be coated or plated on its surface must be very clean so as to be completely plated and firmly, and hence, it should have no oils, wax, paint, rust or other debris on it. Some metals do not plate well onto others, and a third or intermediate plating of a metal may sometimes be used so as to help facilitate the desired or final metal plating. Chromium (chrome) or gold plated metals makes them very resistance to rusting (ie., oxidation). To help facilitate electroplating, the metal to be plated can generally be placed into the electrolyte as dissolved (ie., atomized) particles. **Proper plating equipment and safety equipment needs to be considered: tanks, electrolyte, voltage, time, gloves, ventilation, eye protection, clothing protection, etc.**

The science of chemistry and electricity often coincide, especially for the general topic of **electro-chemistry**, and it is good to know at least some of both, but they are also different in their fundamental goals. Both of those sciences are beyond the scope of this basic mathematics book. however the topic of electricity has been given a larger discussion in this book due to its widespread use and interest by the average person. A recommendation is that a chemist should take a basic electricity course, and an electrician should take a basic chemistry course. The sciences of biology, agriculture, materials and many other fields of science will sometimes include and-or require at least some basic chemistry knowledge. An educational course for a field of science and-or work will often include the necessary and-or minimal and most useful knowledge needed, and without the need for a full chemistry course. Batteries, both primary (non-rechargeable) and secondary (rechargeable) batteries are results of electro-chemistry. Electroplating is one result of electro-chemistry.

More About Electricity And Electric Charges

Below are brief articles which contain selected key points of common knowledge, and consists of common brief facts, standards, some history and noted people, and so as to also not be excessive in detail and beyond the scope of this book and its practicality. The reader of this book may skip over these until another time needed and-or casually glance at the highlighted topics. They were included in this book so as the reader may have at least some practical familiarization and knowledge of these topics so as to be able to have a more robust, scientific understanding and use of how math can be applied in a practical manner, and to understand and-or possibly have some discussions with others. The science articles below are mostly about electricity, energy, flight, space, and measurement conversions**.** Each topic could fill a book by itself, and it is often not practical for all the knowledge, thoughts, examples and ideas to be presented in any one book, and you are therefore encouraged to seek more about any specific topic that you need to know more about. The way knowledge is presented, expressed or written, such as a derivation of a formula, also plays a role in how it is perceived as interesting, necessary, and-or easily remembered, such as being an important step for further knowledge.

Amber is a hard, petrified form of ancient tree sap, and was used to deliberately create some noticeable and interesting effects; usually showing the attraction and-or repulsion of small (electrically charged) objects after friction rubbing it with other objects. The words "electricity" and "electron" as we know of today is from the Roman word "electrum" that was used for the Greek word "amber". Today, we, now know that the friction (surface contact rubbing, or "drag" force) of the amber and-or other material(s) essentially applied enough energy to free some electrons, creating (highly mobile) electric charges, and charged objects (objects containing a net charge of either positive or negative).

William Gilbert (1544 - 1603), from England, made a study of many materials such as minerals that can be electrically charged and-or be affected by it, including a thin, lite-weight metal pointer in the year of 1600 which can turn its direction in the presence of electricity rather than magnetism. In particular, he studied electrical attraction, and his device was the first **electroscope**. He thought that the electric charge was a form of energy due to friction and the resulting heat increase of the object. He was the first to understand that the Earth had two magnetic poles, one called North, and the other called South due to their opposite directions on Earth. In the 1740's, **Benjamin Franklin**, from America, discovered that there are two distinct types of electric charges or "fluids": positive and negative. Current flow was then later conceived by Franklin to be from the positive side to the negative side of a battery, and this is modernly called **conventional current flow**, and this actually helps theoretically and mathematically such as that a voltage or energy potential greater should have a mathematical positive sign, meaning greater than 0 and-or nothing, and is a reality and of value. It was unknown during Franklin's time that electron particles usually carry (electrical, charges, with kinetic) energy and electric charge, and that they actually flow (id., repelled from other electrons of the same charge) from the negative side of a battery to the positive side of which they are attracted to. This is called as "electron and-or charge flow" of current. Since electrons have a negative charge, they are often simply called as being negative charges and-or carriers. In most circuit analysis situations, this strict technical or scientific truth of which terminal the charge or current flows out form does not need to be considered in most practical cases. Today we know that in a wire, it is electrons that are flowing or moving, and positive charges (protons) do not.

Once a practical way was found to make and-or store electricity or charges, such as with Volta's battery in about the year 1700 AD which made the study and use of electricity very practical, the revolution in electronic discoveries, understanding and applications grew exponentially where newer technology grew rapidly by using existing discoveries and technology so as to make further discoveries.

At about 400 B**.**C**.**, an ancient Greek known as Democritus called the smallest bit of matter of any particular substance an atomos (the indivisible or "uncut" part left after repeatedly cutting something into smaller and smaller pieces, and yet each piece is "unaltered" and still has the same (ie., one) physical properties). In the 1600's, the similar word of: **atom** was used for this smallest individual bit of a substance. The word "atomic" is in reference to atoms. Not long after **Volta** invented the battery in about 1800 so as to generate electricity, in about 1803, **John Dalton** (1766-1844)**,** a chemist from England, and in the first modern and scientific use of the word, called this smallest bit of a particular element as an "atom" of that element, and that each one is identical for a particular element. Some elements had atoms which behaved differently (such as observed in different gases) and must be somehow constructed differently and have a different mass because the same volumes of different elements weighed differently. Dalton was the first to consider that **compounds** (several elements that compose or create a unique substance) are composed of two or more elements bonded together so as to create the smallest fundamental particle of a compound which is called a **molecule** (since the word "mole" is an old word for mass, "molecule" means "mass part", "a part of a mass", "a small particle of mass") of that compound substance. For example, a water molecule ("H2O") is the smallest piece or particle of the substance called water, and each molecule of water is composed of hydrogen and oxygen atoms bonded or joined together - technically via electric forces and the "sharing of electrons" in a molecule. A molecule is significant particle in the field of chemistry science. Dalton started his atomic quest after theorizing that gases must be made of tiny moving particles that have mass and kinetic energy so as to put pressure upon the inner surfaces of its container, much like that of an inflated, expanded and (internally) pressurized balloon that is filled with some type of gas. Dalton is usually credited to starting the theory of atoms and their atomic structure, or simply: "atomic theory". Dalton is a contemporary to Avogadro.

The **electron** particle was discovered by **John Joseph Thomson** ("J.J." Thomson , 1856-1940) in the year 1897 when experimenting with a **William Crookes** (1832-1919) cathode ray tube (CRT). **Crookes Tubes**, invented in England in about 1869, were hollow glass shapes containing a vacuum, a metal cathode terminal at one end [used as a source of the mysterious cathode (energy) rays, later to be found as being electrons] and a metal anode electrical terminal at the other end. A high voltage (created using a Ruhmkorff induction coil, somewhat like a transformer) potential difference at each end of the tube is required for the mysterious "rays" to be produced and flow, and with the cathode name being assigned the negative (-) terminal, and the anode being assigned as the positive (+) terminal. A small amount of gas ions in the (near) vacuum assisted the flow of current from the cathode to the anode. The electrons will gain kinetic energy as they are attracted and travel to the positive electric force at the anode, and the stronger the vacuum, the less collisions within the tube as it travels. When the electrons collide with the gas or tube anode (+) end of the glass tube, a glow of light will be produced, and this is due to a "fluorescence" (ie., a visible radiation of light) which is due to the ray energy induced upon or applied to a weak gas, surface and-or element, and then being re-radiated as light from energized ("excited") atoms of that certain element(s). This glow of light indicated that invisible and mysterious (electromagnetic) rays from the electrified cathode to the anode were involved and causing it. A metal plate between the cathode and anode of a Crookes-Tube will block the "ray" beam (electrons) and cast a shadow of that metal plate onto the glass tube at the anode side, and this also verified that the rays were in fact emitted from the cathode end, and a Crookes tube is often called a **cathode ray tube**. A magnetic (magnetic field) could bend and-or direct the electron ray's or "beam" and this showed that ray or beam had some type of magnetic property like a metal. Crookes tubes would eventually lead to the discovery of the vacuum tube electronic amplifier. Years later, an enhanced version of a cathode ray tube would be used as the (output, viewer, image) display, reproduction, simulation ,"picture" or "image" screen of a television (TV) (radio signal) receiver.

In 1897, **Karl Ferdinand Braun** (1850-1918) from Germany, made a version of a Crookes tube and it had a phosphor coated screen which improved the visible effect to more of a (longer in time duration) after-glow or "glow in the dark" effect. Braun is also credited with the fundamental basics of the electronic **oscilloscope** ("oscillation" or wave, electronic signal, "scope" (viewable) or imaging) device where an electron beam could be moved about with a magnet (magnetic field) and-or high voltage (electric field) deflector plates. The movement of the bean across the screen was effectively done by using a (mechanical) moving mirror to reflect the vertical screen image (a glowing dot representing the value or amplitude of the input voltage or signal), and gives the appearance of a horizontal (left to right) movement with respect to time. This device visually displays a fluctuating (changing) voltage waveform shape and its (relative) amplitude (strength, as indicated by the size or height of the displayed wave) with respect to time of the input electronic signal. Later, **Jonathan Zenneck** (1871-1959), from Germany, made significant improvements to the oscilloscope in 1899 so as it be completely electronic (non-mechanical), and therefore having no slow moving parts to severely limit its maximum input and frequency, and it had a corresponding higher (ie., "faster") display (viewing screen) frequency, and it was practical. Zenneck's oscilloscope used more electrostatic focusing of the electron beam, and electro-magnetic control or deflection ("sweep", movement and direction) of it periodically onto, across (ie., horizontally rightward), and up and down (ie., vertically) the CRT viewing screen. Besides having access to a volt, amp, and-or ohm meter, an oscilloscope is one of the most useful instruments for the practice of electronics. An oscilloscope is often used for calibration, optimization, measuring and testing of AC signals and-or circuits. Today, there are several free low frequency (ie., audio range, < 20000 hertz) oscilloscope programs for computers, and of which a soundcard input overload safety probe may also need to be made or acquired, and there is also relatively inexpensive and portable digital oscilloscopes available from many electronics stores and-or websites.

The concepts of Crookes tubes would later play a (scientific knowledge) significant role in the concepts of (thermionic, where a heated cathode is used to release the electrons more easily and at lower necessary supply voltages) vacuum tube diodes (1904) or "(electronic, 2 total (1 input, 1 output) "lead" wires) one-way current flow valves", and 3 lead triode amplifiers, phototube light sensors and photo-multipliers [such as needed for TV cameras], oscilloscopes (first invented in 1897 by **Karl Braun** and who also discovered the **point contact**, "solid state", (galena) crystal **diode**, such as that used for radio use years later so as to hear the audio part of the radio wave received.). It is of note that this was not a lab grown crystal or diode, but more raw and natural, and not as reliable or easy to use. Television (**"TV", CRT - cathode ray tube**) tubes or "(image viewing) screens" are fundamentally based on an oscilloscope's type of beam control and of as mentioned, the oscilloscope is based on a Crookes Tube device. Most ("digital")TV's and computers today as of about the year 2010 have low cost, low power LED diode arrays as their viewing screens, however, the analog oscilloscope is still available and also used for very high frequency signals. Inexpensive oscilloscopes and-or kits are available for lower frequencies such as audio and AM radio signals.

Crookes tubes were also used during the discovery of X-ray [essentially, high frequency and energy, rf (radio frequency), penetrating particles] production as discovered in 1895 by the German professor **Wilhelm Rontgen** (1845 - 1923) after noticing that some of the mysterious, yet unknown (= X) rays (called Rontgen rays) from a Crookes tube were also passing through a thin layer of paper and-or metal near the outside wall of the tube. Today these rays are usually called **X-rays** and are often used in hospitals and dentistry so as to help have a reasonable internal (photographic, light and X-ray sensitive) image of internal things such as bone and metals which are not normally seen. X-ray radiation is created when an electron having a high velocity and kinetic energy, such as created in a Crookes tube, collides with certain materials. To create this high velocity, high kinetic energy electron(s), a high voltage is used. At about the same time, **Nicola Tesla** and others were also investigating X-rays and X-ray imaging (ie., photography). Rontgen noticed that these rays would easily pass through some materials, and also affect a photographic or light sensitive plate material as if it was a type of visible light . In 1895 Rontgen took the first medical X-Ray of his wife's hand, and too this day (year, 2022), X-rays's and-or X-ray imaging are widely used in the practice (ie., use of) and research done in the medical and scientific fields.

In 1953, a special type of X-ray image to study crystals helped discover the helical (twisting) of two main parallel rails and inner connecting rungs (like ladder steps) of the DNA structure that is often described as the "code" or "blueprint" or "instructions" of all life. The field associated with the structure of crystals is called crystallography. The structure is similar to a ladder that has been twisted or spiraled along its lengthwise or long direction. **DNA (DeoxyriboNucleic Acid)**, a special and very long type of molecule tightly packed into every cell, particularly its nucleus or governing part, of the body and is responsible for all the chemical (chemistry) structures and processes in the body, and heredity from both of a persons biological parents. In 1882, the structures called **chromosomes** were discovered by **Walther Flemming** from Germany, and later found by others to be responsible for "traits" or inheritance, and the next question was how is this actually done. The word chromosome means "colorful bodies", as seen through powerful microscopes. When a baby receives the chromosome structures from each parent, these mostly contain a copy of the DNA from each parent. The baby will then actually have a unique or combined DNA that is composed of both of its parents DNA. Each new cell of the growing baby will then need a replica (ie., a copy) of this unique DNA of the baby, and this is accomplished by a process called cell division. The DNA structure or code is composed of using 4 possible (chemical) molecules joined together as one rung or step of the long (many steps) DNA sequence, "chain" or ladder". Because of the preceding facts, certain sequences or lengths of DNA are said to contain the specific genetic data of a particular part or process of an organism or person, and these unique sequences or segments of DNA are therefore called genes. The main researchers to first see the previously proposed fundamental structure of life, now called DNA, were **James Watson**, **Francis Crick** and **Maurice Wilkins**, while in London, England. **Rosalind Franklin,** a chemist with some biologic studies and an expert in X-Ray crystallography study, is credited to actually taking the difficult to obtain, x-ray photograph(s) of DNA (deoxyribonucleic acid, aka "the [fundamental] building block of all life") in 1952 which showed its (atomic) molecular and (double) helical (twisting or spiral) structure. An image of the small object(s) being viewed is accumulated or "built up" by the diffraction of the electrons from the object and then onto a photographic sensitive plate. In more recent times, electrons with a reduced velocity are rather used than x-rays. Nonetheless, as of the year 2024, imaging improvements still need to be made in this field, and of which will yield many new discoveries. What we call DNA today was first noticed as a mysterious substance in the nucleus of a cell in about 1870 by **Johann Friedrich Miescher**, from Switzerland, and he naturally then called it "nuclein". **Albrecht Kossel**, from Germany, found that nuclein is actually an acid and called it **d**eoxyribo**n**ucleic **a**cid (**DNA**), and he identified the five nucleotide basses: Adenine, Cytosine, Guanine, Thymine and Uracil in 1881.

In 1888, **Philipp Lenard** in Germany, previously noticed some of the effects seen by Rontgen. Later, Hertz and Lenard are credited to discovering the photo-electric effect where surfaces, such as a cathode, would emit ("photo-induced") electrons or ions when struck by electro-magnetic radiation such as light and-or a radio waves. This effect can also be verified such as when shining UV light onto initially charged, similar electric force repelled and-or separated plates of an electroscope, and then the plates will get closer together as the charge and-or (similar polarity) electrostatic force gets less and can eventually be depleted. He also notice that the speed or (kinetic) energy of the electrons emitted depended on the wavelength (ie. frequency) of the incident light striking the surface, and not necessarily the intensity of the energy of the light. **Albert Einstein** would later suggest that this is due to that light or photons are composed of individual, discrete "packets" (ie., pulses) of energy, and this lead to the concept called quantum (energy) theory which is basically the study of individual, discrete, packets or steps (levels) associated to energy and matter such as electrons. Photons (ie., light particles) are essentially a form of a mass-less, electromagnetic (ie. "RF" - radio frequency) radio wave energy and vice-versa, that is, radio waves are a form of light, but having a frequency so high or so low that we cannot see it with our natural vision, much like how we cannot hear high frequency audio waves called "ultrasonic" audio frequencies. **Solarcells** function and create energy due to the photo-electric effect. The reverse of the photo-electric effect is the **(Arthur Holly) Compton Effect** (in 1923) where electricity can give energy to and free electrons in a conductor, and the electrons will release that energy as a photon of light. The Compton effect also applies to traveling electrons (rather than photons) that strike a metal and its electrons will release some of their kinetic energy as photons or RF radiation, including x-rays. Charged particles from mostly our nearby Sun cause the colorful (usually green) **aurora** lights near the poles of the Earth as these particles collide with gas atoms and-or molecules high up in the atmosphere, especially above the Earths magnetic pole region, and they become **ionized**, and the energy gained from each is eventually released as a photon of light (much like how Geissler tubes function) when the electron falls back into its steady natural orbit. The realization of this surly came after the discovery of the gas discharge tubes and the resulting electrical discoveries. The aurora lights are also called the ”northern lights” and ”southern lights” and they are a glowing, light green color usually.

Max Planck (1858-1847), from Germany, is credited (for his scientific contributions from 1900 to 1918) to creating the concept of quantum (discrete bits of energy, quanta or particles of a total quantity) theory which is proven by the fact that the energy of a photon (a light "particle") is directly related to the frequency of the light and which also determines the color of the light seen. When an electron gains kinetic energy, it will eventually release that gained energy as a photon (ie., photo, light, radio-frequency (rf), radiation, energy particle). Considering the rainbow or spectrum of colors, a particle or photon of red light has less energy than a photon of violet light. The color of a heated object such as a piece of metal can therefore be used to determine its temperature. When the mass conducts or radiates this energy away, it will reduce in temperature. Since wavelength is equal to the reciprocal of frequency, light or photon energy is therefore inversely related to the wavelength of the light. **Planck's constant (h)**, discovered in 1900 while studying a black-body's emitted radiation, is **h=6.62607015 (10^-34) Joule-seconds.** **j-s** is a unit for the total energy created and-or used during 1 second of time. h = (energy of the photon or light) / (frequency of the photon). Mathematically: (energy of a photon) = (h)(frequency of the photons) with units of Joules. The higher the frequency (pulses or cycles per second) of light or photons having energy, the higher the total amount of energy transmitted per second. This is similar to the concept of **duty cycle** or total percent of "on time" for a pulsed signal with respect to the "off time" of it during a unit of time, say 1 second. Note that just like there can be no real negative distance, there is also no real negative value for photons. Planck's constant is considered as the smallest, discrete ("quantized", "quanta") unit of energy. 1 quanta = h, and it is essentially the energy of 1 photon at the speed of light. The amount of energy of 3 quanta is: E = 3 h = (h)(3) joules. The unis of h are both: j-s and joules per cycle = joules per hertz. The Planck length is considered as like a boundary between known practical science, mechanics and energy, and quantum science where many things are yet unknown.

$d = v\,t\ ,\quad W = c\,t = c / f\ ,\quad f = c / W$: d=distance, v=velocity, c=velocity of light, W=wavelength, f=frequency

$$\text{Planck's constant} = h = \frac{\text{(energy of the photon of light)}}{\text{(frequency of the photon)}} = \frac{Ej}{Fhz} = Ej\left(\frac{1}{(Fhz)}\right) = Ej\,c / W \qquad : W = \text{wavelength},\ c = \text{speed of light}$$

$E = h\,F$ with units of Joules , and $E = h\,c / W$, and $E = mc^2 = h\,f = h\,c / W$: with units of joules

Since (h) is a constant, if the frequency of the light doubles, the energy from that light will double. $P = E/t = J/t$

From distance = (speed)(time) = (speed)$\left(\frac{1}{\text{frequency}}\right)$, and for electromagnetic radiation, speed = speed of light =c and distance traveled = the wavelength = W , we have:

$$W = c\,t = \frac{c}{F} \text{ meters} \quad \text{and} \quad F = \frac{c}{W} \text{ hz} \quad , \quad \text{therefore,} \quad E = h\,F = \frac{h\,c}{W} \text{ Joules}$$

Photo-sensitive conductors are certain elements (such as selenium) or materials that have a change in electric conduction or resistance when light hits their surface. These are also called light-sensors and-or photo-resistors, and the effect is called the photo-conductive or photo-resistive (a resistor that is photo or light sensitive) effect. **Solar cells** produce or create (ie., "generate") a voltage (ie., energy, and the induced or resulting electric current or flow is composed of free, unbound and moving electrons) from light energy striking and transferring some of its energy to the surface of the solar cell. This effect is called the photo-voltaic effect. A solar cell is more sensitive (generates more current and power) to a certain frequency range of light rather than being a pure linear (ie., equally sensitive) light sensor or and ideal electricity generator for any frequency of light.

One famous experiment with light is known as the "double-slit" experiment where a strong light was directed through two close horizontal slits (long thin openings) in a thin piece of material, and then it was shown onto a surface at a short distance away. Two vertical stripes or bands of the light were then seen on the surface as expected, but there was also other bands and gaps (spaces) between them which indicated both constructive (additive, strengthening) and destructive (negating, reducing) type of wave energy strength interference (affecting each other, and resulting in a net total) action going on. This interference pattern is similar to what happens with water or sound waves being in phase (ie., at the same time, and combining their intensity or amplitudes) or out of phase (ie., 180 degrees, completely opposite, out of phase or timing, and resulting in 0 intensity) with each other. This implies electro-magnetic waves such as light have a wave-like property during creation, transmission and reception. This was first noticed in 1801 by **Thomas Young** (1773-1829) from England**.** The results of this experiment and the nature of light are still somewhat controversial and are still debated during this writing in this book at the year 2022. A wave implies a frequency and an amplitude of something that is varying, either an amount of particles and-or energy. Another idea is if the main wave is rather composed of many (discrete) pulses or square waves of energy. Considering all this, a photon of light and-or radio frequency (RF) energy can perhaps be describes as an infinitesimal particle, bit, or quanta, of a RF wave.

In 1927 Clinton Davisson, Lester Germer, and George Paget Thomson (independently) showed that a beam of electrons that was directed to and scattered away (ie., diffraction, reflecting) from a crystal surface would produce a similar intensity pattern as X-rays and other electromagnetic waves, hence the new thought that light could also have some type of similarity to (atomic and discrete, quantized, integer, countable steps) particles such as electrons. This concept is called the **wave-particle duality of light.** A particle of light (technically a radio [rf- radio frequency] or electromagnetic-wave of energy) is called a photon, and is said to be mass-less and a packet of energy. Strong gravity can bend light slightly which indicates that light may have some type of mass, even if just temporary during part of the wave, and-or there is some type of "rf energy diffraction (bending)", much like light diffraction, due to an electromagnetic field about a star, etc. The electron diffraction pattern also gives a magnified view of the crystal structure of the object so as it can be photographed and studied. **Clinton Davisson** (1881-1958), from America, is therefore given some of the credit to the development of the **electron microscope** which can effectively magnify many times (ie., a multiple of) more than even the highest power optical microscopes available, and the resolution is therefore much smaller so as to see (ie. "resolve") smaller objects and details. The electron microscope discovery and creation is rooted in a (1926) theory by the German physicist **Hans Busch** that a magnetic field(s) could be used as an electromagnetic focusing lens for a beam of electrons. Some of the already known concepts of how the oscilloscope worked surely helped this effort. The first basic electron microscope was then created in 1931 by physicist **Ernst Ruska** and **Max Knoll** who was an electrical engineer who helped build it. Electron microscopes are more complicated than optical microscopes, and therefore, they are more expensive, and they usually require special preparation of the object being viewed, but it is still worth it, and even outdated and-or replaced electron microscopes are sill sought after. An electron microscopes does not use penetrating x-rays that pass through the subject and onto an x-ray image sensor, but rather uses electrons to be directed to and then reflect off from the subject and onto an electron image sensor. Electron microscopes use high voltages to create a high static electric field to repel electrons and give them kinetic energy and direct them. So that the electrons do not collide with air and loose their kinetic energy, they are made to through a vacuum that the subject to be magnified and viewed is also in. The resulting images of the subject are also displayed in gray-scale (various shades or intensities of gray [a mix ratio of black and white] color, and which is commonly called black and white images). Because electrons are so small, they can reflect off of the smallest of pieces of matter that optical microscopes can not even "see" or resolve due to their lower magnification, hence electron microscopes are capable of very high resolution. Because electrons are relatively much larger than photon (light and-or e-rays) particles and-or waves, they do not sense and return any color information, and that is why the images produced form an electron microscope are gray-scale images. These images can later be subjectively colorized so as highlight the main points of interest and-or similar items in the image. A complete electron microscope system is expensive, but a business and its personnel specializing in electron microscope imaging can be accessed and "rented" if needed and at a much lower price than buying an electron microscope system and employing its personnel just to have a few images needed. Electron microscopes have constantly been improved since the first one was made, and especially for having better image clarity. It is possible to image atoms with some modern electron microscopes.

Attempts at creating **television** (tele is a word prefix for "remote" or "at a distance", ex. telephone, TV = television = "remote vision", a "transmitted vision or image", electronically sending or transmitting images so as to be viewed at another location) has a long, varied and incredible history of electro-mechanical devices and people such as **Paul Nipkow** (1860-1940), from Germany, who created the Nipkow disk in 1884 that had a spiral array of holes to "scan" (horizontally, and vertically) during a bathing of a subject with bright light, and of which its reflection was then sensed with an electronic photo (ie., light) sensor.

The first electronic imaging (ie., camera) tube for television was called an **iconoscope** (ie., **an image viewer**, electrical device, an electronic camera) and was created by **Vladimir Zworykin** (1888-1982), from Russia, while working with **RCA** (**R**adio **C**orporation of **A**merica) in America. Some of its initial concepts were conceived in about 1923, but it took until about 1931 for it to be demonstrated as fast and practical. In 1929, **Philo Farnsworth** (1906-1971), from the United States of America, is credited to the first completely electronic, demonstrated, fast, non-mechanical television system. He created both an electronic camera or image sensor called an "image dissector (tube)" in 1927, and also the image viewing screen, for a complete and functional electronic TV system. It could be considered that Crookes (ray, electron, "cathode ray", CRT) tubes is where TV (television) and many other modern scientific discoveries such as the electron atomic particle, were initially rooted in, discovered with, and-or based upon. When a thin beam of electrons strike the phosphor coating on the viewing screen, a visible dot will glow due to the energy it absorbed and some is re-radiated as light. The intensity of the electron beam will determine the intensity or brightness of the visible light seen on the viewing screen, and this is the method used to display an (picture) image with a "black and white" (ie., shades of gray, gray scale, not mono-tone single color) television (TV) receiver and-or screen. Scheduled television (TV) broadcasts generally begin in 1928 to the relatively few people and-or business who could afford a (relatively rare) TV receiver, and until the production of TV receivers became relatively affordable for many after WW2 and-or 1945. Color television cameras and receivers began to be manufactured by RCA (Radio Corporation of America) in 1953, and it took until after 1965 for the large and popular TV transmitting stations to mainly transmit color TV signals, however, the less expensive "black and white" TV's could still view these signals as "black and white" (ie., greyscale) images.

It could be said that both Farnsworth and Zworykin independently share the majority of credits to the initial invention of the first practical electronic television system. The people and companies involved in the development of the first practical electronic TV system occasionally filed suit against each other. The **iconoscope** was eventually favored as being more practical than the dissector tube. The dissector tube was (independently) proposed as early as 1925 in Germany. Farnsworth also invented the "**electron multiplier**" during his TV research. With this vacuum tube device, each incoming electron can strike a charged plate that had a high voltage potential applied to it, and therefore having a high electrostatic force upon the incoming electron(s) and causing them to accelerate with more kinetic energy, and this will cause a few other (hence multiple) electrons to be emitted from that plate when it collides with it. These plates are usually placed at angles and in a series arrangement so as each electron, perhaps nearly undetectable or impractical to, can be easily sensed when there is many (ie., a multiple, "amplified") of them for each single and initial instance. Some devices which use an electron multiplier are some spectrometers which can determine what element(s) a substance is by the light and-or wavelengths emitted form it, and the electron multiplier has found use in some night-vision (low light ,sensitive) cameras.

A **movie** (ie., moving or motion pictures) is a recorded series of consecutive ("still") images or pictures of a subject with either the subject and-or camera in motion (ie., moving about). A good quality movie recording process will record enough needed images, pictures, "stills" or "frames" per second so as to have a seemingly continuous viewed subject having natural motion, and without any apparent "missing image gaps" or chronological distortion. A movie recording when viewed or "played back" should also appear as consecutive images having a nearly or seemingly smooth, unbroken, natural looking motion or sequence of the subject that was recorded or "filmed" (ie., onto photographic or light sensiteve recording film media, often formed into "reels" ie., a roll of recording film media, containing many consecutive individual photos or "stills"], but modernly as digital or electronically recorded, stored and viewed images). The movie concept most likely got inspired from what is known as a "**flip book**" which is a consecutive series of stacked drawings or photographs so as to effectively show objects having a reasonable, natural motion and time used, and it was only a matter of time until the technology allowed movies to be made and played back for viewing in a practical and pleasing manner, and for wide distribution for many uses. Before practical television (TV) was invented and-or (TV) video recorders, many of the first TV transmissions were that of per-recorded movies shown on a screen for the TV camera and radio transmitter to view and then broadcast to many, and often repeatedly as needed, such as once a day, or once a month, etc., and relatively inexpensively much like radio broadcasting with per-recorded audio.

Like "motion pictures" or movie technology that was made practical by Thomas Edison, TV technology also uses a minimum (ex., often 15, 30 or 60 frames) number of images, "pictures", "stills", or "frames" to be recorded and viewed so as to appear as being continuous and real. Movies are recorded onto a (photographic, light sensitive) film and are (re-) viewed on a viewing screen by projecting the images onto it with a light passing through the film images and onto the viewing screen. This frame rate was first done in order to reduce the amount or length of costly photographic film, but still be fast (frames per second) enough to record and view moving image as continuous, unbroken movement due to a concept called "persistence of vision" where we essentially can not perceive the slight "gap" (omission of any images) between each frame and-or perceived motion. Having a relatively low number of frames per second also effectively reduces the (fast, high-speed) electronic frequencies (cps, hertz) needed for a television system with electronic image camera, image recording, image transmission and image viewer (TV, screen). Each frame or image displayed on the TV viewing screen is essentially quickly built and-or displayed using many vertical rows from the top of the screen to the bottom that are each composed of many horizontal dots or picture elements ("pixels") side by side. This all happens very fast so as to produce 60 complete images or frames shown on the viewing screen each second. 15 frames per second = 15 fps is acceptable for when the subject is not moving much, after all, consider a picture on a wall that is not moving, and this would be like a TV system having just 1 frame per second to viewed subjects that are not moving much.

How long does a reel of 35mm (in width) movie film have to be for a movie that is 1 hour = 3600 seconds in time length or duration? If each movie frame or still was 24mm high, and its dividing border space or gap was 2mm from one frame to the next, the total frame height or length is 26mm, and if there were 30 frames per second, the answer is:

movie film reel length = (total number of frames) (total length of each frame) =
movie film reel length = ((frames / second)(seconds)) (each frame length) =
movie film reel length = (30 frames / second) (3600 seconds) (26mm / frame) = 2808000mm = 2808 meters = 2.808 km

Long movies were usually divided into several film reels to be practical, and with each real being perhaps 20 minutes in duration.

(movie frame and-or viewing) Aspect Ratio = (frame width) / (frame length) , Ex. 35 mm / 24 mm =~ 1.4

People tend to see things more horizontally than vertically, mainly due to their eyes being horizontally side by side, and that is why each frame is wider in the horizontal direction. Television displays also have an aspect ratio > 1.

Popular film width sizes: 8mm : The film and the camera were relatively inexpensive for "home quality" movies, and relatively inexpensive to process into the positive image from the negative image. These reels of film could allow a few minutes of image recording, and of which could be done in various and short time segments of interest. The cameras for 8mm film were relatively small, lightweight and inexpensive. The projector machine, that projects the film image onto a distant viewing screen were also relatively inexpensive, but still costing several times more than the camera. There is also inexpensive hand operated mechanical viewers that project the film onto a small nearby viewing screen of several square inches. 8mm movies were usually silent, but if someone had a tape recorder available, then some movie sound was possible, and if it could be synced (synchronized, meaning at the same time) or timed correctly with the movie.

16mm : For reasonable quality movies , somewhat inexpensive, but lesser quality than 35mm.

35mm : A common size for most film movies made to be of high quality. The professional movie studios often used 35mm film, and made color movies at a time even when TV was still black and white. Even when color video tape was available to record, such as for television, many movies for the movie viewing theaters were still made using 35mm film. These movies could then be rerecorded or converted for television.

> 35mm : Before electronic or digital photography, 35mm film, still image cameras were the most popular type of camera. "Studio, professional quality", "portrait film" and-or older "plate or large format" photographs used negatives that were large, perhaps 4" =~ 10cm for each dimension. Compare this to a modern (2022) "smart phone's" digital camera with an image sensor that is perhaps 2mm x 2mm and produces good quality, inexpensive images consisting of many pixels - actually several mega-pixels or millions of pixels for high image resolution, detail or precision.

Television signals can be stored on magnetic ("reel") tape recorders and were developed in the early 1950's by the **Ampex Co,**, and this made it possible to store (record, save) electronic images from an electronic ("TV") camera, and later broadcast (transmit to a wide area and-or population) and-or rebroadcast the recorded images as a common TV radio-wave or electro-magnetic signal. Metal wire, initially for crude and-or voice recordings only, and then magnetic tape for high-quality audio recording were already available by then since the 1930's. TV signals can be transmitted through the air like a radio-signal, and-or through direct wire connections such as for "cable TV". This direct connection usually results in a received TV signal with much less electronic and atmospheric noise in it, therefore producing a more pleasant viewing image and-or experience for the audience. For business use, thinner (1 in or 2 in) width tapes were available since about 1960, and for home use, even thinner width tapes were available in protective cartridges by about 1970 and these were essentially two reels, side by side in an easy to handle and use flat-like plastic box. Today many of us take these technologies for granted as part of everyday life.

With all the new technology becoming available in the 1950's and 1960's after the invention of various transistor devices, it was only a matter of time until first portable, consumer **digital camera** was created. A digital camera does not store picture data (ie., elements, pixels) onto light sensitive film which changes its shade or color depending on the amount and-or frequency of light striking it, but rather uses an array of many very small individual light (photon, "pixel") sensors and "color filters" (here, [Brice] Bayer filters), and then stores that light information or data from the sensors as binary numbers in electronic memory - perhaps 8 memory bits or 1 byte (0d to 255d) for each RGB color. This invention then lead to the digital movie camera. By 1972, some satellites, such as the (NASA) Landsat 1 launched in 1972, which could sense multi-frequency [infra-red (ie., heat, temperature), green, red] bands, so as to create decent resolution (~ 75 feet / pixel, on average) images such as of Earth's natural and agricultural resources on the surface. To sense and record the images, a scanning mirror and light sensor system, rather than a high count image sensor array that were available later. This imaging system was managed and developed by **Virginia Norwood** and some others - many working for the Hughes Aircraft Co. in the USA which researched and manufactured much technology. The small number of light sensors used and their data could then be used to build, transmit and eventually print a much larger image at a radio communication receiving stations on the Earth. Norwood also worked on the 1966, Surveyor 1, Moon lander's communication system. Her knowledge of both math and science surely helped her and others later so as to make many discoveries and products.

With modern digital and computer technology, analog audio, movies and video (ie., visual ,TV) tapes can and should be converted to a digital form for extended storage, safety backups, and-or for proper distribution. They can also be digitally edited and-or enhanced if needed. There are people and-or services who specialize in these types of conversion and-or enhancement processes, and it is well worth the cost for preserving precious and-or valuable memories and-or data.

To view digital images, they can be printed and-or viewed on an electronic digital image viewing screen (ie., a monitor and-or TV screen). It is possible to view digital images on a analog TV screen or monitor if the digital data or images is first converted to the input signal needed for those devices, such as a TV radio signal. Many pre-1990 computers, such as the VIC-20, **Commodore 64**, Amiga, and video-game systems, such as the Atari and Sega products, and with their internal digital (ie., binary data) electronics, often used a commonly available, standard TV receiver as the output display. It is an interesting note that some Sega products could connect to the relatively new internet, often freely. Before low cost digital storage or memory, relatively low cost magnetic (tape, disk) storage was rather used to store digital data, and you can research the KCS Standard which was one of the digital to analog conversion schemes or methods to do so, and which is also similar the SSTV (**S**low **S**can **TV**, picture, image) method that some hobbyists, amateur radio communicators, and some older weather satellites still transmit and can then be received by a radio tuned to the proper frequency, and then processed by a computer so as to create an image on a computer screen.

By 1975, reasonable images sensor arrays began to be available to the general public and utilized for image creation and recording (storage of information, data). The most common images sensor arrays are the **CCD** (charged-coupled device, invented in 1969 by Bell Labs with **Willard Boyle** and **George Smith** while creating electronic memory for storing binary [ie., "digital"] data) image sensor which essentially uses the capacitance of each tiny metal-oxide semiconductor (MOS) capacitors placed in an array configuration so as to sense (ie., a photo-sensor) and store the amount of light striking each one as an accumulated amount of created electric charge, and due to the photo-electric effect where the kinetic energy of a photon of light can create a free electron - hence more electric charge. The coupled concept is a description of transferring charge from one capacitor to the next. The amount of light in each capacitor is a true analog value, and this value will be converted to a representative and approximated binary value, say from 0 (ex., for black) to 255 (ex., for white) such as for 1 byte of data per pixel of the image sensor. Values between 0 and 255 would represent a gray tone or shades of the image, essentially the light intensity of. For creating the first color sensing camera, color filters would be used later. Various other image sensors have been developed since then: NMOS in 1985, and CMOS in 1993. CMOS is similar to CCD, however **CMOS** (Complementary-MOS) in an array (ie., a horizontal and vertical "grid" arrangement or placement), and uses more active electronics such as better amplification of each photo-sensor or pixel's signal. The first (semi-practical, crude) portable digital camera for potential public use was demonstrated in 1975 by **Steven Sasson** while working for the Eastman Kodak Co. which is well known in the film and camera industry, and making the camera relatively inexpensive and very practical with its Kodak Film and-or cameras. This camera stored digital images as binary (ie., digital) data on a magnetic, floppy disk. Since the CCD concept was realized, several companies have made significant contributions to digital imaging, and by 1983 video (ie., electronically stored movie, no film) recorders using CCD image sensors were becoming relatively affordable and were made by the Sony Co. in Japan. The data from these video cameras was stored onto magnetic tape rather than onto expensive movie film which stores an actual image. Magnetic tapes are relatively inexpensive and can be reused by directly recording over the previous information.

The **first completely digital camera** for commercial sales was made in 1988 by the **Fujifilm Co.** in Japan, and which stored the digital data of each image on a digital (ie., non mechanical, electronic only) memory device. By about 2010, many mobile telephones were low cost and included a decent quality and resolution digital camera, and these phones could also access the internet system ("WEB") so as the user could transfer the images and-or a movie to another phone and user. Modern phones also offer text-messaging which is very handy for deaf people and-or for people who can not immediately answer a ringing phone, and would rather read any text messages when they are more able to do so. As of about 2020, short duration audio recordings can also be easily included in some of the more advanced text messaging systems, and this also helps those who would rather not type much and-or have the time needed so as to enter a typed in text message.

Before arrays of tiny photo-sensors were created for imaging, the most basic image (such as a photo, or page of printed text) scanners rather used a single photo-diode for the image sensor and moved it horizontally across the page and gathered (ie., "sampled", "took", "scanned") and stored a single line of image data. The higher the number of samples taken, the higher the number of data values, and the higher the resolution of the image. The sensor or paper was then slightly adjusted to be a short distance away so as the next horizontal line of data could then be taken. When these lines of data were printed one after the other onto paper, the result was a created or representative image of the source image.

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High Voltage And Gas Filled Glass Tubes

The first light to be deliberately or artificially produced by man, besides light from a fire, was probably produced from the friction of two fast moving dry sticks (wood from trees, having a high carbon element and low moisture content) producing red glowing embers and-or (yellow colored) fire (hot gas particles), and sometimes as visible bright sparks created by two rocks colliding. Later, candles were created that used oils and waxes as the fuel, and they produced a soft yellow-white light. A few thousand years later, man began tinkering with the mysteries of electricity - first with various methods of friction induced static electricity and charges, and sometimes producing both visible "sparks" (quick pulses of light) and-or felt as (electric) shocks. With the invention of the electric battery which can accumulate (slowly increased its stored amount) electric charge, particularly electrons, it was noticed that a bright spark of light could be produced that was very similar to that of lightning from the sky, and then realizing that lightning must also be due to the same cause, that is, lighting must be electric in nature, and contains electric charges. Lightning can be considered as a big, long and powerful electric spark. People noticed that with a higher battery voltage (ie., electromotive force or energy, emf) applied, a longer spark could be created across a greater distance, and surely lightning between the ground and clouds must be due to a very high voltage. But what was producing the actual spark of light? If the air was taken away, would a spark still be seen? It was found out later that no spark would be produced or seen when air is not present such as in a glass vacuum tube, but still, the electricity or current would still flow unseen through a vacuum if the voltage or electric field strength was high enough for it to happen. With air or some other gas, and applying a voltage that was high enough to overcome the internal resistance of the gas between the two electrodes or terminals, a glowing (faint light) gas could be created, and such as seen in the Geissler Tubes which are discussed next. Surely, electricity was doing something to the air or other gas so as to cause it to give off or radiate some light. It was discovered later that when an electron gains kinetic (ie., motion, speed) energy, such as due to a nearby electromotive (electricity causing motion) or electric force (ie,, in a nearby region or "field" of influence) caused by other charges (charged particles with their electric force), it can then release that gained energy as electromagnetic energy (such as light) when it collides with another particle and-or settles (ie., "grounds") into an orbit about an atom and electrically neutralizes that atoms net electric charge from being positive. The electron releases its gained or excess kinetic energy as light or radio frequency (rf) energy, but it could also include heat or thermal energy. For an analogy, consider how when a weight is dropped and it eventually looses its kinetic energy when it lands or "grounds" itself, but its gained energy (potential energy, or potential ability energy, from working and lifting it to a certain height and storing that energy in its location, and this is somewhat similar to how an expanded spring stores potential energy due to its configuration]) is transferred elsewhere such as radiated sound energy, and friction and heat, etc. during the impact.

Crookes tubes are based upon the glowing novelty (curiosity, fascination and amusement), neon-like gas tubes previously invented in 1857 by **Heinrich Geissler** (1814-1879) from Germany. These tubes (see [FIG 234] for some very basic drawings of some electronic devices), and its operating principle are sometimes called "gas discharge" tubes or lamps, which basically means an electrically charged gas lamp that emits light. Geissler tubes could also be used to detect a high voltage nearby and-or connected to one or two of its protruding leads (leading into and-or out of the device), electrodes or terminal wires. At both ends of the sealed glass tube, a high voltage was externally applied (ie., connected) to the internal metal electrodes placed within a low pressure gas that could be **ionized** (produce free electrons that broke away from their atomic (force) bond with the atom) by the applied voltage electric field and create many movable charges (electrons from the negative cathode moving towrd the positive anode, and the created positive ions moving tord the cathode) so as to help the conduction of current and-or through it. The excited (energized, higher energy) electrons created would eventually recombine with positive ions and produce a glowing fluorescent (ie., an absorbed and readmitted or secondary light radiation, but has less energy, hence a lower frequency in terms of color and-or intensity) light of a specific color that was determined by the specific type of gas, such as neon, used in the tube. As a side note to these man-made things, some creatures, such as some bugs and some fish, in nature can also produce light such as in what is called as bio-luminescence, such as (flying) lightning-bugs that can radiate some intermittent light and can be seen from a distance of up to about 100 feet away during the night.

To create a high voltages, such as for Crooks tubes, the previously invented (mostly for novelty and curiosity) friction (or "triboelectric") machines for creating (ie., separating) and-or storing a large amount of electric charges were used such as a **Wimshurst machine.** This is a hand powered rotating disk with several radial positioned conductors, and was a (kinetic) friction device to give (ie., transfer) energy to and create free electrons.. This machine, was actually invented about 80 years after Volta made his Voltaic or galvanic pile, and is based on the concepts of making electricity or charge with things like amber, etc. via hand powered rubbing or friction, and generally without any electricity or charge storage being considered. This machine was created in about 1880 by **James Wimshurst** (1832-1903) from England, and is a high voltage static electricity generator that accumulated (positive and negative) charges on two metal surfaces (usually metal balls) separated a short distance away from each other so as to observe a visible spark when the voltage created was high enough to overcome the resistance of the air between the positive and negative terminals, and therefore achieve the "breakdown voltage" needed to pass the charge through that air. A visible spark (like a miniature lightning bolt) is usually a temporary (short time duration) electrically produced light due to high voltage excited air ions or particles. As indicated previously, when an electron "lands", or "grounds" itself after falling into orbit about an ionized (missing an electron) atom, the electron releases its gained excess kinetic energy as [electromagnetic] light energy. For air to conduct electricity, the air and its natural resistance needs to "breaks down" by applying a very high voltage (or voltage potential difference) to it and it then becomes more conductive with ions and electrons, and particularly so when the charges are flowing though an effective low resistance region of air. Note that for a spark to occur, a high voltage is necessary, and by Ohm's law, a high voltage or potential difference will cause a high current and power, but here, the discharge of current and the resulting light created is very brief in time, and essentially a brief, high power (E/s = J/s) light pulse was created.

Before the Wimshurst machine was invented, there were several previous studies and inventions by others on the topic of generating static electricity or "charges". First, the fascination and study with amber and its attraction and repulsion (now known as due to the two types of electric charges - positive and negative) effects are about two to three thousand years old. Later, things such as a hand ball of (hand powered) rotating sulfur could be used to obtain a more consistent and practical electric charge. In 1831, and knowing that a wire and magnet moving perpendicular to each other could induce or create a voltage and current in a wire, **Michael Faraday** invented a hand powered electrical charge generator called a Faraday Disk or Faraday Wheel that used a fixed in position, rotating disk of metal such as copper that rotates in the magnetic field of a nearby permanent magnet, and it produced a relatively low voltage, but substantial DC current of electrical charge (ie., electrons) could be created due to the low resistance of the disk. This device would later inspire other devices that created (ie., generated) either static electricity, electric current, and motors. In 1865 **Wilhelm Holtz,** from Germany, invented an electrostatic machine that he called the Holtz electrostatic induction generator, and this also used a rotating disk. Rather than using a magnet like Faraday did, the Holtz machine used friction upon a non-metalic disk. Many other people soon made improvements to this device and its electrical working concepts, including James Wimshurst. for his electrostatic generator, and later, the **Van de Graff generator** created in 1929 by **Robert Van de Graff** from America, and which used an electric motor to move a friction belt, rather than using a hand turned ("cranked") disk machine. The Van de Graff generator could create a very low current, but very high voltages in the millions of volts of which could then be used for atomic particle research such as for experiments with electric charges and-or electrons. The spark seen and heat of the air created is very brief in time, and its energy and-or power does not exceed that which created it.

It is of note that a spark can not be created in a vacuum (ie., lack of air or other gas), and this indicates that the electricity is somehow affecting the medium (ex. a gas, such as air) in the vacuum. Today, we know that when an energized free (from the nucleus attractive, positive force) electron which has gained (kinetic) energy will eventually fall back into an orbit about the nucleus, it will release that gained energy as electro-magnetic energy, and for this example it will be light and other (invisible) RF energy.

The construction of the **barometer** ("weight meter") to measure air pressure is somewhat similar to, and has a construction much like the temperature gauge created by **Galileo** that used a glass tube placed vertically in a pool of water, and the height of the water in the tube would rise or fall in a direct relationship to the temperature. The modern form of a barometer was developed by **Evangelista Torrricelli** in 1643 and used (dense, heavy) the height of a thin column of mercury instead of water as the air pressure indicator, and this also allows a barometer device to be much smaller and more practical. There is a device to dispense drinking water into a container for animals, such as for cats and dogs, that uses the same principles as these types of barometers, and its amazing to see how an amount of water can be suspended higher in the sealed container than that of the lower water level in an open container or pool at the ground level, and without falling downward or flowing out due to its significant weight. During a storm of rain, the local (ie., nearby location) air pressure can also drop (ie., decline, reduce, decrease) in value due to an increase in the air temperature creating less dense, higher pressure air, and more dense, cooler air will try to move toward and into. Today, we know that mercury is considered as a toxic substance and should be avoided, and therefore, it has become difficult to purchase without good cause and legal permission.

Not too long after the invention of the barometer to measure air pressure was invented, **Jean-Felix Picard**, (1620-1682), from France, noticed in 1675 that there would sometimes be a visible "glow" (ie., like a fog [fine water mist, drops] illuminated by a candle) of light in the (near, almost) vacuum region in the top of the glass tube of the barometer when the mercury in it was agitated (ie., shook, stirred), and then by touching the glass with some static electricity which has a relatively high voltage value of a few hundred volts. This was the first publicly known instance of what would later be called a **gas discharge tube, bulb or lamp**. A modern example of a gas discharge tube or bulb is a neon ("indicator" and-or "night-light") lamp which is usually a dim, orange colored light. Electric charges were exciting the trace of mercury vapor ions in the top of that tube, and electrons settling into an orbit around atoms would release their excess, gained energy as some visible light This simple and accidental discovery, curiosity and novelty would eventually inspire and lead all the way to the inventions of arc-lamp, Geissler Tubes, Crookes Tubes, and later, vacuum tube amplifiers several hundred years later and which inspired modern electronics, This discovery was therefore a very significant and important discovery, much like how Edison discovered the (electronic) diode action in a light bulb experiment and didn't have much use of it at that time, but it did lead to the vacuum tube amplifier years later. Picard also created many more advanced devices so as to obtain more precise (ie., precision, more accurate, "finer", greater resolution) measurements for astronomy and various geometries (or "geodesy") of the Earth, land features, and map making (cartography), and these were also the forerunners of the more modern devices which have slightly better accuracy. He worked with other scientists and astronomers who made several important geological and astronomical measurements. Picard is credited to a calculating a newer and more precise value of Earth's radius.

Otto Von Guericke (1602-1686) from Germany invent the first practical **vacuum pump** in 1650, and this allowed experiments and devices that needed a substance-less volume or space, which is called a vacuum (vacated, evacuated, empty) region, and so as to produce a high quality experiment results, and to be practical. This pump consisted of a piston (much like a rod and-or "plunger") inside a cylinder (here, a rigid or solid tube shape) and having one-way directional openings or "valves" controlling the flow (input and-or output) of air so that it is removed from the vacuum region being created when the plunger is pulled outward, and then discarded into the air when the plunger is pushed inward. This pump could make high quality vacuums in rigid (ie., strong, non-flexible) containers. He then discovered that when the air (a gas) is removed from a container, that its internal pressure is also removed, and then the outside air pressure will still exert a great force (14.7psi) everywhere upon the surface of that container, and directing it to the much lower pressure region inside, and it therefore tries to compress (ie., "squash") it and may collapse it. This fact also makes it more difficult to separate the parts of some containers of which a vacuum is within. Guericke also made an electric charge generator out of an iron rod which held a rotating sulfur sphere of which friction was applied, and so as to give that sphere some charge which could then attract or repel nearby small objects like amber would.

In 1705, **Francis Hauksbee** (1660-1713) from England, and who was Isaac Newton's assistant, would use a vacuum pump to do some electric experiments. He made a hand powered rotating glass sphere like Guericke, but one that had a vacuum and with a trace (very small amount of mass) of mercury gas vapor or atoms in it. Guericke would see a visible **glow** when it was agitated, and particularly, when the surface of the rotating sphere was touched, creating some static electricity charge and forces, and-or touched by another (statically) charged object, and this is very similar to that of the historical, barometer glow effect discovered in 1675 by **Jean-Felix Picard**, and which was mentioned previously in this book. These discoveries and technologies would later lead to **Geissler tubes** in 1857 that also created a gas discharge or glow when a high voltage and low current is passed through a trace of gas in a vacuum, rather than using (high voltage) static electricity to excite or energize the gas. Hauksbee's glowing sphere would later lead to the invention of electric **fluorescent tube lamps** ("lights"), where much of the invisible radiated (UV light) energy from the glowing mercury gas is converted to a visible (or "fluoresced") white light by using a thin, phosphorous (a natural element), paint-like coating on the inside of the glass tube. The phosphor powder essentially absorbs the UV light energy and converts it to re-radiated visible light which is also lower in frequency and longer in wavelength than UV light.

In 1729, **Steven Gray** (1866-1733), from England, found and demonstrated that static electricity (ie., static = non-moving, stored charge) can essentially move or "flow" from point to point along a (static) electricity (ie., electric) conductor such as various types of materials including cloth, rope, and metals and to then attract or repel other lite-weight objects near the end of a length it. Gray's work is also based on some of the work of his friend **John Desaguilers** (1683-1744), from England, study of electricity, and who made the words "insulator" and conductor". What Gray also discovered is the first form of distant or remote communication using electricity, and before the much longer distance, (non-static, but current flow operation) single-wire, electric telegraph communication system developed over 100 years later in **1837** by **Samuel Morse**, and of which was just a few years after the invention of the electronic buzzer in 1831. Volta's electric battery (1799/1800) and Morse's code system made this form and method of (long distance) communication both possible and practical enough to be placed in many cities, and for fast and inexpensive communication, and therefore having much social value. In 1816, **Francis Ronalds** (1788-1873), from England, made an early form of a (static electricity) telegraph, and being 8 miles long, and proposed it could be used for long distance communication. In 1845 Ronalds invented the first instance of time delayed photography for recording the data and-or measurements from scientific instruments, and this was essentially an early and crude form of a movie, although not so continuous as a real movie such as by Edison, having a continuous appearance of motion. **Carl Gauss** and **Wilhelm Weber**, both from Germany, tested a simple telegraph system for nearly a distance of two miles, and in 1833 - just a few years before the more practical Morse telegraph system. Their system used a galvanometer as the receiver and-or indicator. It could be said now that having a battery and a wire, that it was only a matter of time since Volta's battery until electricity was used for practical communication.

In 1831, **Joseph Henry**, (1797-1878), from the United States of America, while experimenting with electro-magnets (essentially a coil of wire with electric current flowing through it - often called a solenoid, while wrapped around a piece of iron). he invented the **electronic relay** (ie., an electrically controlled switch where a low current can switch on a higher current, and-or isolate current in two independent circuits from interfering with each other). In a relay, the magnetic field from the activated solenoid will attract a (springed) moveable conductor to make contact with another fixed metal conductor, and thereby completing an external electrical circuit. A technologically similar related device he created was the **electronic buzzer** or sounder, and is modernly known as an "(electric) doorbell", signal and-or sound alerter or sounder. This device is an electro-mechanical oscillator. In 1850, the telegraph sounder for the telegraph system was one of the first practical uses for an electro-magnet.

The relay was later was used by others, such as for a version called a Telephone Sounder (ie., a electro-mechanical device to make a sound) that was invented by **Alfred Vail** in 1850 for the electric telegraph system created by **Samuel Morse**. Essentially, this device is a telegraph receiver, and is much like the opposite of a telegraph sender that has a user operated (input or signal) switch called a "(telegraph) key" to send a pulse of current (and-or information) to the distant receiver. The current in the telegraph wire would pass through an electromagnet in the receiver and temporarily cause a piece of metal on a pivot to be attracted to the electromagnet, and then that metal will gain energy and move and produce a sound when it struck another piece of nearby (fixed in position) metal. A buzzer or bell also includes this principle, but the moving piece of metal also opens the electric circuit and stops the current, and so as to essentially reset itself (sometimes with the assistance of a spring or gravity) so as to make another possible pulse of sound. An electro-mechanical buzzer is essentially the first electro-mechanical oscillator. If the moving piece of metal would strike a tuning fork or bell, it can produce a pleasing tone, such as used for a doorbell. **Joseph Henry** also invented the concept of self inductance when two coils are in series and being near each other, so as to magnetically affect and induce each other. Some electronic oscillators (with triode-tube or transistor amplifiers) use a self inductance or "tapped" coil so as to obtain and reuse a small amount of the output power by sending it back into the input of the oscillator so as to keep it functioning without any other input or starting signal needed.

Another device, besides the Wimshurst machine, that can create high voltages is a **Ruhmkorff Coil** which is basically a pulsed operated (input, first, primary) coil of wire that will apply its current created high magnetic field through the many more turns (of wire) of a nearby secondary (output) coil. Since the secondary coil has many more turns of wire, and therefore, a higher voltage is induced or generated in it. The ratio of the ("magnetically [energy] coupled") number of turns in each coil will determine the increase or gain factor of the input voltage. Since the output energy or power (energy/time) cannot be greater than the input energy or power, the maximum current available will decrease (ie., demagnified, be divided) by the same factor as that of the voltage gain or increase. This decrease in current occurs naturally due to that thinner wires have a higher resistance and therefore do not pass as much current. Power = (voltage)(current). It is also possible to use many low voltage single voltaic cells connected in series so as to have a high total available voltage being equal to the sum of their individual voltages. A Ruhmkorff coil can produce a high voltage to power a (glowing) Crookes tube. After the necessary initial "breakdown" or "arcing" voltage is applied to a Crookes tube, it will begin to conduct current, and thereafter, the applied voltage can actually be reduced so as to reduce the current flowing through it and so as to prevent damage to it and to reduce wasted power. The Ruhmkorff coil was invented in 1857 by **Heinrich Ruhmkorff** (1803-1877), initially from Germany, and later, England. This device is essentially the first practical or useful low input voltage, DC (steady or direct current) to high voltage pulsed DC transformer and-or oscillator. The output is actually a form of "half-wave" or one-direction AC, that is, the voltages rises and falls like AC does, but is much faster at reaching the peak or maximum value. With this device or **transformer**, a few thousands volts can easily be produced and this is very similar to how an automobile's (car's) high voltage spark is created so as to ignite the compressed gas mist inside the engine cylinders (ie., compression and combustion cylinder tubes or containers made of metal) which causes it to **combust** and release its stored energy in a brief, but powerful (joules/second) high energy pulse, and this eventually applies mechanical rotating power to the wheel(s) of that automobile so that it can move forward.   Though a one-cylinder, one-piston engine is possible, such as for a small remote controlled airplane, most automobile engines have 4 to 8 pistons so as to reduce mechanical vibrations by firing each piston one after the other in a repeated succession. This also effectively applies more (ie., a multiple of) power to the wheels over an instance of time, and improves cooling and system efficiency. Multiple cylinders will also reduce the frequency of combustion in each cylinder, and by the number of cylinders used, and this will provide a greater longevity of use for each cylinder. Larger cylinders and pistons are used in the combustion engines for very high power devices such as large truck vehicles and ships.

Ruhmkorff based his high voltage transformer on the first known induction coil or transformer created which was made by **Nicholas Callan**, a priest in Ireland in 1836. and which also produced a very high voltage from a low input DC voltage battery. Callan was influenced by Michael Faraday's discoveries. These coils produced a high voltage and spark across a short distance called a "spark-gap", and would later be used by others to study and transmit radio waves, and even be used to produce the high voltages necessary to produce X-rays for hospitals and dentists. Callan's interest in electricity was inspired by **William Sturgeon** (1783-1850), from England, who invented the first electromagnet in 1824 which was made from an insulated coil of wire wrapped around a piece of iron, and then connecting the ends of the coil to a battery so as to have electric current flow through the windings ("turns" or "loops") of the coil so as to produce an electromagnetic field about it. The more turns of wire or loops in the coil, the greater the accumulated magnetic field. The accumulated sum of the magnetic field of the complete coil is due to the (smaller) magnetic field of each length segment of 1 turn of wire in that coil. The turns essentially concentrates the magnetic field of the entire length of a long wire into a smaller and practical volume. The intensity or strength of the magnetic field produced is directly related to the number of turns of the coil and the current through it. A longer length of total wire to make a coil, the greater its resistance, and the less current through it given a certain voltage.   Increasing the diameter of wire can reduce the total resistance of that coil given the same number of turns, however the volume or size of that coil is now larger.   Thicker wire is also more costly. Due to all these parameters of a coil, and a desired magnetic field strength, a practical trade-off or compromise needs to be found and-or calculated.

The Electric Motor

Sturgeon also made the first practical electric motor. This motor did not rely on electrostatic forces of attraction as previous electric motors did, but rather relies on the magnetic forces of attraction, and could therefore produce much more movement and-or torque so as to do and-or move things. In general for a give amount of power (P = VI) input to a motor, a higher torque or rotational force motor requires a higher current so as to create a stronger magnetic field and its resulting force to create (rotational) motion of the rotor (ie., rotational axis and-or body structure) in that motor.

Note that an electromagnet does not actually need an iron core, but the iron or steel core improves its performance since the magnetism does not have to travel through the air which does not pass magnetic fields as easy as the metallic iron. Air is then like a resistance to magnetism, and iron is a conductor of magnetism. Some common examples that use an electromagnet is in an electro-mechanical buzzer or sounder, audio speakers, a crane to pick up scrap iron metal pieces, and a relay that is essentially an electro-mechanical switch. Note that although a DC electric current can cause and keep a steady valued magnetic field about an electromagnet, a DC electric current cannot be used with transformers when there is a need to transfer an AC signal from the input coil to the output coil, and this is because the input current must also be AC or changing in value and not DC which has a steady, unchanging value of current and this will not induce any voltage and current in the output coil. In short, besides for a brief, temporary or quick magnetic pulse during current "inrush" or application creating the magnetic field pulse or rise, a steady magnetic field strength of any value does not induce a voltage or electric current, but it is rather a changing magnetic field strength that does.

To help understand how a common electric motor (ie., electrically powered) works, consider two magnets on a flat surface with the same magnetic poles or magnetic polarity facing each other. If magnet (1) is pushed closer to the other magnet (2), magnet (2) will be repelled by the magnetic force from magnet (1), and then it will move away from that magnet (1). This is an example of a linear or non-rotating motor (such as a solenoid) and its motion. In a rotating motor, magnet (2) would be affixed to a rotating shaft called the rotor which rotates. Magnet (2) would then move in a circular direction or motion rather than in a linear (line-like, straight) motion. Here, one or both magnets may be electro-magnets rather than "permanent magnets" (non-electric, no current needed) made of iron metal and-or ceramic material.
Michael Faraday (1791-1867), from England, and who first discovered in 1831 that even a single wire with electricity (ie., electrons, current flow) going through it would produce a magnetic field around it, and after noticing that it would make a compass pointer (as a magnetic field and direction sensor) change direction. The more current, the greater the magnetic field produced and the greater the compass pointer deflection - hence the compass become a current sensor and meter. When pulsing current through this wire, Faraday noticed that another nearby wire circuit would then also have a pulse (temporary) of current flowing through it.. Faraday called this phenomenon of creating or inducing a (changing in value) voltage and current due to a (changing in value) magnetic field, particularly its strength, as **electro-magnetic induction**.

Faraday also invented the first simple (but not yet practical) electric motor in 1821 and which produced some mechanical motion from the applied electricity. **Moritz Jacoby** would improved upon all the previous motors and made the first useful electro-magnetic motor in 1834. Before the electro-magnetic motor, there were the scientifically interesting, but not of much practical use at the time, electro-static motors created by people such as Andrew Gordon (Scotland), and Benjamin Franklin (United States of America, U.S.A.). These electric motors used the acceleration and velocity caused by accumulated electric charge which has an accumulation of electric or coulomb forces which can affect and-or apply energy to other nearby electrically charged objects.

Anyos Jedlik, (1800-1895), from Hungary, invented an electrical connection called a **commutator** of an electric motor in 1827, and it was significant for improving the direct current **(DC) electric motor**, and of which needs to rotate in only one direction. His invention allowed motors to be powered by DC (direct current, single direction, steady voltage, such as from a battery) power. He also made a small "toy car" with his motor which surely inspired cars of the future.

The first high power motors for machinery were invented by **William Sturgeon** in about 1832 and **Thomas and Emily Davenport,** from the Americas, in 1837. Experiments with primitive electric vehicles began at about this time at about 1834 such as one developed by **Robert Anderson** from Scotland, who made the first electric carriage (ie. "car"). The Davenport's are also given some credit to the first electric car at about the same time as Anderson. **Antonio Pacinotti** is credited in 1864 to the next significant advancement in the DC electric motor, and this is the **ring armature** for the coils of wire, and which improved the efficiency and performance of the motor. Some other people who made further and significant improvements to the electric motor will now be mentioned. In 1856, **Werner Siemens** [siemens is now a unit of electrical conductance = G = 1/(resistance in ohms)], who invented the needle-letter telegraph system in 1843] made some improvements to the electric motor. Siemens and Halske in 1872, Jonas Wenstrom in 1880, and Frank Sprague in 1886 would later invent some further and significant improvements to the efficiency of the electric motor. As of the year 2022, an electric motor having a maximum efficiency of 70% to 80% is common if the output load (ie. resistance) and-or power used is about the same value, otherwise the motors efficiency = (output rotating power) / (input electrical power) is reduced due to the wasted power from not optimizing the load and designed performance or capabilities of the motor.

The **AC electric motor** took longer to develop and to became a reality after the DC motors became practical. AC motor design also took longer to improve since it required AC generated electricity to power it, rather than DC power from a commonly available battery. An AC powered motor does not need a commutator (ie., a mechanical, electrical connection via a piece of thin flexible bent metal, or spring pressure upon carbon (electrical contact) "brushes") so as to keep the flow of electricity and magnetic field in the same desired polarities so the motor will rotate in the same (CW or CCW) direction. There were many developments to the AC motor, and by many people and-or companies, but **Nicola Tesla** and the **Westinghouse Company** are generally given credit to the basic modern design of an AC powered motor in 1882. Rather than a commutator as in a DC motor causing the flipping or changing of the polarity of the magnetic field so as to cause more movement of the rotor, the fluctuating polarity of the AC power supply voltage will automatically perform the changing of the magnetic field polarity so as to cause more movement of the rotor in the same direction. Since a changing magnetic field strength induces current in a wire or coil, an AC motor is also called an induction motor, and it is sometimes said that the stator has a (apparent, or simulated) rotating magnetic field to cause the rotor to spin.

Zenobe Gramme discovered the **AC electric generator** in 1871 after noticing that a DC electric motor, and in a reverse type of use, can also produce electricity without modification]. If a rotating force is applied to the rotor of the DC motor, and causing it to rotate, it will produce electricity at its two terminals and through a circuit connected to it. The faster it is rotated, the higher the voltage and current produce, hence the higher the input and output power. An AC electric generator, perhaps at a distant electrical power station, can be used to power an AC electric motor. Though an electric motor can be used to produce electricity, such as for some small, homemade electric power stations, a generator is usually designed so as to have more efficiency at making electricity than an electric motor. In particular, it is the high current in a motor with relatively few windings or turns on its coil that creates a high magnetic field and resulting torque needed for many (mechanical power or force needed) power applications. an electric motor will generally produce a lower output voltage than a real electricity generator. For a electric generator at a power station, a high voltage is rather created by using many more turns of wire in its coils so as to transmit high voltage electric power efficiently via long distance electric power transmission wires.

It took until 1854 for a **lead-acid based and rechargeable battery** to be created by **Wilhelm Sinsteden** (1803-1891), from Germany, and it was later improved in 1859 by **Gatson Plante** (1834-1889), from France, and this then facilitated many advances in various electronic devices and energy storage to come later, such as with electric motors in many types of devices and for many uses, and for example, an electric water pump, electric transportation vehicle, and various electronic and-or radio communication. This type of battery has relatively large surface, electrode plates made of lead metal immersed in sulfuric acid that is a compound made from hydrogen and sulfur. The acid with its hydrogen ions (ie., protons) and lead will chemically form lead sulfate, hydrogen ions (ie., protons), and two free (unbounded to an atom) electrons which can then be used to travel through an external circuit as electrical power. This chemical reaction is reversible by charging the battery.

In 1887, **William Morrison** (1855-1927), from Iowa State in United States, is credited to the first usable (for transportation for relatively short distances [a maximum total of about 45 miles] over unpaved roads) modern electric car that used a (lead-acid) rechargeable battery. Rubber covered tires (ie., wheels) were not invented until 1896, and this vehicle was truly a horseless carriage, and and having a similar wooden type of spoked wheels as used for covered wagons that were drawn (ie., pulled) by horses. This transportation vehicle that did not require a horse, boat or a steam-powered train soon began to create much interest in it and of the future improvements to it in the minds of many people. In about 1890, Morrison also made significant internal construction improvements to the electric rechargeable battery by using a fiberglass layer, and where the total amount of energy storage and-or power output available (ie., its energy storage density) per weight of the battery was significantly improved. This made electric power storage, and usage very practical. This also increased the number of total charging and discharging cycles possible with a rechargeable battery, hence improving the battery and-or power stability, and "battery life". Battery power stability, is essentially how long it can maintain at least its rated voltage and current ability, and before it drops below its ratings. Morrison greatly improved the rechargeable battery's stabililty and reliablility with a relatively low cost improvement to it.

While on this subject of transportation, the **Studebaker Co**. in U.S.A. made the first practical electric car (automobile) for the American general public in 1902. This was at a time when gasoline was a limited produced item, relatively expensive, and not yet available in many locations, and most public transportation roads were still dirt covered and unpaved with a hard surface. The batteries in these cars was a lead design and was therefore heavy and reduced the efficiency of the car. A few years afterwards, in 1904, Henry Ford began making the Model-A automobile. In 1913, the (Henry) Ford Motor Co. became a factory for mass producing relatively inexpensive gasoline powered automobiles, and this greatly diminished the public's desire for an electric car or horse-and-carriage for personal transportation. The "Model T" car was a very popular and relatively affordable design, and it was the initial basis to many future designs in the entire automobile industry. Many dirt roads soon became paved roads for better travel and efficiency. Trucks could now transport many and heavy items to many miles away and with greater speed than a horse (an animal engine) powered vehicle. As of about the year 2000, with advances in lite-weight electric batteries to power all types of devices, the electric car is once again on the wish-list of many people. During the same year as Ford's Model-A car was available, the Wright brothers created and flew the first practical plane in 1904, and which would revolutionized long distance transportation and made it practical.

Generation of electricity without using a battery:

Heinrich Emil Lenz, from Estonia-Russia, in 1834, created **Lenz's Law** which describes the voltage polarity and direction of magnetically induced currents of which happen due to the relative and perpendicular (90°, right angle) motion of a magnet and wire. The "Right-Hand Rule" considers conventional positive (+) to negative (-) current flow, and with your right hand thumb going in the direction of the wire and current, and with your other fingers of your right hand then wrapped around the wire is the (clockwise) direction of the magnetic field about around that wire. The symbol L, in honor of Lenz, was first used as the unit of measure of inductance, and others may use H (Henry's), instead of L, as the unit. Still, many inductors are still noted using an L symbol, but measured in henry units.

In 1841, **Michael Faraday** discovered (Faraday's Law of) **electromagnetic induction** by applying a voltage (and resulting current) to two coils placed closely together; essentially a transformer, and which are said to have "mutual inductance" in that they actually affect each other with their magnetic fields. As mentioned, Faraday also discovered that a moving magnetic field (and-or wire, relative to each other) such as from a magnet moving near a wire or coil of wire, will produce (ie., induce or generate) a voltage across that coil's terminals (end points, connections or leads) and create a proportional amount of resulting current through that coil and circuit connected to its ends. Electrons can move (ie., be given kinetic energy [ie. joules, and V=J/C]) by electrostatic forces or magnetic forces. Moving charges likewise produce both an electric field and magnetic field. Faraday's discoveries about magnetism and electricity (magneto-electricity or electromagnetism) are crucial for generating electricity and using electric motors, both solenoids [non rotating, but for linear (push and-or pull) motors providing non-rotating, but rather straight motion], and rotational or spinning motors. The unit of measurement of charge storage or capacitance is now called Farads (F), and was given that name in honor of Faraday.

In 1865, **James Clerk Maxwell** (1831-1879), from Scotland in the U.K (The United Kingdom of Britain) was greatly influenced by the work and discoveries of Michael Faraday, and he took electro-magnetic (electricity and magnetism) science to new heights and future possibilities with his four electromagnetic "Maxwell equations" of which were later simplified and credited to: **Maxwell**, **Faraday**, and **Oliver Heaviside**. These equations showed the mathematical relationships between electricity. magnetism and light, and even considered the transmission of electro-magnetic energy (ie., radiowaves) when they are considered as a type of light and-or vice-versa. Heaviside invented the coaxial cable in 1880 for short or long distance, efficient, and noiseless (low electromagnetic interference) transmission of signals through this special (outer "shielded", inner cable) "wire", "cable" or conductor. Maxwell's work influenced Albert Einstein with his discoveries. One equation by Maxwell relates the amount of created magnetic flux (ie., magnetic field strength) of a coil to the amount of input current (Amps) and the number of turns (windings, loops) of that coil. Maxwell also deduced that light is an electromagnetic (ie., a radio) wave (ie., has a frequency like a wave or oscillation). In 1861, Maxwell made the first, but crude, color photographic method, and where three color (green, blue, and red, or RGB) filtered images of the subject were made onto three separate, (transparent) glass plate, black and white photographs. The amount of greyscale (or sometimes spelled as grayscale) of the image on each photograph would effectively indicate and record the intensity and-or color of the light of the subject. If bright light was then passed through each photograph, and then that light being passed through the same color filter that was used to created it, the three separate color images can then be combined, by "overlaying" (to be at the same position or location) the project images to coincide, be combined with, or "on on top of the other", into one multi-colored image of the subject. Besides Heaviside's work on Maxwell's equations, he also invented the coaxial cable in 1880 for short or long distance, efficient, and noiseless (low electromagnetic interference) transmission of electric communication signals. This internal two wire "cable" is shielded from the natural elements such as rain, and often by using some type of plastic. Within that insulated outer surface coating is a two wire cable. There is an outer radius, metal "shielding" wire wrapped about or covering an insulated, inner [signal] "wire", or conductor. Both wires can be used as part of the circuit to conduct the signal, and the outer wire may sometimes be "grounded" so as to help reduce unwanted radio interference and-or noise such as from storm lightning in the area. More distance radio (ie., rf) noise will be weaker in amplitude and will be easier to reduce, tolerate, filter out and-or eliminate.

Crookes tubes are based upon previously invented Geissler (gas) tubes, but Crookes tubes do not contain any gas so as to glow, but use a pure or high vacuum with no internal gas desired. The glow at one end of a Crookes tube is rather due to electrons striking a thin phosphor (sensor) layer. Crookes tubes also demonstrated that electricity could flow through a vacuum that is essentially an insulator or very high resistance that does not conduct or allow current using practical, relatively low voltages. For very high or pure vacuums, the (vacuum) breakdown or conduction voltage is about 80kv per centimeter. Since there is no air or other substances in a vacuum, there will be no glowing gas or spark produced and seen as the electricity passes through a vacuum.

Once the electron particle was discovered, it was then fairly understood what electricity and-or electric current was actually composed of. Since electron particles are associated with negative charges, there must also be corresponding opposite or positive charged particles to attract them with their charges and forces. This opposite type of electric charge would later be indicated and found by observing Geissler ("discharge") tube experiments where holes were put in the cathode plate, and some new type of "anode rays" or particle went through those holes in the opposite direction than electrons did through the Geissler tube. This was first noted in 1886 by **Eugene Goldstein** (1850-1930) from Germany**,** and later understood to be positive charged ions (atoms missing an electron and its electric charge, hence having a net positive charge) composed of positive charged, fundamental particles called **protons** by **Ernest Rutherford** in 1911. These proton particles are in the nucleus of every atom and contribute to nearly all of an atoms physical matter which is measured and expressed numerically as its mass size in standardized units called grams or kilograms. The amount of mass of an electron is considered negligible as compared to the mass of all the other particles (protons and neutrons) of an atom. **The mass of a proton is 1836 times more than that of an electron**. Because of their very small size and low inertia, electrons can travel more freely between atoms, hence they can more easily travel through substances such as metals, especially in the presence of a (electromotive, emf) force attracting or repelling their low, easy to move mass:

The **mass of an electron** has been calculated to be about: **9.1093837 (10^-31) kg** =
9**.**1093837 (10^-31) (10^3 g) = 9**.**1093837(10^-28) g

The high voltage (ie., electric field of force or electric potential energy) in a Crookes tube actually repels and-or emits and repels electrons at the cathode, and the anode (ie., the positive terminal) at the other end of the tube will attract, via electric force, and accelerates the electrons to a very high speed and they will gain high kinetic (ie., motion, movement) energy and will produce light or (invisible, and higher in frequency) X-ray electomagnetic radiation when they collide with some (hard, dense) objects such as the metal anode in the tube. Geissler's, and Rontgen's (who discovered X-rays) research probably considered the previous discovery, in 1785 in London by **William Morgan,** a glow being created when a high voltage was applied in a glass tube that had a slight vacuum. Nicola Tesla is also known as a contemporary (ie., at about the same time, and studies) to Wilhelm Rontgen, particularly in the filed of X-rays production and recording onto light (or rf - radio frequency, radiation) sensitive film, such as for medical purposes.

Electrically charged protons and electrons attract each other with a force that is called an electric or electromotive force. It is much stronger than their gravitational force of attraction which is considered negligible for the very low mass particles. When the particles are close to each other, their electric forces are stronger than Earths gravitational force. The force from a magnet on a nearby iron object is also stronger than Earths magnetic field and force. Protons and electrons are two of the three fundamental particles (protons, neutrons and electrons) of an atom, and are the known as fundamental particles and-or units of electric charge and-or force. When these particles are not in motion, they are static (stationary, resting, as opposed to dynamic, kinetic or moving), and this charge or force may be called an electrostatic charge or force. If the particles are moving such as an electric current, the force that causes them to move is more correctly called an electromotive force (emf).

"Unlike charges" and-or charged particles, such as protons and electrons which have "unlike" (non-similar) or opposite charge, will attract each other like the two opposite poles of a magnet, and so as to naturally and eventually balance (settle, rest, equalize) the charges and have a net charge of 0, such as in a non-ionized or neutral atom with a balance of (ie., equal) electric charges in it when it has the same number of electrons and protons. Each element has a unique number of protons, neutrons and electrons. Generally for a given element, both the number of neutrons and both the number of electrons will equal the number of protons. An electrons attraction to a proton and-or the repulsion by other electrons, will cause (ie., induce, force) a "free" or unbound (from its atom) electron to move or flow towards a positive charge, such as when it is moving in an electric (electrically conductive path such as a metal wire) circuit to the positive voltage polarity and terminal side of a battery. This direction of current flow is for the true, physical and actual flow, but "conventional" or "common" flow of electricity, or electrical energy and-or power, is considered from positive terminal to the negative or "ground" (considered at 0 remaining potential) terminal of the voltage source, and this does aid with the math of it. For most common circuits, there is not problem using conventional flow and-or analysis. For things like Crookes and other vacuum tubes, the actual direction of electron flow matters more than having a (generalized) current flow and-or direction.

1911, Rutherford, who discovered the electron particle of an atom, theorized that the proton and electron composed a larger particle (called a nucleus of the atom) with a balanced or neutral charge state and called this unknown particle the **neutron** particle, and that an atom must have a large center or "**nucleus**" of mass. Rutherford and his team of scientists then experimented by sending fast moving alpha (protons, positive charged) particles at a very thin layer of gold foil, and because most of the alpha particles went through the foil and struck a fluorescent screen made of zinc-sulfide and could then be seen as re-radiated light energy, he theorized that the atoms of the foil had mostly empty (no matter) space between them. Since alpha particles have a positive charge, they should also be attracted to, and combine with the electrons in the gold foil, but it was noted that this alpha particle was sometimes changing or swerving away from its expected straight direction after leaving the foil, hence it must be colliding with and-or deflecting (changing) its direction due to a positive, electrically charged particle at the center of the atom. He theorized that electrons must be very small, and must compose some space or volume of an atom, and that they must then be attracted to, and orbit the nucleus.

Neutrons are very similar (but actually just slightly greater) in mass to that of protons, however, they don't have any electric charge like a proton or electron. They are not influenced by an electric field, and are only slightly influenced by a magnetic field, but are rather influenced only by kinetic force or collision. Later, in 1932 **James Chadwick** (1891-1974), from England, discovered a particle that did not have any electric charge, hence neutral in charge, and therefore called it the **neutron**. Neutrons do not attract or repel each other, and it is thought that they therefore do not have a "rotational spin" on its axis, and that they are attracted to the proton by a mysterious "strong atomic [nuclear] force". In the 1960's it was theorized and discovered that protons and neutrons are composed of even smaller particles called quarks. Each proton or neutron is composed of 3 quark particles. In 1979, even smaller particles called "gluons" were discovered and are thought to hold ("glue") the 3 quarks together. The concepts of quarks and gluons is generally not needed and-or used in the practice of electricity and electronics. If a nucleus has an unbalance (excess or lack) of neutrons, the atom is called an (unstable) **isotope** of the stable or balanced form of the element, and it can then split or decay into other lower mass atoms while releasing or radiating energy in the form of radioactivity, and then becoming (more) stable. Nearly all of the mass (real physical material, or substance) of an atom is due to the total mass of its protons and neutrons.

A proton is said to have and be the 1 fundamental unit or particle of positive (+) electric charge and force, and an electron particle is said to have and be the 1 fundamental unit of negative (-) electric charge and force. Neutrons have no charge. Protons will repel each other, and electrons will repel each other. Electrons and protons will attract each other due to their opposite electrical charge. Since the electric charge of a proton is equivalent to that of an electron (e), the electric charge of a proton may be expressed as ($+1e = e$), and the electric charge of an electron may be expressed as ($-1e = -e$). Note that the mass of an electron is much smaller than the mass of a proton, and yet an electron has an equal, but opposite, electric charge as that of a proton's electric charge.

Just like magnets having a magnetic (force) field (area of presence and influence), or field of force about them which can induce, attract or repel other magnets and (magnetic, iron) particles, charged particles also have an electric (force) field about them which can induce, attract or repel other charged particles. Because of an object just having a non-zero electric force, these electrostatic particles and their charges can also cause (induce, force) other particles and their charges (particularly electrons) to move or flow (attracted to, or be repulsed away) without any electrical (ie., electron) path or conductor between them such as a wire. The **electric field strength** value is much like that of a magnetic field strength value and a gravitational field strength value, in that its (force) **field strength or effect decreases in value by the square of the distance away from it**, hence this is a mathematical inverse-square law relationship of the field strength and distance squared in value. The force of gravity between two objects is relatively weak compared to the sizes of masses involved. The forces due to electricity (ie., charges) and-or magnetism are much greater between the same two masses.

In an uncharged, electrically balanced or neutral object, the net charge is 0e. It is as if it has not gained or lost any charged particles and the object is electrically neutral without any extra charges. If an object is said to be "negatively charged", it has a net negative charge because it has lost some of its positive charge. If an object is said to be "positively charged", it has a net positive charge because it has lost some of its negative charge. Charges are quanitized (ie., counted) as integer sized steps and values only, and that there is no fraction of a charge. Charges can be added together to produce a net total effective charge. If an object has 4 positive electric (e) charges and 7 negative electric (e) charges, the net effective charge of that object is: (+4e) + (-7e) = (+4 -7)e = -3e. Since the net charge is negative in sign, that charge is due to an excess of 3 negative charges, hence electrons.

An electroscope is a simple device to sense, observe (see), and possibly measure, the presence of a net electric charge of an object. The device is a glass jar with a very thin, lite-weight (low mass) metal foil sheet folded in half and held in position by a wire from the lid and external metal terminal or input connection. When the metal foil sensor is charged due to the presence of another charged object, it will expand due to the electric charge repulsion of the like charges on it. It is even possible to indicate and measure the type and basic amount of the net charge. The basic concepts of an electroscope were partially invented by **Charles DuFey** (1698-1739), from France, in 1721, and who also discovered that there is two types of (static) electricity, now known as two types of electric charges, and that the same type of electricity (on charged objects, including metals) cause a repelling, and the opposite type of electricity (on charged objects) causes an attraction. Charged objects can also induce or cause an opposite charge region in an uncharged object, and hence cause an attraction as if it were already oppositely charged. This effect can also be seen with an electroscope instrument when the two plates repel each other due to having the same type of charge on them.

Benjamin Franklin (1706-1790) from the United States of America (U.S.A) made several basic or fundamental discoveries about static electricity. Franklin noticed and proposed that a spark is similar to lightning, and vice-versa, and that it is due to electric charges. Franklin invented the concept of a metal "lightning rod" on the top of a high structures so as to divert (electric) a dangerous strike lightning away from that structure and down through a high conduction wire placed into the ground. The rod was made to have a pointed tip so as to help concentrate electric charge and force of attraction at the tip. Franklin made the first battery or array of connected and stored charges so as to have more of it for experiments and to possibly do useful things with it. The charges were stored in glass jars with two metal plates, and were called "condensers" (charge accumulators, hence "condensed" charge in a location) and are commonly called today as "**capacitors**" (ie., store a capacity, quantity or an amount of charge), and were first known as **Leyden** Jars since they were invented in that city, and in about the year 1745. Franklin discovered that many of these storage devices placed in parallel would store more charge, and called this capacitor arrangement as a "battery". In 1746, Franklin discovered that electricity would flow from a (having more, or "positive") charged object to that of a lesser (lacking, or "negative") charged object, and that the total charge of both objects would always have the same total, hence there is a "conservation" (ie., remains the same total amount, the amount is conserved and none is lost) of charge. Franklin had a theory that there is only one type of electricity or charge carrier, and today we know this a electron charges and flow.

A simple electric communication system (as conceived in the 1750's) can be made with electricity conductors (such as wires) and electroscopes used as the communication sensors or receivers, and a mechanical analogy to this could be a rope placed between two distant bells to be used for a communication system. A step above this system would be to use a battery, on-off switches, (transmission) wire, and electric buzzers or lights as the receivers. This system or circuit is similar to, and very useful as that of a modern electric doorbell (invented by **Joseph Henry** in 1831) and of which the unit of measurement called Henrys (H), for inductance (L) was named after, and it was also similar the **electric telegraph**, communication system that was made practical in 1837 by **Samuel Morse** (1791-1872), from the United States, by utilizing a single electric wire. Morse, who studies included communication for the deaf, also helped developed the **Morse Code** method of transmitting and receiving distant communication by using a series of short (a "dot" [**.**]) and long (a "dash" [_]) electrical pulses to construct and represent a specific letter of the alphabet and so as to effectively build each word, letter by letter. At the distant receiving station, perhaps hundreds of miles way, these pulses were received, (sometimes) recorded, collected and then decoded or reconstructed back into letters of words and written down by the telegraph operator so as to create a written "telegram message" (or simply a "telegram") to be delivered as mail to the receiver of it. This was a significant step in fast, low cost communication, and it helped society advance more efficiently and rapidly. The telegraph did not replace the mail system, for it was, and still is needed for the many and long messages and packages, whereas a telegram was often limited in the "length" of it in terms of the total number of characters and-or lines permitted by the station, and therefore, the messages were often short and "to the point" or issue, much like that on a postcard (mail). The Morse code is given an article in the "Late Entry's" section in this book.

Rubbing (friction) can "charge up" some objects. Rubbing two objects together can free some electrons from their atomic orbit or bond so as to produce ("free", movable) charges or "electricity". Larger, more massive atoms of elements have a weaker bond or force applied to their more distant outer electrons, and they are more easily freed using less force such as electric force. An object that has more negative charges (free electrons) than positive charges (positive ions, positive ionized or charged atoms) is said to be negatively charged.

In 1832, and not long after Michael Faraday discovered electro-magnetic induction, **Hippolyte Pixii** (1808-1835), from France, and inspired by Faraday's crude non-coil design of an electricity generator, created a hand cranked and powered by arm force, rotating machine which can be turned or rotated so as to generate (AC, Alternating [in two directions] Current) electricity. **This electricity generator machine converted or transformed input mechanical energy and-or power to electrical energy and-or power.** His machine rotated a magnet near a coil of wire. With this, he invented a practical A.C. **electric or electricity generator** machine or electricity generator. This simple concept and device was actually monumental in terms of the electricity advancements it would inspire and be used to power. **Andre Ampere** gave Pixii the idea of using a mechanical device (now called a [electric] commutator) so as the current would be caused to flow in just one direction, somewhat like DC does, but here, with AC involved and always changing in value, and the generated DC will also vary in value, magnitude or amplitude, but not in two direction like AC does.

A **dynamo** is an advanced type of **electricity generator** created by **Werner Seimens** (1816-1892), from Germany, in 1866, and which used electro-magnetics instead of permanent magnets. These electrified coils (electro-magnets) have a higher magnetic strength or power, and therefore, the output electric power was (theoretically) greater if the generator's rotor (ie., its rotating or rotational part) was spinning as fast as that without electrified coils. It should be of note that generators, for either mechanical or electrical output power, can contain mechanical input and-or output gears so as to alter the rotational speed and torque (force) needed for the rotor and a stable (ie., a constant value of voltage and current, or power), regulated electricity production or output. Seimens created many useful electrical inventions, and his name is used for the SI unit for conductance in his honor, and supersedes the mho unit created by George Ohm. Before the ohm unit of resistance, the Siemens reference unit of resistance was slightly less than 1 ohm as we know of today, and by about -0.05 ohms = five-hundredths of an ohm, and was determined by a specific length, diameter, and temperature of a tube filled with mercury liquid which can conduct electricity. Since mercury is now understood as being a toxic substance, this unit is no longer used after about the year 1890, but rather the ohm unit is now used for the resistance to the flow of current and-or electricity. Perhaps you will find an old book where the Seimens unit is mentioned, and the above reasoning will help you comprehend and-or convert its value to its modern equivalent having ohm units.

How Electric Current Is Generated In A Generator

A changing magnetic field, such as through the coils in a spinning electric (electricity) generator, can be used to cause ("induce" or generate) electrons to be free (from their atoms) in a wire (or coil of wire) so as to then have an electric (electron) current (flow, movement) in a circuit. The magnetic field force will give kinetic energy (ie., J/C, joules of energy per coulomb of electrons, and this is defined as Voltage) to the electrons in the wire of the spinning coil (or sometimes the magnets are rotated, rather than the coil), and an external electric circuit will allow those energized electrons to flow. The (alternating, and in a sinusoidal wave) voltages produced from a generator and-or output from a transformer can be very high and dangerous, and can easily force a relatively low current (due to high resistance) through long distance wires to the destinations of use. At the destination(s) for public use, the voltage available at the electric power transmission lines is reduced ("stepped down") by a device called a **transformer.** An electric (power = voltage and current) transformer is basically two electrically separated, but magnetically coupled (magnetically influenced) coils that will convert a high voltage, low current input, to a much safer and lower voltage (about 220v, 50hz in Europe which requires less current per same amount of power needed, or about 120v, 60hz in many American countries, and with 127v in Mexico) output and with a higher current ability than that normally found in the long transmission wires. Long transmission wires have a certain resistance per meter or mile and therefore have an additive or accumulative resistance and would waste (as heat) much of the power generated and available for use. When the resistance is higher, the current is reduced, and the power loss becomes more significant because power is related to the square of the available current value. The voltage on a long distance transmission wire may be about 10kv to 20kv and is extremely dangerous, lethal, and can easily cause electric arcing (ie., like lightning, electrical discharges), shocks, damage and-or fires.

Power = P = (voltage)(current) = V **I** = I^2R = V^2 / R. Power is directly related to voltage (V) and the square of the current (I), but is obviously, inversely related to resistance (R) which will limit the current and then limit the power available to do things such work, and of which was initially defined so as to be measurable as: work = (force)(distance); a force applied through or over a distance. The units of work are the same as the energy needed or used to that work, and that unit is called joules (J). The higher the resistance, the less the current, energy or power available for use because it was wasted as heat (ie., friction energy) through that resistance.

The first hydro-electric (or hydroelectric) power was created by **William Armstrong** in England in 1878. Flowing water has kinetic energy and can apply pressure to the rotating blades of a turbine, much like how wind-electric power is generated. The turbine is connected to the electric generator via a shaft, however gearing may also be used. By this time of 1878, using hydro-power, water wheel power, and wind power for industrial purposes was already commonly used in many industries. The first wind-electric generator was made by **James Blyth**, of Scotland, in 1887. **Charles Brush**, from the USA made a large and practical wind-electric generator in 1888.

Experimental Long Distance Power Transmission

Before Tesla and others, **Lucien Gaulard** (1850-1888), from France, and **John Gibbs** (1834 - 1912) as a financier, from England, conceived of a practical way to transmit power using electricity and long distance power wires or lines in 1881 and of which the Westinghouse Co. would later buy in 1885. Gaulard and Gibbs invented a different type of transformer and-or design that could handle large amounts of power, and of which William Stanley of Westinghouse Co. would later improve upon. This transformer was of course an AC electrical power transformer, and this was also made to reduce (or step if needed in a house, factory or business) the high transmission voltage down to a lower, practical and more safe level. Later, others would improve upon Gaulard's transformer and replace some metal parts with layers of laminate metal so as to reduce internal, eddy [side, edge of, opposite] current, power losses, and specifically as heat energy.

By using transformers as part of power distribution, this is a much more efficient version of using high voltage (AC) power lines for an electrical power distribution system that we still use in these modern times, and it is mostly credited to **Nicola Tesla**, and who also made many other significant advances such as in motor design, particularly, inventing the (AC induction) motor that can be powered by AC power. AC means alternating current that is created from an alternating (sinusoidal, reversing polarity ("back and forth") periodically) voltage and current source created, often in a spinning electricity or "electric power" generator. A generator is much like a motor, but a public electric utility generator is designed to produce output power (= voltage x current) that has a form of a high value of voltage and a low current such as for the transmission of it. An electric (powered) motor is rather designed to use that generated power in the form of a much lower voltage and a higher amount of current. When more current is available, an electric motor can develop a stronger magnetic (force) field in its windings or coils and so as to have more output torque ([mechanical] rotating force), rather than high speed, to rotate and do things such as work, perhaps for an electric drill or water pump.

At a distant city from the electric power generation plant, the high voltage output at and from the power station is "stepped-down" (ie., reduced) to a more manageable voltage by using a ("step-down") transformer. A common AC power supply for households is 110-120v single (1) phase AC wave with a frequency of 60hz, such as used in Canada, Mexico, U.S.A, and many other western countries, and with most of the other countries using 50hz, 220-230v single phase AC wave for their households. It should be noted that the higher the frequency of AC electricity or signal, also means more power or energy per second available, and so with the lower frequency 50hz AC electricity, the voltage generated must then be higher so as to deliver the high needed power, and that thinner, lower current ability, but lower cost wires can be used. As mentioned previously, electric motors and generators are nearly identical, and some homemade generators may be constructed by modifying an electric motor, particularly the coil(s).

Nicola Tesla, (1856-1943) was an immigrant from Serbia in eastern Europe, and then to the United States of America in 1884, and worked as an engineer for the popular and significant (inventor) Thomas Edison Company., and later with the Westinghouse Company. Tesla made many significant improvements in many electric devices and concepts, and made discoveries such as with ("2-phase" or 2 electric sine waves) alternating current (AC, 2-phase, 4 wire, and 3-phase, 6 wire system in 1888) powered electric motors, radio (electric radiation and transmission to a receiver) and remote control via radio, xrays, high-voltage coils and transformers, etc. Tesla was a master of the electrical-magnetic transformer so as to convert or transform the voltage and current of electrical power (Power = voltage x current). A "Tesla-coil" is essentially a very high voltage transformer than can easily produce many visible sparks of light, much like man-made lightning bolts. Due to Tesla's initial discovers and inventions, he is generally credited as the main creator of the modern AC long line and distance electric power distribution system to factories, businesses and households. The key to this system, as indicated previously, is using "step-down" transformers so as to convert a much higher generated voltage to a more practical and safer lower voltage. Voltage can be thought of as the potential or kinetic energy of the electrons in the current, and-or as a measure of the force applying energy or power to the electrons and pushing and-or pulling them through the conductor metal (ie., wire). The more electrons or current, the more energy or power being transferred.

A contemporary to Nicola Tesla, named **Mikhail Dobrovolsky** (1862-1919) of Prussia (Poland-Russian, once a German held region) soon took Tesla's 2-phase discoveries further, and is credited to inventing a differently constructed 3-phase AC generator, and a 3-phase, 3-wire AC transmission or distribution system in 1890. In these generators, there are three coils and where each coil produces one of the three phases or sine waves, and these coils are physically offset by (360°/3) = 120° This is what most of the modern world still uses as of the year 2024 for its electric ("grid", public serving utility) household (ie., home, residential, personal) AC power. The three coils used inside the transformers for this generation and-or distribution system used an internal "star" (or "Y" shaped), or "delta" ("D" shaped, triangle-like connection) connection. A three phase generator is more efficient at producing electric power than the two phase generator, and since the average amplitude or voltage of all three of the AC waves or "phases" at any one instant is greater, and the output power (P = VI) is therefore greater. Note that a typical household or home is still normally only supplied a single phase (usually 110V or 120V peak in the USA) of AC electricity from the local area, (voltage) step-down or reduction transformer. A circuit connection of 220V peak is sometime available for a high power machine such as a washing machine or dryer. Dobrovolsky is also credited to the "squirrel cage" AC motor design that uses segmented metal pieces instead of larger pieces, and so as to reduce the unwanted (magnetic field caused) "eddy" (ie., side or residual) energy storage and resulting wasteful (back-emf [ie, a reverse voltage] caused) current reduction, heat and other losses, and this improves the efficiency of the motor and power available. It is also of note that the frequency (ex., 50hz, 60hz, or some other and-or adjustable frequency) of the AC power to an AC motor will determine the fundamental speed (ie., RPS, RPM) of that motor. In general, a 3-phase, AC electric motor has three coils connected in such as way that a two pole (N and S) magnetic field is created and which will effectively rotates due to the applied AC power, and this will then rotate the rotor shaft of that motor.

A contemporary to Nicola Tesla, named Karl Steinmetz (aka., **Charles Steinmetz**, while in the USA), (1865-1923), from the Prussia area that is now located in Northern Germany and Poland, made many practical discoveries about AC electricity, and made some of the equations for it to be more practical and learnable. He gave us the understanding of the phenomenon of (magnetic) hysteresis in electro-magnetic motors which contained magnets. Hysteresis is when an electro-magnet briefly retains some of its same magnetic polarity after the current has switched to the opposite direction such as what commonly happens with AC (Alternating, changing direction, Current) electricity. If hysteresis can be reduced, it will improve the efficiency of motors. He is honored by the IEEE (**I**nstitute of **E**lectrical and **E**lectronics **E**ngineers). Charles was born with a form of dwarfism and a genetic [DNA]) physical handicap and-or (bone, osteo) disease. After his education and several initial discoveries, he went to the United States of America in 1889 and worked as an engineer for the (Rudolph) Eickemeyer Company that specialized making (AC) transformers, and then worked with the General Electric Company so as to further develop both the science and application of electricity.

Edith Clarke (1883 - 1959), from the USA, was one of the first women to be recognized and (formally) be certified as an electrical engineer by the IEEE, and is honored for her practical work, calculation charts, and significant educational writings about AC electricity, circuits, and AC power transmission so as to help make them more understandable and practical for many to study and comprehend.

There are many electricity generators available and are intended to be powered by certain forms of input energy such as wind, water, and hand cranked, and each type of generator has a certain maximum rated output power (Pw = V I) . There are small portable ones which hikers, campers and remote locations without grid power can use. Some of these can charge a capacitor and-or batteries so as to store energy for when it is needed, such as for an emergency radio and-or light. It is even quite possible to rotate a fairly large electric generator of a few thousand watts output by using an automotive (car) wheel that is being powered by the engine, and this powered wheel is not on the ground, but a few inches above it so as the car will to remain in place for safety and to stay connected to the generator. It is possible to use some electric motors as generators of electricity.

Many of the people mentioned in this book about electricity, and many others not mentioned, are mentioned in a new book called: **The Lightning Tamers , by Prof. Kathy Joseph** as its author**.** It is a book about their lives and times, and how it then interacted and contributed with their important discoveries about electricity. It is a rare type of book to create and publish, and she has made it available for us so as we may better understand the science of electricity from its origins. This book is available on some popular e-commerce store websites. She also has a Youtube.com video channel, and a website called KathyLovesPhysics.com

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RADIO

Radio is a word derived from the radial-like (radius from the center) direction of the outward transmission or radiation of electromagnetic energy from a central source such as from a metal wire antenna with electric energy applied to it. Visible light is one form of electromagnetic radiation and is of the (radio or electromagnetic) frequencies that are visible to the human eye as colors. Most radio frequency (RF) waves are invisible radio waves such as used for the electronic transmission of audio (sound) such as for a radio receiver and its audio speaker, and-or for visual images as for a TV receiver and its viewing screen. The telegraph and-or phone system requires a direct electrical connection (with a metal wire) to the people communicating, but radio communication (ie., "wireless" communication) does not require a direct electrical connection, and is therefore available and cost effective for many more people, and-or for all at the same time such as with standard TV (television). This discussion is mainly an introduction to radio, and some of the technical or specifics of it are discussed further ahead in this book.

Ever since electric sparks were created, invisible and unknown radio wave energy was also automatically created and transmitted, and it was also not known at the time that these waves could pass though non-metallic things such as a wooden box or wall that is not too thick and-or made of metal. It was also not then known of how to sense or receive this invisible radiated (ie. radio) energy. A high voltage spark of light through the air is visible, but was there also some invisible (radio) waves also being transmitted at the same time? In 1888, **Heinrich Hertz** (1857-1894), from Germany, detected the first invisible (or radio) wave, by using a receiver (ie., antenna) that was composed of just a single loop of wire having a very small spark-gap and (initially) placed just a few feet of distance away from the electric spark-gap The transmission spark (which created the (invisible) radio wave energy) used was made using a high voltage induction coil, which is a type of transformer that can also raise the voltage of the input signal, and so as produce a high voltage output signal, but actually at a lower maximum amount of current since the energy output cannot be greater than the energy input. That is, (energy out) = (energy in), and-or (work out) = (work in) or (power out) = (power in) if there are no losses of power along the way. Modern radio transmitters and-or receivers do not use a spark-gap, but rather use much more modern and sensitive radio wave circuits. The electric spark and spark-gap in this type of application became known as the (electric) spark-gap (electricity, energy, radio-wave) transmitter.

Hertz probably got his idea while considering Faraday's discoveries and that an electric current pulsed wire and-or coil could transmit a magnetic field, of which each (transmitted) pulse can be detected, sensed or received by a (nearby) magnetically sensitive compass. Would a spark be an indication of some type of charge or energy being radiated? Hertz then made a receiving antenna to find out. The metal antenna received and collected just enough (a small fraction) of the transmitted energy so as a small spark was visible across its own spark-gap or end terminals. This discovery started a revolution in electric oscillators (ie., electric signal [voltage, current, frequency] oscillators that use or are powered by electricity) and wireless communication such as with radio, and eventually TV, and as of the year 2022, it has resulted in what are called "smart (tele-) phones".

One of the first practical radio wave detectors or sensors created and used was called a **coherer**. It was invented in 1890 by **Edouard Branly** (1844-1940), in France, and the scientific fact of fine metal dust particles being affected by a high voltage had been known for about 50 years previously, but it was of no practical use yet. To imagine this effect, consider how amber and static-electricity attracts small lite-weight bits of material. Branly made the first practical (ie., here, useful for something) coherer device, and used it to help sense or detect a received, relatively weak, radio signal collected by the nearby antenna attached to one of the coherers two end terminals. Branly's coherer used iron dust particles. It was basically loosely packed metal bits that would cohere (ie., combine, cling or stick together) together in the presence of (electro-magnetic) radio wave energy transmitted by a (on-off, radio pulses) high voltage, spark-gap radio wave transmitter. This then created a less resistive, more conductive path through it - presumable through an oxide layer after it electrically overcame its high resistance and conducted, and therefore behaving like a electrically activated, remote controlled switch that then allows electricity to conduct or flow through it to the rest of the electric circuit. The circuit may include an electric buzzer (sound) device, or the pulse heard in sensitive earphones, and-or be recorded with a Morse paper tape recorder which used a wound reel of paper to record the received (pulses or radiowave energy) Morse Code encoded signals and information. The coherer device then had to be physically tapped by the radio operator so as to reset it so as to be in the receiving mode again. Later, an automatic tapper circuit with a (electr0-magnetically shielded) relay or electromagnet that could force and move a small metal tapper (ie., a "decoherer") was used to automatically reset the coherer back into receive mode. After about this time, others made improvements to the coherer device, and they were notably made by **Oliver Lodge** who used it to study radio or "Hertzian waves", **Sir J.C. Bose** (iron and mercury, liquid coherer, and which did not need a decoherer, and was fast), **Nicola Tesla**,, and **Guglielmo Marconi** used a coherer for the first wireless (radio-wave) telegraphy system. Marconi's radio-telegraph system was made practical by using Bose's improved type of coherer.

Crystal detectors (ie., various forms of solid-state diodes) and the vacuum tube diode would eventually replaced nearly all of the mechanical-like coherer devices, and because they were much faster and sensitive to the weak (low energy) radio waves received, and could later demodulate (ie., retrieve) the information and-or audio that was transmitted by the aid of a radio wave. A coherer was susceptible to lightning and other radiowave noise (ie., [electromagnetic, electronic,"rf"] interference) that would cause unwanted radiowave detection and-or information losses along the way.

In 1903, **Reginald Fessenden** (1866-1932), from Canada and America, developed an electrolytic detector (basically, a bare or uninsulated thin platinum wire with its point or tip in a liquid acid) and called it a bare wire detector called a "barretter". This radio wave detector was more sensitive, fast, and could actually be used to demodulate voice-signal, amplitude modulated (AM) radio waves much like how a solid-state diode demodulates AM radio signals today. The technology of AM radio is given a brief discussion in this book. Besides the crude-like coherer that as discovered previously, Fessenden's solid-state-like discovery showed that various elements and their interaction with radio waves and-or electricity could detect radio waves, however it was still a mystery of what was exactly happening.

The technology of radio wave transmission and reception was improving rapidly after 1900, and rather than keep using pulsed radio waves without much frequency tuning and control, and then also receiving much interference of the radio signals, radio waves began to be tuned to be a specific frequency and were automatically and constantly made and transmitted due to fast electrical oscillation (repetition, here of making radio waves at a certain frequency) or oscillator circuits. Amplitude modulation (AM) of radio waves uses an input or control signal, such as an audio signal, to directly vary the output or broadcast signal's amplitude or power, and according to the amplitude of that input signal. AM began to be known as "continuous wave" radio and having one main, limited or narrow frequency bandwidth (ie., frequency range) associated with it. If a transmitted radio wave a has tuned, set, main, specific or fundamental ("carrier") radio frequency, then the radio receiver circuit must also be "tuned" or adjusted so as to receive that specific radio energy or wave frequency, and for the maximum or best energy transfer from the transmitter. Fessenden was greatly involved with the initial development of **radio telegraphy** (ie., Morse code via radio signals, such as for ships, and before voice via radio) communication and AM radio (communication) of which was first demonstrated in the year 1900 as being able to transmit human speech (audio).

Sound Recording

Thomas Edison (1847-1931), from the United States of America (USA), and his company made many significant advances in technology, such as for the first practical electric light in 1879. Since Edison used a filament within a vacuum, rather than regular air, it did not degrade or (slow) combust like the carbon electrodes in an arc-lamp. Edison also invented a completely mechanical, audio (ie., sound) recording and playback machine in 1877 called a **phonograph.** It was sometimes called a "talking machine". A phonograph could both record and mechanically play back sound vibrations and-or imprinted "bumps" on the surface of the recording media such as aluminum-foil, and later, wax was used which allowed a better quality of recording and later, reproduction of the recorded wax cylinders for mass production and distribution. These mechanical machines were hand-cranked powered, and nearby, loud sound was recorded and without strict timing and-or revolutions per minute. To hear the recorded audio at its proper frequency, the recording needed to be played back the same amount of revolutions per minute (rpm) as it was recorded with, and it was often about 120 rpm (ie., 120 rev / 1 min = 120 rev / 60 s = 2 rev / s). A recording standard was later recommended and used, and it was 160 rpm which improved the higher frequency recording range up to about 5000 hz., making it more suitable for accurately recording high frequency musical instruments. At those times, due to the lack of electronic amplification for both recording and playback, a sound needed to be relatively loud to record it, and sound collectors shaped like funnels were used to concentrate sound so to increase its intensity or pressure for the mechanical recordings. Years before Edison's sound recording and playback machine, there were some crude inventions and recordings of sound waves (ie., air or mechanical vibrations) and seismic waves (ie., earthquakes. seismograph recorder) onto paper using ink, and with no invention yet invented at that time to play (ie., convert to audio) and hear these recordings. The first sound recorder of this type was made in about 1857 by **Edouard-Leon Scott de Martinville** , from France, and the recording device is called a phonautograph. Paper recordings are still needed and used in the modern world so as to mechanically record many types of data and their values (ie., typically their signal amplitude) and the time they happened. Recordings of electronic and-or digital input signals to created a recording (ie., in electronic memory) composed of the data are also becoming widespread, and that a microprocessor or computer device is needed to display the data or results on a screen, and vast amounts of data can be easily analyzed by a fast computer so as to produce a result. Most likely, Martinville's recorder inspired Edison to make his sound recorder and playback machine. Before Edison made his version of a spring motor to power a phonograph and at the correct speed in 1898, there were several other previous researchers in this endeavor for the phonograph such as: Edward Amet in 1891, for a speed regulated spring motor, and Levi Montross in 1896.

In the times of Edison, the concept of "motion or moving pictures" to photographically record and display objects in motion was around for many years, but he made the first practical **motion picture camera,** and of which began to set the standards for motion pictures or "movies"**.** He also invented a greatly improved **microphone** (converts an input sound energy or signal into a corresponding or simulated electric energy signal) in 1876 which was then used to significantly improve the existing, but relatively crude telephone system created by Alexander Graham Bell. Edison is generally credited to the invention of the carbon microphone in 1877 which also allowed larger (ie., in amplitude, louder) audio signals to be used with the telephone and other devices. Noted credit to the invention of the carbon microphone at about the same time as Edison is also given to David Hughes from England, who also invented a telegraph system that printed telegraph messages onto paper tape, and to **Emile Berliner** (1851-1929), from Germany-USA, who worked with Edison in 1876 and created a better carbon microphone in 1878 that Alexander Graham Bell would use to significantly improve his electric telephone system. Berliner would later, in 1887, invent the ("Gramophone" phonograph) flat or disk phonograph recordings (commonly called as just "records"), and he also invented a significant record playback reproduction method for mass distribution of audio recordings or records. This technology was a vast improvement over the relatively soft and degrading wax recordings made previously by both Edison and Bell. Berliner created a process to record the audio into a thin wax coating on a metal disk, and then electroplated it, and this resulting metal disk would then last much longer than (flimsy) wax cylinder recordings and any possible (degraded) re-recordings of it. This metal disk was not actually played back, but was used as a type of mechanical die or imprint that could then be used as the source or mold to stamp (ie., press, imprint) exact (but reversed) copies of itself onto a heat-softened plastic that would soon harden firm when cooled. These recordings would last much longer than wax would when a vibration sensor called a "pickup needle" was physically dragged through its groove (ie., a long indentation) as the recording was played back so as to hear it. These disks were to be heard or played back using a rotational speed of 78 rpm (revolutions per minute), and have a recording that had a maximum of about 5 minutes of time duration or length. These disks originally were 7 inches in diameter, but later, and mostly, the 10 inch and 12 inch diameters were used. As of 1948, and created by the **Columbia Record company**, similar 12 inch diameter types of records having a longer recording and playback (ie., "listening") time and having improved sound or "audio fidelity" quality were called LP (Long Play) records. These records were to be played back at 33.3 rpm using a sensitive electromagnetic needle (physical-mechanical vibration "pickup", sensor and transducer to an electric signal) so as to then be heard louder by using an electronic sound (signal) amplifier. Each side of these records had a recording and-or playback time of about 30 minutes maximum, and therefore these records could contain a total of: (30 min / side)(2 sides) = 60 minutes of audio. If the rotational speed (ie., rpm) and-or listening or "playback speed" of a recording is changed by a factor of (n), all the frequencies of that recording will be changed by that same factor of (n). For example, frequency of (x) cycles per second will become (n)(x) cycles per second (ie., cps = hz).

In 1948 the **RCA Victor Co.** invented the 45 rmp, 7 inch diameter plastic record, and which was often used to sell "single" (ie., one song per side) recordings of which were often the most desired or popular songs found on the more expensive LP records. These relatively small disks were also used with coin-operated (ie, pay first) playback machines, often called a "Juke Box", that had many available songs on these small disk records inside it, and for a coin fee, it could play a desired song using its internal record player and audio amplification. As of the year 2022, a few of these original machines could still be found working, and usually inside a restaurant, bar or night-club. They are quite collectible, especially if they contain old recordings. A popular and-or song being promoted was usually placed on what is called the A-side of the record, and the B-side of the record was generally a song that still needed to be heard, but it often was not promoted and -or it did not have much of a chance of being a popular or "hit" record, but there are some exceptions to this.

The available magnetic type of audio recording technology, and available before the magnetic tape technology, was called "**wire recording**", and where a long and thin iron wire on a spool could stored a corresponding magnetic image or recording of the input sound. This wire was moved very close to and past a "recording head" (ie., an electric to magnetic transducer), and the wire became locally magnetized there, and having a magnetic representation of the input audio signal (amplitude [amount or level of power] and frequency) at that location on the wire. This technology surely inspired the magnetic tape, "(tape) reel-to-reel" recording technology later. Wire-Recording was invented in 1898 by **Valdemar Poulsen** (1869-1942), from Denmark. This technology improved greatly over the following years, and especially with advances in microphones, speakers and electronic recording and amplification. This was the first form of electro-magnetic recording technology and of which would later find much practical use for recording or storing audio and-or video (TV signals or recordings) onto magnetically sensitive tape, and for storing vast amounts of digital (ie., computer data) onto magnetically sensitive "floppy" (flimsy, soft) or hard metal disks before the modern (2022) times of very high capacity digital (ram) memory made from small (ex., 1 sq. cm of area) integrated (IC) circuits on pieces or "chips" of silicon.

The first practical magnetic **tape recordings** and technology were initially invented and created in about 1928 by **Fritz Pfleumer** (1881-1945) in Germany, and it became affordable and popular in the late 1940's due to its high quality recording and playback sound, and its long total time possible to record and-or and playback sound. The word tape used in this context is a thin real or roll up of media that is plastic with a coating of fine iron particles which can be magnetized and record magnetic signals. This also made it practical for the average recording studio and-or person to record and edit sound. If a mistake was made, the same exact tape could be erased and-or directly recorded over, again and again until it was satisfactory. These recordings could then be used as the audio source so as to then make a relatively inexpensive (plastic) record (ie., recording) for the public. Cassette tapes and the associated audio recording technology (ie., portable playback machines), having a narrow tape width were a very popular, very low cost recording format in the late 1960's to about 1990. (Magnetic) tape recording technology places the electronic audio image or representation onto a long paper and (usually) plastic tape that has been coated with a fine dust of iron particles. This recording tape was much wider than a thin wire, and therefore, it had a higher possible audio quality, especially with the advances in audio sensor-transducer or microphone technology and electronic amplification.

In about 1982, recordings began to be made onto **Compact Disks (CD's)**, and this technology which is a mixture of the older (rotating) disk-record technology and the newer digital technology, and here, the (audio and-or computer digital data) information and playback sensor is a small LED laser. The CD or "optical disk" recording medium is specially constructed, thin (1.2mm) plastic disk having a diameter of 4.75 inches = 120mm. **James Russel** (1931 -), from the United States, is credited to the initial invention of the CD concept and its technology, and the **Sony Co.** and **Philips Co.** jointly created the production standard called the "Red Book CD standard". This audio standard is 16 bits (ie., 2 bytes long or "wide", having an equivalent decimal steps or range of values from 0d to 65535d) of data for each audio sample and storage of it, and a total of 44100 (ie., 44.1kHz) samples per second. Each sample is essentially the voltage level of the audio signal and-or waveform at that brief instant it was sampled and-or measured. The sample value is about twice as high as the theoretical 20kHz maximum frequency of human hearing, but it is necessary so as to still be able to effectively detect and record those high frequencies with reasonable quality. Lower, and most, frequencies in common music sound would naturally be recorded with much higher accuracy, representation or quality than the very high frequencies. It is of note that recording studios will probably use a higher frequency for the digital or analog to digital sampling and then reduce it to 44.1kHz for the CD standard. The CD standard also allowed a maximum of 74 minutes for each separate track of stereo (L and R recordings, Left and Right, dual, two tracks) audio recording and-or playback time. 74 minutes = (74min) (60s/min) = 4400 seconds. These CD's are also capable of storing 640Mb, or slightly more, of (digital, computer) data such as images, documents, programs, etc. Using a laser, a CD's digital (ie. a logic and-or numeric 1 or 0) data is actually burned through the inner foil layer as either a tiny microscopic hole or absence of one onto a "track" (ie., the spiral and location of data) or memory address location within the disk of the CD medium. Although much of the finalized, usable invention of CD optical recording and playback technology is credited to **James Russel**, some ideas about optical digital recording were already being thought of by some others working for the **Philips Co.** as of the late 1950's, and also by **David Gregg** (1920-2001) while working for a **Western Electric** partner company called **Westrex**, and he even filed a patent for a videodisc concept in 1962. Russel began his optical recording research in about 1965 and by 1973, he developed a prototype version of it and offered it to companies to consider producing it. Many personal computers before about 2022 had CD readers and-or writers ("burners"), but their use is fading because of the newer, completely "solid-state" (having no moving parts to wear out), all digital medium, and having a very high density (many bytes) or amount of memory. This memory is RAM [rewriteable], Random Access Memory) data IC's (integrated circuits "chips") inside many electronic devices. A RAM chip is inside the relatively small USB (Universal Serial [data] Bus) memory sticks that are often called "thumb-drives" for digital data storage. **DVD** (Digital Video Disk) recordings are an advanced form of CD data storage, and a DVD could contain up to several times the data as a CD could.

As of about the year 2000, music and-or other (computer or digital) data can be recorded, stored or saved onto non-mechanical memory IC's (ie., integrated circuits, or "chips") devices, and one common method and-or portable device is called **USB** (Universal Serial Bus having a standardized (bit by bit) serial data connection so as to store data to or from a computer) memory card, a "thumb drive" or "memory stick". A particular modern USB memory stick can hold from 1 GigaBytes (GB, = 1 billion bytes) to perhaps 128 GigaBytes of binary and-or digital data. If this device gets lost or stolen, much valuable data, perhaps proprietary, company, employee, family, friends and personal data is then at a security risk, and you may wish to use your computers software (ie., perhaps a Windows program(s)) so as to secure folders of data on your device(s), and to also make and store "backup" copies of your data. If possible, include a readable, plain text file in the data on the device, and which includes your phone number so as the device may be returned to you, and-or write or engrave your number on the device.

A dynamic (or "magnet and coil", electro-magnetic) microphone was invented in 1877 by **Ernst von Siemens** in Germany and it is very similar to what we know of today as a small loudspeaker, both electrically, physically and mechanically with its suspended, movable coil of wire around a fixed in position permanent magnet. The first practical **loudspeaker** used a suspended or movable coil (about the magnet) and cone, and it was made in 1925 by **Edward Kellog** and **Chester Rice**. In general, it turns electrical current into a varying magnetic field which will interact (attract or repel) and moves in relation to the fixed magnet of the speaker. **Edward Wente** is also credited to inventing the loudspeaker at about the same time. To make speaker, inductor, generator and motor coils, a special wire called **magnet wire** is used, and it is made from copper or aluminum metal and coated with a material to prevent it from electrically shorting with other parts of the coil and-or wire. It is called magnet wire since it is used to develop a magnetic field in it and without it remaining so as it can be used in high frequency circuits. The dynamic loudspeaker and dynamic microphone have similar construction.

Today (as of the year 2021), **condenser microphones** with great audio input sensitivity (ie., able to detect small signals) have mostly replaced the use of the more rugged carbon and dynamic microphones. This microphone consists of a capacitor with two plates, and with one plate that has low mass and weight and is very sensitive to sound (audio) waves and could easily vibrate or move in unison to sound. Because a plate of this capacitor can move, its capacitance value will vary and which will then cause its output electric signal to vary at the same frequency and amplitude as the received audio. The condenser microphone was first invented by **Edward Wente**, from America, in 1916. As of about the year 1995, condenser microphones are in great use as for computers and phones. There are also other types of microphones available, such as the (thin, metallic) ribbon microphone, and which is sometimes used in the recording industry.

Before the invention of the first audio tube amplifier, some principles of the carbon microphone were sometimes used to make audio (low frequency) amplifiers before the invention of tube (or "audio" or "[electronic] valve") amplifiers. The concepts of the change in resistance and-or conductance of the microphone so as to amplify sound would also later be found as part of the understanding and functioning of the semiconductor in a transistor amplifier. Before the transistor, hearing aid sound amplifiers used a small **carbon amplifier**.

How Edison's Electric Lightbulb Made Electronic Signal Amplification Practical

In about 1875 while Edison was experimenting with his **electric light**, he notice an effect that the electricity (ie., current, later to be called as electrons or electron flow) from the hot filament would flow to a (voltage applied) charged metal plate inside the tube or light-bulb, and also having a much higher current when the plate had one voltage or charge polarity than the other. After applying a reverse polarity voltage connection, this resulted in no current flow, hence there was a very high resistance to the flow of current in one direction, and a low resistance to current in the other direction or polarity. Edison therefore essentially created the first man-made electronic "one-way" (ie., direction of flow) valve or diode for electric current, and he did patent it, but he had no immediate use for its application since he was more interested in perfecting the electric light. This "**Edison Effect**" (ie., technically a diode with the **diode effect**) would later be improved and used in a radio receiver as the first practical and reliable (vacuum tube, "tube") diode or **(**thermionic [thermal, hot filament, electron radiator], electronic, not mechanica**l**) "valve" that was practical and formally presented in 1904 by **John Ambrose Fleming** (1849-1945), from England, and while working for the Marconi Company, and which he had helped make the equipment for the first radio transmission across the Atlantic ocean in 1901**.** Before this time, Fleming once worked for the Edison Company in the United States of America and was well aware of the discovery and basic knowledge of the Edison Effect. The input signal to this diode device is applied to a hot filament wire, here as cathode (-) side, and which radiates electrons (here, by thermionic (heat) emission, kinetic energy and electrostatic repulsion by other electrons) as current to a positively charged anode (+) plate in the same vacuum. This diode allowed the practical rectification and-or detection of high frequency radio waves as needed for radio reception, and since this was before practical semiconductor (PN) diodes were understood well and manufactured for practical use,and without the need for properly setting a sensitive "cat-whisker" detector diode. This diode tube is essentially a descendant of the Geissler Tube and Crookes Tube. It is unclear if Fleming new about the invention of the mercury vapor lamp technology invented in 1901, and the mercury arc rectifier in 1902 by **Peter Hewitt** (1861-1921), from America. This was not a sensitive rectifier like the Fleming Valve (diode) is to weak radio signals, but was rather for a more practical (electronic, not mechanical) method of the rectification of high AC current from electricity generators into a high DC current. Again, even this is a descendant of the tubes made by Geissler and Crookes.

Before the electric diode or (electric) "valve" was invented and made practical so as to maintain and-or create a voltage polarity and-or current direction, mechanical (diode-like) methods were used such as the commutator which was a mechanical or physical electrical connection near the rotating shaft or axle of rotating motors and-or generators. Though these could safely handle large amounts of electrical current, they were slow and impractical in terms of being useful for the high (AC) frequencies used radio signals, and not then being able to rectify them to be a form of DC.

Before Edison's incandescent electric light, many already knew that electricity through a material or that a high voltage arc could produce light, but Edison's electric light filament (a hot, light radiator or emitter) was placed in a vacuum so as to not burn (slow combustion) quickly. His filaments of choice was a carbonized wood fiber made from bamboo, and a very thin platinum wire. In 1875, **Joseph Swan** (1828-1914), from England, independently created a practical incandescent light bulb at about the same time as Edison, and his filament was a piece of carbonized plant fiber thread, and then later a drawn cellulose thread. Edison and Swan actually created a company to manufacture electric incandescent lamps (ie., lightbulbs, light-bulbs, light bulbs, "electric lights") which consisted of the best research, manufacturing technology, cost, power required, and the lifespan (average time of use before failure) of the filament and lightbulb. Swan would also invent the use of nitrocellulose plates for creating photographic images, and this eventually lead to movie film reels made from a thin (ie., a film) layer of nitrocellulose. Edison's carbon filament construction was later greatly improved by one of his science and business associates who was an African-American named **Lewis Latimer**. Various types of tungsten metal filaments began to be invented, initially in Austria, Europe by **Alexander Just** and **Franjo Hanaman** in 1904, and in United States by **William Coolidge** in 1910 with his own tungsten filament study and construction, and where he then greatly improved the X-ray tube for medical use in 1913. Tungsten has a very high melting temperature, and is therefore ideal for bright, high current, electric lighting, and high voltage produced X-rays.

Soon after the invention of the vacuum tube diode in 1904 by Fleming, **Lee De Forest** (1873-1961) from the United States of America (USA) created what would become the first reliable and practical electronic signal amplifier called a **triode** in 1906. "Tri" is a word-prefix meaning 3, and the device is much like a 2 lead diode tube, and with another lead and plate (actually a "screen" or plate with holes in it) in the middle of it. The triode was sometimes called an "Audion" or "Audion tube" (ie., audio tube, an electronic audio amplifier tube) a few year later in about 1911 since is was thereafter often used to amplify weak electronic sound (audio) signals (AC voltages and current, electric signals representing the actual audio signals (V,A, frequency, and waveform) so they could be heard as louder in volume. The first triode amplifiers had low gain (perhaps < 10), however putting them in series (ie., "cascading" them), one directly after the other like a chain of amplifiers and-or repeated amplification or multiplication, they would effectively multiplicity the signal gain (amplification) together and act as like one powerful amplifier. If two of these gain of 10 triodes were placed in a cascade amplification mode, the total gain would be: (gain of first triode) x (gain of second triode) = 10 x 10 = 100 , and this is about the average gain of a modern (power, high current capable) transistor. The triode used an electric current heated (hot) cathode (ie., a "filament") so as to give the electrons on it more kinetic energy so as to be easier to be released and repelled by the polarity of the cathode having negative polarity, electromotive force (emf) from the battery supply. The electrons would then gain more kinetic energy while being attracted and traveling closer to the (positive charged, polarity) anode or "plate" terminal of the tube. A small input signal to be amplified was applied to the terminal connected to the triode tube's internal metallic "control screen" or "grid" that is located between the cathode, hot filament (-), and the anode plate (+). The grid is then used to (electronically, electrostatically [electric forces]) control (ie., set the amount of) a much larger current (of electron flow) through the entire device, and this process is what can be called: external electronic current control. With some ("bias" or working) current already flowing through the cathode to the anode, a small or weak AC signal can then be applied to the control grid of which the electric field there would greatly affect nearby electrons, and that input signal can then be used to create an amplified version of itself by modulating or controlling the amount of current flow from the cathode to the anode terminals. When no current is flowing from the cathode to the plate, the tube essentially has a high resistance like an open circuit, and the full supply voltage will be across its terminals. When current is flowing through the tube, it will have a lower internal resistance. For maximum amplification of an input AC or sine waveform, the voltage across the tube is set using an external resistor in series with it to half the supply voltage, and this is called (voltage) biasing (ie., settings) the device, and this is also done with transistor amplifiers which came years later starting in 1947.

The triode and other similar electronic amplification technologies have been a significant technological achievement for mankind till this day because of there prolific usefulness. The triode with both its diode and amplification ability was very useful for improving radio communication [receiving (reception), and-or transmitting (sending)], and for many other devices and products such as record players with loudspeakers, and the internal electronics of both radio and TV receivers (ie. a "TV set"). The triode tube amplifier made radio communication (transmission and reception) practical for the average household to obtain a practical radio receiver and listen to the (demodulated rf) audio (sound, entertainment and news) for extended lengths of time. See [Fig 234] for a basic drawing of a triode tube.

By about 1920 transmitting sound using radio waves was commonplace and radio receivers became less expensive due to price competition, wide-scale manufacturing and availability of triodes and radio-receivers. Vacuum tube amplifiers would later be superseded by the small, low power loss, inexpensive, more reliable transistor amplifiers that were first created 28 years later in 1948. Vacuum tubes (amplifiers, etc) were still in common use for many devices such as in some television (TV) receivers made the 1970's, and even then, most transistorized TV's still had a CRT (cathode ray tube) as the viewing screen up to about the year 2000. As from about the year 2010, most TV's use "digital screens" that use low power and consist of "(LCD) plasma" or LED arrays to reconstruct a decent electronic approximation of the television camera images on the ("pixelized", ie., dots) viewing screen.

After inventing the **triode tube amplifier**, **Lee DeForest** is also credited to inventing the first demonstrable and practical optical and electronic technology of recording an audio or sound signal directly onto movie film images, and in the year 1928. This method is commonly known as "sound on film". Before this time, movies did not have sound, and are sometimes called as "silent movies", or they had an external audio recording on a record which then had to be synced (properly calibrated, aligned, in step or timing with) carefully so as to have the correct and corresponding timing with the subject and-or actors. Sometimes a piano player would play some ambient music while the movie was playing, and-or there were visible, on-screen titles, captions and credits overlayed onto and then into the final movie film images, and-or appended to the start or end of a movie. Before De Forest's work and reality, there were some other (more crude, but inspiring) sound for movies ideas and patents such as by Charles Fritts (1880's), and Ries in 1923.

In 1912, **Edwin Armstrong** (1890-1954) from the USA, while experimenting with a radio receiver with a triode tube amplifier, he invented what is called positive (ie. enhancement, additive) feedback or "regeneration" (ie., reusing), and where a small amount of the (same wave phase) output signal of the RF tuner and amplifier circuit is electrically recombined or joined with the input signal at the control grid (or signal input grid) of the triode amplifier. With this feedback, the total gain of that amplifier was many times more than what the triode could otherwise produce without it. Now, instead of using earphones to hear the sound, large diameter, high power "sound reproducers" or "loud-speakers" could be used. Today these are simply called as "speakers". If the feedback was strong enough, a radio wave frequency, electronic **oscillator** or "signal generator" could then also be created which is often a self-starting oscillator and circuit without the need for any input signal to start the process of making a (tuned, specific frequency) waveform, and this is called an **Armstrong oscillator circuit and-or concept.** This greatly improved the field of radio communication, for before this time, continuous-wave (sine waves) were limited to a relatively low maximum frequency due to that they had to be mechanically made using high speed alternators, or by relatively crude, low frequency electro-mechanical oscillators which worked well for musical organs (electric piano's).

Armstrong was one of the contemporaries of engineers who invented the concepts of superheterodyne for radio receivers. Heterodyning (ie., to create a difference by force) was previously developed by Reginald Fessenden and is a type of advanced tuning concept that is more stable, efficient, but it requires more circuitry such as to combine (ie., "mix") the received AM carrier signal with a fixed high rf frequency sine wave so as to produce a fixed or constant "intermediate frequency (IF)" signal that is the difference (ie., the "beat frequency") of those two combined signals and which is then demodulated (ie., to remove) so as to have just the desired audio signal. **Heterodyning** allows the radio receiver to be more sensitive to the (weak) received radio waves, and more "selective" by offering a more precise (ie., a limited or narrow frequency bandwidth or range) tuned signal with little interference from other radio stations that might be using a transmission frequency just a little higher or a lower than the one desired or selected. Armstrong also found a way to stop the high frequency oscillations in a regeneration receiver, hence this allowed even more amplification, and therefore, he invented the super-regeneration concept and receiver.

In 1933, **Edwin Armstrong** made the first practical "wideband (bandwidth, frequency range)" **FM** (Frequency Modulated radio) transmitter and receiver which has nearly no audible noise, such as from "static electricity" interference caused by lightning which is commonly heard as noise during AM (Amplitude Modulated radio) reception which is based on the amplitude or strength of the received radio wave energy, and lighting will greatly affect that strength, both increasing and decreasing it. FM (Frequency [changing slightly] Modulated radio waves) is often used for "high fidelity" (low noise, "high quality") radio communication. Frequency modulation is when the input information or audio signal slightly can vary the carrier or main radio frequency to be slightly higher or lower in value, and which corresponds to the input signal's amplitude. In an FM receiver, the frequency changes in the tuned or desired ("channel") frequency are converted back into changes in audio amplitude. In general, most FM transmitters are designed to be of lower transmission power than **AM** transmitters, and therefore have a shorter range or distance of radio transmission. This also prevents possible reception interference from other distant (FM, wideband, frequency range for each channel) stations that may be assigned to and using the same frequencies. The standards set by the **FCC** (Federal Communications Commission of the USA) allow AM radio signals to have a maximum allocated frequency bandwidth (ie., total change in frequency) of 10khz and with 5khz below the main carrier, center or "channel" frequency, and 5khz above it. FM radio signals are allocated a 200khz frequency bandwidth and with 75khz below, and 75khz higher than the main assigned carrier frequency. Since FM broadcast stations and their signals and are allocated a wider frequency bandwidth than actually needed, the full audio spectrum of 0hz to 20000hz is easily included into its rf signal, but this also greatly limits the total number of (nearby) stations allocated to the complete, but limited, FM frequency bandwidth communication allocation.

Some "primitive, solid-state-like" or semiconductor discoveries and devices were made years before the first transistor amplifier was discovered, and some of these are discussed next.

In 1874, in Germany, Professor **Karl Braun** first noticed an electrical diode action in certain (semiconductor) crystal materials. A diode is the name given to any device that only allows current in one direction, and is the electric equivalent or analogy of a special (mechanical, force sensitive and operated) valve that is used to permit only one-direction of a gas or fluid flow in a pipe (conduit). This type of mechanical valve is often a "ball-valve" due to its internal construction with a ball that will essentially block the pipe and fluid flow if that fluid goes in the opposite direction for some reason, such as some pressure applied to it someplace. The first non-manufactured, natural and-or primitive, solid-state light emitting diode (LED) was discovered by the British scientist **Henry Joseph Round,** (1881-1966), from England, in 1907. Round worked for Guglielmo Marconi who made many advances in radio (ie., "wireless") communication. Round is also noted as independently discovering the triode amplifier tube, and then positive feedback independently at about the same time as Lee DeForest and Edwin Armstrong. In 1907, Round noticed what is called today as "electro-luminescence" (electrically caused) light being created when a voltage (and a resulting current) was sometimes applied to some locations on certain crystals of minerals such as silicon carbide. Round noticed it was possible to make various colors of light depending on the crystal used, and blue light was one of those colors of which would later prove difficult to manufacture, but it was also surely known to be very possible due to Round's work. Round's observation and discovery was significant, for today, we have LED's in nearly all electronic devices such as in electronic (TV, computer, phone) viewing screens composed of thousands of miniature LED's acting as picture elements (ie., a pixel, an image piece). Surly, a thought of those time must have then been if the yellow glow from a lightning bug was electro-chemical in nature.

Today (2025, and years previous) LED's can be purchased discretely or individually as what is called "LED strips" of multiple LEDs attached to a thin strip of material so as to easily place them on a wall, etc. These LED's are usually placed in parallel, and with each having a current limiting resistor in series to it, but depending on the design and applied voltage of the circuit, sometimes multiple LED's in series might compose each parallel branch. These will usually be powered by either a small, low voltage electric grid "wall transformer", or sometimes directly plugged into the electrical grid, or sometimes battery powered of which a voltage regulator is then usually used to ensure a constant voltage and current to the LED's. Since LED's are much more efficient (ie., light produced / input power in watts) than incandescent lamps, the cost of lighting a room or area will be much less, and more reliable. There are now electrical components called an "active resistor" that can regulate the voltage and therefore, the current to a LED(s), such as if the applied input voltage is changed. This device is essentially a small voltage and current regulator, and for example, it ensures that only 3.3V and about 20ma of current is applied to a white LED.

In the early days of public radio service, many ("solid state", metals of certain elements and-or compounds) minerals were often used for a radio wave detector device. A common detector device was called a point-contact-surface-diode and was informally known as a "cat-whisker" diode. This semi-man-made diode construction includes a thin, hair-like metal wire so as to make contact with the crystal at just a small specific point of which was determined by the radio-operator or listener as having good detection and-or diode action and audio reception. The concept of a special boundary or "junction" (depletion) layer between each element in the crystal and-or cat-whisker was mostly likely started here, and which lead to further understanding of the diode and the development of modern and practical solid state diodes, and then later, the BJT (Bipolar diode Junction Transistor) transistor that has its technology and understanding that is largely based on the concepts of the diode.

Since a diode can only conduct in one direction, it can effectively remove the entire reverse polarity and-or current direction part of an AC wave or signal. This is called AC to DC conversion or **rectification**. The (half ac-wave, half-wave, or half-rectified) output from the diode will still have a general form of the AC input signal in terms of the varying amplitude of that portion of the wave that was effectively passed through the diode and not blocked. Because half the AC signal is blocked 50% of it and its available power is blocked by the diode. To output a complete and-or steady DC voltage and-or signal is have a full-wave or complete AC rectification. This will allow most of the input AC available power to be available as DC power. To accomplish this will require the use of an energy storage device such as a **capacitor** that is charged up to the maximum voltage of that input AC signal, and then to release that power or charge when the diode is not conducting. The larger this storage capacitor is, the more charge it can hold and the longer it will take to be released through a circuit, and this effectively "smooths" or "levels" the DC output voltage to be a more consistent or constant value with a low value of AC "ripple" or voltage fluctuations.

An AC power source can be made using a DC power source such as a battery that has its current output being controlled and varied by several high power, "current pass through" transistors that are in series with that battery. The amount of current passing through these transistors is being controlled by the output signal from a small power, AC, sinusoidal oscillation circuit. There are premade DC to AC converters available for purchase at many and various types of stores, and these are often marketed as **(AC to DC) power inverters**, and some are capable of creating a few watts to over a thousand watts of AC power. For sensitive electronics and "just in case", a pure sine wave output inverter is better and more expensive to have than a modified sine output wave inverter. Many power inverters are for converting a common 12V automobile or car battery input power to be a 120Vac output power which is commonly used for and called as household or grid power in the USA and some other western countries. Since the output voltage peak is higher than the input voltage, a voltage "boost", "step-up" or increase is needed, and this is accomplished by using step-up transformers in the inverters circuitry. A common car battery has an amp-hour (aH = current x hours) rating of about 50aH, and this is usually not a "deep-cycle" or "deep-discharge" battery. For example(s), if the battery is still at least 12v, it can supply 50 amps of current for 1 hour of time, or 25 amps for 2 hours of time, or say 10 amps for 5 hours of time. Pmax = V max A max = (12v) (50A) = 600Wmax. rated output or load power. A typical **car alternator** (ie., electricity generator) can create about 600W of electric power at about 14V output to charge this 50 aH battery. Do not have a load of current that exceeds the maximum current of a batteries aH rating. In this example it was 50A max, otherwise higher values and heat can damage the battery. To overcome this problem, and so as to have safe, and larger power available, use a battery with a higher aH rating, and-or similar batteries in parallel. More current means thicker wire will be needed. These wires are usually very short from the battery terminals to the power inverter terminals. When the inverter is in use, it is recommended to also have the alternator charging the car battery at the same time. A 24v, 36v and 48v inverter system will require more batteries, but requires much less current, and therefore for a longer time of use at a given load current. This book has several articles and examples about electrical power inverters, and so you may do an electronic search for them when needed.

Before the transistor was developed, the solid state (non-vacuum tube, etc) diode made from germanium semiconductor metals was developed and manufactured to a high degree of quality and reliability, and was often used as a microwave frequency radio wave detector in **RADAR (Radio Detection And Ranging)** systems for airports and the military so as to detect and display on a screen, the directions and-or location of (metal) airplanes in the sky due to the reflection back of some of the transmitted microwave radio wave energy from a transmitting RADAR station. This process is kind of like shining a flashlight in the dark and seeing what direction and angle that a distant metal coated mirror is located when it reflects the light back to you. RADAR for both weather and plane location began being studied and developed in the 1930's.

Pre-transistor-like devices

Since the development of the two-lead diode tube rectifier, and the three-lead triode tube for amplification in 1906 with its third lead between the two end leads or terminals, it most likely made people consider if there was some way to place a third terminal between the two layers of a (point-contact, "cat whisker") semiconductor diode, typically used in radio receivers, so as to provide some radio signal and-or sound amplification. There have been several claims of such a semiconductor amplifier device(s) having been known about and-or made in about the year 1926 and later, and this was well before the formal creation of the first practical, man-made germanium diode for sale, and then the transistor in 1947, and of which also relied on the newly available and pure semiconductor metal of germanium. These (crude, unstable and-or impractical) (pre) transistor-like devices must of surely inspired the future refinement and development of the transistor. Germanium has largely been replaced by silicon being used for semiconductors since the 1960's. Any such (pre) transistor device must of been impractical to use for most experimenters, but it could of inspired some semiconductor research for creating pure and practical in size semiconductor metals for diodes, and the science of doping them with other metals to slightly enhance their ability to conduct and to create a (pn) diode junction..

The **transistor** is a solid state amplifier. Some people may say that the vacuum tube amplifier was the first solid state amplilfier. It is a word based on "transfer and resistance" whicih loosely means "changeable resistance". A transistor construction is somewhat based on the solid state diode. A **FET** transistor (**F**ield **E**ffect **T**ransistor) has a (first) semiconductor layer or (flow, current) channel with a terminal at each end, but it also has a (second) opposite polarity or type of semiconductor layer surrounding that channel. The overall consturction of a FET is then also somewhat like a common two terminal resistor who's resistance can be controlled and-or varied by the third or input "gate" terminal. The gate is not connected directly to the channel but is actually insulated from. The electronic control is rather done electrostatically. - either enhancing conduction (ie., less internal resistance), or reducting or depleting conduction (ie., more internal resistance) through the channel layer. The **BJT transistor (Bipolar Junction Transistor)** mentioned below, the current flow is not controlled by electrostatic forces, but rather by a small input current.

The **BJT transistor** electronic device was invented in 1947 at Bell Labs in the state of New Jersey, USA. The first working transistor amplifier is credited to **William Shockley, John Bardeen and Walter Brattain** in late 1947 while working for Bell Labs**.** A bipolar junction transistor (BJT) is a small, three layer (NPN type or three layered construction, or PNP type or three layered construction) metal semiconductor(s) material device that functions as an electronic signal controller (ie., an on-off switch), or an AC signal amplifier, and which has replaced most vacuum tube amplifier devices since it is much smaller, much cheaper, and much more efficient since it wastes far less power as heat or thermal energy.

The first BJT transistor used germanium PN junctions, and in 1954 silicon PN junction transistors were invented at Bell Labs. Most modern electronic devices after 1954 have used lower cost silicon transistors which offer several other advantages than that of germanium transistors, and particularly having much less reverse bias "leakage" current. The size of a common low power transistor is slightly less than (1/5 = 0.25) square inches - excluding the wire or "leg" leads. Power transistors can be say a square inch in size. The size of a single transistor on a large (count) scale integrated circuit (IC or "chip") can be very small, nearly microsopic in size as of the year 2025.

The two main types of transistors are the FET, (electric field, electrostatic) Field Effect Transistors, and the (BJT), Bipolar Junction Transistors which can be described as a two-layer (solid state) diode of identical semiconductor material and with a thin (control, or "base") layer made of opposite (electric charge, and carriers) polarity semiconductor material placed between them. A transistor can be tested as being good if the two diode sections are tested as two good diodes that will conduct in one direction and not in the other direction. Essentially, transistors control the flow of current through it with the control layer called the "gate" as for FET's, and the "base" as for BJT's. In general, FET's require higher load voltages to operate than BJT's, and this is due to the (Source to Drain) channel material and its resistance in the FET.

An FET is basically controlled by its input voltage, and a BJT is basically controlled by its input current. It is of note that a single FET generally does not come close to amplifying as much as a single BJT could, but the FET has the advantage of having a very high input resistance or impedance, and therefore it requires almost no input signal current to function, and therefore nearly electrically isolates that transistor's output circuit from that transistor's input circuit. An FET is usually functions faster than a BJT, and therefore a FET is often chosen to be used in very high frequency circuits such as for radio frequency (rf) amplification of the weak radio waves received by a radio antenna. Due to its high input impedance an FET is an ideal choice for the output signal from a tuning or oscillation circuit, whereas, if a lower impedance device was used it would actually reduce the performance of that tuner and or oscillator by effectively lowering its (tuned) impedance and the outputs signal's maximum voltage.

In 1948 **John Bardeen** is credited to inventing the **Insulated Gate Field Effect Transistor** (IG-FET or IGFET), and which its specific technology design lead to the more practical Metallic Oxide Semiconductor Field Effect transistor (**MOSFET , Metalic Oxide Semiconductors, FET**) invented by **Mohamed Atalla** (from Egypt) and **Dawon Kahng** (from Korea) working for the Bell Labs company in 1957. MOSFETS eventually became widely used in both low power, inexpensive (cost per transistor), "high density" or transistor count per unit area, integrated circuits, and for high power transistor-like device applications such as for high current switching and signal amplification due to their fast speed.   It is of note that the BJT was created in 1947, and before the theoretically "more simple" (yet more difficult to produce, and having more consistent device reliability) FET was created and made available between 1957 and 1959. The Junction Field Effect Transistor (**JFET** which is often called as the common FET) manufacture was being studied and developed by several people and companies between 1950 and 1953.

Compared to BJT's, JFET production and device standards have been much more of an problematic issue at first, and it took until many years later to gain a high device reliance from each JFET created, such as a consistent source (S) to drain (D) resistance of a specific (ie., manufactured, registered number, type) JFET of which some circuit initially utilized (and then required) in its design, and for replacement if needed.

Besides being a signal (ac wave) amplifier, the transistor is also used as a simple switch, a fast electronic on-off switch or (solid state, fast, no coil) relay (ie., an electronic equivalent of a water on-off valve), for logic and signal control via electronic ("digital") circuitry, and for data storage in computer or digital memory devices. When used as a switch, the control signal such as the base current in a BJT is at a high enough level so as the collector to emitter resistance becomes the lowest value it can be, much like the low resistance of a closed switch. The transistor is said to be "full on" or completely on and conducting as much as possible. When the control signal or current is 0A, the transistor is not conducting and is said to be "off" and its effective resistance is very high, much like that of an open circuit or switch. In terms of (on-off switching) speed, as of the year 2022, BJT's are still capable at operating at a higher frequencies than MOSFET's, and in both cases, the smaller the device, the more faster it can be due to less internal capacitance, and the smaller the distance that the charge carriers have to travel through the device during the full on (conduction) to full off (no conduction, high resistance) states of it. JFET's can operate at high frequencies and are becoming close the (switching and frequency handling) speeds of many BJT's as of the year 2022.

Before transistors were finally created, some of the initial patented concepts and-or theory of the (future) FET transistor, such as for a possible electrostatic control with semiconductor materials as the conductor were conceived in about 1925 by **Dr. Julius Edgar Lilienfeld** (1882-1963) who was an immigrant from Austria-Hungary to America. Because his conceptual device and-or idea required high purity semiconductor materials, he did not make any known example or public demonstration, and his idea was therefore remained as a theoretical one, an inspiration, and to become realized. Lilienfeld is also credited to inventing the **electrolytic (ie., liquid electrolyte) capacitor** in 1931. These capacitors are said to be "polarized" (have a positive (+) and negative (-) electric polarity plates and indicated terminals and-or current direction) due to the materials used for the plates and the liquid-paste electrolyte between them. These capacitors have a very thin (essentially, electrically or electrostatically polarized so as to store electrons) oxide layer for the dielectric material, and have a large capacitance (ie., charge and-or energy storage) for its volume and-or small size, and are therefore used in many electronic circuits where a high capacitance using a small sized capacitor is needed. Because of this very thin layer of construction, the maximum safe or "breakdown" voltage between the two plates of these types of capacitors is generally a low value. Like the FET's theory of operation, a capacitor relies on an electric field to function, and of which most likely got Lilienfeld thinking about a device (ie., the FET) to control the flow of current using static electricity or charges. Perhaps some of his reasoning is also based on some of the concepts of Crooks tubes, and the glass tube, triode amplifier. Generally, electrolytic or polarized capacitors are not recommended for AC circuits such as audio (electric) signal circuits which constantly change polarity - particularly if the applied voltage changes between above and below ground = 0v, and particularly so for the high power circuits and a ("plain") non-polarized ("ac") capacitor should then be used. **DANGER: Applying reverse bias voltage to a polarized capacitor can cause it to eventually**

explode. It is of note that the term electrolytic capacitor seems to have been used for (low power) liquid (diode) rectifiers since about 1913, such as for example having an aluminum plate that will develop an oxide layer, and an iron plate for the other terminal, and with an certain dissolved electrolyte substance to aid current conduction between them. To build a satisfactory oxide layer, it was done slowly using a low amount of current, and by first using a series load of say a lower power lightbulb(s). The oxide layer could also have been initially started using a chemical solution and process. These diodes have a relatively high reverse bias or leakage current, and several liquid diodes could be put in series so as to reduce that. Lilienfeld's capacitor could then be said as being a solid electrolytic capacitor. To prevent electric shocks and-or injury, do not touch a capacitor that was not discharged first. For a time reference of technology, the relatively high-tech (for those times) diode tubes were formally invented many years previously in 1904 by Flemming, and this goes to show that many new concepts and improvements are still possible in the realm of "lower-tech" or "previous-tech" devices.

A simple descriptive analogy of electronic amplifiers such as a triode - "tube amplifier", or transistor ("solid state amplifier") is to first consider a tiny easy to operate simple electrical switch (or possibly a mechanical pipe valve) that controls the (on or off) power to a machine that requires a high current. Now consider if that switch had the capability to be someplace between on and off, such as in a class A, transistor amplifier circuit, and can vary (increase or decrease) or limit the amount of power to that machine at some level between full off, to full on. In this center or "bias" (operational) position or state, and with no AC signal applied to the input or control pin or lead, half the total possible DC power of the battery or supply voltage is passing through the transistor, but there will be no AC output signal output until there is an AC input signal. Due to amplification of the input signals, even small changes in the input signal or switch position can control huge changes in the output power. This is very similar to what happens in a light dimmer switch which is basically a variable (adjustable) resistor controlling the amount of output power to the light. For a tube amplifier, the control is the center "grid" (a metal screen), and for a transistor, the control is the base (as for a BJT transistor), or gate (as for an FET transistor). The gate material of a FET is insulated from the main current or conductance channel that is constructed between the source and drain leads, and therefore the FET has very high input impedance [ie., AC effective resistance] with almost 0 current needed for it to operate like a switch or (low-gain) amplifier, and it is rather a (electric charge) voltage operated or controlled device. The FET device is ideal for amplifying weak radio signals, requires almost no current since it has a very high input impedance, and without reducing the efficiency of the radio's tuner circuitry. The main result of these electronic amplification devices is that a small input signal can control and-or vary (ie., modulate, affect) the transistors conduction or performance ability, and it creates a larger or amplified output AC signal that is essentially an exact and amplified (ie., multiplied, or a magnified image) copy of the smaller AC input or control signal.

The increase or multiplication factor that an input signal is amplified by an amplifier is called the gain (A), and which is simply the ratio of the output signal to the input signal, and which is numerically how many times more or greater that the (amplified) output signal is than the weaker input signal.

Voltage Gain = voltage amplification = Av = (output voltage) / (input voltage) mathematically:
(output voltage) = (Av) (input voltage) as for an FET , likewise: (output current) = Ai (input current) as for a BJT

It is sometimes necessary to amplify a signal that was already amplified, and the resulting gain is the product of the two gains of the series connected ("or cascaded") amplifiers: Total signal gain or amplification of this system can be shown to be simply the product of the gains within that system, hence a gain of another gain:

(Gain or Amplification 1)(Gain or Amplification 2) = (A1)(A2) = At. and

$$\text{Total Gain} = At = \frac{\text{(output signal)}}{\text{(input signal)}} \ , \quad \text{(output signal)} = At\ \text{(input signal)}. \quad : \text{use voltage, current, or power for signal)}$$

If designed properly, two transistors can be specially connected one after the other (ie., cascaded) and so as to effectively increase the resulting gain, and which is the product of the gain of each transistor. Special 3-terminal transistors have even been made that do this more easily and they are called **Darlington transistors**. This construction effectively functions as a single, very high gain transistor that is generally unavailable. Many Op-amp (or opamp) IC's with their characteristic high input impedance can also create high gains with their on-chip amplifier(s). One of the most common

Darlington transistors is the MPSA13 with an AC signal gain (ie., hfe) of about 10000. This amount of gain is very useful for small signals such as weak radio signals, however any noise is also amplified this much also unless it was filtered out first and-or later.

In digital ("1's and 0's") electronic circuits, transistors are usually used as very fast and simple electronic ("on and off", or "1 and 0") switches for logical (truth) decisions and results, performing mathematical operations using binary values, and-or for storing the bits (ie., digits) of data and-or binary numbers to be used for mathematical calculations and their results.

Semiconductors

As previously mentioned in this book, some preliminary solid-state or semiconductor electronics before the invention of the transistor actually goes back to the first effective discovery of effectively a **light emitting diode** (LED) by the British scientist **J.H. Round** in 1907. He noticed some "electro-luminescence" (electrically caused light) being created when a voltage (and a resulting current) was applied to certain minerals (such as silicon carbide) crystals composed of certain elements. Many of these (solid state, metals) minerals were often used for a "cat whisker" (a thin hair-like metal wire for point of [surface] contact, electrical connector) part of a radio wave "detector" (ie., technically a half-wave rectifier) device. A radio wave detector used in the first radio receivers was effective as an (ac) radio wave (audio signal) extractor or demodulator, and has a one-direction (dc) diode-like action, hence a basic (ac) to (dc) signal converter. A theory of a special boundary or "junction" (depletion) layer (such as an oxide layer) between each element in the crystal and-or the cat whisker wire, was mostly likely started here, and which eventually lead to further understanding and man-made production of the diode and further development of modern high speed and-or power diodes, the three-layer (NPN, or PNP) BJT transistor amplifier - diode-like. layered construction, and even the modern and efficient light emitting diodes.

In 1894, **Dr. Jagadish Chandra Bose** (1858-1937), from India, was the first to discover and use (what was to be eventually called a semiconductor) metals (here, naturally formed "metallic crystals") as a radio wave detector, particularly with (high frequency, generally direct line of sight) **microwaves** which have a very high radio wave frequency and therefore also have a very short wavelength. Unlike the first coherer, radio energy or wave detector, this device was very fast and with a with high frequency ability, and it did not need to be physically reset after each pulse of energy. In 1901 he patented the first crystal **(**modernly called a **"solid-state", semiconductor**, and here with **galena** [a natural lead and sulfide (sulfur) compound], (semiconductor) metal radio detector, and of which had a thin metal wire as the other metal necessary for the detector (ie., diode or rectification) action, and general electrical conductivity required for a circuit. Today, we understand more thoroughly that a (necessary) junction and-or a (electric field caused, and a current) barrier region between the two metals is created. Perhaps Bose was considering the mystery of the two different types of metals used in the construction of a single voltaic cell of a battery. Two years before Marconi's radio-telegraph communication system, in 1895 and inspired by Hertz's previous experiments with both creating radio waves and the detection of, Bose was the first pioneer in **microwave** (have a short wavelength) radio-wave creation, transmission and detection (via a solid-state metallic detector that he invented). Bose made microwaves using a specially designed electric spark-gap, transmission and detection system. He was the first to create what can be called as radio signaling, and which can be thought of as (electronic) remote-control or switching remotely at a distance without a wire. Like Hertz, he associated and studied invisible radio waves as being similar to that of visible light waves which can be directed, reflected, but microwaves can also pass through non-metalic materials like walls between the rooms of a house. Though his galena device (fundamentally called a radio-wave or microwave detector) worked, the physical properties and concepts of semiconductors and diodes were generally unknown at that time, very limited in use, and that it just happened to work so as to be useful. This mysterious, magical, and very useful device inspired others to research these types of devices (ie., diodes) and materials so as to improve upon radio wave detection, and then later to be used for voice demodulation or audio detection (particularly by isolating or filtering it from the received radio signal [electric wave energy, voltage, current, frequency] so as it could be properly heard and not effectively lost due to effective out of phase signal cancellation) such as for AM (Amplitude Modulated) radio wave communication and for many other electronic circuits. After many years of research and technology, high frequency signals such as microwaves are used in RADAR (to locate distant planes via radio energy reflection), communications, and to heat many food products in a microwave-oven.

Greenleaf Whittier (G.W.) Pickard (1877-1956) from the United States of America is credited to the first study of identifying and using many types of semiconductor metals, often in a natural crystal form, and which when deliberately joined together, they and their **junction** (surface area connection) have a diode action necessary for radio wave receivers that are often called "**crystal radios**" so as to distinguish them from the vacuum-tube (electronic amplifier) radios available. He also patented the first **silicon crystal** as a radio detector in 1906. This book contains a basic and common crystal radio circuit of which you can also research, experiment and improved upon. You may also research both long-wire and (portable) loop (directional, coiled) antennas and their circuits so as to improve the amount of radio energy collected so as to be able to hear weak and-or distant stations. The farther a radio signal travels, the weaker or lower its intensity and-or energy becomes per unit of area. This is due to the energy expanding like a cone shape outwards, and also encountering air molecules in the atmosphere. AM radio reception can often be improved simply by turning the radio and-or antenna to a direction having the best audio performance, hence often in the direction of the transmitting antenna or radio energy source.

Many types of solid state, "cat whisker" diodes were put into use at about this time, especially for the use of radio reception, and when considering the high cost of vacuum tube amplifiers and the relatively large amount of energy needed to make them function, and the ("warm-up") time to heat up the filament to its normal operating temperature so as to emit electrons. The commonly used semiconductor mineral used for "amateur" radio receivers ("crystal radios") was often lead-sulfide ("galena", a lead and sulfur compound, PbS, and found or minded as an ore). A thin wire was the other terminal or contact used to (audibly, via the carbon headphones [speakers]) locate and set "sweet spots" in the mineral that would provided the best or a reasonable diode rectification action and the best possible, loudest, highest power, audio quality output. The diode blocked either the negative side or positive side of the incoming (ac) radio wave so as the audio portion could be heard (with the sensitive earphones) when the (high frequency) AM radio wave with the sound information in it are not effectively canceled out in the (low frequency operated) earphones when not using the diode. The high frequency radio wave cannot be heard in the headphones since the headphones are too low in natural resonant frequency and effectively filter out or remove the high frequency radio wave, but the lower frequency audio (AM) wave or fluctuations will still be heard.

Pure lead will not create a diode action, but lead-sulfide (called galena metal) does, and this started much research of what was happening microscopically and-or at the atomic level so as to produce a lower conduction, and diode (one-way, direction) action to current movement or flow. Lead sulfide does not conduct electricity as good as pure lead, and hence it was called as one of the semiconductor materials now known. First of all, elements such as sulfur and phosphorous are non-conductive elements, so it is reasonable to expected that mixing a conductor and non-conductor together creates a semi-conductor. Because two different metals were always part of the construction of a diode and-or for the diode action it indicates that something is probably happening at their junction or contact region, often with an oxide and-or depletion layer. Today after much scientific research, we now know that the diode action of two metals has to do with the applied voltage across the diode or its junction region, and which then creates a static charge region, either causing an abundance or a lack of the two types of charge at the junction region, and which then affects conduction and-or resistance of the diode.

Galena is a lead crystal (atomic bonding with a geometric arrangement and structure) structure that has some trace (about 13.6% by weight) amount of sulfur atoms mixed (aka, "doped" = containing just small "trace" amounts of only, impurities, contaminants, additives and-or enhancements for a particular purpose) in it. The ratio of the mass (and corresponding weight) of lead atoms to that of the mass of sulfur atoms in a galena crystal is about 1 to 1. This does not mean that there is an equal number of atoms of lead and sulfur, but rather there are an equal number of atomic particles (protons and neutrons) from the atoms of each element. Since their mass are the same, the weight of each will also be the same. Galena is essentially a naturally made equivalent to a modern day N or P type doped germanium or silicon crystal used for semiconductors such as diodes and transistors. Note, the materials (ie., elements) used and the percentage of "doping" in modern semiconductors may not be so similar to that of natural or man-made galena, and is in fact much less of a percentage due to the nature of silicon and germanium crystal structures. Galena has a bandgap of about 0.4 electron-volts (eV). A (energy) **bandgap** of a specific material is the amount of energy (with units of Joules) needed to remove or free an electron (that is usually, relatively far from the nucleus of the atom) from an atom and promote the conduction of electricity (the flow (ie., current) of charges) in and-or through the material. This negatively charged electron moving about will essentially leave behind a hole in the atomic crystal structure, and that atom then has a net positive charge. When an electron fills the hole, its gained bandgap amount of kinetic energy is then released as emitted light energy such as seen in an **LED** (Light Emitting Diode, and of which a common Red LED has a relatively low bandgap of about 2 ev). Many man-made semiconductor materials are doped with phosphorous to create an N-type semiconductor with an excess of electrons in its crystal structure, or with boron so as to create a P-type semiconductor with a lack of electrons, and therefore having an excess of holes in its crystal structure. This doping greatly improves the conductivity of a semiconductor material, and it also gives it a main (N or P) polarity and sensitivity to an applied voltage polarity. Both N and P type semiconductor pieces are used to build diodes, transistors and other semiconductor devices. Electrons in the farthest orbit about an atom's nucleus are called valence electrons and the electric forces of attraction are weaker at this distance, and it is then more easier to free a valence electron than any other closer electron.

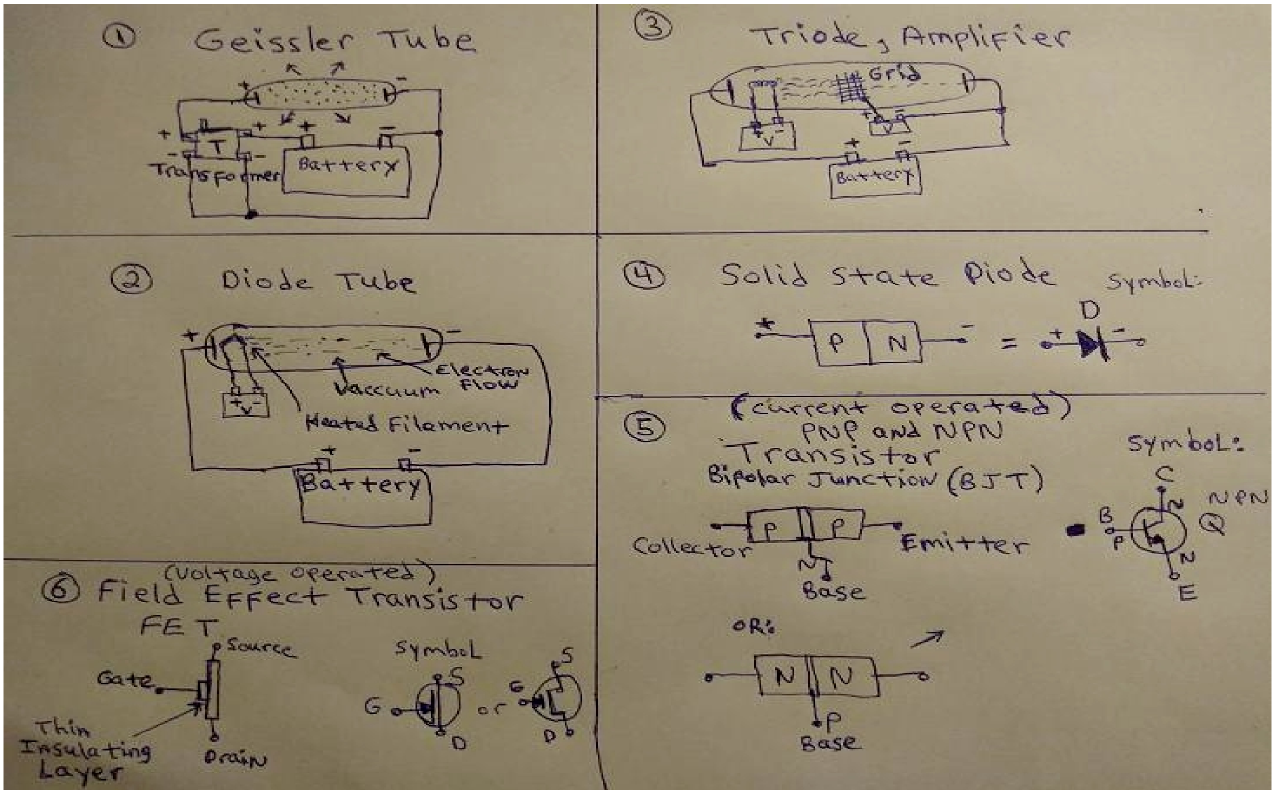
Solid-state diodes (non-tube, metallic, passive - no external (bias, setting, turn-on) power needed to make them function) began to be studied in the 1870's, and this was before the era of having pure, refined silicon and germanium semiconductor metal. Some common examples are the **copper-oxide rectifier** (about 1925), and the **selenium rectifier** (1933), and these were often used to convert (ie., rectify) AC to DC for powering electronic devices. The construction of these devices usually included some other metal(s) and-or (pn junction) plates. Just like with modern silicon diodes, these early types of diodes could be placed in series so as to increase the total rectifying power (VI). Solid-state diodes required no warm up time like a tube diode, and they did not waste power (as heat) like a tube does.

If you are interested in homemade diodes, look up the zinc-oxide diode, and the aluminum-oxide diode. These are relatively easy to make, and the zinc-oxide diode has a very low turn on voltage of about that of a germanium diode often used in a (crystal, experimental, low-cost) radio receiver. The germanium diode has the advantage of being housed in a small protective glass case, and is relatively inexpensive, less fragile and more reliable. Modernly, new precision manufactured types of zinc-oxide diodes are capable of very high frequencies of many Gigahertz. Some zinc-oxide diodes are also LED versions. There are several websites that you can search for and know of, such as: **sparkbangbuzz.com** that have many interesting homemade diode and other electronic projects. Zinc metal is commonly available as dry-cell battery cases, and the protective covering of steel and of which is then called **galvanized** steel, generally meaning electrically modified steel since it prevents oxidation (rusting). A homemade diode will generally be of the fine wire or "cat whisker" type, and of which a diode conduction spot is to then be found by trial and error and then set. Some diodes have a metal to a (doped) semiconductor junction, and are called Schottky diodes, and some diodes have a junction of two (doped, pn) semiconductors. **Schottky diodes** also have a low forward or on-voltage of about 0.25v if the current is low (a few milli-amps), and these are ideal for low signals such as radio signals. **1N34A** (by the Sylvania Co. in 1946) germanium diodes are **still recommended for crystal radios** since they will generally produce a more faithful output (and audio signal) of the intended audio due to their low "turn on" or forward voltage. 1n34 types are also generally only for low frequency signals due to their relatively high junction capacitance, and therefore, their longer turn-on and turn-off times. A heated hot and cooled a piece of certain types of thin steel, such as some utility knife blades, can form a very thin, "blued" colored iron oxide, or perhaps a selenium dioxide layer or coating on it, and this makes quite a good Schottky diode for radio reception, and having a forward or on-voltage of about 0.2v to 0.3v. Some alternatives to the**1n34A** are the AA117, AA119, AA143, AAZ15, AAZ17, BAT46, 1N270, 1N695A, 1N949, 1N3287, 1N3592, 1N3773, OA47, OA90, OA91, and especially the **1N60 or 1N60P** germanium diode with a slightly less forward bias voltage needed. It is of note that diodes also do conduct with a lesser extent with a voltage less than their rated turn-on voltage, and the values mentioned here, are for their rated turn-on voltage where they begin to conduct well and the diode has a lower effective resistance to current. Some diodes can be, or may need to be biased slightly so as to improve the resulting electronic audio signal in a radio receiver. This can be done using a voltage, say between 1.5v and 9v, and a series connected high variable resistance, say 500k ohms or 1M ohms in a voltage divider configuration, and then connected this circuit across the diode. If would be wise to also include a series resistor, say 47k ohms, as a safety resistor to prevent high current damage to the potentiometer or variable resistor when it is set at a very low resistance setting.

In 1946, the first manufactured, practical, reliable and commercial (point-contact, "whisker" (ex. a very thin tungsten wire)) diode was made available to the public and industries. This diode is actually a pre-Schottky diode (which most likely inspired Schottky to study it) and a favorite of crystal radio receiver makers due to its low turn-on or forward voltage needed, about 0.35v, and is the 1N34A germanium diode. A few years before this time, these diodes were available to only the military, and were used in RADAR (Radio Detection And Ranging) systems to detect planes and their location(s) in the sky. It was a purified germanium (metallic, crystalline) point-contact (whisker) semiconductor diode. 1N21 (year

1940), 1N31 and other similar diodes were typically used in (high frequency) RADAR systems for finding airplane locations (direction and distance). This diode is technically a type of (Walter, 1886-1978, from Germany) **Schottky diode** which is a metal (wire point, whisker) and semiconductor junction diode, and has a low "turn on" (forward, conduction or barrier) voltage of about 0.35v (or less, and when the current is kept low, otherwise it will increase) and it has fast turn on (conduction) and-or turn off (non-conduction) switching time and can therefore be used for very high frequency circuits. It is a very practical and efficient diode to use with weak radio signals such as that received by many radio receivers. A common Schottky diode is the **1n5817** which has a max. rated current of 1A. and a max. reverse bias voltage of 20V. The 1n5819 is similar but with max. reverse bias of 40V. The 1n5622 is similar to the 1n5819, but it has a 3A max. current rating. It is of note that it is possible for some electronic tinkerers to make a practical homemade diode with various metals and-or semiconductors such as zinc-oxide diode that could have similar electrical characteristics of a machine manufactured germanium diode. Schottky and Erwin Gerlach invented the ribbon microphone in 1924, and which are generally said to be as sensitive or greater than that of some quality condenser microphones.

Russell Ohl (1898-1987), from the USA, while working for AT&T Bell Labs in 1939 is credited to purifying silicon to over 99% purity, and so as to eventually make the first practical and commercial "(PN) solid-state" diode. He is also credited with the modern understanding and concepts of the necessary diode barrier-junction (**PN**) region, and the concepts of solar cells to create electricity (as of 1941) using the light sensitive properties of silicon and the barrier region between the P and N joined pieces. A P or positive region essentially has a hole or gap in its crystal structure, and the N or negative region has an extra electron in its crystal structure. Light striking the barrier or junction region would create a voltage (ie., v = J/C, electronic energy so as to move electrons (ie., a current) with electric forces). These silicon **solar-cells** created by some engineers at **Bell Laboratories** ([Alexander Graham] "**Bell Labs**") in 1954 initially had a relatively low efficiency (about 5%) of converting solar light energy into electric energy (moving or flowing [like a water current] electrons with kinetic energy), but it was still very promising and also useful for electronic light sensors and small electrical devices and batteries. A solar cell is sometimes described as a large PN junction diode. PN refers to a positive (P) electrical polarity and-or majority current carrier type, and negative (N) polarity and-or majority current carrier semiconductor type of (purified and lightly "doped" (added in), metallic) materials. Solar-cells were used to power most of the man-made satellites sent to outer-space starting in 1958. The PN solid-state discoveries would lead Bell Labs to begin research for a solid state electronic signal amplifier based on the practical solid-state diode, its manufacturing process, and the basic concepts of the vacuum tube amplifier having a current control grid in the middle. This "solid state" (semiconductor) amplifier device invented a few years later in 1948 would be called a transistor. Many other similar (germanium or silicon, PN junction(s)) electrical devices, including the first practical and commercial **LED (Light Emitting Diode)** light soon followed these (seemingly more advanced) discoveries. Below is a crude, but helpful basic, and not to scale drawing of some of these electronic devices. [FIG 234]



The first practical (reliable, easy to use, commercial) **LED** (light emitting diode) light was invented by **Nick Holonyak** (1928-2022) at a General Electric Co. laboratory in 1962. The light emitted was red in color. These LED's required about 2 volts and 10mA of current to function reasonably. If the current is increased slightly (up to a few milliamps (mA) more, maximum), the light intensity will also increase due to the increase in power sent to it. The power (voltage x current) to an LED can be controlled or limited by using a resistor with an appropriate, calculated ohms value of resistance. The price of these LED's was initially very high and therefore not worth using with most electronics devices. As of the year 2020, a bag of 100 red, yellow or green LED's can be had for about $5USD. 7 or 8 segment LED numeric displays are composed of 7 or 8 individual, long LED's of which each can be individually turned on or off by a circuit and so as to display a number such as for calculators, watches, meters and other various counters. The general construction of an LED is very similar to that of a typical PN diode,and with using a correct voltage polarity (+v anode , -v cathode) to function, except that the LED is also designed to produce and radiate much more visible light. Holonyak is also credited to inventing the first visible **laser diode** in 1962. The Monsanto Co. began producing relatively low cost red LED's in 1968. These LED's required 1.65v forward voltage, and about 20mA of current. Red LED's are mostly used for circuit operation and-or as indicator lights, and sometimes for when low light levels are needed, such as for astronomers at nighttime.. Before white LED's were available, yellow LED's could be used to make a homemade reasonable efficient and long lasting electric light. Astronomers, if needing to see what they are doing, will use a red light so as to help maintain their eye's light sensitivity to faint light at night and which is commonly called their "night vision". Single LED's are typically produced as either a 3mm or 5mm housing width about the actual LED diode chip.

Before the technology was created to make white-light emitting LED's, a common goal was to make LED's having a particular color such as often: red, amber, yellow, and green. Later, a low efficiency blue LED's was roughly developed in 1972 at the RCA co. in the USA, and then a much more efficient design was created in Japan in 1992, particularly by **Shuji Nakamura**, and of which eventually lead to the very useful white LED. All LED's usually use a specific alloy layer of gallium and another element. Having a white LED, it can then be covered or coated with any color desired.

White light emitting LED's were always desired and were finally produced in 1996 and became widely available at about the year 2000, but they were still relatively expensive and were often used in high quality, efficient (lumens/watt) flashlights. At about the year 2010, white light emitting LED's became cheap in cost so as to be used as a part of many electronic devices including lights (ie., lamps). The advantages of LED's over other forms of electronic lighting is that they are cheap (low cost, sometimes 100 white-light emitting LEDs for just $10 USD as of the year 2020), small, require low power (ex., a voltage of about 3v to 3.3vmax rated (although up to about 4.2v is possible but may shorten the usage time or function life of it) and a typical current of about 20mA to 30mA max for a white LED, hence a power of about: **Pw max. = 3.3v max. X 0.030A max = 0.099W =~ 100mW**) to be used, produce more light or lumens per watt of input power and are therefore more efficient at converting electrical energy to light energy. Two 1.5v batteries in series is sufficient to light and-or power a white or most other colors LED. **Lumens** is the measurement of the light intensity per unit area (1 ft^2, or 1 m^2) at a standard reference distance (1 ft, or 1m) from the light source, and hence it is usually called the brightness level of the light being transmitted to the object being illuminated (ie., radiated, transmitted to) by that light. Compared to incandescent lamps, LED's produce a relatively low amounts of heat per amount of power input and-or light produced. A modern (2022) LED might be rated between 100 and 200 lumens per watt, whereas an incandescent (ie., thermally generated and radiated light) light bulb might be rated at just 10 lumens per watt, hence an LED can produce at least 10 times more light per watt. Because LED's are more efficient, they will produce more photons or light and which is rated as lumens per watt. LED's can be 45% efficient on average, whereas an incandescent light is about 15% efficient on average. This means that for an LED, 45% of the input energy is converted to light energy, and (100% - 45%) = 55% is wasted as heat energy. As of the year 2020 most light-bulbs designed for household lighting (some costing as low as $1USD) and high brightness streetlamps now use LED light-bulbs. White LED's can be covered with an opaque colored coverings and-or decorative shapes. The emitted color of LED's is sometimes expressed as "Degrees K" = "Degrees Kelvin". Incandescent lamp light color is about 2000°K, and has a slightly yellow-red hue, whereas a white LED light color is about 4000°K. Higher temperature LEDs will be rated at about 5000°K and have a slight blue hue, however, some people may not like this due to potential eye problems.

In general, white LED's, and LED's have a relatively low maximum reverse voltage of which they will usually be damaged, and this is only about 5V regardless of how small the current is, but it is very possible to protect an LED using a standard diode (such as an 1N4007) in reverse polarity and then placed across and-or parallel to it, and which will turn on first to 0.7V, and which also has a series resistor to limit current through it.

There are some modern,specialty LED's. Some LED's transmit invisible, low frequency infrared light, such as for sending remote control signals, and some LED's transmit high frequency, ultraviolet light which is relatively dangerous and precautions must be observed. Some LED's have a convenient built in current limiting resistor so as the LED will function when connected directly to a specific voltage. RGB LED's have three led color chips in them with a corresponding external lead. Some LED's have a small internal IC (integrated circuit) that will cause the LED to blink periodically. Small LED's are known as "rice lights" and are mostly used for various decorations, however a small flashlight can still be made using them. The value of light in an emergency is priceless. Some LED's are lasers that transmit a single frequency (ie., monochromatic light) of high power light though a very narrow (small) transmission angle and-or beam, and precautions must be observed so as to prevent damage and-or injuries. Depending on its power, a use and license may be required.
Alexander Graham Bell (1847-1922) from the United Kingdom, and later Canada, along with his electro-mechanical assistant Thomas A. Watson, invented the **telephone system** in 1876, which electrically converted (electronically simulated as an electrical signal composed of voltage and current) and transmitted an input sound (audio, voice, air pressure wave) signal using a long wire electrical circuit connection to a distant sound reproducer (ie., electrical transducer [causing a energy transformation, hence a "transformer" of sorts] or audio "speaker"). Bell's "talk piece" or microphone was a crude "liquid microphone". About a year later, Bell would significantly improved his telephone system by using Edison's and Berliner's more efficient and sensitive carbon microphone (input [mechanical, force or pressure] sound signal converter or transducer to an electrical signal). The invention of the telephone was initially based on some of the concepts of the already existing telegraph system and its technology. Both the telegraph, and particularly the telephone are extremely significant for both short and long distance communication. These device became very important, both personally and socially valuable. The telegraph communication methods were also adapted for use with the wireless radio communication systems, particularly with the (initial) use of Morse code before radio could transmit audio information well. The telephone system quickly became very practical and desired in businesses and households for people to communicate due to its ease, widespread availability, low costs, general privacy and without the need to learn Morse code to communicate such as generally needed for the telegraph (remote-graphing and-or sounding [audio bell or

buzzer]) system.

Alexander Bain, a clock-maker from Scotland, was the first to conceive how images could be transmitted via the telegraph system in 1843. His idea considered that an image or photograph could somehow be scanned (horizontal) line by line and dot by dot (ie., "pixels" = picture elements), and this concept was much like how a modern image is both scanned to be transmitted and also seen on a television screen. **Frederick Bakewell**, from England, succeeded in transmitting a crude image via the telegraph system in 1847. The first transmission of high resolution, fine detailed images and-or text was sent using a device called a **pantelegraph** that was invented by **Giovanni Caselli**, from Italy, in 1856. The first photo-electric system for image transmission, and which used a (selenium) photocell for the image and-or light intensity sensor was invented by **Arthur Korn** from Germany in 1901. By 1924, images could be transmitted wirelessly via radiotelegraphy.

Elisha Gray (1835-1901), from the United States, was a telegraph engineer like Bell was. At about the same time as Bell's telephone patent application, he independently filed a patent application for a telephone ("remote-sound", electronic communication) system. Gray was one of the founding members of Western Electric Company. Gray invented the first practical type of "**fax**" (facsimile = "making similar") machine and system called the **telautograph** in 1887, and which could reproduce detailed handwriting over the telephone system, and was more accurate than the ones previously invented in Europe that needed a complex system of a rotating drum, and it needed perfect (synchronized, coincidental) timing so as to have a reasonably accurate reproduction or facsimile (modernly called as a "Fax") image of the source image transmitted via the telegraph system. Gray is also credited to making what could be called the first practical electronic musical instrument and-or system in 1874. It was called an electro-harmonic or music telegraph. Since speech could not be transmitted over the telegraph system for various reasons such as the technology was not yet available or impractical, Grey considered transmitting musical tone signals (particularly their frequency's) using via their equivalent or representative electronic signals. With this invention, Grey invented a form of (mechanical) vibration microphone or "sound pickup", and a crude form of a loudspeaker which has an electromagnet to make the sound vibrations from the input electric signal. This instrument was basically an electric organ, and where each note and its corresponding current variations or "fluctuations" at a particular frequency were automatically determined by a corresponding, thin metal reed that could vibrate at a particular natural resonant frequency. Each acoustic and-or electric representational tone of it was therefore produced by what could be called a "(metal) reed oscillator", and which was powered by an electromagnet, and much like how an electric buzzer is operated as a type of (electro-mechanical) oscillator. What was Grey initially thinking? Grey most likely wanted to replace the "dot" and "dash" electronic Morse code pulses with two different and discernible audio tones at the receiver, thus improving the telegraph system for some people to decode the pulses more accurately.

Alexander Graham Bell was born and raised in the United Kingdom but immigrated to Canada, and then to Boston city in the United States of America (USA) just to the south of Canada. Bell often worked with educating the deaf and helping their community. Bell made an improvement in sound recording and playback by creating a modified version of Edison's phonograph system. Bell also invented the photo-phone where sound could be transmitted using sound modified ("modulating", varied, here the intensity) light waves transmitted through the air to an electronic sensor or receiver placed at a short distance away. This was one of the first significant uses of the newly discovered (by **Willoughby Smith** (1828-1891), from England, in about 1873) light or photo-sensitive, photo-conductive electrical materials. The light sensitive or reactive material used was **selenium** which is a metallic element that can vary its electrical (ie., electricity) resistance in direct relationship to the amount of light intensity upon its surface. Selenium is said to be a **semiconductor** material that does not conduct electricity very well like a typical metal, but still conducts better than a resistor. Semiconductors are usually sensitive to light, heat, etc., which will proportionally vary their electrical properties such as its electrical resistance when some type of energy is applied to them. It is said that the "band-gap" or amount of energy (joules or "electron volts") needed for a semiconductor atoms to release (from their orbit and force of electric attraction) or give off electrons and conduct electricity is slightly greater than that of a standard electric conductor or metal. The temperature level of something very hot can be sensed by using a probe that has temperature sensor (ie., a heat to electricity "transducer" or converter) made of a semiconductor material. In about 1879, **George Carey** (1851-1906), from America, conceived of a crude (electronic) camera system having an array of selenium photo-sensors that could sense the light levels of an object and so as to send the (crude, low resolution) image of it electronically (via many wires, one for each photo-sensor), hence it was a type of (crude or primitive, wired, direct connection) television. The image could be recreated and seen by using an electric light for each light sensor used, but probably a electro-mechanical solenoid operated flap could be used instead of a light, but this would not be as fast. His device was experimental, and it surely inspired many in the future. An image sensor(s) is now common in (digital) image scanners (copiers) and cameras of which have made photography very practical and affordable, and especially for quickly sending an image with modern electronic digital communication such as the internet ("web").

Charles Fritts (1850-1903), from America, essentially created the first solid-state electrical generator or **solarcell** in 1883. He created this device when he coated selenium (a semiconductor metal) with a very thin, nearly clear layer of gold on the light receiving side, and it had a metal plate on the other side so as to make a second electrical connection for it to function in an electrical and-or conduction circuit. It had a low efficiency (less than 2%) of converting sunlight energy to electrical energy. Surly this device must have eventually inspired some of the later solid state research, construction, manufacturing and technology such as with the diodes, transistors and other metals used for modern solarcells. His PV (photo-voltaic [pv], photo or light energy to electrical energy) solar-cell fabrication would be called today as a form of a Shottky (semiconductor to metal) type of solar cell where one terminal or solarcell layer was not a semiconductor metal, but rather a conductive metal. In theory, an area of a square meter of his pv-cells could provide about (1000w/m^2)(1.5% efficiency) = 15W (ie., 15 joules of energy per second) of power, hence it is enough to charge a battery over a period of time, and which can then deliver a much higher amount of power (energy / second), and in particular, a high current for a shorter amount of time when needed. P = energy / time = (voltage)(current) = VA

Before the PV device invented by Charles Fritts, **Alexandre Edmond Becquerel** (1820-1891), from France and who was the father of Henri Becquerel who made discoveries about radiation energy, previously made the initial discovery of the **photo-voltaic (PV) or "Becquerel Effect"** in 1839. With this effect, a generated voltage (emf) is created due to light energy, and therefore, a current through a circuit is also produced. His light sensor and-or (energy) converter used two platinum wires in a silver compound solution. It must be mentioned that at those times, the technical understanding of how these (semi-conductor, layer, junction) devices worked was mostly unclear, and a mystery until the research by Albert Einstein years later, and then by Bell Labs who created the first transistor (used for switching or amplification). The basic process of the photo-voltaic effect is that if a photon of light has enough energy, it can collide with and knock an electron out from its orbit about an atom of certain materials and-or electrons with a low enough (energy) "band-gap" that is equal to or less than the energy of that photon of light.

Thomas Seebeck (1770-1831), from Germany, discovered the **thermo-electric effect** in 1821 when using two different metals and heating their junction and noticing a small current flow. A thermo--electric generator converts heat energy to electric energy, and is therefore much like an electric heater in a "reverse" type of operation or function. This current flow is due to the voltage created at the metal junction(s) of which can also be connected in electrical series so as to create a larger voltage and current - much like how stacked (ie., in series) voltaic cells in a battery can produce a higher voltage and current. Seebeck's device had a low conversion efficiency and it was of little practical use other than for a very good temperature sensor (micro-volts per degree Kelvin) = (uV / 1K°) which is usually called a **thermocouple** device or sensor so as to measure temperatures, and it is an interesting topic for further future research, discoveries and products. The Seebeck coefficient (v / °K) for (PN) semiconductor and-or alloy junctions is many times that as compared to regular or common metal junctions, and arrays of these junctions can be assembled so as to create a device called a **thermopile** which has a higher output voltage and-or current. This discovery is also called the Peltier-Seebeck effect, and modern, more efficient ones are used in some more modern electrical devices (perhaps a simple radio receiver) such as to make some useful electrical power by using "waste heat" energy, and-or to create a cooling effect from the input of electrical energy. This use of the device in this reverse type of way is called the **Peltier Effect** and is essentially the reverse of the Seebeck effect. In one version of these devices, heat is applied to one metal junction or side, and cooling is applied to another metal junction or side, and the output voltage and current will increase with an increase in the temperature difference of the two metals and-or sides. If a voltage and current is rather applied to a **thermopile**, one side will increase in temperature and the other will decrease in temperature depending on the voltage polarity used for each side, and it is then possible to use it to make a small insulated refrigerator container to keep things cool inside it, and where the warm side has a reasonably large external heat-sink/radiator to help increase the temperature difference and efficiency of the device. The transfer of heat to one side is due to the heated electrons which have gained thermal energy and transformed it to their kinetic energy and will then carry it to the other side of the device which will then become warmer, and the other side where the electrons were form will become cooler.

About 20 years after Charles Fritts's solar cell technology, **George Cove**, from Nova Scotia in Canada, was the first to produce some small (about a square meter or more), relatively cheap (using zinc and other metals for the semiconductor) and practical solar-electric (pv = photo-voltaic) generator systems in about 1904 and later, and they had an appreciable energy conversion efficiency (estimated at 5%), and this is a reasonable value, but it was much less than the modern (as of 2021) 17% to 25% (more complex) solar-cells and -or solar-panels. Cove's PV (solar-electric) system was a great achievement at those times, but overall, it wasn't needed for the average person and home, especially if they already had convenient and high power electricity supplied via wires form the electric utility company. Also, clear skies, shadeless open space, and aiming it directly at the (apparently moving, changing position) Sun is always needed for best use and highest efficiency. In 1980, the average price/watt of solar-electric energy was about $20USD/watt. In 1986 the average price per watt began to be relatively affordable for the wealthy at about $10USD/watt. As after about the year 1996, silicon solar panels were in great demand by the general public, and they were becoming relatively cheap at about $4 to $5USD/watt generated. In about the year 2010, about $1 per watt became common, and due to the increased demand, manufacturing abilities, lower cost of transportation for supplies and distribution, and competition. This price is very reasonable and practical for the average person to afford some form of solar-electric energy. As of the year 2022, it is still hoped that the use of solar generated electricity will greatly increase to several times more, especially in terms of portable energy, energy availability and security, especially for critical applications such as for hospitals and vital industries such as to produce food products and medicines.

The first practical phone system, and many of its concepts are still available and-or being used today as of the year 2022, was created by **Alexander Graham Bell** and it used DC rechargeable batteries at the local phone company to power the telephone system. Each phone company or station had some employees called "telephone operators" who would take your request to be connected or (electrically) "switched" ("telephone operators" were also called "**switchboard operators**") by using a small cable or wire called a "patch cable" to connect the callers phone electric circuit directly to another phone that they are "calling" so as to communicate with someone using their own phone transceiver (transmitter and receiver). A small (magnetic, DC) generator inside each phone would be used to "ring" (by powering and sounding an [oscillating] electric buzzer or bell), signal or alert the telephone company operator of a phone communication request, and the operator would listen to your verbal request and then connect (via a short "patch cable") your phone circuit to the desired recipient's phone circuit and then "ring" or alert them via an electronic bell sound so as to "answer" or "pick-up" their phone and communicate. Years later, and made practical in the 1950's (but actually invented several decades before in about 1896 by **John** and **Charles Erickson** and **Frank Lundquist**, with their "automatic telephone" system which included electric "rotary dialing" (ie., for the assigned phone number of the recipient's phone), and automatic (ie., electronic) connection switching to each desired person, home, or business containing a phone(s) that was assigned its own unique "phone number" (technically a unique switch(s) setting at the local telephone station) when the phone system became automated so as to directly connect the caller and recipient phones, and without the need for the assistance of an operator. Here, a desired phone number was first entered or "dialed" using a "rotary-dial" - a circular mechanical switch that is selected and moved and set by the callers finger. This switch mechanism sent a series of electric pules to the phone station for each digit of the phone number entered. Still, if operator assistance was needed, perhaps when trying to find a phone number assigned to someone or someplace, a person could simply dial a "0" to connect to the operator. Rotary phones began to be replaced in about 1963 with "push button" or "tone" phones which sent fast and unique electric-audio tones to the phone station, and which when electronically decoded, it represented the phone number being requested or called. Many modern phone systems also use the special number of **911** to remember and quickly call for emergency assistance. Most phones today as of the year 2022 are wireless or radio telephones called as "**cell phones**", **"mobile phones"** and-or "**smart phones**" due to all their internal computer circuitry and abilities such as having access to the Internet communication system. Older phones that used a direct wire connection to the phone system and company are now called as "land-line" (ie., using wires or "electric lines") phones.

Preparing for communication problems.

During some local disaster of any sort, telephone communication, especially with cell phones, becomes problematic and-or unlikely, and you should consider obtaining some backup communication such as a radio transmitter of some practical sort. Learn how to use and maintain it, and tell other members of your household and-or community how to use a specific communication system. Write a guide on paper(s) if need be. These communication systems can be powered by a previously charged battery(s) and-or solar panel(s) electrical power system. Knowing what radio frequencies to use will be very helpful, and this book mentions some radio frequencies, especially with the CB (Citizens Band, public, legal, no-license required, but here are still several rules to follow) radio system. **In general, and in many countries and-or locations, radio receivers of any sort are legal without a license, but please check first so that you do not break the local laws and risk prosecution.**

More About Batteries

The word "battery" is actually another and older word for an "array" (one or several) of similar things, such as individual electricity generating voltaic cells that can be used to store (ie., "be charged up", accumulate) or create (by chemical energy) free electrons as potential electrical energy. This energy can then be transferred or delivered via conduction wires to another location and-or electric circuit, and in the form of an (electric, electron) current (flow, moving) that has (kinetic, in motion or active) electrical energy and-or power (voltage and current) to do things. Electric energy is called "electricity". Electrons have or carry (kinetic) energy with their momentum that consists of their small mass but high velocity. The movement or flow (or "current", from one point to another, somewhat like a water flow or current moving to a lower or ground level, and then having less energy) of electrons is naturally called a current of electrons. **Capacitors** (basically two plates of metal that are very close to each other) can accumulate (build up, increase or increment the amount of, store, "condense") and store the flowing electrons (ie., electrical charges) and then release those stored electrons as electrical power (P = V x I) much like a battery does. A single voltaic cell and-or a battery of cells voltage depends on the materials (ie., metals, chemicals, electrolyte) and construction used, and the amount of energy it can safely store and-or release is mainly determined by its size or volume. Common cell voltages (v, volts) are 1.5v for chemical, non-rechargeable batteries such as from these smaller to larger (volume) standard types: AAA, AA, C and D. There are also 1.2v similar sized rechargeable batteries, especially if a lithium type, however it must be then determined how useful this lower voltage and power will be for any particular device, unless that specific device was initially designed to use 1.2v rechargeable batteries such as the lithium types. Three 1.2v batteries in series can create a 3(1.2v) = 3.6v battery,and this is quite adequate for a white led which needs at least 3v , and about 20ma to 30ma to function as a lamp (an "electric torch" or light source)

Liquid fuels can have a energy density or concentration stated and-or rated as for example: (joules of energy / gram or Kg of fuel weight). Batteries can be stated and-or rated as having an energy density of (joules of energy / gram or kg of battery weight).

Lithium types of batteries have a higher energy density (Joules / Kg of battery weight) or (aH / Kg of battery weight), or more often as: (Wh / Kg) = J/Kg as compared to most other (non- lead-acid) rechargeable batteries which have a rating of about 40Wh/kg. In general lithium batteries have at least twice that of lead-acid batteries. A common and popular (as of about the year 2020, due to its portability, size, weight, and energy storage ability) **lithium** (internal metal material or construction), rechargeable battery is called an **18650 lithium-ion battery (LI-Ion)** due to its physical dimensions or size, and has a rated average and-or expected working voltage of about **3.7v (typically 50% charge or power left)**, but many may be safely charged up to a maximum of **4.2v**. A typical 18650 battery at 4.2v will have a fairly high rate of voltage reduction till it reaches about 3.7v at where the rate is of voltage reduction is a gradual downward slope till about 3.2v. When this battery is drained ("0% rated charge left") of its energy when down to about **3.2v**, and no less than about **2.75v to 3v**, it is usually considered drained and useless of its rated energy and-or circuit design needs, and the battery needs to then be recharged before more is drained and degrades the battery by internal crystal growths and corresponding higher internal resistance which reduces voltage, current and therefore power available. For comparison, a 12v lead-acid car battery is considered drained at any less than its 12v rated value, and yet 50% of its charge and-or energy still remains in that battery, and this should not be used since the battery can then get damaged inside. At less than about 3v, a lithium battery has given nearly all of its internal charge, energy or power, and the internal resistance of the 18650 battery has also increased significantly and will offer even much less output power due to the internal power loss, and this heat created can damage the battery and-or shorten its life and the number of further charging and discharging cycles. A **Lithium Iron Phosphate battery (LiFePO4 or LFP)** cell has a slightly less voltage rating than a regular lithium-ion battery, and that value is **3.2V** with a max of about **3.7v**., but some may charge to about 4v for when the batteries are in an array and so as to all be "equalized" to having the same voltage more effectively to say 3.2v or so. A LiFePO4 cell should not be allowed to go lower than **2.5v**, and hopefully the battery system has a **battery management system (BMS)** to prevent things like this. The BMS will usually equalize each cell on the battery. If 4 LiFePO4 batteries are connected in series, the charging voltage is then about 4 times that of 1 cell, hence: 4(3.2v) = 12.8v (and usually generally rated as a 12v battery, but may state 12.8v on the battery, and with the lowest safe voltage at 10v). Note that a lead-acid battery charger designed to charge lead-acid batteries to a max. of about 14.7 v will be too much voltage for the LiFePO4 batteries, hence they will need a charger that is capable and-or designed to do so, say generally to **13.5V max** (some people say this is actually below 100% of total possible charge, but it is safer and that the battery type maintains its voltage quite well as compared to a lead-acid battery. In short, charging to say 14.6V (if the BMS and-or charger even allows it) is excessive and the battery will drain quickly from that peak voltage so as to be about a consistent 13.5v), however some chargers can also be set to charge to specific voltages, etc. Also, although maximum discharge currents can be a high temporary (in time amount) value for most batteries, the charging current is still recommended to be only about 20% or less of the Ah or "C", current rating of the battery. Some advanced battery chargers, and some are inexpensive, have a setting for the specific type of battery that will be charged. Some chargers can essentially "reset" the BMS so as to allow the charging of a battery that has been drained to low, and the charging current should likewise be low and slow so as to not overheat the battery and damage it in the process.

A damaged battery and-or one which went too low or high in voltage is a fire hazard. Do not open or puncture a lithium battery for it short circuit (ie., unwanted electrical contact, low resistance, very high current) may get hot and begin a chemical fire that does not need oxygen and will ignite or combust even in water. Store batteries in a proper, certified and-or recommended container(s), and insulated each battery and-or container from each other and any other potential fire hazard. If possible, place a hot or damaged lithium battery into a metal container that had some sand or dry dirt in the bottom of it, and cover the battery with more sand, and then place the lid on the can, and move the container to an open area so as to prevent a larger fire. Contact the police and-or fire department if you feel the situtation needs it.

LiFePO4 batteries have less thermal issues and more recharging cycles than that of Lithium-Ion batteries, however Lithium-Ion batteries can store slightly more energy than that of LiFePO4 batteries.

Placing identical batteries in series (s) will increase the output voltage, and placing identical batteries in parallel (p) will increase the total energy capacity, current and-or time of use. For example, (n) batteries in parallel will extend the use time by the same factor of (n). Experienced electrical minded people can make a custom made battery pack, but extra caution must be observe when dealing with lithium batteries. One caution is to prevent overcharging (ie. > 4.2v), over discharging (ie. < 3v), and a short circuit load, and a **battery management circuit and-or module is recommended**.

For **charging and-or recharging lithium batteries**, there are commonly available, simple to use, small size, inexpensive circuit boards such as the **TP4056** (very common as of the writing of the book, 2025), **TP4069**, **TP5100 IC (**similar to TP4056, but for example, it has a 2A max charge current, and it can also charge two batteries in series)**,** and **CD42** which is a 5V USB to (3.7v to 4.2v) lithium battery (or parallel batteries pack to increase maH or aH capacity) with 5v regulated output. These circuits have various parts, ports (sometimes a 5v USB port), input and output battery protections (if needed, and-or when the lithium cell does not have any built in protection circuit) can be purchased so as to easily charge (typical, 3.6v rated, to nearly 4v max) lithium batteries more safely. These circuit boards have a USB input power terminal or jack port, and you may need to check if its standard size or miniature size, etc. The default (constant, regulated) charging current is set by an on-board current setting (or "programmable") resistor so as to be 1A max. and this is usually sufficient, however when charging smaller lithium batteries at only 10% of their rated max. (constant) current capacity, say 0.1A = 100mA charging current, the on-board current setting resistor should be calculated and increased in ohms so as to reduce the current to that desired value. Some of these circuit boards and-or lithium battery protection circuits also have a low voltage cut off feature so as to protect the lithium battery from being drained too low at about 2.5 or 3v. Note that a white LED operates at about 3v or slightly higher (say between 2.5v to nearly a 5v range - if the current is controlled to less than about 30mA, and 20mA is often the recommended and test current for its light output rating - about 1 lumen from a 15° beam angle of the LED). These common lithium batteries are about an inch thick and 3 inches long, and many can be electronically connected together in series so as to create and increase the total output voltage, or be connected in parallel so as to increase the maximum current availability and-or a longer time of use before that battery is drained of its energy (electrons and their charge) and needs to be recharged so as it can be used again as a power source.

Connecting sets of **similar** batteries in both a series and parallel configuration, both the total voltage and total current can be a multiple of just one single battery, and therefore the maximum stored energy and-or the maximum output power ability will be a multiple of just one battery. When a battery's stored charge (electrons) and energy is draining through the circuit, the voltage across its terminals will reduce since V = J / C. Any energy or power lost in the circuit is equivalent to the energy or power delivered, drained or removed form the battery. Once this voltage level of a battery has reduced to a certain rated, "safe" low value, the battery should be recharged before using it so as to always have the maximum possible energy available, and-or to prevent damage to that battery.

As of the year 2020, most electronic devices can now be designed to use rechargeable batteries as their power source, and so as to the reduce the incredible waste, time and costs of standard non-rechargeable batteries. The most common rechargeable battery in use as of the year 2020, contains a metal mixture that contains some lithium metal on one of its internal plates. Lithium is classified as a "rare-earth" metal and is therefore correspondingly expensive. There is always research being done so as to create rechargeable batteries either cheaper and-or with a higher energy density.

As of the year 2020, a typical "lithium battery" of any amp-hour (aH) rating has only about 0.3 grams of pure lithium metal per 1 aH of battery rating. This mathematically corresponds to 1 gram of lithium =~ 0.035oz of lithium per 3.3aH of battery rating. 1 oz =~ 28.35g of pure elemental lithium metal, as of the year 2020, will cost about $10900 USD which is about the price of a good used automobile or about 5.8oz of precious gold. The price per gram of lithium is about: $384USD/gram, and so it is well worth recycling it. Lithium-ion batteries have a "life" (usefulness) of about 400 charge cycles (full charged to fully discharged) on average, but can have many times more "partial charges" if the battery is only partially discharged and (slowly, low current, say (1/10) the max rated output of 1C , hence 0.1C.) recharged, and which is recommended so as to prevent crystal-like growths within the battery that will limit its (energy, charge, electrons) future storage capacity and "life". A typical constructed, currently popular, 18650 lithium-ion rechargeable battery has an amp-hour (aH) rating of about 2000 maH = 2 aH, but it may be more for some brands. This battery is constructed like a long, rolled up capacitor with two plates or electrodes, and of which due to its length, its internal resistance increases, and so as to decrease it, the plates are made to have a larger area for larger batteries with a higher aH or Wh capacity. A **"protected lithium-ion" battery** has a small internal circuit (often called a Protection Circuit Board - PCB) so as to prevent over-charging (>4.2Vmax), over-discharging (< 3V, and with about only 20% of its total internal charge remaining) and sometimes may have short circuit, current limiting and-or thermal protection, and these batteries will cost more, but it is still a good value, and these are a slightly longer (by a few mm) 18650 battery with a "button top" instead of a "flat top" regular 18650 battery. Most lithium battery charger devices (ie., a charge controller") and-or batteries include some battery protection circuitry (**BMS B**attery **M**anagement **S**ystem), but do not assume an electrical device, being powered by unprotected batteries as having any battery protection circuitry within it, and so it is wise to purchase batteries with the protection circuit already built in. For extended life or charge cycles, it is generally not recommended to charge lithium batteries to its max. possible 4.2v, but say 85% of this value which is about **3.5v**. If you are powering low power devices such as a LED flashlight,then 3.2v will do alright, and it will extend the battery life even further. If you are placing batteries in series to power a device, then you must account for what voltages are in effect with this system, and how to charge this battery. Sometimes each battery can be charged separately, or in series, or in parallel. It is best to charge only similar batteries of the same capacity and voltage rating if they are to be used to create a larger battery with a larger capacity or power output rating. Similar batteries in parallel will have a max. current rating of about the sum of the rating of each battery. If possible monitor the voltage (usually when operating, with a power load) of a battery, and small, inexpensive LED voltage displays are available, and of which can be occasionally operated by installing a push button switch.

It is possible to make a homemade, portable, and-or emergency power supply for an homemade (electric) **arc-welder** by connecting about three 12v large, high current batteries such as automobile batteries in series with thick wires that can transmit high amounts of current due to its low resistance. Technically, 1 to 4, 12v batteries will work, but lower voltages produce a lower amounts of current and-the corresponding generated heat to melt metal with will be lower. Higher voltages above 48v are generally not used for arc-welding. Common "jumper cables" (wires) for a typical automobile batteries are rated at about 400A, such as to power the starter motor (ie., "cranking" or [piston, movement cycle] "turn-over" so as to rotate the pistons and motor for needed air and fuel compression and then fuel ignition and a started, self sustaining motor) during a brief amount of time, however to charge an automobile battery, usually much less current is used and for a much longer period of time (ie., "slow charging"). A **spot-welder** is designed to give a brief high pulse or amount of current so as to quickly melt two pieces of metal together in a small localized area A spot-welder is often used in the process of electrically connecting batteries together using thin strips of nickel metal. An option to this method is to purchase specially made battery holders that can hold the entire amount of batteries needed, and which allow connections to be made by wire soldering or already have the connections made as part of the battery holder.

On the amount of power used from a battery:

A battery may be rated by the recommended, maximum amount of power that can be safely drawn (taken, removed) for one hour of time so as to keep it from getting hot. This value does not consider any high power drains of energy during just a brief amount of time; perhaps a minute or less.

Since 1 hour = 3600s, 1wH = 1 watt of power applied for a total of 3600s = 1 joule of energy used per second, and for a total of 3600 seconds, which may be intermittent and actually occur during a much longer period of time, and sometimes with a high amount of energy or joules being used, and sometimes with a low amount of energy or joules being used.
1 wH = (1J/s)(1H) = (1J/s)(3600s) = 3600 joules of energy. :1 watt-hour = 1wH : total energy available or used

To help rationalize this value, consider the energy needed and-or stored by lifting a mass of 1Kg a height or distance of 1 meter: E = joules = (force in Newtons)(distance in meters) = (ma)(meters) = (1Kg)(9.8m/s^2)(1m) = 9.8J =~ 10J (ie. 10 Joules per kilogram, and per meter of distance = ((10J/1Kg) / m) = 10J / 1Kg-m). If the distance was 1000 times more, hence 1Km of distance, the energy needed and-or stored would be 1000 times more at: (1000) 10J = 10000J.

Likewise 1 kilowatt-hour = **1 kwH** = 1000 watts of power applied for a total of 1h = 3600 seconds. 1kwH = (1000J/s)(3600s) **= 3600000 joules of energy** = 3.6 million joules of energy = 3.6 MJ of energy = 3.6 mega-joules of energy = 3.6 (10^6) Joules .

Watts = energy / second = Joules / second = J/s . V = J / C , J = VC and A = C/s , C = As , using substitution:
J = VAs = (power)(s) = **(watts)(s)** , also note above that: s = C / A = (total charge) / (rate or amt. of charge flow)

1 watt-hour = 1 wH = (1 watt)(1 hour) = (1 watt)(3600s) = (1 watt = VI of energy or power) for (1 hour or 3600s of time)
Note that 1 watt-hour of energy can be obtained by many other ways such as:
1wH = (0.5w)(2) = (0.5J/s)(7200s) = 3600J

1 kwH = 1000 wH = (1000 watts)(1h) or= (1000 watts)(3600s) = (1000J/s)(3600s) = 3600000J = 3.6MJ of energy

A lithium-ion battery rated at 3.6v, and 2000 maH = 2aH of current = 2 amps of max. rated current for 1 hour of time, will have a power rating of: P = (V)(I) = (3.6v)(2aH) (3.6v)(2a) for (1H of time) (3.6v)(2a)(1H) = 7.2wH = 7.2 watt-hours. Since 1 hour = 3600s, 7.2wH = can be thought of as: 7.2 watts applied for a total of 3600 seconds = (7.2w)(3600s) = 25920 watt-seconds = 25920 (J/s)(s/1) = 25920 joules in total. Some simple examples of using this rated power value would be using 1 watt of power for 7.2 hours, or 7.2 watts of power for 1 hour.

On the (output) current and-or power rating of a battery:

A battery may be rated as how long it can supply an amount of current an at the battery's rated voltage. This does not mean the a battery is completely drained of its stored energy, and in general it probably has at least half of it left. If the battery is not of the rechargeable type which can be damaged if it is drained of too much energy and-or for too long, that semi-drained battery and its energy can possibly be used for circuits that require a lower voltage. Two lower voltage batteries can also be placed in series so as to have a larger net or total available voltage equal to the sum of voltages.

1 aH = 1 amp-hour = 1 amp of current flowing or delivered for 1 hour of total time of which cold be intermittent.
1 aH = 1 amp-hour = 1A for 3600 seconds of time = 1 Coulomb of charge for 3600 seconds.
Voltage = joules of energy per coulomb of charge = V = J/C. Mathematically, Energy = VC = J = Joules.
Power = energy / s = J / s = VC / s = V (C/s) = VI with units of watts.

Small batteries with a low energy storage capacity and a current ability less than 1 aH are then rated with a lesser unit value which is usually mill-amp hours = maH. Ex. 200maH and 500maH. Sometimes you may find a battery rated as perhaps: 1500 maH = 1.5aH

A 5aH battery can supply a current of 5A for 1 hour.

The rated amount of amp hours of a battery does not mean that the output current being drawn by the load is this value, or must be less, but the amount of amps drawn could be a higher value, and as long as the battery's power rating is not exceeded at any one instant. If the battery's data sheet is available, it may state a maximum continuous current value. Many batteries also allow for a brief (less than a few seconds), high current (often rated) that can be drawn from it.

For example, a cay battery may be rated as 80 amp-hours of current train. 80 aH. means that up to 80 amps of current can be safely drawn for 1 hour of time, and before the battery needs to be recharged. Since a car battery is rated at 12v, if 80 amps of current were drawn by the circuit and pushed out of the battery at this voltage pressure (emf), the maximum rated amount of power that the battery can deliver at its rated voltage is: P = VI = (12v)(80A) = 960 watts of power = 960 joules of energy per second = 960 J/s. If the load circuit is drawing more than this amount of the battery's rated power, then that battery will be drained quickly (probably less than 1 hour of time) and-or damaged due to high heat.

Ex. A 20W device can be powered by a 10 aH rated battery for how long? aH is the product of the continuous current rating and time rating of that battery. The current drawn or needed from this device will be determined by the specific applied voltage to it. If the supply voltage is 12v, then the continuous current drawn for this device is (I = P/V = 20W/12v) = 1.67A Note that 1 aH = (1A of current) for (1 hour of total time of use), or= perhaps (0 .5A) for 2 (hours) etc, and where the product of the (total time) and (total allowable amperage) will equal the (a)(H) = aH rating. 10aH / 1.67A = 6.25 hours. This mentioned battery having a rated output of 12v and a 10aH rating will then have a wH rating of: (V)(aH) = (12v)(10aH) = 120Wh rating. 120Wh / 20W = 6 hrs of total time of use.

Related to the above topics is this article further ahead in this book: A general example of a car or automobile battery and (electrical power) inverter system

Testing the maximum current rating of a battery:

If a battery is rated at (n) Ah, then it should be tested for a few seconds if it can deliver that rated (n) amps of current or whatever rated or stated value it has the max. current. A current and-or amp meter will be needed for this, however a voltage meter can be used to indirectly measure it by measuring the voltage across that metal path, and then dividing by the resistance of that piece of metal used: I = (V across metal) / (Resistance of that metal). The resistance of the metal bar used should be low, and the bare should be thick enough not to glow hot and-or melt. In general, if a battery can only deliver 80% or less of its rated current, it should be replaced if more current is required by the devices(s). A lower current output means a higher internal resistance and power loss within the battery, and it may get hot and-or possibly start a fire or get shorted internally and potentially explode. At the max. rating of current output, say for a 12V battery, and of which is often charged to a higher voltage so to account for its internal voltage drop when it is connected to a circuit, its voltage should be at least 12V or slightly higher during that maximum current test.

For a given battery powered electronic device, such as a flashlight ("electric lamp"), help the consumer and-or user know how long a device will last with a given battery. To know this value of time, time duration test(s) can be performed by powering and-or using the device under certain conditions, such as the air and-or device and battery temperature used, etc.

A special note on using lithium batteries:

As of the year 2020 there are other battery types being considered and-or developed, and they use less or no rare-earth metals, and which could then be an alternative to using the various types of relatively expensive lithium-ion batteries. As of about the year 2020, there are similar size rechargeable lithium batteries that sometimes can be used to directly replace the common or standard sized alkaline batteries. These (usually **1.2v**) lithium batteries are usually slightly lower in voltage by about **0.3v** than the commonly available **1.5v** alkaline battery types in many stores, and this might be problematic for some electrical devices it is to power, especially if the lithium batteries are to placed in series and-or parallel so as to have certain value(s), particularly the voltage, otherwise, they are generally good to use and will save much money, and are therefore recommended. If you will purchase lithium batteries, be sure that you also have and-or purchase a lithium battery charger device.

Automobile ("car") batteries have a standardized, average useful voltage rating of 12v, and most can be charged up to be a maximum of 14.4v. The min. voltage is actually about 12.6v. The average of these two voltages is (12.6v + 14.4v)/ 2 = 27v/2 = 13.5v Since the internal resistance of a battery can build up and reduce output power, the output voltage of a battery should be checked under load (drawing moderate current, ex. with the headlights on) so as to measure the actual "running" or "working" output voltage level of that battery when its internal resistance to current is in effect with the load circuit. This is a reason why a battery is charged slightly above its rated voltage value, and so as to remain longer at its rated voltage value during use with energy discharging from it. Each of the 6 cells connected in series inside a lead-acid 12v rated automobile ("car" [cart]) battery can be charged up to have a voltage of 2.4v maximum. (2.4v)(6) = 14.4v For a 6v rated lead-acid battery with 3 cells in series, then the maximum voltage of that battery is: (2.4v)(3) = 7.2v

Automobile batteries are usually the lead-acid type, are very heavy due to having (dense and heavy) lead metal plates in it, and are not designed to be much below 12v at which point the battery can be damaged by sulfate crystal growth (from the sulfuric acid electrolyte) and-or it may then have problems recharging from the electricity generator in the automobile, and it is best to slowly recharge that battery using a low current if this is the case, and so that there is more time to effectively reverse the problem. A damaged battery may have a reduced current output, and may not be able to charge up to its rated voltage, hence its total power output will be reduce.

The liquid electrolyte between the plates of each cell in a lead-acid battery helps the creation of free electrons and current conduction through the battery without shorting out the plates of the cell with a direct connection. Automobile batteries are designed with large lead plates and so as to provide low resistance and to release a large amount of power (energy, joules, Power = (V)(I)), and particularly in the form of a large amount of amperes of current in a brief amount of time so as a high torque, electric "**starter motor"** can "turn" (move) the cylinders in the engine and cause the initial combustion to get the engine functioning without any further need for the high starting power. Once the car's engine is functioning or "running", the battery and a high voltage step-up **ignition coil** is still needed to produce a high voltage spark in the combustion cylinders when a piston in the cylinder routinely compresses the gas-air mixture near the spark-plug terminal. The spark will ignite the compressed fuel and cause it to combust, and the resulting hot kinetic gases will push the piston downward, and this motion and force will eventually be transferred to the wheels of the car via the mechanical rotating drive train or shaft, and causing the wheels to rotate which then move the car.

A battery will be recharged as needed via a (diode(s)) rectifier and voltage regulator circuit, and while the automobile engine is working and the generator (or "AC alternator") is recharging that battery. When an automobile is not in use, it is recommended to shut the power off to all unnecessary devices being powered by its battery so as to prevent the battery from being drained of its stored energy. If it gets drained, it is possible to "jump start" or recharge that battery by using high power "**jumper wires**" for the connection cables that are usually red and black in color and made of thick wire that connect that drained battery to the same polarity terminals of the battery of a running automobile or battery charger. These cables are relatively inexpensive and should be stored in the trunk of the automobile so as to have them available when in need. Portable and rechargeable "battery packs" can also be purchased for emergency use for when a battery is drained. Note that it will take about 400 "cold cranking" amps or more, but briefly in time, start an engine using a lead-acid battery, and that **most lithium batteries are not capable of providing this and starting a car motor**, and since it may damage that battery which is not designed for this amount of current through and out of it. Automobile battery chargers that use "household" 120Vac power can be purchased and can charge an automobile battery up to 14.4v at reasonably high currents if there is no overheating of the battery such as if it is internally damaged. If enough time is available, relatively low currents of just a few amps, say 1A (for low aH batteries) to 5A or 10A (for high aH batteries), is recommended to charge a drained battery. The recharging current to an automobile battery will also decrease as the battery voltage increases, and this is due to that the difference in voltage potential between the charger and the automobile battery is getting less, and there will be less effective voltage available to charge the battery, and this is generally why it takes so long to charge a battery. Some lead-acid batteries use water as part of their liquid (acid) electrolyte and it could become low in volume over time and use, and it then needs more water, and rather than use "tap water" with various minerals (ie., elements) in it that can reduce the function of the battery, try to use pure water only, and this can be purchased as **distilled or "evaporated" water** that is available at many stores.

As of the year 2021 there have been many recent developments in battery materials, construction and better storage ability or capacity. It is promising for the future when people and businesses are becoming more focused on "green (renewable, non-fossil fuels)" energy and energy security so as to have enough electric power available when needed, and this will require more localized power storage and-or generation stations, and of which will also include homes generating electricity from solar panels on their roofs.

As of the year 2024, the retail or store price of a lithium-iron-phosphate (LiFePO4) battery is about **$3.5 USD per aH** = (10000mAH) for batteries less than about 50aH, and is about **$2 USD per aH** for greater aH batteries. For an example, a 3.2V battery of this construction could power a 25mA white light LED for: (1000mA / 0.025) = 40 hours , hence almost 2 days of continuous use, however, if it is on for only a few minutes at a time, there could be many times more days of available use of that light circuit, perhaps 30 days. Be sure to buy a charger also. About the price in terms of rated Wh = (rated voltage)(rated aH current), Ex. 1 Wh of energy from a lithium battery is about (12V)(50aH) = 600Wh. If the price of this battery is $100 USD, the price per **Wh** is: (price) / (Wh) = $100 / 600 = $**0.17 USD / 1 Wh** For comparison, this is about the year 2024 cost for 1KwH of electricity in the USA, and which is a thousand times more power, but if the battery can be recharged say 1000 times then the lifetime use of that battery is then about $0.17USD / 1 Wh. Larger aH batteries are commonly about 10% to 20% lower in price per aH and wH due to that the sale price is essentially due to bulk size and-or energy capacity. The main problem with larger batteries is their increasing size and weight, and more difficult portability.

Some further lithium and-or rechargeable battery considerations:

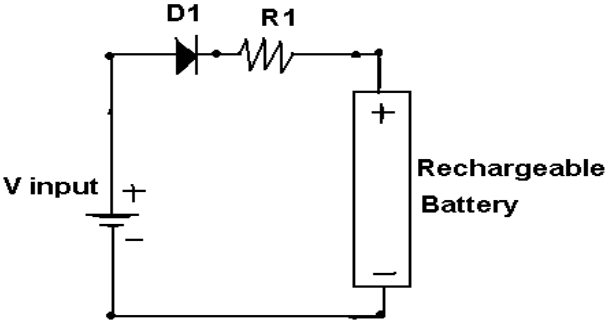
Typical Max. Voltage: 4.2 v
Average and-or Rated Voltage: 3.7 v
Min. Voltage (3 to 3.2 v)

Look on the battery label or data sheet for its mAH charge and-or storage capacity:

Ex. A lithium battery may be rated as: 3.7 v , 550 mAH , 2.4 Wh . 3.7 is the rated and-or typical (ie., average) voltage of the battery. 550 mAH = 0.550AH is the maximum (and safe, low heat) current drain, and is available for 1 hour total, before the current is reduced to a lower value since the voltage of the battery will be lower. 2.4 Wh is the total rated power available consistently. Pw = (VI) = (4.2V)(0.55A) = 2.31w calculated =~ 2.4 Wh rated. 2.4Wh can be thought of as 2.4W for 1 hour, or even 1W for 2.4 hours, and so on, and the product of (watts)(hours) is always 2.4Wh for this specific battery.

In general, the maximum voltage of a lead-acid car battery is about 1.2 times at of its rated 12v value, hence about 20% more than its rated voltage. For example, the max. voltage for a 12v rated lead-acid battery is: (12v)(1.2) = 14.4v. For a typical lithium battery with a 3.7v rating, the factor is slightly less at about (1.14) such as for the 18650 battery, here, the max. voltage is: (3.7v)(1.14) =~ 4.2v . This is about 14% more than its rated voltage.

Here is a simple, homemade rechargeable battery charging circuit example. [FIG 234A]



In the above circuit, V input is a DC source and-or supply of power (V and I), but it could also be an AC source if it is converted to a DC source and-or steady, constant power with components such as a voltage regulator (best, since it eliminates the internal resistance of the battery into the charging circuit), and-or capacitors and-or diodes such as with a "bridge rectifier". The source may very well be a 5V USB supply. The diode represents voltage regulation and-or a voltage drop such as 0.7V across a silicon diode, and which then leaves (5.0 - 0.7v) = 4.3v to charge the battery. If the supply voltage is higher by about 0.7V, you can use several diodes in series to reduce the supply voltage to that needed for the specific battery being recharged. A diode also prevents discharging the battery through the voltage source if it that voltage source gets lower than the voltage of the battery and there is then a new difference in voltage (ie., emf or [voltage, energy] pressure difference). The resistor chosen is to reduce the maximum charging current to the battery to say 10% to 20% of its rated capacity for heat, fire and possible explosion safety reasons. **(Imax of circuit being powered by the battery = Vmax / mAH)** , and for charging the battery safely us a resistor: **R = Vmax / (10% of Imax)** = 4.2v / (10% mAH). Ex. R = 4.2v / 0.1A = 42 ohms, and a 47 ohm resistor is a common and close value to use. Pr = I^2R = (0.1)^2 (47ohms) = 0.47W, hence a (1/2)W = 500mW = 0.5W power resistor is needed, or possibly use two, 22 ohm, (1/4 W) resistors in series, or two, 100 ohm resistors in parallel, etc.. Charging a battery that was drained to a low voltage is a fire hazard due to a relatively high internal resistance that will dissipate heat into the battery., and it is recommended to use a very low current if it must be done. For most circuits, consider an on/off switch an indicator light(s) such as an LED(s) in parallel with the supply battery with a series resistor to limit its current to a safe value. As the voltage of the battery being recharged increases, the current to it will decrease due to the "back or reverse emf" making the net potential difference applied and current lower, and it is then possible to then reduce the value of R1 to a lower value - perhaps using a resistor(s) switched in series or parallel to it., and so as to increase the current to the battery to its 10% or 20% (charging and discharging) rated **current capacity rating** (ie., "**C-rating**") value. Ex. 0.2C = 20% of its continuous current rating. Note the C-rating is different than, and usually much lower than the temporary maximum "surge" current rating of a battery. Many charging and-or discharging modules (pre-made circuits) have a constant current output and other battery protection methods such as overcharging prevention, low voltage shut off protection, short-circuit protection. Larger batteries will usually have a larger C-rating and can store more energy per unit of volume and or weight.

How fast the voltage (ie., potential energy) of a battery is reduced depends upon the amount of energy drained and-or used form the battery. Lithium batteries are said as not having a (plate or electrolyte) memory and-or required need to fully discharge slowly and recharge slowly, but still a lower charge and discharge current say 10% to 20% of the maximum rated capacity is recommended to reduce the internal heat (ie., power loss, due to resistance) within the battery, and therefore to prolong the life and number of charges (ie., charging and discharging cycles) that the battery will have. It is recommended to not drain the battery to less than 30% max, however, 50% max. is a more typical value of its rated total capacity - and that for a lithium battery is **not** (3.7v x 50% = 3.7v x 0.50 or= 3.7v / 2 =) 1.85v, but is rather about 3.5v and where the internal resistance of the battery has increased, limiting its current ability and increasing its internal heat and power losses. Keep lithium batteries away from both internal heat (due to current) and external heat sources (such as direct sunlight energy). Store lithium batteries at about 50% to 60% of their rated energy or power capacity (here, rated ability). In general, the smaller the volume a battery of the same given materials, the smaller its charge and energy capacity. While a lithium battery is discharging, its rate of change in voltage is relatively a low as compared to other battery types, hence it maintains is working or rated voltage quite well. A lithium battery can output a higher percentage of its total internal charge carriers (ie., electrons). **Considerations**: A load device needs a certain amount of power (ie., P = VI VA) to operate or function, and this is product of the current and voltage to it, and of which must be maintained for it to function, and this 50% (or some other value) "charge left" concept is rather 50% of the total time available for powering that specific device. **When the voltage of a battery is less than its rated value, the maximum output current is then less than its rated value, and the battery is then considered as drained ("0% available" , due to the battery now having both less than the rated aH and Wh available) or below its rated value**, and a connected load which depends on these (max.) rated values and-or power delivery may cease to function unless some voltage tolerance was designed (ie., by an electronics engineer) considered and "built into" its construction. A battery which is useless for some devices may still be able to power devices with lower power needs, however, this may eventually damage a battery unless in an emergency situation where every watt of power is needed.

Some common cylinder shaped lithium batteries, their identifier number and size, and max. current rating:

11640 = 560maH typ, , 14500 = 750maH , 18250 = 650 maH , 18350 = 1200mAH typ. ,
18650 = 3000mAH typ. = 3AH (common, and an example lithium battery) , 21700 = 4500 mAH

Capacitors store electric charges (electrons on one plate, and positive ions on the other plate) and their energy on its two closely spaced, thin metal plates (ie., surfaces, with area for the charges to fit) with the aid of the electrostatic (or electric field) force between them. Since the charges no longer have kinetic energy are not moving, there is no magnetic field created or involved in the storage of static or stationary electrons. The kinetic energy of the electrons has been transformed into electrostatic (field) energy which is a form of potential energy. Since the charges on a plate are similar, they are in repulsion to each other after being forced onto the plate with an electromagnetic force of repulsion by the same polarity, and attracted to the plate by the electric field of opposite polarity on the other plate. These stored charges can then be thought of as like a spring that has been either compressed or stretched and having stored or potential energy to do things. Capacitors were also mentioned previously in this book. For comparison, an **inductor** (ie., a coil of wire) will temporarily store the kinetic energy from electrons into a generated magnetic (field) energy around that coil.

A thin layer or coating of material called a dielectric is often placed between the plates so as to increase the capacitance ability and rating (measured in Farads units) of a capacitor. The dielectric essentially allows the electric (force) field to pass through it, but should also prevent the electrons from passing through it as a ("leakage") current. One possible dielectric material is ceramic, and then these capacitors are simply called as ceramic capacitors. One plate will hold electrons, and the other will hold positive ions. Positive ions are atoms that have lost an electron, and therefore have a net positive charge. A capacitor has a capacitance or capacity rating (in units of Farads) which is a measure of its ability to store energy, and it does this by a transformation and storing the kinetic energy of the electrons into electrons that will be held in position (somewhat like a compressed spring storing energy) by the electric (force) field (emf) of the positive and negative of charges on the other plate.

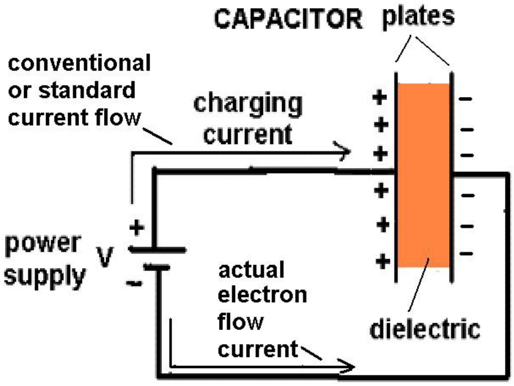
A capacitor will store its charge if there is no longer a complete circuit attached to the capacitor that allows electrons to flow and drain its potential energy, or that the voltage potential (emf, force or pressure) across its terminals has not decreased, creating a net potential difference and which would allow charge to flow towards the lower voltage potential and drain the energy in the capacitor. Remember that electrons have the same charge and will repel each other unless they are held in position by another force, and that force is the electrostatic (charge) force of the charges (ie., charged particles) on the other plate and-or the electric force from an external voltage source.

An older word for "capacitor" is "condenser" (ie., charge storage via electric forces, concentrating charge). A larger area of the plates will allow more charges to be stored on those plates, basically because there is more area or "room" to do so, and with less repulsion by the same type of charge. The charge, particularly moving or flowing electrons, is accumulated onto the surface of the plates when a voltage (emf) is applied across the plates. Even though electrons repel each other they can still be placed on a plate if there is enough room and there is enough voltage (emf) to force them to be on the plate. The amount of charge and-or discharge current is determined by any external resistance (R ohms) to that capacitor, and this will also increase the time of maximum charging and-or discharging of the stored charges as a current. A higher resistance to the flow of current means that the current and power (joules/second or energy/second) output of a capacitor will be lower to a load. In a circuit, the voltage (emf) or energy in a capacitor will decrease as each stored electron leaves the plate and takes some energy with it, and then the current to a circuit will also decrease because there is less electrons available and at a lower voltage applied to those left on the plate of the capacitor.

In general, the larger the capacitor and-or the larger the same type of capacitor construction and materials, the more charge and energy it can store. Having a larger capacitance in a circuit is not always desired, and particularly for high frequency circuits which require a capacitor with a fast charging and discharging time period, such as to reach the peak supply or signal voltage in a brief or fast amount of time. It is a fact that the smaller the rated capacitor, the easier that a higher frequency (ie., AC, or [DC] varying, pulsing) signal can then effectively pass through it via the electrostatic forces on the plates of the capacitor, and which will create the resulting charge (ie., electron) or current flow of that input signal. A small capacitor connected to circuit ground (ie., having a low voltage or 0v potential) will effectively cause electrons or current to more easily flow there, and this can be used to filter out or remove unwanted high frequency signals such as some electronic noise and-or high frequency oscillations causing a disturbance in a circuit. Because a small capacitor can more easily pass a higher frequency signal, it likewise effectively blocks the lower frequencies more, and a small capacitor is therefore a low-frequency signal (V, I, and frequency) remover or filter.

Capacitors have a maximum voltage rating which can be applied to and-or across the plates at which a "voltage breakdown" can happen and then it allows conduction through the dielectric and to the plates of the capacitor, and therefore "shorting (no more capacitance, very low resistance) out" and-or permanently damaging the capacitor and possibly the rest of the circuit, and even cause a possible explosion of the capacitor due to some internal hot gasses created. Often a capacitor with a voltage rating of the (input voltage) + (half of the input voltage or more) is chosen to be in a circuit, and this gives some circuit damage prevention. For example, if a capacitor will have 5v potential difference across its plates, then use a capacitor that has a voltage rating of at least: 9v + (9v / 2) = 9v + 4.5v = 13.5v or higher.

A capacitor can be thought of as a type of battery, but it has various other uses such as for blocking direct current, and for use in **electric oscillators** (create a periodic voltage and-or current, AC waveform, hence an AC signal generator), frequency "tuners" and-or filters, and even an AC voltage divider since capacitors have an effective resistance due to how it reacts (ie., [AC] reactance) with an applied AC signal and its frequency. In the circuit below, the capacitor is essentially in a parallel connection with the voltage source or supply, and that capacitor will charge and will be the same voltage as the voltage source, and then the current flow will stop because there is no difference in voltage potential (emf) for any current to be flowing. For example, if the supply voltage is +5v and the capacitor charges up to +5 volts and effectively in the opposite or reverse polarity direction, there is a net voltage difference of 0v, and which will cause no flow of current. A resistor could be placed between the power supply and the capacitor so as to reduce the charging current, perhaps to a safe level, and this will also cause an increase in the time it takes to charge that capacitor up to the supply voltage. The limited amount of current due to the resistor will also decrease even further due to the increasing "back-emf" or (reverse polarity) voltage (emf) of the capacitor. A formula for the voltage or current of a charging or discharging capacitor is shown ahead in this book. [FIG 235]



Considering the common conventional current flow analysis from the positive terminal of a voltage to the negative terminal of a voltage or potential difference, and which could be thought of as positive or real energy flow analysis, the positive terminal of the battery will repel and force like charges which then will gain kinetic energy and will also be forced and-or be repelled onto the positive plate of the capacitor as potential or stored energy. Likewise, opposite polarity charge will be stored on the opposite side of the capacitor. The unlike charges across the capacitors dielectric material will also attract each other due to their electric or coulomb forces, and the capacitor will store those charges, even if the input voltage is removed.

Super Capacitors can have a capacitance value in many Farads, however,their rated voltage is relatively low, perhaps just 3 or 4 volts maximum, enough to power a 3 to 4 volt LED, however, placing several identical capacitors in series can increase this working value, but it also reduces the total effective, series capacitance. Capacitors can have a high rate of charge and discharge currents as compared to a battery, and that the charge time of a capacitor is also much quicker. The (RC) charge and-or discharge time constant can be low if R is very low and C is very high, and this means a high current is also possible, and this means a high amount of power is possible (P = watts = VI = joules of energy / second).

To create practical **radio waves** at a specific or tuned frequency for transmission and reception, a stable (ie., non-fluctuation in frequency) **electric oscillator** circuit is required. To receive those same radio waves being transmitted at a specific frequency (such as that being transmitted from a certain radio broadcasting "station" and its transmitting antenna), an electric circuit called a "tuner" in a radio receiver is "tuned" or adjusted to accept just that particular frequency and reject, block or remove all the unwanted frequencies, particularly those higher and lower in the desired frequency and-or a limited band, bandwidth or range of nearby frequencies necessary to contain and convey the source information. A tuner circuit is actually a type of (frequency) filter circuit that allows frequency rejection and-or the selection of a particular output signal and frequency.

An oscillator circuit is designed to (electrically) resonate at, produce or generate a specific AC signal or frequency, and which usually has a (natural) sine waveform. A simple tuner circuit contains just an inductor (L) and capacitor (C) placed in a parallel electrical connection, and which they becomes both a filter and-or resonant (ie., oscillator) tuner circuit. This tuner circuit will allow (pass) a certain radio frequency and electronically reject all other undesired frequencies by sending them to the ground or lower (ie., 0V) voltage potential via a lower impedance or reactance path than the rest of the circuit. The frequency filtering will remove unwanted frequencies from being heard and-or interfering with the radio station and information broadcast that you want to hear. The parallel LC tuner actually has its highest electrical impedance (ie., a (frequency dependent) reactance or effective resistance to the AC signals) when the desired frequency of the input signal is equal to the specific natural oscillation and-or "resonant" frequency of that tuner circuit. The result is that the voltage of the desired frequency will be at a maximum value at the tuners output, and then it will be available to the rest of the circuit such as to a diode and earphones in series with each other in a radio receiver circuit. The energy from the desired or tuned signal radio antenna will give some energy to keep the oscillation happening, and so as it does not decrease in amplitude and stop.

There are many ways to make both mechanical (force, energy powered) oscillators and **electronic oscillators. The basic definition of an oscillator is something that oscillates or repeats itself constantly or "over and over" or "again and again", and hopefully at a constant rate or frequency (how many repetitions or cycles / time) during a particular amount of time such as a 1 second time of duration, length or unit.** In terms of electronics, an electronic signal and-or frequency generator contains an oscillator so as to be tuned to and-or create an electronic (varying) signal (V and I) having a desired frequency.

Some high frequency oscillators, such as in the relatively modern and-or advanced "quartz oscillators" commonly used in clock or "watches" so as to keep a high precision of the current time of day, or as for radio communication and test equipment and computers (for the synchronized processes and steps of a computer program(s)) use a tiny man-made, cut and shaped, frequency calibrated **quartz crystal** (silicon dioxide) oscillator circuit that is tuned or set to have a very high natural or physical electrical and-or mechanical resonant frequency due to its small size. This electrical oscillation behavior is effectively equivalent or comparable to that as if the crystal had both a small internal capacitance and inductance, "tank circuit". Crystal oscillators can have a high accuracy and with an error (+, -) of just a few seconds a year, and this is a very small fraction of the total amount of seconds in a year. The total amount of seconds in a year (365**.**25 days) is: (3600 s/hr)(24 hr/day)(365 days/year) = 31,557,600 seconds / year.

The crystal in a crystal oscillator will physically deform, compress, and-or bend slightly like a compressed and expanded spring when a voltage (ie., energy, emf, electrostatic force) is applied to or across it. This was discovered in 1880 by **Jacques and Pierre Curie** (husband of **Marie Curie**) of France, and it is called **piezoelectricity** (piezo is a prefix and old word for pressure, and as in a electricity generated by force or pressure). During a decompression (ie., a lesser force and-or pressure applied), the crystal will release its gain energy as a sinusoidal voltage at a very stable or constant frequency determined by its physical characteristics, dimensions or size, particularly after cutting it so as to be a certain desired frequency. Crystal controlled oscillators were first built in about 1917, and became practical around 1920. By 1950 large synthetic (ie., man-made) quartz crystals were being produced and having a very high quality were then being cut and polished (fine tuned, shaped) so as to produced crystals for high frequency crystal oscillators. Without these crystals is highly questionable if we could have much modern (2022) technology such as satellites, computers, digital clocks, and "smart phones".

Before modern, more accurate timing was developed by using crystal (or "atomic") oscillators was used for timing and electric powered clocks, the methods for timing were the position of the Sun and stars, sundials, water clocks (ie., raising fluid level in a calibrated container), and **pendulum** (a mechanical, swinging, oscillating mass) and spring ("wind-up", stored energy) powered clocks. The study of timekeeping is called **horology** (a word based on an old Greek word "hora", and means "hour" in English), and some impressive kits (projects) are available so as to build a wooden geared clock which has a pendulum to keep fairly accurate time. In 1637, **Galileo Galilei** developed most of the mathematical relationships for the swing time (ie., a periodic or routine amount or period of time) equation of a pendulum and considered the device for time keeping, and which included an (mechanical) escapement gear for it to function properly, but **Christiaan Huygens** (1629-1695) from Denmark and a contemporary to Isaac Newton, actually built the first pendulum powered clock in 1658, and it kept a highly accurate time of day. The escapement mechanism (much like a lever system controlling the movement of a gear) is used to control and-or allow some energy to the swinging pendulum which gradually looses some kinetic energy due to friction, and it also advances the gears of the time indicators ("dials", "pointers", "indicators", "hands"). The source of energy used to power the escapement is from the weight of an object that was initially raised to a height and thereby given some stored potential energy, and which will then slowly reduce in height and loose some of its potential energy when it periodically transfers its (falling) kinetic energy as kinetic energy to move the clocks escapement mechanism and time indicators or pointers. During each swing of the pendulum it would force and permit the advancement or slight rotation (ie., temporarily escape from being held in position) of the escapement wheel or gear by one tooth. The ticking sound heard in mechanical clock is due to this escapement system. A geared escapement system for timing mechanical devices had been previously known before Galileo, and since the 1300's. The oldest escapement mechanism is known is the crown (gear) wheel and-or verge escapement mechanism which was also used for mechanical devices such as clocks before Galileo's and Huygens more accurate pendulum clocks.

A **pendulum** is a mechanical (force powered, non-electric) oscillator. It contains a mass, usually called a bob, that is held in position by a length of a string or wire. The mass at the end of the pendulum can freely swing (ie., move) side to side and has a (desired) constant time or period of movement during each swing cycle or oscillation. This time period is determined by the length of the string. The longer the string, the longer the arc length or distance of motion of the pendulum mass object (or "bob") and the greater the distance of travel and time of travel, and therefore, the slower the oscillations and longer period time, and less frequency of those cycles per unit of time. The pendulum will cycle through converting its gravitational potential energy (after a person raises it a height) to kinetic energy as it swings. It is interesting to note that the mass and-or weight of the pendulum does not determine its period of oscillation, and the reason for this is that masses of different amounts of weight will fall at the same speed due to being influenced by the same gravitational acceleration (g = 9.8m/s^2 for Earth). Different sized masses and-or weights will also reach the ground or lowest part of the pendulum swing cycle at the same time when raised to and dropped from the same height. A larger pendulum mass will contain or store more kinetic energy ((mv^2)/2) and have more momentum (p=mv) so as to keep swinging, but a heavier weight may also cause more friction due.

Pendulum Swing Period = Ts = (2)(pi) (square-root of (length / g)) : theory, and accurate for small angles of swing. Credited to **Huygens** in about 1673.

In the above formula, length is measured with units of meters form the pivot to the center of mass and-or weight, and time is measured with units of seconds. Note that: frequency = (1 / Period) = (1 / Ts) hz or cycles per second. For a practical example, a pendulum with a string that is 1m long will have a (total swing, cycle) period of 2s to reach the same location it started from, and-or 1 second to swing from say left to right. A pendulum with a string that is 0.2485m = 24.85cm = 24cm + 8.5mm, (nearly 25cm), will have a (swing and-or oscillation) period of 1s. Note that 25cm = (1/4)m. If the string length increases by 4, the period increased by 2, or doubles, and this is the square root of 4. Because (g) = gravitational acceleration is in the period equation of a pendulum, a pendulum can also be used to measure the local gravity in an area. A (very near, metric) standard of 1s unit of time, and-or 1m unit of length can also be verified by a pendulum.

Although it is fairly easy to find the local (north, south) latitude by observing and measuring the angle that the North star (Polaris) is above the local horizon line, it is not so easy determining your local (east, west) longitude. It would take an accurate clock set to Greenwich local time in England. At you local longitude, when the Sun is the highest in the sky, it is local noon or 12 P.M. If the accurate clock shows 1P.M., this is still the local time in Greenwich, and you are 1 hour behind that time, and therefore, you are 15° west of Greenwich.

Some oscillators are composed of just transistors, resistors and capacitors, however some radios use crystal oscillators for the frequency transmission and reception. Transistorized oscillators are mainly based on the fact that a capacitor takes a certain amount of time to charge up when a resistor is in series with it. The first type of oscillators used for tuning (or selecting, adjusting, setting, choosing, allowing) a specific frequency radio waves are called **(LC) tank circuits** or **(LC) tuned or tuning circuits**. This circuit is simply a capacitor and inductor electrically (with a wire) connected in parallel to each other. When energy is applied to this circuit, it will resonate or oscillate current (electrons) "back and forth" from the inductor to the capacitor, and then back to the inductor and to repeat this process "over and over" periodically at a constant amount of time and frequency called the (natural or electrical) resonant frequency of that circuit. Smaller valued capacitors and inductors will have a lower time constant (RC and-or RL) for charging and discharging, and the natural frequency of that oscillator will be higher. The radio receiver and its tuner will be discussed below and in various places in this book.

For a periodic signal and its waveform from an oscillator or repetitive circuit: time = 1/frequency , and frequency = 1/time, hence the time and frequency values are mathematically reciprocal in value to each other. Time is the number of seconds to complete one complete cycle or repetition of the wave, and frequency is the amount of cycles per second of that wave. This will happen for as many cycles as possible and before the energy losses in the resistance of the metal connection wires essentially deplete the energy left or available, and the output voltage and current of the oscillator gets lower and lower in strength (ie.,amplitude or peak) until it is practically useless to the rest of the circuit. To keep the oscillations going and at a strong amplitude, a resupply of (external) energy to that oscillator circuit is needed. A transistor can be pulsed on for a short duration so as to supply some energy to an oscillator so as to keep it going. Due to the electric current or flow of electrons with their electric charge and kinetic energy, the inductor (ie., coil) stores its input energy in a created magnetic field, and the capacitor stores its input energy in a created electric (electrostatic) field or potential.

It is possible to build an electronic oscillator circuit without an inductor. One such circuit is called a (transistor) **multi-vibrator**, and it basically has two transistors, two resistors (R), and usually of the same value, and two capacitors (C), and usually of the same value. One transistor, resistor and capacitor group is connected to the other group a parallel and reverse type of configuration such that when one capacitor is charges up, one transistor will be on and the other transistor will be off, and until the charged capacitor is discharged, and this cycle continues over and over at a certain frequency and-or time period determined by R and C, and which is also known as an (RC) time constant.

The tuned LC (inductor and capacitor) oscillator circuit of a radio.

The simplest of (metal, semiconductor) **crystal radio** receiver consists of just a long antenna wire to receive some radio energy which is radio frequency (rf, high frequency) electromagnetic radiation, and with one end of it connected to the top of the inductor and capacitor - LC-tuner ("tank circuit"). Don't confuse this type of radio and circuit as having a quartz crystal tuner in it, but it rather uses a silicon crystal diode to help demodulate (ie., remove, extract) the low frequency audio out of the high frequency radio signal. The LC tuner-oscillator or transformer connected to the antenna will also have an "earth" or "ground" wire from the bottom of the LC-tuner or transformer and going into the soil so as to be able to make a complete radio wave energy receiving circuit. The ground connection will also conduct the (relatively small) induced or generated current, particularly of unwanted signals of higher and-or lower frequencies to be directed to the ground and effectively removed. Connected to the output of the LC tuner-oscillator is a (semiconductor metal, a crystallized or "crystal") sensitive diode which is then connected to (high-impedance, highly voltage sensitive, rather than current and-or magnetic force driven) earphones with one of its wires connected to the ground connection at the bottom of the LC-tuner so as to complete that circuit of (very small) current flow. To improve radio reception, the antenna can also be "tuned" to collect the desired radio frequency by adjusting its length and-or direction, and this will increase the collected radio signal energy, and then the output power available to the earphones will increase. It is also very possible to use an adjustable capacitor and-or inductor for the LC-tuner circuit, so as to make the LC tuner a variable frequency tuner or selector. This will help improve the section of more stations and their transmitted frequencies. When using a simple radio receive, such as a homemade crystal radio, strong signals, especially from nearby stations may greatly "drown-out" or interfere with the weak signals of other stations. There are many known ways and-or new ideas always being experimented with so as to improve radio reception, particularly for a crystal radio. Some will use a battery or solar-cell so as to give the radio and-or headphones a little more power to work with, and to have an improvement in the output audio level or volume. For some, it is a serious study, and for others, it is rather a novelty to have or try. The resonant or tuned frequency of a LC tank circuit can be shown to be:

$$\text{Fhz} = \frac{1}{\text{2(pi) (squareroot (Lh Cf))}} = \frac{1}{\text{Ts}}$$

: natural or physical resonant or tuned frequency of a LC tuner
L = the inductor, its inductance is measured in Henry's (h) units
C = the capacitor, its capacitance is measured in Farads (f) units
T = total time in seconds of each repetitive cycle or period ,
F is inversely related to the product of (LC)
This formula has a basic derivation in this book.

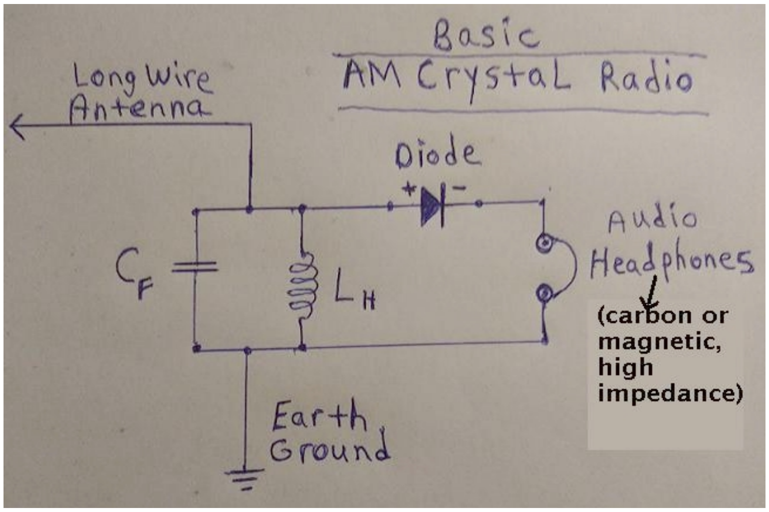
$$\text{Fhz} = 0.159154943 / \sqrt{LC}$$

, from this, you can solve for L or C needed for a certain frequency:

$$\text{Cf} = \frac{0.025330295}{\text{(F^2 hz) (Lh)}} \quad \text{and} \quad \text{Lh} = \frac{0.025330295}{\text{(F^2 hz) (Cf)}}$$

From the above formulas, it is clear that the oscillators resonating and output frequency is inversely related to both the capacitance and inductance values. For a given frequency, capacitance and inductance are also inversely related in value.

To "tune" a radio is to adjust the resonant frequency of the (parallel inductor and capacitor) LC tuner-oscillator or LC (signal, energy) "tank" circuit so as select (choose) a desired or particular radio frequency and filter (here, remove) all the other unwanted radio frequencies. The word "tune" is from the older process of tuning or adjusting an instrument such as adjusting or modifying the tension (force) in a string of a violin. The string will then have a set or fixed desired natural or physical frequency of vibration or oscillation. Shortening the length of that string will cause the natural or physical resonant frequency of that string be higher in frequency. Increasing the length or size of the string will lower its natural or resonating frequency. The ends(s) of the vibrating string are anchored to or connected to the body of the musical instrument. Vibrations and-or the energy of the string will be transmitted or conducted to that musical instrument's vibration "soundboard" or (energy, frequency, vibration) resonator surface, and these vibrations will then be converted into (compressed, pressurized, more dense) air vibrations that are radiated or transmitted outward to the less dense air about it. These compressed and vibrating air waves will radiate or travel outwards from that source, and can then be heard as audio (sound waves) and having the same frequency(s) as that of the vibrating string. [FIG 236]



For experimenting with an AM, crystal radio circuit as shown in the above drawing or schematic, it can be pre-made, homemade and-or purchased as a kit to build. This circuit shown above is considered the minimal and basic AM radio receiver circuit, and of which several helpful improvements have been made later. The following discussion is of the typical common parts, and in particular, of the typical capacitor and inductor values used for the tank or tuner circuit so as to be tuned approximately (via the above calculation) to a common 1Mhz AM radio frequency. Due to this receivers low tuning selectivity (ie., rejecting other close, unwanted and-or interfering radio signals and-or frequencies) issues, you will probably hear some stations "overlap" or interfere with each other, nonetheless, you should be able to at least hear one nearby, "strong" (high power and-or close distance) station so as to verify it is working and-or to help your circuit and reception experiments continue. There are many nice books about understanding and-or making and-or improving crystal radios. Some radio receiver circuit kits include all the necessary parts, including the high impedance piezo-crystal earphone. This type of earphone is a very voltage sensitive, miniature speaker that can flex and-or vibrate when a small voltage is applied, and the amount of current needed is very low, almost 0A. A piezo-crystal has a very high impedance since it is nearly a non-conductor material, and a piezo speaker or buzzer works via the supply voltage creating electro-static forces which will then physically bend the crystal material, hence causing a electro-mechanical vibration of which can then be transmitted to and through the air as sound.

A , the **A**ntenna, 10 ft to 100 ft of insulated "hookup" (connection, or speaker) wire strung off the ground and not during any possible high power (voltage and current) lightning, of which the antenna will essentially become a lightning (directing) rod. During a storm or potential storm, disconnect the outside portion of the antenna several feet from your dwelling, and having several feet of unconnected wire and-or gap (space). Do not connect the antenna or any other other part of the radio circuit to an electric power or telephone line which can then cause injury and-or fire. Because of the preceding facts, it may be best to place the antenna indoors, but if it is long, it will have to be bent so as to make it fit, and-or raised up near the ceiling so as to not cause a potential problem for the movement of people and pets. If the antenna is outdoors, you must disconnect and place it away from your dwelling if lightning might strike it, and it is best to not give the lightning a complete path to the ground and-or electric charges, and behave like a lightning rod.

In theory, the antenna length should be equal to the wavelength of frequency of the radio wave, and be perpendicular to its direction of travel, so as the antenna has maximum interception, reception or signal collection and efficiency of converting the received (rf) energy to electricity energy as the antenna, and much like a musical string, can electrically resonates at the same frequency as that of the desired radio frequency.

Modern portable radios have a relatively small antenna that is only a few feet long and-or is part of the tuning coil wrapped on a **ferrite** (fine ground iron oxide particles mixed with a binder and possibly some other metal(s), ex. ceramic magnets) **rod** which may also be part of a transformer-like construction. Because of this method, the (AM) radio should be physically turned to the direction of the transmitter and radio wave energy for best efficiency and performance. Most portable radios greatly amplify the very weak input signal, and that a ground wire is not absolutely necessary. It is possible to tune a long wire antenna for the best efficiency at a desired frequency by using something such as a variable LC tuner circuit, and this would be electronically and effectively adjusting its length so as to increase the radio circuit's (rf) to audio conversion efficiency. A ferrite rod or "loopstick" antenna is designed to convert the magnetic field energy of the received RF (radio frequency) signal and-or energy to an electric signal energy for the radio, whereas a vertical wire antenna is designed to convert or transform the electric field energy of the RF energy to an electric signal energy for the radio circuitry to utilize and amplify. In general, a loop antenna with no additional core material is also a directional antenna, and it can also help eliminate some unwanted radio interference from other directions. The fine grain ferrite material used for electronics can quickly change their magnetic polarity,unlike a regular magnet. Ferrite can pass a magnetic field easilly, but it is not conductive to electrical current, including eddy currents such in a solid metal transformer design, and which can get hot and waste electrical power, and-or be dammaged.

L and C comprise the basic (frequency) tuner or selection circuit. Somewhat oddly, this circuit is desired to have a very high impedance to the selected or tuned frequency, and for the power matching purposes of the diode and earphone circuit it is to be say about 50K to 100K ohms. The high impedance will then direct some of the signal through the diode and earphones to be heard as sound, rather than all of it going into the ground, low voltage potential, first.
The signal an antenna receives can optionally be filtered by an inductor and-or capacitor so as to remove higher and-or lower frequencies that may interfere with the best reception of a desired frequency. Antennas can also be tuned to simulate a proper match for a desired frequency and its wavelength, by using an inductor or capacitor so as to simulate a longer or shorter antenna.

L , the **I**nductor, is often just a fixed value, but it could have selectable outputs "tapped" along its length so as to tune in the desired radio frequency and-or close frequency range and then fine tune with the capacitor. For this circuit, use **~ 220uH.** 220 uH = 220 (10^-6) H = 0**.**000220 H Use about 100 turns of thin, 30 gauge, enamel coated insulated, copper wire wrapped around a 1" = ~ 2.5cm to 2" =~ 5cm cylinder form of a non-conductive material such as plastic, wood or cardboard. A coil with a larger diameter, but with the same number of turns will have a higher inductance, and therefore, a much shorter length of wire on that coil can be used so as to obtain the needed inductance level.
A typical inductor for a crystal radio will use from about 20 to **30 AWG** multi-strand wire, but solid will work also. Some advanced designs use just a selectable portion of the wound inductor coil, and this is done by creating some taps or connection points along its length, and of which the diode can be connected to by a wire. This will affect the tuning and-or frequency range of the receiver, and could help improve signal strength (ie., sensitivity) and selectivity. Other designs might also magnetically couple (ie., much like a transformer function, isolated, no direct connection electrically) the antenna and-or some "feedback" coil to the receivers antenna and-or tuning coil

C , the **C**apacitor, can be fixed in value, but is often adjustable to as to select the desire frequency: 115 pF
Or by using a variable capacitor that has a range of 0 to 240pF. 115pF = 115 (10^-12) F = 0**.**000,000,000,115 F
Others have simply rounded this value to an average value of about **220 pf**. It is quite possible to make a capacitor out of a piece of writing or typing paper, glue, connection wire and aluminum foil for the plates that are less than a square inch each. It is possible to trim this capacitor with scissors, and as long as it is not "short-circuited" or "shorted-out" when the two plates are in contact with each other. A fixed value ceramic capacitor(s) will also work.

The parallel inductor and capacitor are electrically connected or "hooked up together" so as to be in series with the incoming or received radio signal energy, and are performing the function of a frequency band-pass signal filter that can reduce or effectively reject the amplitude of frequencies that are higher or lower than the desired ("tuned") frequency. The desired frequency will be adjusted or "tuned in" for best reception, and it is of note that other frequencies, both somewhat higher and lower will also pass through the radio circuitry, and this frequency range composes what is called the **bandwidth** (ie., range of frequencies, BW) of that LC (energy, oscillation) tank circuit and-or tuner. Many radio stations are assigned so as to have a maximum transmitted bandwidth so as to not interfere with other stations and their assigned bandwidth. This bandwidth also allows the inclusion of the audio signal or "information" along with the main or fundamental carrier, radio frequency, and the radio receiver (ie., the tuning circuit) needs to have a wide enough bandwidth so as to properly receive all the information clearly. Tuners that are said to have a high quality or **Q-factor** have a more narrow bandwidth or less of a frequency range. The Q-factor of a tuner also helps determine the selectivity of a radio frequency and-or the rejection of other unwanted frequencies.

A LC tuner circuit for a radio will resonate at a particular resonant frequency (Fr), and it will resonate longer when the Q-factor of it is high, and this occurs when the load impedance (ex. R, for resistance) is also high so as not to drain (ie., "damping", reducing) any energy from the oscillations and a resulting further decreasing amplitude.
Qf = R sqrt (C / L) = (Fr / Bandwidth)

D , a small signal or low power **D**iode. A silicon diode is often used today, but a germanium diode is more sensitive and efficient with less signal power loss, and is more practical for a "crystal radio receiver". The diode is also called a "rectifier", and for this radio it removes one half of the AC radio wave signal received and so as the audio or "information" portions of each cycle of the AC wave do not effectively cancel each other out after being converted to sound in the headphones, and where out of phase sound waves or half cycles will be effectively canceled out and have an amplitude of 0. The natural resonance of a piezo-crystal earphone designed to work with audio frequencies, cannot vibrate much with high frequency, (rf) electric signals, and so as to efficiently converted them to sound. These high frequency sounds can not even heard by the human ear anyway. It needs to be noted that the diode has a fairly high signal impedance (ie., resistance) to the low voltage (and current) rf-input input signal from the antenna and the output from the LC "tank" circuit, frequency selector, filter or tuner. A Schottky diode is possible since it has a low voltage drop when used with low currents, however, it may have a relatively high reverse "leakage" current. You may wish to do some research and-or find diodes with a very low forward voltage drop.. Received (rf) signals (voltage, current) from the antenna are often very low in voltage, and therefore, a very low current is being generated and flowing in the circuit. Having a lower diode impedance allows more of the received or input (rf) signal's voltage to be applied to the headphones and produce a louder sound. Perhaps the most popular diode for a crystal radio is the germanium **1N34 or 1N34A**, however nearly any other type of diode can be used, but some diodes may require a small forward bias voltage so as to set the diode near its conduction voltage, and since the voltage available from the received radio signal is too low to even allow those diode to function (ie., conduct, "turn on") and pass a signal through it. Reducing the available voltage will reduce the volume of the sound output from the high impedance earphones. A simple method to bias a diode is to use a 1.5v battery cell and an adjustable resistor, say 50K to 100K ohms, in parallel (ie., across) the diode so as to limit the current to a very small value. Some adventurers have even placed two or more diodes in parallel to get every drop of power (here, mostly in the form of a voltage) to the (piezo) headphones.

Headphones , Headphones are essentially small speakers that have only a relatively small amount of power output and are therefore placed close to your ears. Headphones, like a loud-speaker, are an electrical energy to audio or sound energy transducer or converter, and can therefore be considered as a type of (energy) transformer. **WARNING**: If the sound is too loud, and-or for too long, hearing damage will usually occur.

For crystal radios, use high impedance carbon, or (fine wire) magnetic earphones which contain a high impedance to a low impedance transformer (2 coils). A relatively inexpensive "crystal (piezo)" earphone can also be used.

You can also use a high input impedance, electric audio amplifier for the sound or audio. and this will also amplify the weak, input audio signal to be used for an 8 ohm impedance, loud-speaker or 16 ohm headphones output. The output signal of the high impedance amplifier or circuit may even be sent ("ie., "connected") to a computer's "line-level [a certain std. max. amplitude]" signal input. A "line-level" input typically has about a 10k ohm input impedance, and about 1mv max. peak to peak, maximum input signal before (amplitude) "clipping" or limiting of the signals waveform takes place and the signal gets distorted. The audio signal from a (non-crystal) radio can also be sent to a computer amplified ("electret") microphone input (~1k ohm input impedance, and ~100mv max. peak to peak input signal). The computers audio circuitry can sometimes amplify these input signals and-or send an output signal to an amplified speaker system.

For the crystal radio's tuner circuit to remain at high impedance to the input signal, and for maximum circuit efficiency, anything connected to that tuner or LC circuit needs to be a very high impedance since it is essentially connected in parallel to it and will reduce the impedance of the tuner like how two parallel resistors effectively creating a lower resistance. Then for maximum power transfer to the headphones, those headphones will also have to be a high impedance type so as to match that of the high impedance tuner and diode. For maximum power transfer from one impedance to another, those impedance's should "match" or be the same value, and this concept of "impedance matching" is used for the concept of maximum power transfer and vice-versa. Some advanced crystal-radios may use a high impedance to low impedance transformer to match the circuit impedance before and after the transformer such as to the headphones, and so as to have greater efficiency and some more audio volume. Using common 8 ohm to 32 ohm headphones will not work for this circuit since the output power of it is too low to physically move or vibrate these speakers so as to make sound (air pressure waves), and there is also an impedance mismatch which will reduce the electric power transferred to the earphones. Some crystal radios may include or be experimented with using a **high value resistor in parallel with the crystal earpiece** so as to help reduce excessive charge buildup on the earpiece and to help it perform better in terms of current flow. Experiment with the best sounding value, generally about 50K ohms, possibly higher to 1M ohms, or use a variable resistor. Some radios may also use a low value of capacitance in parallel to the crystal earpiece so as to more easily (ie., due to having a lower impedance) pass high frequency noise and-or RF through it to the ground rather than through the earpiece.

G , **G**round, a wire going deep into the ground, such as being connected to a metal radiator and-or water pipe.

Connection or "hookup" wires are quick connection wires, and are therefore helpful for quickly building, testing and experimenting with all kinds of electronic circuits. These conductive wires have a small spring clamp (or "alligator clip") at each end so as to easily connect (or disconnect) and hold the wires and-or components electrically together without using soldering and-or desoldering so as to make another circuit test. These wires can be purchased relatively inexpensively, and having a few dozen for general circuit experimentation is recommended. The inexpensive types are flexible, plastic-rubber coated, but have a relatively thin internal multi-strand metal conductor, usually copper, and are therefore safe for only low currents, say less than 1 Amp. 1 Amp or less of total current (**I**t) is typical for many low power circuits.

The LC parallel "tank circuit" in a crystal radio is used as a oscillator and tuner. It will oscillate at its natural electrical frequency determined by the capacitor and inductor. At the desired adjusted or tuned frequency the impedance (ie., ac resistance due to the ac signal reacting with that component) of this circuit is at maximum. Consider this thought experiment: If the input frequency is very high above the tuned frequency, it will more easily pass through the capacitor and to the ground. If the input frequency is lower than the tuned frequency, it will more easily pass through the inductor and to the ground. In both cases the signal is essentially lost as it travels more easily (ie., less resistance) to the ground before passing through the rest of the circuit. While the resonant signal and-or frequency is in the tank circuit having the same natural resonance, the signal will oscillate back and forth by charging one component (L or C) while discharging the other and so on. Due to some resistance still present, some energy is lost and the signal will diminish in amplitude, however more signal from the antenna will boost it back up.

An inductor and a capacitor can be placed in series and this will create a signal filter that will pass a certain frequency much better than others, and because the impedance of the inductor and capacitor will have the same value of impedance at one specific frequency, but the two impedances will be opposite in phase or polarity.

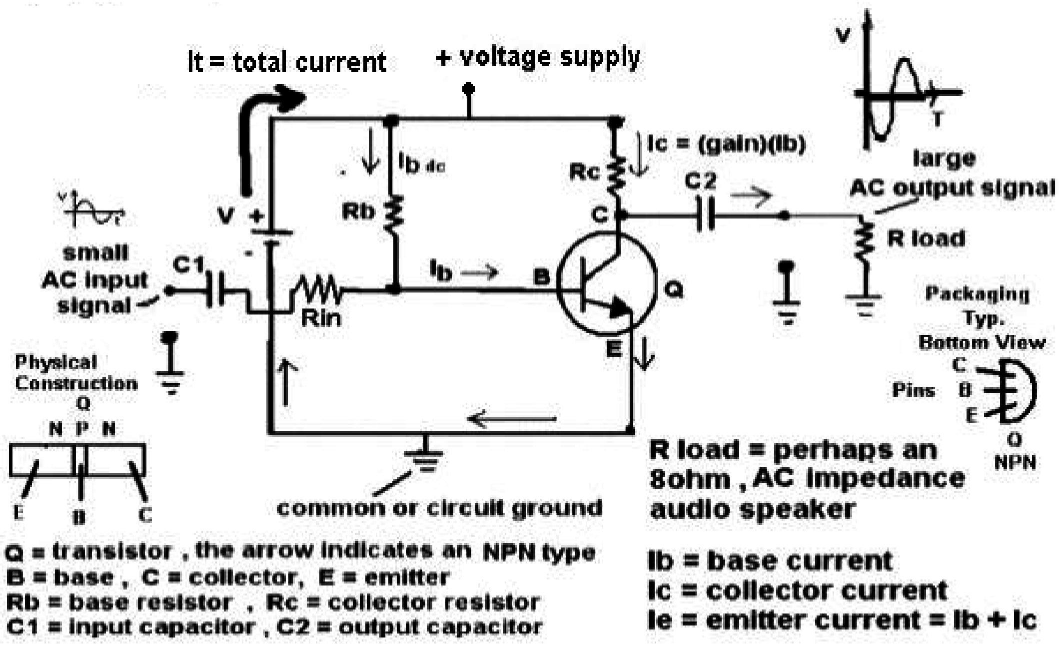
High frequency noise can often be eliminated in a circuit by simply connecting a small capacitor to ground, and of which it will have a low impedance to high frequencies, and those signals will take the path of lease resistance or impedance to ground. An inductor can be placed in series with a circuit so as to reduce and-or block high frequency signals. A capacitor can be placed in series with a circuit so as to pass only high frequencies. The specific values of the capacitors and-or inductors must be calculated and-or experimented with, and it also depends upon the frequencies in question that will be encountered, unwanted and-or desired in the circuit.

Modern crystal radio kits and-or designs often include some type of amplification (tube, transistor, or IC chip(s) such as the popular **LM741** [for small or weak signals, high gain op-amp] and-or the **LM386** [audio amp] and-or impedance matching for a commonly available 8 ohm speaker and-or headphones for the sound output, or for the line-input or microphone input of a device such as a computer so as it can be used as a higher output power sound amplifier. The cutoff-frequency where the gain of the LM741 is reduced to only 1 is about 1Mhz near the center frequency of the AM tuning band or range of frequencies, and so this may not be the best choice to use for high frequency RF signals like common AM radio signals, but is rather best for audio frequencies and applications. The reader will have to research what op-amps or transistors (BJT's and FET's) are capable of amplifying RF signals from the antenna and-or the output of the tuning circuit and-or filter.

In these modern times, there have even been low cost, relatively simply to use IC's specifically made for AM radio receivers, and which then require a minimal number of extra parts such as a power supply (ie, a battery 1.5v to say 3v), a tuning capacitor, a tuning coil on a ferrite rod - sometimes called a "loop-stick antenna". The most common and modern (2024) IC for this is the **TA7642**. It is sometimes called a "radio on a chip". This IC looks much like a common three terminal or lead transistor, and inside it contains several RF amplifier stages for high gain, hence making the radio very sensitive to the received low power signals. The sound output can be heard with an earpiece (piezo, magnetic) or boosted with an audio frequency amplifier stage so as to be heard with a 8 ohm (AC impedance or effective DC resistance at 1000 hz) rated speaker.

A BASIC TRANSISTOR AMPLIFIER CIRCUIT [FIG 237]

In the circuit below, the voltage supply depends on how much electrical power you want to control. For starters, say its a 9v battery, and then mainly the resistor values used will be determined by that 9v value and-or desired current. Typically, the DC base current, or DC bias-current, of a common, small single transistor used for audio amplification is about 1mA. With USB devices very common as of the year of this writing - 2024, you might try designing a simple amplifier that is supplied by a 5V dc USB power supply and-or port.



Rin = input resistor, fixed or variable to set input level, this can also be a potentiometer (adjustable, variable voltage potential setter, usually via an internal resistance and voltage divider) with one terminal connected to the ground, and so as to divide the voltage of the input signal so as to have the desired level to the transistor base from the center terminal of that potentiometer. A safety resistor can also be put before the potentiometer, and provide some transistor and-or potentiometer safety in case the input signal has too much current and-or when the potentiometer is set to a low or 0 resistance which can damage the potentiometer and-or other part of the circuit.

An electronic amplifier will amplify (increase, gain, magnify) the power of an input signal such as an electronic, AC waveform equivalent, representation or facsimile of a sound waveform. Power = P = VI = (voltage)(current). The increase in power can come from an increase in output voltage and-or current of the input signal. Since the input signal controls the transistor amplifier, the output signal of it will be of the same waveform shape and frequency (cycles per second) as that of the input signal. This transistor amplifier circuit and-or configuration is called a "common emitter" since the emitter is common to both the input circuit and output circuit. In brief, Rc and Q (the transistor) form a voltage divider circuit and as the input current to the base (ie., control terminal) of the transistor increases, the transistor will conduct more and its resistance will decrease, and this will cause the voltage across it to decrease and the voltage across Rc will increase. If the current to the base is decreased, the opposite will happen. For transistor (signal) gain or amplification values:

Gain (A) in voltage = Av = Vout / Vin , Ex. If Av = 10 and Vin = 0.1v = 100mv, Vo = Av (Vin) = (10)(0.1v) = 1v
Gain (A) in current = Ai = Iin / Iout
Gain (A) in power = Ap = Po / Pi = (Av)(Ai)

The values of the components in the circuit are determined by many factors such as the input and output impedance needed so as to have maximum power transfer. An FET transistor has a high input impedance, and rather just amplifies the input voltage. The transistor (Q) in the above circuit is a **NPN** polarity, BJT (Bipolar Junction Transistor) and is mainly used to amplify the input current. NPN means a P or Positive carrier type semiconductor layer placed between two N or Negative carrier type semiconductors layers. B-E junction and-or terminals to it, should test like a good diode, and the B-C junction and-or terminals should test like a good diode - conducting in one direction and not the other direction and-or polarity of the voltage applied.

As of the year 2020, small plastic transistors for small input and output signal (voltage and-or current, ac waveform) amplification can be purchased inexpensively at just a few dollars (about the price of a 5lb bag of fine ground wheat flour) USD for 100 of them. When they were first produced in the 1950's, they cost perhaps the equivalent of the year 2022: $10USD each, often due to that they were partially "hand made". Many circuits designers also try to use the same transistors having the same ID numbers, dimensions and operating characteristics, and which are also commonly available. A "generic" type of circuit and transistor, such as that shown above, will usually permit the use of many similar types of transistors with similar electronic characteristics (ie. "parameters") such as its gain = Ai, highest frequency (ie., fo = frequency cutoff where the gain is reduced to only 1), maximum collector to emitter voltage = Vce, maximum current = Imax, maximum power handling capability Pmax or Watts. High power transistors generally have a low gain (ie., input signal amplification) of perhaps 20, while small signal transistors can have a gain of 75 to 200. The most common BJT transistors for hobbyists are the: **2n2222 npn** , and **2n3904 npn**. Their respective pnp complement are: **2n2907** and **2n3906**. The most common low power JFET transistors for hobbyists are the: **2n3819 n channel**, and the **2n5457 n channel** which has a complement of the **2n5458 p channel**. The 2n3819 has a (gain = 1) cutoff frequency (Fo) of 700mhz. A general purpose low power MOSFET is the **2n7000 n channel**. If a device is not available, a substitute can be used if it has similar electrical characteristics, parameters or settings.

In general, a replacement part of the 2n2222 transistor is the 2n3904, BC547, and BC548. A substitute for the 2n3906 is the BC557.

A transistor is much like a sensitive electronically operated valve or switch such as when the transistor is fully on (low resistance, high conductance) to the flow of current when there is enough input base current applied to do so. A small amount of input current will cause the transistor to have a higher conductance through it. A relatively small amount or range of input current can therefore control a large amount or range of current output with the aid of a transistor. When a transistor is used as a switch, this is similar to the concept of an electronic relay that pulls and closes a lite-weight metal switch when a small current creates a magnetic field in a coil of wire. With the switch closed, much current can flow through it and onward to the circuit being powered. A purely electronic transistor is faster than a electro-mechanical relay, has a longer durability or "life" of use, and has no audio and radio-frequency (rf) noise due to possible sparks. A computer's CPU (Central Processing Unit) is an integrated circuit (IC, "chip") that contains and uses transistors as (binary, two states - on or off, an "on" transistor = 1 and an "off" transistor = 0) switches to control and perform basic logic, simple math, conditions (ie., comparisons, evaluations), and data access so as to perform the steps of a program.

When a transistor is not operating as a switch, but somewhere in-between full on and off, the transistor can be used to amplify an input (varying) AC signal. A small amount of input current at the base or input lead of the transistor can cause or control a larger amount of current to flow from the collector to the emitter. The resistors (Rb and Rc) are also chosen so as to set the initial operating voltage and-or current settings of the transistor, and this is sometimes called "biasing" the transistor. Sometimes the current gain (ie., the amplification) of the transistor can be set and-or stabilized to be fairly consistent by using a resistor(s) in the circuit.

For the class-A amplifier circuit above that uses just one transistor, that transistor will amplify the complete waveform of the input AC signal, and rather than just half of it per transistor as for a (higher output power) class-AB type of amplifier circuit which requires both a NPN type and PNP type of transistor. The bias settings of a class-A amplifier are such that without any input signal applied, the output DC voltage across the transistor is set at half of the supply voltage. Ex. For a 9v battery power supply, the DC voltage measured at the collector to emitter terminal will be 9v/2 = 4.5v This will allow the largest possible voltage change, range or swing of nearly 9v less about 0.6v needed to keep the transistor turned on, and in the linear region of amplification. The voltage changes across the transistor will also be at the output, and all due to a changes in the AC input signal. If the input signal gets too large, a distorted output wave will be created wince the transistor has a limited, maximum output voltage, here +,-4.5v.

The output signal or waveform of this amplifier shown above is also out of phase with respect to the input signal and-or its waveform, and by 180°, and is said as being an inverted signal. This is generally not a problem, and if needed, the output signal can be inverted again using another transistor amplifier, and a small portion of this signal can be used as a feedback (or return) signal to be combined and-or added to the input signal as positive feedback. Negative feedback can be used to reduce input signal noise, such as if a small capacitor allowed only the high frequencies to be the feedback to the input.

How does this signal inversion happen in a single stage or transistor amplifier? Signal amplification by a transistor happens since when the input signal voltage increases or rises, it puts more current (ie., electrons that can carry charge) into the base or central region of the transistor and it then can conducts more, and therefore, the transistors internal resistance (R) decreases. The voltage drop across the transistor will decrease since V = IR. Since the output signal is taken from or across the collector and ground, its voltage will also decrease.

The capacitors (C) shown in the circuit above are used as "(AC signal) bypass" and-or "(AC signal) pass through" capacitors which will block a steady DC current, and also keep the bias settings of the transistor constant by isolating it from any external (ie., such as input and-or output) circuit DC power. The capacitors also allow AC signals to flow into and out of that amplifier circuit. Capacitors and-or inductors (such as a "choke") can also be used as signal filters that can block (not allow) and-or pass (allow) certain unwanted (such as noise) frequencies, and for example the filtering depends on the (Farad, charge storage) value of that capacitor used and the frequencies of the AC or varying signals. In general, smaller capacitors tend to block low frequency signals and pass high frequency signals.

A "load" resistor or device can receive or "draw" some of the available signal (v, I, power) from the transistor amplifier circuit. This "load" is also considered as a "load" (quantity or value) of current, hence a "current load" or quantity. A common load device for an electronic audio signal amplifier is an 8ohm speaker and-or coil in it. An 8ohm speaker is a common or standard value that a circuit can be designed for. Technically, 8ohms is the impedance or effective AC resistance of the coil when a 1000hz frequency, electric and-or audio signal is applied to it. This is due to the magnetic field and "back-emf" (in reverse polarity and-or opposing that of the source) voltage created in the coil when there is a change in current such as during the applied AC signal's waveform. The DC resistance, or the "wire resistance" of this same speaker coil is about 6 ohms, and the impedance of that coil will rise as the frequency of the input signal increases.

Since a class-A amplifier is biased to be always ON (functioning) with current flowing thorough it, a current on-off switch should be used so as to not waste any battery energy or power when the amplifier is not in use.

The above circuit for a transistor mostly considers the necessary DC (steady, fixed, V, I, R) biasing or operational settings. What happens to the AC signal is somewhat of a different nature. We want to consider both the positive and negative halves of the AC signal, and not just the positive half of the AC wave (ex., a sine wave). A simple description of what happens is that the incoming AC signal to this amplifier will electronically "sees" or "encounters" the battery with its low internal resistance, and which can then be considered as 0 ohms in ideal conditions. This effectively shorts the positive and negative "rails" or conductors together for the AC signal. For example Rb will then effectively be connected across the input of the AC signal, and Rb will also essentially be across or in parallel to the transistor. Rb will then also be part of the AC signal's encountered input impedance of that amplifier, hence it will also be a part of the AC biasing of that amplifier. Consider if the DC biased transistor had a very low internal "on" impedance or resistance, then Rc is also essentially connected across that transistor in terms of the AC signal. For a class-A amplifier circuit as shown above, and for the most signal output, the value of Rc could very well be set to the average value of the transistors internal (collector to emitter) resistance. Rc will also help determine that amplifiers, AC signal's output impedance seen or available for any circuit connected to it, and of which should be matched for best power transfer, to or by any connected device for best power and -or signal transfer. A typical value for Rc may be about 1000 ohms = 1k, but it may be as low as 10ohms, or as high as 50k ohms. A typical value for Rb may be about 4.7K to 10K ohms, and since the DC bias base current to a transistor is relatively a small value, say about only 1 milli-amp (mA) or less. For a high input impedance amp with this simple circuit, you can try setting Rb at say from 100K to 1M ohms.

The author has found that the electronic sound signal from devices such as a modern **mobile phone** can be amplified by using an inexpensive, common 5mm to 5mm male earphone jack plug connected from the phone earphone jack and to the microphone input jack of a computer so as to be amplified and heard by the computers amplified speaker(s).

In the above transistor circuit, if Rb is made perhaps 100k ohms to 1M ohms, the base to emitter junction will be biased to a very low voltage and hence this will not be conducting very much and will be a high input impedance, and this is ideal for a low voltage radio signal to be amplified. This type of amplifier is usually called an RF (radio-frequency) or high frequency amplifier and-or a high (input) impedance amplifier.

More Advanced Uses Of Transistors And Their Manufacturing Technology

Soon after the discovery of the transistor and having affordable, stable and practical versions of it, the transistor began to be used in many circuits and many with new designs, and where a vacuum tube is no longer needed which made many circuits price prohibitive to even consider for manufacturing for the general public. The first transistor and "pocket (size)" and-or portable radio was the **Regency TR-1** and it was available for purchase in 1954. This radio does not need a ground wire since the transistor RF (radio frequency) gain (ie., amplification) is high so as to amplify weak signals, and the battery acts as the ground. The first transistor based computer was made in 1955 and was a closet sized, "mainframe" type of computer, but it was still much smaller than the room-sized vacuum tube versions available at the time. It required much less power, and it was also much faster and reliable.

The first integrated circuit (IC, or "chip") with tiny transistors and other circuit components began to be independently conceived in the late 1950's by several people, and much credit is given to **Robert Noyce** (of the Fairchild Semiconductor Co., and later, the Intel Co.), and **Jack Kilby** (of the Texas Instruments Co., "TI") due to their work for the process of the miniaturization of silicon based components such as transistors. Kilby, along some other members of an engineering team at the Texas Instrument Co. is also credited to the first all electronic pocket-sized calculator in 1967, called the **Cal-Tec Calculator** and which had several individual integrated circuits in it, and of which each one would calculate just one of the four common math functions (ie., to add, subtract, multiply, and divide). For the result, up to 12 digits of precision, it was printed on a thin (about 1 cm wide) strip of (thermally or heat sensitive) paper. By the early 1970's, new editions of it were being sold as the **Pocketronic Calculator** which was manufactured by the Cannon Co. in Japan. By 1972, the Texas Instrument Co. began to manufacture calculators that had a small sized LED (ie., electric light) numeric display for each digit of a number to be displayed.

The IC invention and miniaturization process eventually allowed the creation of a "computer on a chip" IC. The first IC microprocessor was the **Intel 4004 microprocessor IC** and it was available in 1971, although the military and space industry (such as the U.S.A, NASA and the Apollo program) was considering and-or already using a microprocessor on a chip for several years, but this was generally of for specific and limited use. The Intel 4004 had over 2000 transistors on a single IC or "chip". This IC and its technology and fabrication eventually lead to the modern (2022) "Pentium IC", computer processor designs which have a magnitude or two more of transistors, and are also much faster and more efficient in terms of both a lower power requirement and less wasted heat energy. A company in Japan called the **Busicom Corp**. wanted the Intel company to make several IC's for an all electronic calculator, and **Marcian Hoff** suggested they simply put all the IC's on a single chip so as to reduce costs. Engineers, **Federico Faggin** (previously from the Fairchild Semiconductor Co.) and assisted by **Masatoshi Shima** (from Busicom Corp.) designed the ("4 bit") Intel 4004 microprocessor for the calculator, and which was said by some as also being capable of having a (very practical) square root function ability. A later financial agreement with the Busicom Corp. allowed Intel to share the proprietary rights of this IC it put a lot of money and effort into, and which was also needed so as Intel could make newer microprocessors that were more or less based on it. About the same time as the Intel 4004 IC , Intel also created the 8008 IC which had some advantages over the 4004, like having much more RAM memory, but it also had some disadvantages to the Intel 4004. Future versions of the microprocessors would implement the best qualities of both of these microprocessors into the 8080, 8086 and 8088 IC microprocessor which became part of the low cost and practical personal computer (PC) revolution available or sale to the general public in about 1974, and from various companies.

Light that we can see is essentially a type of (electromagnetic) radio wave and-or energy having a very high

frequency. Light energy is said as being composed of tiny bits or packets of (electromagnetic, (rf)) energy called photons. "rf" means radio frequency, and-or electromagnetic radiation. Light is now often considered as a visible "rf wave" and as being a part of the entire "rf spectrum (band, range) of frequencies" which include (invisible, unseen) radio transmissions and heat radiation.

Our eyes have specials cells designed like very small (rf) antennas that are naturally tuned to sense light and its intensity and color. If the (rf) frequency is too high or too low, we cannot normally see or sense these radio waves with our eyes, however, special sensors and-or cameras can be used to effectively see these frequencies of light.

Theoretical particles of light are called photons (from the word "photo" which means "of light") and each has some (electro-magnetic, field) energy. Each color of light has a particular frequency of its wave energy. White light is a visible construction or composition of light waves of many frequencies that are otherwise seen as different colors such as red, blue, green, etc. The visible result of these colors or frequencies being mixed together is a bright or white light because all those simultaneous waves of light effectively (visibly) simulate a much higher frequency of light seen. As a basic verification example, consider how two different colors of light can be mixed (combined) together to produce a third and different color of light. When white light strikes an object, the reflected frequency or color of it that we see is due to the fact that all the other light frequencies were absorbed by the atoms on the object and converted to mostly heat energy.

There are some people born who can not see much color and-or differentiate between certain colors, but modernly, as of about the year 2010, there are special glasses available which can help some of those people see some colors.

Some rare discoveries indicate that ancient Greeks had some primitive magnifying lenses. These are called **magnifier lenses** or glasses. Their rarity is mostly likely due to that clear glass manufacturing was rare and-or that larger clear crystals were difficult to find and-or shape. It is not too difficult to imagine that many were broken after several years of time by accident. In nature, a drop of water is similar to a magnifier or vice-versa, and its curved surface surely indicated what shape of clear lens is necessary so as to make something appear bigger, and which eventually lead to both the telescope and microscope image magnifiers, particularly with two or more lenses.

Sir **Issac Newton** began to study light and optics in the late 1660's and was one of the first people to scientifically study and publicly demonstrate (in 1672) that a beam of white light can be separated into a rainbow of 7 main colors by using a specially (triangular) shaped, thick piece of clear glass called a (light or optical) **prism**. A rainbow or spectrum of colors of white light can sometimes be seen being emitted (ie., refracting and-or reflected) by some crystals or gems such as quartz and diamonds when in the bright sunlight, and sometimes when sunlight is reflecting from ice (frozen water) or snow (small frozen ice crystals, crystallized water). He then showed that these separate colors of light can then be combined back into a beam of white light by using another prism. Newton proposed that light is made of particles instead of waves. Today, as of the year 2020 with the popular "wave-particle duality" theory of light, it is not too difficult to imagine light as being a wave of light particles, much like a sound pressure wave that is a wave of air particles having a varying density (due to varying force or pressure upon the air from the source of the sound) or amounts, and of the same wave frequency as the source of that sound. A color is due to that it is a certain frequency of electromagnetic and-or radio frequency (rf) energy, and therefore, that energy has a periodic waveform and can be called a wave. If light passes directly (ie., perpendicular to its surface) through a flat piece of glass, that glass will not separate the colors and-or frequencies into a rainbow or spectrum.

A magnifier and-or telescope lens that was ground and polished so as to have a high quality and without image distortion will have no apparent "**chromatic aberration**" (ie. color and image distortion) which is a prism-like effect of separating the received light from the image and into a visible (chromatic, color steps and-or separation) rainbow of colors, and particularly due to the outer edge or rim area of the lens where the incoming light has a greater angle to bend, especially when the focal length is short and the colors will not have a great distance to converge back into white light. The amount of diffraction or bending of the light in and out of the lens will also vary slightly by the frequency (ie., color) and-or wavelength of that light. In general, the red (long wavelengths) and blue (short wavelengths) will be separated more than the middle RGB frequency green wavelength of which a single lens is generally "tuned" for. Corrective lenses can reduce this. Denser glass will bend or refract light more when it is entering and-or exiting the glass, particularly when the light transmitted is not perpendicular to the surface of the glass, and then the density of the glass essentially creates what can be described as a resistance and-or friction with the light and it will slow down and bend. Special glass lenses called "Extra low Dispersion (ED)" lenses contain additional natural elements and have a higher refractive index for the red and blue colors, and therefore have less chromatic aberration. **Spherical aberration** or "shape aberration or distortion" is when the lens is simply made or ground with the wrong curve shape, and there is then more than one focus point created rather than a single focus point and-or "focal plane (area)".

A simple magnifier lens can be used to project an inverted (ie., "upside-down) image of an object onto a white surface such as a piece of paper, and this is the essence of a camera device that is used to make photographic images. In a camera, an image is made when the light coming from an object or subject passes through the lens and to be focused onto a light sensitive material (film, or a photo sensor array) and recorded or stored as an image of that subject. The first cameras often had a pinhole for the lens, and which only allowed a small amount of light into the camera, and was therefore a slow camera system, but it provided a sharp or well defined image, and also allowed a large distance or depth of field that was in focus. Many simple lens systems, including a the pinhole opening and even many large telescopes initially will show the viewer an inverted or image of the subject. This has to do with the way light rays from a subject will pass through a lens and are diffracted (ie., bent by some angle).by that glass and shape of the lens. If need be, another lens can be used to upright an inverted image.

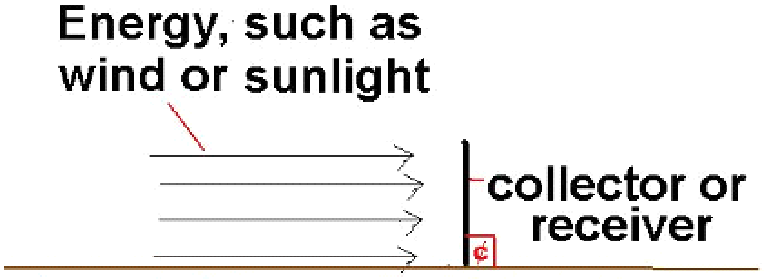
Today, as of 2020, the costly light sensitive paper or film material has been mostly been replaced by a light sensitive electronic image sensor in an (electric) digital camera. Here, digital means that the camera image senor saves the light or pixel (picture element, a small part of) data of the image as binary (1's and 0's) numeric data in the cameras memory and-or a file. Digital cameras have made photography very inexpensive, quick and practical for the average user. Most modern "smart" phones (ie., telephones) have a built-in camera(s) and internal computer circuitry so as to access the (electronic) internet communication and information, and-or run useful programs (ie., "apps", applications) such as a photo editor, perhaps to increase the brightness and-or contrast of the photo so as to make it more presentable and-or printable.

As mentioned previously, the light energy not reflected off an object is usually converted directly into heat energy in the atoms of the object. If a photon or "particle of light" has enough energy, it can collide with an electron and free it from the electric force of an atom by increasing the kinetic energy of that electron, and this creates electrical energy and this is the basis of how **photo-voltaic** devices such as solarcells function. Current in a circuit is electrons that are free to move and have kinetic energy, and can be influenced or directed to flow via an electric force connected to that circuit.

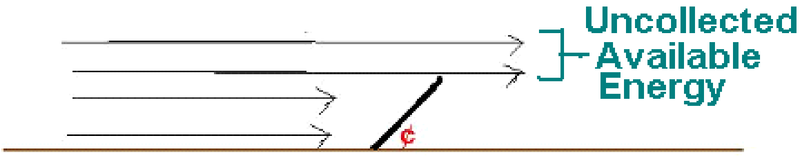
A photon or particle of light energy striking a **solarcell's** atoms can have enough energy or force (the application of energy) to free an electron from the atom it is bonded or electrically attracted to. Solarcell is similar to and associated with the words: Photo-voltaic (light energy to voltage, electrical energy or power), Solar-voltaic and cell such in reverence to the device which Volta called a cell and which produced a voltage and thereby a current of electrons. The function of a solarcell is to convert solar (from the Sun) or sunlight energy into electrical energy. Free or "loose" electrons have gained some (kinetic, moving) energy from the photon collision and can then be used in an electric circuit so as to do useful work (applying energy to other things, and through a time length). Note that an electron is not destroyed when its energy is used up, but it will be slowed down and then it will be easily attracted to and reside in **positive ion.** A positive ion is an atom that has a net positive electrical charge because it is missing an electron and has a "hole" or empty position for an electron to reside. Positive ions are created by electric or chemical processes and-or forces, and are also used for other electric and-or chemical processes such as for charge storage in capacitors and-or batteries.

One electron may not have a noticeable amount of energy to do a noticeable or practical thing, but a very large amount of free electrons and their summed up total energy can have a high power ability and do useful things such as work when most of their kinetic energy is converted to another form of energy such as heat or light. **Most (silicon based) solarcells or any size have a typical rated voltage of about 0.5 volts when functioning in a circuit.** It is of an advanced technical note and in the realm or study of maximum power transfer that solarcells do have a small internal resistance when illuminated and producing electricity via a complete circuit, and if many are placed in series, their total resistance increases proportionally, and for a load to have maximum power to it and a low internal power loss in the solarcell(s), it is best that the load resistance equals that solarcell's or solarpanel's internal resistance. The internal resistance or **"characteristic resistance" (Rc)** of a single solar cell can be found from: Rc = Vmax cell / I max. cell Ex,. for a typical 6" x 6" solar cell, 0.52v / 5.2a out max = 0.1 ohms effective internal resistance The total resistance of a typical 12v rated solarpanel with 36 solar cells in series is then: Internal source resistance = Rs = (18)(0.1o) = 1.8 ohms. Internal voltage drop or loss and-or internal power loss will increase with an increase in the circuit or load current drawn. Power loss max for a solarcell = I^2R = (5.2a max. for this solarcell ex.)^2 (0.1) =~ 2.7w). A typical solarcell can only create a limited and-or maximum amount of power and current: I max = (Vmax / Rs). A shaded solarcell, solarpanel, and-or areas of either will cause a high resistance region (area), and the cell(s) will obviously not produce as much output power (output power = input power - internal power losses and-or conversion losses), but will then loose that power internally, and of which can damage those cells(s) by excessive electrical heating and power dissipation. **In general, larger solarcells, especially in the width dimension have a lower internal resistance, and a higher output current, lower internal voltage loss, and therefore a lower internal power loss.** An **ideal voltage source** has a very low to 0 ohms of resistance and therefore has no internal voltage loss and produces a relatively high current.

The maximum amperage (the amount of electron flow or current per second) or current that a solarcell can produce (in a circuit, with 0 or no resistance [ie., short circuit current rating] to the flow of current) depends mostly on the surface, receiver or collector area of that solar-cell and the effective intensity of the (Sun, solar energy) light striking the surface of the solar cell. When the solarcell is perpendicular (90°, right angle, straight-upon) to the Sun light rays, the effective surface area of energy collection is at its maximum value and the generated electricity will be at its maximum value for that system. It could be said that in this condition, the solarcell and its surface is not tilted (or angled) away from the Sun and its light. [FIG 238]



A. Collector is perpendicular to the energy input.



B. Collector is not perpendicular, and is less than 90°

In the above figure, the effective collector area, Ae is: First, let At = total collector area, and h = heicht = side opposite ¢, Sin ¢ = (side opposite the angle) / (hypotenuse) = (h) / (At) = Ae / At , mathematically in this drawing and-or analysis: **Effective collector area = Ae = At Sin ¢** . If the tilted collector is made longer so as to collect and-or apply more of the available energy, its area will then be increased, and the energy density ratio of: Area / joule in, and-or Area / watt out will then be lower. than the perpendicular collector which has a smaller area. A larger solar collector also has a larger internal resistance. For the kinetic energy of the light to free electrons and make an electric current, the "collision" of the energy onto the surface of the solar collector needs to be direct in order to have the highest transfer of energy efficiency.

The low 0.5 typical output voltage of a solar cell can effectively be increased or "summed up" when they are electrically combined or connected in series (like a row) so as to make higher voltages. Solar-cells in series are electrically connected in a charge attraction, opposite electrical polarity connection, from one to the other. This is very similar to connecting (identical) batteries in series so as to have a higher usable output voltage (V = J / C) available, and where each battery is increasing or boosting the kinetic energy of the electrons passing through it. A higher voltage (ie., electric force, "electrical pressure", "electrical potential") will also create a higher current flow in the same circuit. Another way to increase the generated output current flow of a solarcell is to increase its surface area. Since this is not possible for the average person to do, a practical and effective way to increase the surface area of a solar cell is to electrically connect them in parallel to one another, and with all their positive terminals connected together, and all their negative terminals connected together. If the solarcells are (and should be) the same size, type and have the same electrical characteristics (ie., "ratings", output voltage and current), the output current of parallel connected solarcells will be the sum of current from each solarcell, but the output voltage will still be 0.5v for parallel solarcells. As indicated, when connecting solarcells together, either in parallel and-or series, to help avoid potential solar-cell damage, such as over-heating, solarcell damage and safety problems, it is recommended to use (identical) solar cells of the same type, ratings and size. The wires carrying the output current from a solarpanel (series and parallel connected solar cells, a "battery" or array of solarcells) and-or battery should have a maximum current rating that is greater than or equal to the maximum amount of current that will be flowing through it.

Because sunlight energy has a maximum amount of energy per square meter, a solarcell collecting that energy is limited to receiving a certain maximum amount of energy, and therefore its output electrical energy will be limited to a certain maximum amount of watts which is determined by the energy conversion efficiency of that particular solarcell and-or solarpanel. An 18% to 20% energy conversion efficiency is a typical value as of the year 2020. For example, if an 18% efficient solar panel receives 100 watts of sunlight power, then (100w)(0.18) = 18watts of it will be converted to electrical energy and power (energy/time, voltage and current), and most (100% - 18% = 72%) of the rest will be converted (and usually wasted) to heat energy, and some will be reradiated or reflected. If heat energy is desired in the form of warm air, a **solar air heater** can heat air for free after it is either purchased or made and installed, and with the sunlight and energy shining on its black, heat-absorbent collector surface inside an insulated rectangular box-like shape, and located a few inches beneath a layer of tempered glass. **Solar air heaters are very efficient, often 80% to 90% efficient at converting solar light energy directly into heat energy.** A solar air-heater is also less expensive (perhaps, half the price) than a solar electric panel of the same size. Due to the high conversion efficiency of a solar air heater, it is not recommended to use a photovoltaic solar panel to make heat (warm air, such as to heat a room or house) with an electric air heater of which requires high current and will quickly drain your energy storage (ie., accumulated charge, electrons, current) batteries. Clear glass or transparent plastic will allow the lower frequency, near-infrared solar radiation to pass through it, and this is ideal for warm greenhouses where plants are grown in cooler climates and-or weather and seasons of the year. Glass will block much of the higher frequency light and-or energy such as ultraviolet (or ultraviolet) light (UV).

If a solarcell is rated at 0.5Vdc and can produce 4A of output current, the theoretical or effective internal resistance of that energy source can be found using Ohm's Law: I = V / R , therefore, R = V / I = 0.5v / 4A = 0.125 ohms = (1/8) ohms = "an eighth of an Ohm". If the solarcell is producing less than 4A due to that there is less sunlight on its surface, its internal resistance will be higher. Here is how to calculate the output power: The maximum amount of output power of this example solarcell energy source is: P = (V)(I) = (0.5v)(4A) = 2watts = 2 Joules of energy per second.

A **solar panel** (or **solarpanel**) is simply a convenient array of electrically connected solar cells so as to increase (ie., multiply) the available output power which is many times more than that of a single solarcell. Many modern silicon solar cells are in the range of about 18% to 24% efficient, and are often priced according to their efficiency, and the 18% type is very common as of the year 2021. That is, they will convert 18% of the light energy into electric energy, and therefore waste (100% -18% =) 72% of it when its converted to heat (molecular vibration) energy on the surface of the solar cell and-or having a low (light to electric) energy conversion inefficiency to begin with and is therefore incapable of a higher efficiency. In terms of efficiency, they would make a better solar heater than an electric generator, but the usefulness of electricity in our modern world of electronic technology (devices, machines, lights and phones, that need and use the power of electricity to function) is important enough to tolerate some energy losses or waste. Energy availability or security is important, and reducing wasted energy or losses by creating more efficient solarcells and devices is an ongoing challenge.

The available and-or maximum amount of solar energy per square meter is about 1000W/m^2. Because of cloudy skies, sunlight incidence (application, input) angle, fixed positioning as opposed to more efficient solar (Sun) tracking or aiming, internal resistance, wire and battery resistance, diode(s), voltage regulators and converters, etc., the actual power available from a solar panel, especially the current, will be reduced below the maximum possible value and-or the solar-panel's rated output value. To help overcome these types of problems, and have a more consistent or stable amount of expected power, the solar panel should be designed and created so as to produce a slightly higher voltage or output power, say 30% more than what is actually needed due to an estimated 30% energy loss or waste in entire solar electric system. Often a solar-panel is designed so as to have an output voltage that is 1.5 times higher than it is rated. For example, a 12v rated solar panel is often designed to produce (12v)(1.5) = 18v open circuit without any load or device being powered. Once a load or resistance is connected to a solar panel, the voltage of the solar panel will drop in value, and this is due to that its internal resistance no longer has all of the generated voltage across it.

To power devices (ie., [current or power] "loads") that require "home power" or "utility power" voltages of about 120v, when using a rated 12v solar energy system or solar electrical generator system, (SEGS), a converter called a power **inverter** can be used, and this is a 12vdc to 120vac converter. An inverter often has a transformer to "step-up", increase or "boost" the input voltage, and at the cost of reducing the output current. Remember that power output cannot exceed power input. If the voltage is increased by a factor of 10 in a step-up transformer, the maximum, safe current output of that transformer is reduced by that same factor of 10. The inverter is connected to a large battery(s) that has already been charged up to about 12v or slightly higher. A diode in series with the battery prevents it from discharging its current back into the solar panel when the output voltage (emf) of the solarpanel becomes lower than that of the battery, such as during cloudy weather and at nighttime. Some solar panels may not have this diode since the charge controller usually has one. Some electric power inverters may still function at voltages slightly lower than 12v, perhaps as low as 10v.

A solar panel that is rated at 20% max. efficiency of the incoming or incident light energy can convert or transfer a maximum of 20% of that solar energy (ie., sunlight, photon energy) into electricity. If the flat plane or surface of the solar panel is directly perpendicular to the direction of the incoming or incident light energy of the typical available solar energy of about 1000 W/m^2 on a bright sunny day at the equator, it can convert that 1000 W/m^2 into a maximum of:
(0.20) (1000 W/m^2) = 200 watts/m^2 of usable electricity energy. At higher angles or latitudes from the equator, the solar energy available will be less. The solar panel will be rated and sold as a 200 watt (max power, = joules/second) solar panel, and here, it is when the offset or difference angle of the incident rays of solar energy to the perpendicular surface of the solar panel is 0°. That is, the solar panel is directly facing, aimed or "pointed to" the Sun and its light energy in the local area. When the Sun's position in the sky changes and-or the solar panel is effectively tilted away form the direct incident light, then less energy is available to it and it will collect less energy since the available (to the incoming light) effective collector area is now actually less, as if using a smaller solar panel. The output voltage will still be near 0 .5v, but the output current will be decreased by a larger factor value as the (incidence or "normal") angle increases, and is especially noticeable starting at about 30° away from the solar cells 0°, perpendicular (or "normal") reference line to the incoming sunlight rays. A basic theoretical formula relating the actual output power to the sunlight offset or incidence angle is:

(actual output power) = (COS offset angle) (max. rated output power of the solar panel)

Ex. If the solar panel is rated at 200W max. output, and its surface is tilted 30° away from the direct, 0° offset, sunlight, the output energy is calculated to be about:

(COS 30°) (200W) = (0.866) (200W) =~ 173 Watts maximum panel output, and before any other system losses.
If that solar panel was directly facing the Sun, the output energy of that solar panel would be:
(COS 0°) (200W) = (1.0) (200W) = 200W

With modern solar cells and efficiency techniques, if the plane of the solar cell is not exactly perpendicular, and is tilted away by up to say 15° to 30°, the output is not reduce too much. Many solar panels on roofs are set at a fixed angle throughout the day as the Sun appears to travel from the eastward to the westward direction. It is possible to mount a solar panel on a sun tracking motor that has light sensors (perhaps small solar cells, LED's, or photo-resistors) so as to follow or "track" the Sun's current position in the sky. This will maintain a more steady or constant amount of power output throughout the day, and to achieve maximum efficiency and power from your solar electrical generator system (SEGS). During the summer months, direct sunlight on the solar cells can heat them to a hot temperature and their efficiency (current output) is reduced, and because of this, it is often said that solar cells and panels work (produce more power) better when it is cooler in the winter months.

As mentioned previously, to help overcome various power losses and still have enough electric power available, solar panels are constructed so as to have a rated output voltage of about 1.5 times the advertised voltage. To accomplish this, more solar cells are connected in series so as to produce 18v with no load or circuit applied to or using that power. Connecting solar cells in parallel means more total surface area and more current generation without the associated power losses due to solar cells connected in just a series configuration where the total resistance of that circuit increases for each solar cell used. Connecting solarcells in series will not increase the current because the effective or total resistance of each solarcell in series is then a multiple of the resistance of one solarcell, and this high resistance will limit the current to just that from one solarcell. Placing solar cells or panels in parallel is also problematic since one may be at a different output voltage and-or resistance than the others and they will electrically conflict to some degree. In that situation, it is best to always use an output power diode on each solar panel so as no current can then flow back through it in the reverse direction. The preferred type of diode to use with a solar energy system is of the Schottky diode variety which has a low forward or "turn-on" voltage and power loss, but any high current (perhaps 6A to 10A max.) power diode should work reasonably for relatively low power systems. You will have to use a diode capable of safely handling or passing the expected maximum current, and power loss in it - Pd = (Vd)(Id), for a solar electric generator system (SEGS).

Solar cells and-or solar panels can be compared by various factors such as the advertised or rated solar energy to electricity conversion efficiency which is typically about 18% for the (lower cost, blue color) high quality amorphous ("multi-crystal mix") silicon solar cells, and 22% for the more modern (year 2020 , black color) "mono-crystalline", single or solid crystal silicon. Solar cells and-or solar panels can also be rated and compared in terms of maximum power, voltage, and current output, or possibly even amps (current) or power output per square centimeter of surface area. Solar cells and-or panels can also be rated and compared in terms of price per output watt. On a clear day, the sunlight energy per square meter is about 1000 watts per square meter of Earths surface area when directly illuminated by the overhead Sun and its sunlight energy. Since there is (1m x 1m) = (100cm x 100cm) = 10000 square centimeters per square meter, that is about (1000 watts / 10000 square centimeters) = 0.1 watt / cm^2 = 100 milliwatts per square centimeter. If the efficiency of the solar cell or panel is 20%, then the wattage per square centimeter is reduced to just: (0.1watt)(0.20) = 0.02 watts = 20 milliwatts (mW). The (max.) current generated per square centimeter is therefore: I = Pw / Vv = 0.02 / 0.5 = 0.040A = 40 mA per square centimeter in ideal conditions. An area of (1W / (0.20W / cm^2)) = 50 cm^2 will produce 1W of power.

As of the year 2020, the total cost of a pre-made solar panel with an aluminum frame and a tempered safety glass protective layer is as low as $1USD a watt. Some 100W lightweight, more portable solarpanels do not have the aluminum frame and heavy glass, but still have a protective plastic layer coating are as low as $50USD, hence 50 cents per watt. These somewhat fragile solarpanels can be "stiffened" or "firmed up" and protected with a lightweight wooden square frame on the back and sides, and-or attach a "tilt stand" so it can be aimed more directly at the Sun. You can search websites such as the Amazon.com and-or Ebay.com for solarpanels and other supplies for a solar electric generator system (SEGS), but locally in your area there may be a store such as Harbor Freight, Lows, Walmart, and Home Depot, etc. selling these products and having experienced workers there who may help you choose and-or install a solar-electric system. Many stores also have an associated internet website where you can browse many items both shown and not shown at the store location(s), and to then make purchases "online" via the Internet, and possibly have the items delivered to our house. Sometimes shipping is "free" or relatively inexpensive. A purchase helps both you, your community, the store's business and the product's manufacturing company so as the supply and demand of beneficial things can continue.

Though the cost of solar energy photo-voltaic panels has decreased significantly as of the year 2023, there is still a cost associated with all the other equipment and-or labor needed to install a complete solar electric system. When you consider energy security and-or availability when it is needed, the cost is well worth it and could be considered inexpensive and-or paying in advance for future electricity, and even if just to charge the battery in a flashlight so as to have light to help do things. Many portable devices as of the year 2020 use rechargeable batteries, and so as to also reduce the expense of purchasing more batteries, reduce wasted raw materials, and prevent batteries from being put in "garbage dump landfills".

To make a complete and functional basic solar electric generator system (SEGS), this is what is often

recommended:

1. **Solar Panel** A solar panel (solarpanel, or solar-panel) is an energy collector, and specifically, it is a solar or light energy collector. On a more technical level, it is also an energy converter or transducer since the energy of the light (ie., the energy of the photons) is converted to another form which is electrical energy.

   This panel may include a blocking diode so as the battery or other electronics does not send power back through that solar panel and causing the battery to drain power through it and-or preventing damage to the solar panel. This high-power diode capable of handling the current generated at the solar panel will have about a 0.7v drop or loss through it, or sometimes less if it is a Schottky diode with a lower forward voltage drop, and therefore a power loss in it that is equal to Pd = (voltage across the diode)(current through the diode). Ex. (0.7v)(5A) = 3.5W. A solarcell will have a "soft"- still a relatively high internal resistance and begin to conduct at about -5v reverse bias, much like an LED diode - and of which a series, reversed biased or connection diode with it can prevent current through it..

   Most solar panels are designed to generate a voltage that is about **1.5** times their rated output voltage. For example, a 12 volt rated panel will usually be designed to produce an open circuit (no load or current) of (12v)(1.5) = 18v. This will allow for an internal solar panel power loss or drop due its internal resistance having a voltage and power loss, and it will allow for other losses in the other parts of the solar electric generator system (SEGS). **A charge controller will usually include a blocking diode.**

2. **Charge Controller**. This is much like a voltage regulator which takes an input of 12v to 18v and creates a steady output voltage of 12v to13.7v which is noted as good for charging 12v lead-acid batteries. Some also have current charging options and control. Some may offer convenient 5v USB (Universal Serial Bus) power supply output on two pins or wires so as to charge up phones and power some devices such as LED lamps. Besides calling it a voltage regulator, a more accurate name would be a battery charging controller. USB versions 1.0 to 2.0 have a 5v output with a +,- 5% tolerance or "window", hence 4.75v to 5.25vmax USB3.0 has 5.5v max. Note that USB version 1.0, 1.1, and 2.0 have a 0.5A max current output. USB3.0 has a 3A standard max. output. You may research USB specifications. USB-A, USB-B, and USB-C are the names given to the connectors, and of which are also male and female ends. Adapters and-or special "mixed" cables are also available and common. Some charge controllers may already include a "blocking" or safety diode. Some charge controllers offer a 12v dc rated fixed output such as used for automobile and RV (Recreational Vehicle, campers, etc) appliances. For lower values of DC power, a DC voltage regulator can be purchased so as to have a reduced DC voltage, and an example is with the LM317 IC chip that is an adjustable DC voltage regulator, and it is discussed in this book. A charge controller is more properly called a battery charging controller. **NOTE:** Before connecting the solar panel to the charge controller, first connect the charge controller to the battery, and it often must also have a high enough voltage to be sensed by the charge controller as being a valid voltage to charge the battery. After connecting the charge controller to the battery, then connect the solarpanel to the battery. A 12v consumer device can then be connected to the charge controller's 12v regulated output port/jack, and it will also function if the Sun is not shining onto the solar panels creating power, and the power will be rather provided by the charged battery connected to the charge controller. A charge controller and-or power inverter may also electrically disconnect the battery from being discharged too much below its rated voltage and-or charge storage capacity, and since it may damage the battery by various chemical processes and mineral growths.

3. **Battery**. A battery can be used to accumulate ("build up") and store energy, and in particular, electrical energy such as from a solar panel. This energy can then be used when it is needed, such as during power outages ("grid down", or loss of the supplied power from elsewhere), and the "energy security" (ie., availability) is well worth it when it is needed and there is no other source(s) available. Small batteries are more portable.

The most popular type today in the year 2020, is some type of lithium-ion or lithium-iron type. These batteries are relatively lightweight as compared to a lead-acid battery. LiFePO4 is a Lithium-Iron-Phosphate battery. Lithium-ion batteries can be discharged much more than standard and heavy lead-acid batteries, and are a form of "deep-cycle (charging, and discharging)" battery where nearly all (about 90%) of its stored energy can be used without worry of battery damage. While discharging, a lithium-ion battery will maintain much of its rated voltage. It is recommended to recharge batteries as soon as possible and to try to keep them at their peak voltage so as to prevent internal crystal growths, and to have maximum power when needed. Recharging a battery should cease when the maximum rated peak voltage of the battery has been reached so as to prevent damaging it. Lithium batteries rated 3.7v can usually be charged to 4.2V (max.).

SLA is Sealed Lead-Acid battery and is similar to a standard lead-acid battery, but is also made to be a "deep-cycle" battery. Typically, a 12V rated lead acid car battery is charged up using 14.4V (max.) and this is also the recommended maximum applied voltage to it. A 6V rated lead acid battery would be charged using a voltage of half of that to: (14.4V / 2) = 7.2V (max.). These batteries are sealed so as to prevent the electrolyte from evaporating and-or leaking out if the battery is tipped over.

In general, the max. voltage of a battery is about 1.2 times that of its rated value,and which is about 20% higher.

Some other battery constructions and types are GEL (like a jelly) and AGM (Absorbed Glass Mat) and these are said as being more user friendly than regular lithium batteries. GEL batteries have a higher energy density than lead-acid batteries, but somewhat less energy density than lithium-ion batteries. Portable devices and tools often have an attachable battery pack that is Nickel metal based. NiMH (Nickel metal hydride) or NiCd (Nickel and Cadmium) and these have a higher energy density than lead-acid batteries, however not as high as lithium-ion batteries.

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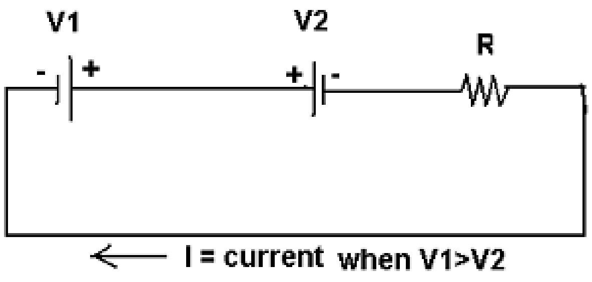
A lead-acid "car battery" type will work, but its battery drain considered as safe before potential battery (sulfate [sulfur] buildup from the sulfuric acid electrolyte, charging prevention or "life" reducing) damage is much less than that of a deep cycle or lithium battery. In general, the bigger the battery, the more maximum rated power or watts = (energy / second) available, heavier (if lead), and the higher the cost. A lead-acid battery is often lower in price than other types and of the same capacity, and can safely provide a high current when needed, such as to power a starting electric motor for a car.

A battery after being charged up over time, will usually have many times of power available than the solar panel(s) can produce per second and are rated for. A battery and-or a "battery-bank" (multiple batteries) are needed for running high power devices such as a motors and other devices, and especially at night when the Sun is not shining and delivering light energy to the solar panel(s).

Charge a battery at a recommended amount of current, say 10% to 20% of its amp-hour rating. A battery will get warm while charging because it has an internal resistance to the current flowing through it. As the battery increases in voltage, it will effectively become a "back-emf" (ie., reverse voltage) and where the net effective sum of voltages in series creates a less effective supply voltage and-or voltage difference to be applied the charging battery, and this will also reduce the amount of charging current to the battery and increase the charging time. Still, this is a safe way to charge a battery. To then get more current into that charging battery, the charging resistance to it can be reduced, but do not increase the voltage supply of the charger as an attempt to get more current into a battery since this may lead to overcharging the battery, and it will then have a higher voltage than the safe rated maximum voltage that it can be charged up to.

[FIG 239]



V1 = supply voltage , emf

V2 = charging battery or capacitor, emf, but since its polarity is reversed than that of V1, it is considered a reverse and-or back emf.

When V2 = V1, Vt or the effective charging v or emf = = V1 - V2 = 0, and I = 0V / Rohms = 0A

R is current limiter, for a safe amount of current. $I = \frac{\text{Effective charging V}}{R} = \frac{Vt}{R} = \frac{(V1 + (-V2))}{R} = \frac{V1 - V2}{R}$

4. **Inverter**. This is also called a power inverter. It will take the 12vdc to18vdc output of the solar panel and convert it to common household power of about 120vac, as used in the USA. Like the energy storage batteries, inverters come in all manner of sizes, maximum rated output power (watts), and corresponding prices. The trade off of increasing the voltage using a transformer-like voltage step-up system like an inverter, is a decrease in the available current, but the available power, less some small internal losses from the inverter, will be close in value to the input power. If the voltage boost or increase is 10, the current output will decrease by the same factor, hence it will be a tenth (0.1) of that at its input. An advantage of this is that cheaper, thinner output wires from the inverter can be used. With just one large capacity 12v battery, an inverted can be used to temporarily run high power devices such as tools (drills, saws, etc.) that contain high power motors which require a high input power to function properly. Two 12v solar panels connected in series and two 12v batteries connected in series can create a 24v solar electrical generator system (SEGS) of which the current, from the battery to the inverter, needed for a certain amount of power will be half as that of a 12v system, and this will reduce the cost of the inverter (especially if its a true or pure sine wave output type having thick wire coils for higher currents and power output ability) and have less (here, half) the battery drain because the voltage is now an increased factor of the power used, and therefore the current can be less, and at a safer and cheaper value with thinner wires needed to deliver that amount of power.

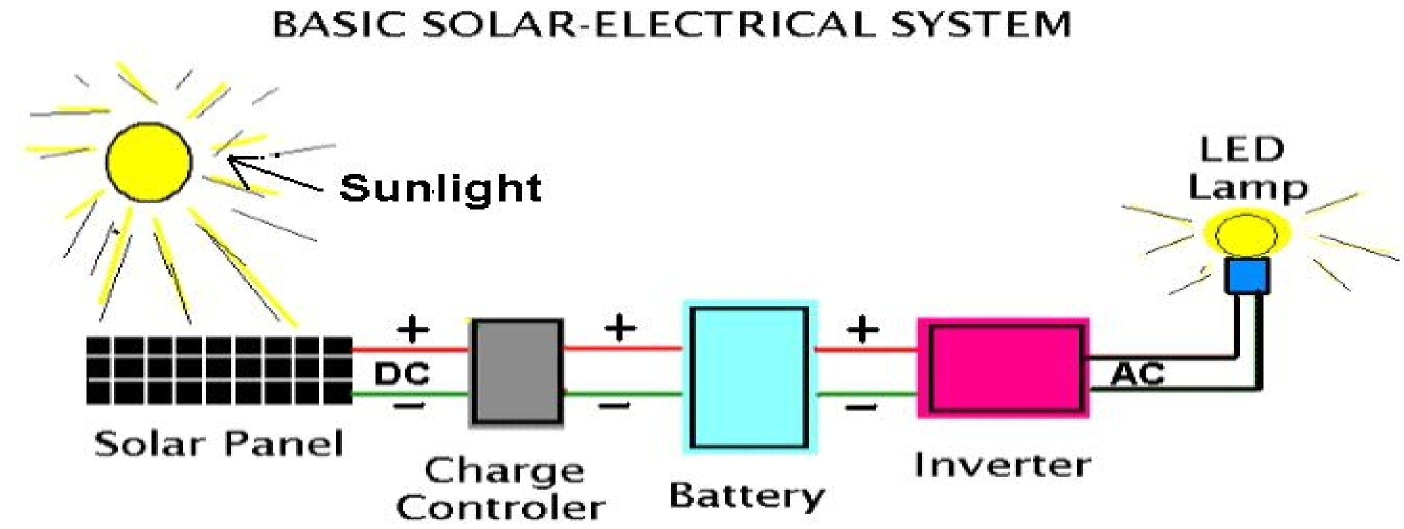
   If a DC to AC 120v inverter is rated at 600 watts, the maximum output current is: From: P = V I ,
   I = P / V = 600w / 120v = 5 amps. Since the inverter essentially boosts the input voltage of 12v to 120vac, a factor of 10, the output current will be reduced by a factor of 10, hence to use 5A of current at the output, then (5A)(10) = 50A from the battery will need to be input to the inverter. The cable to do this will need to be rated at 50A or higher, and particularly if it is long, rather than just a short distance of say 3 feet. The battery will also be needed to be rated as being able to safely deliver 50A max, hence use a 50aH rated battery.
   Pin = (12v) (50A) = 600w = Pout = (120v)(5A)

   When first connecting the battery to the inverter, the capacitors in the inverter may be drained of charge and there will then be an inrush of current from the battery which can cause damage to you and the inverter system. To help prevent this, a procedure called "**pre-charging**" can be done. For a 12V rated system, use a 10 ohm to 20 ohm power resistor with a power rating of 20W or more. Place this resistor in series with the negative wire and-or terminals, and keep it in place for about 20 seconds. **When sparks are possible**, **please use protective eye wear**. Be sure that the inverter is not powering any devices or "load" when doing this procedure. For each 12v higher solar electric system, use an additional 10 ohms to the total resistance value needed. Just the same, please considering discharging an inverter before handling and-or storing it, and this can be done by connecting the resistor to the terminals of the inverter.

   Using small "wall transformers", is a way to step-down the voltage of 120v to other voltages necessary. A DC to DC converter can lower the 12v input from a car battery, solarpanel or other electric power source so as to output another DC voltage level, say 5V, and this device is based on a DC voltage regulator circuit. The cheapest DC to DC solution is a low cost LM731 premade circuit, and another option would be a "switching" type of converter that can maximize power transfer, but those are more expensive, and may introduce noise into sensitive electronics including radios. Some of these types can also boost a voltage.

5. **Wire**. Usually, the wires or cables to connect the parts of a solar-energy system come with the components purchased, however, having some 10 to 14 gauge copper wire can be used in most cases, depending on if the maximum current needed is only about 20A or less as for 10 AWG (gauge). For lower currents, then a lower gauge wire can be used. A 100W solarpanel usually produces about 6A max. of direct current (dc) when the output is shorted (ie. directly connected together, practically 0 ohms load resistance) or sent only through an ammeter (current meter) when measuring the output current. For low power devices, some crafty people have managed to use speaker wire having or exceeding the necessary power handling needed.

[FIG 240]



(Typ. 12v to 18vdc max.)
Rated for A 12Vdc System   For A 12Vdc Battery   12Vdc Rated   12Vdc To 120Vac   120Vac LED Lamp

Solar cells (or solarcells) can be cut to be various smaller sizes from a larger full-sized manufactured solar cell, and in particular, often for a certain value (and without excess, waste) of current needed since the voltage output will always be about 0.5v no matter how big or small a solar cell area is. The area size needed, often for device portability is a reason to cut solarcells to a smaller size of area with particular dimensions, and this may also be done so as to increase the output voltage using several small series connected solarcells, usually of the same dimensions. For example, to power a white-light LED and without concern about storing any power, about 3v voltage is needed and about 20mA of current so as to have a reasonable amount of light from it. In order to have 3v, a series of 6 equally (same area) sized solarcells can be placed in series. (0.5v)(6) = 3v. The area of each solarcell is such that is will provide at least 20mA of output current. Since the solarcells are in series, that same 20mA of current will flow through each and also through the LED in that series circuit. Due to some resistance in the circuit, cut the solarcells slightly larger than needed and ensure (measure) that 20mA of current is actually flowing through it. Solarcells are often cut into smaller solar cells of a certain size needed and so as to also not have any wasted solarcell left over. Solarcells are very thin, brittle and difficult to cut without using something such a laser (high powered, concentrated light beam, energy) cutter, and it is recommended that you first obtain them already cut into smaller sizes if they are needed, and-or to have available for future projects.

As indicated above, the most common types of solar cells are very thin, about one-eighth of a millimeter thick ~ 1mm/8 =0.125mm = 125um, about the thickness of an index or playing card, and are very brittle due to being a rigid glass-like crystal structure and will easily crack if they are bent and-or too much pressure is placed on their surface. For comparison, it is slightly thicker than a human hair which is about 0.1mm = 100um = 0.0001m thick. To cut solar cells is difficult, and special lasers or saws and methods must be used. Silicon dust is **toxic** to breath and therefore a mask and eye protection must be used when cutting them and cleaning up any solarcell dust. The best cutting methods seem to be with high power lasers. For most people, the best and modern low-cost (actually often cheaper than building a solarpanel) option and recommendation is to buy a completely made solarpanel(s) and other parts for a solar electric system of the size and power needed. Some full sized solarpanels are heavy and are not very portable for a person to carry, while others are small and inexpensive, and can even store energy in an internal battery bank until it is needed.

A solar cell area that is cut into a smaller area will should have an output current of:

$$\frac{A\ \text{cutcell}}{A\ \text{uncutcell}} = \frac{I\ \text{cutcell}}{I\ \text{uncutcell}}$$

: same ratio , I cutcell is the current from a solarcell that was cut to a smaller area.
If a solar cell is cut into (n) equal areas:

$$\frac{A\ \text{uncutcell}}{n} = \frac{I\ \text{uncutcell}}{n} = I\ \text{cutcell}$$

How big should a solarcell be cut for, or purchased to charge a given battery? Since it is usually recommended to charge a lithium type of battery at a maximum of 10% of its rated amp-hour (aH) rating, a solarcell or solar electric system only needs to have a maximum output of 10% of this aH rating. It a battery has a 1 amp-hour rating, the charging current at 10% of this value is therefore, 1A (10%) = 1A (0.10) = 0.1A = 100mA. Use the above formula to solve for the corresponding area of the solarcell(s). Don't forget that a recommended diode will also drop the available voltage of the solarpanel by about 0.7V, and therefore, reduce the available current to your battery and-or device. To overcome this, design the solarpanel so as to be 0.7v higher than needed, and probably a bit more at 1v since two solar cells will produce 1v. Another option is to use a higher voltage solarpanel and a voltage regulator and current limiting and-or safety resistor to charge a battery, and this method also eliminates the need and expertise to cut solarcells.

Some new solar panel designs use a "half cells" (half-size of usually a full 6" x 6" solar cell) size, or perhaps "quarter sized cells" and so as to get a higher voltage from a smaller solar panel that has many more solar cells in it. A theory is that this will also reduce the power loss in each cell a little since the internal resistance will be less, and therefore making the solar panel a small percentage more efficient. Just remember that output current is charges (electrons), and that is what is actually needed and used to charge (accumulated, to be stored) a battery so as to store power. If the charging current is low, it will take that battery longer to charge. Pre-cut solarcells for homemade solarpanels are very useful for projects.

If the output current is lower than expected in full sunlight, the solarcell might have what is called micro-cracks. You can determine if a solarcell has micro-cracks by shining a bright light through it from the back side, and then observing for them. Sometimes a dealer may have received and sold these accidentally and-or at a reduced cost, or possibly they were returned and then resold. This is a reason to purchase a pre-made solar panel.

Some newer solar panels may include some solarcell(s) or solarpanel **bypass diodes** that will conduct current around any (hopefully just temporarily) shaded cells so as to prevent those shaded cells from getting hot, creating a power loss and possibly being destroyed, and then ruining the function of the entire solar panel. Bypass diodes will also prevent a parallel solar panel(s) from sending current through an shaded solarpanel. If your solar panel does not have these, then prevent any shading on the solarcells in a solar panel, such as not letting people stand too long between the solar panel and the sunlight. These diodes can also be installed by an electrician if needed.

Here is a photograph of a basic solarcell kit that can be purchased so as to make a homemade solarpanel by an experienced hobbyist. The photo shows a stack of 3in by 6in solar cells, the dark solar cell surface with many thin, lengthwise conduction wires on it, and the back of a solarcell with its electrical connections. Also shown is a 40ft spool of thin "tabbing wire" so as to connect each solar cell in series, and a 4ft spool of wider "bus wire" which can carry more current safely. A large 6 amp diode is shown next to a U.S. penny coin (0.75in =~ 19mm wide, diameter) as a practical reference size and measurement. Some kits come with some type of soldering "flux paste" or "flux pen" which greatly helps solder adhere to the wire and-or metal surface it will be jointed to, and you are encouraged to obtain, learn and use this for a more professional result. A 60W soldering iron is typical or average for many soldering projects. Delicate electronics like IC's, small transistors and capacitors will use about 40W with a pointed tip, and performed quickly so as to not overheat and ruin them. A flat tipped soldering iron is good for various flat tabbing wires. **Generally, though this**

seems like the right and economical thing to do, these kits are intended for knowledgeable and-or experienced builders, experiments and-or for some type of custom solar panel designed for a certain area size and-or output voltage, current and-or power level. For most people without the desire and-or expertise, resources, tools and ability, it is highly recommended that they rather purchase premade solar panels, or lower current portions of, because they will be cheaper (as of say 2020), less problematic, less worry, and more reliable. It is almost impossible as of the year 2025 so as to make a solar panel cheaper than what is already availa\ble, and 99% of the time, with much, much leass trouble and headache to build at home. Portability also needs to be considered, for standard solar panels are bulky (ie., large, unwieldy - difficult to move or carry) and generally heavy, mostly due to their size, tempered glass, frame, and often a clear protective epoxy resin. Lite-weight, lower power, portable solar panels are now commonly available. These can be placed within more protective transportation and-or storage cases such as plywood if needed. Here are some parts of a solar panel kit. Although, it helps to have extra power when needed, consider the solar panel wattage only needed for a device(s) since to waste created energy is not desired either. For example why need a 100W heavy panel when only a 30W panel is needed at best, and especially if it is for portability. [FIG 241]

Technically, the large surface area of a solarcell is much like the junction of a common p-n diode. A common p-n diode actually produces a small amount of light (usually infrared) since its area is small. If the p-n diode is illuminated by light, a reverse voltage and current will be created. A solarcell has much more junction area than that of a p-n diode, and therefore, it will produce much more current. It is even possible to create a very small amount of ("glowing") visible light from a solarcell if it is connect to a battery in reverse bias, and of the minimal voltage needed.

Solarpanels are a good way to power and-or recharge portable electric devices and-or power storage batteries. If you need to carry a solarpanel, you will need to then consider the lite-weight portable versions rather than a heavy solarpanel designed for the roof of a house or some supporting structure. Having something is said as being better than nothing. You can even purchase portable radios and flashlights that include a small solarpanel for recharging, either built in and-or external to the device.

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COULOMB'S LAW (of physical force between electrically charged particles and-or objects)

This topic and article is about the physical force of attraction or repulsion of electric charges and-or of charged objects containing those (net or total effective) electrical charges. An equation was created by **Charles Coulomb** of France in 1785 so as to express the value of the electric force and in relationship to those charges (Q1 and Q2) and the distance (d) between their effective center points of attraction or repulsion. In short, the electric force of attraction or repulsion is measured as a being a common physical force with units of Newtons (N). From this equation, an amount of charge can also be determined by the force and distance between those two charges. The charges for this equation are also considered as static, resting, non moving, so as to avoid the magnetic field and forces created due to moving current.

$$F = \frac{k(Q1)(Q2)}{d^2}$$

: Q=charge , Charges Q1 and Q2 are measured in units of Coulombs (c) of electric charge.
k (or Ke) is the Coulomb's Law Constant = 1/**[**4(pi)Eo**]** =1/**[**(12.566371)(8.854(10^-12)**]**C^2/Nm^2)
Eo is the permittivity of a vacuum, and is also called the "electric constant"
with units of Farads / meter when used in terms of the capacitance of a capacitor.
Permittivity is a measure of the ability of the substance to allow the electric field.
k =~ (8.98755179)(10^9) C^2/Nm^2 =~ (9)(10^9) C^2/Nm^2 for air.
The unit for this value for k is: (Nm^2/C^2) , k and then F will decrease in denser mediums or substances between the charges. **k** is sometimes called the **electro-static force constant**.
F = Fe = the electric (not gravitational) force of attraction or repulsion in units of Newtons = N
(d) or r = radius from the centers of the net charge, and has units of meters (m).

On the atomic scale, the distance between the nucleus and electron of a hydrogen atom is about 52.9pm =~ 53 (10^-12)m = 53 picometers, according to the (1913), **Niels Bohr** popular atom model. This value is also equal to that electrons orbital radius about the nucleus (center). This distance is also considered as the (average and-or probabilistic) minimum distance that an electron can be about an atom, but for other elements which have other electrons in "higher" or farther radius orbits, their distance from the nucleus will be more than 53 pm. Electrons in more distant orbits have a weaker (Coulomb) attractive force applied to them from the nucleus which contains protons which have an opposite and positive electric charge, and of which is even lessened or weaker from all the other electrons present in that atom and having some repulsive force. Loosely or weaker bound (by force) electrons in farther orbits will require less energy to be free and then have kinetic energy, and this is what electric current is composed of. 53 pm is called the **Bohr radius**

Coulomb found and verified these mathematical relationships while experimenting with charged metal covered balls (and possibly, sometimes using metal plates) near each other and then measuring and calculating the force of their attraction or repulsion with a sensitive torsion (twisting force, here, using a thin hanging string) balance with a thin "pointer" (ie., a needle, indicator) or rod on it to indicate the amount of twisting force. A coulomb (C) is a practical unit of measure for a specific quantity of charged particles, such as electrons, and-or their net charge. These particles are difficult to see, count and-or numerically quantify with a number, and the Coulomb unit is very helpful because it is practical. Others have also studied the electric force using capacitors with closely spaced, large metal plates of which can accumulate and store charges (ie. electrons, or positive ions (atoms)), and this would actually be somewhat like a type of (electric charge sensing) galvanometer with larger plates that move in the presence of an net electric force due to the net charges.

If two uncharged (or de-charged) objects such as hollow metal spheres are identical, and if one is given a net charge, and due to like charges repelling (via electric forces), half of its charge can be transferred to the second metal sphere upon electrical contact which allows the transfer or conduction of the charged particles and-or their charge. The result is that each of these objects have the same amount of charge, and without actually knowing exactly how much other than that they are now both equal in charge. It could also be said that the original sphere held twice the amount of charge,and this could be expressed as: (100% charge amount) / 2 = 1/2 = 50% = 0.5 charge amount on each sphere. The sum of charges on each sphere is: 0.5 + 0.5 = (1/2) + (1/2) = 1.

If an amount of electric charge is divided in half by Coulomb's reasoning and method, this smaller amount can also be divided in half with another uncharged sphere, and the resulting charge on the last sphere is (1/2) / 2 = (1/4) = 0.25 of the starting amount of charge as that on the first sphere. According to Coulomb's electric force formula, when the amount of electric charge is reduced or divided by 2, but the original or starting force value of it is was not divided by 2, or in half, such as the charges were, but the resulting force was rather divided by 4 according to his experiments and measurements, and this is how he constructed his formula. Consider mathematically, and regardless of distance: A half of a half is: (half)(half) = (half)^2 = (1/2)(1/2) = (1/4) = (1/2)^2 = (0.5)(0.5) = (0.5)^2 = (1/4) = 0.25 .

Coulomb naturally knew that the greater the distance between those charges, the less the force between them. This is like the effect that the greater the distance from a source and-or its initial and total energy, the less sound and-or energy received. The same can also be said about a candle light. The less distance between electric charges, the greater the force between them. The relationship between force and distance was found to not be a linear or direct mathematical relationship. but rather a reciprocal (here, due to a division to represent a force reduction if distance increases) or inverse of the mathematical square of it, relationship, and this is why (1/d^2 or 1/r^2) is used in Coulomb's formula. Consider candle light that will pass through a square shaped opening representing 1 unit of area (= length^2, and will be projected onto a wall at a distance away. If the projection is 4 times larger, and-or the distance has doubled, the amount of light energy per unit area is now only (1/4) of that at the square shaped opening area. The total light energy is still the same, but it is less intense per unit area now, and this is also due to an inverse squared relationship.

Note that Coulomb didn't initially find the charge of a single electron or proton, but rather used the net or total effective amount of force from the charge of many electrons and-or protons called a Coulomb (quantity or number) of charge, and this amount is considered as a (manageable, measurable) unit of charge and is called the Coulomb (C) unit. Coulomb knew that the force was not only directly related to the quantity of each net effective charge (Qc) on each object, but also to their mathematical product of the charge on each metal surface. As a thought-example, a charge of 5, and another charge of 5 will sum to 10, but their product as a simplified numeric force value is (5)(5) = 5^2 = 25, hence resulting in a much higher force, much like an amplified force. This force is also an equal and opposite force as according to the known concepts of force.

As indicated previously, Coulomb also knew that the force was (inversely) related to the square of the distance between the charges, and this causes the force to decrease rapidly as the charges are separated more, and this is similar to the force of magnetism between two magnets, and the force of gravity between two masses. Coulomb also discovered the mathematical inverse square law of the force due to magnetic fields. These equations are very similar to that of the law of gravity or gravitational force and its equation that was previously developed by Sir Isaac Newton.

The concept of electrons and a way to measure current flow was not yet known or available, but what Coulomb did was to first charge a metal sphere, and then use that to charged sphere to charge another metal sphere, and in practical theory, the total quantity or amount of the electric charges is equal to the sum of the charges, and here, because of the way it was done, each sphere has half the total charge, but now each sphere has the same amount of charge on it and will repel each other with the same amount of electric force. Each would also have same force upon another charged object, and which indicates that those spheres do indeed the same amount of electrical charge. Given a fixed distance between two charges, we see that a quantity of charge on an object can be indirectly measured by the force (F) it would have on another quantity of charge (Q) on another object. As for the quantities of charge being multiplied together so as to find there resultant force value, this was done by measurements of using known relative amounts of charge and then measuring the (equal and opposite direction) resulting force. The resulting force was found to be related to the product of the two amount charges. It should be of no surprise that a small amount of charge has a lower amount of electric force ability associated with it, and that a large amount of charge has a higher amount of electric force ability associated with it. It is then also of reasonable consideration that a smaller amount of electric charge will have a low amount of influence upon a larger amount of electric charge, and that in an opposite manner, a larger amount of electric charge will have a high amount of influence upon a smaller electric charge. The resulting influence upon each other is a force that is an equal and opposite force upon each other. Unlike two forces exerted in the opposite direction upon an object, this resulting force is not simply, and mathematically directly related the sum of the two amounts of charge, but rather their product which is much like the product of the two sides of a rectangle, and the result being its area (ie., an analogy to the resultant force). Q1 and Q2 both affect each others force ability or potential, and are both a factor of each other and of the resultant equal and opposite force.

Coulomb was rather measuring charge as he knew it as something that caused a force and was involved with electricity, rather than as being a charged microscopic particles known today as electron particles, and that each has an electric charge and are therefore called charged particles or simply as equated to being charges. Coulomb did not know the number of these charged particles a unit or charge such as the Coulomb, but he knew that an amount of charge can be repeatedly divided. into smaller charges which have smaller forces associated with them.

Ampere created a unit to measure the flow of electric charge, which is called a current of charge or simply current:

1 Ampere = 1 Amp. = 1A = the amount of current that will flow through a resistance of 1 ohm when a 1 volt potential difference or emf (electromotive force) is applied to it. And by Ohms law:
current = I = voltage / resistance = V / R

current = $\frac{\text{1 volt}}{\text{1 ohm}}$, we see that a unit for current is needed, and this is what is used:

Since an amount of (flowing) current should also be measured as a flow rate of electric charges, A unit of 1A of current was defined as 1C of charge passing a point per 1 second:
And:
1A = $\frac{1C}{1s}$ = $\frac{\text{1 Vv}}{\text{1 Rohm}}$ also: 1C = 1A-s = the total amount charges in a current of 1A of current for 1 second. 1C = 1Amp-second

In general electronics studies, electron (e) is the smallest charged particle or mass. A net amount of total charge on an object is due to the (algebraic) sum of these individual charges: positive charges such as (net) positive ions (atoms that lost an electron(s)), will have a positive sign (+), and negative charges such as electrons will have a negative sign (-):

Net charge of an object = positive charges + negative charges

1 C = 6.24 (10^18) electrons. : 1 coulomb (number) of charge(s) and-or its carriers such as electrons
A coulomb of electrons or charges, and:
This is derived below. For consideration: 1C / second = 1 Ampere . By this definition, an Ampere (A) is a rate of flow of (an specific amount of) charge past a point per second, and-or the quantity of charges past a point per second.
Amps = Coulombs of flowing charge / second . 1A =1Qc / 1s = flowing charge / time

Dividing both sides by 6.24 (10^18) we can find the amount of charge or the number of coulombs that 1 electron is:

The charge on 1 electron is : **1 electron charge = 1.6025641 (10^-19) C** : a small fraction of 1 coulomb of charge

Total charge, Q = (number of charges)(charge of each charge) = (number of electrons)(charge of each electron) ,
Total charge = (number of charge carriers)(charge of each carrier) now consider:
If we assign 1C as the value of this total charge: 1C = (number of charge carriers)(charge of each carrier)
If each charge carrier is an electron particle, and is given its individual charge amount the same unit as C:

Total charge = Qc = (number of charge carriers)(1.602)(10^-19)C which can be expressed as
1Qc = 1C = (number of charge carriers)(1.602)(10^-19)C mathematically:

number of charge carriers = $\frac{1C}{1.602(10^{-19})C}$ = **6.24 (10^18) charges or electrons in a coulomb= (1C) of charge or electrons**

We can assign 1C of charge carriers = 6.24(10^18) elementary charges and-or electrons = 1C of electric charge.

Extra: The ratio of 1C of charge to 1 mol of charge is: $\frac{1\text{ C}}{1\text{ mol}}$ = $\frac{6.24\ (10^{18})}{6.022141\ (10^{23})}$ = 1.036176 (10^-5) : a ratio
0.0000136176 , hence small

Taking the reciprocal of this value, we will have the ratio of 1 mol of charge to 1C of charge =~ 96500 : a ratio
That is: 1mol / 1C = 96500 , and mathematically: 1 mol = 96500C , It could be said that in 1 mol of charge, there are about 96500 coulombs of charge. Extra: 96500 is called **Faraday's Constant** (Used in chemistry, electrolysis)

This can also be calculated as:
coulombs of charge in 1 mol of charge = (number of particles in a mol of charges or particles)(charge in an electron)
coulombs of charge in 1 mol of charge = (6.022141)(10^23)(1.6025641)(10^-19) = 9.6509 (10^4) =~ 96500C

Extra. Ex. Since 1A = 1C / s , If a current of 1A is flowing, how long will it take to total 96500C = 1 mol?
Setting up a proportion type of equation: (1C / 1s) = (96500C / Xs) , Xs = 96500s = 25.805h

Ex. If 1A of charge is flowing per second, how much charge will have flowed in 2h?
First, 1A = 1C / 1S , 2h = (2)(3600s) = 7200s . Equation: (1C / 1s) = (Xc / 7200s) , Xc = 7200C

Coulomb developed his Coulomb's Law or electric force formula with the aid of the previous (about 1687) understanding by **Joseph Priestley** (1733-1804), from England, of charged objects repelling each other, and having a force inversely related to the square of the distance between those objects. Priestley is generally more famous for discovering oxygen gas in 1774 after heating up mercuric oxide to a high temperature, and he discovered some other gases later. Priestley found that oxygen gas increased combustion and-or fire, and for developing a way for people to carbonate water in 1767, so as it will contain dissolved carbon dioxide (CO2, a molecule with 1 carbon atom and 2 oxygen atoms) gas and retain it in a fluid when that fluid is pressurized. This carbonated water or ("bubbly") "soda water" would later lead to the more desired flavored and sweetened ("fizzy", due to the many small textured bubbles) soft-drinks that we have today which are informally and simply called "soda" or "soda-pop", probably due to the similar bubbles found when using "baking soda" (sodium bicarbonate) with water. The labeling on the soda beverage may indicate carbon dioxide as carbonic acid. This soda is made when it is a cold liquid and CO2 molecules and microscopic bubbles are forced into it under pressure at slightly higher than the 14.7 psi normal air pressure, and at about 18 psi. When the container with some internal pressure and the carbon dioxide gas is opened, the internal pressure (about 50psi when the soda is at room temperature) is then reduces to 14.7 psi and the dissolved compressed gas will start to "come out of the solution" (ie., dissolution) and expand into bubbles and rise to the surface in the water. This will then reduce the carbon dioxide level and "fizzyness" or "effervescence" of the drink, and the drink is then said to be "flat" or bland, and not as enjoyable. This reason to keep soda drink or beverage as cool as possible before and after it is opened is because an increase in temperature causes an increase in pressure and the faster the (pressurized) CO2 is released out of the soda solution into the lower pressure region outside the container it is in, and that will result in wasted soda.

In a circuit, the number of charges (ie., electrons) passing a point will increase as the time of that flow of current increases. The total amount of these charges that passed a point would simply be the product of the number or amount of flowing charges per unit of time, and the total time, hence:

total charges = (flow rate of electrons or their charge)(time elapsed in seconds) : total electron charges and-or electrons

If the flow rate had units of coulombs (C) of electrons or charge per second:

total charges = (charge / second)(time elapsed in seconds) = (Coulombs/s)(s) = Coulombs of charge

Therefore, if you know the current amount, or simply the current, and the total time, you can calculate the total number of electrons that flowed passed a point in a circuit during that time:

If 1C of electrons passes by a point in a circuit in 1 second of time, the flow rate is 1C/s and this flow rate is defined as:
1 ampere unit of current = 1A. Amperes = Coulombs per second

1A = 1C / 1s : 1 amp is or has a flow rate of 1 coulomb of charge (ie., electrons) per each second of time elapsed

If electrons carry or have the charge, 1C of charge is also said as being the amount of charge in 1C of electrons.

total charge = (rate of charge)(total time) = (charge rate)(total time) = (coulombs/second)(seconds)

1C = 1A x 1s : 1 coulomb of charge or electrons will pass by a point in a circuit that has 1 amp of current applied for 1 second of time. In general: **C = A s and A = C / s**

Ex. If the current was 2A applied for 10 seconds, the total number of electrons or charges that flowed was:

(2A)(10s) = 20 amp-seconds = (2C / s)(10s) = 20C = 20 coulombs of electrons and-or their charges

1C = 1 coulomb = 1 amp-second = "an amp for one second" = "an amp of current flowing for one second"
= "a current of 1A for 1s of time will be a total of 1 coulomb of charge and-or electron charges"

From: A = C / s , we have **s = C / A** so as to determine how long it will take an amount of charge to move past a point when the flow rate of it is A. , A = C / s is essentially a flow of charge (time) rate = the amount of charge per second

Here are a few related equations in terms of energy:

$$1W = \frac{1J}{1s} = 1V\,1A = 1V\,\frac{1C}{s} = I^2R = \frac{V^2}{R}$$

: 1 watt of power = 1J/s is delivered and-or used up = 1 volt moved 1 amp of current. From this, we can derive:
Note mathematically: **J = VAs = (watts)(s)**

$$1V = \frac{1W}{1A} = \frac{1J/s}{1C/s}$$

Simplifying this fraction, we have:

$$1V = \frac{1W\,s}{1A\,s} = \frac{1J}{1C} = \frac{\text{energy}}{\text{coulomb}}$$ per , and 1C = 1J / 1V , and (1C)(1V) = 1J = W-s

1J = (1C)(1V) can be thought of as:

1J = (1V)(1C) = energy = work = (force)(distance) = Nm = W-s

V = J / C = energy / coulomb : voltage is shown here to be the energy per coulomb of charge

Electromotive force (emf) applied to charges will give that charge a change in its kinetic energy which is its movement energy, or to give the charge potential energy which is stored energy. Potential energy means the potential ability to use that energy so as to do work. Energy stored in a battery is potential energy.

If a coulomb of charge has 5 Joules of total energy, its potential energy is said to be 5V. 5V can be thought of as 5J/C. If that coulomb of charge used up or lost 2 Joules of energy in a resistor, it now has 3 Joules of energy left for that coulomb of charges, hence 3V, and it is said to have changed its potential energy by 2V. 2V can be thought of as 2J/c. This difference or change in energy and-or the voltage is often called a potential energy difference or simply the potential difference. The potential energy of stored or static charges can also give nearby charges (ie., electrons) some motion (kinetic energy) due to how charges repel or attract.

For charges to flow in a circuit between two points, there must be a potential energy difference between those two points which is essentially a voltage or energy difference between those two points. Current is said to flow with or from a high potential to a lower potential as the energy in the charge (electrons) is depleted to 0J such as when the electrons recombine with a positive ions (atom missing an electron) in the battery. Ex. A 5v battery is connected to another 5v battery using the same terminals on each, and the net pressure upon the charges in the circuit is (+5v) + (-5v) = 0v, and therefore there is no net pressure or force (emf) is upon the charges to influence them and cause them to flow due to electric forces. Simply stated, there is no potential difference for current to flow.

An analogy to electrical potential (energy) difference (the net electric force, or net emf) would be two similar containers of the same height connected by a pipe or tube at the bottom of each one, and with a substance, say water, and at different heights or depth in each one that is creating a different internal pressure at the bottom of the container than the other container. It could be said that there is a (net) difference in pressure between those two containers. When those containers are joined together with a pipe or conduit (ie., "conductor" or wire as for electricity), the difference in pressure (ie., stored energy) will cause the higher pressure substance (water, or current in terms of electricity) to flow to the container at a lower pressure. The flow will actually stop if and when the pressures are the same or "equalized" and there is no longer a pressure or potential (to use that [stored] energy, to do work) difference.

Another analogy for a voltage potential difference is with a mechanical balance scale. If the weights on both sides of this scale are the same, their will be no difference in force and therefore, there will be no difference in torque so as to cause the scale to rotate or move downward on one side. If the weights are different, their corresponding torques (ie. rotating forces) will no longer be the same and scale will then move by the available, torque due to the net difference in those two weights and their torques.

It will take energy from the battery to force current through a resistance, and this can be described as wasted energy and there will be a loss in voltage (ie. energy, J/Qc or= J/C) across that resistance, and which is also called a voltage drop. Likewise, this can also be described as charge (electrons) containing (kinetic) energy, and it is passing through a resistor (r) which can be described as a friction or collision force in the opposite direction of its motion, and that charge will loose some of its (kinetic) energy (J = VQ or= VC) in that resistor, and this will be called a voltage loss or drop or a potential energy loss or drop:

Vr = Vloss = joules / coulomb. This is usually an energy loss, unless it was for intended heat, etc.,

Since force is the application of energy, and voltage is a measure of the potential energy per coulomb of charge, voltage is often called potential energy, potential difference, electrical force, or electromotive force (emf), and since more potential energy means more (potential) force and ability to do things such as work.

An electric energy source is often called a voltage (J/C) power supply or power source, and its output voltage is often called electromotive-force (emf) which applies energy to the electron charges. Since 1**C**/1s = 1A, a 1 Volt power source can give 1 Joule of potential energy to 1 **C**oulomb of (forced, induced flowing, current) charge in 1 second of time. A current of 1A can therefore deliver 1 J/s of energy = 1 watt. **A resistance that looses and-or wastes 1 joule of energy per second when 1A = 1C/s flows through it is defined as a resistance of 1 ohm.** The voltage loss or drop across this resistance is said to be: voltage = energy/coulomb =J/C = 1J/1C = 1V. Hence J = VC. If 1joule of energy is wasted per second, then that amount or rate of energy usage is:

Power = P watts = J / s = (VC) / s = (V)(C / s) = (V)(A) = (voltage) (current) mathematically:

Joules = (watts)(seconds) =w-s and 1J = (1 watt)(1 second) = 1 w-s : one watt of power usage for 1s of time
Given an amount of voltage applied across a resistor, it will be its resistance value that will limit and determine the current through it. If there was only a very low resistance in a wire, the amount of current induced into the wire would be a high and dangerous (heat, burning hazard) value. Energy from the battery would be quickly applied to a large amount of the electrons and therefore that energy is quickly drained from that battery's supply of its total available energy. Though a resistance is thought of as energy or power wasting thing, it can limit the current or voltage, and therefore power to a safe value where needed, particularly if the supply voltage is higher than needed.

If the resistance (R) doubles then the current through it will be half. Current (A) and resistance are inversely related:

nR when $\frac{A}{n}$: mathematical relationship between R and A when there is a change in either and when the voltage is the same. Here, (n) is a multiplying factor.

Ex. In terms of the factors involved: $2R = \frac{A}{2}$ or= $0.5\,A = (1/n)\,A$ OR $2A = \frac{R}{2A}$ or= $A = \frac{0.5R}{A} = (1/n)\,R$
(here, n=2)

From also **V = I R or= A R** , and given a certain or fixed voltage, we see that the relationship between R and A is an inverse relationship, and they are not necessarily reciprocals, and that if one changes by a factor of (n), the other changes by a factor equal to the reciprocal of that same factor, hence by (1/n).

More force will be needed to push a certain amount of current through more resistance, hence a higher amount of energy is needed, therefore a larger voltage potential: V = J/C will be needed to force the same amount of current through a larger resistance. In short, the electrons will need a higher kinetic energy. If the resistance (R) increases by a factor of (n), then the voltage needed for a given amount of current must be increased by that same factor. If the voltage is not increased, but left the same value, then the current will be reduced by that same factor:

$$V = (A)(R) = A\,R\,(1) = A\,R\,\frac{(n)}{(n)} = \frac{(A)}{(n)}\,(n)R = (n)A\,\frac{R}{(n)} = (1/n)\,R$$: here A = I = current

If the voltage (V) or voltage force applied across that same resistance is doubled then twice the current (A) will flow. Voltage and current are directly related.

nV leads to nA
$nV = n(A)(R)$, when R is constant: $nV = (nA)R$ or when A is constant: $nV = (A)(nR)$

If 2 volts is across a resistor, it fundamentally means V = 2J / C of energy per coulomb is lost through that resistor when forcing each 1C of charge through it. If the current or charge rate is 3A = 3C/s , then the total energy lost through that resistor per second can be found using a proportion:

$\frac{2J}{1C} = \frac{xJ}{3C}$ and $x = \frac{(2J)(3C)}{(1C)} = 6J$: hence the energy wasted in 1 second is (V)(A) = J/s = watts

As for the specific value of this resistance (R) that passes 3A of current per second through it with a voltage of 2V applied across it, this can be thought of as 3C of charge that looses 2J/C of energy per second:

$\frac{2J}{1C} \times \frac{3C}{s} = \frac{6J}{s}$ = six joules will be lost each second = 6J/s = 6 watts : P = (V)(A)

and:

$$R\text{ ohms} = \frac{V}{A} = \frac{\frac{J}{C}}{\frac{C}{s}} = \overset{*}{\frac{J\text{-}s}{C^2}} = \frac{(6J)(1s)}{(3C)^2} = \frac{6\ J\text{-}s}{9\ C^2} = \frac{2V}{3A} = 0.667\text{ ohms}$$

: * see the note below
This can be interpreted as: 0.667 volts per 1 amp of current are needed to force that current through that value of resistance.

Js = Joule-seconds : this can be thought of as an amount of energy applied for a number of seconds, hence like a measure of the total amount of work or force applied to move that charge through that resistance. The resistance formula with (J-s / C^2) units indicates a measure of the energy needed, used or lost per coulomb squared of charge so as to force and move the charge through it. This specific and unique equation for R has a similar form to:

From: Pw = VA = (A R) A = A^2 R therefore: $R = \frac{Pw}{A^2} = \frac{\text{total energy}}{\text{current squared}}$

This ratio could be thought of as the energy needed and-or lost per unit (ie. amp) of charge or current as it travels through that specific resistance.

* From current = I = charge / s = Coulombs / s = C / s , and V = J / C ,
C = Is = J / V , therefore, C^2 = (C)(C) = (Is)(J/v) = IsJ / V , and
Js / C^2 = Js / IsJ / V = V / I , and since V and I are inversely related by a constant value of the resistance, this becomes: **V / I = R** : **Ohms Law** , and-or=: **I = V / R**

Joules (J) are the defined and standard units of energy which is defied as the ability to do work. The units of work are therefore the same as the energy that it took to do that work, and which are Joules.

Energy = Work : fundamentally , in quantity

Energy can be described as the ability or potential to do work = Work is the result of energy being applied. **Force** is the application of energy: F = (mass) (acceleration). **Work** is a measure of the transformation and-or transfer of energy, hence the energy used up and-or required to do something, and it is equivalent to the amount of energy involved: Work out = Work in = Energy used, and both have the same units of Joules. Work = (Force) (distance)

Energy in, or done = Work out, or done :and both have units of Joules (J).

1J = 1 Joule of energy is defined as the amount of energy used when applying a constant force of 1N through or over 1m of distance.

1J is also the amount of energy used up or needed to increase the temperature of 1cc = = 1mL = 1 gram of water, by 1 degree Celsius temperature. Joules are the accepted scientific units for heat, energy, and work (essentially the total of needed, "used up" or transferred energy).

To power a 2 watt light for 5 seconds, the total energy required and-or used is:

Here, showing various related equations, so as to have a greater understanding:

Total energy = (energy rate)(time) = (energy used per second)(number of seconds) = (power)(time) =
(watt)(seconds) = (2 watts)(5 seconds) = (5)(2)(watt-seconds) =
= 10w-s = 10J/s x s = 10 Joules of energy

The charge on an electron (e) is the natural and smallest fundamental electric charge unit. In other words, 1 electron is 1 unit of electrical charge. The charge of an electron in terms of, or with coulomb (C) units of charge will be shown below. Current is composed of electrons and-or their electric charges.

The specific charge of just one electron was found and averaged through many experiments by people such as **J.J. Thomson,** and then later by **Robert Millikan** with his famous **"oil drop experiment"** in the year 1909. In brief, this experiment used electronically (electric, electrostatic force) levitated tiny, spherical oil drops of a known, calculated volume and mass, and therefore knowing the number of atoms and electrons [charged particles] in that mass. These tiny drops were placed between two metal plates with at a high voltage difference applied to them and which created a high electric (force) field between them so as to levitate (held in a steady position in the air) those charged drops of oil, as if weightless and-or not affected by the downward force of gravity. The amount of electric force to levitate a drop is equal to the amount of the force of gravity upon it, hence its weight. Since this (physical, a force) levitation was due to an electric force, this force is essentially a result of Coulomb's Law and-or the equation for the electric forces of two charged objects. With this equation, and the force known, the quantity of charge of 1 oil drop could be calculated, and then divided by the number of electrons in that oil drop so as to find the amount of charge for just 1 electron. After many experiments, he calculated an average value for the charge of 1 electron to be about:

Charge of an electron = 1 electron charge = -1e = -e = Qe = - 1.602176634 (10^-19)C

Charge of an electron or proton. The electric charge of a proton atomic particle is the same as that of an electron, therefore having an equal value of a fundamental charge unit, but opposite in sign with a positive sign (+).

The total charge of a mol of protons divided by the charge of a single proton is equal to Avogadro's Number = (Na)

The total charge of a mol of electrons divided by the charge of a single electron is equal to Avogadro's Number = (Na)
Mathematically:
(charge of an electron)(Na) = (charge of a mol of electrons) = (96485C of charge) : with Faraday's Constant
Hence there are about: (96500 C / mol of electric charge) = (~96500C / mol of electrons)

Extra: A mol of Atomic Mass Units = 1 mol of amu = is defined as having a total amount mass of 1 gram. Protons and neutrons are equal in number in each atom, and they are also equal in mass. 1 gram of atom matter has 0.5 grams of protons and 0.5 grams of neutrons. 0.5 grams of matter corresponds to half a mol = (mol / 2) of protons, neutrons or amu. The amount of mass form the electrons is negligible. Because of the above concepts, and that hydrogen atoms do not have neutrons, 1 gram of hydrogen atoms will have 1 mol of protons.

Charges tend to concentrate on sharp points, angled edges, and corners of metal, and therefore, there is a higher chance of arcing ("sparks", "lightening") there when the emf of the total charge exceeds the breakdown voltage of air and distance to another object, and then lets a current to flow from those spots. Sometimes electrical objects and-or insulators are deliberately made round in shape so as to help distribute the charges more and therefore reduce the chance of arcing. A **lightning rod** to protect structures (houses, building, towers, etc.) from lightening has a pointed or relatively narrow piece of metal high in the air above a structure so as to divert a possible lightening strike away from the structure and into the ground around the structure. The lightning's energy will be diverted into the ground via a high conduction wire attached to a reasonably deep (several feet, maybe 1 meter), metal rod(s) and-or metal plate(s) in the ground.

Not too long after **Volta** created the ("volta pile", "voltaic pile") battery in about the year 1800, and which is a relatively easy and reliable way to make usable amounts electric power more available and practical. **Georg Ohm,** in 1827, discovered by observation and measurement the fundamental mathematical and physical relationship of voltage, current, and resistance of an electric circuit or path that the current is passing through and delivering energy. This relationship is known as Ohm's Law. It is nearly intuitive, easy to remember, and is the most useful formula about electricity: **I = V / R** : Current (I = A, or Ia) is directly proportional to the applied voltage (V) or (electromotive) force to move that current, and is inversely related to the amount of resistance (R) in the circuit or path that impedes the flow of current. The units for current (I) are amperes (A, named after Ampere). The units for the voltage is volts (V, named after Volta). The units for resistance is Ohms (ohms, named after George Ohm):

$$I = \frac{V}{R} \quad , \quad \textbf{therefore mathematically:} \quad V = I\,R \quad \text{and} \quad R = \frac{V}{I} \quad \text{: Ohm's Law}$$

A common analogy to the flow of current (moving electrons, or "flow of electricity") and Ohm's Law is with water flowing through a pipe. The pressure (ie., force) due to the weight of the water at the start of the pipe corresponds to voltage (ie., emf, electro-motive force). A force can apply energy to things such as cause water and-or electrons to move after transferring some kinetic energy to them. A larger amount of force can move a larger amount of water or current. The moving current of water corresponds to electric current. If the pipe is wide (ie., its diameter) enough, a large volume or amount of water can flow per second through that pipe, and this corresponds to the amount of current per second or amperage (amps). The pipe itself corresponds to a wire (ie. electrical conductor, conduit, or "path"). If the pipe somehow gets narrower, it is like increasing the reverse force, pressure or resistance in the opposite direction to the flow of a current, and the rate of water flow will be reduced. The **flow rate** = amount per time = (gallons or pounds) / second , correspond to: (current = electrons or coulombs of electrons) / second = **I** = A = C/s. The height and weight of the water at the start of the pipe determines the pressure (force / unit area) and this corresponds to the voltage (joules/coulomb , = J/C) or "voltage pressure" of a circuit.

Electric current will naturally take the path of least resistance, and which can be thought of as the least opposing force, resistance (such as friction) or pressure to its flow. In brief, water will flow through an unclogged pipe much more than a pipe with debris in it, or is a clogged preventing (blocking) the free movement or the flow of the water. A small passage hole, or nearly a clog in the pipe will represent a very high resistance or "open circuit", and which results in none or a minimal flow of current. A broken pipe is analogous to a short circuit where electric current is dangerously flowing in the wrong location.

Electricity is also found in many electro-chemical processes in the human body. A spark is the visible light produced from electric current passing through and interacting (ie., colliding) with air atoms and releasing some energy as light. Air is usually a non-conductor of electricity, except when the voltage (potential, force, emf) value across it is high enough so as to overcome ("breakdown") that resistance and force current to "break through" a high resistive barrier, either with a physical hole or causing that barrier substance to conduct electricity. Once a small current is flowing, it will help widen and-or reduce the resistance of the path in and through the substance. The air will then behave like a "short circuit" (direct or short, low resistance path) conductor with many free charges (electrons and positive ions) flowing and carrying energy. In this state of very low resistance, the current flow can be high and dangerous. The break-down voltage value of a non conductor such as air depends on the distance of air it will have to travel through. More distance means more resistance, and therefore it will have a higher breakdown voltage level or threshold. The breakdown voltage of air is about 30kv = 30 thousand volts per centimeter of distance. = 3000 volts per millimeter. This breakdown voltage is sometimes called the dielectric strength of the specific material.

Verifying the existence of atoms, molecules, and the value of a mol.

It was previously mentioned in this book that Dalton in 1811 named atoms are the smallest, individual and unique particles of an element. Since that time, it was difficult to convince others of the existence of individual atoms and molecules. In 1827, **Robert Brown** (1773-1858), a botanist from Scotland and later England, noticed the random-like vibrations and-or (short) movement of small particles (living = organic, and-or unliving = inorganic) in water while he was looking through a microscope, and called this motion as **Brownian Motion**. The motion of the small particles was due to collisions from (invisible to the eye and microscope) water molecules which have various (ie., random) velocities and corresponding kinetic energy and directions due to previous collisions with other water molecules. This collision of a particle by a water molecule would give the small particle, such as pollen, motion and-or change its direction. Water molecules can also receive and gain kinetic energy, usually from thermal energy, and these molecule can move and will collide with anything in its way including other water molecules and-or larger particles.

Albert Einstein would study Brownian Motion and made some further discoveries of Brownian Motion in 1905. Starting in about 1908 and later years, Jean Perrin would study Brownian Motion further and made some advances. From the equations developed from all these studies, a relatively close value for the number of particles in a mol of particles was also made and which could be compared to other values calculated for a mol elsewhere. Avogadro's Number = Na = 1 mol. From this value, the mass of the particle (atom, molecule, etc) could be made from: (total mass) / Na For example, from 1 gram of hydrogen = 1 gram of hydrogen atoms = 1 gram of protons, and we can then find the mass of 1 proton. 1 gram of hydrogen / Na = 1 gram of protons / Na = mass of 1 proton

As mentioned previously in this book, the mass of a neutron is nearly equal to that of a proton, and in most practical cases, they are considered the same value, however, the true mass of a neutron includes the additional mass of an electron as part of its mass. It is now understood by scientists that neutrons come from atoms that were affected by extreme force and pressure where a proton and electron forms into a neutron.

The concepts of Brownian Motion have also been applied to other areas of science and-or statistics. Brown formalized the word of "nucleus" within a biologic cell, and made significant advances in microbiology with his microscope instrument. Today, it is possible to view rough or low resolution images of atoms by using advanced forms of electron microscopes that can magnify up to ten million times. Note that since individual protons, neutrons, and electrons (which orbit the atom at a very high velocity) are much smaller than the nucleus of the atom, that it is most likely that they will not be individually resolved by any microscope, except for a "cloud" of many moving electrons about the nucleus, and that can be imaged. Atoms also vibrate due to thermal radiation energy striking them. The general image resolved when viewing atoms is usually the surface layer of the crystal lattice arrangement of atoms. It is possible for people to purchase and-or invest with others in an electron microscope, and even a used one in working condition is of significant scientific valuable in terms of what it can do to see small things that optical microscopes cannot due to that the long wavelength of visible light is used to transmit the image (shapes, colors) of the object to the viewer's light sensitive eye or camera sensor after striking the object and reflecting from it. To resolve smaller objects, a smaller wavelength is needed, and that can be achieved by using many electrons which have a small particle size, or by using higher frequency electromagnetic waves such as x-rays. The **DNA** helix (ie., a double spiral, like a twisted ladder with two side rails with rungs connecting to them in between) structure of life was found using an x-ray imaging microscope which projects the reflected image of the object onto photographic film as the sensor. He words "helix" and "helical" are older word meaning "spiral" which means "twisted" or "coiled",especially to make a longer structure.

NATURAL RADIOACTIVE ELEMENTS

It is thought that heavy elements which have a high amount of mass were created at "the beginning of time", energy and matter such as during the "Big-Bang", and-or then also created from the high pressures created during exploding stars of which may also have previously concentrated some heavier elements at its highly pressurized and dense core location. Earth too is thought to have some heavy elements that have sunk to its core and are then of much higher concentration, and that some of it can eventually reach to the surface via volcanic activity or plate tectonics (ie., surface, land movements). Radioactive atoms on the surface of Earth contribute to the natural background radiation (from Sun, stars, mineral ores) we experience. Heavy element concentration near Earth's surface, either in mineral ore and-or atom form. is relatively low, and the amount of radiation from these that are radioactive is correspondingly relatively low and without issue.

In 1896, **Antoine Henri Becquerel** (1852-1908), from France, is credited to the formal discovery that some elements such as uranium, and without external stimulus or energy applied, produced "spontaneous", high-energy, penetrating rays of invisible light called radiation. To help comprehend this invisible light, energy or radiation concept, if you were to shine a light or laser from left to right in a vacuum (ie., having no reflection and-or light scattering due to particles of mass in it) front of you, you would not see that beam of energy, but it is there. A heater or heat radiator is a metal structure designed to make and radiate heated air, such as to heat a room. The word radiation means an outward release, emittance, or transfer of energy, and which extends "radially" in 360° or spherical-like pattern from the source of that radiation. Heat energy is generally invisible to the human eye, but special cameras can convert this invisible form of (light and-or rf) radiation it receives into a visible image on a screen.

The radiation Becquerel discovered was invisible (like heat is) and traveling outward like invisible light rays from its source at the center of that radiation. This "invisible light" was discovered using a light sensitive photographic plate, and is somewhat similar to how X-rays are produced from a Crookes tube were discovered a year earlier by Rontgen. His contemporaries at the same time and study were **Pierre and Marie Curie** (1859-1906 and 1867-1934 respectively), from France and Poland respectively, who discovered many things about radiation while working in France, such as the glowing light (ie., a form of radiation) emitting and highly radioactive unknown element they eventually purified (essentially filtered away from other unneeded substances) and-or concentrated it together into a larger amount and called it radium. This radium was refined - concentrated (unwanted debris removed) and isolated from the hard, black mineral called Uraninite ("pitchblende"), which contains the element uranium. The isotope, radium-236 has a half life of 1600 years and is considered as the decayed form of uranium of which the isotope form uranium-238 has a half life of 4.5 billion years. Unstable atoms which have an excess of energy and-or particles, can decay or change into another atom after releasing some energy and-or particles. Radium will decay into the element lead. Uranium, radium and lead are metals. She also discovered the element polonium in the pitchblende. Becquerel was the first to formally discover a radioactive element or material, although a few people have previously noticed some effects that uranium caused on photographic plates. Becquerel essentially discovered both nuclear or atomic energy and radiation. The SI unit (Bq) of measuring the rate or intensity of radiation is named after Becquerel. 1 Becquerel or Bg is the decay of or within one nucleus of an atom per second, hence it is also a rate of decay. **1 Bq = 1 decay / 1s**. This decay will cause the release of a pulse of energy which is the emitted radiation or radioactivity.

1 **Curie (Ci)** is 37 (10^9) = 37 billion Becquerels per second, such as emitted from 1 gram of radium-226. How much radiation life forms have absorbed per unit of time generally has other defined units such as **grays** , **ergs**, and **rads** (a word from the word "radiation"), and this and other related topics of radiation is beyond the scope of this book, and you may research these.

1 gray = 1Joule of radiation energy absorbed by 1 Kilogram of matter = 1J/k. :by Louis Gray (1905 - 1965), from England
1 gray = 1 **sievert** = 100 rads). : by Rolf Sievert (1896 - 1966), from Sweden

A combined or total amount of radiation of 1 gray = 1 sievert = 1000 milli-sieverts = 1000 mSv = 100 rads = 100 rem during 24 hours = 1 day of time could cause radiation sickness. A cumulative total of about 100 mSv of radiation per year, such as for nuclear energy workers, and hopefully even less, such as the natural background radiation level (about 3 mSv total / year, or 0.333 rem / year) is generally considered as tolerable to the body and will generally not cause long lasting damage to the body. A cumulative total of about 1 sievert / year = 1000 mSv / year is considered dangerous and can cause cancer in some people, especially if that dose or amount occurred in less than a year and-or was concentrated in one region of the body.

ergs, are an older unit for work in the **cgs** (centimeters, grams, and seconds) system of standardized measurements, and it may sometimes be mentioned in older literature, and 1 joule =10 million ergs = (1)(10^7) ergs. 1 erg is the work and-or energy used when a force of 1 dyne of force is applied to move an object 1 centimeter. This is about the energy needed to move a small dime sized coin its own distance. From: energy = work = (force)(distance) = (ma)(d) = fd = mad ,
1 dyne of force = (m)(a) = (1g)(1cm/s^2). **1 dyne** = (1)(10^-5) Newtons = 0.00001 N = 0.000010 N of force.

Seek medical attention if you or someone you know has been exposed to dangerous amounts of radiation. Keep others away from dangerous amounts of radiation. Inform authorities.

Some radioactive elements and-or materials have much more radioactivity than others. Obviously, the more mass of a specific radioactive material or element, there will be more atomic decays or changes, and therefore, more total radiation per second. **Anything with radioactivity needs to be treated seriously and with a high amount of caution, and it is best to shield any amounts of it in thick lead containers having a label (often with a skull and cross-bones image indicating a serious health concern) as to what elements are contained within, source, date, use, etc.** When radioactive elements were first noticed and-or purified, they were prized as both novelties due to light and-or energy without electricity, and scientific advancements, and little was known of their penetrating and life damaging radiation danger, and particularly when the radioactive substance is close to the body and is therefore higher in energy intensity, breathed in as dust, and-or ingested (ie., swallowed). If someone is not allergic to iodine, and if available at the time of need, **potassium-Iodide** (molecules) can help prevent the body from storing radioactive iodine particles, especially in the thyroid gland which naturally absorbs and requires (non-radioactive) iodine for body health. There has been a suggested dosage per day is about 130mg per 154lb adult person. After division by the numerator or denominator by the other, this fraction has the equivalents of: (1mg / 1.185lbs) = (0.844mg / 1lb) , and a dosage for a specific person and their corresponding weight can be calculated as:

Dosage according to the weight of a per person / day = (weight in pounds) (0.844mg / pound) , for example, if a person weighed 100 pounds = 45.3592 kg, their daily dose per day based on their weight is:

$$\text{Dosage per day, by weight} = (100\text{ lb})\left(\frac{0.844\text{mg}}{1\text{ lb}}\right) = (100)\,(0.844\text{mg}) = 84.4\text{mg}$$

For each radioactive element, it will have a certain half-life time for its (decreasing, decaying) radioactivity intensity, and will produce less nucleus decays because the radioactive substance or source is becoming reduced in (radioactive) mass, and is being changed to another (lighter weight) element, and therefore producing less radiation. There is an example in this book about calculating half-life time, etc. Nuclear radiation is dangerous at high (Bq) levels which increase the chance of that radiation to ionize or break apart atoms and-or molecules, and therefore causing more destruction within cell structures of life. In a living body, cells that compose it naturally die and are replaced, however if too many cells are damaged at once, such as by radiation, the health risks are much greater and even life threatening. This radiation and ionization concept, applied with the careful guidance of scientists and doctors is sometimes used as one of the methods to help stop cancer cells from growing further, but sometimes these types of methods to treat cancer may produce other side-effects and-or problems.

The main types of radiation, and in increasing strengths, are called: **alpha** (relatively large in size, atomic particles of matter, lower energy and-or speed protons and neutrons, and can be blocked by some thin materials like paper, a few inches of air, or surface of skin, and are dangerous over time if ingested (swallowed into the stomach and-or inhaled into the lungs) and not removed from the body), **beta** (high speed protons [beta+ radiation] or free electrons [beta- radiation], atomic particles of matter, can penetrate deeper into the body than alpha particles), and **gamma** (very high frequency, high energy electromagnetic RF = radio frequency energy like light is, high penetration, high danger, ex. an x-ray) radiation. Gamma radiation can travel a great distance through much matter. To significantly reduce gamma radiation to low levels, a thickness of about 1 foot of lead, or 7 feet of concrete, or 14 feet of water is needed. The denser the element and-or the thicker the shield is, the greater it will absorb and block radiation from traveling further. Generally, using just a few inches of these materials and-or an inch of steel will reduce gamma radiation by half. Combinations of materials and-or dirt, such as below the ground level, will also provide shielding from gamma radiation. It is also of note that high speed electrons striking metal can produce x-ray radiation which is a high frequency RF energy and-or radiation. There are many radiation detectors which are known commonly and-or generally as "(Hans) Geiger counters". A geiger counter can detect alpha, beta, and gamma radiation. These are more formally called "**Geiger-Muller** counters (~ 1928)".to detect electromagnetic radiation and-or particles of energy. They generally work by detecting (secondary) ionizing radiation (usually free electrons) produced in the air or some other gas, and which has a high voltage (ie., high energy, V=J/C) and can produce a brief pulse of electric current, however some other types may have a crystal which when struck by the radiation will then produce a small amount of light (ie., a photon) which is then greatly amplified by an electronic photo-multiplier (ie., photon or light multiplier - a type of free electron creation and chain reaction [ie., for more electrons created] circuit) so as to determine if radiation (usually gamma) was present. The pre-geiger tube technology is credited to **John Sealy Townsend,** (1868-1957) , from Britain, in about 1900, and who studied the electrical conduction of gasses.

Ernest Rutherford (1871-1937), from New Zealand and a British-kingdom scientist, discovered the proton in 1911 by observing hydrogen ions which is just a single proton of a hydrogen atom and without an electron. Note that a hydrogen atom does not even have a neutron counterpart of its proton, hence a hydrogen atom without an electron is a positive ion and that it is composed of just a single proton. He also classified and named the alpha and beta types of radiation, and conceived the half-life formula for radiation decay and intensity. Rutherford also discovered the gas and natural element of radon after experimenting with radium. Rutherford found that a radioactive element can change and-or decay into another element such as radon gas produced from radium.

1. Just as with common light or radio energy radiation, the farther a person is from the source of nuclear radiation, the less energy received per unit of area, and-or, the less intense it gets according to the inverse square law of electromagnetic transmission and reception. The intensity of radiation at a certain distance from the source can be indirectly measured by the intensity of the ionized radiation or particles produced by that radiation, such as what happens within the tube (ie., the radiation sensor) of a (Hans) Geiger (radiation) meter or counter. A received (ionizing) pulse or instance of radiation will then produce or create some free electrons which is technically a small current of charge or electricity, and which can then be greatly amplified to produce a more useful amount of current or pulse of electricity through a meters and-or a signaling light or audio speaker.

Directed energy beams such as a laser bean do not follow the inverse square law concept of energy radiation, and the intensity of the beam does not diminish or reduce by a great extent as it travels further from its source. This is very helpful for outer-space communications where the signal power is often relatively small and the distances are great. "Pulsed" laser or radio wave beams is a way to transmit a very brief and circuit safe amount of (intense or concentrated) energy, and of which can travel and be received even much further than a standard laser beam. As a simple example, consider: Power = energy / time , with units of watts. 1 watt = 1 Joule / 1s , but if that same amount of available energy can be released in a quicker (ie., shorter, less of a duration) or faster amount of time, perhaps in one-thousandth of a second = 1s / 1000 = 0.001s, the pulsed output would be equivalent to a much more powerful: 1 Joule / 0.001s = 1000w = 1000J / 1s for a brief, but effective moment of time. High power lasers, with many used for cutting metals, use this concept of having a high available and useful amount of power for a brief (ie., "pulses") amounts of time.

Some long distance spacecraft, such as the one that went to Pluto, used a radioactive substance as part of an electricity generator, and since being far from the Sun, there isn't much sunlight energy to power all the equipment on it such as the camera, computer, and radio-data communication system by using just typical sized solar panels. In general, these "atomic batteries" or "nuclear batteries" will convert the heat generated during radioactive decay into electricity by using many semiconductor PN junctions in series and-or parallel electrical connections and-or configurations near the heat source. Some small power capable nuclear batteries can use the emitted nuclear radiation directly rather than the thermal radiation (ie., heat, molecular energy) so as to create electricity.

A Cloud Chamber

A mentioned above, cosmic rays are from stars and other astronimical events such as collisions, and these rays will mostly be composed of particles such as protons. These protons will usually strike an oxygen atom or other gas atoms in the air in the atmophere and cause a secondary emission of radiation.

A **cloud chamber** is a device made of glass, and containng a gas - usually alcohol molecule vapor, and of which when a particle collides with a molecule of it, it will leave a visual indication, and the result is a visible trace line ("streak") or curve of the collision(s). This cloud chamber can therefore detect the presence of ionizing radiation or partilces, and-or it can be used to study the behaviour of those particles, hence it became a tool or device to study (essentially invisible) particle physics with. The types of particles that can be deteced are: protons, electrons, alpha rays, beta rays and muons.

The cloud chamber was invented by **Charles Wilson** (1869-1959), from Scotland-England, in 1911. Wilson used water vapor instead of alcohol vapor. These vapors are very small condensed gas particles (ie., atoms, molecules) forming a larger particlces, and of which are still seen as very small particles to the observer. If this gas particles is struck by a cosmic ray for example, the gas particle will loose an electron and become ionized (ie., having a net positive charge) and attract other gas molecules and essentially form a visible line, essentially by having several gas molecules nearer to each other in succession.

Others people have improved Wilsons cloud chamber and-or created other systems to see particles, and some use alcohol vapor instead of water vapor..

The following discussions are optional to the reader of this book. These discussions contain common knowledge about flight and modern technology so that the reader may have a basic awareness and understanding of them, and some examples of how having some math and science knowledge is very useful in many fields of study and-or work.

How Balloons And Planes Can Fly, And The History Of Flight

This book topic is mainly used as a practical example of the progressive advances in science and technology - usually aided with some amount of math along the way. The reader may skip over this till a later time since its not needed for further use in this book. If a person has interest in these or other similar topics, it soon becomes clear that also knowing some math will be useful. The selected discussions are relatively brief and to the point so as to keep the style of this book, and if you are further interested, you can research each topic elsewhere, such as by visiting your nearby bookstores for specific books about each topic(s).

Since ancient times, it was well known that warm and-or hot air rose upward or higher in the air and sky. For example, this can be felt as warm air rising up and away from a candle. The warmest air of a house would often be located in an upper room and especially near the room's ceiling if it was not vented or released outward. Warm and-or hot water or liquid also rises upward and resides on the surface of the cooler liquid below it. A practical question would be how does this actually happen, and the following discussion(s) will help answer this question. A related topic and question is why does oil always rise upward and-or float on water, and even when they are both at the same temperature. The quick and short answer to much of these questions has to do with the density of the substance(s) or object, and the pressure or forces acting upon it.

First, a brief discussion on the topic of gasses and pressure so as to help bring some understanding to the technical issues of flight.

Joseph Priestley (1733-1804), from England,, discovered and published in 1774 the findings of a substance we know today as **oxygen**. Just a few years before this, **Carl Scheel** (1742-1786), from Sweeden, made a similar, but published later, discovery of a "fire gas" or some substance in air (a mixture of gases that we breath) that helps combustion (ie., fire, burning) occur or happen. Scheel is also generally given some credit for the discovery of some other elements such as barium, chlorine, manganese, molybdenum, and tungsten. Priestley's friend and contemporary **Antoine Lavoisier,** (1743-1794), from France, determined this gas to be a unique (chemical and fundamental) element and called it **oxygen** which is a word based on the Greek words for "acid (strong or sharp)" and "gene" (that which creates or produces, "genesis", "generator", etc). Later discoveries by others showed that the element **hydrogen** (formally discovered by **Henry Cavendish** (1731-1810), from England, in 1766 and named by Lavoisier in relation to the word water (hydro) where the gas is part of the water and oxygen molecule: H2O) rather than oxygen, is actually the main substance or element for creating acids. **Robert Boyle**, from Ireland, previously noticed this gas being produced as bubbles during a certain experiment (iron in an acid solution) and resulting chemical reaction in 1671. Hydrogen is the most abundant element in the universe, and is particularly found as a major element of stars. Lavoisier then also made advances in the knowledge of oxidation and combustion. Priestley also discovered some other gas compounds, particularly some nitrogen gas compounds. Nitrogen gas was discovered in 1772 by **Daniel Rutherford**, (1749-1819), from Scotland. Nitrogen is a useful "inert gas" meaning it does not generally cause any chemical reactions with other elements and-or molecules, and is non-flammable and non-combustible.

Henry Cavendish determined that water is a compound of hydrogen gas and oxygen gas, and after combusting the gas mixture and noticing water drops. Shortly after **Alessandro Volta,** from Italy**,** created the first battery in 1799-1800 which permitted a portable, reliable, and steady (direct) current flow and rate, and at a desired voltage, the concept of **electrolysis** (using electricity to cause chemical reactions by using electricity to reduce or break apart matter, and-or bind matter such as for electroplating one metal onto another) or electro-chemistry was discovered in 1800. Electrolysis will separate a water molecule (ie. a compound of two or more atoms) into its atom parts and therefore create hydrogen gas (at the cathode or negative terminal or electrode), and oxygen gas (at the anode or positive terminal or electrode) from it when a voltage (about 1.23V (ie., 1.23 joules of energy / coulomb of charge) minimum, and any excess power may be wasteful if not designed correctly) is applied in and across the water with two metal electrode surfaces. Plain or pure water does not conduct electric current (ie., electrons and-or net positive atoms called ions) very well, and therefore some "electrolyte" (such as a small amount of salt, an acid or base) substance or particles are added into the water so as to create a (ion, solution, positive particles able to move) solution that can conduct electric current much better than pure or plain water. The electrolyte solution should also not conduct very well like a standard conductor or wire because that would then effectively "short circuit" the power, particularly the voltage difference and electric forces needed to propel particles needed for the electrolysis action on or from the metal or graphite electrodes. During electrolysis, electrons flowing (ie., the electric current) will add or remove electrons to or from ions and atoms in the electrodes (including objects used as an electrode(s)) and-or the electrolyte between the electrodes.

Hydrogen is flammable and is used in some safely designed "high heat micro-torches" and "fuel boosters", otherwise, hydrogen gas is dangerously explosive (highly, fast, combustible, resulting in much energy being released in a short amount of time, hence high power, explosive) in the presence of oxygen and when ignited ("lit", initiated, started, initial necessary energy input) by a flame, and this gas mixture is often called "Brown's gas" or "HHO" such as created during the electrolysis of water. Brown's gas is usually made only when immediately needed, and using short tubing, flame-back arrest or prevention filters, and a lack of pockets or regions where the gas can dangerously accumulate and cause a larger explosion. Often, the produced gasses are sent through a water-bubbler system as part of the flame-back prevention and safety system. By using some type of separator wall and-or collectors at the electrolysis process area, it will keep the hydrogen and oxygen gases separate, and they can be used for various purposes and-or storage, such as hydrogen balloons and breathing oxygen. The minimum voltage for this electrolysis is 1.23v (ie. 1.23 Joules / 1 Coulomb of charge), but 2v to 5v is more typical. Higher voltages can be used, but it can be a waste of energy and-or cause "thermal runaway" where the temperature increases conductivity and eventually leading to electrical problems. The amount of current drawn depends on the plate area, concentration or amount of electrolyte in the water, and it also depends on the temperature.

Electroplating uses the basic concepts of electrolysis for when a metal object is given a thin coating of another metal, atom by atom by using electricity through a conductive solution with the two metals also being used as the electrodes or terminals. This may be done to put a protective layer on another metal such as gold which is a metallic element that is highly unreactive to the presence of other substance and therefore has very low tarnishing or oxidation, and is therefore highly useful for electrical components and-or physical contact surfaces or joints. Often metals are coated with the relatively cheap metal zinc to prevent rusting (oxidation). Electroplating is also used for decoration and-or crafting needs. There has been some speculation if the ancient, possible crude or basic battery looking device modernly called the "Bahgdad battery" was used for some electroplating and-or novelty use, after all, several of any battery in series connection will increase the available energy or voltage, and any battery's in parallel connection will increase the current. There is no historical record or knowledge of any battery being manufactured and available for use to the general public, unless it was privileged knowledge and being kept secret, or simply too limited or insignificant to record. Much valuable knowledge is said to have been destroyed in the fire of the library of Alexandria (Egypt) historical incident and loss in 48BC.

Michael Faraday is often credited to the formal or scientific understanding of electrolysis, and created Faraday's Laws of Electrolysis, but there has been some limited use of it before then, but it wasn't understood and practical until the invention of the electric battery ("voltaic pile) by Volta in about 1800. Shortly thereafter, experimenters discovered electrolysis. Electrolysis has lead to the isolation (from rocks or "ores" containing the element(s) in practical quantities) and discovery of several elements such as the metallic elements: sodium, calcium, gallium, boron, lithium, barium, magnesium and potassium. **Humphry Davy** (1778-1829), of England, discovered several of these elements (sodium and potassium) by using electrolysis on substances at about the year 1807, and so as to break apart and refine substances such as compounds and oxides of elements into their more pure or elemental parts. This is a fine example of how technology can effectively improve, enhance and-or lead to other discoveries and technology since it was just a few years after the invention of Volta's battery which made the study of electricity more available and practical to many. Another similar leap in technology was with Geissler and Crooke's Tubes, and then later, the vacuum-tube and transistor electronic amplifiers. Michael Faraday initially worked for Humphry Davy. Davy invented the "trough battery" for his experiments, and where the two different metal plates (such as zinc and copper) were rather placed horizontally in series, rather than vertically in series like Volta's pile, and fully submerged in the (stronger, here diluted sulfuric acid) electrolyte solution.

Some of his battery's were also electrically (with wire or metal bars) connected in series (like a chain) so as to create a very high voltage (electric potential energy and-or force, and-or available energy difference to cause current to flow). With his large voltage supply, he also created the carbon arc (electric discharge through air, creating light) lamp that had carbon electrodes separated by a short distance in the air, and which then created a very bright light when an electric current was flowing through the air from one electrode to the other due to the high voltage applied. This was technically, the first (semi-practical) electric lamp (ie., that which can produce visible light energy by using electric energy) created before Thomas Edison made his study and discoveries so as to make a much lower voltage necessary, more practical (for the average person and household use) electric lamp about 70 years later in 1880. The light seen from Humphry's arc lamp is similar to that seen when welding (ie., gluing or joining) metal parts together with a molten metal which will soon becomes solid and rigid (strong) when the high energy and temperature is no longer applied.

The elements discovered by Humphry Davy using electrolysis are: Sodium, potassium, calcium, magnesium, boron, barium, strontium, and he also verified chlorine (a gas) as an element.

It is of note that some elements such as sodium are very reactive with some other elements (such as water) and are therefore not found in a pure metallic form in nature, and are rather found in ionic form and-or a (metal) salt form. Davy used the electrolysis of sodium hydroxide (NaOH) and found sodium metal, but modernly the electrolysis of molten sodium-chloride (ie., NaCl table salt) is used. Sodium metal cannot be stored in water (moisture) and-or air, and is stored in oil, and like some other elements such as mercury and the radioactive elements, it is generally illegal to have unless a permit for a specific use is granted.

Three ways to inflate a balloon:

1. Less dense, energized hot air. 2. More dense, compressed air. 3. A lighter than air, gas

Heated gas such as air molecules will acquire kinetic (moving) energy and move faster, and when these high speed, moving particles within a container strike its inner surface or wall, this will cause an expandable container to expand, "inflate", "balloon outward" and increase in volume size. If a (rigid) container cannot expand, the gas will retain its higher pressure state in its is more limited volume or space until it is somehow cooled and decreased in energy. Each "excited" or energized (increased in energy, such as kinetic energy) and moving atom or molecule of a gas has kinetic energy that can then apply a force or pressure on the inner surface of which it is in. This is how a hot-air balloon is inflated, and so as to have a larger volume, but containing less dense air, and therefore less air weight than the common cooler air at standard atmospheric pressure. As the kinetic energy of the moving particles of air is relatively, slowly reduced and transferred to the cooler balloon surface, and then to the air around the balloon, the balloon will decrease in internal pressure and will slowly deflate. If a hole develops in the balloon, and particularly near its top which has the hotter air and most pressure, it will deflate much quicker.

A key to remember when the word pressure is mentioned, is to always remember that it is caused by a force which is the application of energy. A ridged (strong, non-expandable) container may have a gas or liquid having a high pressure (and potential energy) within it. It took energy to compress that gas or liquid to have a high pressure, and it is stored much like a metal spring that was compressed, but here, the spring is made of a gas or liquid particles. This energy if not used, is then stored energy and-or "potential energy" that can be potentially (possibly, probably) used when needed.

As for inflating a rubber party balloon with cool air, this is done with air that is compressed and therefore it is more dense. Compressed air inside a rigid cylindrical tank can be purchased, or it can be made by a person blowing and forcing more and more air into a balloon until it becomes compressed within it. To keep the internal pressure and pressurized air inside the balloon, the balloon is tied closed, otherwise the compressed air will effectively will be forced out by that higher internal pressure, and it will travel into a lower pressure region, and the balloon will then deflate or decrease in volume and have the same air pressure in it as upon it, hence a pressure differential or relative pressure of 0 psi. When more air is forced into the balloon, it will cause the air within it to be pressurized (ie., greater than 14.7 psi, normal sea level air pressure) with the force upon it and the moving air molecules will compress closer together like a compressed spring. This effective "spring" of compressed ("pressurized") air will try to release its stored (potential) energy and apply a force upon the inside surface of the balloon which causes the balloon surface to expand outward. When a balloon rise into higher, less dense, less pressurized air (ie., < 14.7 psi) it will expand if the internal pressure is greater than the outside air pressure upon the surface of the balloon. If the air pressure is half, the volume of the balloon will double. According to the **gas pressure law** which considers the relationships between volume, temperature, and pressure of a gas:
$P1\ V1 = P2\ V2 = k$ = a constant value. Mathematically: $P1 / P2 = V2 / V1$ and these are reverse (or inverse) ratios, since the corresponding values of pressure and volume are inversely related. As the gas volume of a balloon increases when it expands into a lower pressure upon it, the internal pressure will decrease unless it is maintained, such as by heating it up to give it kinetic energy.

The air pressure of an deflated (non-inflated, expanded) elastic party balloon that is to filled by a person blowing into it will initially be at standard air pressure of 14.7 pounds per square inch (psi) on both its external and internal surfaces. To inflate (expand it internally) the balloon with a persons breath pressure, the air pressure upon its insider surface must be greater than 14.7psi for that balloon to expand into the now effectively lower air pressure of 14.7psi. This balloon will not rise upward vertically in the surrounding air since it now contains compressed air that is greater than 14.7psi, and is therefore denser and heavier in weight per unit volume than the surrounding air per unit of volume, and that type of air will actually "sink" (decrease vertically) in the normal air. **If the balloon is inflated and has less dense, high energy air in it, it will rise vertically such as like a bubble rising upward in water that is more dense and heavier per unit volume. More dense per unit volume, higher pressurized water will sink beneath the bubble area and will force the bubble upwards.**

Heating a gas(s) such as air will also make it less dense (ie., in amount) per unit volume, when the atoms or molecules of the material with an increased kinetic energy are moving faster and expanding outward into a larger volume or shape.

That air is then said to be "more lite" or "not as heavy" in weight per unit volume. Since ancient times, people knew that hot air, such as heated air from a fire, rose or lifted in a upward or vertical direction. Why does hot air rise upward? Heated air is less dense (per unit volume) than unheated air. This will effectively make it rise upward to a lower or equivalent pressure region as denser and heavier cooler air moves sideways and downward around it due to the constant force of gravity upon it and takes (ie., displaces) or "fills in" the balloons older position as the balloon moves into and displaces any denser air above it if the balloon does not have much weight, and this will be discussed further ahead.

It was just a matter of time from the first fires made by nature and ancient mankind until a small, lightweight hot air balloon type of structure called a Chinese Lantern was invented and which was made out of lite-weight paper material. These devices were sometimes used in the night as a signal for an event. Because they use a small flame for the energy or power source to make heated, less dense air, they are a potential fire hazard.

Matter that is said as being "lighter" is due to that it is, on average, less dense per unit volume of that matter or mass (the measure of matter). Less dense means it has less matter per unit volume, and less matter means it will also have less weight (force due to gravity acting on or applied that matter). Denser things have more mass per unit volume and therefore will correspondingly (and proportionally) weigh more per unit volume, and are said as being heavy or heavier in weight. This weight or force of the air up in the sky will cause a pressure upon the matter below it. The denser (weight, more force, more pressure) and more heavier air will effectively "sink" around and cause the less dense and less weight matter per unit of volume to be forced out (here upward) or "displaced" from its position, and cause it to move upwards towards the less dense air.

If the air pressure within a balloon or bubble remains the same as it rises upward, the size or volume of that balloon or bubble will actually increase due to the decreasing pressure upon its outer surface as it rises into a less pressure region.

When water freezes when being equal to or below 32°F = 0°C, it is one of the few substances that expands slightly when it is frozen to be a solid and no longer a liquid. This happens because these molecules that are loosing energy and joining together to form a rigid crystal structure with a slightly larger volume than that of the water. Ice will therefore actually be less dense (and "lighter") per unit volume than liquid water. Ice (ie., solidified water) is much harder than liquid water, but it is slightly less dense. If a piece of ice is placed in liquid water, the ice will effectively rise upwards (vertically), much like a less dense hot-air balloon would in the air, or a bubble of air in the water. By the force of gravity that applies some energy and induces motion, colder, more dense air or water will then sink to a lower level than less dense, warmer air and water. That which is less heavy in weight on average will be naturally forced via the pressure (from the weight, force) upon it to essentially move upward, as the more dense and heavy substance takes its position. This can be conceived as exchanging positions due to density, weight and pressure. Ice will float on water, and the relatively small amount, or percentage, of ice that is above the surface of the water is due to that ice has weight and is only slightly less dense than water. A unit and-or given volume of water will expand by about 5% in volume after it freezes. Now consider removing that 5% amount of volume, and we see that a unit of volume of ice then weighs about 5% less than the same volume of water. Since ice is much more dense than air, the ice will not be caused (by a net force due to a pressure difference) to move and rise upward and into the air to a lower air pressure region like a hot-air balloon would.

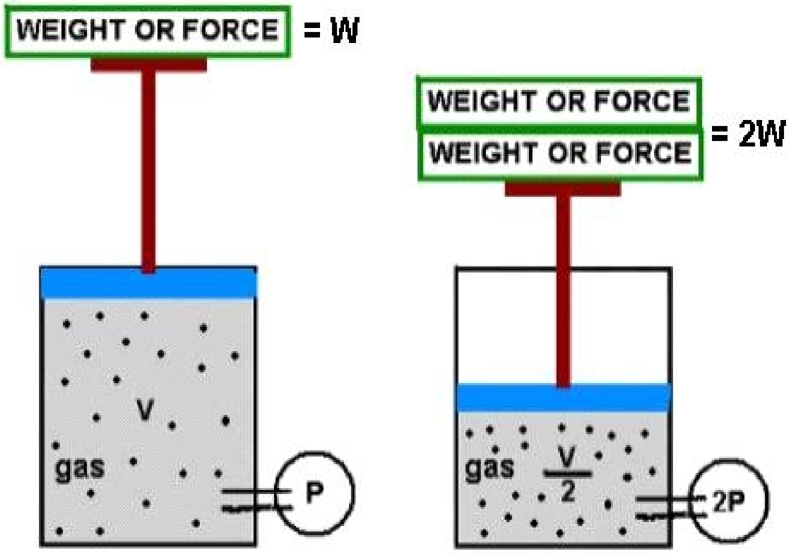
Air or any other gas that is twice as hot in temperature will have twice the kinetic energy, and therefore it will have twice the pressure or force upon the inside surface of a balloon to inflate or expand it. If the container is solid, rigid, non-expandable, then volume will not increase, but the pressure of the internal gas and its inner container wall or surface will increase when its temperature is increased. Excessive high pressure within a container can potentially cause a container to burst open abruptly and cause damage and-or injury. A lesson is to check how much maximum internal pressure a container is rated for before it is filled and-or (possibly) heated.

For a balloon filled with heated gas that is energized and expands in all directions, it could be said that the volume (**and not the mass and-or its weight**) of the gas has doubled when the temperature doubles. The space between the atoms or molecule of the heated gas has increased. If the container cannot expand, and is rigid, then the pressure of the gas will increase when its temperature is increased.

For a balloon, when the heated gas looses energy and looses temperature, the pressure upon the inside surface of the balloon will decrease, and the balloon will get deflated. For a balloon that was inflated with compressed air, it will generally remain inflated to the same volume for much longer and when the (internal, external) temperature difference is minimal and-or held constant. The volume of an inflated balloon can change if the outside, external air pressure on it increases or decreases, and-or the temperature changes such the temperature and pressure of the gas in the balloon changes. If an air filled balloon is placed in water, the internal volume of that air will decrease as it goes deeper into the water where more water pressure is upon the surface of that balloon or "bubble". The air in that balloon will get more compressed the deeper it goes in the increasingly pressurized water.

For a balloon with heated gas such as air, the density (mass per unit volume) of the gas will be less, and this will effectively cause the balloon to rise upward when the higher pressure, more dense air takes moves in to its place or "displaces it" with force. The higher air pressure at the bottom of the balloon will also force the less dense balloon upward into a lower pressure region. Technically, the air near the top of the balloon is slightly less dense and at a lower pressure than the air just beneath the balloon. For the balloon to rise, the upward pressure or force acting upon a balloon must be greater than the downward weight, force and pressure upon the balloon. At sea level, the air pressure per square inch is about 14.7 pounds per square inch = 14.7psi = 14.7lb / 1 in^2. Air is compressed more near the surface of the Earth and where the height and weight of the above "column" of air above it is the largest. Likewise, in a reverse way, the higher up into the sky, the less air pressure and density. The weight or force of this column of air or water can move into and displace less dense (per volume) objects (such as a bubble in water or a balloon to go up in the air) upward into a lower pressure and density region. It is also said that an object displaces (ie., occupies, replaces) the substance around it, and for example, it may be expressed in tons of weight of water displaced, rather than the volume of water displaced by a ship. The weight or pressure of the water displaced by a ship will causes the less dense on average ship (even if metal) to effectively remain floating on the top of the water rather than sink. The weight of just the metal parts of the ship are still denser and heavier than water, and will cause the lower portion of that ship to be submerged below the waters natural level until it **displaces a volume of water having an equivalent weight (or force) as that entire weight (or force) of the ship**. Since the weights are equal in value, the masses involved are then also equal in value.

For a strong or rigid container containing a gas, if the volume within that container can be compressed to be half, then the volume of the gas is also halved, but the gas pressure is doubled, and if the temperature of the gas is controlled to be the same. The gas is then said to be (more) pressurized or compressed. Raising the temperature of the gas will also raise the internal pressure of that gas, even if the container volume is rigid and remains the same value and does not expand in volume. Much of these concepts are often noted as being discovered by **Robert Boyle** (1627-1691) from Ireland, and later, in England in 1662, and is called **Boyle's Law** or the **Boyle-Mariotte Law**. This law states that pressure (P) and volume (V) of a gas are inversely proportional to each other. See [FIG 242].



If the container is not rigid (strong, unchangeable) and can expand in volume, an increase in temperature of a gas inside will cause an increase in pressure of that gas and it will expand the volume of the container. The expansion or movement will stop if the temperature does not increase further because the internal pressure will then not increase further.

The above figure shows that when twice the weight, force or pressure is applied to a gas such as on the right side, that the volume of it will be half because the gas molecules are compressed or pressurized closer together. A pressure gauge will display twice the pressure. In a reverse manner, when a volume of a gas is doubled and still having the same mass (ie., amount, weight), the internal pressure of it will be half. This relationship is true when the temperature of the gas is constant (ie., the same). Hotter gas has higher kinetic energy, hence it exerts more pressure upon itself (internal pressure) and the inside surface of its holding container.

(Jacques) **Charles's Law** published in about 1802 in France states that when the pressure (or weight as shown in the above figure) of a gas is constant, that its volume is determined by the temperature of that gas. A heated gas will have more kinetic energy and will want to expand the volume of the container it is in. Charles's and the **Robert brothers** also developed the first hydrogen gas, manned balloon in 1783. Hot air will expand the volume of a balloon. The pressure of the gas in the balloon is at the same pressure as the air pressure (14.7psi), and even if it has an opening at its bottom of it to let the hot air gas rise up into it.

Boyle's Law and Charles's law are two fundamental **Gas Laws** that consider volume, pressure, and temperature, and of which can affect the others two values unless one is held constant. This book briefly discusses these laws here, so as to have a practical, minimal understanding for chemistry, and for hot air balloon and-or airplane flight.

According to Boyle's Law, the product of pressure = (p) and volume = (v) is a constant (k), and for a specific type of gas (which has a certain density) used, and at a constant temperature. Though (k) is a constant, it rather has a specific and constant value for each specific system. For a certain gas (and its natural density) at a certain or constant temperature, and at or under (ie., subjected to) two different pressures and volumes, the equation is:

(p1)(v1) = k = (p2)(v2) : **Boyle's Law** , p = pressure , v = volume, and when the temperature is constant
Note for ex.: v2 = (p1 / p2) v1 : if p2 > p1 , their ratio is <1 , and then v2 < v1

Since the product of each side is a constant of (k), and if one factor of a side of the equation increases, the other will decrease by the same factor. Therefore, the factors of (p) and its corresponding value of (v) are inversely related.

Also note from this equation, mathematically and in reality, the "inverse ratios" of both pressures (p), and both corresponding volumes (v) of a gas:

$$\frac{p1}{p2} = \frac{v2}{v1}$$

: for these values, please consider a non-expandable, rigid container having a constant volume and holding a certain amount or volume of the substance such as a gas within it.

If one ratio (ex., p2/p1) went up by a factor 2, the other corresponding ratio (ex., v2/v1) will decreased by a factor of 2. If pressure doubled (increased by a factor of 2), the corresponding volume was halved (decreased by a factor of 2).

For the volume and pressure of a substance in a rigid container with a constant volume: At standard pressure (14.7psi, at sea level) and temperature (72°F) or STP conditions, air will exert a pressure of 14.7psi any place on a surface, including a container of any constant volume that is then sealed, and the air within it will still have the same pressure of 14.7psi. This rigid container and-or volume can have more air placed within it such as due to a pressure such as from an external air compressor forcing more compressed air inside, and while using a one-way ball valve so that the internal, pressurized (ie., having higher pressure) air cannot flow back out and then decrease the internal air pressure. In this type of situation with a rigid, constant container volume, both the internal air pressure and the internal volume or amount of the air gas matter will increase, but the volume of the rigid container will not. If the internal pressure (force/area) doubled, perhaps as displayed on a pressure gauge, the volume or amount of the internal gas has doubled. This volume or amount of gas per volume is called the density of the substance or gas, and is calculated as: density = mass / unit volume = mass / volume . For a compressible substance such as a gas, pressure and density are directly related. For example, if the pressure increases by a factor of 2, the density of that gas will increase by that same factor of 2. Another consideration is that for non-gas substances such as many liquids and metals, they can have a large pressure applied, and yet their density does not change by much or at all, and these substances still have the same density = mass/volume. The reason for this is that gas particles (atoms, and or molecules) have much more space between them than liquids and solids do, and when a pressure is applied to a quantity or volume of this gas, those particles will be forced to move closer together, and which results in more mass per unit of volume. Liquids and solids have particles that are closer together and held by positive and negative electro-static forces due to electron sharing at close distances, particularly with a metals that have more of a dense, closely packed and uniformly arranged atomic structure which is often called a crystal. Liquids are often said as being held together, particularly at its surface, by "surface tension (forces)". If a liquid is cooled enough, it will become a solid, hence denser. An exception to this is that when water freezes to a solid and becomes ice, its crystal structure actually expands larger than the container volume, hence its density actually decreases. density = mass / volume. If the volume of the substance increases, its density will decrease. This is why ice will rise up in water and then float, except that only about the upper $9.08\underline{7}\%$ of the ice structure is above the water due to that the density of ice is almost the same as that of water, except that ice is slightly less dense than that of water by about $9.08\underline{7}\%$ =~ 9.1% = roughly 9%. Liquid water at standard temperature and pressure is defined having a density of 1g/1cc = 1g/1mL. Ice has a density of 0.9167g/1cc = 0.9167g/1 mL . (1g/cc) / (0.9167g/cc) =~ 1.090869423

The molecules or atoms of a gas can be compressed closer together so as to have a compressed gas. The density (mass / volume = mass / unit volume) of a gas is directly related to the pressure on it. If the pressure doubles, the density of that gas will double, and vice-versa mathematically. (density / pressure) = a constant for a given gas , hence:

$$\frac{d}{p} = \frac{(\text{density})}{(\text{pressure})} = \frac{(n)(\text{density})}{(n)(\text{pressure})}$$ = equivalent ratios or fractions, = a constant value = k for a certain or given gas. : **when the gas pressure increase by (n), its density will increase by (n)**

For the physical and mathematical relationship between the volume and temperature of a gas:

$$\frac{v1}{T1} = k = \frac{v2}{T2}$$: **Charles's Law** , v = volume , T = temperature

The ratio of v to T is a constant (k) for a certain gas,
and when the pressure (force/area) is held constant.
Note for ex.: v2 = (T2 / T1) v1 , If (T2 > T1) , then (v2 > v1)

The Gay-Lussac's Law (1808) is that when the volume of a gas is constant, that the pressure of that gas is directly related to the temperature of that gas:

$$\frac{p1}{T1} = k = \frac{p2}{T2}$$: **Gay-Lussac's Law** , p = pressure , T = temperature

The ratio of p to T is a constant (k) for a certain gas, and
when the volume is held constant
Note for ex.: p2 = (T2 / T1) p1 , If (T2 > T1) , then (p2 > p1)

Air can be compressed to have a high "compression ratio" of the initial or total volume of air to the final or compressed volume of that same air. For a gasoline fueled combustion engine, the compression ratio is about 12 to 1 = 12:1, and for a diesel fueled combustion engine the (air, volume) compression ratio must be higher at about 20:1. Because of this, the air is more concentrated or dense and the fuel mixture becomes more combustible due to the denser structure of the air (specifically the oxygen atoms which promote combustion) atoms, and **Diesel engines** are said as being more efficient and need less fuel per mile of travel or for use in other machines, and mainly because diesel fuel has a higher energy density than gasoline. Things that compress air to a higher pressure will also force the water vapor (ie., steam, humidity) or moisture in that air to condense together into water droplets or "moisture" of which can be an issue to take into consideration.

When a gas is under a higher pressure (think of compressing or squeezing a balloon filled with air into a smaller volume or space, and the pressure of the air inside increases), there is also more atomic collisions among the (moving) gas atoms and the sides of its container, and the temperature of the gas will increase proportional (ie., by the same mathematical factor) to the pressure applied to that gas. In short, the pressure and temperature of a mass of gas in a fixed volume are directly related and proportional. This concept is called the: (gas and temperature) **Pressure Law**. The density of a compressed gas will increase, and if the heat is removed,it is possible to make a (dense) liquid form of that gas.

Here is the **ideal gas law = PV = nRT** **:**a practical generality only **,** P=pressure in atm , V=volume in liters , n=number of particles = amount of substance in moles , T=temperature in Kelvin. R=ideal or universal gas constant = about (0.08201 (atm)(L/(mol K) = **(**Avogadro's Number**)** **(**Boltzmann's Constant = particle energy /°K**)** = 6.02214076 (10^23) (1.380649)(10^-23 J/°K) =~ 8.3144626182 J / (°K mol) , hence **R = the total energy of 1 mol of the particles /°K**

If n=1mole of gas, and P = 1atm , V = 22.4L and T = 32°F = 273.15°K , R = PV / T =~ 0.08201
22.4L = 0.0224 cubic meters =~ 0.791 cubic feet =~ 1367 in^3
Setting up a proportion type of equation composed of equivalent fractions so as to find other amounts of mols:
For ex: 1 mol / 22.4L as= x mol / 1L , we find x mol = 0.044643857 , hence: 0.044642857 mol / 1L

There is a derivation in this book of: **(Pressure)(Volume)** = **PV = nRT = Energy in Joules units**
This **ideal gas law** is also called the "**combined gas law**" and the "**generalized gas law**" due to it being based on the equations and-or laws of Boyle, Charles and Gay-Lussac. For a given temperature, P and V are inversely proportional in value, and if one increases by a factor of (n), the other will decrease by that same factor of (n).

There is an example of using the gas laws in the Extras And Late Entries section of this book.

The compression piston in a Diesel type of combustion engine for vehicles briefly compresses the (fine mist) air and fuel mixture (previously created in the carburetor) to a very high pressure, and therefore, its temperature rises to a very high value and automatically ignites the air-fuel mixture without the need for an electric spark, or (electric, hot) "glow plug". At the core of a **Diesel** engine , created in about 1893, is a concept based on a very old and amazing method to light or start a fire which is often called a "fire piston". This fire starting concept was invented around 3500 years ago in the southeast Asian island areas such as the Philippines. With this device, a volume of air in a cylindrical container is compressed by a hand powered piston or plunger, and the air becomes heated to a high temperature by that high pressure. This hot, self igniting and combusting air is used to create a nearby hot glowing ember particle in a small piece of cloth or fiber that can then be used to create a fire to do beneficial things. This ancient concept was used and adapted in about 1893 by **Rudolph Diesel** (1858-1913), from Germany, when he made his combustion engine. The fuel this combustion engine used was refined (ie., filtered, processed) petroleum (ie., crude, unrefined oil from beneath the ground) based, but using various plant based oils is also possible. It could be said that the combustion engine is one of the results, derivative or product due to the "**industrial revolution**" phase of the world where much science, technology and communication assisted each other and they all began to increase exponentially, and until this day in 2022. The steam engine machine is considered as starting the "industrial revolution" in the 1700's. The steam engines of James Watt and previously builders had less than 5% efficiency. Modern steam engines and systems can be 40% efficient or more. Efficiency = Power Out / Power In , and with values being less than 1.

Fueled powered combustion engines could then be used for vehicles and machinery, and inspired the transportation revolution with automobiles ("cars" , ie., carrier, carriage, cart, and a form of chair) and paved (flat, hard), vehicle efficient roadways for quick and low cost, practical transportation to locations not easily or possibly accessible by other methods of transportation. This combustion engine replaced many of the smaller steam engines, however most trains would still use a very powerful (boiling water, high pressure, high energy and power) steam engine for many years into the future. For his automobile, Diesel also used some of the concepts of steam engines such as the piston, rod, valves, and the rotating wheels of a steam engine. Both steam and combustion engines convert pressure energy into mechanical energy such as for the rotation of a wheel or gear. There have been some crude attempts of internal combustion engines shortly after the year 1800, such as in 1807 by **Joseph Nicephore Niepce** (1765-1833) and his brother Claude from France, but it was not practical due to the type of plant based powdered fuel used which was not a liquid hydrocarbon, petroleum based fuel which contains much energy potential. Still, the engine showed a great concept of using (slow, pulsed) combustion to then give a propulsion or movement such as for a small boat. Niepce is most often historically credited to the **first photograph** and the basic photographic creation concepts in about 1825. These images were crude by today's standards, however this is how modern (analog, true, "film") photography (image or picture recording) and ("film") movies started, and which now includes very practical, inexpensive digital (a close approximation of the analog) images. Before Niepce, some others in the 1700's have experimented with light sensitive chemicals, such as various silver based solutions, and made crude outline images of various objects. Though it is possible to use a simple convex magnifying lens in a telescope or camera, a practical, less problematic glass lens was first used in a camera in 1832. **John William Draper,** from England, is one of the earliest pioneers or creators (ie., photographers) of quality photography in about 1839, but before this, a pinhole was usually used as a type of lens for great depth of field and high sharpness (low image distortion, no chromatic and spherical aberration) or clarity, and this lens which only allows a small amount of light into the camera and onto the photographic paper which at those times required a much longer exposure of light onto the photo sensitive material, and a very steady (still, not moving) subject to photograph, otherwise, a (visual) image "blurring" (distortion) of the resulting image of the subject recorded onto the photograph would occur. Since the pinhole opening is not an actual glass lens with spherical (ie., causing focus plane issues, out of focus areas) and chromatic (ie., for color ray divergence issues) aberrations, the depth of field is said to be infinite. It is of note that a pinhole was used since antiquity to study light and-or its conceived rays reflected from the subject and through the pinhole onto a surface of which can be adjusted in distance from that pinhole, and hence adjust the magnification of the image. The dark area for viewing the image created was often called something like the "**camera obscura**" [ie., a small room without illumination or light], and in here, the image can be viewed and-or traced onto bright paper. With photography, the image is rather recorded onto light sensitive paper. With a pinhole or simple lens, the image is rotated or "flipped" 180°, both horizontally and vertically, and this can be easily verified by following the rays of light from the subject and through the pinhole. It is of note that by using a light source and object, such as your hand, that the image outline defied by the shadow (ie., shaded, unlit portion) or shadow-image can also be traced onto paper and-or magnified by adjusting its distance from that light source. Before "fixed" or permanent images were made into film, **Johann Schulze** (1687-1744), from Germany, made a light sensitive solution nitric acid and silver (ie., silver nitrate) and which would temporarily record images via a chemical reaction using the light energy, and surely, this must of inspired Niepce who would later find a way to make the recorded image (ie., photograph) more permanent. When light shined on the solution, it became darker. It is of note that the recorded image is then naturally created and-or stored as a negative or ("reverse brightness") image where a bright object is recorded as being dark, and a dark object is stored as being as bright. If you were to then photograph this negative image, you could then obtain the positive or true image.

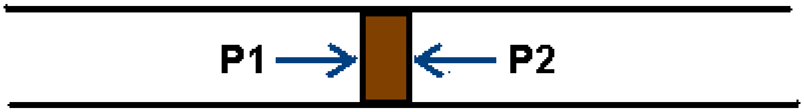
Carl Benz (1844-1929), from Germany, created the first practical automobile (ie., "horseless carriage" or simply a "car") for the public in 1885. It used gasoline fuel and worked well, but it was relatively crude when compared to **Henry Ford**'s "user friendly" Model-T car designed later 1908. Ford's was surely inspired by the Benz car. Carl was the husband to **Bertha Benz** who is credited to making the first long distance (about 60 miles in total length or distance) car ride in 1888. This distance and relative ease of travel was incredible for those times when a horse and carriage form of transportation was still very popular, and it brought the car into the minds and "wish-list" (hopes and desires) of many. She is also credited to some practical improvements to the design of the practical car. Relatively few could afford and properly maintain a horse, even till this day as of 2025. There is also a sanitation issue when a large number of horses are used within a city. Some of the modern "horse rides" for amusement in a city can probably be replaced by low cost Model-T replica cars, and so as to create a more nostalgic city theme.

To help understand how velocity can increase as pressure decreases consider this. If there is a difference in pressure (ie., force) in two containers of water, the water will be forced to flow in a pipe from the higher pressure container to the lower pressure container. The net difference in force or pressures upon the water will allow the slow water at the region of the pipe opening to gain kinetic energy and move, hence its speed will increased through that pipe and into the lower pressure region.

Here is another helpful description of the inverse relationship between the speed and pressure of a liquid or gas in a pipe:

Why is pressure inversely related the speed of the liquid or gas in a pipe?

One way to understand this is to consider a pipe that you are blowing into, and it has a sealed end. The air will get forced into it and get compressed or pressurized, and it will apply that pressure to the sides of the pipe as like trying to inflate it like a balloon shape, but not much change in volume will happen because the pipe is rigid (strong) and requires much more force or pressure to change its shape. If an opening of any size is then put into that pipe, such as a hole opening at its end, the pressurized air will force the air to the lower pressure region outside that pipe through the opening and where it is traveling at a higher velocity (possibly through another pipe) in that less pressure region. The compressed air in the original container or pipe will become more and more decompressed or depressurized and will no longer have energy to apply a force or pressure so as to move air, and the air in that pipe will eventually be at the same pressure as the outside air beyond the opening of the pipe or container. [FIG 243]



In above figure, when the pressures identified as P1 and P2 are equal, the object in the pipe will have no movement, hence no velocity. The net force applied to the object is 0 and its net velocity is 0. If P2 is now less pressure, there will be a net pressure or force upon the object in the pipe due to P1 being greater than P2. This net pressure or force due to the (remaining) difference in pressure will cause the object to move rightward in the pipe and to the less pressure region. If the difference in pressure is small, the velocity of the object will be small. If the difference in pressure gets larger, such as even when P2 gets even lower in pressure, the velocity of the object through the lower pressure region of the pipe will increase. For an analogy to the above pipe and pressure system, consider a lever or "balance-beam", perhaps for a scale, that has the same weight (ie., force) on both sides of the pivot, bearing or center point of rotational movement, and therefore, the beam or lever does not move (here rotate) in any direction (up or down, vertically) since there is no difference in the weight or force on either side of the balance point so as to create a force difference or imbalance to apply energy and induce motion. With an imbalance or difference in the forces, the greater it is, the greater the net force and the resulting velocity of movement.

Generally, the pressure in a container is evenly distributed and acting upon the entire inside surface of the container, and with that pressure trying to expand it, and the pressure trying to compress the material in it. If the container surface has a "weak point" that is not as rigid or strong at the majority of the container, it is possible that a hole will develop in that region and the internal pressure will then decrease due to the loss of the pressure and-or the pressurizing substance within it. A common example of this is when the inner tube of a bicycle wheel is punctured and all the compressed air within it, for example having 20 to 50 psi (pounds per square inch), is released to the outside air that is at a lower pressure, typically 14.7psi standard air pressure at sea level.

Ex. A weight or force of 10 was applied in one direction, and 10 in the other direction such as either left or right horizontally, or in the clockwise or counter-clockwise direction of rotation. We can assign a positive sign (+) to values for one direction, and the negative sign (-) to values for the other or opposite direction. Using a subtraction or difference operation so as to find the difference in these two (vector or component forces, and with direction) forces:

Net force upon the mass that can move = algebraic sum of those forces = difference in those force

Ex: (+10) + (-10) = +10 (+) -10 = +10 -10 = 0 = a net force, or difference of 0

Ex: If P1 = 3 and P2 = 10 : : ex. 3psi and 10 psi respectively

(P1) + (-P2) = 3 (+) -10 = 3 -10 = -7 = a net force or difference of -7, and the object in the above pipe will have a net, effective, resultant or useful force of 7 applied to it, and the negative sign indicates that the direction that the object will move will be in the left direction which is the same direction that the net force is being applied to it.

The difference between pressure and force

From: **pressure = P = (force / area) = F / A , force = F = PA Pressure can be considered as like the average force per unit of area, actual or a conceived distributed force on an surface area, and force can be considered as the total sum or multiple of the (same) pressure(s) applied to the entire area.** Higher effective pressure upon and in the water per unit area will also apply more force to the water at that area and pipe, and this potential or pressure energy will transfer more kinetic energy to that water at the output pipe at the bottom of the tank and its velocity will increase and flow faster into and in the region of lower pressure due to a pressure imbalance which could be thought of as a force or energy imbalance, or potential or static pressure energy difference. This could be thought of as that the energy stored in the compressed or pressurized water is now converted to water with an increase in momentum and-or kinetic energy and is now not as compressed when subjected to a high pressure, but now is decompressed and with a lower pressure and faster speed.

Though pressure applies a force and (potential) energy to the sides of a pipe or container, there is no transfer of (actual) energy to the side of the container if it is rigid and does not expand. If it did expand, some this could be imagined as expanding a spring which also requires energy, but then stores the energy applied. A expandable container could also possibly burst open and release (ie., transfer) that internal pressure energy to the outside, lower pressure region. If a container does expand, its volume will increase and the internal pressure will decrease.

Faster moving liquid or gas has more (kinetic) energy per volume of it because its kinetic energy (KE=mv^2 / 2) is more. An increase of applied **force (a measure of the application and-or transfer of energy)** causes an increase in pressure which causes an increase in the momentum (p = mv = momentum = (mass)(velocity)) of the gas or liquid if it is able to move from that higher pressure region.

The velocity of the gas or liquid is inversely related to the cross sectional area of the pipe. If the cross-sectional area of a pipe segment changes by a factor of (n), the velocity of the gas or liquid will change by (1/n), or in other words, the velocity of the gas or liquid will decrease by (n). For example, if the pipe area (A1) decreases by a factor of 2, n = (1/2) = 0.5 and A2 = (A1 / 2) = 0.5A1 = n A1, the velocity of the substance will increase by (1/n) = (1/0.5) = 2, and the kinetic energy of the substance will increase by 4 since KE = mv^2 / 2 = m(2v)^2 / 2 = 4 mv^2/2 = 4 KE

Faster moving gas or liquid has more of its energy stored in its velocity and motional ability (momentum = p = mv), rather than as a compressed gas or liquid storing (potential) energy, and therefore, for a moving liquid in the output pipe, there is less ("static") pressure applied to the inner sides of that pipe. This is the reason why the pressure is reduced if the velocity of the gas or liquid increases.

An airplane wing has less air pressure on its upper surface when there is fast moving air flowing over it. This concept is formally called **Bernoulli's Principle or Bernoulli's Law** and was discovered many years before the invention of the airplane. For the airplane, when fast air passes upon the upper surface of that wing, there is then less air pressure (force/area) and-or force upon the upper surface of the wing than on the lower surface, and the net difference in these two forces or pressure on each side of the airplane wing is the net or effective force applied in the upward (perpendicular to the direction of planes travel) direction to wing of which is attached to the plane and also transfers its upward force (ie., [upward, vertical] lift or lifting force) to it If the wing is tilted slightly upward, some compressed or higher pressure air beneath the wing also improves the lift force, but the increase of the cross sectional area of the wing in the direction of the plane movement also increase the "drag" or resistance to forward motion of the plane and causes it to collide with the air and loose some kinetic energy, speed and fuel efficiency (distance/Volume). A plane can use smaller adjustable wings called **"flaps" or ailerons** to change (horizontal and or vertical) directions due to the created forces such as drag, and this is much like how a **rudder** of a boat in water is used to change direction or steer the boat. The drag on one side of the boat will slow that side of the boat more than the other, and will effectively cause a turning or (usually partial) rotation of the boat about that slowed, lower (angle) velocity side closer to the center of rotation if the rudder is fixed into a position or angle and not reset (to 0 drag) to the desired, straight direction of travel. Setting the ailersons at a certain angle can also help provide some extra **compression lift** forces during the relatively slow speed during the take-off (ie., "lift off") from the runway to flight. This lift is somewhat like a plane going up a ramp that directs and-or forces it upward.

Faster moving liquid or gas has more energy per volume of it because its kinetic energy ($KE=mv^2 / 2$) is more. An increase of applied force causes an increase in pressure which causes an increase in velocity and momentum ($p = mv$ = momentum = (mass)(velocity)) of the gas or liquid. Faster moving liquid or gas means a high flow rate or volume of it per unit time being transferred from one location to another. **Flow Rate** = (accumulated amount of substance) / time
Flow Rate = volume / time , or mass / t. For a pipe, it is reasonably obvious that the amount of fluid input into the pipe equals the amount of fluid output of the pipe, and that the flow rate in equals the flow rate out, even if there is a restriction or constriction in the pipe, such as a smaller diameter section of pipe where the velocity of the fluid will increase and which therefore has or maintains the same flow rate as the input and output.

Flow Rate In = Flow Rate Out or: Flow Rate In = Flow Rate Out : "conservation of flow"
volume in / time = volume out / time mass in / time = mass out / time

If a pipe is then restricted, and so as to have the same flow rate and-or volume rate at the input and output, the velocity of the gas or liquid substance is naturally or automatically increased, and this will then result in the same volume of the substance flowing per unit of time at that point in the pipe, hence the same flow rate.

A pipe constriction, smaller diameter or cross-sectional area pipe is often called a **Venturi tube**. A common example for the use of a Venturi tube and its effect is found in the carburetor (or "fuel vaporizor", "atomizer") device of a gas or fluid combustion engine, and so as to provide a low pressure region of which the gas stored in a tank (a container, a volume) can be naturally forced up into by normal and effective higher air pressure upon it in the tank, and then it is turned into or converted to a fine mist mixture of fuel and air created by the high velocity force of the incoming air, and so as to promote a more powerful and efficient fuel combustion in the engine. A Venturi tube section of a pipe, such as on an airplane, can be used to determine or measure the pressure difference of which also determines the velocity of the substance (here air) flowing in it which is equal to the velocity of the airplane.

For an analogy to a Venturi tube, a two gear system can be considered, and with a larger input gear than the output gear. The smaller output gear will turn at a faster rate (angle or rotation / time) and store energy as kinetic energy of a smaller mass but at a higher velocity, and this is analogous to the higher velocity or speed of the substance in the Venturi tube section of a pipe. Since the input energy and-or power, must equal the output energy and-or power, there is a reduction in torque (ie., a force) in the output gear, and this is analogous to the reduction in pressure (= force / area) of the fluid in a Venturi tube. A decrease in cross-sectional area of the pipe will cause both an increase in velocity so as to maintain the flow rate, and a decrease in the (local) pressure in that location.

Due to the force of gravity, fluids can naturally flow "down hill" (at an angle below the horizontal level) in the less dense atmosphere, and-or through a pipe without any extra power needed. If the liquid, such as water, is deep, such as in a deep storage container placed at a height above ground (ie., an altitude of 0), the water at the bottom of that container is at a higher (static) pressure (ie., force) and it will be accelerated to a fast velocity in a pipe due to that pressure or force. As the water drains or moves away or out from the container and is less deep, there will be less pressure at the bottom of that container and into the pipe.

Another formula for flow rate in terms of velocity is: **Flow Rate = Av : (Area) (velocity)**

$$Q = \text{Flow Rate} = \frac{(\text{Volume})}{t} = Av = (\text{Area})(\text{velocity})$$

: Q for "Quantity rate".
This can be verified from:

$$\text{Distance} = \text{Length} = L = \frac{\text{Volume}}{\text{Area}} = \frac{L^3}{L^2} = L^1 = L = v\,t$$

, mathematically:

From: $\frac{\text{Volume}}{t}$ = A v , we have: $\frac{\text{Vol.}}{A}$ = vt = L , as shown above

Another related equation is: Flow Rate = pAv and this is derived here:

Qv = Flow Rate = V / t = "volume-metric flow rate" , or also, Qm = "mass flow rate" = m / t

density = p = m / V , therefore: m = pV : V = Volume , here, p = density and not momentum

Qm = m / t = pV / t = pL^3 / t = $\frac{pL^2\ (L^1)}{(\ t\ )}$ = pAv = (density of substance)(travel Area)(velocity of substance)

At two locations in a pipe with different cross-sectional areas, pressures, and velocities, the flow rate is still the same constant value, and Q1 = Q2. The density of the substance is also the same.

Q1 = Q2 : Q = flow rate

(p)(A1)(v1) = (p)(A2)(v2) : flow rate of a substance in a pipe with different cross sectional areas
Dividing out the common factor of (p=density) from each side:

(A1)(v1) = (A2)(v2) Due to Q being constant, A and v at a given location are inversely related and are not generally reciprocals of each other.

Since the flow rate Q remains constant in a pipe, if the cross-sectional area (A) changes by a factor of (n), the corresponding change in velocity (v) will be the reciprocal of this (n) value, and which is therefore: (1/n) :

Av = (n)(A) (v/n) or= (A/n) (nv) : the product of Av is constant for a constant flow rate

$$Av = \frac{n}{n}\ (Av) = nA\,\frac{v}{n} = \frac{A}{n}\ nv$$

If the known flow rate in a pipe has changed, and usually to a lower value, then there is a restriction or blockage in it, and-or an opening, "hole" or "leak [leakage]" at some place along its length which changes the internal pressure(s) on the pipe and within the moving substance, and this will indicate a leakage of some sort.

If there is compressed air in a container, and having a high pressure, and it is used to fill a balloon, that compressed air which had 0 speed, as like a liquid at 0 speed since it is not moving or flowing, will get decompressed as it moves out of that container and toward a lower pressure region. Once an (expandable material, stretchable) balloon starts to inflate, the air in it will begin to get compressed and pressurized so as to apply pressure to the inner side surface of that balloon and will cause it to expand (in volume) or inflate as it stretches outward from the internal pressure. The expansion will continue until that internal pressure equals the input pressure and there is no net difference in pressure to permit a flow of the input air, or the possible output air from the balloon such as if it had a greater pressure.

If you were to blow air into a rigid or strong container that does not expand, the pressure of the air within the container will increase up to the pressure of the air input, and that increasing pressure will reduce the speed of the air being input into the container until that input air speed falls to 0. If an opening was then put into that container, the velocity of the air being forced out will be initially high due to that internal pressure, and then the higher (than atmospheric) pressure within the container will decline to the value of the pressure on the outside of the container, such as the standard air pressure at sea level (14.7 psi).

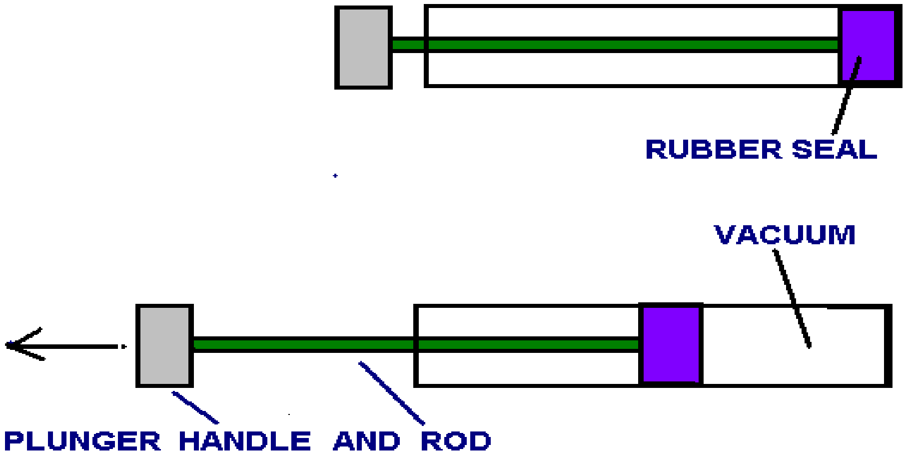
Considering **density = mass / volume**. Compressed air is "packed or squeezed", more dense air, and hot air is less dense and lower weight air and is therefore like decompressed air having less pressure. What keeps the hot air balloon inflated or expanded is the high kinetic energy of the air that is in the balloon as it applies pressure to the inner sides of the balloon. Though the denser air will inflate a balloon, it will not allow the balloon to rise upward since it is actually heavier in weight air than the air outside the balloon. Heated air is less dense, and therefore has less weight and is forced to rise upward like a bubble in water due to the downward and surrounding pressure of the water it is displacing. For the balloon, it will rise upward as the more dense, heavier and higher pressure air moves in and takes its place or "displaces it", effectively forcing it upward to a lower pressure region. Hydrogen gas can be used to inflate a balloon, however this gas is not heated because it is combustible, but simply is a low density, low weight gas that is put into the balloon as a compressed gas, much like how a person can inflate a balloon with their breath of compressed air. If a balloon is inflated with cool, regular compressed air and then sealed , it is possible to heat that mass and volume of air from within.

As a hot air balloon rises into the air, the pressure on the outside of the surface of the balloon becomes less, and that pressure within the balloon will then cause that balloon to expand even further if the temperature and pressure of the hot air inside remains the same and didn't cool down to a lower temperature as it rose upward into the cooler atmosphere.

While on this topic of pressures, and before more discussion about balloon flight history, a related topic to pressure is about a **vacuum**. The word "vacuum" is based on the word and meaning of "vacate", "evacuate", or "vacated" which means the lack or absence of something in it such as a gas (often air) and its pressure. If it helps your understanding and analysis, a vacuum can be considered a like a region of low pressure, and of which a high pressure region or substance will try to enter. A "vacuum-cleaner" is a machine that people can use to clean up debris on a floor surface by creating a low pressure (vacuum) region in the device of which the outside air (at normal pressure) and debris will be forced into that region as if the machine itself pulled, forced, "sucked" (inward, towards itself) or drawn it in. The debris will be captured and-or collected by a filter system that allows air to easily flow and pass through it, but not dust and-or other larger debris. This air flow is necessary so as to maintain air and debris movement into the machine, otherwise the machine would not function as a vacuum if there is a "clog" or blockage impeding air and debris flow or movement. The air pressure created by a vacuum machine is less than that of the normal air pressure of 14.7psi. A pure or complete vacuum has no gas or liquid substance in it, and therefore it can not possibly have any gas pressure from within it. The pressure within a ("pure", "perfect", "real", "true") vacuum is said to be 0psi, but for a home-use machine or vacuum, it only needs to be somewhat less than 14.7psi. The outside air pressure applied upon a vacuum container is still 14.7psi, and it will collapse due to that outside pressure if it is not a rigid structure that cannot bend and-or be compressed easily inward,, and it will take much more pressure and-or force for that to happen. Outer-space is generally considered as a perfect vacuum with some light and radio wave energy, and possibly some debris, constantly passing through it and which are not subjected to an external gas or liquid pressure. There are many uses for vacuum environments, such as for canning or jarring certain foods, or for manufacturing certain products where air would cause a contamination or more difficult process, such as in vacuum tube devices such as amplifiers.

A person using a straw to suck up a fluid from a cup, so as to have a drink of it, is essentially creating a low pressure region (less than 14.7psi) when they suck on the end of the straw, and the air or fluid is pushed (ie., forced) by the normal air pressure (14.7psi) upon and in the fluid to move upward into the straw and towards the lower pressure region in a persons mouth. When a person breathes inward and-or outward, it is due to the pressure differences created in the lungs and body.

Here is a basic form of a vacuum device or machine: [FIG 244]



In the above figure of a plunger (ie., a compressor) and-or suction (ie., a vacuum, decompressor) device when the handle is pulled out of it, a vacuum and larger volume with a lack of air or very little air is created. The little bit of air that was initially there, at say 14.7psi, is now all within a much larger volume, less compressed, and therefore under less pressure on itself and the side walls of the device in the vacuum region. The rubber seal is expandable and expands or pushes tight enough against the wall of the container so as to maintain a vacuum or compression (such as when making a compressed air or substance) forces. After pulling the plunger out a little into the normal air pressure region it becomes apparent that it requires more force to do this (besides any friction and-or compression forces of the rubber seal). The required force by the user is due to that the normal air pressure (or force) of 14.7psi must initially be overcome with an equal amount of force, otherwise the normal air pressure will force this plunger end to move back into the plunger device because the vacuum that was created there is at a lower pressure. If the plunger had an opening, tube and-or pipe on the right side of the image shown, it could be used as a vacuum device so as to effectively suction up or "draw" a gas or liquid into a low pressure vacuum region (volume) when the plunger is pulled outward from the device in the leftward direction.

The vacuum created with the above device is a "high vacuum" with a very low amount of material remaining in it such as some air atoms that may have been put into it when initially putting the plunger into it. An opening or an out-only, one-way ball valve on the right side of the plunger container above will allow the plunger to be easily pushed into the container since there will no compressed or pressurized air causing a resistance to its motion and the full seating of the plunger. This opening can then plugged so as to use the device as a vacuum, and with a ball valve, this will happen automatically as the plunger is pulled out of the container.

The device is also very similar to that of a medical, fluid injector or remover called a syringe or "needle" that is often used by a doctor or nurse so as to give a patient some medicine. The medicine dose (the prescribed amount) needed plus a little extra is drawn into the vacuum area by the aid of the plunger, and then the plunger is tipped vertically and physically tapped so as to allow any possible air bubble to rise to the top of the liquid so as to be ejected out of the plunger and through a thin, hollow needle (essentially a rigid metal tube with a sharp point at one end) with an inner hole along its length. The slight waste of medicine is acceptable due to the dangers involved. When the plunger is forced in, the fluid or gas will get pressurized and compressed slightly (ie., reduced in volume, denser, and more so for gasses than liquids) in size, and if there is an outlet (opening) to a lower pressure region having less force acting upon that gas or fluid, then that fluid or gas will move (via the applied force, pressure [compressing the material together] and some kinetic energy given to that fluid or gas) into that lower pressure region and become depressurized and be at the same pressure as that of which it was sent out into. The above device is also the basis of a mechanical air pressure gauge to measure the internal air pressure of vehicle's tires or inner tubes.

When a person breathes in, their diaphragm muscle pulls upon their lungs and expands or increases their volume while creating a lower pressure, partial vacuum region in the lungs, and then the higher, normal outside air pressure will then move into and fill in to occupy. When we "breath in" or "inhale" air; a process called "inspiration", that air in our lungs will get pressurized until it is at normal air pressure. When we "breath out" or "exhale" air; a process called "expiration", our muscles begin to compress and deflate the volume of our lungs. This increases the pressure of the air within them of which then has (kinetic) energy to travel towards the (relatively) lower pressure region of normal air pressure (14.7psi) outside our bodies.

The above device shown in the figure, when used in a reverse manner as a compressor or "pump", is how inner-tubes, tires, balls and various floats are inflated and expanded in volume. Tires and-or inner tubes of a tire are inflated to a safe and effective rated psi (pounds (of pressure) per square inch) value for assisting (avoiding friction and wasted energy) the movement and momentum (p=mv) ability of a vehicle. This will help the tires avoid much friction and the resulting wasted energy - usually as heat, into the road surface and tire. Tubes and-or tires, etc., will have a one-way valve that is much like a ball-valve with a pressure spring upon it so as to keep the pressurized air within the tube once it is put in using that same valve. **Over-inflating** a tire and-or tube, or anything else being inflated, can apply a high pressure above its safe rated value on its inner surface and damage the tube or tire, and-or cause a **dangerous, outward explosion**. Please check the literature and-or the side of the tire and-or inner-tube for the recommended, minimum and maximum air pressure.

If the above device is used without general modification as a compressor, so as to compress the internal air when the plunger is pushed in, the pressure of the internal air or other substance will increase. This compression will "squash" the air together into a smaller volume and therefore, it will create more air per unit volume, hence the air density (mass / volume) increases. The pressure or force compresses the air closer together and there will be more air atom collisions (especially since its now denser air) and the temperature of that air will increase. This is how a fire piston device works so as to heat a small piece of material to make an (combusting, glowing) ember so as to make a useful fire. The internal heat energy will eventually dissipate or be transferred through the wall of the container if the surrounding air or other material is cooler. During this process, the internal volume will not increase from the increase in temperature if its a rigid or strong container. If the container could expand like a hot air balloon, the heated air would expand the volume, and then the internal pressure will decrease in this process. If the heater internal air cools to that of the surrounding air, the balloon will be deflated.

A small air compressor can fill a huge tank with compressed gas but it will take a long time, depending on its volume much like filling a large container ("tank") of water with a small pipe and low flow rate. It does this by compressing a small volume of the air or other gas into an even smaller volume of gas having a high pressure. This pressurized gas is directed through a hose or pipe and into a rigid tank being pressurized. At someplace within this compressor system, there is a "one-way" valve(s) that allows the pressurized gas to flow in only one direction such as into the tank and prevents the pressurized air withing the tank or container from cumming back out, and will be there when needed. As mentioned previously, one type of valve for this is called a "ball valve". A can of "spray paint", that allows the spray of a fine mist of paint onto a surface, has an internal gas and paint contained and at a high internal pressure, and this paint and gas material can come out when a valve is pressed, releasing and decreasing its internal pressure into the lower, normal air pressure of 14.7 psi. As an average value, a can of spray paint may initially have an internal pressure of 75 psi.

To see a very easy and practical demonstration of air pressure, water pressure and vacuum, take a plastic bottle that has a screw type lid on it and put a very small hole near its bottom area using a metal pin. Fill (at the normal dispenser opening) the bottle with some water and then put the top on or cover that opening with your hand. If the outside air pressure is greater than or equal to the water pressure at the pin hole, no water will leave through that hole. If you remove the top or your hand, the air pressure will fill or displace any (low pressure) vacuum created and apply more pressure to the water in that bottle and force it out through the pin hole. Some containers are deliberately constructed so as to have a small (air pressure, inlet) hole near its top so as to prevent a vacuum from forming at the top of a liquid and preventing it from flowing well out from it. Some non-fluid barometers (for air pressure) or general pressure scales are construct so as the small movement or distance produced by changes in pressure is magnified by a long lite-weight lever so as to be more visible have more precision ("preciseness" of measurement).

A **siphon** is a device and-or concept that can help raise a liquid or fluid in a container and into a pipe or tube and raise it to a height above that fluids surface level, and then direct it to an outlet at another location that is physically below the supply or input liquid height. The siphon is first "primed" (ie., to prepare, such as the tube or pipe filled) by filling it with the source liquid, either by suction (ie., a vacuum with lower pressure region) or by filling the (closed outlet end) pipe with liquid. Once primed, opening the outlet end so as the liquid will move out from it due to its weight and gravity will create a lower pressure region in and at the pipe or tube opening in the supply fluid container and causing more fluid to be naturally pushed (if it is already at some pressure) or sucked into the pipe, fluid or tube region that is at a lower pressure. A siphon is a good method to irrigate (to apply water) farms when possible and needed. This type of siphon functions much like a hand operated or motorized water pump. The word "sip" is a derivative or form of the word "siphon", and a common use of it is to take a sip of a drink from or through a straw (a small tube used to assist some people when drinking fluids). A siphon will create a low pressure region, and therefore, it will cause a pressure difference upon a substance such as air or water pipe. This pressure difference will cause the (higher pressure) water to move toward the lower pressure of the siphon which could be a water pump. This water pump has a limited amount of power available to create the siphon, and it will be useful on level surfaces when moving water rather than trying to move it upward or vertically which will require more energy. The more water to be moved, the more massive and heavier it is and more energy and-or power will be required to do this, and this will usually result in a larger, more powerful water pump or siphon. It takes less energy to pump a unit volume water on a level surface than upward or vertically. If the energy is limited and-or a smaller pump is used, a smaller volume of water, and which has a corresponding small amount of mass and its resulting weight per cubic volume, per second (ie., the flow rate = volume / time) can be used, and the pipe and-or hose can be thinner in cross sectional (ie., diameter) area. How much wight of water or other fluid can a person siphon vertically? It depends upon the amount of "siphon power" (ie., input energy to cause the pressure difference and-or resulting application of it, hence to induce a [proportonal amount of] force upon the water so as to move it) they can create. A "self or free running" siphon requires no additional energy once initiated to function, but it can only lift 14.7 pounds of water per square inch of cross-sectional area of pipe due to the natural 14.7 psi of the atmosphere or air pressure at sea level and upon the opening of the pipe and-or the water there. It take a difference in pressure to cause the water to move, and even if the water has a certain amount of weight and-or pressure and kinetic energy already associated with it.

A calibrated **barometer** ("baro" means weight or pressure, hence a pressure meter) can sense and indicate any changes in its local air pressure value and displays the current air pressure. In short, a barometer is a (air) pressure gauge for measuring the natural air pressure due to the above air column. A mercury (column or filled) barometer is sometimes called (Evangelista) **Torrricelli** (ie., units of tor=mm of height, units) barometer because he invented it in 1643 in Italy. He was a student of Galileo. This barometer uses a rigid glass tube sealed at the upper, top end and is filled with liquid mercury (Hg) metal. Mercury, with a higher density per volume and weight than water, is used since if water as used, the device would need to be much taller (about 33.8 feet = 10.3m, and this is derived in the topic of Air Pressure And Barometer Values Derivation, further ahead in this book) since water has less density, and therefore, less weight and less pressure (at the bottom, deepest part) per unit area of a (vertical) column of it. The tube is then tilted vertically (ie., "upside down") and will drain slightly into a bowl filled with mercury which can be used as a sensor and be affected by the air pressure upon it. The mercury will drain into the bowl until there is no difference in the force (ie. weight) or pressure (psi) of the column of mercury and the air pressure (that is on average, 14.7psi) upon the mercury in the bowl. When the pressure of the column of mercury equals the air pressure, say 14.7psi at sea level, then the pressures will be in balance and no further movement or flow of that column of mercury will take place and reduce the height of it. There will be a slight vacuum (without air or other substance) of space created near the sealed, top or upper end of the tube. This space helps the mercury have some room to move upwards. If the local air pressure increases, it will push down on the pool of mercury and cause it to apply more pressure to the mercury in the glass tube and cause it to raise upward in height, or decrease height if the local air pressure decreases.

With a barometer, it is possible to calibrate it and use it as an altitude sensor and altitude (height) indicator since air pressure (or "air weight", or "air density") decreases with altitude, and this was discovered in 1646 by **Blaise Pascal** (1623-1662) from France. Other liquids, such as water, can be used to make a homemade barometer, but then that barometer needs to be larger (especially higher) in size and is therefore impractical. Normal air pressure is defined as 14.7psi = 1atm = 1013.25 millibars (mb) = 101.325 kPa , or a height of 760mm = 76cm = 29.92in. =~ 2.5ft. =~ 30in of mercury (Hg). This pressure is when there is 760tor = 760mmHg =~30inHg of a column of mercury (Hg). 1mb = 0.01451 psi. 1 psi = 68.93mb. 1 in. of Hg = 33.86mb During stormy weather, the height of the mercury may reduce by 1 inch or so, and this indicates a lower air pressure, and during a hot and dry day, the height may increase by an inch or so, and this indicates a higher air pressure. Surly, hot, less dense air which can expand a balloon, is at a higher pressure due to its excited or energized state or condition with its atoms having a higher thermal (kinetic, motional) energy and can apply more force to the inside surface of the balloon. pressure = force / area = weight / area. Since pressure is expressed as a ratio, if the numerator and denominator are multiplied or divided by the same value or factor, it creates an equivalent fraction, and yet the pressure rating is still the same. Due to this reason, the column of mercury can have any cross sectional area, such as for example, 3mm, and it is rather the density of the liquid used and the height of it which determines the pressure:

P = pressure = (force / area) = (mass)(acceleration) / (area) = (weight / area) : units of **Pascals** = Pa
$1Pa = 1N / m^2 = kg / m s^2$

P = pgh = pressure of a vertical column of an element = (element density)(gravitational acceleration)(height) =

P = (mass / volume) (g) (h) = $(mass / L^3)$ (g) (L^1) = $(mass / L^2)$ (g) = (mass / area) (g) = **(weight / area)**

work = (force)(distance) = (ma)(h) = (mg)(h) = **mgh** = Kinetic Energy joules , to raise an object in gravity = KE
KE = work = $\mathbf{mv^2 / 2}$ **= Fd = mad = mgh = (weight)(h)**
Here, KE = GPE , joules

P = pgh = (density)(g)(height) = (m / Vol)(g)(h) = $\frac{mgh}{Vol}$ = $\frac{\textbf{stored or potential energy}}{\textbf{Volume}}$: joules / volume mathematically:

PV = pghV = **mgh** = joules of potential or stored energy : stored or potential energy in a vertical column of fluid
Extra: For a given amount of energy, and PV being two factors of it, and if one changes by a factor of (n), the other will change by the factor of (1/n).

Blood Pressure Values

Normal and-or average human blood pressure is usually noted as: **120 / 80** (often said as "120 over 80"). When expressed more technically with units, it is actually: 120 mmHg / 80 mmHg relative to standard pressure, hence these are the increased, change or greater values than standard air pressure. mmHg means millimeters of height of a mercury barometer or pressure gauge or meter (here meaning a measuring device and not a distance). In terms of PSI, this corresponds to: 2.32 psig / 1.55 psig , and where psig means pounds per square inch greater than atmospheric pressure, hence its a relative pressure. This is because our bodies are always being compressed by the typical 14.7psi of the atmosphere. The numerator, upper or top value is called the systolic (or systole, an old Greek word having a meaning such as "contraction") pressure (P=F/A) on the artery walls when the heart is pumping or "beating" (ie., heart muscle contraction), and the denominator, lower or bottom value is called the diastolic (or diastole) pressure when the heart is at rest, not pumping or moving blood throughout the body, and is actually refilling with blood fluid for the next pumping cycle. In general, a person is said to have some high blood pressure if the systolic pressure is greater than 120 and the diastolic pressure is greater than 80. If the systolic pressure is greater than or equal to 180, that person has a dangerously high blood pressure and should seek medical assistance. Blood pressure is also mentioned at other locations in this book.

Extra: A (water) **ram pump** is a purely mechanical (ie., non-electric) pump that can operate using the kinetic energy (KE) of the fluid such as water being pumped. A stored mass of water above a reference height has stored energy called gravitational potential energy (GPE = mgh). Moving or flowing water has movement or kinetic energy: $KE=mv^2 / 2$, and which increases as it fall downward under the constant force of gravity (g) constantly acting upon its motion and accelerating its velocity (ie., speed) Since energy is conserved, except for some system losses, we ideally have: energy out = energy in, and so as KE increases, GPE decreases. Total energy = GPE + KE. The more mass and velocity of the flowing water per second of time, the more power (watts = joules of energy / second) that flowing water will have. Since this device is a pump, it can force (application and transfer of energy) and move (ie., lift) water up to a height above itself, but not higher than the distant, initial source of that water such as a stream of flowing water - weather a natural and-or artificial source and flow from a dam (essentially a pool or reservoir) of water.

If you were to take a container filled with water, and a hose connected to the bottom of it, and then bend the end of that hose upward in height, the flowing water will stop flowing at the surface level height of that source of that water, and therefore not any higher. A water surface at sea level on Earth has a (air) pressure (P = F/A , and F = PA) upon its surface, and at 14.7psi.

The (automatic) ram pump is also called a **hydraulic pump**, and it was invented in 1796 by by **Joseph Montgolfier.** Montgolfier and his brother are also celebrated as for creating the first manned flights in hot air balloons. The ram pump has an input for the flowing water, and output of flowing water. It has two internal valves where one is called waste value (ie., wasted, non-pumped water, but can otherwise be used for something), and the other valve is called the (water) delivery value or output valve. These valves periodically (ie.,cyclically) open and close one after the other. The system is "primed" or stated by allowing some water out of the waste valve, and then shutting it, and there will then be a higher water pressure in the input pipe due to the kinetic energy of it then having a value of 0 Joules. This higher pressure and stored energy will then open the delivery value. Higher pressure water has more energy, and the flow of the output water can go higher than the height of the source of that water. Usually the output pipe or tube usually has a smaller diameter than the input pipe or tube, and so as to pump the water even higher since it there will be less weight to force and move given available (temporary, pulse of) energy applied to it. Once the output water pressure cannot keep the output value open, it will close, and the water pumping cycle automatically repeats. A ram pump essentially functions due to the "water hammer" effect of quickly stopping the flow of water by using a (flow, movement) valve, and then hearing and-or seeing the mechanical stress it induces with that pipe.

Flight With Hot-Air Filled Balloons

Before parachutes, gliders (ie., like a kite - an air-plane without an engine) and air-planes with motors, the first form of "human flight" or transportation through the air was made possible using hot-air filled balloons.

Hot-air balloons are capable of lifting a gondola, vessel or container having some reasonable (heavier than air) weight and-or travelers in it, and upward into the sky, and may even drift (move without any needed balloon power) and travel with the aid of the forces of the wind upon the balloon surface. Wind is a movement or flow of air gas and its composing atoms of mostly nitrogen and oxygen, hence that air has kinetic energy which can apply a force and transfer (kinetic) its energy to other objects such as a "wind generator" that can make electricity. Most of these balloons will have an on-board heat source so as to produce any needed heated to heat the air within the balloon and make it less dense and less heavier air which will effectively allow the balloon to rise further upward. It is the heated air in the balloon which gives it its expanded and-or round shape due to the pressure upon its inner surface. When the heated air in the balloon looses energy and cools down, the balloon will start to "sink" or move downward in altitude (height), and more heat will be needed again to heat the internal air so as to maintain a desired altitude in the sky and-or to keep traveling further.

Though a hot-air balloon may have a few (lite-weight) metal parts in its entirety or whole volume, a hot air balloon structure, on average, is still less dense per unit volume of air, and therefore it also has less weight per volume than the surrounding air, and this will effectively cause the balloon to rise upward into a less dense region above it as the denser, heavier, more pressurized air forces it upward like a bubble in water. For a balloon to lift a weight or object upward, it must supply the conditions necessary for an effective upward lift force to be greater than that weight of that object. More technically, the (heavier than air) vessel or weight is suspended or held by the balloon, and it is rather the pressures and-or forces acting upon the hollow, less dense balloon volume that initially causes it to move upward and which with then will pull (ie., a tension force) the suspended weight upward with it.

The **Montgolfier brothers** worked by making paper materials for people, and noticed that when it was constructed to be a shape of a hollow lightweight bag (essentially an open-ended, 5-sided box-like shaped container made of fabric and-or paper cloth and glue), it would float upwards when heated air was allowed into the bottom opening. This is very similar to how the small, lite-weight, low power, ancient Chinese sky (floating upward, "flying") lanterns functioned, however, there is no knowledge or public record of them being large and practical enough so as to carry people by the time of the Montgolfier flights. We can still ask today: "Were the Montgolfier brothers inspired by the (ancient) developed Chinese sky lanterns?", and the answer would be "probably", but occasionally, things have been discovered independently and in such a way that it both verifies and increases the total known and useful body of knowledge about it. The first recorded account of a kite (a plane-like surface lifted into the air by the wind, and then held stable by a person holding a string attached to it) is in China, and is generally considered to have been created and flown there at about 500BC. Both kites and balloons have shown that various types of flight are possible. Since ancient times, ships or "sail-boats" have been using fabric (cloth) sails so as to collect the wind's (moving air) kinetic energy and transfer it to the forward movement and-or kinetic energy of the boat. A sail can be considered as a type of pre-kite since antiquity (for many thousands of years). The word "kite" is rooted in the words: "bird", "flight" and "sail". The word "sail" is rooted in the words "sea" (ocean, water) and "veil" (cloth).

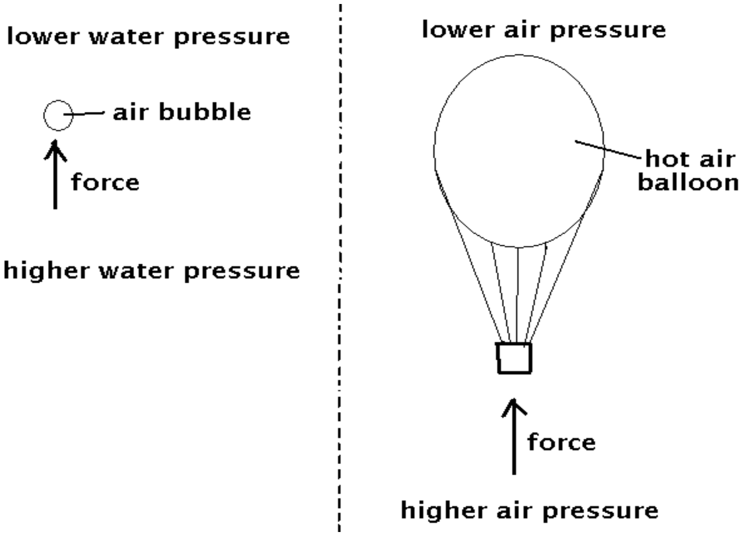
The first travelers on a large enough hot air balloon were a sheep, duck and rooster. These hot-air balloons, capable of lifting a substantial weight, such as for a one or two people, were first developed and tested by the **(Joseph-Michael, and Jacques-Etienne) Montgolfier** brothers in France in the year 1783. This was an surely an incredible event, and had peoples minds filled with joy, inspiration, and the wonder of science and the things yet to come, and which eventually lead to other flying objects and vehicles such as gliders, airplanes, rockets, satellites, and the Moon landings 186 years later.

The word "flight", "flying" or "to fly" is the common description of how a bird, bug (ex. a fly), creature or object such as a balloon is seen moving ("flys", or is "flying") in and through the air. "Flown" is the past tense of the word and meaning of "to fly" or "flying". Even the word "flown" can be associated with the word "flow" (movement) of a current of water.

After the initial discoveries and tests, the next logical step for the Montgolfier brothers was to just increase the balloon's size (ie., it's internal volume) and strength so as to have enough air displaced so as to essentially allow a larger, upward and more practical (ie., more useful) upward force to carry or lift some objects, animals, and people. These balloons would be made from a lite-weight cotton cloth material with paper glued over it so as to seal it from air leaks which would eventually release the internal hot air and its pressure and that will deflate the balloon and reduce its altitude.

The famous United States of America (U.S.A) statesman, inventor and one among the first electric pioneer's, **Ben Franklin** witnessed the first manned (by **Jean-Francois-Pilatre De Rozier**, and **Francois Laurent d'Arlandes**) balloon flight while in France on November 21, 1783, and then some of the other initial balloon flights later, and with some filled with hydrogen gas which is even less dense and lighter than heated air and will effectively allow a greater upward force (lift) than just hot air alone, and it can therefore lift more weight per (same) volume of gas, and without the need to have an on-board air heater. Some of these flights went several thousands of feet high and traveled more than 10 miles away. People who knew of these balloon flights (and even later, such as with the first airplane and rocket flights) were amazed and considered them as a wonderful technological discovery and advancement for mankind. Ben Franklin soon told **George Washington**, the President of the country of United States Of America, and in 10 years later, George Washington became privileged to witness the first manned balloon flight launched in United States, and this was on January 10, 1793 at Philadelphia city in the state of Pennsylvania (PA). The balloon landed eastward in the state of New Jersey. The pilot was Jean-Pierre Blanchard, and who previously was the first person to cross the English Channel on January 7, 1785 via a balloon, and while delivering the first air-mail from England to France. Many practical applications or uses for balloons were soon found such as: travel, mail, package or cargo delivery, observation, photography, signaling (communication - visual [lights] or radio), recreation, and high altitude research of which eventually lead to space flight research.

A lighter than air balloon rises upward due to a similar process that happens with an air bubble in water, or warm water that is less dense than cooler water, and it rises upward to the lower pressure water surface. Another example is when the less dense than water, piece of ice rises vertically (upward) in water. The simple answer is that the air bubble is less dense and-or lighter than water around it. The internal pressure of the water bubble, due to its "water column" and weight above it and-or at its sides will move (due to its pressure or force) into (ie. displace, by force, to take the place of) the position of the less dense, less weight bubble, and replace (opposite of displace) or fill its current location, and the bubble will be moved or forced to a less dense, less pressure region above it. [Fig 245]



Though a balloon's construction material is heavier than air, its volume displaces (resides in its original place and-or volume ["space"]) the air around it like how an air bubble in water displaces the water around it. Regular air in the balloon will not let it go upwards, and it needs to be filled with something lighter (less dense and weight) than the surrounding air and the air pressure upon it. The hot air, hydrogen (a very dangerous, flammable or explosive gas) or helium gas in a balloon is lighter than the cooler, denser (due to air pressure) air around it. The word "lighter" here means of less dense, having a lower density, and therefore it weighs less per unit of volume, and is said as being "light-weight" (or "lite-weight" to distinguish it from visible light) which is the opposite of something that is said to be a "heavy-weight" (heavy-in-weight).

Being less dense or lighter in weight, such as oil that rises upward and floats on water, may not a sufficient or full explanation as for how something like the bubble or balloon would move upward. When a bubble of air is below the surface of the water, there is water pressure (ie., a force) on that bubble, and is generally the same value all around that bubble when the dimensions of that bubble or other container is not vertically large in volume where the pressure would be greater on a deeper part of it. The deeper that bubble is, the more water pressure and force on it since there is more water above it like a heavier stone (vertical) column on a piece of foam, sponge, wood or a weight scale. The bubble displaces water and there is now the pressure due to this water displacement upon it. Since there is nearly the same force in every direction upon the bubble, the bubble should not move, but it does, and where to? Rather than being forced to a deeper depth where water is more compressed and dense, and with greater water pressure upon it, the bubble is rather forced upwards and moves into an area of lower water pressure or force since it is essentially the "easiest" natural place and direction it can be moved to with less resistance (due to less pressure and-or force) to its movement. Even a hollow metal, such as steel (ie., many times denser and heavier than water) ball can rise upward like a bubble and float near the surface if the total average density and-or weight of it is less than that of the volume of water that it displaces. It can also be said that the volume of water that was displaced by that hollow metal ball was heavier than that hollow metal ball, and so the hollow metal ball rises upward, rather than downward, and at the same time, water of higher pressure or weight (under the constantly applied force of gravity promoting movement) flows around it to fill its previous place at the same time that it rises upward with some gained kinetic energy and displaces water again and so on. This process repeats until the ball is at the waters surface and ceases movement upward because its average density and average weight is too much for it to rise up any further, such as up into the much less dense air.

A less dense substance will rise upward, and-or be forced upward, in a more dense substance. A common example of this is oil that will rise up to the surface of water because the oil is less dense than water. The weight of oil is also slightly less than that of water, and water will essentially sink around it and displace the oil and cause it to move upward.

Objects that rise upward in water or air are said to be **buoyant** (pronounced correctly as "boo-yent", and more commonly as "boy-ant") and-or have buoyancy. A buoy ("boo-ee") is a floating structure in water that is usually held in position by an anchor or heavy weight at the ground beneath the water, and so as to visually help guide or direct boats as they travel near the shore or other structures in the water that are a collision and-or boat grounding hazard due to a low water depth. The total amount of upwards force or "lift" (ie., lifting force) upon a balloon or bubble can be increased by using a gas with less density and-or by increasing the size or volume of that balloon(s) or bubble(s). In a very similar way, a heavier than water boat can float on water due to the total effective water pressure or force on the bottom of that boat. Due to the weight of the boat, the bottom of a boat will go into the water to some depth until the total upward force of the water pressure equals the entire downward weight of that boat. The hollow shape of the bottom of the boat structure gives buoyancy (floating ability) to the boat portion that is beneath the water as that boat portion essentially has less density per unit volume than that of the water it went into and displaced or "pushed aside". In general, and as mentioned previously, a hollow metal ball, and a solid metal ball of equal size will displace an equivalent volume of water, but the solid ball will sink in water since the effective (ie., average) density, and therefore its weight, is greater than that of the water it displaced, and the force of gravity and resulting weight (ie.,force or pressure) of the ball upon the water below it will be greater than the upward buoyant (water pressure) force upon it, and that ball will sink or move vertically downward. The hollow metal ball or boat shape will rise upward and float if the weight of it is less than the weight (ie., a force, a pressure, the buoyant force) of that of the displaced water, and this is similar to the average or resulting density of the volume of material (here a volume consisting of a hollow metal sphere and air) being less than that of the density of water it displaces.

The buoyant force upon an object is rather due to the water volume displaced and the average density of the object, and not its weight. The effective buoyant force acting upon a submerged object , or its effective weight in the water will be the value of: **effective weight of a object submerged in water = (objects weight - buoyant force)** , if the buoyant force is more, the object will rise and have a negative weight or upward force.

One of the greatest scientist of antiquity, **Archimedes** from the city of Syracuse in Italy, in about 250BC, discovered that the upward force called the buoyant force upon an object in water is equal to the weight of the volume of water that the object displaced (ie.,is in place of and-or pushed aside). If an object, such as a hollow plastic ball is above the surface, and you slowly push it under water, you can feel this **buoyant force** or buoyant pressure increase as the object is caused to push back against your hand in an equal and opposite force. **There is a formula for the calculation for the amount**

of buoyant force for a given volume and-or object in the Extras And Late Entries section of this book. This buoyant force is due to the weight and pressure (force / area) of the water surrounding the ball. More accurately, it is due to the water pressure difference on the upper portion and lower portion of the object. The weight of a cubic foot of water displaced by an object having a volume of 1 cubic foot (7.48 gallons) is 62.43 pounds (lbs) of weight. The weight of a liter of water = 1L = 1000g (gram weight, of 1000cc = 1000 ml of water) = 1kg = 2.205lbs =~ 10N. If you had a lite-weight and rigid box structure with 1 cubic foot =~ 28.317L of empty volume in it, it would take about 62.43 pounds =~ 278 N of downward force so as to make it submerge below the water surface level. Due to this upward (buoyant) force, this also means that an object submerged in water will feel less heavy (ie., weigh [the downward force due to force] less) by an amount equal to the weight of the volume of water it displaces and-or the (upward) buoyant force. In short, the water pressure is essentially forcing the object upward to a region of lower pressure, hence against the downward pull of gravity, and the object will then effectively weigh less in the water, but still have the same amount of mass. It is also as if the force of gravity was less upon it. With the known values shown above, you can use a proportional type of equivalent fraction equation to solve for the buoyant force or volume. Ex. 1ft^3 / 62.43 lbs = 2 ft^3 / x lbs, after solving: x = 124.86 lbs

The concept of displaced water is also a very effective way to calculate the volume of an object by measuring the equivalent volume of water it displaces, which can be determined by the difference of the height levels of the water that it is placed into - such as a calibrated measuring container. The (mass, rather than weight) density (p) of the object can be found by dividing the mass of the object by its volume. p = m / V . Note: weight = effective force = (mass) x (acceleration) = mg, and therefore, mass = weight / g. Therefore, **density = p = m / V = weight / g V** . Note that: **weight = mg = Vpg**. On Earth's gravity, (pg) is a constant for a given substance. Knowing the density (density = mass / volume) of the object can also be used to determine the specific element that the object is composed of, such as determining if a king's crown is composed (ie., containing, consisting) of pure gold which has a certain known density of matter and-or weight (= mg) per unit of volume. Many modern scales will automatically convert the weight of an object to its equivalent mass value, and then display that amount of mass.

Ex. For 1L of water displaced, the weight of it, and the resulting total buoyancy for a structure of equivalent volume:
(pg) = (1kg / 1L) (9.8m/s^2) = (9.8 kg-m / L s^2) , and since L was used for volume instead of m^3:
Vpg = L (9.8 kg-m / L s^2) = 9.8 kg m/s^2 = 9.8 Newtons = 2.205 lbs = Force = ma = Weight

Due to being capable of displacing a large amount of water that is relatively heavy in weight, underwater balloons with a suspended basket or rope beneath it are very practical for helping lift heavy weighted objects to be propelled (ie., forced) to near the surface of the water for retrieval. For this to happen more easily, a deflated balloon must first be taken beneath the water to the object's location and then filled and expanded with a volume of air so as to displace water. The larger the volume of the air filled balloon beneath the surface of the water, and it displacing a larger volume of water and its equivalent weight, the more total effective buoyant force or actual "lift" force it can produce. This process mentioned can also be achieved by using several smaller sized balloons connected to the object, and the total lift or buoyant force will be sum of the lift or buoyant forces from each individual balloon used.

A research submarine can remain at a certain depth of water, and without naturally rising upward into a less depth of water, or naturally moving downward to a deeper depth of water. If need be, it can remain at a depth by placing enough water into its (water) ballast (ie., additional weight used for some stability and control) tanks or pipes to where the average density of the submarine structure has the density of the surrounding water - about 1g/cc = 1kg/1L. In other words, the submarine will have the same weight of the water it displaced. If it weighed more, it would sink, and if it weighed less, it will rise upward. To make the submarine rise up to the surface, the water in the ballast tanks is simply replaced with low density and weight air, and via a pump. Likewise, to submerge to a certain depth, a certain amount of water is let into the ballast tanks so as to make the submarine heavier and denser per unit of volume, and therefore reduce the effective buoyant force upon it.

A hot-air balloon may use some ballast weight such as sandbags (bags filled with sand or dirt) to help set a maximum altitude, or to possibly give it some extra altitude if needed by discarding some of it. Note that a balloon will not keep rising upward like a bubble in water will. This is due to the fact that the density and pressure of the air is getting much lower at higher altitudes, and the buoyancy force, here equivalent to the weight of the displaced air, will not be enough to move the balloon and its corresponding weight further upwards unless it can displace more air by increasing its volume, and which is generally unlikely. Due to this, high altitude balloons will require a larger volume balloon, and where the pressure of the balloons internal gas will cause it to expand and displace more ("thin", low density) air.

The amount of the upward buoyant force or "effective lift" is due to the amount of displaced water and-or the difference in water pressure on the upper and lower surfaces of the object. Due to this, even at great depths and pressures, the buoyant force on an object remains the same since the difference in pressure upon its surfaces is still the same as that of when it is in water not as deep, and where the water pressure is lower. There is an article for calculating the amount of the buoyant force upon an object in water in the Extras And Late Entries section of this book.

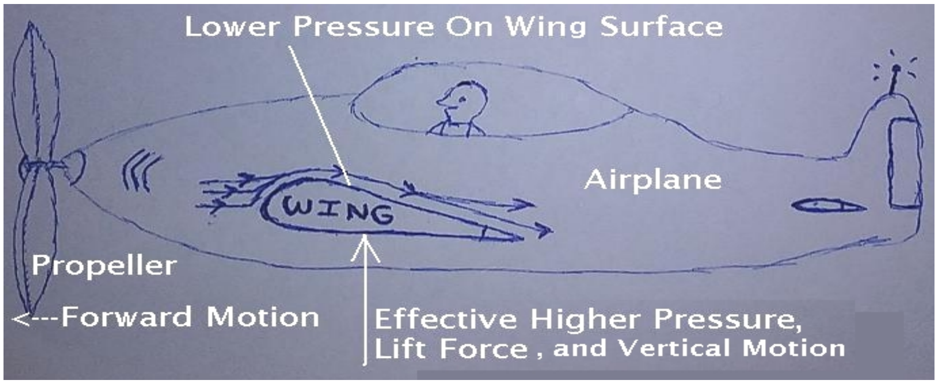
By the second half of the 1800's, experiments of machine powered balloons ("airships") were beginning to be made by several people, and first using steam powered (paddle wheel, shaped) propeller (for speed and-or various directions) in 1852, by **Jules Henri Giffard** in Paris city, France. An electric motor was first used in 1883 by **Albert and Gaston Tissandier**. In 1884, in Paris city of France, **Charles Renard** flew the first practical and controllable, electrically powered airship. **Alberto Santos Dumont** (1873-1932), from Brazil, made the first practical balloon with a gas powered engine and tested it in Paris city of France in 1898, and this balloon was propelled forward by using a spinning propeller which effectively pulled this airship forward. He was also an early and significant pioneer in airplane flight soon after the success of the **Wright brothers** first combustion ("gasoline", "petrol") engine powered, controllable (ie., steerable) and sustained airplane flights. Dumont is generally credited with making the first plane with wheels (rather than curved wooded skids or ski's) to assist with take-off and landing of the plane. The Wright brothers initial flights used a weighted catapult mechanism so as to give the airplane some initial kinetic energy and-or power, particularly some initial speed, and wheels would of been useless in the soft sands at Kittyhawk located near the Atlantic ocean shore in the state of North Carolina. Wheels offer low friction, and therefore low energy losses, and practicality on a hard surface such as hard dirt and-or a paved runway of hard material such as asphalt and-or cement.

In 1848, **John Stringfellow** (1799-1883), from England, created a steam powered, unmanned airplane, but it was relatively crude, under-powered and very limited in the amount of time available for sustained manned flights. It essentially made short "hops" or distances rather than have a sustained flight, but it did indicate and helped inspire the possibilities of future advancements and powered manned flights. Stringfellow was inspired by the earlier, in 1842, "aerial steam carriage" idea or concept created by **William Henson**, also from England. Henson and Stringfellow received a patent for a steam powered airplane design. Both dreamed of a future with transportation of passengers using airplanes. It is of note that balloon knowledge and flight was already in existence since the 1780's, and surely the concept of a large kite (ie., a glider) carrying a person began to be considered, and that an engine of some sort for power would be helpful. Their airplane was under-powered and could not sustain (upward, vertical) lift due it to being under-powered, low speed and not having a proper and adequate wing design or curvature so as to more easily create a lift force for sustained flight. Still, it was a promising starting point for other inventors to continue their research on machine powered flight. Between 1848 and 1932, there were many attempts at a steam powered airplane for this was the generally the era before a lite-weight and powerful combustion engine could be manufactured. By 1933, about 30 years after the Wright brothers first practical flights, the **Besler brothers**, from America, were the first to create a steam powered airplane capable of sustained flight. Their airplane consisted of a modified airplane with its combustion engine replaced with a steam engine.

Samuel Langley (1834-1906), an American scientist, demonstrated the first (steam) powered, heavier than air, unmanned flying machine (airplane) in 1896, but it could only travel relatively short distances (or "hops") of a few thousand feet due to a lack of sustained power of the design of the on-board, lite-weight steam engine. Nonetheless, it was a great technological achievement for the time, and it pointed directly to a high possibility of future manned flights, and must have been some inspiration to the Wright brothers and their airplane soon to come a few years later. In 1903, he created a manned, piloted version with a combustion engine of high power for those times, but the flights were unsuccessful - probably from not having a long enough runway to develop a sufficient initial airspeed and the resulting lift force. His airplanes also had limited steering, landing and reuse capabilities. Previously, in 1891, Langley invented the **bolometer.** A bolometer is very sensitive light or energy meter, and is made from a sensor [usually a seimi-conductor] that is sensitive to various RF and thermal radiation, and of which will increase its temperature and change its resistance. This can be sensed when the sensor is part of an electronic "bridge curcuit". This meter could measure very small or minute changes in temperature, infrared (ie.. "heat" energy) light (electromagnetic energy) and other forms of (energy) radiation (ie., transmission). Changes of the input light, radiation level or intensity can likewise also be sensed and measured. The word "bole" is from an ancient Greek word for a ray of light, energy and-or a thrown object. Langley also established an accurate **(railway) time standard** for the United States eastern railroads, cities, etc., and this was based on the position of stars as Earth rotates on its axis per day. The exact time was communicated via using the (fast, electric) telegraph system. Some places in Europe were already doing this at this time, such as in telesscope observatories. Many locations and facilities are named in honor of Langley, such as the NASA Langley Research Center, and Langley Air Force Base.

The inevitable march toward manned flight finally culminated on December 17, 1903 at the windy and relatively soft sandy hills near Kitty Hawk city in North Carolina state of the United States of America (USA). **Orville and Wilbur Wright,** commonly identified today as the "**Wright Brothers**" from Dayton city in the state of Ohio, brought to, and flown the first-ever manned (as the pilot and passenger), practical, self-contained (on-board) powered flying machine or "**airplane**" capable of sustained flight using the power from the on-board engine. They also experimented, designed and built this powered aircraft and since they were already crafty with fixing, modifying and improving bicycles before this time. The bicycle at these times was already becoming very similar to the modern designs and practicality. In geometry, a plane is essentially defined as an endless (ie., infinite), thin and flat (ie., no thickness) surface. This airplane machine essentially created and relied (like a balloon and glider does) upon a difference in air pressure, but now applied to the "wing" surface as the airplane moves forward, so as to create a net pressure difference and resulting net force beneath the wings so as to lift the plane and pilot (driver, controller) upward (vertically), albeit, even if just at a low incline angle above and with respect to the horizontal or level flight line considered as 0°.

To make the Wright Brothers flight more possible, a lightweight (here, mostly aluminum metal) liquid fuel ("gas") combustion engine was built by their machine shop employee **Charlie Taylor**, and was necessary to give this airplane power for forward movement so as to essentially have sufficient air flow of higher speed and lower air pressure along the upper surface area of the wings. It needs to be noted that the air effectively and relatively travels faster over the upper surface of the wing due to the entire airplane propelling itself and going at a fast velocity. A wing that is tilted or angled slightly upward from the forward direction of motion will also have some increased air pressure on the bottom surface of the wing, and it helps to improves lift, but this then also begins to increase drag and therefore reduce efficiency and a compromise angle for that plane needs to be found for the best fuel and efficiency. This wing tilt or angle with respect to a horizontal line at its lowest point on it is called the "**angle of attack**" [or (wing) "attack angle", into the forward motion, into the wind]. The net or resulting difference in air pressure or force upon the upper and lower surfaces of the wings is naturally called a (vertical) **lifting ("lift") force** (applied to the bottom surface of the wing) so as the entire plane can be forced or lifted vertically (upward in height) and-or perpendicular to the direction of flight. After their airplane flew or worked, the Wrights constantly improved their airplane design and made it more controllable (steerable), powerful, and efficient. The Wright brothers sister, Katherine, helped her brothers promote aviation and became a worker at the Wright Company which made airplanes at Dayton, Ohio. In 1909, Wilbur Wright, as the pilot, dazzled the spectators as he flew his airplane around the New York city area and Statue Of Liberty. Wilbur, born in 1867, died in 1912 from illness, and Orville, born in 1871, continued onward with things about airplanes and the business of aviation. It is interesting to note that the propeller and forward motion of the plane will effectively supply a current or flow of air across the wings so as to create a lift force due to the air having a higher velocity and lower pressure on the upper surface of the curved wing. The propeller is essentially collecting some air in front of it and pushing it back towards the wings at a high velocity. This will also cause an equal and opposite force in the opposite direction, and which will then cause the plane to move forward with the same force of the wind it is pushing back towards plane and its wings. [FIG 246]



A noted good attempt, similar to **Samuel Langley's** attempt of creating and flying an airplane at about the same time frame as the Wright brothers, was from **Richard Pearce** of New Zealand. With some more assistance and resources, he could of made a plane that was practical enough to be publicly and repeatedly demonstrated. Some of his knowledge and ideas were important and eventually became helpful to other airplane makers, but his claims were generally not publicly documented by him at the time. Even the Wright brothers did many of their test flights in secluded locations so as to help keep them private from potential competitors. One concept Pearce used tor his crude and under-powered airplane was that of using the previously known concept of ailerons used for balloon airships. Ailerons ("flaps") are small movable wing extensions or segments of the wing itself, so as to help control the direction and-or stability of a flying vehicle. His flying machine and concepts are considered as a very good attempt of powered flight, but his airplane didn't have a sustained flight capability since it could only be flown in "hops" (short distances). He later acknowledge the Wright brothers first practical and independent flying achievements which include a special curved wing shape to effectively create a lifting force which is due to the difference in air pressure (a force) upon the upper and lower wing surfaces, and with the higher force (ie. usually, normal air pressure) being applied to the bottom of the wing and forcing the wing and plane vertically.

In 1916, the first airplane manufacturing company was created by the Wright brothers and was called the Wright Company. It briefly joined together with their rival, the Glen L. Martin Company, so as to improve new airplanes. Later, the Wright Company became known as Wright Aeronautical and focused on the manufacture of more powerful airplane engines, and specifically, improved engines based on the original 1901, high power, **Manly-Balzer radial engine** created for Langley's failed experimental aircraft just before the Wright brothers success. This company later joined with the (Glen) Curtiss airplane company and became the Curtiss-Wright corporation. In 1916, the Boeing company started making seaplanes that could take-off and land on water, and this helped to overcome the problem of the limited number of airplane runways (long "take off" or launch surfaces where enough speed and lift is first developed) available at the time, and especially near more remote destinations in need of supplies and where no runway could possibly be built. Boeing currently builds many of the large passenger airplanes with jet engines such as the 737 airliner. The word of "jet" is a short form of several old French words, mainly the verb word of "jettison" which means to toss out, or to force out and away (ie., "thrust"). "To jet about" is a phrase that means to move quickly. A jet is also a word modernly used for an airplane that has a jet engine. A liquid under high pressure will jet outward from an opening. As of the year 2022, the Curtiss-Wright company is still in operation and making various technology products. After the business relationship with the Wright Company, the Glen L. Martin Company continued onward and would manufacture both airplanes and rockets, and later, this company joined with the Lockheed company and became known as Lockheed-Martin.

The first human to walk on the Moon was **Neil Armstrong**. He was born in 1930 in Ohio state (where the Wright brothers are from) and was already flying an airplane in 1946 before he had a car (automobile) license. Orville Wright lived long enough, up to Jan 30, 1948, to know of jet powered airplanes, and the increasing dreams and possibilities of advanced airplanes and rocketry. Because Neil was good at flying planes such as a high speed, high altitude rocket plane to the edge of space, and of which requires a pressurized space-suit, he was chosen to be an astronaut along with several other airplane pilots trained for space flight. Neil took some small samples of the Wright brothers first airplane to the Moon and back in honor of the Wright brothers and their achievement for all of mankind. Neil mentioned that he never met Orville in person and that the Moon landing was made possible due to all the teamwork and cooperation of many people. Neil mentioned to the effect that he was just in the next available job position, chance and luck, location and time so as to make the first human footstep on the Moon, and of which represented all of mankind landing on the Moon with him.

Lift or lifting force is due to a difference of force due to the difference of air pressure upon an object such as an airplane wing, and can be easily demonstrated by blowing fast moving air directly above and past a flat strip of paper about 1 inch wide and several inches long on top of your hand, and held by thumb and index finger. The paper and its surface where the moving air is applied to should move in a perpendicular (ie., 90 degrees) direction to that of the moving air, and in the direction away or opposite from the opposite side of that paper where the air pressure is (naturally and-or commonly: 14.7psi) slightly greater than that of the faster moving air on the top of the paper. On the top of the paper the air pressure will become less than 14.7psi. This net or total difference in air pressure (force/area) effectively places an upward net force on the bottom wing surface. For example if the bottom air pressure on the paper is 14.7ps, and the top air pressure is 10psi due to the fast moving air over it, there is a net air pressure or force of (14.7psi - 10psi) = 4.7psi on the bottom side of the paper. This force which causes the plane to be lifted vertically (higher) in the air is naturally called a "lift force", or simply called "lift". The faster the plane moves horizontally in the forward direction, the faster the air moves over its surface and the lower the air pressure on it, and the more lift force created since the net difference in force upon the wing (top and bottom) surfaces is now larger. Planes with a low speed will therefore take longer to move higher and to be at a certain altitude (height). When the lift or upward force created can no longer be greater than the (downward) weight (a force) of the plane, such as when the plane has a low speed, and-or the air density is low due to a high altitude, the plane cannot go any higher. This happens when the airplane is at an altitude where the air is "thinner" or less dense, and-or due to a low speed of the airplane not creating enough lift force. In this condition, at level or horizontal flight, any instantaneous gain in altitude due to a small amount of lift, is then lost by the equivalent instantaneous drop in altitude due to gravity. To maintain level flight, the upward lift force created at and upon the wings must equal the downward force of gravity, and which is the weight of the plane and anyone or anything else on-board.

The net difference in air pressure, and which creates the lift force, upon the upper and lower wing surfaces is generally not much in value, say only 0.1lb psi, but if the wing surface is large enough, the total lift force will become larger than the weight of the plane and allow it to move vertically upward. As a reminder: **Pressure = Force / Area and F = P A**

Effective lift force = (total lift force on the wings) - (total weight of the plane, passengers, etc.) : 1 ft^2 = 144 in^2
Effective lift force = (pressure difference in psi) (area of the wings in in^2) - (weight of the plane and passengers)

Ex. Effective lift force = (0.1 lb psi)(3ft x 20ft) - (500 lbs)
Effective lift force = (0.1 lb / in^2) (36 in x 240 in) - (500 lbs) = (0.1 lb / in^2) (8640 in^2) - (500 lbs) =
Effective lift force = 864 lbs - 500 lbs = 364 lbs

As indicated previously, any object, whether a lite-weight balloon or a fast and heavy plane will require some amount of energy to maintain its altitude because Earth's gravity and force is always acting upon it and causing it to accelerate and move towards the Earth at: a = g = 9.81m/s^2, hence the object will go lower in altitude, and even if maintaining a constant horizontal velocity. Heavier objects with more mass and weight will require more energy to both arrive to and maintain their altitude or (horizontal) level flight. To overcome the effect of gravity and maintain level flight, the lift force must provide an upward acceleration of the object by the same value as that of gravity, hence 9.81 m/s^2. For a balloon, the buoyant force of the atmospheric pressure will provide the lift force on the (average and low density) "lighter than air" balloon. Heavier objects with more weight will require a larger amount of lift force so as to obtain an upward acceleration of 9.81 m/s^2. Planes with longer and-or larger wings, and-or being faster will generate a higher amount of lift force upon its wing surfaces. If this lift force is constantly applied as the plane drops by only a very small amount of height or altitude distance per fraction of a second, the corresponding amount of energy required to move the mass of the plane this short distance upward or vertically will be relatively small: Work = Energy = (Force) (distance). As a plane takes off from the ground, much energy is initially needed and-or used to lift it to that height or altitude of flight. It must be remembered that the drag force upon a plane also slows the airplane's velocity and is a waste of fuel energy, and this must also then be replaced by using some fuel and its energy so as to maintain velocity and-or altitude. In general, the upward lift force must be equal to the weight of the object or plane so as to maintain level flight or the same altitude. Relatively slow moving solar or bicycle powered, and-or other experimental planes, usually have a very long and lite-weight wingspan so as obtain the necessary lift force.

As a object such as a ball is thrown horizontally, the force of gravity is always acting upon it, even when it is rising upward, and pulling it vertically downward. Its velocity (v = d / t) or rate of the change in distance with respect to the change in time will then not be a constant value. As the ball rises vertically, it will be constantly decreasing in the amount of its vertical velocity due to gravity. When the ball has reached its highest altitude or height and appears to briefly stop in motion, the force of gravity will still keep acting upon the ball and accelerate it to have a higher velocity, and it will then move quicker and quicker towards the ground (ie., in the vertical direction) since the force of gravity is constantly acting upon it. If the ball was also thrown with some initial horizontal velocity, its flight path or trajectory will be that of like a curved arc, and is in fact a **parabolic shape or arc** of which a parabolic or quadratic equation (similar to: height = y = - at^2 = - gt^2) can be used to mathematically represent and study an objects motion through a gravitational field. In a vacuum with no air resistance to reduce the kinetic energy of the balls motion, the maximum height of the ball will occur halfway between its starting point and ending point on a horizontal or .evel surface.

Seemingly "weightless" spacecraft or satellites orbiting the Earth are sill in the nearby presence or gravitational field of Earths gravity force which causes a constant gravitational acceleration (of the amount of: a=g = 9.81 m/s^2 at the surface and-or low (radius distance) orbit heights above of the Earth and-or from its center point). The (apparent) "weightless effect" is due to that the observer and other objects are all falling at the same speed, and it then appears as everything is not moving faster or slower than the observer or vice-versa, hence as if those items are motionless and-or weightless.

Another fairly common example of where movement can be commonly observed due to pressure differences is by aiming the wind from a fan across the surface of a movable door. This will create a low pressure region on that surface and will cause that door to be pushed or moved in a direction that is perpendicular to its surface and toward the lower pressure wind direction on its surface.

Aviation Discoveries Before The Wright Brothers First Manned Airplane Flight

The glider (unpowered, no engine, manned (piloted) airplanes, somewhat like a kite) makers, and later, the Wright brothers naturally based some of their work on the previous knowledge of others. Scientists and-or inventors have no need or desire to reinvent the wheel, so to speak, but rather try to understand it more and-or to improve it. The Wright brothers also used their own imagination and observational discoveries noticed in nature, such as with the flight ability of bird wings and their movement, various "propeller" seeds of trees, flying insects, various types of fish (ie. their tail-fin being the "rudder"), boat sails, windmills, hot-air balloons, ("box") kites (ie,, with wing surfaces), and (non-machine powered) gliders (essentially large kites, capable of a lifting a pilot and sustaining a reasonably long horizontal movement and direction, particularly in the direction of any oncoming wind). Before the first self-powered airplane flight, the Wright brothers made and experimented with manned gliders beginning in 1900 at Kitty Hawk in North Carolina state, and it would only be a matter of time and experiments such as with new wing designs and their increased lift efficiency found with the aid of a small wind tunnel they had built. A **wind-tunnel** is structure that consists of a powered fan blowing (ie., moving) a fast and controllable, steady current of air (ie., wind) at a known speed and into and through a long rectangular box and expelling it over the wing models and-or airplane shapes and surfaces. The amount of lift can then be measured. Thin streams of colored smoke over the wing surfaces can help visualize the air movement and-or turbulence with that specific wing shape and-or angles it is set to. The wind-tunnel was first created in 1871 in England by **Francis Wenham** and **John Browning**. The Wright brothers used patience, persistence, calculations and technology (such as the engine itself) until that first successful airplane could be made and then flown. Unlike a balloon at the mercy of the near random windspeed and direction of the wind, the airplane allows people to fly in any direction needed, and to any practical landing location (ie., a designated flat, "runway" surface). The Wright brothers first airplane had a similar design to the 1896 biplane (bi-wing or 2 main wing surfaces. so as to slightly (about 20%) increase the total lift force) glider made by **Octave Chanute and Augustus Herring** from the United States of America, however the Wrights used a wing shape with its upper surfaces deliberately made slightly curved (sometimes called "camber") since it was first noticed by **George Cayley** who, many years previously, made the first scientific study of how things could possibly fly, and of a specially curved wing surface shape called an "**airfoil**" (see FIG 246) that improved lift. Cayley is also further discussed below in this book. The Wright brothers eventually perfected their glider to the point that it was practically a controllable airplane without an engine to sustain longer flight. Chanute and his knowledge also provided some help to the Wright Brothers before their first flight. Many modern **windmill** wings or blades for the generation of electricity also use an airfoil shape so as to help increase the efficiency of the energy collection and conversion of the available wind power to available electric power. In brief and without consideration of the airfoil shape, for a windmill ("wind generator") blade to collect the wind and its kinetic energy, it cannot be flat or at a 0° angle because it would collect no wind energy, and it also cannot be perpendicular or 90° to the wind for its force would be simply pushing the blades in the direction of the wind, and rather than rotating them. A compromise blade angle must be found by experiment, and obviously a 45° angle is a reasonable place to start, but it will usually be lower than that. The best angle will produce the most output power for a given windspeed. If the blade is at a too low of an angle, then the input energy collected will be low, and some of its energy will be lost when it is deflected or diverted at an angle. If the blade is at too much of an angle, then the wind force applied will not rotate the blade as much, and the drag (air resistance, friction) caused by the side of the blade colliding with the air as it rotates at a high number or revolutions per time will also reduce the speed of the blade and the power output. The **Betz Limit** is a theoretical, **59.3% =~ 60%** maximum limit to the amount of the input wind energy collection possible by all the windmill blades of a wind turbine. Due to other losses in the wind-energy system such as mechanical friction and electrical resistance, the final output power of the wind turbine, wind energy collection system will usually be less than the Betz's limit, and at about **30% on average**.

After the first successful airplane flight, the knowledge of flying, airplanes and their manufacture expanded quickly and greatly all over the world, and with many people still being naturally skeptical and unconvinced of the manned-flight news until they have personally seen photos of an airplane in flight, and-or actually seen an airplane flying nearby. By the end of just the first year of airplane flight, the Wright brothers were flying for several miles of distance in their airplane at a relatively low altitude since their plane had a very low amount of lift force available. By as early as 1910, some planes such as the Wright Brothers popular "Model-B" plane were already capable of flying up to and over a mile (5280 feet) high (altitude) in the sky. The study of flight, airplane design and efficiency has continued ever since their first flight in 1903.

The first practical, flown manned gliders were developed by the **(Otto and Gustav) Lilienthal brothers** from Germany, in about 1891, and which was only a few years prior to the Wright brothers first airplane flight. Nonetheless, gliders were very important for (mechanical powered) airplane development, and are a fun way to fly. In modern times, some highly experienced skydivers at a high mountain location, or jumping out of an high airplane, put on a suit called a "wing-suit" that has a wing-like shape and some gliding capability which gives them an incredible fast, bird-like flying sensation and view of which is often recorded for others to experience later such as via a video. At some point with sufficient altitude left in their journey, these fast moving skydiving pilots will deploying a parachute so as to slow them down and land safely.

About 100 years prior to the Lilienthal gliders, **George Cayley** (1773-1857), from England, in about the year 1799, made many of the fundamental discoveries of ((effectively) "lighter than air") airplane flight by using various wing designs including the necessary "airfoil" shape of the wing(s) that the Wright brothers would also use later for their gliders and first airplane. He recognized lift (upward force) upon that airfoil or airplane wing shape. Cayley also recognized thrust (the force that provides forward propulsion or motion) and drag (the effective force in the opposite direction to the forward thrust direction, and due to air collision and air movement resistance on a surface, particularly the front forward edge of the wing and plane shape). A larger surface area in the direction of flight will create more air particle collisions and this increases the net air resistance to fly through, and reduces the maximum possible (such as if there was no drag) forward thrust or force, and therefore reducing the airplane speed, wasting time and energy (available or stored in the fuel source). Cayley probably got some incentive to do his aeronautical studies after the first manned balloon flights 15 years earlier. Even if the air is not moving, it is as if it was and has kinetic energy and collision force relative to the motion of the airplane. This collision force is called a drag force or sometimes as a reduction or an opposite (direction) force.

For the airplane to have any upward vertical movement, the total effective lift force generated and available must be greater than the weight (a downward force due to gravity force and its acceleration upon a mass) of the plane. If the lift force equals the weight of the plane, the plane can maintain and have a constant altitude (height) in the sky. Reducing the lift forces, such as when the plane is slowed down, will make the plane descend such as to make a landing. A plane can change its horizontal direction of travel by using a rudder (a "flap" [flying, planar], surface) in a similar way that a rudder of a boat is used to change its forward direction. Cayley also made some simple or basic constructed experimental gliders to help verify his fundamental discoveries and ideas. Cayley discovered that the upward lift force was directly related to the square of the (relative) speed (ie. velocity^2) of the wind and-or the surface (ie. the plane wing) it flew over.

Due to his significant contributions, Cayley is considered as the founder of modern aerodynamics or aeronautical engineering. Before Cayley's study and discoveries of flight and lift, **Isaac Newton** discovered that the drag (ie., the effective resistance force opposed to forward movement) force upon an object, such as a rectangular plate placed in a stream of flowing water, was directly related to its area size and the square of the speed of the fluid, air or object. Upward "lift" or force on a flat plate or plane surface is also related to the square of the speed (v) of the fluid or air (sometimes considered as a low density fluid) over the wing surface. A pointed shape has less drag force upon it since the force is not then perpendicular to its surface, but rather at an angle to it, and is then also spread out over a larger area. P = F / A. This area still needs to be as thin and small as possible and practical, and the ideal ("bow", "nose" or "forward") shape of the moving structure in air or water is a compromise shape of which can be described as that of a combination of a flat plane, a cone and a sphere, and it is actually an "airfoil" shape that is used for airplanes. Boats, automobiles and objects are also designed to have a low drag resistance when needed. If you hold a flat object in moving water or a air, you will see that when you hold it perpendicular to the wind or water, that the (resistance, drag, collision) force is the greatest amount possible, and that it is best used for things like water-wheels to power grain milling and other machinery, and for water turbines (somewhat like a large geared wheel) that can collect and transfer as much mechanical energy as possible so as to make electrical energy (electricity) from a (rotating, coil and magnet, electric) generator. The amount of drag to expect for a given shape has been experimented with and the resistance force measured due to the air or water. From this force value, another value is calculated so as to then have a comparative, relative rating called a "**drag coefficient** (c)" for each wing shape. As you can imagine, a flat, perpendicular surface will have a high (c) rating of about 1.0, and a thin planar, level surface has a rating of about 0. A semi-sphere shape has a rating of about 0.45, and an airfoil has a rating of about 0.05 which is very low. Formally, the drag force (Fd) is mathematically defined and-or calculated as:
Fd = c p A v^2 / 2 , where: c = drag coefficient , p = fluid, or "medium" (substance) density , A = cross sectional area of resistance, v = flow or relative velocity with respect to the objects motion. Doubling (2) the velocity of the object or fluid, and-or the effective relative velocity change will increase the drag force by 4.

The faster the plane goes, the faster the air across the upper surface of the wings and the air pressure upon it will decrease, and the pressure difference on the lower and upper wing surfaces will be more, and of which results in a greater lift force. A common formula used for understanding and calculating the (generated or created) lift is:

$$\textbf{Lift Force} = \frac{(\text{coefficient of lift})\,(\text{air density})\,(\text{relative velocity}^2)\,(\text{upper surface area of wings})}{2} = \frac{(Cl)\,(p)\,(v^2)\,(As)}{2}$$

The coefficient of lift (Cl) number is mostly determined by the **attack angle** (ie., upward tilt with respect to the horizontal line) of the wings, and the wing curve and-or shape. The generated or created lift force produced must be greater than the total weight of the plane system so as to provide a net or resulting vertical lifting force applied to the plane. The above equation for lift force does not include a pressure or pressure difference variable upon the wing surfaces for the calculation effectively includes it via the other variables used. Depending on speed, an attack angle up to about 15° at low speed might be employed to increase the air pressure on the lower wing surface, otherwise if greater, it increases the drag forces. As indicated previously:

Effective lift force = (Total lift force created) - (Total weight of the airplane, passengers and any other loads)

The lift force due to the attack angle or angle of attack is sometimes called the **compression lift** force, and where its angle naturally causes the plane to go upwards and toward a lower pressure region, much like a propeller (ex. windmill, plane) does after wind strikes it and slightly compresses it and imparts some kinetic energy into the propeller or blade to then move and-or rotate. Modern propellers or the blades on large windmill, wind electric generators may be designed to provide a lift force so as to help reduce the drag (ie., air friction) force as it rotates.

A **glider** is a plane without any engine, and it is launched from a high location such as on a steep mountain, or it is towed by another (engine powered) plane to a high altitude and released. A lightweight glider (or "glider plane") can stay aloft in the air for a reasonable long time (ex. 20 min. typical, up to about 1 hour if the conditions are more favorable) as it slowly descends to the ground and-or is given some new lift force and a increase in altitude by some wind over the wings, or "thermals" that are updrafts or currents of warm air rising upward from the ground region to under the plane body and wings. An nice idea is to have a small light weight rocket or jet engine that could assist a glider gain speed and-or altitude, and-or to help travel a further distance to the runway landing zone if needed.

An **ultra-light aircraft** is usually a relatively inexpensive airplane of cheaper (lite-weight) construction where the pilot is often not fully enclosed within an aircraft body ("shell", container, insulator) which would provide more comfort from the elements (wind, cold, rain, etc.). By law, there is a maximum weight limit to these types of planes which can often be flown without a license and-or much training. This weight limit could reduce construction, comfort and-or safety measures, and perhaps the maximum engine power and-or speed should rather help decide the weight limit.

A **paramotor** (a parachute type wing shape, and motor) is a backpack-like, portable, lite-weight combustion or electric engine powered flying paraglider which is essentially a parachute-glider that uses its cloth-like fabric wing, somewhat similar to a parachute, to enable the lift process. Paramotors are generally not very fast, but can travel reasonably far and at reasonably high altitudes. Paramotors require only a very short runway, perhaps less than 30ft, and can safely land in a clearing (absent of trees, brush, fences, wires, stones, water, etc) at the on the top of a mountain after a few minutes of flight from the ground level. A paramotor is generally much less expensive than a regular airplane and-or ultralight aircraft. Having an airplane pilot license is sometimes not required, but pre-flight training by a certified paramotor flight instructor is recommended and-or required by local (ie., location of flight) laws. Due to that there may be other planes in the sky, there are further laws, rules and-or recommendations to follow, such as what altitudes to fly at, and-or not to fly at, etc..

Today, experiencing a form of flight and-or being a pilot is relatively cheap with modern remote controlled (RC) small electric engine(s) helicopters (non-passenger "**drones**") having an on-board, live radio-wave ("wireless") video for steering, location, and-or "filming" (image recording) capability via a small and lightweight camera. This is a virtual ("nearly", "as if", virtually, simulated) flying experience. Being small helicopters, they require no runway and can be launched almost anyplace and at anytime after charging up its power battery. These pilots and planes often need to observe any local laws created in regard to their use, and to fly fair and safe. I would personally recommend blade protectors or guards ("standoffs") to help prevent device damage and-or injuries to others. These devices are also used for many practical and valuable applications such as for search and rescue, movie making, aerial photography, data acquisition and map making.

Several human powered lite-weight airplanes have been designed and successfully flown at low altitude (less than 100 feet high due to having only a low "air speed" and-or forward movement, hence a low lift force). These would of made the Wright brothers proud since they use human leg power, much like riding a bicycle, and so as to turn a long propeller. The drawback is that a very long wing surface (about 112 feet) is then needed to create enough lift force. The longest verified flight as of the year 2020 was about 71.5 miles of distance in about 4 hours of time. The average speed of this flight was therefore: From distance = (speed)(time). speed = (distance / time) ~= 71.5 miles / 4 hours ~= 17.<u>9</u> ~= 18 miles per hour. The pilot and human leg force power for the airplane flight and record was **Kanellos Kanellopoulos** from Greece in 1988. He flew the plane from and between two islands: Crete to Santorini Island of Greece in the Mediterranean ocean, and for a continuous distance of **118km =~ 73 miles**. He was previously a cyclist in the 1984 Summer Olympics sports events, and so he had a good physical condition, strength and duration to achieve this. With comparative reasoning, a person of typical or average physical health could probalby fly this same plane a few miles.

Several solar-electric powered and piloted airplanes have also been designed and successfully flown and they require wings longer than most small commuter airplanes that have a combustion engine. These planes are therefore rare, costly, and impractical, but it is still good to know that it has been tried, possible and accomplished.

Before extreme high altitude airplanes, jets or "rocket-planes" were made, the only way to test peoples (mankind's, pilots) abilities in very high altitude flight near the edge of space was with high altitude balloon flights. The data from all these high altitude flights was, and still is, very useful for all the future manned high altitude rocket-planes and spaceflights. Amazingly, as early as 1800, the year Volta invented the "Voltaic pile" (ie., battery), balloon flights were already over 4 miles high and near the zone of where supplemental oxygen is typically needed for most trained high mountain climbers.

In 1932, in an internally air pressurized craft ("gondola", "capsule"), Swiss scientists **Auguste Piccard and Max Cosyns** were the first to reach 10 miles high (= 52800 ft). Piccard is also credited later to the creation of the **bathyscape** (strong container or vehicle for deep water, essentially a "**submarine**") which is used for exploring deep water, and with the inside occupants still at a normal air pressure for normal breathing. Piccard has therefore made many additions to the knowledge for survival in both low and high pressure environments. His son **Jacques Piccard**, and **Don Walsh**, from United States of America are among the few explorers to ever reach the deepest part of the ocean which is in the **Mariana Trench** region known as 'Challenger Deep' (named after a ship that discovered this very deep area) in 1960. This deep location is in the west Pacific Ocean near the island of Guam. While a typical space capsule or container only has to endure a relatively low (air) pressure, say 14.7psi, from the inside and upon its inner surface, the Trieste bathyscaphe ("deep-ship)", that is much like a specialized submarine vehicle, had to endure the extreme danger of thousands of pounds per square inch from the outside water pressure at the depth of about 35800 feet (6.78 miles deep). This depth is greater than the height of Mt. Everest (29032 ft ~= 5.5 mi high). The pilots were safely located in a thick steel sphere located on a submarine vehicle called the Triest which gets its name from an Italian city where part of the vehicle was made. A sphere shape can withstand much more pressure than other shapes because it helps distributes any outside forces over the entire surface and structure of that sphere. Its circular window was made from a transparent plastic material, and its thickness was cone shaped so as to give it high strength, and while held in and against the sphere wall as a support. Plastic has some ability to flex rather than crack or break like glass might when under high external (here, water) pressure. During the Apollo 11 mission in July 1969, Jacques Piccard and a group of other people studied the Gulf Stream current and ocean bottom in a current drifting submersible called the Ben Franklin while traveling along the eastern U.S.A. coast from Florida to Nova Scotia. Because of the extreme (inside and outside) pressure difference and danger at great depths, and the need for a strong (and heavy) vehicle, much fewer people have went to the deepest parts of the oceans than into outer space, and much of the oceans still need exploring, even if by using unmanned vehicles. The Gulf Stream current is a relatively fast moving current that typically moves at about 4 miles per hour =~ 6.44 km / hr .

By early June 1957, manned balloon flights were near the edge of space at a height of 100,000 feet (about 19 miles) as for Project ManHigh, and so as to know the feasibility of future manned space flights and the spacesuits that would be needed. The pilots of the two first and notable flights to about this altitude were **Joseph W. Kittinger**, and then **David Simons**. Their craft was like a miniature one-man modern space craft which was pressurized (to simulate Earth's normal 14.7psi atmospheric pressure at ground level) and contained: oxygen, heating, food and water, communication, flight instrumentation (data) displays, photographic systems, sensors, lighting and electrical systems. In 2012, **Felix Baumgartner** ascended to the highest manned balloon flight of 127851 feet (about 24 miles), and then he performed the highest parachute jump while wearing an insulated spacesuit having a "life-support" system such as oxygen to breath and some internal air pressure, hence much like a spacesuit for an astronaut.

A **parachute** (based on the words: "resist-falling") is much like just the top hemisphere or portion of a balloon made of strong cloth, and with the passenger and-or load suspended below it by strong fibers. Its large and curved surface provides much upward drag force against the downward pull of gravity as it descends in the air, and it will reduce the speed of the descent of the person or object to a relatively safe value.   The invention of the parachute is nearly intuitive and-or common sense based on the idea of an umbrella that prevents rain, wind and-or snow from landing upon an object or person, much like a small and portable roof, but it took till about the 1450 AD to be scientifically realized as being capable with a relatively heavy human passenger. **Leonardo DaVinci** is credited to a designing a more practical design in about 1485, and which was later improved and used by **Louis Lenormand**, from France, in 1783, and it was therefore, available for use for the hot-air balloons invented just a few years later in France, and later for the airplanes.

SPACE FLIGHT

Let us begin at some of the notable scientific concepts and discoveries for science, astronomy and space.

Robert Hooke (1635-1703), from England, theorized that the motions of the planets could be like that of a pendulum bob swinging or rotating about an imaginary center and force. This motion describes an ellipse (or possibly a circle - a special instance of an ellipse) shape, and that the same forces acting on the pendulum bob, object or weight must also be influencing the motion of the planets. One force was causing it to curve so as to stay in its circular motion and not fly away from its orbital motion. He later told **Isaac Newton** these things and it would later help Newton develop the value for Earth's gravitational acceleration (g) due to the constant force of gravity upon objects pulling them downward, and then later, to develop the force of gravitational attraction (Fg) of two masses.

In about 1660 Hooke made some observations and measurements and created an equation known as **Hooke's Law** that describes the amount of force needed to compress or stretch a spring, and of which temporarily deforms its atoms alignment, and it can store the input energy for later use as potential energy. For relatively short distances (x) of deformation and where it is linear, the amount of force (F) needed is linear in value and will be a multiple of the amount of distance moved. **F = kx** , and where k is a constant value, sometimes called the springs stiffness constant, that is determined by the spring's material (ie., what metal(s)), construction (ie., thickness, diameter, turns per inch) and then testing it with a known weight or force, and its units are force per unit distance: **k = F / 1x.** Note that even a flexible or bendable piece of metal or wood can also act as a spring. Note that stretching a spring too much will cause it to permanently deform and not function properly as expected, and the equation no longer applies unless that spring is somehow repaired. The distance moved by the spring is mathematically: **x = d = F / k**, and the more ridged or stiffer the spring, the higher its mechanical constant (k), and the less distance it will move when a given or certain amount of force is applied to it, and this spring may be described as being "stiff", and being harder to compress and-or to expand.

In 1665, Hooke was also the first to use a mid-power **microscope** (a device or instrument that magnifies the image seen of small objects) he designed and it had about 50 "power" or magnification and he could see very small creatures and large cells that are essentially invisible to the unaided human eye in terms of detail. He made up the word "**cell**" (ie., small room, area, container, chamber, body or region structure similar to closely spaced bee cells in a bee hive [dwelling, home, shelter]) as being the common or basic micro-structure of all organisms. The average diameter or length of a cell is **20um**. Shortly after this discovery, **Antoni Van Leeuwenhoek** (1632-1723), from Denmark, made high-power microscopes using a small melted sphere a few millimeters in diameter made of clear glass as the lens, and they had about 300 "power" or magnification, and he effectively created the field of micro-biology. He discovered bacteria, and that (microscopic) sperm (from male) and eggs (from female) are the basis of reproduction in many animals. The average length of a bacteria is **5 um**. Reproduction or growth by the division of plant cells into two identical cells was discovered in 1835 by **Hugo Von Mohl** (1805-1872), from Germany, while observing (plant-like) green algae. Before Leeuwenhoek, there was some (low-power, about 2 to 3 power) single lens, magnifier "glasses" or lenses since antiquity, perhaps inspired by the rounded shape of a drop of rain which can slightly magnify the object under it. Credit to inventing a modern type of microscope is generally and independently given to **Zacharias Janssen** and **Hans Lippershey** from the country of Netherlands (often called "Holland") during the years of about 1570 to 1590. Today, it is recommended that a microscope have a minimum of 400 power ("400 X") so as to identify and study cells effectively. To identify many bacteria species, it is recommended to use a 1000 power microscope, and in particular having a higher power objective lens, say of 40x, for greater initial detail rather than just a higher power eyepieces lens. Today's modern and high quality microscopes available are relatively inexpensive as compared to large and costly telescopes with the same optical power, and they are much more portable and useful to most others, such as to study blood samples for various problems or diseases that need medical attention. If you have a clear glass sphere and-or a clear glass "marble", you can use it as a crude microscope; perhaps 30 power, but it will have image distortion at the outer edges due to it being an out of focus area of the subject when it is not directly in line with the center or area of focus.

The first virus (a Latin and ancient word meaning "poisonous", hence dangerous, the spread of it is often noted as being "viral") as we know of today was noted in 1892 by the **Dmmitri Ivanovsky**, from Russia, while investigating what substance was causing a disease in some plants. **Martinus Beijerinck**, a biologist in Holland actually coined the word of "virus" in 1898. A virus is a complex (DNA and protein) molecule that can replicate and multiply within the infected host cell, and this will eventually damage that cell. To view a virus requires the high magnification of an electron microscope.

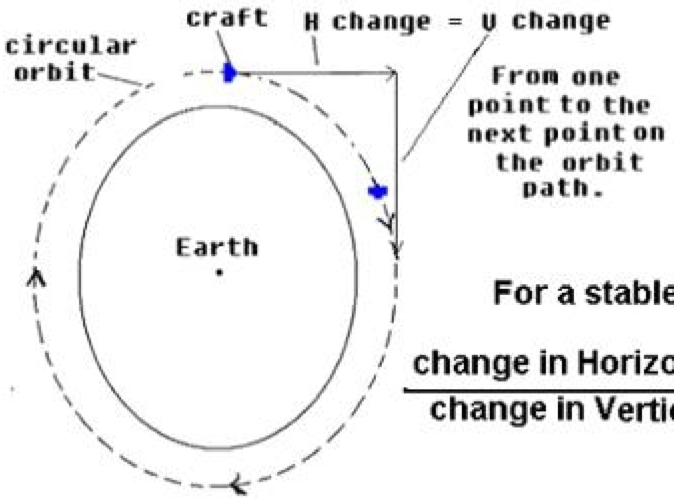
The world celebrated scientist, philosopher, and astronomer **Isaac Newton** was the first to conceive how masses such as the Moon can maintain a fixed orbit (travel around, go about) about another such as the Earth. He created a (logical) thought experiment, commonly called today as "Newton's Cannon" (ie., an imaginary cannon ball, fast projectile) in about the year 1687, and this hypothesis (something probable, and yet to be proven) states that if an object (such as a high speed cannon ball projectile) was made to go fast enough over the surface of the Earth, it would **orbit** (go completely around) the Earth. It would do this by effectively overcoming (but not actually cancel or stop) the constant downward gravitational (gravity) acceleration force of the Earth that causes it to decline in altitude, and then it will no longer orbit the Earth when it reaches the ground. At this orbit speed, it's vertical fall rate (height or distance per unit of time) towards the Earth would be equal to the downward (beyond and-or over the horizon) curvature, or decline rate (in height or distance per unit of time) of the Earth's surface that is below the craft or object.

For a simplified example of an orbit: If the craft was falling at 10 feet per second toward Earth, and the Earth curved downwards by 10 feet in that same amount of time, then the craft will effectively maintain a constant altitude and its orbit. For orbiting, the horizontal and vertical movements of the object or craft are constantly and instantaneously happening at the same time and rate, and the result is a smooth circular-like orbit or curved path of the craft or object about a central point such as the center of the Earth. For a simplified example to help comprehend all this, consider if the craft falls 1 foot due to gravity, and that the Earth's surface has curved downward by 1 foot during same amount of time, hence the orbit altitude is maintained a constant value.

The orbit mentioned above would be circular about the Earth's center point of gravity. If the craft or object does not have enough horizontal velocity for a specific altitude of orbit, the craft will not orbit the Earth indefinitely, but rater it will eventually be pulled downward in a normal or standard parabolic (ie., a quadratic or a squared variable equation, From: $g = 9.8 m/s^2$, $d=vt = (at)t = at^2 = gt^2$) motion by the force of gravity. If gravity was somehow stopped, an orbiting craft would no longer be pulled downward and be in its circular-like orbit, but it will continue in the direction that is tangent (ie., a tangent line, perpendicular, 90°) to that orbit and radius line to the center of the Earth.

To launch a spacecraft into orbit eventually requires a high, effective horizontal velocity so as to maintain orbit. Soon after a stationary rocket with the spacecraft (in and-or on it) is launched vertically, it will be steered to climb higher towards space at about a 45° angle with respect to the ground or horizon for much of the flight, and then later, to a lesser angle so as all of its remaining fuel energy is converted to mostly kinetic energy and velocity of the spacecraft in the horizontal direction (ie., parallel to Earth's surface), rather than the vertical or higher direction, and until the desired orbit altitude is reached.

The downward or vertical drop (downward distance) over (ie., below) the horizon (horizontal, level or straight line) of the Earth at its surface (ie., 4000 miles, the radius from the center of the Earth) is about 8 inches per mile. To orbit the Earth near its surface, a horizontally traveling object will need to likewise drop by the same value of 8 inches per mile. For higher altitudes of orbit, both the force of gravity and the curvature (ie., vertical drop per horizontal distance) of the Earth gets less from the center of Earth. At higher orbits, the gravitational acceleration caused by that amount of continually applied force of gravity is less. Here, in more distant or higher altitude orbit, the orbiting object will then fall (vertical drop distance) a lesser amount of distance per amount of time, perhaps just 1 inch per mile of travel in its orbit, and therefore, that object can and must actually go slower in the theoretical linear, horizontally direction so as to maintain its altitude of orbit. Below is an analysis of a circular orbit. [FIG 247]



For a stable orbit:

$$\frac{\text{change in Horizontal distance / time}}{\text{change in Vertical distance / time}} = 1 = \frac{dh}{dv}$$

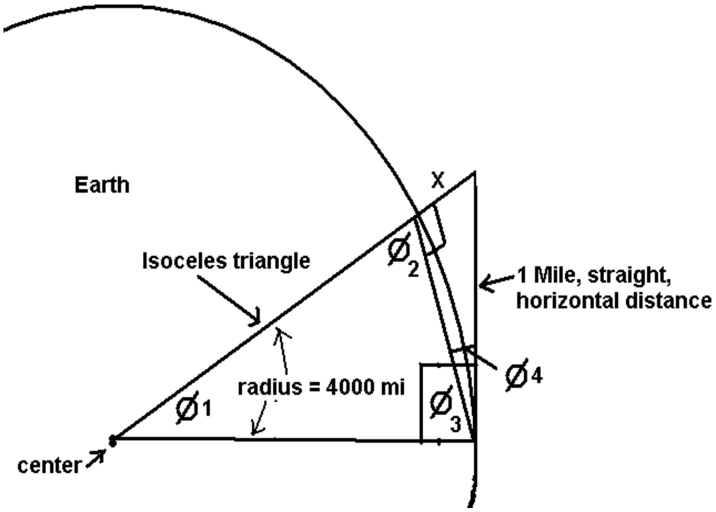
The amount of vertical drop per unit of time depends on the gravitational acceleration = (change in velocity / change in time) at that altitude.

From the above equation for a stable circular orbit: (change in horizontal distance) = (change in vertical distance)

At low Earth orbits, say less than 1000 miles, the gravitational acceleration due to the Earth is only slightly less than at ground level.

Another equation similar to that shown in the image above is:
(change in vertical distance / time) / (change or drop in the curvature of Earth directly below the horizontal / time) = 1

Here is a derivation of the 8 inches per mile drop in vertical height below a straight or level horizontal line located directly on the Earth's curved surface, and that the specific vertical drop from that line is due to the ideal (with no hills or valleys) amount of curvature (below Earth's level, straight horizon) of the Earth's surface at that radius (here, 4000mi from the center point of Earth). [FIG 248]



In the figure above which is not drawn to scale, but having exaggerated parts so as to view the concept better, (x) represents the distance of the drop below the straight horizon line. This vertical drop is due to the natural curve or curvature of the Earth away from and-or below the horizontal or tangent line. Though the Earth's surface appears to be flat on average to a person near the surface level, a true, level or horizontal distance of 1 mile away from any point, the Earth will have curved and-or dropped in elevation by 8 inches. In the above drawing, we will be solving for (x) which is equal to 8 inches.

The triangle construction shown on the left, within the Earth's circle, is an isosceles triangle with two equivalent sides equal to the radius (r) of the Earth, and two equivalent base angles of: ¢2 = ¢3. This inner isosceles triangle is also shown as part of a larger right-triangle construction where:

tan ¢1 = opp. / adj, = 1mi / 4000mi = 0.00025
¢1 = arctan (tan ¢1) = arctan (0.00025) = 0.014324°

Since the internal sum of angles of a triangle is 180°, for the isosceles triangle:

180° = 2 (¢2) + ¢1 , after solving for ¢2: ¢2 = ¢3 = 89.992838°

At the indicated larger right angle: 90° = ¢3 + ¢4 = 89.992838° + ¢4 , after solving for ¢4 , ¢4 = 0.007162°

At the smaller indicated right triangle: sin ¢4 = opp. / hyp. = x mi. / 1 mi mathematically:
x mi = sin ¢4 (1 mi.) = sin 0.007162 (1 mi) = 0.000125 mi

0.000125 mi = 7.92 inches ~= 8 inches = $0.66\underline{7}$ ft : only at earths surface, an altitude of 0ft, and a radius distance
= 20.32cm = 0.2032m of about 4000 miles from the center of Earth.

Extra: According to this 8 inch value, an observer looking along the horizon at ground level (and without any increase or decreases in the altitude of the object), the top of an object less than 8 inches in height at 1 mile away would not be visible. If the observer rose their eyes to 8 inches above the ground level, then that object would be fully visible.

A 12 inch = 1 ft. drop corresponds to 1.5 miles. At 10 miles distance, the drop from level is 6.67 feet.

If the spacecraft drops in altitude by (x) feet per second, and the Earth's curvature or surface directly below the spacecraft drops by (x) feet per second, it will be the same rate vertically, and without any difference in the rates, and the altitude of the orbiting object or spacecraft will effectively remain the same value.

The larger the distance or radius from the center, the greater the circumference, and for a given angle, the corresponding or subtending segment or arc of the circumference is longer but actually gets "flatter" (less of a curve) or more straight-like or more linear or line like. It could be said that the curvature at a farther radius value is less per degree, and therefore, there will be less effective drop below the horizontal line per mile at that radius or orbit distance, and the craft will not have to go as fast to effectively regain the altitude it lost due to gravity, and of which is actually less for farther radius orbits.

Extra: The **radius of the Earth** is about 4000 mi =~ 6537 km, and the **radius of the Moon** is about 1080mi =~ 1738 km. The ratio of the Moon's radius (Rm) to the Earth's radius (Re) is about: 4000mi / 1080mi =~ 3.7 Using the same calculation as above and using the Moons radius (Rm), the surface of the Moon will curve away, and horizontally below from a point or location at 2.4444 ft / mile =~ 29.3333 ft./mi. Hence nearly 2.5ft or about 30in per mile. The ratio of the surface declinations (D) per milie is about: 29.3333 in / 8in =~ 3.7 which is the same ratio as the radius values, but using (larger value / smaller value) or reverse ratio here, and this fact can also be used to simplify or estimate this value for say Mars. From this analysis, we find: Re / Rm = ~ 3.7 = Dm / De , and mathematically: Re De = Rm Dm

Due to that the force of gravity upon two objects or masses also depends on the distance between them, a higher altitude spacecraft will have less force of gravity constantly applied to it, and therefore, it will not accelerate as much downward or vertically, and the value of the gravitational acceleration at that orbit altitude will be less, such as shown here:

Note: Circular orbit distance or altitude from the center of the Earth = Radius of orbit = Ro =
Distance from the center to the Earth's surface (Ro) + The Altitude of orbit above the surface of the Earth (**Ae**)

Earth's radius , (Re) + **Ae** , (Altitude of orbit above the surface of Earth)
Ro = 4000mi + Ae : Ro = radius of orbit, from the center of orbit

Altitude (mi)	g (m/s^2)	Altitude (mi)	g (m/s^2)	: and Ro = Re + Altitude above Earth's Surface : Altitude (Ae) is the measured from Earth's surface
0	9.81	1000	6.25	: If: Ro=5000mi = 4000 mi , (radius) + 1000 mi , (Ae)
100	9.33	2000	4.33	
200	8.87	4000	2.426	: = 8000 mi = 4000 mi + 4000 mi, from (r) Earth's center.
300	8.47	8000	1.075	: = 12000 mi = 4000 mi + 8000 mi, from Earth's center
400	8.09	20000	0.268	: g = 26.8 cm/s^2 = 0.0268 m/s^2
500	7.73	25000	0.183	: r = 29000mi = 4000 mi + 25000 mi , g = 18.3 cm/s^2

Due to the inverse square law of forces, if the distance increases by a factor of (n), the result is a force value of: force / n^2. distance2 / distance1= n. At Ro = 4000 miles from the center of the Earth, g = 9.81m/s^2, and at twice this distance which is: Ro = 8000 miles from the center of the Earth which is 4000 miles above the ground level, n = Ro / Re = 8000mi/4000mi = 2, the value of (g) is: (g) = 9.81m/s^2 / n^2 = 9.81m/s^2 / 2^2 = 9.81m/s^2 /4 =~ 2.45 as shown in the table above. If the radius of orbit doubles, (g) decreases by 4 as expected for an inverse. square law.

(go) at a certain radius from the center of Earth = (g) at Earth's surface / ((orbit distance from center, mi) / 4000 mi)^2
go = (9.81m/s^2) / (Ro / Re)^2 : A formula for the gravitational acceleration at a given radius of orbit of Earth
go = (9.81m/s^2) / n^2 :n = ratio of actual orbit distance above Earth's surface to the center, to Earths radius
This is due to that the force of gravity is an inverse square law mathematically.

For an object to remain in a stable orbit, the force of gravity (Fg) upon an object (with mass = M2) must equal the centripetal force (Fc) upon that object. The inward or center-seeking force is called the **centripetal force** upon an object, and it can be thought of as being the equal and opposite force of the outward-force called the **centrifugal force** pushing the object away or outward from the center point. Centripetal force is due to the force of gravity. When these two forces (F = ma) are equal or "in balance" with no net difference, the object will not move closer to, or further away from the center of orbit, but will remain in a stable or constant (circular) orbit and altitude and-or distance. As noted previously, the force of gravity is less for higher orbits. If you were to attach a string to an object and move or "swing" it as if in an orbit about you, the inward (to and upon your hand) tension in the string keeping it from "flying off or away" due to centrifugal force, would be the centripetal force.

When: Fg = Fc : When the are equal: Fg - Fc = 0 : = no net difference = not net force

$$\frac{G\ M1\ M2}{r^2} = \frac{M2\ v^2}{r}$$

: G = universal gravitational constant, **M1 = mass of Earth** in kg
M2 = mass of orbiting object in kg, v = velocity of orbiting object in m/s^2 ,
r = radius of orbiting object from Earth's center, in meters
= radius of Earth + altitude of orbit
Re = 3959 mi = 6381 km =~4000 mi =~ 638000 meters
: If the forces are the same, the accelerations must be equal to (g) at that alt.
: F = ma = Fc = M v^2 / r , hence (a) = v^2 / r and (m/s)^2 / m = m / s^2
: v = sqrt [Fc (r) / M] = d / t = 2(pi)(r) / t = Ce / t

Solving for the velocity of this orbit in the previous, main equation:

$$vo = v = \sqrt{\frac{G\ \mathbf{M1}}{r}}$$

The av. Period (T) or time of each orbit can then be calculated as:

$$T = \frac{\text{Distance}}{\text{Speed}} = \frac{\text{Circumference or orbit length}}{\text{Average speed of the orbit}} \text{ , ex: } = \frac{2(pi)(Ro)}{v}$$

Other equations for centrifugal and centripetal force:

You may also wish to view the topic of **Orbital Motion Of Objects** given later in this book.

First: v = w r : (linear velocity) = (angular velocity) (radius) and (angular velocity) = (linear velocity) / (radius),
v^2 = w^2 r^2

Fcentrifugal = m a = **m w^2 r** and **Fcentripetal** = m a = **m v^2 / r** : such as for an natural orbiting object

From the above equations, we can equate: a = m^2 r = v^2 r , and from this and the a previous equation, we can find that:: v^2 / w^2 = r^2 and v / w = r and w = v / r

checking: a = v^2 / r = w^2 r^2 / r = w^2 r

The mass and-or weight of an orbiting object does not determine its needed or natural orbital velocity, however it does determine its kinetic energy and-or inertia (ie., resistance to changes in movement, hence a resistance to changing direction and-or speed, f=ma, a=f/m). All size of mass in an orbit at a certain altitude will fall towards the center of gravity (such as Earth's center), and at the same rate due to the same gravitational force at that altitude. Consider that objects of various mass and-or weight when dropped at the same height will reach the ground at the same time since the force due to gravity upon them is the same. Orbit velocity, and orbit altitude or height have an inverse type of mathematical relationship. The farther from the center of orbit, the lower (g) is and the lower the distance dropped (vertically) per degree of orbit and-or unit of time. Farther orbit distances of a mass have a larger orbital path (such as a circle, C=2(pi)(r)) to travel about the center of orbit, and therefore, it will take longer to orbit at a given speed, and especially at a lower speed. For a given angle about a center of orbit, such as the center of a circle, the (arc) distance traveled is directly related to the distance (r) from the center of orbit or reference, and it is also said that the "arc of curvature" (ie., or "departure from being straight") is inversely related to (r). For a very large circle or orbit, the "arc of curvature" is very low, and that orbit segment can appear to be practically a straight line with an imperceptible arc or curve about a central point, such as the center point of orbit. Note that arc of curvature is not the same concept as arc length which is a segment length of the circumference of a circle, and that this length depends on the angle and radius distance from the center.

In the equation above for orbital velocity , (G Me) is a constant = Ke, for Earth orbits, v = squareroot (Ke / r), and if (r) changes by a factor of (n), v = squareroot (Ke / nr) , and v will change by a factor of squareroot (1 / n). Ex. if n=2, the change in orbital velocity is squareroot (1/n = 1/2) =~ 0.707 =~ 70% of the first velocity.

The force due to gravity (g) at (r) is inversely related to (r). g2 = g1 / r^2

Obviously, it will take more energy and work to lift a larger mass to a larger height or altitude of its orbit:

Work = (force)(distance) = (ma)(distance) = (ma)(height) = (force)(altitude) = Joules of energy

It will also need enough momentum and-or kinetic energy (KE = mv^2 / 2) to maintain its ("free-fall" over the edge of the Earth) orbit, rather than just loose altitude due to gravity. Once an orbiting spacecraft has enough KE for orbit, it will also have enough velocity to maintain that orbit regardless of its mass, and without requiring more energy since the energy losses are very minimal, practically 0 while orbiting Earth. At low orbits, there is an increased chance of colliding with some gas atoms such as of air, and this is much like a very small friction force that will transfer a very small amount (ie., Joules) of the orbiting craft's kinetic energy to that air, and which causes the craft to loose a small amount of velocity and then orbit altitude, and if this keeps happening, eventually the craft will need to go back to its intended, higher orbit, and via a force or thrust, usually created by its on-board rocket engine(s).

The **International Space Station (ISS)**, as of the year 2021 has an altitude of about 250mi =~ 400km, and has an orbit velocity of about 4.76 mi/s = 7.66 km/s = (4.76 mi/s)(3600 s/h) = 17136 mi/h. Its velocity is often spoken as (roughly) 5 mi/s. (g) at this altitude is about 8.75 m/s^2 = 28.71 ft/s^2. It's orbit radius is about 4000mi + 236mi = 4236 mi, and the surface of the Earth drops below the imaginary horizon line (at this altitude and-or radius) at about 7.55in/mi, and this is a less rate than at the surface of the Earth as mentioned previously in this book, at about 8 in/mi. The arc of curvature is less at this radius distance, altitude and-or orbit. Its orbital period is about 1.5 hours which corresponds to 90 minutes. During 1 day = 24hrs, the number of orbits would be: (total time / time of each orbit) = 24hrs / (1.5 hrs / orbit) = 16 orbits. If each orbit is about: C=(2(pi)(r) = (6.28)(4200mi) = 26389mi, the total distance traveled per day is: (number of orbits per day)(distance per orbit) = (16 orbits / day)(26389 mi / orbit) = 422230 miles / day . Also, 422230 mi / day =
= 422230 mi / 24hr = 17593 mi / hr = 17593mi / 3600s = 4.887 mi / s = (distance / time) = v = velocity of the ISS orbit. The ISS has an orbit inclination of about 51.6 degree with respect to the equator. With this angle, much of the Earths surface will eventually be directly below it. The gravitational acceleration (ge) value in a low Earth (altitude) orbit, is slightly less than that at sea level of (9.81m/s^2), however since both the spacecraft, people and objects in it are falling at the same rate relative to each other, it then appears as if everything is weightless, non-moving, and unaffected by the actual force of gravity upon them. Skydivers jumping out of a plane, and before deploying or opening their parachute to slow down and land safely can also experience this weightlessness effect, except that the drag or friction forces due to the air on larger objects is greater and this effect is not a great as that in space.

Even when a spacecraft is in a low orbit about the Earth, which is called a "Low Earth Orbit (LEO)", for it to go to a high orbit, or even to the altitude of the Moon, it must still overcome the gravity of the Earth, and much energy will be needed. This energy is most often in the rocket fuel which will be combusted by the rocket engine so as to produce the propulsion or thrust force. Some special elliptical orbits about Earth, and also using "flyby gravity assist" of the Earth so as to increase the kinetic energy of the spacecraft can also be used to save fuel, but the time to reach the Moon would then take a longer distance and time.

If a spacecraft does not maintain its required orbit velocity, it will not maintain its current orbit. If the spacecraft slows down, perhaps by using a rocket engine to provide a force in the opposite direction to its forward travel, it will then not have enough (outward) centrifugal force (Fc) to balance (ie., 0 net force, 0 difference) the gravitational force Fg upon it and pulling it inward towards Earth. Fg will then be greater than Fc. The altitude of the spacecraft will decrease, and it will usually have some remaining horizontal velocity if it was not completely slowed down in horizontal motion to 0 mi/s. The spacecraft will then also have a downward vertical component to its motion and it will accelerate toward the Earth due to the net **e**ffective downward force (F**e**) of gravity (Fg) now acting upon it **:** (F**e** = Fg - Fc).
In order for the spacecraft to slow down and make a relatively soft landing at say 0mph, all of the high kinetic energy of its orbital kinetic energy must be expelled or removed, and much of this energy will be lost as friction (resistance to movement, and converted to heat) and-or pressure with Earths (air) atmosphere, all due to the high speed of the reentry flight. Some kinetic energy (particularly the velocity factor) will also be lost with a parachute's upward drag force or resistance with the air. Some spacecraft even have a rocket motor or engine to reduce its downward motion and kinetic energy during landing, and some spacecraft may deploy a type of "air bag" (ie., a type of strong material, inflated balloon) at the bottom area of the spacecraft so as to help it make a softer landing. A spacecraft could reside and travel in a lower orbit if it can be made to have the natural orbit condition of Fc = Fg, and at a lower orbit, the velocity needed for a given mass will be higher since the force of gravity (Fg) is higher at lower altitudes. A higher velocity means a higher Fc.

Since a high velocity spacecraft has a high amount of both kinetic and gravitational potential energy, the spacecraft needs a high temperature thermal shield due to the high temperatures created by friction with the air upon reentry and-or landing on Earth. This shield protects both the astronauts and spacecraft. When it lands, all of the energy (KE and GPE) it had in orbit will have mostly dissipated as heat, and reduced to 0 joules.

The longer that astronauts are in space, they will usually develop some slowly worsening health problems while there. On-board larger spacecraft such as a space station, there is often various exercise machines and-or routines. Due to the effect of feeling no gravity and some atrophy of muscles and bone mass, they can also develop problems such as standing and walking for awhile after they return back to Earth and its apparently (to the astronauts) much stronger force of gravity force. Over the next days and weeks, they will need to exercise slowly so as to develop a typical amount strength.

The famous author **Jules Verne** (1828-1905), from France, was probably fascinated by the more modern telescope views of the Moon, advances in technology, and considered Newton's theory of orbiting in about 1865 while writing his book called 'From Earth To The Moon', and then afterwards, writing another book called 'Around The Moon' in 1870. In 1902, before the first practical airplane was invented, and inspired by Verne, the first science fiction and outer-space type movie called 'A Trip To The Moon' was created in 1902 by **Georges Melies** (1861-1938). Melies also created several new technical, film processing (ie., editing, "special effects" such as appearing and disappearing things, etc.) and-or filming (ie., movie making) and movie concepts for the motion picture industry. Melies also made the movie called 'The Impossible Voyage'. These books and movies surely helped inspire the imaginations and creativity of people, and the future of science and flight. Shown in the movie, the passengers entered a large projectile and were launched out from within a large cannon (a propulsion and aiming tube for the projectile). This idea was most likely based on Verne's books which seems to consider the previously known (Isaac) "Newton's Cannon (or projectile)" theory about how to orbit the Earth. While returning from the Moon in the year 1969, Neil Armstrong (the first person on the Moon) mentioned Jules Verne. Verne's book had essentially and reasonably predicted some similar details of the actual ventures to outer space and the Moon. Verne also wrote many other highly popular science-fiction books, some of which were later made into popular movies such as: 'Around The World In Eighty Days', 'Journey To The Center Of The Earth' and 'Twenty-Thousand Leagues Under The Sea'. A contemporary to Vern was **Herbert George (H.G.) Wells** (1866-1946) who wrote books such as: 'The Time Machine' in 1895, 'The War Of The Worlds' in 1898, and 'The First Men In The Moon' in 1901. These writers took actual, known scientific explorations, discoveries and facts, and mixed them with their own "vision" (ideas about the future), imagination, dreams and fantasy into a "mixed literature" called science fiction (fictitious , meaning probably "fake", "made-up", "exaggerated" and often with some facts being real or true so as to make a known and believable connection with reality), and of which can even inspire real science, invention and research in an increasing, circular-like growing or continuous manner. Some of the earliest and significant science-fiction, outer-space adventure type of movie series were: Buck Rogers (literature by 1929, movies by 1930, TV by 1950), and Flash Gordon (literature by 1934, movies by 1936, TV by 1954). Some other popular, and relatively modern science fiction writers are: **Arthur C. Clarke**, **Isaac Asimov,** and **Robert A. Heinlein**. Arthur C Clarke who wrote books such as '2001 A Space Odyssey', was also an early visionary of man-made satellites for (re-transmission) relaying communications on Earth. **Robert A. Heinlein**, was a writer and visionary who included many science facts and social ideas in his science fiction writings. **Isaac Asimov**, a writer and visionary that promoted the concept that robots should only be made to help humanity. In late 1958, the military launched a large area communication satellite which was mainly for proof of the concept, and other experiments. About three years later, **Telstar-1** was the first public communication satellite, and it was launched on July 10, 1962, and which soon inspired a popular instrumental song called Telstar in late 1962. Telstar was placed in a relatively close to Earth, elliptical orbit, and therefore had to be tracked (ie., moved to aimed at) more by receiving antennas here on Earth, unlike some of the more modern communication satellites placed into a very high altitude, **geostationary orbit** (ie., has the same rotational or angular velocity as Earth's axial rotation) about Earth. The Telstar mission demonstrated that its technology was practical for public communications such as relaying some distant (wireless) TV (electronic TeleVision) and telephone (electrically transmitted) audio signals.

A modern large and heavy rocket can enter a high velocity (about 17100 mph =~ 4.75miles/s after starting out at 0 mph and then gaining more and more velocity when the rocket engines are providing force), low altitude (about 255 miles high, such as for the ISS - International Space Station) or low Earth orbit (LEO) in about 8 minutes to 9 minutes of time - hence an average of 8.5 minutes = (8.5min)(60s/min) = 520 seconds. It is sometimes mentioned how many seconds of rocket fuel and-or the resulting thrust (ie., forward force) was needed or used, and it can be about 9 minutes maximum = (9min)(60s/min) = 540 seconds (maximum). Once this fuel source reduces to 0, and-or is shut off from reaching the engine, the thrust force created and applied to the rocket will be 0, and the rocket will no longer accelerate and gain speed (ie., velocity). Note that for a rocket to orbit the Earth, it must enter orbit at parallel (ie., a 0° angle, level, horizontal) to the Earth's surface, and for much of the trip to space, the rocket travels at an average angle of about say 45° relative to the ground reference plane (ie., horizon). In a way, this is much like using a ramp or inclined plane so as less force is needed when the change in height or altitude is less per unit of time, but it will then take more time to reach the intended altitude. With a given thrust or force value, the rocket will gain more and more velocity as expected due to a constantly applied force creating the acceleration (ie. gain in speed, due to more energy being additive to the rocket) of its movement, but there is also an increase in acceleration (a = ch v / ch t) due to the very lower density, "thin" high atmosphere, and which effectively applies a lower friction, resistance or drag force in the opposite direction of the thrust and-or flight. As the amount of heavy rocket fuel and its corresponding mass and weight is significantly reduced, the thrust from the engines will apply its force to a smaller rocket mass. From F = thrust = ma, we get: a = F / m , and if the total mass (here of the rocket) decreases for a given or constant force (F) from the rocket engines, the acceleration of movement and-or velocity of the rocket increases.

If the velocity of an orbiting object increases, such as due to a temporary uses of its rocket engine(s), its orbit will then become an elliptical (like an egg or squashed circle) shaped orbit. If the velocity of the object increases too much and is greater than or equal to the "**escape velocity**" (the minimal velocity to escape from the attractive force of the local gravity and-or to leave orbit), it will break free from its natural gravity force held orbit and travel away from the Earth mass it was orbiting, and this is necessary to happen for the object to travel to the Moon or planet and possibly orbit and-or land there. The minimal velocity for this to happen is called the (gravitational field) escape velocity. The escape velocity for Earth is about 25000 miles per hour for low Earth orbits. The force of gravity, or gravitational attraction and acceleration, decreases the farther from the center of mass. Near the surface of Earth, the constant gravitational (g) force and acceleration rate of the Earth's mass is about 32 feet per 1s^2 = 9.81m/s^2 = "one G". At a higher altitude or orbit, the force of gravity is less and therefore, less orbital velocity is needed since the downward pull toward the Earth is less and objects will fall less distance in a given amount of time. Note that there is still significant (about 0.9g) gravity upon low orbiting (<1000 miles high) craft such as the ISS (International Space Station), however it only appears as 0 amount of force or a negligible (micro gravity) force due to the "free fall" effect. If this craft, people and things in it are falling at the same speed together and their relative velocity to each other is 0, and then things (mass) will generally appear as being not moving or falling, but rather appear as "weightless" or still (ie. not moving vertically) in motion. If a spacecraft slows down with force created in its rocket thrusters to reduce its forward motion or momentum, or encounters friction ("drag") due to the object or craft essentially colliding at high speed into some of Earths atmosphere (air) atoms, it will reduce its altitude and come out, or "fall out", of orbit and return to Earth since its horizontal speed is no longer enough to effectively regain (due to the downward curvature of the Earth) the loss in altitude and overcome the corresponding (gravitational, gravity) drop in vertical height per unit of time as it orbits the Earth.

If an object, such as a communication satellite is orbiting at the same angular velocity or rotational speed as that of the rotation of the Earth which is 360° per day = 15° per hour, it is said to be in a **geostationary orbit** when it is always orbiting above the equator plane of the Earth. The object will appear (if it was large enough to be seen) up in the sky as being steady or still, and it will always be positioned directly overhead at one point on Earth's surface, and-or at the same angle and direction in the sky as seen from other locations. This is ideal for satellite communication since the communication antennas can remain fixed and does not have to constantly track (follow) the satellite so as to move the communication antenna so as to keep aiming the antenna for the best or strongest communication signal. A fairly common example of this in many countries as of the year 2025 is for fixed in position and aiming, circular-like, parabolic satellite TV antennas-receivers seen on rooftops of homes and buildings. A **geosynchronous orbit** (ie., synchronized or "in sync" with Earth's rotation on its axis), perhaps at an angle relative to the equator line, is any orbit having the same angular velocity as that of Earth's axial rotation, and if it is at an angle with respect to the equator line, it will not then not actually be stationary above a constant location or point on the Earth's surface, and it will always appear as moving.

For a, and like any other stable orbit, and the centrifugal force upon the object equals the gravitational force upon the object. A geostationary orbit needs to be a circular, and not an elliptical orbit. A point, object or person on the circumference (about 25000 miles around) or surface of the Earth at the equator will go about the center point of Earth at an angular or rotational velocity of 15°/1h, and-or a corresponding linear velocity of (circumference / time) =
25000 miles / 24 hours = about 1042 miles per hour = 0.289352 miles per second = 1528 feet / second. As the latitude on Earth increases, this velocity will decreases. Technically, the Earth rotates once on its axis during a "solar day" or "**sidereal day**" which is almost 24h, and is actually 23h, 56m, 4s of time length. A natural geosynchronous orbit of 15°/1h angular or rotational speed about Earth is usually located at about 22236 miles = 35785 km above Earth's surface (or about 26236 mi =~ 43333 km from Earth's center, and of orbit) at the equator due to Earth's mass and gravity there, and the velocity to maintain it is about 1.86 miles per second which is much lower than that of low orbiting spacecrafts that are just a few hundred miles in altitude above Earth's surface. Since the Moon orbits the Earth, the Moon is actually in a "free fall" orbit about Earth due to the gravitational attraction between the Moon and the Earth, and it orbits the Earth about every 27.32 days. Note that due to Earths axial rotation and the Moons rotation about the Earth, it takes a longer time at about 29.5 days for each same phase (ex. full or new moon) of the Moon to be seen by a person at the same location on Earth. A helpful equivalent fraction: (Ce=25000 mi) / 360° = x mi / 15° , x mi =~ 1042 Here is an equation derived in

this book for the **average velocity of an orbit : vo = squareroot [(g at that radius distance) (radius of orbit)]** , and this value is about 1.91 miles / second =~ 6865 mi / hour for the geosynchronous orbit altitude mentioned. Notice that mass of the object is not used in the formula. The specific high altitude distance of the geosynchronous orbit is chosen so that the spacecraft or satellite will orbit the Earth at 15° per hour, hence 360°, and only once (1) per day = 24h unlike low orbit craft orbiting many times (about 16) per day at the required high orbit velocity. Each circular orbit has its own unique radius (r), a unique corresponding velocity, and therefore, it also has a unique corresponding time period (P) of orbit. As (r) increases, v and P decrease. There is only one circular orbit about Earth that has a natural period of 24 hours. At high altitude or radius orbits, the satellite can communicate to a very large area on Earth, and where any possible adjustments to its orbit will not require much distance, energy and time to correct.

A company recently created (about 2021) by **Elon Musk** (born 1971, in Africa) has currently manufactured a (stationary and-or portable) satellite transceiver system called **Starlink** which is for relaying and-or improving communication services such as for the popular Internet (electronic, digital data composed of "1's and 0's") communication system for undeveloped, difficult and-or distant regions on Earth. Though each Starlink satellite launched for this system is relatively expensive, the users transceiver system is relatively inexpensive and has a relatively low monthly fee for the many subscribers or users of which will eventually add up to pay for the system. He has helped employ many people, and with his earnings from other businesses, he has branched off into the exciting field of space and rocketry of which also employs many people, and so as to help benefit all the people of Earth in some way, either now or in the future. As of the year 2025, Musk is still creating more efficient and powerful rocket engines, and launching spacecraft with people into space, and many small communication satellites into a relatively low altitude orbit about Earth on his SpaceX company rockets. He is currently associated with several other popular companies such as Nasa, Paypal (for digital payments using the Internet communication system), Ebay (a popular selling store website on the Internet), the Boring Company (for tunnels), Tesla (for electric automobiles, and power storage systems such as electric batteries). In 2025, Musk began working as the director of a new U.S. Government agency created by President Donald J. Trump in the year 2025, and it is called DOGE (Department Of Government Efficiency).

Many of today's rockets to space use various types and combinations of solid and-or liquid fuels such as gasoline, nitrogen oxide, methane, liquid hydrogen and liquid oxygen. On March 16, 1926, the first successful liquid fueled rocket launch was made by **Dr. Robert Goddard,** (1882-1945) from the state of Massachusetts in America, and who was a scientist and teacher, and who theorized about rocket construction and space flight. His liquid rocket concepts began in about 1909, and his rocket fuel was gasoline and liquid oxygen that was then mixed together in the rocket engine during launch. Later, he would launch his rockets at a more practical and larger test range (ie., a large unused land area) site at Roswell, New Mexico. The best rockets he made did not go very high (about 2 miles high), but it was still enough to prove they had potential abilities such as high-altitude research, and perhaps, for space research. Using liquid fuel allows the fuel flow to be controlled with pumps and valves, whereas it is much more difficult to do this practically with a solid rocket fuel. Rocket engines need to produce a controlled ("controlled explosion") combustion of just some of the total amount of rocket fuel, and so as to produce a practical (non excessive) force in just the (forward, travel) direction of the rocket body or mass. Goddard is credited to the first modern methods of steering (guiding, guidance, aiming control, stability such as with gyroscopes) rockets, and creating the modernized concept of using multi-stage rockets that are essentially a rocket attached to the top of a another rocket. A heavy stage can be discarded once its internal fuel is used up, and this will significantly improve the thrust and efficiency of the remaining rocket engines since they rocket weight will be less. Goddard utilized the previously known concepts of liquid cooling, and made the first practical method to cool the rocket bell, nozzle or exhaust cone from being disintegrated by the very hot, high kinetic energy exhaust gas particles after being combusted. First he used water, and later he used the much colder rocket fuel being circulated in its nozzle structure before being used to create combustion and thrust force in the rocket engines. As a youth, and even before plane flying was proven to be practical in 1904, Goddard was highly interested in all forms of flight such as with birds, kites, and hot-air balloons. In 1908 Goddard thought that cameras could be sent to and back from the orbit of other planets. Goddard, in 1912, was also one of the first pioneers in vacuum tube construction and its uses at about the same time as its first certified original inventor - **Lee DeForest.** Goddard's rocket studies and discoveries would eventually lead to the development of rocket-engine powered airplanes ("jets"), jet airplanes with liquid fuel, and then rockets to outer-space. For a historical note, "**blackpowde**r" is a solid, granule fuel developed for small rockets and-or various projectiles and some machines was invented in **China** in the early 10th century AD, and today, it is used in some permit only (due to the potential dangers of fire and-or injury) "fireworks" which are colorful, decorative light and sound displays launched up into the sky during a celebration of a festive occasion(s).

The first hot gas, and here, particularly a primitive steam engine that could produce (here rotating) motion via the force and-or pressure of the hot (water steam) gas is credited to **Hero of Alexandria** in the first century AD. His (steam) engine is commonly known as the **aeolipile** which can be translated to English as "wind ball", "wind sphere", "wind spinner", and possibly a "wind engine". This engine eventually would eventually lead to things such as steam trains for transportation on rails (ie., metal rail roads), and turbines (ie., mechanical energy transfer blades, or energy transducers) for electricity generation. Rocketry is also a creative hobby, and some people joyfully build the available kits for purchase and launch them in relatively safe, open (no nearby man-made structures and-or dry flammable grasses) and permitted locations. Some of the rocket contemporaries to Goddard are **Hermann Oberth** (1894-1989) of Hungaria-Germany, and who liked the Jules Verne books, and he later wrote some of his own books about spaceflight and rocketry, **Konstantin Tsiolkovsky** (1857-1935) of Russia, and **Robert Pelterie** (1881-1957) of France were also pioneers of both theoretical rocketry and space flight which helped to motivate people and science in various ways. Oberth was also involved with the first (solid fuel) rocket plane tests in 1929, and was a mentor (ie., a teacher) to **Werhner Von Braun** (1912-1977), (aka, ["Verner"] "Von Brown" as an English language derivative or interpretation) of Germany. Braun and some other members of his rocket team became prisoners of the Americans (USA) and Russians (the Soviet Union at that time) in WW2 (1939-1945), and some of their rocket knowledge would become usable and cost saving for their space programs such as those of **NASA** (National Aeronautics and Space Administration, of the USA) who would create the large rocketry and spacecraft industry to take mankind to the Moon. Since his youth, Von Braun was a joyful and artistic visionary of the future of spaceflight, space research and exploration so as to benefit humanity. He was probably inspired by Jules Verne, Goddard, and the amazing leaps and exponential growth in science and technology of those times he was living in and so as to benefit humanity. With having some rocketry knowledge, he was like many young men, drafted (ie., ordered) into the military service for his country, and for him, so as to create new large, powerful and accurate rockets - some of whose infrastructure and manufacture were performed by "work details" that were assignments given to some Jewish civilian prisoners (many even Germany born citizens) during World War 2 (WW2, 1939-1945 Europe, 1942-1945 USA), and of which some were also scientists, etc. A few years before the war ended in May 1945 in Europe, and in August 1945 in Japan, some of the parts of these used rockets were eventually found in England, Germany, and France, and were then studied by scientists. A lesson learned from WW2 is that old tribal ways of ancient mankind may resurect themselves from time to time and that blame rather rests on humantiy itself, and that it needs to be aware of these possibilities so as to prevent them. Various cultures of humanity offer a healthy variety of sorts, often dictated by climate and local conditions, resources and infrasturcture, and they should be acknowledged, embraced and helped if need be.

The first practical **jet engine** for (jet) airplane, high speed propulsion is credited to **Frank Whittle** (1907-1996), from England, in 1930, and the planes with jet engines would eventually be simply called as "jets". The basic concept of a jet engine is to use a highly combustible fuel mixed with compressed (ie., high density) air so as to created a high combustion, semi-explosion of hot gasses having high kinetic energy which then produces the resulting forward or thrust force upon the plane. In 1941 Frank hoped his recently built jet plane would be able to shorten WW2, and meanwhile, the Germans were already testing and using a recently built (solid fuel, not liquid) rocket engine (and not a jet engine) plane. One rocket plane was briefly tested in 1929 with **Fritz Opel** as the pilot, and it had a modern-like design based on the primitive rocket car and rocket plane flights by others. Opel also tested a liquid fueled rocket plane in 1929. Another significant pioneer in Germany's and Austria's early, pre-war rocket programs was **Max Valier** (1895-1930), from Austria, and who was one of the founders of the Society For Space Travel in 1927, and of which Wernher Von Braun would join. **Pedro Paulet** (1874-1945) from Peru tinkered with rocket fireworks and conceived of, and possibly built an unverified, primitive liquid rocket engine in the late 1890's. He then also conceived of a rocket plane as early as 1902, but it was never built, and that any practical, self-powered airplane flight of any type had not even been achieved yet. In August 1941 the American **JPL** (**J**et **P**ropulsion **L**aboratory) were first to use small solid fuel rockets attached to the airplane so as to assist and shorten the take off length needed, and this is termed as **JATO** or "(rocket) Jet Assisted Take Off" **John Whiteside Parsons (aka, Jack Parsons),** (1914-1952), from America, was the leading chemist and a founding member, along with a few others, such as **Frank Malina**, of JPL, and then later with the **AeroJet Co**. Parsons invented a formula mixture so as to make solid rocket fuel much more stable for storage, safer when working with, more powerful and reliable. By early 1942 liquid fueled rockets were also being tested and used for JATO. One of the early members of JPL was **Qian Xuesen** (**Tsien Hsue-shen** is an Englishized spelling and pronunciation), (1911-2009) who was a visiting Chinese professor in America, and who would be one of the founding members of JPL and so as make some helpful calculations. He was eventually given a US military rank of (honorary/temporary) Colonel in WW2 so as to help deal with the seized German rocketry tech, and after returning to China many years later in 1955, he became the founder of the Chinese rocketry and space program. It is of note that the USA and its Allies of WW2, greatly helped both Russia and China in WW2, and that both Japan and Germany were eventually allowed to govern their own countries again after being occupied and guided for several years.

By early 1942, the Americans began testing a liquid fueled (rocket) jet plane engine that used Whittle's jet engine design. By 1944, the still fairly secretive jet plane became a commonly known reality and future hope of human flight. In general, jet planes can obtain higher speeds than that of propeller driven airplanes, but then are less fuel efficient at lower altitudes where the air is denser. The first passenger jet plane built specifically for public or civilian transportation, and called today as "airliners", was ready for service by 1952 in England, and it was called the **Havilland Comet 1**, and it could carry a maximum of 44 passengers in reasonable comfort. This plane included the required internal air pressurization needed for high altitude flights, and which is a more fuel efficient flight. This plane could fly at a maximum altitude of about 40000 feet, hence well above the highest mountain - Mt. Everest. Its maximum speed was about 460 miles per hour. The first propeller driven, civilian passenger planes came at about 1936, such as the **DC3** in the USA, and there are some modernized DC3's still flying as of the year 2025. **Rocket engines** are much like a jet engine, however, rockets carry their own oxygen - usually liquid oxygen which is also then of a high density needed for high fuel combustion and the resulting high amount of forward thrust force.

On **October 4, 1957**, Russian scientists used a rocket to launched the first Earth orbiting man-made satellite called **Sputnik-1**, which in an English language translation, and which means "path maker", "space traveler" or simply: "Satellite 1", and this remained in orbit and transmitted routine radio pulses on about the 20Mhz and 40Mhz radio frequencies for 3 months before reentering Earth's atmosphere and burning up (melting to vapor) and-or breaking apart due to its high speed and the high amount of air friction and the corresponding heat energy created. This historic satellite mission also finally proved the basic concepts of orbit that were first scientifically proposed or theorized by Isaac Newton (ie., in "Newton's Cannon", thought experiment) many years previously. In just a few months after the successful launch of Sputnik-1, the United States successfully launched a satellite called **Explorer-1** into orbit about Earth on January 31, 1958. The rocket used was a military rocket converted for the benefit of mankind, and was launched from Cape Canaveral in the state of Florida. Soon afterwards during the same year, **NASA** (the **N**ational **A**eronautics and **S**pace **A**dministration) was formed so as to manage all things related to space for the United Space and some of its partners.

In just three and a half years later after the success of the Sputnik-1 satellite, the first man to be sent to outer-space and orbit (just once) the Earth is (astronaut/cosmonaut) **Yuri Gagarin (1934-1968)** from Russia on **April 12, 1961.** This flight quickly verified many aspects of human spaceflight, and Yuri became an instant global celebrity. **Gherman Titov** was the next person sent to space and was also from Russia. He was launched just a few months after Gagarin, and on August 6, 1961, and was the first person to have several orbits (17) of the Earth, and was first to do many common human and scientific research activities in space. At about 90 minutes per each of Titov's orbits: about (17 orbits)(90 min / orbit) = 1530 min = 25.5 hours = 1 day and 1.5 hours, and therefore, he was the first person to orbit the Earth for over a day. About a month after Yuri Gagarin's venture, and on May 5, 1961, **Alan Shepard** (1923-1998), from the United States of America (USA), was the first American astronaut to go above the altitude considered as the start of space, and his (non-orbital) flight lasted for 15 minutes, and at a maximum altitude of 116 miles, and he walked on the Moon ten years later in 1971 during the Apollo-14 mission.

In less than two months after the success of Sputnik-1, the President (the highest position for an elected leader) of the USA, **John F. Kennedy**, formally began the Apollo Moon program on May 25,1961, and so as to land a human on the Moon by 1970. This was a dream for many years since ancient times and the more modern times of Isaac Newton's scientific discoveries and theories. Modern (late 1960's) technology finally made it a possibility to begin to try it, but much work in rocketry, space travel, communication and many other things still needed to be created, improved and tested first. Each lesson learned would help all future rocket and space missions, and often, even help the general society so as to have a better life due to advancements in science and technology. Many of the achievements during this program would essentially pay for themselves many times over by helping the people of Earth. A good example is with weather forecasting satellites and their photographic imagery. A single advancement in technology often leads to an exponential growth in other technology.

On February 20, 1962 **John Glenn** became the first American to orbit the Earth. He later became a member of the U.S. Congress, and at 77 years old, and he flew aboard the Space Shuttle spacecraft in 1998 which orbited the Earth for several days. The famous actor **William Shatner** (b.1931), from Canada, who was an actor in the celebrated science-fictional, cherished outer-space movies made for TV broadcast starting in the year 1966, and it is known as the Star Trek movies, and he flew to a 66 mile high region of outer-space for a few minutes during a non-orbiting (sometimes called "sub-orbital"), 10 minute rocket flight on October 13, 2021, and when he was 90 years old. He was in the rocket capsule with three other people, and he had then become the oldest known astronaut in the world. The rocket was made by a rocketry and space company called **Blue Origin** and which was founded by **Jeff Bezos** (b. 1964), from USA, and he himself even flew on his own rocket upward to the border of space. Bezos founded the popular **Amazon Company** which is a popular "online" (ie., internet, "web", electronic media, communication) shopping website.

On June 16, 1963, **Valentina Tereshkova** (b. 1937) from Russia (aka, the Soviet Union, previously) became the first female astronaut. The spacecraft she was in made an incredible 48 orbits of the Earth.

Since airplanes require a difference in air pressure (or force) to cause a lift force, airplanes can not then fly in the vacuum of space above Earth's atmosphere which has no appreciable amount of air for wing surface pressure differences and the resulting lift force to be created. Different methods of propulsion and steering (aiming, direction) such as from rocket engines are needed in space. An advantage of space travel is that once a craft is in motion, such as in orbit around the Earth, and with little or no air resistance or drag force to reduce its forward velocity, it will keep moving without any extra rocket fuel or energy needed to keep it moving at the same orbit speed.

Speeds of several miles per second are common for spacecraft such as for the **Apollo-11** craft moving at about 7 miles per second on its way to the first manned Moon landing on July 20, 1969, and this is considerably more than the 0.64 miles per second that the Moon orbits about the Earth. **Neil Armstrong** and **Edwin ("Buzz") Aldrin** were in the Moon landing craft called "Eagle", and **Michael Collins** was the pilot in the Moon travel to, orbiting and return to Earth spacecraft called "Columbia" (Command Module for the mission) . Each astronaut mentioned was born in 1930; the same year planet Pluto was discovered using two telescope photographs of the stars, and with each photograph made at different time and then comparing both for any visible changes and-or movement. The launch of Apollo-11 to the Moon was on July 16, 1969, at 9:32AM, Eastern U.S. time, from Florida state, and returned to Earth eight days later on July 24, 1969 in the Pacific ocean. The Apollo-11 moon spacecraft ("Eagle") landed in the Moon's Sea Of Tranquility ("a mare" [marine, sea], a relatively flat, dark basalt (ie., a hardened lava stone) region) on the Moon at 4:17 Eastern U.S. time. The astronauts collected rock samples and walked on the Moon's surface for about 2 hours during their 20 hour stay there before firing its launch engine so that their landing craft could go back up to the orbit of the Columbia craft, and then dock (ie., attach, connect, join to) to it and return back to Earth.

Shortly before the Apollo-11 mission, and just 64 years after the Wright brothers first airplane flight in 1904, and 11 years after the first orbit of Earth with Sputnik-1, and 281 years after Newton's 1687 orbital theory, the **Apollo-8** mission craft was the first manned craft to arrive at and orbit the Moon on December 24, 1968 and three days after launching from Florida state of the United States. For the spacecraft to be fully "captured" by the Moon's gravity and to then orbit it, the spacecraft had to use its rocket motor in a reverse direction so as to slow its forward velocity to just that needed to orbit it at about 70 miles high in altitude from it. Some of the initial verses from the historic book of Genesis from the Old Testament Bible book were read and broadcast to the people of the world on December 24, 1968 by the astronauts in their spacecraft. This first manned mission to the Moon was profoundly important for all the later missions. Their craft didn't land on the Moon, but it was needed for proof of concept, telemetry (remote or distant sensing and-or measuring), communication, photographs, and many other initial tests, modifications and improvements needed before the future landing missions to the Moon. This mission could be thought of as mankind's rehearsal for the Moon landing. The world celebrated vanguard (first, leading) crew was **Frank Borman, Jim Lovell, and Bill Anders**. Their mission included lunar photography images so as to find some practical landing locations for the future landing missions. They orbited the Moon for a total of 10 times during nearly a day of Earth time. All of the Apollo missions that went to the vicinity of the Moon would eventually land relatively softly in water (aka, a "splashed-down" landing) in the Pacific Ocean, and usually near various retrieval ships having cranes and helicopters. Several minutes before landing, the spacecraft is caused to reenter the Earth's atmosphere at a calculated angle (set for enough time and atmosphere distance to safely slow down in velocity via atmosphere (ie., air) friction, and to also not to skip off the Earth's atmosphere and back into space), and so to reduce its kinetic energy and slow the spacecraft down from about 5 miles per second in outer-space to about 20 miles per hour (= 3600 s) or less at landing.

In March of 1969, the **Apollo-9** mission tested the lunar landing craft and the spacesuits for the manned Moon walks while in orbit about the Earth. In May of 1969, **Apollo-10** was a similar mission, but it also tested the lunar (Moon) lander at a height of only about 10 miles from the lunar surface. Through several years of time and Moon landings until the last Apollo space program Moon landing on December 19, 1972, a total of 6 craft and 12 people landed and walked on the Moon's surface from the Apollo-11 mission and up to Apollo-17 mission. Many scientific instruments such as a (Earth based) laser-light retro-reflector (ie.,essentially a efficient mirror), a SiesMonitor (a vibration sensor, Moon-quake sensor) were placed on the Moon's surface for studies to be performed when needed - such as to actually measure and calculate the distance to the Moon. Rocks and dirt were collected during all of the 6 manned landings, and this material was brought back to Earth to be studied until this day, and by more modern and advanced instruments and analysis concepts. These missions were made possible by the accumulation of knowledge, technology and talents of many people since ancient times to these modern times (2025), and as a recognition and goodwill gesture for the people of the Earth, a piece of Moon rock that was brought back from the Moon was honorably given to each country of the Earth at the time (135 total countries), and to each U.S. state (50) and partner territory. A few meteorites found on Earth have since been verified as being from Moon's surface, and most likely due to impacts from large and-or fast asteroids collisions then sending some surface debris away the Moon and into space. In general, it is illegal for people to own, purchase or sell any Moon rocks or part of that was brought back from the Apollo missions. A sample of Moon rock can actually be felt by tourists at the Johnson Space Center in the U.S. state of Texas, and at the Kennedy Space center in the U.S. state of Florida.

Outer-space has no pressure (0 psi) since it is a vacuum and generally void (empty) of any gasses (such as air) or fluids to induce or apply any constant, physical force and-or pressure upon an astronaut or spacecraft. The air pressure used within the astronauts spacesuits as they walked on the Moon during an EVA's (Exra-, or External Vehicle Activities) was a relatively low value at about 4.9 psi so as to reduce the total amount of force necessary to move the arms, legs and fingers inside a relatively thick and "pressure stiffened" (like the surface of a balloon constantly pushing [ie., a force] outward and trying to retain its shape) spacesuit. The required amount of oxygen needed for them to breath properly was supplied from (cold, condensed) liquid oxygen tanks in the spacesuit's backpack structure, and their exhaled carbon-dioxide gas was removed from the inside of their helmet (solid head covering) area. A spacesuit is also a temperature insulated and temperature controlled environment to sustain life, and also has radio communication installed in it. With a lower pressure in the spacesuit, astronauts must also avoid any decompression problems when they enter a region of different pressure such when they go back into their pressurized spacecraft (LEM, Lunar Excursion Module, the Moon landing craft). An air-lock is a small room that an astronaut can go into and close so as to be slowly adjusted to a desired air pressure such as perhaps 4.9psi, or the 14.7psi standard air pressure [STP]). If pressure changes happen too fast, dangerous forces, damage and health problems can result. Within the living quarters of the Apollo lander spacecraft, the pressure was maintained at about 5 psi, and which is a low value, however the amount of oxygen to breath was made to be higher than normal. Replaceable canisters containing **lithium hydroxide** was used to chemically remove ("scrub away") the carbon dioxide exhaled from the internal air of the living cabin of the spacecraft.

The rockets and spacecrafts of the Apollo space program were some of the first applications to use the newly developed, small (about a square centimeter of area) sized **i**ntegrated **c**ircuit (**IC**) on a single "chip" (ie., a small piece) structure. The Apollo crafts used these for many functions such as data acquisition, and some (automated, electronic, logical [ie., truth], decisions - and here made electronically) computerized steering guidance and control functions. Basically, an IC chip that is less than a square inch in size can contain many miniaturized, low power components such as transistors to quickly process electronic signals. Before the transistor electronic amplifier, a vacuum tube amplifier was required, and many of these would require a large amount of space, expense and power, and would therefore be impractical for even a small spacecraft. RAM (Random Access Memory, both readable and writable (to set and-or change the "1's and 0's") memory chips were still not yet available for the Apollo missions, and so an array of many hand-wound small magnetic cores (ie., ferrite material, hollow, ring shaped) wrapped with thin wire were used for the RAM (Random Access Memory) and ROM. ROM (Read Only Memory) is"hard wired" or hard-coded, and are not generally erasable and-or re-writeable) was available on IC's but the total amount of bytes (ie., data space and-or addressable memory or data locations) available were few. Many new, more advanced devices, materials, electronic and computer technologies were initially developed and-or proposed for the Apollo space program during the 1960's and early 1970's. This technology would later be used to develop powerful and affordable electronic hand calculators, **p**ersonal **c**omputers ("PC's"), and many other electronic devices in the next decade(s) following the Apollo space program.

Planning for the (reusable, relaunchable, cost saving) **Space Shuttle** Rocket program actually began before the last Moon landing with Apollo-17 in December 1972. The Apollo space program concluded with the (1973-1979) **Skylab** Space Station, and which was also like a test and step leading to the first **International Space Station (ISS)**. The Space Shuttle was first sent to its low Earth orbit in 1981 and **Astronaut John Young** (1930-2018) was the pilot, and who previously flew to and orbited the Moon with Apollo-10, and then walked on the Moon during Apollo-16. During this Apollo mission, the astronauts were given the news of the upcoming Space Shuttle program. The space shuttle was attached to the body of a large main booster rocket that used liquid fuel, and this rocket was also assisted by two smaller, solid fuel rockets and the liquid rocket engines of the Space Shuttle vehicle.. The space shuttle vehicle also had some smaller engines for steering in space and for reentry. Due to its high speed, the Space Shuttle spacecraft only needed small wings for the necessary lift force and steering during its reentry from orbit and space flight, and it could land like a plane on a very long runway and reused to save much costs. The ISS (International Space Station) was mainly built by using the incredible abilities of the (reusable, heavy payload lift, cost saving) Space Shuttle rocket which had a large cargo bay, and which also took and placed the world celebrated (Edwin) **Hubble Space Telescope** (**HST**, 1990, Nasa, Esa) and many other satellites into an orbit around the Earth. The first part of the ISS was launched on a Russian rocket in 1998, and the last part was launched in 2011. As of the year 2025, the HST is still continuing its space photography and allowing many new discoveries. As of the year 2022, for the ISS, its construction and technology has become relatively outdated and problematic, and it may be deliberately de-orbited and replaced by a new space station(s) within perhaps ten years after 2021.

In 2022 the celebrated **James Webb Space Telescope (JWST)** was launched and put into operation taking images with about twice that of the (camera image) resolution of the HST, and has a much larger objective mirror (here, an array) that can effectively sense much lower light levels. This telescope is in a special parallel-like orbit relatively near to Earth as Earth orbits about the Sun, and at about 1 million miles more distant from the Sun than the Earth is.

The first man-made spacecraft to reach the vicinity of the Moon was made by Russian scientists, and it was called **Luna 1**. On January 4, 1959, it came to be just a few thousand miles from the Moon's surface, and along its journey, it measured the "solar wind (of microscopic)" particles and transmitted this data back to scientists on Earth. It did not go into orbit about the Moon. **Luna 2** impacted the Moon on September 14, 1959, and without orbiting it first. The scientific instruments on Luna 1 indicated that the Moon did not have a magnetic field. Luna 2 had scientific instruments to measure magnetic field strengths and radiation levels during its journey to the Moon, and like Luna 1, they indicated that the Moon has a very low, insignificant magnetic field and strength about it. Due to the weight of all the other scientific instruments, Luna 2 was not equipped with a photograph system. **Luna 3** was the first to photograph the Moon's unseen, "far side" on October 7, 1959. These photographs had low resolution due to the (limited) photographic and digital technology available at those times for satellites, but they did show some prominent features (the larger mares and craters), and were therefore very important to all of mankind who have never been able to see the far side of the Moon until then. This mission was during a "new moon" phase, and where the "near side" of the Moon that we always see was dark (ie., not illuminated by the Sun), and its far side is then fully lit by the Sun when the Moon's orbit position is generally directly inline to, and between Earth and the Sun. It could also be said that the Moon's far side is periodically lit or illuminated by the Sun, and just as much as the near side is, and as the Moon rotates on its polar axis during a Moon day of time. After taking the photographs during a half-orbit of the Moon, the Moon's gravity was then used to ("gravity assist") cause, direct or steer Luna 3 back towards the vicinity of the Earth, and so as the stored Moon images could be electronically transmitted to the relatively close receiving antennas on Earth. This advanced level of flight control or orbital maneuvering was also a great first accomplishment for rocketry and space research.

Two months after Luna 1 was launched by Russia, United States scientists attempted lunar photography with a Moon "flyby" (ie., just passing by, close to, and then continuing onward in its journey such as a Solar (Sun) orbit, or perhaps out of the solar system such as for some modern spacecraft) on March 4, 1959 with the **Pioneer 4** spacecraft, but no photos were taken due to a system failure. Later, the United States launched the Ranger 3 spacecraft, however it only made a successful Moon arrival and then impacted (ie., collided, crashed) upon the Moon's surface on April 26, 1962, and without transmitting any data due to system failures. **Ranger 7,** from the United States, was much more successful, taking the first good resolution photographs of the Moon as it approached and impacted the Moon's surface on July 31, 1964. On February 3, 1966, **Luna 9**, from Russia, was the first lander to successfully land on the Moon without crashing into its surface, and it also took some photographs ("photos") and transmitted them back to Earth via (data) radio communication. **Luna 10 became the first man made, artificial satellite to orbit the moon on April 3, 1966**. It made 460 orbits and mainly measured solar radiation levels, but it did not have a camera system to take photographs and transmit them back to Earth. **Surveyor 1**, from the United States, landed on the Moon on June 2, 1966 and took many photos and transmitted them back to Earth. It is of note that Surveyor 1 went directly to the Moon without orbiting the Earth first. Parts of Surveyor 3 which landed on the Moon on April 20, 1967 were brought back to Earth for analysis during the Apollo 12 mission which landed very close (~ 200 m) to it on November 24, 1969. **Lunar Orbiter 1**, from the United States, was the first satellite to photograph the Moon while in orbit about it, starting on August 18, 1966. A total of 5 Lunar Orbiters were launched at later dates up to late 1967. Depending on the crafts altitude, these good to high resolution photographs would be used to locate possible landing sites for the Apollo program that would land people on the Moon in the late 1960's and early 1970's so as to gather a variety of rock samples, perform scientific experiments and measurements, and to take high resolution photographs of the surface. The image resolution varied throughout the program, orbiter and orbit altitude, and each image pixel represented (ie., actual image resolution, resolveability) a range from about 6 feet to 200 feet of distance of the lunar surface.

Before about 1970, many of the missions to the Moon by Russia, and also the United States were of very limited success and-or were often considered as failures, but they were also leaning experiences which various types of knowledge was gained and used to improve future missions. As the years went by, technology greatly improved, and this also allowed a greater probability of a mission success and-or science data returned.

The main rocket used for the Apollo Moon missions was called the Saturn-5. This rocket was usually composed of 3 rocket-stages or segments, and it carried liquid fuel. It was about 360 ft. tall, had a diameter of 33 ft., and weighted about 6.4 million pounds. The first or lower and most powerful stage of this rocket had a thrust of about 7.7 million pounds of force, and this resulted in a net initial available thrust force of: (7.7Mlb - 6.4Mlb) = 1.3Mlb. As fuel is used up in the rocket engines, the available net thrust force available increases. To electrically power the main Apollo rocket, command craft to the moon, a hydrogen and oxygen fuel-cell (a type of galvanic battery with a liquid electrolyte) was used to create electrical power (voltage and current, Pw = Vv Ia) with a typical conversion efficiency of about 60% when properly designed and made, and the landing craft to the Moon surface used a battery. Note that solar panels to generate electricity for these crafts was possible at the time, but it would also create some extra difficulties (size, aiming, deployment, etc) for these specific missions. A total of 24 people were sent to the vicinity and-or an orbit of the Moon. The Apollo-8, Apollo-10, and Apollo-13 spacecraft went close to the Moon vicinity, but did not land on the Moon. Each Apollo, Moon landing mission had 3 crew members, and the smaller landing craft contained 2 of those same crew members. After 6 Moon landings, a total of 12 people have walked on the Moon as of the writing of the book in 2024. These missions were very costly and dangerous, and the results were valuable teamwork, science, technology, data, experience and a further understanding of the Moon, rockets and further spaceflights such as for Earth orbiting space stations and spacecraft to orbit and-or photograph other planets, the Sun and asteroids. Apollo-15, Apollo-16, and Apollo-17 missions included an electric, four wheeled vehicle for two astronaut passengers, and it was called the **Rover**, and so as to easily travel up to a few miles away from the **Lunar Lander** to collect and return a larger variety of Moon rocks, etc. The rover also reduced the problem of accurately landing very close to the desired location.

As of the year 2022, scientific spacecraft have reached all of the known planets in the solar system, including Pluto and some asteroids, and much precious data and photographs were collected and sent back to Earth via radio communication signals. At these distances from Earth, the received signals are very weak and just slightly greater than the strength of all the ("background", natural, man-made) radio noise from space and the Earth. Much of this noise is effectively eliminated by using radio antennas that have a parabolic shape, and of which will then focus and highly concentrate the weak incoming radio signals from a relatively small area and-or direction in the sky and space. These collected, reception antenna signals will then be electronically amplified and (noise, signal) filtered to have most of the unwanted noise or bad data removed so as to record and study the desired signal and data (ex. image, photographic) contained within it more effectively and accurately.

Between 1970 and 1976, Russia successfully landed several unmanned crafts (Luna 16, Luna 20, and Luna 24) to land on the Moon, take photographs and bring collected samples back to Earth. **Japan** launched it's first lunar orbiter (Hiten) in 1990. The **European Space Agency (ESA)** launched its first lunar orbiter (Smart 1) in 2004. **China** launched its first lunar orbiter (Change 1) in 2007. **India** launched its first lunar orbiter (Chandrayaan 1) in 2008. **Korea** launched its first lunar orbiter (Danuri) in 2022 with the assistance of the United States. As of the current writing of this book (in 2025), United States, Europe (European Space Agency), Russia, China, and India have sent some (unmanned) landers to the Moon's surface. Many other countries have sometimes given some technological assistance to many of these countries mentioned and so as to be a part of those space missions. Very few countries have a fully capable space program and-or need, but with various types of cooperation, many countries can be indirectly involved.

Space debris (ie., junk, garbage, pieces, parts) from rockets and satellites is dangerous and has become an increasing safety concern due to the damage that it can cause to other space objects such as satellites, spacecraft, and astronauts because it has a high amount of kinetic energy (KE), mostly due to its high orbital velocity (ie., speed), and that energy can be converted to large values of force when a collision happens, and which can do much damage.

> Glass has a density of about 2.5g / cm^3 = 2.5g typical / 1000 mm^3. A 1mm^3 = (0.001cm)^3 piece of glass or sand (silicon-dioxide) has a mass that can be calculated by dividing both the numerator and denominator of this fraction by 1000, and which results in a mass and volume ratio of approximately: 0.0025g / 1 mm^3 = 2.5mg / 1mm^3 . 0.0025g = 0.0000025kg. If this piece of mass is moving at say a (ISS orbit) speed of 28000 km/h = 28000 km / 3600 s = $7.7\underline{8}$ km / 1 s =~ 7778 m / s , its kinetic energy is: KE = mv^2 / 2 joules = (0.0000025 **kg**)(7778 **m / s**)**^2** / 2 joules =~ $7\underline{6}$ joules of energy. If this is set equal to GPE using 1kg mass: KE = GPE = 76J = mv^2 / 2 = mgh = (weight kg)(height) = (1kg)(9.81m/s^2)(h), the height (h) will calculate to that of being 7.76 meters, hence that 1mm^3, 0.0000025kg = 0.0025g = 2.5mg piece of glass moving at 7778m/s has the same amount of energy as 1kg raised or dropped from a height of 7.76 meters and-or a 1kg mass moving at a velocity of. v = square-root of [2 KE j / m kg] m/s = sqrt[2 (76 J) (1kg)] m/s = 12.33 m/s =~ 40.45 ft/s

Some noted first successful flights and missions:

Launched in late 1964 in USA, and reaching closest to Mars on July 15, 1965, the **Mariner 4** space-probe (research spacecraft) was the first human-made craft to arrive nearby to, and (crude, low tech and resolution) photograph Mars. These images showed Mars as having an ocean-less, cratered surface. Mariner 4 came within about 6000 miles of Mars, but did not orbit it, but simply flew past Mars and went into a large elliptical orbit about the Sun. For electrical power, such as for the camera and radio communication with Earth, it utilized solar-panels and a battery. Data was saved onto a (magnetic) tape recorder, and it took about 9 days to transmit the initial volume (about 20, low resolution photos) of data back to Earth since the (digital, binary) signals used a low bit rate (data or bits / time), and which does help reduce some possible communication and-or (send and-or read) data errors. On November 14, 1971, **Mariner 9** became the first spacecraft to orbit Mars.

Mars 2 and **Mars 3** missions by Russia were the first to attempt a landing on Mars. Mars 2 crashed on Mars, and it is thought that Mars 3 landed safely on Dec 2, 1971, but the imagery was not functioning properly. Nonetheless, their main mission, lander platforms were orbiting Mars as artificial satellites which sent many valuable photos back to Earth.

Mariner 10 was the first spacecraft to flyby and photograph planet Venus, and on February 5, 1974, and later, it took some photographs of planet Mercury.

Venera 7, from Russia**,** was the first spacecraft to safely land on Venus and provide some surface measurement data on December 15, 1970, and **Venera 9** was the first spacecraft to orbit, photograph and land on Venus, and starting on October 22, 1975. The lander part of Venera 9 was the first to (briefly) photograph the surface of Venus. Venus is very bright when viewed from the Earth, and generally is seen before sunrise or after sunset since it is relatively close to the Sun like Mercury which is the closest planet to the Sun. Venera 1, in 1961, paved and-or proved the way for many types of deep space travel and exploration to the other planets, however it only came close to Venus. Venera 3 did make it to the surface of Venus, but it crashed landed in 1966. The atmosphere of Venus is much thicker and denser than that of Earth due to having a high percentage (> 90%) of carbon dioxide from volcanoes. The temperature (about 860 °F) and pressure (about 1200 psi) at surface of Venus is very high and this environment quickly degrades the condition and abilities of any landing craft to gather and transmit data before it is incapable. With much thermal radiation from the Sun and having a thick atmosphere, Venus is said to be like a very hot greenhouse.

Pioneer 10, launched March 3, 1972 in the USA**,** became the first craft to flyby (at about 80000 miles) Jupiter on December 3, 1973, and of which it obtained good resolution photographs. The flyby of Jupiter and its gravity attraction also increased the speed of Pioneer 10. **Pioneer 11**, launched on April 5, 1973, also made it to Jupiter on December 2, 1974 and used what is called a "gravity assist" (ie., to change speed and-or direction) of Jupiter (or any other planet, etc.) so as to direct its course to the more distant planet Saturn. Pioneer 11 reached Saturn on August 8, 1979 and took good photographs. Pioneer 10 and Pioneer 11 both continued on a path out of the solar system. By 1979, and due to their electrical power being depleted, communication from these spacecraft has ceased.

Viking 1, and then **Viking 2** soon afterwards, were the first spacecraft to successfully land on Mars and take many photos and obtain other data. The orbiter part of the spacecraft and mission also took many photographs of Mars. Electricity for each lander was provided by an on-board, thermometric electricity generator which converted the heat energy from a radioactive substance into electrical energy. Viking 1, was launched in USA on August 20, 1975, and landed on July 20, 1976. Viking 2, launched in USA on September 9, 1975, and landed on September 3, 1976. Since then, throughout the years, several other landers and-or mobile rovers on wheels have traveled and searched Mars for several miles and have gathered many high resolution photographs and much scientific data.

The Magellan probe, from the USA, produced good resolution (about 100m per image pixel) radar images of the surface of Venus after it began orbiting on August 10, 1990, and gathered images until October 1994.

Some other spacecraft you may research: Voyager 1 (Jupiter in 1979, and Saturn 1980, flybys [non-orbit] and-or for gravity assist]), **Voyager 2** (Jupiter in 1979, Saturn in 1981, Uranus in 1986, Neptune in 1989, flybys). By the year 2023, communication from these two very distant spacecraft has been said as becoming difficult. Being so far from the Earth, the photographs of the stars at this distance from Earth can then help determine the distance to them by using what is called **parallax** which basically can show the apparent, greater movement of closer or nearby stars among the more distant stars in background (ie., farther away). By comparing two images or photographs separated by a large distance makes this use of the parallax method possible, and then trigonometry can be used to estimate the distance to those closer stars. Telescopes on Earth can also use the parallax created when Earth is on opposite sides of its orbit about the Sun - an average distance between both locations is twice the radius distance from the Sun, hence (2)(93Mmi) = 186Mmi.

Clementine, from the USA, Orbited the Moon between 1994-1995 before it malfunctioned, and it made medium-resolution Moon photography of the surface of the Moon, and with an average of about 12 meter =~ 36ft surface per pixel image resolution, and gathered other science data. **Galileo** (Jupiter), **Cassini-Huygens** (Saturn), **Messenger** (began orbiting Mercury on March 2011), **Dawn** (orbited Vesta in 2011, and began orbiting Ceres since 2015; both are proto-planets, however Ceres is a round sphere), **New Horizons** (Jupiter, **Pluto** on July 14, 2015, with a world celebrated flyby and photographs, and since not having enough fuel to reduce speed for an orbit of Pluto), **Lunar Reconnaissance Orbiter** (LRO, Launched in June 2019 by the U.SA, and into a polar orbit around the Moon for high resolution photography and science. The highest image resolution of its digital camera is about 20 inches =~ 1.67 ft. in length or width, hence an area of 20 in^2 = 400 in^2 =~ 2.78 ft^2, hence things smaller and-or details of this area of larger things cannot be seen in the images. LRO has imaged all six of the Apollo landing sites and craft. **BepiColombo** (Mercury, EU and Japan, is scheduled to orbit Mercury in 2025). China space stations: **Tiangong-1** ("**Heavenly-Place**" in English translation, 2011-2016, reentry 2018), and **Tiagong-2** (2016-2017, reentry 2019), and **Tiagong-3** (Launched in 2021) which has an average altitude of 250 miles above Earths surface, and has an orbit inclination of about 42 degrees. Russia space station: **Mir** (1986-2000, reentry 2001). The Space Shuttle has docked (temporarily arrived and connected to) with Mir on several occasions.

Most rocket and-or satellites are launched into a westward to eastward orbit of travel, and of which may also be at an incline or angle with respect to the equator plane of orbit and-or the solar plane. This is especially so since many launch sites are not located at the equator region, and even if they were located there, they may still have a deliberately planned, inclined orbit. It is of note that an orbit cannot be just at a certain latitude about Earth, except for a geostationary orbit which is essentially located at 0° latitude with respect to the Earth. Since the seasonal tilt of Earth is an illusion and-or apparent only, the location of objects in orbit will not be affected by this apparent tilt.

Since the Earth is spinning westward to eastward, an object on its surface effectively has a horizontal velocity toward the eastward direction. We do not notice this velocity due to relative motion and locations of us and other objects on Earth's surface that are also going at the same speed or velocity, and appear to as going at a speed of 0. This horizontal speed will still be present in a rocket heading to space, and so it is beneficial to launch in the westward to eastward direction so as to utilize that initial velocity and-or kinetic energy.

It is possible to have a "backwards" or **retrograde** orbit of travel about the Earth, and where an object travels from the eastward to westward direction about Earth. It is also possible to have a **polar orbit** where the object travels northward to southward during one half of the orbit, and then southward to northward on the other half of the orbit about Earth.

The unaided eye without binoculars or a telescope is capable of seeing satellites travel across the sky as a small dot of light, much like a moving star, and usually in a westward to eastward direction. The light seen is due to the reflection of sunlight from its surface, and when permitting due to the geometry of the Sun, satellite and observer need to be in favorable alignment without the obstacle of the Earths surface, daylight. various "light pollution" issues, and-or cloudy weather at night.

The Basics Of Rocket Science

Rockets can be powered by a variety of fuels, and its rocket engine will be designed to utilize one and-or a mixture of fuels. Many rockets to space use liquid propellants (fuels that propel [move] the rocket). For safety reasons, this fuel is often put into the rocket's fuel tanks (ie., containers, usually circular or cylinder to help distribute internal pressures) inside the rocket, and just before launch (ie., "blastoff", rocket engine ignition, and upward travel). Much of the fuel of the rocket is not actually used to lift the relatively small and-or lite-weight payload into space, but it is rather needed to lift all the weight of the fuel needed to get the payload to a high altitude and high velocity (perhaps 5 miles per second) orbit. It is often said that the fuel combustion in a rocket engine is a "controlled explosion" (ie., a very quick combustion and-or a fast chemical reaction and "burning", creating gas particles with high kinetic energy and heat energy). Here the word "controlled" generally means a slower, manageable combustion and-or explosion so as to reduce heat and vibration. For combustion to occur, oxygen is needed, and this is often supplied by an internal tank of liquid oxygen.

A typical rocket fuel may have a combustion efficiency of about 75%, that is, it will convert 75% of its potential or chemical energy to hot gases (particles with a very small mass) with a high amount of kinetic energy. When the fuel combusts (here, a "controlled" or "slower explosion", as mentioned), hot gasses travel outward at high speed in a spherical direction. About 50% of the hot gas force or pressure created will be used to lift the rocket upward with its total combined kinetic energy of the gas molecules, while the other 50% is simply wasted in the opposite or downward direction of the rocket. It is also of note that the pressure upon the gasses is also less at the open "bell" or cone shaped end of the rocket. With this basic rocket system analysis, the efficiency of its engine system is: (75%)(50%) = (0.75)(0.5) = 0.375, hence about 37.5% of the rocket fuel energy is actually being applied to lift the weight of the rocket. Much fuel will be used just to overcome the inertia of the heavy mass and-or weight of the rocket and begin to move it. Drag forces, much like friction, particularly in the denser lower atmosphere, will effectively reduce fuel efficiency. It is often said that at orbit altitude and its required velocity, that the overall or total efficiency of a rocket fuel and engine system is very low, perhaps 1%. A theoretical energy efficiency of a rocket fuel system could be defined and calculated as equal to:

Total rocket fuel and engine efficiency of a rocket =

$$\frac{\text{(kinetic energy + gravitational potential energy of the mass in orbit)}}{\text{(total fuel combustion energy before launch)}} = \frac{\text{E out}}{\text{E in}} \quad : \text{E out} = (\text{E in} - \text{E losses})$$

For a rocket engine to move the rocket and-or its weight vertically, it must overcome that weight (a force) by a greater force so as there is a net difference in force in the forward direction. The lift force created by the rocket engine is formally called **thrust** (or forward force), and this is similar to the lift force of a plane that causes it to go higher.

lift force = rocket engine combustion vertical force - weight of the rocket :the effective lift applied to the rocket
: lift force = vertical or upward force.
: Note, weight = a downward force

Using these values in a different mathematical expression, we will have a (vertical) **thrust to weight ratio**, and for the rocket to move upward, this ratio must be greater than 1, and the higher this value is, the better. The thrust force must be greater than the weight (a force in the opposite direction) of the rocket for it to move upward. A higher engine thrust force and-or (net) lift force also means that the rocket will have a higher acceleration (a = (f / m) = (change in v / change in time)) which leads to a higher velocity (v = at), and therefore, more distance (d = vt) per unit of time.

In theory, even if the lift force was just 1 pound or 1N of force, perhaps because of the rocket being very massive and-or weighty, the rocket would still move vertically upward, but it would be very slow at first due to the relatively low amount of the net, resulting or effective lift force applied. A low effective lift force creates a low amount of rocket acceleration (a = f / m), and therefore a low rocket speed (velocity = at), and the rocket would probably "run out" (ie., "use up", deplete, drain, not have any left to use) of fuel well before it reached orbit if it did not have enough fuel. From these equations: f=ma , a=f/m , and if the force and-or energy applied low, the corresponding acceleration and will be low. It takes more energy (ie., fuel) and-or work to lift (ie., f=ma, and work = fd = mad with units of joules) a more heavier (due to more mass) object a unit or certain distance. The longer a force is applied to the rocket, the greater the KE (mv^2 / 2) gained by it, the greater its acceleration (a = dv / dt) and current velocity (v). When the force applied stops, the acceleration will stop and the velocity of the object will remain the same until some force is applied to the object again.

rocket engine vertical thrust force   > 1   is needed for the rocket to move or be lifted upward or vertically
rocket weight   : Thrust to weight ratio   ,   weight = a downward, vertical force

A rocket will launch (begin to move vertically) when its fuel is combusted in the rocket nozzle or combustion chamber, and the net sum of the force from each atom and-or molecule (billions of them) with their high kinetic energy that strikes or collides with the bottom of the rocket will be used to created the net (sum, total, resulting) vertical lift or thrust force. After some amount of time, the same combustion or thrust force of the rocket engine will then be used to produce both a vertical or lift force so as to go both higher and also provide more horizontal force so as the rocket can reach a high horizontal velocity needed to ("free-fall") orbit around the Earth.
.
In place of the term: (rocket engine thrust), the term (rocket engine vertical force) can be used, particular when it is rising up into the atmosphere vertically. In place of the term (rocket weight), the term: (gravity induced force on the rocket mass) can be used. For the rocket to move upward or vertically, the upward or lift force must be greater than the downward force of gravity which is the weight of the rocket, hence their difference must be greater than 0. These forces (mathematically equal to f=ma) will cause a corresponding acceleration of the mass such as a rocket, and the vertical acceleration must be greater than the downward acceleration due to gravity. so as the rocket can have a net vertical acceleration and move upward. In other words, the upward force applied to the rocket must be greater than the downward force or weight of the rocket, and for that rocket to move upward or forward into the air. If an object, perhaps a plane, was at a constant altitude, the upward force would then equal the downward force of gravity which is the weight of the plane. The (instantaneous and-or constant) lift force (f=weight=ma) and acceleration (a = f/m = weight/m) generated by the wings must equal the downward acceleration or fall of the plane due to the force of gravity affecting it, and for it to then remain at the same height or altitude. Regardless of the mass and-or weight of the object, the gravitational acceleration in the downward direction is a constant of -9.8m/s^2 near the Earth's surface.

As the rocket depletes (ie., uses up, reduces) its limited amount of fuel, the mass and weight of the rocket will get lower, and the thrust to weight ratio will then be increasing, and this improves the efficiency of the fuel energy. The rocket will then accelerate more and go faster since the same combustion force is now applied to a smaller mass and-or weight. From F=ma we have: a = F / m = (change in velocity) / (change in time). When (m) gets smaller, (a) gets larger when the force (f) is constant. Some rockets may have an additional stage (additional rocket section(s) placed on top of each other vertically), and the thrust to weight ratio may change, and mostly due to the next rocket stage in use utilizing different, perhaps fewer and-or smaller, less powerful rocket engines. The rocket stage(s) with the empty fuel tank are then quickly discarded as useless weight to not keep lifting and wasting energy. As the rocket climbs higher, there is less friction with the air and-or atmosphere because the air pressure and-or density is less at higher altitudes. This friction and-or force against the forward thrust of the rocket is sometimes called the rockets **dynamic pressure**, and at a certain point or altitude during the rocket flight, this friction or dynamic pressure force will be at maximum kinetic or dynamic pressure (often called as the "**Max-Q**" point), friction or drag due to the combination of the rocket's speed and the air pressure and-or density at that certain altitude. Surely the air pressure is relatively low at several miles high, but the rockets velocity is a high value when it is impacting the air molecules which will then gain some kinetic energy from the impact or friction from the rocket, and this causes a reduction of the kinetic energy (KE = mv^2 / 2) and-or speed of the rocket. The rocket must be designed to withstand this pressure and-or impact force at Max Q, and it may "throttle down" (ie., reduce its rocket engine output thrust force) for a few seconds to reduce this pressure encountered. Max-Q will occur at an average altitude (ie., height above the Earth's surface) of about 8 miles.

The pointed tip or "nose cone" of a rocket helps reduces air resistance (ie., friction) or drag, and the pressure (P tip = F fip / A tip) per unit area is effectively reduced. In short, the pointed tip creates a situation where the kinetic energy transfer from the rocket to the air gasses is less, and the impact with each gas particle is less since the collision is not a direct collision but rather at an angle directed away from the length of the rocket.

Though the final efficiency of the rocket fuel system may be as low as 1%, and the cost per rocket launch is relatively high, the social value is very useful, and therefore very valuable in itself, and often financially profitable, such as if the rocket successfully placed a communication and-or weather satellite into orbit.

How much thrust can a rocket engine make? To find this value, the rocket engine is usually tested on a fixed stand so as to check for flaws, stability, efficiency, and the force (ie., thrust) it can produce. The rocket engine is started and the amount of fuel used and the corresponding force it produces is measured and recorded, such as data points on a graph. This data will be used to determine things such as the acceleration to expect [ie., a = F / m = (engine thrust) / (mass of the rocket)] and the resulting velocity of a rocket to orbit, and for a resulting velocity of spacecraft in orbit. By knowing the acceleration (a) of a mass due to a specific engine, a specific and-or required amount of time to ignite a rocket engine to accelerate and change its speed or velocity can then be calculated and determined.

From: (a) = F/m = (change in velocity) / (change in time) , we have: (change in time) = (change in velocity) / (a=F/m)

Rocket engines can also be rated in general terms of how much thrust it can produce for a given amount or 1 unit of fuel such as a kilogram of its mass, or a liter of its volume. This rating is called the **specific impulse (ISP = Isp** , = Impulse Specific**)** of that particular rocket engine system. Different fuels can have different amounts of potential energy available, and this will then affect the specific impulse value and-or thrust force value of that rocket engine. The higher the amount of thrust produced per unit of fuel, say 1kg of fuel, the higher the ISP rating of that engine. Given two identical rocket engines and using the same type of fuel, the one that receives the most fuel will produce more thrust per instance of time, time unit or second, and will therefore have a higher ISP rating also. The higher the rate of using fuel, say kilograms/second, the more instantaneous force produced, but the less amount of time it will be available if the amount of fuel quickly reduces to 0 liters, gallons or kilograms of rocket fuel. The longer (ie., more time) a force can be created in the rocket engine and applied to transfer kinetic energy to the rocket, the more the object or rocket will accelerate and go faster in speed or velocity. For some extra reference, the units of **impulse** (I) are: Ns = Newton-Seconds = Ft = (force applied) (number of seconds the force is applied). If two rocket engines produce the same amount of force, even if they use different fuels and its usage rate (ie., amount of fuel used / time), the main variable for comparing these engines then becomes the number of seconds that the force can be applied to the rocket. Isp is like a measure of fuel combustion efficiency, and also a measure of thrust and momentum per unit of fuel.

Impulse = (force)(time) = F t = (Thrust)(time) = mat = mv : a change in momentum
Specific Impulse = **Isp** = impulse / weight of fuel = mv / mg = v / g = (m/s) / (m/s^2) = **s :s = seconds, a time unit**
Also , Isp = mv / mg = Ft / mg = (F / m) (t / g) = a t / g = v / g = s

Isp = thrust / propellant = thrust / 1 unit of propellant : as an equivalent fraction

Also note this ratio: impulse / mass = mv / m = v : the lower the mass, the greater the velocity given a certain value of kinetic energy

Ex. The engine in a certain rocket is said as having an ISP of 100s , this can be considered as:

1 unit of thrust / 1 unit of fuel = seconds = 1 kg of thrust for or from each 1 kg of fuel, for 100s of total time

For the rocket to rise into the air, the amount of thrust must be greater than the total rocket weight = mg due to the force of gravity (9.81 m/s^2) causing a deceleration, and which is constantly applied to the rocket, and it needs to be constantly overcome until it is into orbit and where the engine thrust is no longer needed. The effective or net thrust force (Fn) applied to the rocket is then: Fn = (thrust - weight of the rocket) , and if this value is low, a low force will cause a low acceleration (a = F / m = [change in v / change in t]), and then the average velocity = (v = d / t) = (at^2 / 2) / t = at / 2 = v / 2) will be low right after launching.

Most modern space rockets can reach a low Earth orbit (LEO) altitude of about 200 miles in about 9 minutes of time due to its high speed or velocity developed later in flight, and where the weight of the rocket fuel is much less than at launch, and the acceleration (a = F / m) is much greater. 9min = (9min)(60s/min) = 450s. Over time, air (a mixture of gas) particles (atoms, molecules) of Earth's upper atmosphere will collide with an orbiting space vehicle, and more so in low altitude orbits, and this friction or drag force will cause the space vehicle to loose some of its kinetic energy, decelerate slightly in the opposite direction, and therefore, reduce its forward orbital velocity. When the orbital velocity required for a specific orbit is reduced, the orbiting space vehicle will reduce in altitude more and more as time goes on. To then undo this situation, the space vehicle will occasionally need to use the energy and force from a rocket engine(s) to raise it higher to its intended altitude of orbit and velocity.

To "escape the gravity" of a planet would be to have an acceleration greater and in the opposite direction (ie., upward or outward from the surface, or radially from the center) than the local and constant, downward or inward gravitational force and acceleration it induces upon a mass such as a rocket. At the surface of Earth, and for low altitude (say, less than 200 miles) orbits about the Earth, gravity acceleration is about 9.8m/s^2. If a rocket engine can supply enough force to overcome the local gravity and weight of the rocket, it can eventually "escape the force of gravity" of Earth. As an object, mass or spacecraft orbits (ie., effectively tangent, parallel or horizontally to Earth at any instant) the Earth, it no longer needs the power of a rocket engine to overcome gravity so as to remain at that altitude. While in orbit, the object such as a spacecraft or satellite is in "free-fall" or "in a perpetual, endless falling" as the curvature of Earth below the spacecraft is also increasing its distance from the orbiting object by the same amount or rate that the spacecraft is falling downwards, and this effectively negates or "cancels" any change in altitude due to gravity, and the orbiting object will effectively remain at the same altitude. Nonetheless, Earth's gravity is only slightly lower in low altitude orbits, and is still constantly pulling an object downward, hence gravity has not actually been "shut-off" to 0m/s^2 during orbit and-or "free-fall".

A discussion of orbiting has been previously mentioned in this book, such as in the topic of Kepler's Laws of orbital motion.

A spacecraft on its way to the Moon from the Earth must take into account that the Moon is also moving as it orbits the Earth, and that the spacecraft must then be aimed not at the current Moon position, but at where the Moon will be at a later time. For the Apollo missions, it took about 3 days on average for the spacecraft to reach the current position of the Moon. The Moon will move or travel a distance and-or angle during that 3 days time and will be further in its orbit about the Earth. While traveling farther outwards from Earth, the spacecraft will still have some circular-like orbital motion about the Earth, and the direct line to the Moon is actually similar to that of a curved line or arc of a circle. While the spacecraft is traveling to the Moon, it is still under the influence of a low value of Earths gravity and force will therefore be constantly slowing it down in velocity. There is a location called a **gravitational neutral point** between the Earth and the orbiting Moon where the gravity (ie., gravitational acceleration) of each upon the spacecraft is the same value: Ge = Gm, and therefore Ge - Gm = 0 m/s^2. At the neutral point, and object with a mass will experience the same gravitational forces from the Earth and Moon: Fe = Fm. When the spacecraft is in the vicinity of the gravity of the Moon, it can then be gravitationally attracted to the Moon and can either go (ie., fall) directly to it or go into orbit about the Moon at the necessary velocity according to its desired orbit height (typically a value between 60 miles to 100 miles high) above the surface of the Moon.

To prevent the human body from experiencing health problems on long duration space travel, say greater than 1 month of time, methods to create **artificial gravity** in a spacecraft have been proposed. In 1952, Werner Vaun Braun proposed a large circular spacecraft (ie., a "space station") having a shape like that of a wheel, and where people and objects in the craft near the outside rim will experience artificial gravity. This craft would have to spin fast enough to achieve this effect. Another method in theory is if a large rocket ship having many floors and therefore much floor and living space, and also having enough fuel can then constantly accelerate at say a constant value of 9.8 m/s^2. Objects in the craft that would otherwise be floating as weightless (but not mass-less), would then be moved toward the floor(s), and they will then experience a force (ie., artificial gravity) upon them, much like how the entire rocket-ship is experiencing a force upon it from the rocket engine. Due to objects and people in the craft having some inertia or resistance to movement, accelerating at a high speed can cause much force and damage to them, and so therefore, the acceleration of the craft and resulting effective gravity should be gradually increased.

The formula for the escape velocity (Ve) from a gravitational, free-fall orbit is:

$$Ve = \sqrt{\frac{2GM}{r}}$$

: Formula for escape velocity (Ve)

G = Universal gravitational constant , and is not the same value as earths gravity (g).

M = a spherical-like or uniform mass being orbited by the rocket or satellite ,

r = radius or distance from the center of the mass being orbited, as (r) increases, the necessary escape velocity will decrease, and this is due to that the force of gravity is then lower. G m/s^2 , M in kilograms , (r) in meters. And:

Letting g = local gravitational acceleration at an altitude or (r) = $\frac{G M}{r^2}$, Ve can be expressed as: $Ve = \sqrt{2 g r}$ =~ $1.414 \sqrt{g r}$

When (r) increases, the effect of gravity upon the orbiting body is less, and it is easier for it to then escape from orbit, and its escape velocity will be lower. Just the same, for an object to go into an orbit about a mass, it must be going at a velocity less than the escape velocity. Another way to consider escaping the effects of gravity is to accelerate away from the mass being orbited by an amount equal to the local gravity at that altitude. At the surface of Earth (where r equals the radius of the Earth) and in a low (altitude) Earth orbit (LEO), if the rocket is accelerating at slightly greater than 9.8m/s^2, then it will eventually escape from Earths local gravity value near the surface. For higher or farther orbits, this value will be lower since the local gravitational acceleration there will a lower in value. Even though an object is said to have escaped the orbit of a body or mass, that object will always be slightly influenced by the gravity of that same mass and every other mass in the universe, and depending upon their mass and distance from the object according to the law of gravitation of two objects and-or masses. Even though the force of gravity may be very weak upon another object and-or mass, a constantly applied force such as gravity or a small amount of thrust from an engine will cause an acceleration and therefore a constant increase in velocity unless it is in a circular orbit about a mass such as a planet or moon. For orbiting conditions, the (inward) centripetal force due to gravity is equal in value to the (outward) centrifugal force of the object wanting to move away from its orbit as it would if the gravity ceased. While in a stable orbit, the net or effective force upon the orbiting mass is 0N: Fc = Fg , and Fc - Fg = 0 Newtons

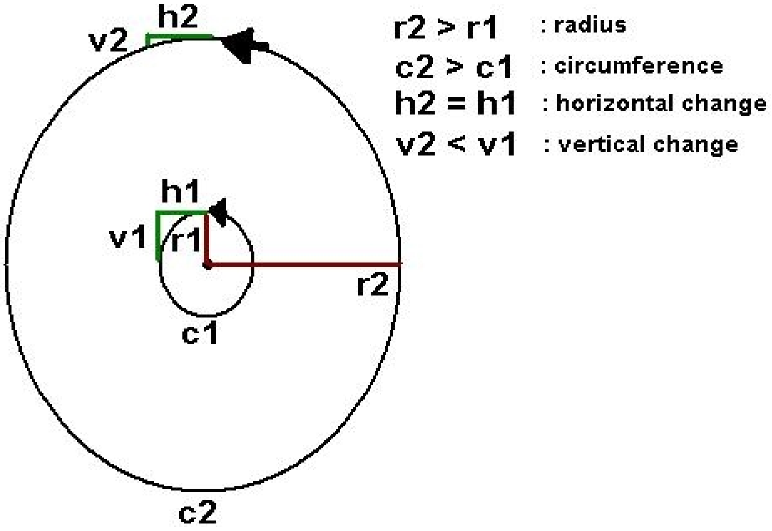
The ratio of the escape velocity to orbital velocity is about the square root of 2 =~ 1.414 , here is the verification:
ve^2 / vo^2 = (2 G M1 / r) / (G M1 / r) , after solving for ve , we have ve = $\sqrt{2}$ vo

Objects in farther natural orbits will travel slower.

An object at perihelion is at its farther distance in its elliptical orbit and it will be traveling slower. This discussion here will focus on natural circular orbits at farther orbit distances and the objects traveling slower. An example of a distant orbit is an object in geosynchronous orbit. Objects in a low altitude or radius from the center of Earth, say less than 1000 miles high, need to travel much faster and regardless of its mass it will be the same value, and this is a reason low Earth orbit objects of any mass need to be on a very fast rocket, and which will require much fuel to be that fast. Although objects in a father orbit will be traveling slower, they will still require much rocket fuel to get to that height (ie., having much GPE = Gravitational Potential Energy), altitude or radius of orbit.

Objects in farther circular orbits will have a lower amount of curvature in their orbit and the figure below shows a method to explain this. If the orbit is very distant and-or the radius is very large, the curvature of its path or orbit is nearly a straight line for a period of time, such as for a minute of time. Though this curvature is slight or low and has a corresponding small mathematical value, the actual radius of orbit and-or curvature associated with it is actually high in value, and this is an inverse type of mathematical relationship, and a pseudo expression for this is: amount of curvature per unit distance = 1 / radius of curvature. Just the same, the effective angle of inclination or declination of travel will be inversely related to the radius of orbit. For an orbit with a low amount of curvature, the angle of declination will be low as shown in the figure below. It could be said that a measure of curvature is the rate of change of the angle and-or height with respect to time or distance. An amount of curvature could also be considered as corresponding to the height of the curve or arc(s) with respect to its two endpoints and their (chord) distance apart. A pseudo expression for this would be: amount of curvature = (height of curve) / (arclength or distance between the arcs two endpoints), and this is similar to a pseudo expression for curvature: curvature = (change in vertical distance) / (change in horizontal distance) = slope of the line connecting the endpoints of the two lines or arc. Note however, with a circle, this slope is not actually a constant value for each same horizontal change, and that is why its a pseudo expression for curvature, but the figure below is enough to give an essence of understanding to the concept. An alternate idea for a pseudo expression for curvature would be the (shortest distance (horz. or vert.)) / (longest distance (horz. or vert)) per unit of time of travel and-or angle

traveled.
[FIG 249]



In the above figure: h = horizontal distance or change per unit of time, and v = vertical distance or change per unit of time. c = circumference of the circular orbit , and r = radius, height or altitude of the circular orbit.

We see that the orbit with c2 has a lower amount of curvature per horizontal distance and-or the amount of rotation angle about the center point of orbit. Observe that when h2 = h1, v2 is much less than v1 due to the farther orbit distance.

In the orbit c1, with the shortest radius, a curve or arc segment of the orbit or circumference is actually said to have a higher amount of curvature, and this actually means a higher mathematical amount of curvature with respect to the small radius value and-or arc length. Given the same arc length of any radius of orbit, if the corresponding radius is small, the radius of curvature is said to be high, and the curve may be thought of as a narrow radius, sharp and-or a large amount of bending of which could also generally mean quickly changing in direction with respect to distance and-or time.

The farther the radius of orbit, the less gravity force constantly being applied to and affecting the spacecraft and therefore, the slower it will be free-falling downward toward the Earth, and the less distance it will fall per unit of time and-or distance moved forward. For farther orbits or altitudes, both the vertical drop of the spacecraft in orbit and the downward curvature of Earth's surface are equivalent in distance, but here, it is a less distance fallen per unit of horizontal distance orbited. Due to this, a spacecraft in a more distant circular orbit will require and have a slower speed or velocity to maintain (ie., keep) that orbit. Since the velocity for farther distanced or radius, natural orbits is less or slower, it will take a longer time to make a complete orbit of the Earth, and this is sometimes called the (cyclic, repetitive, constant) **period of orbit**.

In short, the radius of curvature of an arc, segment or portion of a curve is defined as the radius of a circle having a circumference and-or arc segment of it that corresponds to it. There are various ways and-or instruments to measure the radius of curvature of an arc and-or sphere surface so as to find its corresponding radius and so on. One such tool is called a **spherometer**, and where 3 legs having the same height are set apart in a equilateral triangle formation and will define a flat plane which corresponds to a chord line and endpoints on it, and-or a chord plane, and a central depth gauge can be used to measure the maximum height of that arc from the plane.

If you have a curve segment or arc on a graph and draw lines perpendicular to it, you can see where those lines intersect and you can then locate and measure the distance to that point of intersection so as to have the approximate radius of curvature value, and in fact, this is also a method to find the center point location in a circle. There are also mathematical ways to calculate this radius value given both an arc segment length and the chord length beneath it and-or the height of the arc from the chord or distance line between its two endpoints, and of which is also called the **sagitta** of the curve segment.

To go to a larger orbit having a larger radius and-or to go to another planet.

For a spacecraft to go from a circular, low Earth orbit (LEO) to a higher circular orbit, and here, considered being on the same solar plane as that of Earth or perhaps another plane of orbit, the spacecraft must get a force from its rocket engines so as to do so. At a higher altitude of orbit, the spacecraft will have more gravitational potential energy (GPE). Rather than direct the rocket thrust to be radially away from the center of Earth for a higher orbit, the spacecraft can increase its velocity and effectively "overshoot" its regular orbit and increase its distance from the Earth beneath it, however, this will then put it in an elliptical orbit which must then be corrected if the spacecraft is desired to be in a circular orbit. This particular method to transfer from one orbit to another is formally known as a Hohmann Transfer Orbit. A Transfer Orbit is the formal name for the process and-or location between two stable orbits. It was conceived by **Walter Hohmann** (1880-1945), from Germany, in 1925. More recent discoveries in orbital mechanics such as using what is called a Low-Energy Transfer to change orbit require much less fuel, but will then require much more time to do so. This method is similar to a Hohmann Transfer Orbit, but the rocket uses several increasing elliptical orbits and their altitudes, and then uses some fuel to go into to a final or circular "parking" or stable orbit.

Since a particular radius and-or altitude has a particular natural orbit velocity, and regardless of the mass of the orbiting object, there is a particular natural time or period corresponding to that orbit.. Consider that even: Distance = (Velocity)(Time) = vt, and regardless of the mass. Time = Period of Orbit = Distance / Velocity = d/v , Velocity = Distance / Time. = v = d / t. As a reminder, consider two masses dropped from the same height in a airless vacuum, and they will both reach the ground at the same time, and this is another instance when the mass of the object(s) does not make a difference, and that both will fall in the same force of gravity causing an (gravitational) acceleration of:
32.2 ft / s^2 = 9.81 m/s^2 In the case of an orbiting spacecraft(s), whether in orbit or not, it will fall at a rate of acceleration due to the strength of the local gravity there, perhaps 9 m/s^2 , and regardless of its mass.

When a spacecraft is no longer traveling or going at the natural orbit velocity for that altitude or radius of orbit, its orbit will change to an elliptical one, and it will continue unless it is corrected to be circular. In a circular orbit, the gravitational force of Earth pulls the spacecraft downward in the vertical direction, and there is no horizontal gravitational value of force pulling it forward or essentially backwards when it is slowing it down. While in an elliptical orbit, the Earth's gravity is still mostly pulling it downward vertically, but the remainder of that force is pulling it in a horizontal direction so as to create that elliptical orbit, and as you can imagine, the closer it is to the Earth the greater the gravitational force and the faster (v) it will be traveling due to it accelerating by a constantly applied force (here, the increased local gravity) giving it more kinetic energy (KE = mv^2 / 2). The sum of the total orbiting energy which is the sum of the kinetic and (gravitational) potential energy of a spacecraft in orbit will remain the same constant value, although the actual values are constantly changing by a small amount each, and while one is increasing, the other is decreasing: Total spacecraft energy in orbit = KE + GPE = constant for a constant or stable orbit. When the spacecraft moves away from the Earth during its elliptical orbit path, it will begin to slow down due to the force of gravity still upon it and pulling it toward Earth. The spacecraft will loose speed and-or its total kinetic energy, but it will gain and store that amount of energy as gravitational potential energy (GPE) as it effectively moves to a higher altitude (ie., height) and-or distance in its orbit. The farther it is from Earth, the less the gravity of Earth upon it, but it will still be constantly applied to it and pulled toward the center of the Earth while on its (somewhat circular, elliptical) path or orbit.

According to Newtons's laws and verified by modern space travel, we know that an object in motion will continue in the same direction and speed unless affected by a force that will change them. An example of this to help simulate this on Earth would be a piece of ice first being pushed across a frozen puddle of water, hence ice, and without much surface friction involved. It can also go a great distance after an initial force is applied to it so as to give it some kinetic energy. Due to an initial surface friction, some extra force must be initially used to overcome it so as motion can start. The amount of force applied will then also determine its speed or velocity. The direction the force is applied will determine the direction it will take. If the force is constantly applied, the piece of ice and its speed will constantly accelerate in speed as more energy is transferred to it and it gains more KE. If it were not for the small amount of friction on the ice due to its weight as it travels, it would continue its motion in a straight (ie., like a line) direction. The force of friction upon both surfaces in contact is essentially in the opposite direction to its forward motion, and this force will reduce the kinetic energy (KE =

mv^2 / 2) of the moving piece of ice until it stops its motion. Space travel is similar to this piece of ice traveling on ice, except that there is no friction due to that nothing is in contact with its mass. Remember, the weight of an object is the downward force of an object, and this will effectively increase the actual amount of surface contact when friction is considered. Though most surfaces appear smooth, but when they are viewed under a microscope, they are rather jagged with mico-sized hills and valleys which can then interlock and slow movement when no further energy is input to maintain the kinetic energy of the object.

If a spacecraft was traveling in a forward direction and it used its left side engines so as to make a right turn so as to move rightward, that spacecraft will still have its forward component in its motion, and in fact, it will still have the same velocity in that forward direction, and will now also have a velocity in the rightward direction. To reduce and then remove or cancel out any of the original forward motion and-or direction, and just have the rightward direction of motion, the spacecraft must temporarily use a rocket engine so as to provide some thrust (ie., force) in the opposite direction to that forward motion so as to reduce and-or eventually cancel out any momentum or motion in that forward direction.

How much will an object on Earth weigh on the Moon?

First, any amount of mass on Earth or elsewhere, will still be the same amount on the Moon since the value of mass is universal. 1g of mass on Earth will still be 1g of mass or substance on the Moon. Since the Moon is geometrically smaller than Earth, and its mass is less than that of the Earth, its gravitational acceleration value (a=gm) will be less than the gravitational acceleration value (a=ge) of Earth. The Moon's local gravity is about **(1/6)** = 0.166667 = $16.\underline{7}$% that of the Earth, and an object will then weigh (1/6) on the Moon as it does on Earth. Force = Weight = ma. Just the same, an object on Earth would be said to weigh 6 times more than that on the Moon's surface. Here is the derivation of this value:

First, consider that there could be any other mass near the moon using Newton's Gravitational Force equation. Lets assign this other mass (M2) have a value of 1kg, and this will simplify Newton's equation so as to having just one mass, and that is for the mass of the Moon = M1. After all the Earth is very far from the Earth, and its local gravity upon the surface of the Moon is almost negligible. The height of the liquid (fluid, low density) sea or ocean tides on the surface of the Earth due to the Moon' gravity is a very small fraction, almost negligible, of Earth's radius. For (r), we will use the radius of the Moon.

Fgm = G M1 M2 / r^2 =~ (6.674)(10^-11) Nm^2/kg^2 (7.35)(10^22) kg / ((1738)(10^3)m)^2 =~ $1.62\underline{4}$ N / kg :
1N = (1kg)(1m/s^2) , 1.624N / kg = (1.624)(1kg)(1m/s^2) / kg = **1.624 m / s^2** : slightly more than (1/6) of Earths

A scale calibrated to Earth's gravity, will need to be re-calibrated (ie., correctly set, adjusted) on the Moon since for example, a 1 gram mass will still need to be displayed as 1 gram on the Moon when using a common weight-to-mass scale. The 1 gram of mass will weigh (ie., a force) less on the Moon, specifically (1/6) as that of its Earth's weight, but the scale will need to be display the same amount of mass of 1 gram. To do this, for example, the gram value using this scale would need to be multiplied by 6, either internally or by the user. Weight = W = F = m a , therefore, m = W / a . We see that if (a) or (W) is changed by a factor of (n), that the other will automatically change by that same factor, hence keeping the equivalent fraction and same mass value: m = W / a = W / g = (n) W / (n) g.

$$gm = \frac{ge}{6} = (1/6)\ ge \quad , \quad \text{mathematically:} \quad ge = 6\ gm$$

Force = Weight = (mass)(a) = (mass)(g) : mass = Weight / a

Gravity on the Earth: Weight of a mass on Earth = m ge = (mass) (9.81 m/s^2)

Gravity on the Moon: Weight of a mass on Moon = m gm = m (0.166667) ge = (mass) (0.166667) 9.81 m/s^2 =
= **(mass) ($1.6333\underline{3}$ m/s^2)**

Ex. A 60 lb weight on the Earth will weigh: 60 lb / 6 = 10 lb on the Moon.

How much will a 1 kilogram of mass on Earth weigh on Moon?

Weight on Moon = (1 kg) (1.63333 m/s^2) = 1.63333 N , and a weight-to-corresponding-mass scale on the Moon should be calibrated to display the mass as:

From F = weight = (mass)(acceleration) = ma , m = F / a = 1.63333 N / 1.63333 m/s^2 = 1 kg

On the surface of Earth, 1 kg of mass will weigh (ie, a force due to gravity upon a mass) :
(1kg)(9.81 m/s^2) = 9.81 N =~ 10 N , extra: 9.81 N on Earth / 1.63 N on Moon =~ 6

Extra: A ball having a certain mass and-or corresponding weight is thrown at the same force and angle into the sky from both the surface of the Earth and the surface of the Moon. Since the Moon has (1/6) less gravity or downward force, the ball will travel 6 times farther on the Moon than at the surface of the Earth, and before reaching the ground due to the gravity of the Moon.. That is, the ball will also take 6 times longer in time to land on the surface of the Moon. The horizontal velocity of the ball is still the same at both the Earth and the Moon. The vertical velocity just before impacting the surface of the Moon will also be 6 times as less. The force of the impact of an object dropped vertically on the Moon will be 6 times less than that on the Earth when dropped from the same height. F=ma. For the ball to go the farthest horizontally and vertically, the throwing or thrust angle should be 45°, and where the vertical and horizontal velocity components and-or vectors are both equal in value. If the angle is higher, the horizontal distance of travel would be less, and if it was 90° perpendicular or vertical to the horizontal surface, the horizontal distance of travel would be 0. If the angle is less than 45°, the time until it reaches the surface will then be less and it will not travel any further horizontally.

Extra: The gravity on **Mars**: At the surface of Mars, the local gravity is 3.71 m/s^2, and this is about 37.8%, roughly 38% that of Earth's gravity value at it's surface. This value is slightly more than twice that on the surface of the Moon.

The ratio of the gravity on the Mars with respect or in reference to Moon is:
(3.71 m/s^2 on Mars / 1.6333 m/s^2 on Moon) =~ 2.28 time more

Extra: When two masses are small, if the radius or diameter of one increase by a small value, the change in gravity might be considered as negligible. Now consider two planet sized masses, if the radius of one increases by a small value, say 1 foot, this is actually 1 foot over its entire surface area, and the extra volume of matter and-or mass is then very large in value and has actually increased exponentially (consider Area = s^2, and Volume = s^3). When the mass increases by a factor of (n), the force of its gravity will increase by a factor of (n^2). In other words, the force of gravity changes exponentially when mass changes linearly.

Standard Air Pressure And Barometer Values Derivation

When climbers climbed high mountains in the past and even today, they will quickly find that it is more difficult to breath after being just a few thousand feet high, and even after resting there. This is due to that there is just not enough oxygen in each breath, and that they needed more physical effort to even breath in due to the less air pressure which helps inflate the lungs. With less oxygen in each breath, it is a natural reaction to then breather deeper so as to get the oxygen the body needs. They eventually realized that there was a less amount of air at higher altitudes and more air at lower altitudes, and deduced that the air density and-or pressure is greater at lower altitudes, and less at higher altitudes. This is very similar to the anciently known fact that water pressure is likewise greater at the deeper (ie., lower) depths in the water, however, we now know in more modern times that water, being a liquid and much denser than air gas, generally cannot be compressed any further so as for it to be any denser on the atomic or (H2O) molecule level.

Without knowing the specific value of air pressure, it was notice that in a sealed vertical tube, perhaps 35 feet high or higher, that the column of water placed in an open (ie., no seal, no lid) container of water at the bottom end would then always remain at about **33.8ft** high. This is due to the local air pressure having the force (ie,, pressure) to keep that amount of water suspended and not let it drain out of the tube. That is, the air pressure in the local atmosphere and upon the water surface in the container has enough force upon it so as to keep that amount of water from draining out and-or to keep it suspended to that height in the vertical tube. When the force from the air pressure upon the water equals the weight of the water, there is a force and-or pressure are then in balance and the water remaining in the tube will not drain or move out. This volume of water in the vertical tube will calculate to weigh and-or have a pressure of **14.7 pounds per square inch** (14.7psi) at sea level (ie., average sea water level), and which is the a reference height or altitude considered as 0ft. When the air pressure changes, such as due to a local rain storm, the pressure upon the surface of the water in the container will change, and the level of the water in the vertical tube will then change.

It was later found that if the altitude of this **barometer or** air-pressure instrument, increased, then the air pressure decreased. This indicated that air had weight (ie., force and which can cause a pressure upon a surface, P=F/A), and it is also spoken as the vertical column of air had weight (ie., force due to gravity) per unit of area of the surface and air. It was found that air is denser near sea level and thinner (less) at higher altitudes. As mentioned previously, this was noticed by mountain climbers who have difficulty breathing at higher altitudes. It is recommended for hikers and-or travelers of poor health to not increase their elevation by greater than 1 mile in a short amount of time (say less than a few hours hiking, or less than about 25 minutes if traveling in a vehicle) due to the air pressure change and lower amount of oxygen there which can take some time to help try to adjust to while temporarily (say two hours or less, depending on their health) being at that altitude.

Air or air-gas, or any other gas for that matter, density is proportional to the local air pressure, and inversely proportional to the local air temperature. A simplified expression of these mathematical relationships: gas density = air pressure / air temperature.

At the top of Mt. Everest where the air pressure is about 33% of that at sea-level, and the density of air at the top is also about 33% of that of sea-level. Depending on their health and-or altitude, anyone climbing and-or hiking any mountain may require an supplemental oxygen tank with or near to them. It is recommended for inexperienced climbers, walkers, hikers or other physical activity, to first get accustomed to that activity(s) over an amount of time (days, weeks, years) and experiences, and so as to be more physically and psychologically fit, and to prevent health and safety issues.

The derivation of the standard or reference value of air pressure:

1 cubic inch of water = 0.554 fluid ounces = 0.554 fl.oz : a volume measurement
1 cubic inch of water = 1 in^3 of water weighs 0.578 wt-oz = 0.578 oz = 0.036127 lb : as weight and-or equivalent dry ounces of weight

1 cubic centimeter of liquid water (hydrogen and oxygen molecules) = 1 cm^3 is defined as having a total mass of 1g of mass and-or, 1 gram-weight = 1 gram of force at Earth's surface gravity force level.

$\frac{14.7\text{ lb}}{1}\left(\frac{16\text{ oz}}{1\text{ lb}}\right) = 235.2\text{ oz}$: this is the weight of 14.7 pounds of water

$\frac{235.2\text{ oz}}{\frac{0.578\text{ oz}}{\text{in}^3}} = 406.92\text{ in}^3$: this is how many cubic inches in 235.2 oz of water = 14.7 lbs of water

If each of these cubes having a height of 1 in each, was stacked vertically, 12 of them would make a foot high. Here is how many feet high that 406.92 in^3 would be:

$\frac{406.92\text{ in}^3}{\left(\frac{12\text{ in}^3}{1\text{ft}}\right)} = 33.91\text{ feet}$: the height of a vertical column of water at sea level that will cause or induce a pressure of 14.7psi at the bottom of it due to the force of gravity, and this will also be held back and-or suspended at that height in the tube due to the external air pressure of equal value and-or force per unit area.

1 gallon = 1 gal = 128 fl.oz. : a gallon is a volume measurement, and not a weight measurement

Considering that 1 fl.oz volume of water roughly weighs 1 oz = 1 dry ounce of weight, and this is how the British (England) people first defined and used the word "ounces" for both a volume (ie., "fluid volume) and weight a measurement:

128 fluid ounces of water weighs about 128 dry ounces = 128 oz , or that:
128 dry or weight ounces of a substance corresponds to about 128 fluid or weight ounces of water:

$\frac{235.2\text{ oz}}{\left(\frac{128\text{ oz}}{1\text{ gal}}\right)}$ or= $\left(\frac{235.2\text{ oz}}{1}\right)\left(\frac{1\text{ gal}}{128\text{ oz}}\right) = 1.8375\text{ gal}$: 1.8375 gal weighs **14.7 lbs**
Standard Air Pressure at sea level is 14.7 psi
1 gal = 128 floz or wtoz water / (16oz / lb) = 8lb
1.8375 gal = (1.8734 gal) (8 lb / gal) = **14.7 lbs**

If a heavier and-or denser) liquid, such as mercury (Hg), is rather used than water in a barometer, the height of its column will be much less so as to balance with the external standard air pressure at 14.7psi , and this will make the barometer instrument device smaller and more portable, and therefore more practical.

1 in^3 volume of mercury weighs 0.49 lbs = 7.84 oz and this is 13.56 times heavier than 1 in^3 of water.
(14.7 pounds) / (0.49 lbs / 1 in^3) = 30 in^3 , and by the above reasonings, this corresponds to the pressure of 30 vertical inches of mercury (Hg) = 30Hg. 30in = 762mm.

1" = 1 inch = 2.54 cm = 25.4mm

Stormy weather brings a lower pressure to a region (ie., a local area), and for ex., 28in. Hg = 28Hg = 711.2mm.
Low pressure, warm air will rise high into the atmosphere when cool, more dense air displaces it upward, and the moisture in it will eventually condense into rain drops and-or snow in the cooler temperatures higher in the atmosphere. When cool, more dense air replaces the warm air (such as from a storm having some thermal energy), this will cause an increase in the local air pressure, and it will eventually be at the normal and-or average value after the storm(s).

NEWTON'S LAWS OF MOTION

In 1687, **Isaac Newton** presented his celebrated equation for force: **force = (mass) (acceleration) = ma**

Newton, essentially put together what was already known in some ways, but combined and placed the facts together into a simple and valuable equation form, much like how Ohm later made Ohm's law equation based on what was already known in some ways. Newton knew that to move and-or have a speed or velocity of an object or **mass**, a **force** needs to be applied to it. When a force is constantly applied to an object or mass, it will **accelerate** it. Clearly, these values are related in some way, and Newton wrote his formula for their relationship: acceleration = force / mass = a = F / m , and from this we have: F=ma . The velocity (v) or speed of an object is result of, and rooted in the amount of force applied to its mass, hence the energy and acceleration given to that mass. acceleration = (change in velocity) / (change in time) , and which can be simplified to an average-like value having units of: distance per time^2. For ex.: a = (meters / s^2)

Since: F = ma , if either (m) or (a) increases by a factor of (n), then force will increase by that same factor of (n). If mass is a constant, and if F increase by (n), then (a) will increase by that same factor of (n).

1. An object or mass at rest (not moving, no change in location, a speed of 0 if no motion) will continue to be at that same speed and direction unless a force is applied to that object. The force can be an externally applied force such as a push from a person or collision with another mass, or an internal force due to releasing energy from an internal propulsion system (such as using fuel in a rocket motor). A measure of motion is called **momentum** = p = mv. Momentum is generally considered as a measure of the ability to keep moving and-or doing something. Unlike the pure scalar value of kinetic energy with just a magnitude or value in joules , momentum is a vector quantity with both a magnitude and direction, and of which can be considered as a hypotenuse side of a triangle, and analyzed using the two component (ie., causing, constructing, contributing) momentum vectors considered as the right sides of a corresponding triangle which can be solved for using the Pythagorean Theorem..

2. The speed of an object or mass will usually change when a force is applied to it. The force or impact from a moving mass applies and transfers energy (here, kinetic energy, KE) to that object. This change in speed is called **acceleration** (ie., accelerate, a = F / m) if it is an increase in speed or velocity, and called **deceleration** (a = -F / m = -a) if it is a decrease in speed or velocity. A deceleration will cause a loss in velocity, and may be expressed as a "slowing" of the object. Acceleration is generally given a mathematical positive sign, and **deceleration** which is essentially the opposite or negative of acceleration is therefore given a negative sign. Once the force is no longer applied, there will be no more application of energy to that object, and therefore no further gains or losses in energy (ie. kinetic or motional energy) of that object, and no further acceleration or change in the objects velocity. The object will remain at the same speed and direction, and this may even be a circular or elliptical direction and orbit about another planet or object.

3. When a force is applied to an object, there is an equal and opposite force from that object. This could perhaps be roughly considered as a force or resistance to a change in motion. A simple example of an equal and opposite force would be when a spring is compressed or expanded by a force, and at the same time of doing this, it is also trying to expand or compress back to its original, stabilized size with that same value of force. The energy used to create that force to compress the spring is transferred to and stored as ("potential") energy in the molecules of that compressed spring. The energy in the spring can generally (ie., without other issues) be stored indefinitely and be applied and transferred to another object by its internal and-or potential force.

When an astronaut is in outer-space, that astronaut is traveling at the same speed as the space station and all other objects in it. Surely they are traveling fast in space, but when they travel at the same speed (horizontally and vertically [free-falling downward due to gravity alone]), the difference in the vertical and-or horizontal speeds of the objects and people is 0. It could be said that the astronaut and objects have the same relative speed, and that their actual spacecraft orbit or traveling speed then appears to have no effect on them. While in orbit they will not reduce in altitude due to free-falling downward because the distance they effectively travel over the level horizon and curving (downward) Earth effectively increases their height above the Earth by this same value that they fall downward for each unit of time. If an astronaut applies a force to a (seemingly, effectively) "weightless" (but not mass-less) object in space, so as to give it a push or pull, there is still an equal and opposite force applied back to the astronaut from that object. This force is equal to: force = (mass)(acceleration). As long as the force is applied, there will always be energy transferred to the object and it will accelerate (from F=ma, a=F/m) and increase in speed, and the equal and opposite force will be felt by the astronaut. If an astronaut is braced in or on a spacecraft and has their motion stabilized, it is much easier for them to move and control an object.

Friction, essentially a mechanical-like collision of two surfaces and-or force in the opposite direction that consumes, depletes and-or wastes some of the energy available and-or used to do something like move a box across the floor or rug. For an object, perhaps a bicycle, car, boat or train to continue with the same initial amount velocity and-or momentum, this amount of wasted or lost energy due to friction must be continually replaced, and if not, that object will continue to decelerate in velocity or speed and will eventually stop moving. A typical gasoline or petrol fueled car might loose a total of about 80% of its fuel energy, and-or 50% of its available engine combustion energy to various (mechanical) internal energy losses due to friction such as in its gears and wheels. A typical electric car which has less mechanical parts might only loose 25% of its available battery energy to various internal electrical and mechanical friction losses. A moving vehicle with a given weight, area size, and an amount of kinetic energy and-or power will loose up to about 20% of that energy or power due to roadway friction and wind (ie., air mass which can apply a force) resistance, and of which are sometimes called as moving or rolling friction. Energy wasted due to friction (force) is usually converted to heat energy.

In outer-space, there is generally no friction between two surfaces, and so an object will continue in its motion (velocity, direction) unless it is affected by gravity which is a force upon it.; and technically, space is filled with various amounts of weak or low-level gravity from distant objects (asteroids, planets, stars, galaxies, nebula) in space. When this gravity (ie., a force) is constantly applied to an object, it will eventually change its course (ie., direction) or velocity.

With a force (the application and transfer of energy), an object such as a ball can gain kinetic energy and travel upward (vertically) into the air. During its flight upward, there is a constant force of gravity on that object or mass. This force equals the weight of that mass. For an object to even travel upward, the applied force = (ma) must be more than the weight = (ma)=(mg) of that object. As the object goes upward, this constant, gravitational downward pull upon the object is effectively applying energy and a force in the opposite direction to that of the upward direction of the object. The object will gradually decelerate or loose speed and kinetic energy due to this constant gravitational force applied to it. Since momentum = mv, the object will loose momentum when it looses its speed. When the kinetic energy (KE) of the object finally reduces to 0, and-or its vertical momentum is reduced to 0, the object will stop moving upwards and will then start going downwards since the downward force of gravity is still acting on it and influencing its motion. While at its maximum height, all of the energy to place it there has been converted to what is called as **gravitational potential energy** (**GPE**). As the object falls toward the ground, it will then gradually gain kinetic energy (KE = mv^2 / 2) due to the force of gravity constantly affecting it and accelerating it towards the ground. If the energy losses in this system are small: Energy Out = Energy In , and just before it returns to the initial launch point and lands. If the object had a rocket engine to provide a force equal to its weight, that object would be able to hover (be suspended, not moving) in the air, or to land slowly (a low velocity) when the force created by the rocket engine is slightly less than the weight of the object.

More about momentum and kinetic energy:

If an object if forced or thrown not only upward, but farther away outward or more distant, hence to move at an angle at some value between horizontal (level) and vertical (upward), the object will have both a vertical velocity and a horizontal velocity. Its motion will have both a horizontal component of motion, and which is called a vector (both a ["scaler"] amount, and direction or angle), and a vertical component of motion or vector. Momentum (p = mv) is formally a vector because its factor of velocity can be formally described as a vector. Its motion and-or momentum (mv) will have two component vectors: a horizontal momentum = px = (m vx), and a vertical momentum = pv = (m vy). Its forward motion and-or momentum can then be described as a resultant director of its two component vectors and-or vice versa. These three vectors can be analyzed and-or calculated using right angle trigonometry. For an object moving vertically, its upward or vertical momentum will be changing and-or reducing due to gravity, however, it will regain that lost momentum as it travels toward the ground. The total energy of the object cannot exceed the input energy: Energy Total = KE + GPE , and when either KE or GPE is gained, there is a loss or reduction in the other variable. For example: KE = Energy Total - GPE

Ex. How much energy is needed to make a 1kg mass travel at 100 km/hr , and-or 100km/hr faster?

We see that the question is essentially asking for the input energy, and we are given some factors of its output energy.

First, velocity or speed = $\frac{\text{distance}}{\text{time}} = \frac{100\text{km}}{1\text{hr}} = \frac{100\text{km}}{3600\text{s}} = \frac{0.02778\text{ km}}{1\text{s}} = \frac{27.78\text{m}}{1\text{s}}$

If KE out = KE in , = KE = mv^2 / 2 = (1kg)(100 km/hr)^2 / 2 = (1kg)(27.78 m/s)^2 / 2 =~ (1kg)(772 m^2/s^2) / 2 =

= (mass)(m/s^2)(m) / 2 = ((mass)(acceleration)) (distance) / 2 = (force) (distance) / 2 = N-m / 2 = work / 2 = energy / 2 = 386J

COMMON EQUATIONS FOR MOTION

The study of motion is also called kinematics, and the equations for motion are also called **kinematic** equations or **kinetic equations** since the word: "kinetic" is an old Greek word that means movement. These equations include one or more variables, and often three variables with one being the result: The main variables are distance (d), velocity (v) or speed, and time (t). A fourth variable used for some more advanced calculations is acceleration (a). Some of these equations were previously mentioned in this book, and this is a more general analysis or discussion.

d = distance = length . When an object changed or is changes location from a given reference location, it has moved or is moving to another location. This change can be called and measured as the distance (d) or displacement value of the object. Consider that a car can went along a winding road for several miles of traveling length or distance. It is then of note that on a map, this distance the car traveled is usually not the same as the straight line (ie. the shortest distance) distance between the starting and ending locations. Due to these reasonings, there is then two distance concepts to consider, and where one distance is during the process, and of any length and in any direction, and the other distance is due to the resulting location in a three dimensional, point coordinate (X, Y , Z) system, and having a (linear, line, straight when graphed) distance from the starting reference location, and this distance can also be shown to have a corresponding direction and-or angle on a graph.

If the total distance is due to several smaller distances, either all the same length or of various lengths, the total distance is the sum of those smaller, (travel) segments or partial distances:

total distance = segment1 + segment2 + ... = length1 + length2 + ... = dt = d0 + d1 + d2 + ...

There is a length difference or change in distance between each successive distance segment, and this can be expressed as:

(change in distance) = (dn+1 - dn) , mathematically:

dn+1 = dn + (change in distance) = dn + Δ dn : the triangle means "a change of" or "a change in" and it is also called the delta or change symbol.

A common generalization of this is often:

(change in distance) = (d1 - d0) or (change in distance) = (d2 - d1) , etc.

Also discussed in this article is: velocity (v) or speed (s), time (t), and acceleration (a) and also when it is a gravitational acceleration (a=g) due to the gravity of Earth and when an object is in "free-fall" (in the force of gravity only) from a height. Gravitational acceleration gets its name from that the force of Earth's gravity is (constantly) applied to objects near to it, and that force will therefore (constantly) transfer more (kinetic) energy to that object as long as the force is applied to it, and therefore it will cause the object to constantly increase its speed or velocity. An change that is an increase in speed or velocity is called an **acceleration** (ie., to accelerate, constantly increasing in value) of velocity.

The most common and basic equation for motion is the distance equation:

First, it is obvious that it will take an amount of time (t) to go or travel a distance. If we make this into a ratio, we can find the (average) distance traveled per unit of time:

$\frac{\text{distance}}{t}$ = distance traveled per unit of time = a rate of the change in distance , lets call it velocity (v)
The units of velocity are then those used for: (d / t), and velocity is than describe as the rate of change of distance with respect to a change in time and-or time taken.

While traveling this distance, we will go a certain speed or velocity which is determined by the amount of engine power used. The above formula can be rearranged to show that distance is the product of both velocity (v) and time (t) :

distance = (speed) (time) = (velocity) (time) = v t : this is a linear equation of the form: $y = mx$, and here,
m = speed or velocity = (change in distance / change in time)
= the slope of a line. If the velocity is not constant, the slope will no longer be constant as it is for a true line on a graph.
This also does not necessisarilly mean that the object has changed direction either, and it may be headed in the same direction at just a different speed.

From the above equation we see that mathematically, speed or velocity is the rate of change of distance with respect to time and is the (m) variable.

d = (velocity) (time) = (change in distance / change in time) (time) = m t

$d = v t$ and mathematically:

$v = d / t$: speed or velocity is the rate of the (change in distance) with respect to (time), or more precisely stated as: velocity = (change in distance) / (change in time)

$t = d / v$ **:** time of travel , or = $t = (1/v) d$

The units for distance (d) are usually: miles (mi), feet (ft), meters (m) or kilometers (km)
The units for time (t) are usually: hours (hr) or seconds (s), but may occasionally be minutes (min.)
The units for speed (v) are therefore a distance unit divided by a time unit. Speed or velocity is a rate or ratio value that can be considered as the average distance moved per unit of the time duration considered.

Graphing distance versus time

For some assistance, please view the next FIGURE shown below before and during this discussion.

As mentioned above and in various other articles in this book: $d = v t$ or (distance) = (velocity) (time) has the basic format of a linear equation such as:

(dependent variable) = a (independent variable), and where (a) is a multiplying factor , but may also be a value of (1) and in that special case, the line is a horizontal line with a slope of 0.

$y = mx + b$, and where m is the slope (ie., steepness, angle) of the line that is equal to: $m = (y - b) / x$ =
m = (change in y) / (change in x) , and which can be reduced to the equivalent fraction and-or concept of :
m = (change in y) per (1 unit change in x).

A linear equation such as $d = v t$ will always graph as a line. Here, (v) is the slope of this line.
v = (change in distance) / (change in time) . The higher v is, the more vertical the line will be when placed on a graph.

If a moving object is accelerating, then (v) will not be constant, but will be increasing, or decreasing such as when the object is experiencing a deceleration. When an accelerating object is graphed, it will not appear as a line, but as a curved upward or more vertical during the time it was accelerating, and this is due to that when the velocity increases, the distance traveled for each unit of time will also increase.

a = (change in velocity) / (change in time) = ((change in distance) / (change in time)) **/** (change in time)
a = (change in distance) / (change in time)^2 = v t

$v = a / t$: and if a=0, the object will still retain its last velocity value

On a graph of a curve such as $y = x^2$, and which has an upward curve shape of a basic parabola, it can likewise be said that the rate of change of (y) with respect to (x) is not constant. Here, slope of the curve at any point along it is constantly increasing as (x) increases. It could be said here that (y) is accelerating in value with respect, or in reference to, (x)

.

If there was an initial distance involved:

dt = ending distance or total distance = starting distance + (change in distance)

In this cause you cold assign say d0 as the starting distance of which can then be considered as part of the total distance..

Expressing each distance value with their corresponding velocity and time values:

$$dt = d0 + d1 = (v0\ t0) + (v1\ t1)$$

Given $d = v\ t$, if either (v) or (t) increase by a factor of (n) when (t) is constant, (d) will likewise increase by that same factor of (n), and this keeps the equation in balance. For example, for an object to travel twice (n=2, double) as far or distant during the same amount of time, its velocity will have to double.

$dt = d1\ v1$, after multiplying each side by 2:

$(2\ dt) = d1\ (2\ v1)$

If an object is at rest (ie., not moving), its velocity is 0. For an object to start moving, some force must be applied to it and until a certain desired speed is reached, and then the force no longer needs to be applied. If there is a friction force reducing the kinetic energy and-or velocity, then some small force will then need to be applied. Between 0 and that certain desired speed, the object is increasing in speed when a force is constantly applied. If the force is constant, the acceleration will be constant: $a = F / m$. It is said that its speed or velocity of the object is changing per unit of time, and this is called an acceleration (or acceleration, and increasing of or in its velocity) of the objects motion and-or its speed: When the force applied to an object stops, its acceleration will stop and its velocity will ream its last value and continue to move such as either in an orbit about a planet or other mass, or in a strait line across a smooth floor, or to a distant planet, other mass or location.

ending velocity = initial velocity + change in velocity : this will be shown to be: v = initial velocity + (at)

$v1 = v0 + \Delta v$: mathematically: $\Delta v = (v1 - V0) = a\ t$

The change in velocity per unit time is called acceleration (a). **Acceleration** (a) is calculated as the rate of a change of speed or velocity with respect to the time used or taken, hence the amount or change in time needed to make that change in velocity. For a velocity of an object to change, there needs to be an acceleration of the objects motion, and this is done by applying a force to it: $F = ma$. , and then (change in velocity) = a (change in time) or simply: (change in velocity) = a t

Let: (change in velocity) = (v1 - v0) , if v0 = 0 , (change in velocity) = (v1 - 0) = v1

Let: (change in time) = (ending time - starting time) = (t1- t0) , if t0 = 0 , (change in time) = (t1 - 0) = t1

Letting: v = v1, and t = t1:

$$a = \frac{\text{(change in velocity)}}{\text{(change in time)}} = \frac{v}{t} = \frac{(d / t)}{t} = \frac{\mathbf{d}}{\mathbf{t^2}}$$

with units of: **(m / s^2)** , also note, **v = at**

Note also: **d = a t^2** , $a = d / t^2$, $t = \sqrt{d / a}$

Ex.: $a = g = \frac{9.81m}{s^2}$: (constant) gravitational acceleration of the Earth near its surface, due to the constant amount of force of gravity near its surface. Can be expressed as: 9.81 m / s^2

velocity = (acceleration) (time) = at : from the general equation of: a = v / t

From: force = (mass) (acceleration) = ma , we can then have:

v = at = (force / mass) t = f t / m ,

Clearly, the smaller the mass for a given applied force, the greater the velocity, and the greater and-or longer the time that the force is applied, the greater the velocity. The greater the force, the greater the velocity, and even its only a quick pulse or a brief impact applied to the object.

a = (change in velocity) / t : when t = t2 and t1=0, mathematically:

(change in velocity) = (v1 - v0) = at , if the initial velocity v0 was 0, and letting v = v1, the new velocity is:

v = d / t = at , i f the acceleration was due to the constant force of gravity causing a (gravitational) acceleration;
(a) = (g) of Earth and applied to an object or mass: (g) on Earth = 32.2 ft/s^2 = 9.81m/s^2
a t = g t = F t / m = (Weight)(t) / m , g = W / m

d = vt = (at)t = at^2 or= gt^2 when considering an object dropped from a height and being affected by the constant force of gravity, its velocity will accelerate and-or have an acceleration or constant increase of it. The distance an object will fall is related to the square of the time (t) it is falling, and this will result in an exponential increase in the distance traveled per unit of time.
g = d / t^2 , d = gt^2 = (W/m) t^2

If the object started out at velocity of 0 and was constantly accelerating at a certain value due to a constantly applied value of force, its velocity will be constantly changing; here an increase, and we must then use the average velocity to find the distance traveled since the ending velocity (v1) = (v0 + Δv) = (initial velocity + change in velocity) was not consistent or constant throughout the total time of travel so as to simply use: d = v t, and now due to acceleration being involved into the velocity and distance, we must then use something like:
d = (vav)(t) = ((v1 - v0) t) / 2 = (v1 - 0) t / 2 = v1 t / 2 = v t / 2 when the acceleration is a constant value.
Also, when an acceleration is involved: **d = vav t = v t / 2 = g t^2 / 2 = a t^2 / 2**

If an object had an initial distance (d0) associated with it at t=0:
dt = (d0 + d1) = (v0 t + a t^2 / 2) = (v0 t + g t^2 / 2) = (d0 + g t^2 / 2)

When V0 = 0, and the acceleration is a constant value, the average velocity = vav = v1 / 2 = v / 2 = at / 2 ,
or: **vav = (v1 - V0) / 2** when the acceleration is constant. Also: v = at = 2 vav , a = v / t = 2 vav / t
average velocity = vav is not a constant between various points of travel or motion since acceleration is causing the velocity to constantly increase.

Letting d = v / t = (vav)(t) be expressed as: **d** = v/2 t = v t / 2 = (at) t / 2 = **(at^2) / 2** :
Also:: d = v t = (vav) (t) = (v1 / 2) (t) , we have **v1 = 2d / t** * : where v0 = 0 and t0=0
d = v t = vav t = (a t) (t) / 2 = (a t^2) / 2 **= g t^2 / 2** : or= 0.5 at^2 = (gt^2) / 2 or= **0.5 gt^2**

Ex. **For a free-falling object in Earth's gravity (g = a):**

After 1 second of time, a free falling object will travel:
d = (0.5) g t^2 = (0.5) (32.2 ft/1s^2) (1s)^2 =~ 16.1ft = 4.9 m
v = 2d / t = 2 (4.9m) / 1s = 9.8 m / 1s = 32.2 ft / 1s : note , v = 2d / t = g t = a t

Ex. After 2 seconds of time, a free falling object will travel:
d = (0.5)(32.2 ft/1s^2)(2s)^2 =~ 64.4 ft = 19.6m

v = 2d / t = 2 (19.6m) / 2s = 19.6 m/s = 64.4 ft / s

Ex. After 3 seconds of time, a free falling object will travel:
d = (0.5)(32.2 ft/1s^2)(3s)^2 =~ 145 ft = 44.1m
v = 2d / t = 2 (145 ft) / 3 =~ 96.6 ft / s = 29.4 m / s

Ex. After 4 seconds of time, a free falling object will travel:
d = (0.5)(32.2 ft/1s^2)(4s)^2 =~ 258 ft = 78.4m
v = 2d / t = 2 (258 ft) / 4 =~ 129 ft / s = 39.3m / s

Notice in the examples above that when the time (t) doubles = (2t), that the total distance traveled will increase by **4** due to that the time variable is squared in the equation:
Ex.: (t)^2 = 1t^2 and (2t)^2 = 4t^2 , and these have a ratio of: 4t^2 / 1t^2 = 4

Notice also in the examples above that vav = 2d / t = a t , and that v = d / t = at is a linear-like equation.

From the above equation for (d) = (at^2) / 2, we can solve for (t) with units of seconds (s):

$$\mathbf{t = \sqrt{\frac{2d}{a}} = \sqrt{\frac{2\,d\,ft}{32.2ft/s^2}} = (0.249222\underline{4})\sqrt{dft}\ \ s}$$

: here, dft = distance in feet, but unit-less
: the units of dft and (a=g) must be the same.

If g = a = 9.8m/s^2 was used:

$$\mathbf{t = (0.45175\underline{4})\sqrt{(dm)}\ \ s}$$

: here, dm = distance in meters, but unit-less

From v = a t = g t , we can also find (t) as:

$$\mathbf{t = \frac{v}{g} = \frac{v}{a}}$$

Extra: Setting the two equations for (t) as equal, it can be shown that : $v = \sqrt{2dg} = \sqrt{2da}$

It is clearly seen in the above formula that time is directly related to the distance of travel, and inversely related to the acceleration since a higher acceleration means that a higher velocity and the faster it will take to travel or move a given distance and, therefore, the time needed is shorter or less. The (ft) and (meters) units are shown in the equations above as reminders of what (unitless) value to place there, and that those units are not in the result, and only the seconds = (s) is.

Ex. How long will it take a free-fall object starting out with 0 speed, and to then travel downward a distance of 9.8m = 32.2ft ? (Note that 1m = $3.28\underline{1}$ft and 1ft. = 0.3048m)

$t = (0.451754)\sqrt{(9.8m)}$ seconds
t = 1.414 s : it took longer than 1s because its initial velocity or speed was 0. and:
When t = 1s: d = 4.905m =~ 16.$\underline{1}$ ft when the object starts at 0 speed.
This value is half of: 9.81 m =~ 32.2 ft
At the 1 second point, that the velocity of the object has reached 9.8m/s, and after the object has accelerated to that speed from a speed of 0m/s.
d = a t^2 / 2 = g t^2 / 2 = (9.81 m/s^2 / 2) t^2 = (4.905 m/s^2) (t^2) meters

Ex. If an object of any size mass is traveling horizontally and at any velocity, the force of gravity will constantly give that mass some kinetic energy and cause it to move vertically downward at an increasing speed when that gravitational force is constantly applied. The constant gravitational force will cause it to accelerate its velocity by a = g = 9.8 m/s^2 = 32.2 ft/s^2. The distance it actually moves per unit of time, say after 1 second, depends on its velocity at a given instance in time, and of which the velocity is constantly changing and increasing. Starting at 0s where its velocity is 0m/s, how long will it take for the object to move 1ft = (12 in)(2.54cm/in) = 30.48cm = 0.3048m?

First: When an object is dropped from a height above the ground considered as height=h=0 units, and in the presence of Earth's constant amount of gravity at those heights, it will experience a constant amount of force applied to it before it reaches the ground. Because this force is constantly applied, the object will gain more kinetic energy (KE = mv^2 /2) and therefore, its speed will constantly increase, and this is called an acceleration in its speed or velocity. Since the force (f = ma) is constant, the acceleration (a = F / m) will be constant and not increasing like the velocity does. The change in velocity will increase for each change in time. a = (change in velocity) / (change in time) , ex: a = (v1 - v0) / (t1 - t0), and acceleration (a=g) is constant in Earth's gravity. Since velocity is changing or increasing per unit of time, the distance moved per unit of time is also increasing: (a change in distance) = (change in velocity) (change in time) which can be mathematically reduced or simplified by dividing each side by the same value, and resulting to: d = (changed in velocity) (1t unit) = (change in velocity) (1s) , but since the velocity is constantly changing, we should rather use the average velocity between the starting and ending velocity per unit of time:

d = vav t = v1 / 2 t = ((a t) / 2) t = a t^2 / 2 : = gt^2 / 2 in earths gravity, and object free fall in it

$$t = \sqrt{\frac{2d}{g}} = \sqrt{\frac{(2)\ 0.3048m}{\frac{9.8m}{1s^2}}} =\sim 0.2494 \text{ s} =\sim 250\ (10^{-3}) \text{ s} = 250 \text{ milliseconds} = 250 \text{ ms}$$

Ex. If an object, vehicle or spacecraft started out at 0 m/s and ended at 100 m/s after 10s of time, what was its (assumed constant, such as due to a constantly applied amount of force) acceleration during that time?

Clearly, the object got faster in speed or velocity during that time of travel. Since it got faster and gained kinetic energy, some energy must of been applied and transferred to it, and this is usually done by an external force, or an internal engine producing a force. If the force was a constant value applied during this amount of time, the acceleration also had a constant value during this amount of time. a = f / m and f=ma. When a force (ie., the application and-or transfer of energy) is applied to an object, that object will accelerate and gain speed or velocity and kinetic energy.

acceleration = a = (change in velocity) / (change in time)
= (v1 - v0) / (t1 - t0) = (100 m/s - 0m/s) / (10s - 0s) = 100m/s / 10s = 10m/s^2

ending velocity = (initial velocity) + (change in velocity)
v2 = v0 + v1
v2 = 0m/s + at : from: a = (change in velocity) / (change in time), if v1=0 and t1=0:
a = (v1 - v0) / (t1 - t0) = (v1-0) / (t1-0) = v1/t1 = v / t , and v=at
v2 = v = 100m/s = at

acceleration = ending velocity / (time considered for the acceleration) = v / t = 100m/s / 10s = 10m/s^2

The velocity of the craft after 5s is:
v2 = v0 + v1 = 0m/s + at = 0m/s + (10m/s^2) (5s) = 50m/s

The velocity of the craft after 6s is:
v2 = v0 + v1 = 0m/s + at = 0m/s + (10m/s^2) (6s) = 60m/s = 50m/s + 10 m/s

Notice that after each 1 second of time, that the velocity of the object changed (here an increase) by +10 m/s more. Because the velocity is changing and is accelerating in value, the distance traveled per unit of time will also change and constantly accelerate or increase in value.

Ex. If a vehicle had a mass of 1000kg, what would be the constantly applied force needed to accelerate it at 10m/s^2?

Form: (force) = (mass)(acceleration) = f = ma , we have:
f = (1000kg)(10m/s^2) = 10000 kg-m / s^2 = 10000 Newtons (= 10000N) of force

Extra: Energy and-or Work needed in = Energy and-or Work out = F d
and here it is: F d = (10000N) (distance in meters) j = (10000)(dm) joules

Power in = Poiwer out , P = joules of energy needed or used / second = Pwatts = Pw

For the above example: Pw = (10000)(dm) / s : with units of watts

If a craft (ie., a vehicle) is slowing down in speed, it is decelerating (ie., and (a) will be negative in value, hence (-a)) due to a force applied to it in the opposite direction to its motion and reducing its velocity. Its ending velocity, and-or its velocity after an amount of time is:

ending velocity = (initial velocity) + (change in velocity) , since the change in velocity is a decrease, this is expressed using a negative sign:

ending velocity = (initial velocity) + (- change in velocity) , and which can be simplified as:
ending velocity = initial velocity - change in velocity
v2 = v0 - v1 = v0 - (at) = v0 - at : here for example, V0 might be say 100m/s

If a rocket moving at 30 m/s then had two different values of accelerations, say a1 = 10m/s^2, and a2 = 20m/s^2, how fast would the object be traveling if a1 lasted 5s, and a2 lasted 7s:

vt = initial velocity + (increase1 in velocity) + (increase2 in velocity) : increase = (a change)
vt = v0 + v1 + v2 = v0 + (a1 t1) + (a2 t2) = (30 m/s) + (10m/s^2)(5s) + (20m/s^s)(7s)
vt = (30m/s) + (50m/s) + (140m/s) = (30 + 50 + 140) m/s
vt = 220 m/s

A helpful note about acceleration:

From F = m a , if an object has a factor of (n) times more mass (m), then the force required to accelerate it to the same amount of acceleration will need to be increased by that same factor of (n);

(10) F = (10 m) a , mathematically:

a = (10 F) / (10 a) = F / m

A helpful note about inertia:

If the acceleration applied to the larger mass is still the same original acceleration value as that of the smaller mass, then that larger mass will then accelerate much slower and travel at a slower velocity than the original mass with less value. It could then be said in this comparative situation, that the larger mass has more resistance or **inertia** to motion when the same amount of force is applied. Since motion can be relative, it can also be said that a moving object even has resistance or inertial to a change in motion, and particularly so when its momentum (p=mv) and-or kinetic energy (KE = mv^2 /2) is already high and the force applied to it is in the opposite direction of its motion so as to slow it down and-or reverse its direction.

Due to the inertia of the mass of an object, if the acceleration changes, say increases due to an increase in the applied force, it will take some amount of time **(t)** for the object to reach its intended, or final speed, and of where there is no further acceleration and the object will be traveling at the new intended higher speed. An object such as a ship in water will experience the drag-force (ie., friction [much like micro-collisions], resistance, interference) of the water molecules and it will reduce the craft's kinetic energy and speed due to colliding with the water and transferring some of its kinetic energy to those water molecules and pushing (ie., a force applied) it aside. The ships engine will have to then provided enough constant energy to produce an extra amount of force just so as to maintain its speed at a certain desired and-or constant value. In space, there is usually no drag forces for a spacecraft to encounter, but occasionally, high velocity collisions with some gas atoms will cause a small drag force and corresponding small reduction in speed, particularly when the craft is at a low orbit about Earth, and it collides more frequently with some atoms of Earth's thin (low density) atmosphere. So that a craft does not slow too much in velocity and go lower than its desired orbit it will occasionally need to use its rocket engines to raise it back up higher in altitude to its intended orbit If not, it will eventually fall out of its natural orbit and return to Earth.

If an object is to change (ie., measured as a difference, such as (v1 - v0)) to a new speed (ex., v1 = v0 + change in speed), the larger the desired change in speed, the more time it will take to reach that speed for a given amount of force applied and-or acceleration. Although acceleration can change almost instantly to a new value if a force is applied, the larger the mass of the object, the more inertia (ie., resistance to a change in momentum or motion) it will have and the longer it will take to reach a desired speed or velocity if the force applied is the same original value creating the acceleration. .

From: **Force = F = ma** and **a = F / m = (change in v) / (change in t)** **and-or:**
F = (mass) (acceleration) = m (change in v) / (change in t) = m (d / t) / t = m d / t^2

acceleration = a = v / t = F / m = (force / mass) = (change in velocity) / (change in time) , we have:
(change in time) = (time) = **(t)** = v / a = (change in velocity) **/** (acceleration) = (change in velocity) (m / F)

As can be seen by the equation above, the higher the force and-or acceleration, the less time it will take to change to a new desired speed or velocity, and this is an inverse mathematical relationship. The larger the mass (m), the more time it will take for it to change in velocity. The larger the change in velocity, the more time it will also take to have a desired velocity:

(change in time) = (t) = (change in v) m / F and: (change in v) = F (change in time) / m

The total energy of an object can be described as equal to the sum of its existing kinetic energy and potential or stored energy. If an object is not moving, it has 0 kinetic energy, hence = 0 Joules of kinetic energy (KE).

From: Work = Energy = Joules (J) = (force)(distance) = F d , we have: F = J / d and d = J / F

Total energy of an object: (besides its thermal and other possible energy), hence this value of energy is due to its motion only.

Et = KE + PE : Joules : PE is usually called **GPE** = **Gravitational Potential Energy** if the stored energy is due to its height above the ground, and its mass will be influenced by the gravity (force) field of Earth and its force and associated acceleration, and will change (increase) its amount of speed and, and therefore increase in its kinetic energy it falls toward Earths mass.

KE = Et - PE and PE = Et - KE

As the object falls, it looses its potential energy due to its height and PE value being less, however, its speed increases and therefore, its kinetic energy (KE) increases. The total energy of the object is still the same and is said to be "conserved" (ie., maintained, a constant) since the energy put into that system, such as raising it to a certain height or distance remains stored with that system until some or all of it is transferred ("used up") to KE and-or converted to work ((force x distance) Joules)) when it collides with something.

If energy is used (ie., transferred, work) to raise an object or mas (m) to a height (h) on Earth, its potential energy is:

PE = work = fd = (ma)(h) = mgh , Joules : mass in kilogram units, g=9.8m/s^2, height with meter units, and its kinetic energy is: KE=0 Joules when not moving. If it is then allowed to free-fall under the force of gravity only, all of its potential energy will of been converted or transformed to its (available) kinetic energy just before it reaches the ground at height=0 and then impacts the ground with that energy and its associated impact or collision force = (m)(-a).

Ex. How much GPE does 1kg and-or each kilogram of mass of an object have at 100 km high in altitude have?

GPE = mgh = (1kg)(9.8m/s^2)(100km) = (1kg)(9.8m/s^2)((1km)(100) = (1kg)(9.8m/s^2)(1000m)(100) =
= (1kg)(9.8m/s^2)(100000 m) =~ (1kg)(980000 m) joules =~ 1000000 J =~ 1 MJ

> Since this equation is linear to altitude or mass, an object twice as high or twice as massive will have twice the GPE and-or require that much energy to lift it that high.

KE max = PE max :Just before reaching the ground where PE=0, and all of its GPE energy was converted to KE.

$\frac{mv^2}{2} = mgh$:from this we can solve for (v), and here, it is the object's velocity just before reaching the ground: Note that mathematically, after multiplying both sides by 2: mv^2 = 2mgh , and after dividing both sides by m: v^2 = 2gh = (acceleration)(distance) , checking: (M/s^2)(M) = (M^2 / s^2) = (M/s)^2 = d/ t = v^2 , (here, let **M** = **M**eter units of distance)

Note that: GPE = mgh = (mg) (h) = (weight)(h) = (Force)(height) or= (Force)(vertical distance) = work

$\frac{mv^2}{2} = mgh$, from this we can solve for (v), and here, it is the object's velocity just before reaching the ground:

$v = \sqrt{2gh} = at$: velocity of a free falling object just before impact or reaching the ground. The square root of (2g) is a constant.

velocity (meters/second) = $\sqrt{(2)(9.8)\text{m/s}^2 \text{ (height m)}}$ = $\sqrt{(19.6)\text{ (height) m}^2 / \text{s}^2}$

velocity (meters/second) = $4.427\underline{2}$ m/s $\sqrt{\text{height m}}$: use a height that is an absolute value of meters

velocity (feet/second) = $8.02\underline{5}$ ft./s $\sqrt{\text{height}}$: use a height that is an absolute value of feet
: due to using: g=32.2ft/s^2

$h = \frac{v^2}{2g}$: here, (2)(g) = 2 (9.81 m/s^2) = 19.62 m/s^2 = 2 (32.2 ft/s^2) = 64.4 ft/s^2

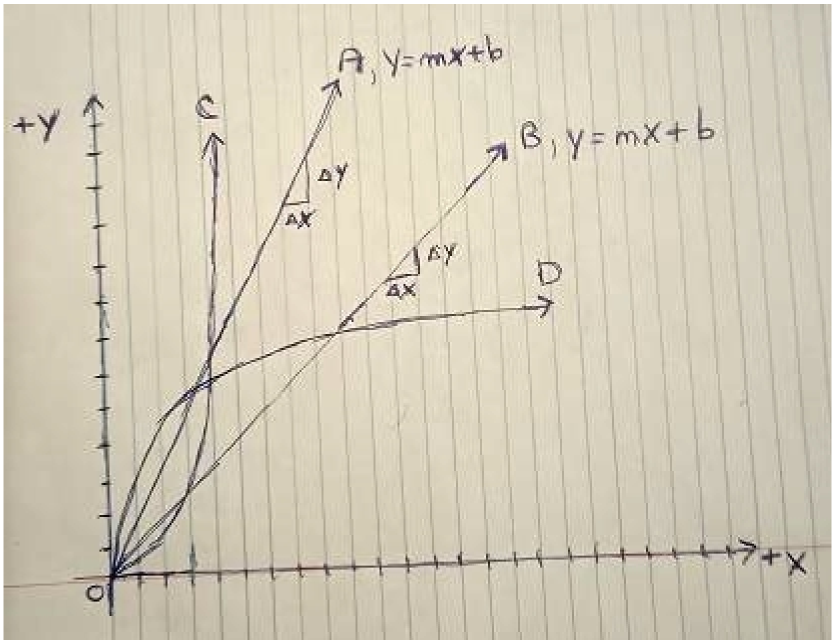
g = v^2 / 2h : "extra"

In general, for a moving object to go slower, it must loose some KE, and to go faster, it must gain some KE. To change speed, an acceleration (or deceleration) of the object must first take place. To cause an acceleration of an object, a force must be first be applied to the object so as to increase or decrease its momentum (p=mv) = KE (mv^2 / 2). Force is the application of energy to an object and can be measured as:

Force = (mass)(acceleration). and acceleration = Force / mass

Using the above formula, mass can be defined as: mass = Force / acceleration , and if there is less acceleration of the object when a given force is applied, the more mass of that object. More massive objects have more **inertia** or resistance to a change in movement, and are therefore more difficult to accelerate it to a given speed or velocity for a given amount of force.

[FIG 249B]



In the above figure, A and B indicate the lines graphed from linear equations. Line A is more vertical in nature and has a higher slope than that of line B. For each unit increase in X, the corresponding change in Y for line A is greater. C indicates a curve such as a parabola or exponential equation, and-or an object that is accelerating and its velocity is getting faster and faster per unit of time which would be the horizontal axis, and velocity as the vertical axis. D shows a deceleration in distance moved per unit of time, and is similar to a graph of a logarithmic or root type of equation, or possibly an exponential equation where the exponent is less than 1, and if it keeps up, the velocity will be 0, and the distance = y will not increase in value over a unit of time, etc.. For a linear equation, the slope is constant, and the line will continue to point into the same direction, and it will not curve towards a new direction. Again, for a line equation:
y = mx + b, the value of b can be 0, and omitted, and even the value of m can be 1 and omitted, leaving the equation y = x.

A Note About Common Weight Scales And Gravity

Most "weighing scales" such as most modern digital scales for home and everyday use, do sense the weight or (gravitational induced) force of the object placed upon it, however, the result displayed is its corresponding mass value. After all, grams and kilograms are (in the modern, technical sense) amounts of mass and not weight. A 1kg of mass actually weighs 9.8 Newtons (of Force) on Earth. In a reverse manner, a 9.8N weight (ie., a force) corresponds to 1kg of mass on the Earth. The machine or "scale" is essentially calibrated (ie., adjusted, set) to automatically divide the weight by 9.8 so as to find the equivalent amount of grams or kilograms, and then it displays that mass result. force = weight = (mass)(acceleration), therefore, mass = (weight / acceleration). People may say that this is the objects "weight", but it is actually the sensed weight converted to and expressed as its corresponding or equivalent mass. It is more correct to say it is the weight of the mass or "mass-weight". Expressed in simple terms, most scales are actually a "mass scale", rather than a "weight scale". Many cooking recipes use grams of certain food substances, rather than say how many Newtons of weight of it are required. Nonetheless, it is possible to have a scale calibrated to just display the "gravitational weight" or force of an object or mass in Newtons of force. Whereas the weight of an object is not universal due to planets having different gravity, mass or the real, physical amount of matter is universal, constant, and does not depend on any specific value of gravity in the universe, such as on another planet. An object with a mass of 1 gram, or 1 kilogram, of matter will always be 1 gram, or 1 kilogram. on all the various planets, or in empty space or anywhere in the universe. Mass is a constant physical property of the object itself, but weight is not since it is due to a force affecting that object or mass, such as the constant force of gravity pulling upon it and giving it an acceleration and weight (force). Mass and weight are physically and mathematically proportional, so therefore, a factor increase in one equates to the same factor increase in the other, and this will always be correctly displayed on the scale. It could be said that the relative changes in the mass and its corresponding weight are identical. If someone says an objects weight has increase by 2, then its mass had increased by 2. A weight to mass conversion scale created for use on Earth would not display correct results on other planets that have a different mass and amount of gravity. The scale would only work if Earth's gravitational force of acceleration (9.8 m/s^2) preset (ie., calibrated as a divisor) in that scale is changed to be that of that planets gravitational acceleration constant.

weight = force = (mass)(acceleration) N , N = Newtons = kg m/s^2 , therefore mathematically:

mass = weight / acceleration = weight / g = weight / (9.8 m/s^2) = kg / (9.8 m/s^2) = lbs / (32.2 ft/s^2)

Ex. 1 gram of mass will weigh (ie., a force due to gravitational acceleration):

First, for the weight or force to be expressed in Newtons (n), the mass must be expressed with units of kilograms (kg).
1 gram of mass = (1/1000) kg = 0.001kg of mass.

Fn = (mass)(acceleration) = (0.001kg)(9.8m/s^2) = 0.0098N : about a hundredth of a Newton of force = 0.01N
The weight of a kilogram = 1000g of mass = Fn = (1.0kg)(9.8m/s^2) = 9.8N : about 10N

If you were to take an object that is calibrated or indicated as 1g of mass and place it on a typical weight (or weight to corresponding mass) scale, it will still display 1g. Under the influence of Earth's gravity, the weight (ie., force) of an object having 1 gram of mass will be: (0.001kg)(9.8 m/s^2) = 0.0098 Newtons or "Earth Newtons" due to that Earths gravity constant is considered and used in the equation. .

A Note About Energy, Work and Power

Ex. How many joules of energy or work does it take to lift a 1 gram of weight (ie. a force under the influence of gravitational acceleration) to height of 1 meter?

work = (force)(distance) = (0.0098N)(1m) = 0.0098Nm = 0.0098 **joules** = 0.0098 (kg m/s^2)(m) = 0.0098 kg m^2/s^2
= 0.0098 kg v^2

This work or kinetic energy applied would be converted to the objects gain in potential energy due to its position above the ground and in the gravitational force field which could then apply a force to and accelerate that mass and convert its potential energy to kinetic energy. If the object was already at a height, it already has some amount of potential energy, and then raising it higher will give it more potential energy.

PE total = (existing PE) + (change or gain in PE) = (X Joules) + (0.0098Joules)

Ex. If it took 1000J of energy to lift an object to a certain height of 2 meters in 10 seconds, the energy used per second is:
power = energy / time = 1000J / 10s = 100 J / s = 100 watts : an average power value

Ex. If it took 1000J of energy to lift an object to a height of 1 meter in 10 seconds, the energy used per second is:

power = energy / time = 1000J / 10s = 100 J / s on average = 100 watts on average

We see that the actual height, mass or weight does not take part in the power equation, but those are rather part of the total energy needed and used. Power is an (average) measure of the total energy usage and not of mass or weight.

Ex. If it took 1000J of energy to lift an object to a height of 1 meter in only 5 seconds of time, the average energy used per second is:

power = energy / time = 1000J / 5s = 200 J / s = 200 watts.

Something, such as a machine rated with a "higher wattage" or power than another machine, can get things done (ie., do work or a task) quicker (a less or shorter amount of time) because more energy is available and-or can be delivered per unit of time or per second. More energy can be used to create more force so as to apply and transfer that energy. More "wattage" (output, available) means more energy available (output), used and-or required (input) per second.

Ex. If a device uses 10 watts of power and is used for 1 minute, how much energy did it use?

From: **power = energy / time , watts** we have:
energy = (power)(time) , joules since 1min = 60s of time:

energy = (10 watts)(60s) = $\left(\frac{10 \text{ joules}}{s}\right)$(60s) = (10)(60)joules = 600 joules of energy

Energy And Pressure

The force applied to or available from an object is: $F = ma = mg$. The localized or concentrated force at the site or area of the impact is called the **pressure (P)** applied from that force: **Pressure = Force applied / area**. , $P = F / A$ = force per unit of area. If a force is applied to a very small area, the pressure and transfer of energy can be very high and more useful for certain things, such as for a hammer to pound a pointy nail into wood, or pointed wedge to split wood. If a hammer is used to apply force to a nail, that force is effectively concentrated, focused, available to, and applied to a small area at the pointed tip of the nail which has a very small surface area. Note that the total energy available at the hammer just before impact is not actually amplified or increased at the tip of the nail. Energy out = Energy in. What is rather amplified is the force per unit area = pressure. Pressure is essentially an amount of concentration of force. This is much like how a large amount of light is gathered, concentrated or focused by a telescope lens or mirror. The total amount of light received is not increased or amplified, but the concentrated light energy into one spot or area makes the light energy per unit area much higher as if it were actually amplified, and objects will then appear as being brighter. It could also be said that the energy density = (energy / area) is now higher. A funnel shape can be used to concentrate audio or sound waves so as they are effectively louder from the source. With the hammer and nail, the total amount of available energy is being applied to and transferred to a small region or area, hence the energy is effectively being concentrated into one spot or area so as to be more useful to do certain things such as making the tip of the nail go more easily into to the wood.

Extra: Letting: P = Pressure , F = Force , A = Area , we have:

$$\mathbf{P} = \frac{\mathbf{F}}{\mathbf{A}} = \frac{(\text{mass})(a)}{m^2} = \frac{N}{A} = \frac{N}{m^2} = \frac{N\ \ (m^1)}{m^2\ (m^1)} = \frac{N\text{-}m}{m^3} = \frac{N\text{-}m}{\text{Volume}} = \frac{\text{Work}}{\text{Volume}} = \frac{\textbf{Energy}}{\textbf{Volume}}$$

: mathematically and:

$$\textbf{Energy} = \textbf{(Pressure)(Volume)} \quad : = (F / A)(V) = \frac{F\,V}{A} = \frac{mg\,(d^3)}{d^2} = mad = \mathbf{mgh}$$

:such as for a column or height of liquid
that applies pressure on a surface,
PE, Static or **P**otential **E**nergy

$$\textbf{Energy} = (\text{Pressure})(\text{Volume}) = PV = \mathbf{pghV} = mgh = KE = \frac{\mathbf{mv^2}}{\mathbf{2}}$$

: p = density of the substance = mass / V ,
Ex.: p = mass per cubic meter = kg / m^3, and:
KE = Dynamic or Kinetic Energy , due to motion
g = gravitational acceleration , h = height

In a pipe with a flowing liquid passing through it, the Energy and flow rate (Volume / time) of that liquid is constant at any point along it, and therefore Energy, Volume (V) and mass (m) and (1/2 =0.5) are constants in this equivalence:

Energy = PV = mv^2 / 2 and the resulting (P and v) relationship we are interested in is this: Considering the conservation of energy, and the relative values:

(constant) = (directly related to P) (directly related to v^2) , if we let (constant)= a relative value of 1.0:
1 = (directly related to P)(directly related to v^2) their relative changes are also reciprocals:
1 = (relative change in P)(relative change in v)^2 : here, each factor is the reciprocal value of the other factor, and is expressed below

If the amount of energy is a constant value, P and V are then inversely related, but not necessarily reciprocal in value. Here, as one changes by a factor of (n), the other changes by a factor of (1/n), and these change factors are reciprocal in value to each other. Due to the above equation, for a given amount of energy, mass and volume (V), P and (v^2) are reciprocally related in value of each other.

Due to this condition, is shows how, Pressure and velocity^2 affect each other, and in a inverse manner since Energy remains the same value. If one factor of a constant value increases, the other(s) must decrease, and this is an inverse mathematical relationship expressing an inverse real or physical relationship. These discussions of **Bernoulli's principle and equation** here can be associated with the discussion(s) about **Bernoulli's Principle** given previously in this book.

For an ideal **pipe (a hollow cylinder)** and fluid, a change in pressure is inversely related to the square of velocity:

$$\text{(relative change in Pressure)} = \frac{1}{\text{(relative change in velocity)}^2}$$

: Ex. If v1 = 5, and v2=10, the relative change is v2 / v1 = 2, the relative change in P = 1/(2^2) = = 1/4 = 0.25

$$\text{(relative change in velocity)}^2 = \frac{1}{\text{(relative change in pressure)}}$$

$$\text{(relative change in velocity)} = \sqrt{\frac{1}{\text{(relative change in pressure)}}} = \frac{1}{\text{sqrt (relative change in pressure)}}$$

If velocity doubles or changes by a factor of **n**=2, the corresponding pressure changes by a factor of: **(1 / n^2)** or= (1 / n)^2 = (1/2)^2 = 1^2 / 2^2 = 1/4 = 0.25

Ex. v1 = 1 and P1= 20 , if the velocity changes to 2 or doubles, what is the corresponding pressure (P2)?

(relative change in velocity) = v2 / v1 = 2 / 1 = 2 :given in the example
(relative change in pressure) = 1 / (relative change in velocity)^2 = 1 / (2^2) = 1/4 = (0.25)

Given a constant amount of (KE), when v or P changes, it affects the other value in a specific way, such as being changed by a specific factor or relative value:

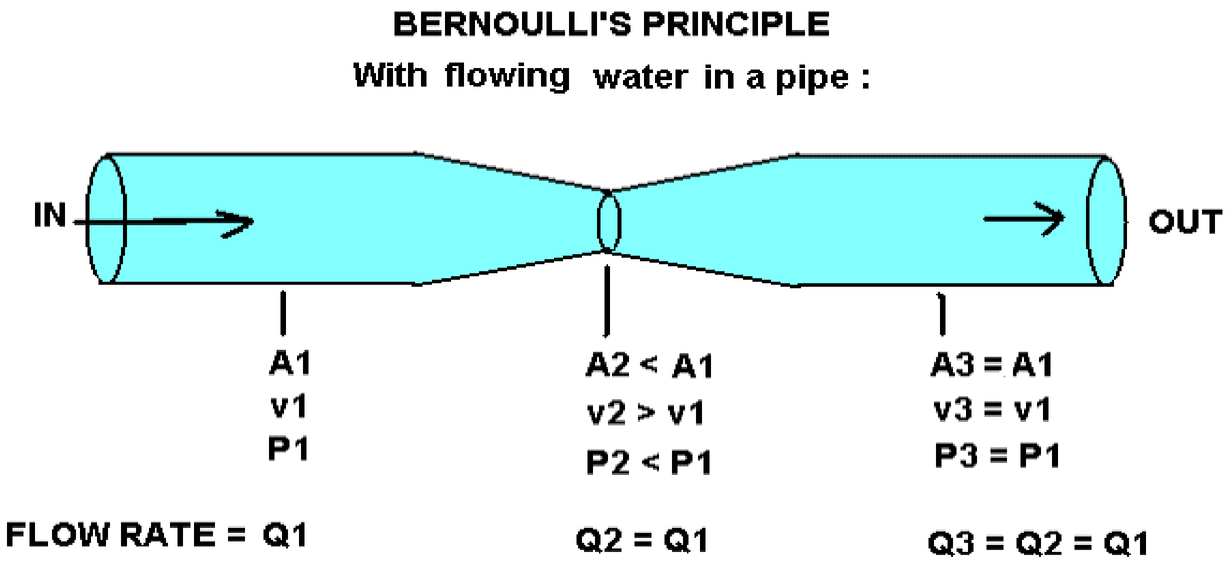
v2 = v1(relative change in velocity) = 5(2) = 10 and
P2 = P1(relative change in Pressure) = 20 (0.25) = 5

As a check, the product of two (relative change) factors should equal 1:
1 = (relative change in velocity)^2 (relative change in Pressure)
1 = (2)^2 (0.25) = (4)(0.25) = 1 : checks

Note that the equation of the form: (1/n^2) has the format of an inverse square law equation, and where the result reduces rapidly when the denominator increases. This is like the inverse of n^2 and where the result increases rapidly as expected for the power of a value greater than 1. Also:

Note that the equation of: $\sqrt{1 / n}$ has the format of the square root of a reciprocal or inverse of (n). If (n) increases greater than 1, its reciprocal value is less than 1 and decreases, and the square root of that value will also be less than 1. Shown in this book is a similar mathematical discussion for Newton's Cradle and kinetic energy.

[FIG 250]



Before we move on to Bernoulli's Equation, there is a way to calculate the velocity of a fluid in a pipe by knowing both the flow rate (Q) and the area (A) at a location along the pipe:

The **flow rate (Q)** expresses the volume (V) of the fluid that will pass a point in one unit of time: **$Q = V / t$** , The Volume of this fluid can be calculated from the equivalent geometric volume within that section of the pipe. Here the pipe is a tube or cylinder shape, and has a circular cross-sectional area and is equal to A = (pi)r^2. The cylinder shape has a height=length=distance that the fluid will travel in that unit of time:

Volume = **Volume of a cylinder** = (cross-sectional area of the pipe or cylinder)(distance) = **V** = Ad
Expressing both volume expressions (V, and Ad) with respect to time so as to have a time rate:

$Q = \frac{V}{t} = \frac{d^3}{t} = \frac{d^2 d^1}{t} = \frac{Ad}{t}$, therefore: $Q = \frac{A(d)}{(t)} = \mathbf{Av}$, and:

$\mathbf{v = \frac{Q}{A}}$ = **velocity of a fluid in a pipe = (flow rate) / (cross-sectional area of the pipe)**

A flow rate can be physically determined by letting the fluid flow into a container, and then dividing by the time it took to reach a certain volume: Q = Volume / time = V / t

$\mathbf{v = \frac{Q}{A}} = \frac{(V / t)}{A} = \frac{V}{\mathbf{At}} = \frac{d^3}{d^2 t} = \frac{d}{t}$ mathematically: : **$V = Qt$ = (flow rate)(time of flow)** and:

$V = Qt = Avt$

When the cross sectional area of a pipe decreases, in order for the flow rate to remain a constant volume per amount of time throughout the pipe and, into it and out from it, the velocity will naturally increase. A high energy or pressure is being applied to a smaller amount of substance at the start to this area and-or pipe section, and then forcing that substance through a smaller area, and a higher force upon a substance means an acceleration and results in a higher velocity of that substance . The mathematical relationship between velocity and area is an inverse-proportional (ie., inverse, and by the same factor) one, such that when one changes in value by a factor of (n), the other will change by the inverse of that same factor, hence (1/n). This can be seen in the above equation if the Volume is constant.

From: $V = Qt = Avt$,
$V = Avt = (n/n) Avt = nA (v/n) t = (A/n) (nv) t$

Note for example, that when the internal diameter of a pipe is half, its cross-sectional area is: A = (pi)(r/2)^2 = (pi)r^2 / 4, hence its area is only one-fourth, and the velocity of the substance will then increase by 4. V = (A/4)(4v) t = V
Since internal pressure is part of the energy per volume of the substance flowing in a pipe, and if this amount of energy is constant, if the velocity of a substance increases in the pipe, the pressure (such as upon itself and the internal and-or external wall at a right angle (90°) to the flow in the pipe) will decrease. A higher velocity means some of the energy per volume of the flowing substance has been converted to more kinetic or motional energy of that substance, and there is less internal pressure energy available, and which can be thought of as stored energy due to compression, like a spring.

Bernoulli's Equation states the that sum of pressures contained within a moving fluid is a constant value, and this is very similar and to, and due to that the sum of Potential (ie,. static, stored) Energy and Kinetic (ie., dynamic, motional, moving, and such as applying force or pressure to the fluid in front of it). Energy is constant in a flowing fluid system. The output energy will equal the input energy, less any losses.

Total Energy = Et = KE + PE .

Energy can be in 2 main forms, and much of this is discussed elsewhere in this book:

1		2
Potential Energy (PE),	and	Kinetic Energy (KE),
static or stationary	,	motional or moving

Potential energy is also called stored energy that can be used when able or needed. It is also thought of as energy due to position such as a compressed spring, or and object lifted to a height and placed there. These object required an initial energy and force to put them in that state (status, condition) or position, and the energy can be released when able or needed. A system can have both PE and KE, and of which sum to the initial or total input energy. Due to the concept of the conservation of energy: Energy out = Energy Out.

Since Energy = Work = (force)(distance) = (Fd) = (mass)(a)(d) Joules, if the distance is equal to a height above reference height, the energy needed to raise it, and or stored in an object is: **(Fh) = (mass)(g)(h) J**. This value of energy may also be added to any amount of potential energy the object may of already had.

When a gas or liquid is compressed (perhaps by a pressure upon it) into a smaller volume, it's own pressure will increase. The energy used to compress that gas or liquid is transferred to the gas or liquid and stored in its structure, much like how a compressed spring can store the energy used to compress it. Some of that input energy is converted into heat energy in the substance. This stored or saved energy is called potential energy. The gas or liquid will condense when pressure is applied and it becomes more dense per unit volume. Density = mass/volume, however water is very difficult to compress, much like a solid.

Water that was raised to a height in a tower or dam by using energy (ie., work), can store (gravitational) that input energy as potential energy (PE), and when that water can flow downward from that height, its energy is converted to kinetic energy and can provide the force to turn a turbine wheel of a mechanical device or an electrical power generator. The kinetic energy and-or velocity of this water will be at its maximum value just before it reaches the bottom of the pipe. In theory, this KE per volume of water at the bottom, will equal the PE per volume of water at the top height, and the total amount of available energy is the same: KE = PE.

Ex. If a water tank up high on a water tower is storing 100J of (Gravitational) PE = Fd = (weight)(height) = (mass)(g)(h). The energy per volume is then: PE / V = (m)(g)(h) / V = (m/V)gh = (density)gh = pgh,
Since p = m/V, m= pV, and the amount of energy stored in relationship to the volume of the liquid is:
PE = mgh = pVgh . For the fluid of water, it has a density of 1g/cc = 1000g/ 1000cc = 1kg/L. g=9.81m/s^2.

If it takes 50s of time for all that water to flow through a pipe to an electric generator, then that stored 100J of PE would be converted to 100J of KE energy in an ideal (low power losses, full energy conversion) system. The rate of using this energy would be:

Rate, or time rate, and power of using this amount of energy in 50s is: $\frac{100J}{50s} = \frac{2J}{1s}$ = 2 watts of power
P = 2w

For an ideal energy system and-or conversion, 2 watts of electrical power would be generated and-or available. Also in an ideal system, the flow rate (Q = Volume / time = V / s) is constant and the available and generate power will be constant, but since the water is draining from the container, it contains less stored potential energy, and less depth (distance from the surface to bottom) of the water as time increases, changes, progresses or passes, and therefore, the water pressure at the level of the outlet pipe is constantly being reduced if that volume drained out is not resupplied into the container.
Since the (water or fluid) pressure in the pipe is less, the force and the application of that potential or pressure energy will also get less, and the available and generate power will be less. As the fluid in the container drains, it will loose both its Gravitational Potential Energy (GPE) and the available (static) pressure energy and-or force, but as the water reaches the bottom of the tower, it will regain, convert or transform that GPE as kinetic energy (KE). If the fluid moved a distance of 1 meter per second in the pipe or middle (a low resistance and-or friction region) of a water steam, it is then said that its velocity = v = distance / time = d / t = 1 meter per second. In theory, and for an ideal system, a turbine or power transfer wheel with a circumference of 1 meter distance, and placed into this flowing water, would turn or rotate once each second if the friction or force against its movement was low, hence it would have a maximum rotational velocity or rate of 1 revolution (360°) or cycle per second =1 rps = 60 rpm.

To calculate the rps value: From v = d / t , v / d = 1 / t = frequency = cycles = revolutions,
ex. cps = hertz = hz

$\frac{v}{d} = \frac{v}{c} = \frac{1}{t}$ = hertz = rps = frequency : c = circumference = 2(pi)(r)
: d = distance
When time has units of seconds (s).

Ex. If the water velocity is v = 1m/s = 3600m/h = 3.6km/hr, and circumference of the wheel is c = 0.25 meters:
rps = velocity / circumference = v / c = 1 (m/s) / 0.25 m = 4 (1/s) = 4 hertz = rps

For torque (effective rotating or twisting force, equivalent to an amplified linear force) applied to this wheel or turbine:

From: Pressure = Force / Area = P = F / A , F = PA.

Torque = (Force)(leverarm) = PA(leverarm) =
Torque = T = PAr : r = radius of the wheel or turbine , and:
P = water Pressure on the wheel
A = surface contact Area between the flowing water and the wheel.

Bernoulli's Equation states that at any location or point in a pipe or moving fluid:

Total Pressure At A Point In A Pipe = Pt = Input Pressure + Dynamic Pressure + Static Pressure

Pt = Total pressure at point1 = Total pressure at point 2

The pressure at a point = $\frac{pv^2}{2} + pgh$ **: (h) is the height above the reference height (considered at 0m). If the fluid is raised a height, it gains potential energy KE) which**

in terms of pressure, can be called potential pressure energy, and this will reduce the available KE or kinetic pressure energy at that point by that same value, and since the total amount of energy at a point is constant. Dynamic pressure is the same as kinetic pressure. This pressure sum and equation is much like a sum of energies and-or forces at a point. p = fluid density

pgh = PE / volume : derived in this book , mathematically, **PE = pgh / V = static pressure / Volume**

As mentioned by Bernoulli's Principle, the pressure within a moving fluid, such as in a pipe, will change if the diameter of the pipe changes. The relationship between the pressure and velocity is an inverse square type of relationship. For a fluid to move, it must have an initial pressure upon or within it to do so, and it will move to the lower pressure region. The initial pressure may be a static pressure form of potential energy due to the height of a fluid stored in a container, and this fluid then has stored or potential energy and the ability to do things.

When a fluid is moving, its potential energy was converted to kinetic energy. When a fluid is moving, it's static pressure was converted to dynamic pressure. The dynamic pressure in a fluid will be less than the initial static pressure because some of that initial energy and high value of static pressure is now part of the kinetic energy and dynamic pressure. When a static fluid begins flowing into a lower pressure region, that fluid then becomes less pressurized itself, but gains velocity and kinetic energy due to the initial force or pressure applied to it and causing it to move. With a given amount of energy, an object will move slower through a region of higher pressure, it is essentially in a region with an increased resistance to movement.

If a fluid is raised to a height it has stored that energy to do so as potential energy (PE):

PE = pgh : p = density of that fluid or substance=mass/volume, kg/m^3 , : pgh is derived in this book
g = Earth's gravitational acceleration: 9.8m/s^2 pgh = KE / volume
h = height

Work = Energy = (force)(distance) = (mass)(acceleration)(distance) = **mad or= mgh** joules
Work = Energy = (weight)(height above a reverence line) = mgh joules

Pressure = **force / area or weight / area** = N / m^2 = **pgh** Pa : Pa = Pascal units of pressure

Since PE = pgh and Pressure = pgh : **,** Pressure is sometimes called "pressure energy". Pressure = force / area , and force is the application of energy, Pressure, such as in a compressed gas or fluid raised to a height, is like a stored amount of available or potential force, hence like a stored amount of energy.

Pressure = PE : Potential Energy (ability to release energy, stored energy)

From: **Energy = (Pressure)(Volume) = PV , Pressure = Energy / Volume = "energy density"**

Energy = (Force / Area)(L^3) = (F / L^2)(L^3) = FL = **Fd** = **Fvt** = work = energy : L = length = distance
Energy = (Pw)(time) = Fd = Fvt , joules , note mathematically:
Energy / t = **Power = Fv or = (Torque)(v)** **:** Power in watts , F in newtons , **linear v in m/s** , and:
Fn = Pw / v = E / vt = E / d and **Pw = Ev / d** and **d = vt = Ev / P ,** hence**: t = E / P :** P is a rate

$$\textbf{Pressure} = \frac{\text{force}}{\text{Area}} = \frac{\text{Energy}}{\text{Volume}} = \frac{J}{V}$$ with units of: joules / m^3 , also: **F = J / d = joules / meter**

Pressure $= P = (F / A) = (J / V) = ma / d^2 = J / d^3$, mathematically:

ma d = (force)(distance) = J = Work = Energy = PV = FV / A **: J = joules of energy**

$P = (F / A)$ = Weight / A = mg / A , $PA = F$ = Weight = mg

Kinetic Energy density of a moving liquid $= \frac{KE}{Volume} = \frac{mv^2 / 2}{V} = \frac{mv^2}{2V} = \frac{pV\,v^2}{2V} = \frac{pv^2}{2}$ = **Dynamic Pressure**
p = density

For the last term above, the units are : $\frac{mass}{V}(m/s)^2 = \frac{kg\,m^2}{m^3\,s^2} = \frac{kg}{m\,s^2}$

We see that Dynamic Pressure is equivalent to the Energy (KE) density of a volume of a moving liquid, and that its equation contains the density (p = mass / volume) of the liquid's. The denser it is, the more energy it will have at a given or particular, constant velocity.

Note: P = pressure = force / area :"force density" , p = density = mass / volume :"mass density"

Dynamic pressure can be thought of as the pressure (= force/area) of the moving liquid pushing (applying force, applying energy) on the liquid in front of it. From the above equation, we have:

KE = (Volume)(Dynamic Pressure) **: "(dynamic) pressure energy" of a moving volume of liquid**
$KE = (V)(pv^2 / 2) = (mv^2 / 2)$: here p = density of the substance (liquid or gas) , and that p can be derived here as: $p = m / V$

E = Work = Fd joules = mad = m (change in v) / (change in t) d = m (ch. v / ch. t) v t = $m v^2 / 2$
This is an average energy value since $v0=0$ and it will take time to reach the velocity of movement.
It could also be thought of as due to an equal and opposite force, say friction, receiving energy, and half the energy input does not get transferred to the mass. If the distance was a height, then: E = W = mah joules or= mgh joules in Earth's gravity of 9.81m/s^2

Bernoulli's Equation:

P = pressure , p = fluid density , v = velocity , g = gravitational acceleration , h = height

At each location in a pipe:

Pt = (Initial Pressure Energy) + (Kinetic or Dynamic Pressure and Energy) + (Potential or Static Pressure and Energy)

Total Pressure At Location 1 = Total Pressure At Location 2 , and:

$$P1 + \frac{(p)(v1^2)}{2} + (p)(g)(h1) = P2 + \frac{(p)(v2^2)}{2} + (p)(g)(h2)$$ **: Bernoulli's Equation**

A related note to consider is if you have a column of water in a container on the floor. The bottom of the container will have a water pressure associated with the depth of that column of water. The deeper the water, the greater the pressure at the bottom. This is static pressure energy. If you were to raise this entire column, container or tank of water up a height or altitude above the ground, such as onto a table or tower, you have used energy to do this, and this energy is stored as potential energy of that column of water. This is static pressure energy and potential energy.

The Basics About Collisions Of Objects

A brief discussion on this topic was previously given in this book with Newtons's Balls, and here is some more to consider:

For an object: **weight = F = mg** , and a constant mass and constant gravitational force induced acceleration (a=g) of the mass will produce a constant weight of an object. weight = mg = (a vertical, downward force) = F = ma

If object1 will collide with object2: : or mass1 will collide into mass2 :

Object1 is moving and has **momentum** (p=mv) and kinetic energy ($ke = mv^2/2$) and will collide (impact, contact and apply force to) with object2 which is at rest. Object1 will then loose some of its energy by transferring it to object2. More specifically, Object1 will apply a force (ie., the application of energy) to object2 and some of its energy will be transferred to object2. Much of this energy will be kinetic energy. As object1 looses kinetic energy and decelerates, object2 will accelerate. During this process, due to the concept of equal and opposite forces, when object1 collides with object2 it will be as if object2 is applying the same amount of force to object1, and in the opposite direction, and therefore object1 will decelerate. The impact force (application of energy) is related to the amount of acceleration (f=ma) or deceleration of an object, and also determines the amount of energy transferred. Remember that: power = watts = Joules/second. If the energy transferred is high and-or the time is a low value such as for an impact, the power can be a high value for a brief amount of time, especially if the acceleration or deceleration is quick such as during an impact or collision. A brief and large pulse or release and transfer of energy can cause significant damage to both objects if the collision is not "elastic" and where much of the input energy is converted to heat.

The force of an impact depends on the total kinetic energy available in the objects and the directions they were moving, such as towards each other, one being still, or both going in the same direction with one being faster and one being slower than the other behind it. The kinetic energy in an object is due to its mass and velocity. Total momentum = p = mv, and total kinetic energy = $mv^2 / 2$ and this equation which is similar to that of momentum is needed since if the velocity of the object doubled, its kinetic energy is actually four times more, but its momentum is only doubled. Momentum can be considered as a measure of an objects motion and to stay in that motion, rather than its stored energy.

The momentum of a mass is: **momentum = mv** : a change in velocity will create a change in momentum, this will not be instantaneous, but will take some amount of time for it to accelerate from one velocity to another.

$$F = ma = m\,\frac{\text{(change in velocity)}}{\text{(change in time)}} = \frac{\text{(change in momentum)}}{\text{(change in time)}}$$

: showing force as (equal to) the amt. of change in momentum (ie., motion ability) with respect to a change in time. Mathematically:

(change in momentum) = (force)(change in time) = (N)(s) = (ma)(s) = ($kg\ m/s^2$)(s) = kg-m / s = (kg) (m/s) = (m)(v) , and where (v) is the change in velocity.

A force (the application of energy), including friction, will cause a mass to change its velocity and-or accelerate (or decelerate) during the time that the force or energy is applied to that mass. The more mass of a moving object, the more potential or kinetic energy it has, and therefore the more energy needed to change it to another velocity. As an example, mass1 is a small mass, and mass2 is a larger mass, and both have the same velocity. Since momentum = (mass)(velocity) = p = mv, momentum1 is less than momentum2. Less energy is needed to slow down a mass with less momentum. If mass1 was traveling faster it could then have as much momentum as mass2.

More energy is needed to slow down a mass with more momentum or kinetic energy. If the force applied to a mass is not much Newtons, or it is applied for only a short amount of time, the total energy applied and transferred (or effectively removed from it so as to slow it down) to it so that it will have energy to do things was not much, and it will then take more time to apply and transfer more energy to slow it down. Still, regardless of the rate of energy use (ie, work done) or transfer of energy, the application (force) and transfer (power=joules/second, watts) of a same total amount of energy will be required to slow a mass with kinetic energy down to a certain speed or stop = 0 speed. In short, the smaller the constant applied force, it will then take longer to slow down an object or increase its speed to a certain value. A smaller applied force means a smaller acceleration of the object being forced. F=ma and nF = m(na), and a = F/m. For a constant value of mass, If the force applied is small, the acceleration of that mass will also be small.

If two objects collide and they both stop their motion, all the kinetic energy is converted to a force. This force may compress and-or deform one or both of the objects. This is the case when an object is dropped from a height onto the Earth's surface which has a large mass with a high inertia, but a small amount of its kinetic energy is converted to sound energy. Another example is when a meteorite collides into a planet or moon, and its kinetic energy is converted to force, and some heat (thermal energy), which makes a crater (ie., a dent or hole shape) after forcing or pushing matter out of the impact area. That matter will gain kinetic energy by the impact and travel outward to an effective "lower pressure region", and away from the point of impact and effective "higher pressure region" due to that force.

The amount of force a moving mass (M1) can apply to another large, difficult to move, non-moving mass (M2) depends on its velocity. A mass with a higher velocity has gained and is storing more kinetic energy, and has more **momentum (p = mv) or ability to stay in motion and its direction**. The more kinetic energy an object is storing in its amount of mass and motion, it can apply more force (the application of energy) and do more work (= energy, joules):

Force = F = (mass)(acceleration) = ma

For a given or fixed amount of mass of an object, in order to have a large impact force and release or exchange of its (kinetic) energy, a high amount of acceleration or deceleration is needed.

To have a high amount of acceleration (or deceleration, in the case of an impact where an objects speed or velocity decreases) the speed of the object must decrease quickly::

$$\text{acceleration} = (a) = \frac{\text{change in velocity}}{\text{change in time}} = \frac{(\text{final velocity - initial velocity})}{(\text{final time - initial time})} = \frac{(v2 - v1)}{(t2 - t1)} = \frac{\Delta v}{\Delta t}$$

If the change in time is small, such as the time it takes to make a quick impact, the acceleration will be a very high value, and therefore, the impact force or "force of impact" will also be a very high value. All the kinetic energy of the object or mass will be quickly expended, used up, and transferred rather than slowly used up over a longer period of time. A high impact force means a high amount of energy was used and transferred, however the total (kinetic) energy of the mass or object is limited to some value, and if much of it was used during the impact, very little, if any, will be then available from that mass to be applied elsewhere.

Since: Power = $\frac{\text{energy}}{\text{time}}$, If the time of applying the force (the application of energy) is a small or quick value, the available or instantaneous power, which can be considered as the effective or average energy and-or force available during that brief or instantaneous amount of time, will be a high value.

With forces and-or collisions (ie., forces being applied to an object), vectors (ie. such as a force magnitude and direction) also need to be considered. For a given force, if the momentum of an object is increased, that amount of force applied to that object will have a less effect. If an object is say going in the (y) direction on an x,y, rectangular coordinate system of measurement. It could be said that the motion of the object does not contain any (x) or horizontal "component (ie., part)" motion and-or direction. and if a object and-or force is applied to it in the (x) horizontal direction, such as during a collision, that object will than have both a vertical (y) motion and-or direction, and a horizontal (x) motion and-or direction. A common and practical example of collisions can be seen with the game called Billiards or Billiard Balls, and which is also known as Pool. In this game, a player uses a pool-stick to apply a force to one ball which will then cause it to collide at a certain angle and-or location (directly or a side of) a second ball, and in such as way as to produce a desired movement (velocity and direction) of that second ball. In general, for this game, a player must "clear the table" of their corresponding balls by making them all go into one of several (here, 6) pockets (ie., holes) on the side of the playing table. Often the first and second ball are not in line with the pocket, and the first ball will have to then collide with the second ball at the appropriate locate on both of their masses so as it will be caused to then travel directly to the intended pocket.

Momentum between two objects or a system of objects is said to be conserved, that is, its sum is a constant value if there are no external (energy, and-or momentum) losses to the outside of the system.

Here is a study and example of the momentum of two objects, and which will collide:

momentum of object1 , momentum of object2 : momentum is considered as: p = m v

p1 , p2
m1 v1 , m2 v2
(5kg) (6 m/s) , (7kg) (10 m/s)
30 kg m/s , 70 kg m/s :

KE1 = m1v1^2 / 2 , KE2 = m2 v2^2 / 2
KE1 = 90 J , KE2 = 350 J

pt = p1 + p2 = 30 kg m/s + 70 kg m/s = 100 kg m/s : total momentum of this system, and it is a constant value when momentum is conserved. If after a collision, if one object gains momentum, the other object has lost that same amount of momentum.

If after a collision, if the velocity of object2 is less, its KE is less and therefore, it has transferred some of its KE to object1 which will then have a higher or " faster" velocity. Let v2 = 5 m/s after the collision. Find v1 after the collision.

p2 = m1 v1 = 7kg 5 m/s = 35 kg m / s

(change in p2) = (p2 new - p2 old) = (35 kg m/s - 70 kg m/s) = - 35 kg m/s : ie., loss of p2 = - (gain of p1)

p1 new = (p1 old + (change in p1)) = (30 kg m/s) + [- (-35 kg m/s)] = (30 kg m/s) + (+35 kg m/s) = 65 kg m/s
or:
p1 = pt - p2 = (100 kg m/s) - (35 kg m / s) = 65 kg m / s.

(change in p1) = (p1 new - p1 old) = (35 kg m/s - 70 kg m/s) = 35 kg m/s

pt = p1 + p2 = (65 kg m/s) + (35 kg m/s) = 100 kg m/s : the same value after the collision
p1 = pt - p2 , p2 = pt - p1

From: p = mv , v = p / m

v1 = p1 / m1 = (65 kg m/s) / (5 kg) = 15 m/s
(gain in v1) = (v1 new - v1 old) = 15 m/s - 5 m/s = 10 m/s

KE1 new = m1 v1^2 / 2 = 5 kg (15 m/s)^2 / 2 = 562.5 J
(gain in KE1) = (KE1 new - KE1 old) = 562.5 J - 90 J = 472.5 J

Obviously, since the objects are now moving at a new velocity, their kinetic energy is now different: KE = mv^2 / 2

Is KE conserved, that is, is their total sum the same after a collision? During a collision, the objects are compressed (even if temporarily, like two rubber balls colliding), and this will cause the masses to increase in temperature. The result is that some of the objects kinetic energy is converted into heat (thermal, increase in atoms and-or molecules motion, hence KE) energy, and therefore, is unavailable to the overall motion of the objects and is lost from their KE. In a theoretical perfectly **elastic collision** , KE is conserved since none will be lost as heat but rather converted to potential energy of the compressed and-or slightly deformed (like a compressed spring) molecules and-or crystal arrangement of the atoms, and given back as a force to induce motion, hence KE.

More equations for distance, velocity and kinetic energy:

For an object that is already going a certain velocity (v1) and then uniformly (through the entire distance) accelerates to another velocity (v2), the distance traveled is equal to the average velocity (Vav) times the time (t) or duration of travel:

distance = (speed)(time) = (velocity)(time) : here (time) is the total time (t) of travel
distance = (average velocity during the acceleration(s)) (time) , (average velocity) = (total distance) / (total time)

$$\text{distance} = \frac{(v1 + v2)}{2}\, t \quad , \quad \text{since } v2 = v1 + \text{change in velocity} = v1 + at \quad , \quad t = \frac{(v2 - v1)}{a}$$

$$\text{distance} = \frac{(v1 + (v1 + at))}{2}\, t = \frac{2v1t + at^2}{2}$$

$$\mathbf{distance = (v1)(t) + \frac{at^2}{2}} \quad : \text{total distance} = \text{initial distance} + \text{change in distance} \quad : *$$
$$: \text{mathematically: } d / t = \text{(average velocity)} = v1 + at / 2$$

$$\frac{\text{distance}}{t} = \text{average velocity} = \frac{v1 + v2}{2} = \text{(starting velocity + ending velocity) / 2} \quad , \quad \text{mathematically:}$$

$$\text{distance} = \frac{(v1 + v2)}{2}(t) = \frac{(v1 + v2)}{2}\,\frac{(v2 - v1)}{a} = \frac{v2^2 - v1^2}{2a} \quad , \quad \text{mathematically:}$$

From the above equation (*) for distance, if the initial velocity (v1) is equal to 0, such as for a still object dropped from a height, then the equation reduces to:

distance = $at^2 / 2$

Work = energy = (force) (distance) = (ma) $(at^2 / 2)$ = $m(a^2t^2) / 2$, joules
Since **$v = at$** , and squaring both sides: $v^2 = (at)^2 = (a^2t^2)$, and substituting this into the above equation:

$$\mathbf{Work = energy = KE = \frac{mv^2}{2} \text{ , joules}} \quad \mathbf{: KE = Kinetic\ Energy} \text{ , motional energy}$$

Note for example, that if a moving object gains a KE value of 10J, then it took, or will take or require, 10J of input energy so as to give it that amount of KE. If you know an objects KE, you can find its velocity from:

$$\mathbf{velocity = \sqrt{\frac{(2)(KE)}{mass}} \text{ , (meters / s)}}$$

For a given or constant amount of energy (KE), if mass increases, the velocity decreases, and this is an inverse relationship.

Energy = Work = Joules = (force)(distance) = (ma)(d) = Fd = N-m = Joules , mathematically:

$$\mathbf{Force = (mass)(acceleration) = ma = Newtons = \frac{Joules}{meter} = \frac{J}{m} = N} \quad \mathbf{: J = N\text{-}m \quad : Netwon\text{-}meters}$$

$$\mathbf{F = \frac{J}{m} = \frac{J}{d} = \frac{Energy}{Distance} = \frac{PV}{d} = \frac{Pd^3}{d^1} = Pd^2 = P A} \quad \mathbf{: mathematically,\ P = F / A \text{ , or by:}}$$
P = pressure , F = force , A = area, V = volume

$$\mathbf{P = \frac{Energy}{(A)(Distance)} = \frac{J}{d^2\, d} = \frac{J}{d^3} = \frac{J}{V} = \frac{Fd}{V} = \frac{Work}{V} = \frac{Energy}{V} = \frac{F}{A}} \quad \text{from this we find:} \quad \mathbf{: V = volume}$$

$$\text{Energy} = PV = \frac{(F)}{(A)}V = P(d^3) = P(d^2)(d) = PAd = F\,v\,t = F\,d = \text{Work} = \text{Joules} = KE = M\,a\,d$$

$$F = M\,a = \frac{\text{Energy}}{\text{distance}} = \frac{J}{d} = \frac{J}{v\,t} \quad \text{, therefore,} \quad \text{Energy} = Fd = F\,v\,t = PA\,v\,t = \frac{M\,v^2}{2} = KE$$

$F = M\,a = P\,A = \frac{KE}{d}$: here showing that Force is the application and-or transfer of energy to another object.
Since $d = vt$: $F = KE / d = KE / vt$, $Fv = KE / t = \text{joules} / t = \text{Power}$

Special note: Since F = PA, if the force is a specific value or constant, pressure and area are inversely related to each other. If one increases by (n), the other will decrease by (n). If the area is a very small area, such as that at the tip of a nail, the pressure value induced by that force will be very high.

$$F = PA = 1\,(PA) = \frac{(n)\,PA}{(n)} = (nP)\,\frac{(A)}{(n)} = (nP)\,(1/n)A = (1/n)\,P\,nA \quad \text{: for a constant amount of force}$$

Considering a gas is in a container, or a fluid in a pipe, Pressure and Volume are inversely related, and are reciprocals of each other if their product is a constant value such as: Energy = PV . We can create a relative analysis and let Energy = 100% = 1, we have: 1 = PV , and P = 1 / V , and V = 1 / P . Clearly, P and V are inversely related to each other, and are reciprocal in value to each other. If one factor changes by a relative value or factor of (n), the other will change by the reciprocal of that factor = (1/n). Given a container with an amount of warm gas which has an amount of volume associated with it, if you press down on a compression rod and piston inside that volume, you will compress that gas. The amount of gas and-or its mass is still the same, but the volume of that gas has decreased, and the pressure of the gas has increased. The gas density = (amount of gas / volume) = (mass / volume) has also increased since the volume has decreased. Note that even though the density of the gas has increased, the mass is still the same value for this system.

Further Kinetic And Force Equations To Consider

$v = a t = \frac{f}{m} t = \frac{f t}{m}$: the velocity an object or mass can get after a force is applied for an amount of time
v=velocity , a=acceleration , t=time , m=mass , f=F= force = ma

$f = \frac{v m}{t}$: the force to apply so the mass has a certain velocity after a certain amount of time
d = v t , v = d / t , f = vm / t = (d/t)m / t = dm / t^2 , t = sqrt (f / dm)

$t = \frac{v m}{f}$: the time to apply a force to the mass so as to achieve a certain velocity
mathematically:

mv = ft : momentum

Work = W = E = F d joules , d = v t = E / F , Given a certain or limited supply of energy, and a certain value of constant force (ma) applied to move an object (m), it will move a distance of (d). Also, let: f = F = E / d = E / v t = P / v

Work = W = Energy = E = F d = **F v t** in joules , **F v = E / t = Pw** = **P**ower in watts = Pw = joules / s
W = F d = E = P t = F v t joules , Force = F = ma = P / v = (E / t) / v = E / t v = E / d = joules / d
v = E / F t = P / F = P / ma , P = mav = m (F / m) v = **Fv , or from: v = Pw / F , F = Pw / v**
Pw = (power in watts) = Fv = (pressure)(area)(velocity) = PAv , F = Pw / v = ma = PA and Pw = P A v

f = **m a** = m (change in v) / (change in t) = m (d / t) / t = **m d / t^2** , **d** = **v t** = **f t^2 / m = mat^2 / m = at^2**
v = d / t = f t / m = mat / m = **at** = P / f , mv = ft , d = f t^2 / m , f = m d / t^2 , t^2 = m d / f
d / t^2 = f / m = a , and: d = v t = a t^2 = f t^2 / m = P t / f , P = E / t = f d / t = f v = f^2 t / m
t = d / v = sqrt (d / a) = sqrt (dm / f) = (E/ f) / v = E / (f v) = fd / mav , and:
P = E / t = J / s = V A = J A / Qc = f d / t = mad / t = mvd / t^2 = f v , f = ma = m v / t , t = mv / f
mv = f t

AN EXAMPLE OF FUEL USAGE AND A MOVING VEHICLE

The average car vehicle weighs about 4000 lbs =~ 1814 kg and has a rated distance to fuel usage of 20 mpg = 20 miles per gallon = 20mi / 1 gal = $32.18\underline{7}$ km / $3.78\underline{5}$ L , and after dividing both sides of this fraction by 3.785, we can reduce this fraction to: 8.504 km / 1L. 1 gallon = 128 fluid ounce = 128 fl.oz. = 3.785L and 1L = 33.814 US fl oz.

Ex. If a car is rated as having 20 miles per gallon, to travel 60 miles of distance will require:
total distance **/** (distance / gallon) = 60 mi / (20 mi / 1 gal) = 3 gallons. If gasoline is $5 per gallon, the total fuel cost is:
(cost per gallon)(number of gallons) = ($5.00 / 1 gallon)(3 gallons) = $15.00

If the car described above is moving at 60 miles per hour = 60 mph =~ 96.6 kph = roughly 100 kilometers per hour, the amount of kinetic energy (KE) it will have is:

KE = mv^2 / 2 = (1814kg)(100^2) / 2 = 9,070,000 Joules

As the car travels on a consistently made road surface, it will loose a certain value of that energy every amount of time and-or distance due to the friction or collisions of the tires ("wheels") and the road surface. To maintain speed and-or the same amount of KE, the car will require a certain amount of fuel so as to accelerate (in velocity) back up to the desired speed or velocity of 100 km/h. The faster the car, the greater the distance it will travel per unit of time, and the greater the friction losses per unit of time, and therefore, it will take more fuel and-or energy per unit of time to maintain a higher velocity.

Actual or current fuel used for a certain situation (ie., a certain vehicle , speed or velocity) can be described as:

(fuel usage) = (x gal / hr) : a fuel to time rate of using the fuel , this depends mostly on the speed , **or**
(fuel usage) = (x mi / gal) , and this is equal to the cars (typical or average) fuel rating
(fuel usage) = (x gal / mi) : this is the reciprocal of the above, but here, it is expressed as a fuel to distance rate

At 60 mph, a car rated at 20 mpg will require a fuel usage = (x gal / hr) of:

From: total = (60 mi / h) = (fuel rating) (fuel usage rate) = (20 mi / 1 gal) (fuel usage rate) , mathematically:
fuel usage rate = **(**60 mi / 1hr**)** **/** **(**20 mi / 1 gal**)** = 3 gal / 1 hr , and-or: 3 gal / 60 mi

This value of 3 gal / 1 hr can also be found using an equivalent fraction or "proportion equation":

$$\frac{20\text{ mi}}{1\text{ gal}} = \frac{60\text{ mi}}{X\text{ gal}}$$, solving for X we have: X = 3 , and if the amount of time was 1 hr, we have: 3 gal / 1 hr

$$\frac{3\text{ gal}}{1\text{ hr}} = \frac{3\text{ gal}}{3600\text{s}} = \frac{0.000833\text{ gal}}{1\text{s}} = \frac{0.107\text{ oz}}{1\text{s}} =^{\sim} \frac{3\text{g}}{1\text{s}} = \frac{1\text{oz}}{9.346\text{ s}} =^{\sim} \frac{28\text{ g}}{9.346\text{ s}} = \frac{1\text{g}}{0.333\text{s}}$$: 1oz = 28.35g

100 km / hr = 100 km / 3600s = 1 km / 36s = $0.0277\underline{8}$ km /s =~ 27.8 m / s =~ 91.21 ft / s
60 mi / hr = 60 mi / 3600s = 1mi / 60s = 1 mi / 1 min = 1 mi / 60s = 0.01667 mi / s = 88 ft / s

For the car to travel at a velocity of 60 mi / hr =~ 100 km / hr =~ 88 ft/s , the amount of fuel usage per second to maintain that speed or velocity will be about 3 g / s =~ 0.107 oz / s = 6.47 oz / 60 s = 6.47 oz / min. The useful (actual combusted fuel portion that is actually converted to forward thrust or movement) energy in 3 grams of fuel per second is what is needed to replenish the KE lost per second due to the weight and friction(s) (road, bearings, etc) at a velocity of 60 mi / hr =~ 100 km / hr . Technically much of the fuel to a typical car engine is eventually wasted as heat due to mechanical friction such as with the "drive train" or "power train" (the mechanical parts to transfer the energy from the engine to the wheels), and wheel bearing friction , and a low combustion efficiency, etc. An average car engine system is only about 35% at producing KE or forward motion from the fuel, and therefore, there is much wasted fuel and-or energy. The miles per gallon rating is the actual result or distance to expect after the fuel and-or efficiency losses.

Other considerations: A diesel fueled engine may have about twice the miles per gallon rating as that of a gasoline fueled engine. This is manly due to that the diesel fuel is denser and has more potential energy per unit of volume, and that the combustion efficient of that type of engine is greater. It is possible to use some types of vegetable oils as a fuel in a diesel combustion engine. A gasoline (fuel combustion) powered electric generator to produce electricity are only about 20% efficient maximum, and only when then having a relatively high electric load of 80% of its maximum rated electrical power output. Due to a diesel engine being more fuel efficient, and mainly due to that the energy density of diesel fuel is greater than that of gasoline, a diesel fueled electric generator will be about 40% efficient or twice that of a gasoline fueled electric generator. For general reference, a typical solar panel is about 18% to 22% efficient at producing electricity, however, the fuel (input energy) is essentially free sunlight for up to 8 to 12 hours per day, and hence it is wise to also charge up energy storage batteries for any system if possible. A battery also permits a temporary high amount of power that a common solar panel is not even capable of.

Extra note: Various fuels for various engines are often made from **crude oil** (**petroleum**) that is pumped out of the ground. Crude oil contains a mixture of various hydrocarbon (hydrogen and oxygen) molecules. This crude oil is heated (typically to about 370°C = °F) so as to cause it to evaporate. This is done inside what is called a **fractional distillation** chamber (ie., like a container vessel). This structure is vertical in construction, and the smaller. less dense, lighter, more volatile (ie., having a lower flash point) carbon molecules will rise further upward in the distiller and will then condense back into a liquid in the upper cooler areas. At the bottom of this chamber will be bitumen which is also known as asphalt or tar, and of which has a density of about 1.0 on average, and gasoline and other fuels will condense higher up at different levels, and to be drawn off by a pipe. Somewhat of a surprise is that the volume of the substances distilled from crude oil is about 7% more than the crude oil used, however the total mass and-or weight will be the same if all of the substances made from it are considered in the calculation. Why is the volume more? It is so because of the longer chains of the connected molecules and the larger molecular structures created in this process, and causing its size or volume to effectively expand in size. For example, if a gram of a substance had a volume of 1cc, hence a density of 1g / 1 cc , and after this same amount of mass expanded in volume to say 2 cc, its (mass to volume) density will then be 1g / 2 cc , and which this fraction can be reduced by dividing both the numerator and denominator by 2, and be mathematically equal to 0.5g / 1 cc, hence actually a lower (mass to volume) density than before the expansion and a corresponding lower weight to volume ratio. The weight to mass ratio, or mass to weight ratio will still be the same value.

Here are some of the distilled products from crude oil which has an average density of about 0.883g / 1cc. All oil products should be handled with care and with gloves and eye protection due to explosion, fire, corrosion and-or toxicity issues. Avoid breathing the gas or fumes of petroleum products. Exposure duration increase chances of cancer.

Product	**Average Or Typical Density**	**Typical % Of Crude Oil Volume**
Bitumen	1.0 g / 1 cc	3% , Used for asphalt or "tar" for making roads, etc. has a "thick" viscosity, hence low and-or slow flow rate, due to the attraction of the specific molecule - somewhat like a weak crystal. Between the Bitumen and Diesel evaporation layers, is various fuel and lubricating oils.
Benzene	0.876	:C_6H_6 , a solvent, such as for paints, and used in some fuels. **A known carcinogenic.**
Diesel	0.83 g/cc	20%
Kerosene	0.82 g/cc	10% Various hydrocarbon molecules from: $C_{12}H_{26}$ to $C_{15}H_{32}$, and this includes both Kerosene, Jet fuel, and some rocket fuel. : mainly a fuel for heating and lighting. **Paraffin** (0.8 g/1cc, alkane, C_nH_{2n+2}, and n has a range from 22 to 27) is nearly similar in density. Kerosene is sometimes called "heating oil" or "paraffin oil" and the word "kerosene" is based on the words for "waxy oil". Kerosene is sometimes used as a degreaser to remove oil off of metal parts so as to clean them. Kerosene, when mixed with oxygen, is highly combustible and is used in jet and rocket engines.

Jet Fuel	0.79 / 1 cc	: Jet engine fuel , a type of Kerosene
Naptha	0.765 / 1 cc	: a solveant, a fuel, used to help make plastics. Naptha has the chemical formula of: C_nH_{2n} , and where n has the range from 5 to 8.
Gasoline	0.75 g / 1 cc	45%, the majority product, and Gasoline, Diesel and Kerosene compose 82% of crude oil. : gasoline = "**petrol**" , a word derived from (liquid) petroleum. Dividing both the num. and den. by 1000, we have: 750g / 1000cc = 0.750Kg / 1L

Hydrocarbon Gas Liquids (HGL) 2% total , and here are some examples:

Butane	0.56 / 1 cc	: C_4H_{10}, is a "thin", low viscosity oil, high volatility and-or evaporation. Can be added to gasoline, and also combined with propane to make Liquefied Petroleum Gas (LPG). Note that Natural Gas = NG is mostly methane, and LNG is Liquefied natural Gas.
Propane	0.493 / 1 cc	: Propane, C_3H_8, is a gas at standard temperature and pressure, but it is often condensed and sold as a pressurized liquid gas in a hard metal tank, and having a l large amount of mass and stored energy. When the valve to the tank is open, it will reduce the pressure inside the tank, and the liquid propane will boil and create propane gas, such as used for heating, cooking and some combustion fuels for vehicles.

Mineral oil is also made from refined petroleum after fractional distillation. This is a clear oil often used to lubricate machine parts so as to prevent friction wear. About 1% of refined crude oil is mineral oil. High grade, clear mineral oil is sometimes used in the medical and food fields as being non-toxic and does not go rancid (essentially oxidize) in taste and smell like other oils can after they degrade from light and-or heat. Some people may have an allergic reaction to ingesting (ie., orally, eating) mineral oil. Paraffin, both liquid and solid forms such as for candles, is a product similar to mineral oil. Vaseline (tm, R) jelly is made from a mixture of mineral oil and wax solids. The main ingredient of baby oil is mineral oil. Mineral oil, a hydrocarbon, does not polymerize or dry to a harder state such as plastic-like coating that linseed and other oils can.

Plastics are long-chain hydrocarbons (ie., many hydrocarbon molecules joined end to end like a chain, and giving the plastic structure some pliability, flexibility, bendability), and which are often made from crude oil (petroleum) liquid into a solid material. Plastics are made applying further chemical processes to some of the refined crude oil products mentioned above, and can account for about 4% of the crude oil volume. Considering average and-or typical values: For harder plastics that sink in water, their density is about 1.28g/cc on average, and for soft plastics, their density is about 0.91, and for the foam types of plastics, such as styrofoam (a type of polystyrene plastic) that is often used for insulation and water flotation products. The density of styrofoam is about 0.1g/cc and which is much less than that of water which has a density of 1.0 g/cc.

Many types of materials such as plastic can and should be recycled when no longer needed. Some experienced hobbyists will first wash, dry and then nearly melt many (relatively soft plastic) water bottle caps and other small pieces of plastic of the same type in a home oven set at about 250 °F, and produce pressed and molded blocks before it cools. The result is thicker and seemingly harder bulk plastic material so as to be used for any purpose needed.

As of about the year 2020 or so previously, the oil industry is said to be at "peak oil", meaning that the known supply will most likely diminish as the years go by, and as the large, natural reserves of crude oil in the ground get removed and depleted. Other problems can reduce the supply of oil and its products, and we need to find alternatives to crude oil. It is difficult to obtain an equivalent energy density per gram, cc or weight of diesel or gasoline, but it is a goal to be considered when manufacturing modern super capacitors, batteries, etc. Where will these electrical energy storage devices obtain their energy from, and it also needs to be efficient, sufficient and with renewed or naturally replaceable quantities so as to be available at a low cost to many people in need? Solar and wind power and-or generate electricity seems to the "green energy" or "free" and renewable (replaceable, unending, reliable) method for now as of the year 2023.

A FLYWHEEL AND ITS MECHANICAL, KINETIC ENERGY STORAGE

A concept and mechanical device that is somewhat related to the concepts of the energy or power of gears or wheels is called a **flywheel.** A flywheel is essentially a rotating wheel, disk or object that can accumulate and store kinetic energy. A "spinning top" novelty device is a type of flywheel. The energy in a flywheel can then be used later to transfer and-or generate some mechanical and-or electrical power when needed. A flywheel can also be used to help maintain a desired rotational speed and-or timing for a machine, and this special version of a flywheel is sometimes called a "governor", and may be used for something such as to help control (ie., govern) the amount of fuel to an engine so as to keep a steady rotational speed of it and-or to maintain a certain speed of a vehicle that is moving forward. Controlling the speed of a machine or technical can be done by controlling a fuel intake valve or possibly an electric switch or resistor. A governor is therefore both a (speed, rotation, power) sensor and controller.

The amount of kinetic (movement) energy a flywheel can store depends on the mass of the wheel and how fast it is spinning (ie., its rotational or angular speed or velocity). Remember that kinetic energy is related to mass and the square of the velocity.

The larger the mass of an object, the more difficult it is to get it to have a certain (linear or rotational) velocity and-or stored kinetic energy value. A larger mass is then said to have a larger inertia (ie., resistance) to movement. The larger the mass of a flywheel, more torque (twisting force) will be required to get it to move (here, rotate). It will also need some extra initial or temporary force (input torque) so as to overcome more friction (on the bearing, pivot, axle or axis) due to the weight of that mass. As long as energy is being input into this system, input force or torque can be applied to the mass of the flywheel. Constantly applying an input force will constantly apply input energy and will cause an acceleration (change in velocity, here an increase), and the flywheel will rotate faster as it gains and stores more kinetic energy.

Torque = Twisting or Rotational Force = (Force)(leverarm) = (mass)(acceleration)(leverarm)

If the mass is increased, the more energy will be needed for it to have a certain amount of rotational kinetic energy. The amount of input energy required to maintain the rotational kinetic energy and-or velocity of the flywheel (or any other object in motion) will be very small if the friction and-or other energy losses (such as using its energy to rotate another gear or wheel) are small. If 100J of energy are lost each second in this system (ie., = 100 J/s = 100watts of power loss, work and-or transfer), in order for the flywheel to maintain its speed and-or kinetic energy value, an input of 100J of energy per second must be constantly applied to this system.

If a large flywheel that being used to store energy is spinning at its desired or maximum rated speed or rotational velocity, the input energy to the (output) flywheel can be temporarily stopped by stopping the input energy wheel or gear connected to it, and the flywheel will still continue to spin and is said to then have a "freewheel", "freewheeling" or continued spinning or motion. This input energy wheel or gear to the flywheel is usually called the **freewheel** or freewheel gear or mechanism, and it works by using a ratcheting type of gear system that can rotate in only one direction when and if it is being rotated so as to apply input energy to the output or flywheel. A common example of a freewheel system is used for the rear or drive wheel of most bicycles, and "pull (cord) start" combustion engines. People who study **mechanical engineering** will consider or include much about all types of gears and mechanisms, and are then able to imagine and-or design a mechanism when needed, such as for some machine at a factory, or for some part of a new product being manufactured.

A discussion and example of how much energy is stored in a flywheel is shown next:

The larger the mass of the balanced (ie., having a true disk shape), rotating flywheel, the more kinetic energy it can store. The larger it is, the more momentum it will have and resist changes such as small values of external force. A larger flywheel used as a governor for a machine will then be more stable at maintaining a desired speed if the energy losses, usually due to friction with the rotation support bearings, are kept at a minimum. Not only does a larger mass have a greater "resistance" (ie., inertia) to changes in motion, any flywheel, such as one called a **gyroscope,** also resists changes in axis tilt and-or direction, and this rotational and-or angular momentum (and-or "angular inertia") is used for maintaining a desired direction of which can even be a reference direction for steering such as for some satellites, etc. For a gyroscope to be in a satellite, it must be relatively small and lite-weight, and it can still have a high (angular or rotational) momentum and-or inertia by having a very high rotational speed.

Ex. A flat disk is essentially a thin section of a solid cylinder or rod. This disk has a corresponding area (A) and mass (m) associated with it. It is not difficult to imagine that a disk with half the volume or circular area will correspondingly have half of that mass and-or weight. If we can find the radius of a disk associated with half its area, then we will also know the radius associated with half the mass and-or weight of that disk. This amount of half-the mass is essentially the average amount of mass of the disk that is rotating and has kinetic energy, and can be considered and analyzed as equivalent to a single conceptual point having the same mass of the entire disk and orbiting (rotating) about the central point and having the kinetic energy of that entire disk.

The area near the outer rim of a disk has the most mass and (linear) velocity than the mass located at the area near the center of the disk, and can therefore store more kinetic energy than that of the slower moving mass closer to the center. Though the rotations per second (ie., angular speed or velocity) are equivalent for any point along a (rotating) radius line, their effective linear distance traveled is not the same, and the linear velocity is greater for points farther out along the radius line from the center. A mass going at a higher velocity will have more kinetic energy. The mass at the center of a disk has a relatively low kinetic energy and storage as compared to the "outer" mass near the rim. Because of this, some flywheels will have a thick outer "rim" of mass so as to help store energy there, especially if space is limited and-or to reduce the total weight and-or size of the device.

Given a circle or disk, here is how to find the radius (r) of a circle or disk that is equivalent to half the initial area (Ac):

From: Ac = (pi)r^2 Using a 1.0m as relative value for r, so as to have a relative type of result, the area is:

3.14159265 m^2 = (pi) (1m)^2 = (pi)(1m^2)

Dividing both sides by 2 so as to have half the area:

$$\frac{3.14159265 \text{ m}^2}{2}$$

$1.570\underline{8}$ m^2 associating this area to the area formula so as to find its corresponding radius value:

1.5708 m^2 = (pi) r^2 isolating r^2:

$$r^2 = \frac{1.5708}{3.14159265} = 0.5$$ after taking the square root of both sides so as to solve for (r):

$$r = \sqrt{0.5} = 0.707107 \text{ m}$$: at 70.7% of the initial radius from the center point, the area and-or mass of a circle and-or disk will be half. Incidentally, 0.707 is also half of the square root of 2, and which equals about 1.414....

If a uniform or complete (without gaps, spokes, and a larger rim mass) disk had a mass of 4kg, a radius of 20cm, and rotated at 5 revs (ie., revolutions) per second, how much kinetic energy is stored in that rotating disk?

At 0.707 of the radius from the center point will be half of the effective disk mass storing the energy.

(0.707)(20cm) = 14.142cm

For the linear velocity of a mass or point (ie., "effective point of equivalent mass") at 14.4cm from the center, we will first calculate its linear distance per revolution, and that is equal to the "effective" circumference of the circle or disk:

c = 2(pi)r = 2(pi) 14.142cm = 88.858cm

In 1 rev, a point at that radius will effectively travel 88.858.cm. If we multiply the number of revolutions by a value, then the distance will increase by that same value:

Let: 1 rev equates to 1 circumference (equivalent linear, one-dimensional) distance
1 rev = 1 circumference distance multiplying both sides by x, for a particular number of revolutions:
(x) 1 revs = (x) circumference distance
x revs = x (2(pi)(r) distance , using the given values, and dividing both sides by 1s, so as to have a rate:

$$\frac{5 \text{ revs}}{1\text{s}} = \frac{5\,(2)(\text{pi})(14.142\text{cm})}{1\text{s}} = \frac{444.28 \text{ cm}}{1\text{s}} = 4.4428 \text{ m / s}$$:an angular velocity converted to its corresponding linear velocity

$$\text{KE} = \frac{mv^2}{2} = \frac{(4\text{kg})(4.4428\text{m/s})^2}{2} \overset{\sim}{=} (4\text{kg})(19.74\text{m}^2/\text{s}^2) = 39.4\underline{8}$$ newton-meters = 39.48 Joules of energy
newton-meters = (F)(meters) = (ma)(meters)

It is clear in the above formula that if the (speed, rotational or angular) velocity of a flywheel doubles, then the kinetic energy stored within it will increase by 4, but it will then likewise then take four times the amount of its previous input energy for its velocity to then double:

KE at v=1 is: mv^2 / 2 = m1^2 / 2 = m(1) / 2 = 0.5m : here, m = mass

KE at v=2 is: mv^2 / 2 = m2^2 / 2 = m4 / 2 = m2 = 2m

$$\frac{\text{KE new}}{\text{KE old}} = \frac{2m}{0.5m} = \frac{2}{0.5} = 4$$, From this, a generalized formula for KE due to a change in rotational velocity is:

(KE new) = (ratio of velocities)^2 (KE old) = (v new / v old)^2 (KE old)

Another, and more general formula for the kinetic energy stored in a flywheel is: **KE flywheel = m w^2 r^2 / 2**
, where m = mass in kg, w = angular velocity in radians per second, and r = radius. Surely, to increase the radius, mass must be increased, but in theory, if the mass is a constant, perhaps by thinning the flywheel and then increasing (r), the KE of that flywheel wheel increase by 4 if r is doubled, and even if the angular velocity remains constant. Note, if r=1, and then doubles: r^2 = (2r)^2 = (2(1))^2 = 4 , and KE will increase by the square of the increase in r.

KE = mv^2 / 2 , v = effective linear or tangential velocity about the center

On a circle: **arclength distance = s = ¢ r** : **¢ = angle in radians , and mathematically: ¢ = s / r and r = s / ¢**
¢ is based on s / r , and is the definition of a radian when s = r , ¢ = 1 radian

v = d / t = s / t = ¢ r / t = (¢ / t) r = w r = (angular or rotational velocity) (r)
d = vt = wrt

KE = mv^2 / 2 = m (wr)^2 / 2 = m w^2 r^2 / 2

Angular velocity may be given in terms of rotations (rots) or= revolutions (revs) per second, and this is analogous to the number of cycles or frequency per second.

1 revolution = 1 rev = 1 rot = 1 cycle
1 rev / s = 1 c/s = a frequency of 1 hertz
1 rev = an angle of: 360° = 2(pi) radians =~ 6.28 radians

If something is rotating at N revolutions per second, that is then N cycles per second, and Frequency = F = N, and the amount of radians per second becomes multiplied by this value:

¢ / t = w = 2(pi) N radians / s = 2(pi) (revs) / s = 2 (pi) F radians / s

A point located at (r) from the center of rotation will travel the length of 1 Circumference / rotation. d = vt = C
After N rotations, the total distance can be found by multiplying both sides by N, and it will be N times greater:

Total Distance for the revolutions = D total = Nd = N vt = N C

At a rotational frequency of (N rotations / s), that point will travel a linear distance of N times faster (v) per same amount of time, and N times farther (d) per same amount of time.

Let's use the second (s) for time:

$v = d / t = \left(\frac{N\ rot}{s}\right) C = \frac{N(2)(pi)(r)}{s} =^{\sim} \frac{6.28\ N\ r}{s} = wr$: with units of: meters / s

$w = \text{angular velocity} = \frac{N(2)(pi)}{s} =^{\sim} \frac{6.28N}{s}\ \text{radians} = \frac{6.28N}{s}$: with units of: radians / second

If we let (N / s) = cycles, rot, or rev per second, F can be used to represent this value:

$v = (2)(pi)(N/s)(r) = (2)(pi)\ F\ r$: F = frequency = cycles per second = cps = rot / s = rev / s
= vibrations per second

$w = (2)(pi)\left(\frac{N}{(s)}\right) = (2)(pi)\left(\frac{revolutions}{s}\right) = (2)(pi)(rps) = (2)(pi)F =^{\sim} 6.28F$

(1 rps)(60s) = (1 rev / s)(60s) = 60 revs after 60s = 1 min of time, and this can be expressed as 60rpm

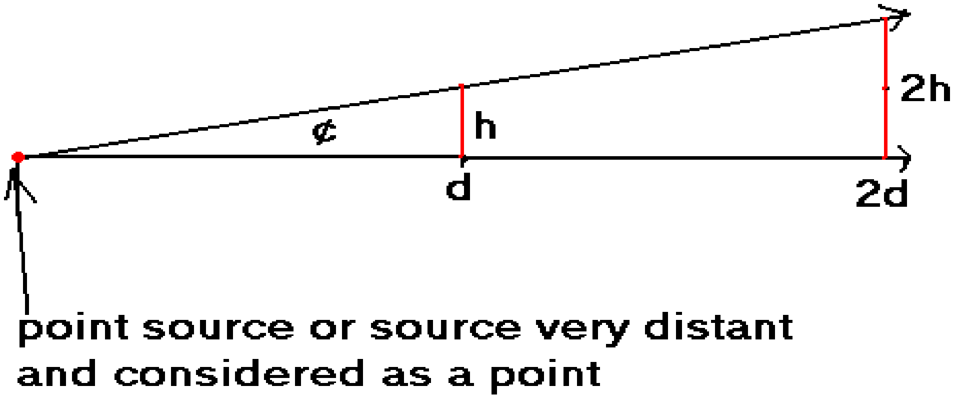
1 rps = 60 rpm : 1 revolution per second = 60 revolutions per minute = 60 revolutions per 60 seconds

Hence to convert rpm to rps, divide the number of rpm by 60. **rps = rpm / 60** . For example:

$120\ rpm = \frac{120\ rev}{1\ min} = \frac{120\ rev}{60s} = \frac{2\ rev}{1s} = 2\ rps$

MORE ABOUT THE INVERSE SQUARE LAW

This is perhaps a simpler verification that as the distance doubles from the source of energy, that the intensity at that distance is reduced by 4. A factor of 4 times less equals a factor of 1/4 = 0.25. [FIG 251]



At twice the distance from the source or vertex point of this right-triangle, the height will double. This is easily verified when you remember the simple and amazing fact that a similar triangle can be created by extending the sides of that triangle, all by the same factor, and here, it is 2. A right triangle, and half of a true distribution angle from a point source is used for this analysis, but the results are still the same. Incidentally, for this figure, if (d) is considered the center of a circle, this figure with similar triangles can be used to verify the fact that an inscribed angle is half of a central angle in a circle.

Now consider that these height values are the diameter of a circle area or side of a square area. At twice the distance, this length doubles and the area would increase 4. If h=1, this value squared would be an area of: 1^2=1. At 2h=2(1)=2, and this value squared would be an area of: 2^2=4. This is (4/1)=4 times the area than at 1d = d from the source. The energy from the source that reaches the area at distance (2d) sums up to the exact same amount of energy at the area located at (d), but it is then spread out becoming a more sparse, thinner, dimmer or weaker concentration, and is then actually 4 times less per unit area as that of the area at (d). 4 times less means a factor of 4 less which equals a division by 4: (1/4)=0.25 , which equals a multiplication by 0.25 = "a quarter of". In a reverse manner, it could just as easily be said that at half the distance to the energy source, that the energy will increase by 4 per unit area.

A simple formula for all this would show the inverse relationship of intensity (I) and distance (d), and specifically the relative distance in terms of the ratio of the distance in question to (d). Also, its more than an inverse relationship, (I) is inversely related to not only the distance, but to the square of the distance. This is the meaning of what is called "an inverse square relationship". In a pseudo-type of equation using a relative value, this information can be initially expressed as :

$I = 1 / d$

1 = 100% of the true, theorized or measured intensity at that location. When distance doubles to 2d, the intensity = I per unit area decreased by 4. This can be mathematically represented as:

(In the above drawing, consider a square area with (2h) as the length of its side. This area is then 4 times more than at (1h), and all the energy or intensity at (1h) has expanded outward into to 4 times the area, with each unit of area receiving less, and is actually (1/4) only.)

$$\frac{I}{4} = \frac{1}{2d}$$

When the distance tripled to 3d, the original or starting intensity decreases by a factor of 3^2 = 9 per unit area:

$$\frac{I}{9} = \frac{1}{3d}$$

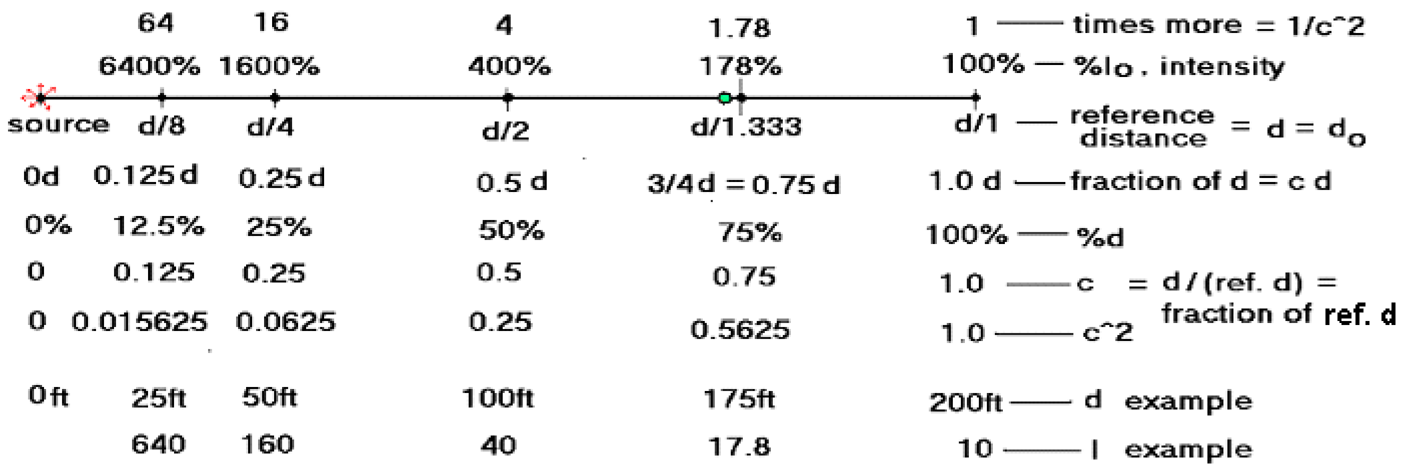
We notice a pattern that (I) decreases by a factor that is the square of the distance factor as (d) increases one unit. Therefore, the resulting intensity, say, I_1, can be expressed as being inversely related to the square of the distance factor (c):

$I_1 = I_0 / (c\text{^}2d)$: I_0 = starting reference intensity level value, I_1 = intensity level value at some distance (d) away.
(c) is the distance (multiplying) factor which is how many times (d) increased.
If we let (d)=1=100%, be the initial reference distance or location, and (c^2d) equal the total relative (ie., as magnification factor from) distance from the reference location, this equation's denominator will become: (c^2)(1) = c^2 = total distance from the reference location:

$I_1 = I_0$ / c^2 **:(c) = (distance in question) / (reference distance) =**
(c) = distance magnification factor or relative distance from the source

Io is the intensity at a reference distance or location and is considered as a relative value of 1=100% , I1 or I is the intensity at the distance in question from the reference location.

As you may see in the chart below that helps visualize this topic, 1/c^2 = number of times or "power" that (I) will increase as the distance to the source decreases. This charge assumes a spherical radiation pattern from a point source, and not the radiation or light from from a laser which has a focused, narrow (low angle, ideally 0°) beam or radiation pattern that can go a much greater distance and nearly maintain the same amount of energy and-or intensity. [FIG 252]



We could assign d_o = ref. distance, and d_1 = the distance from the source , $c = d_1/d_o$

I doubles = $2I_o$ = 200%I_o at : 70.7%d_o = 0.707 d_o = d_o / 1.414 . c=0.707 . c^2 = 0.5

When the distance from the source or measurement doubles, I decreases by 4. Likewise, in a reverse mathematical manner, when the distance at a measurement is halved or divided by 2, I increases by 4.

Ex. If the radiant energy intensity at a distance of 1 mile from the source is measured to be: I_0 = 100 units / m^2 = 100 units per square meter, what is the energy intensity at a distance of 2 miles:

c = (distance in question) / (reference distance) = 2 miles / 1 mile = 2
c^2 = 2^2 = 4

$I_1 = I_0$ / c^2 = 100 units per square meter / 4 = 25 units per square meter

This is also the result if it was said that the distance doubled = 2. 2=(c) is the relative distance factor value. Now consider this reverse-type of example:

Ex. If the energy intensity at a distance of 2 mile from a source is measured to be: $I_0 =$ 25 units / m^2, what is the energy intensity at a distance of 1 mile, or at half (1/2 = 0.5) the distance to or from the source:

c = (distance in question) / (reference distance) = 1 mile / 2 miles = 0.5
c^2 = 0.5^2 = 0.25

$I_1 = I_0$ / c^2 = 25 units per square meter / 0.25 = (4)(25 units per square meter) = 100 units per square meter

It could be said that when the energy intensity increases by 4, that distance is at halfway (1/2 = 0.5) to the source of that energy.

Ex. At what distance from the current location and distance is the intensity level at just 50% = 0.5 = 1/2 = half of the current intensity (Io) level:

$I_1 = (0.5)(I_o) = \frac{I_0}{c\text{^}2}$: in some cases I_0 can also be set or considered 100% = 1. Solving for c, the distance ratio:

$c = \sqrt{(I_0 / (0.5\, I_0} = \sqrt{1/0.5} = \sqrt{2} = 1.414$

Since: c = (distance in question) / (reference distance) then: (distance in question) = (c)(reference distance)

If the initial distance is 3 miles, the new distance where the intensity is at 50% is: (1.414)(3 miles) = 4.242 miles

Ex. At what distance from the current location and distance is the intensity level at just 200% = 2.0 = double or twice of the current intensity level:

$I_1 = (2.0)(I_0) = \frac{I_0}{c\text{^}2}$: in some cases I_0 can also be set to 100% = 1. Solving for c, the distance ratio:

$c = \sqrt{(I_0 / (2.0\, I_0)} = \sqrt{1/2.0} = \sqrt{0.5} = 0.707$

If the initial distance 3 miles, the new distance where the intensity is at 200% is (3 miles)(0.707) = 2.121 miles

If the intensity increased by 400% = 4 , you went closer to, and halfway to the source of that energy or electro-magnetic field, and the distance to the source is then the same distance that you initially went to be closer. As you keep halving the distance to the source, the intensity will keep increasing by 4.

$I_1 = I_0$ / c^2 If I1 is 4 times that of the reference intensity Io :
$4I_0 = I_0$ / c^2 After solving for c:
c = 0.5
c = (distance in question) / (reference distance) = 0.5 : = 50% , this indicates the distance was halved.

If the reference distance is unknown, a relative value of 1 = 100% can be used, and the results will also be relative since (c^1) is a relative (linear factor) value.

Ex. If I_1 was equal to I_0 increased by 50% = 0.5 of its initial reading that was farther from the source, write an expression for I_1:

$I_1 = I_0$ + (increase in I_0) = $I_0 + (0.5\, I_0)$ = Io(1 + 0.5) = Io(1.5) = 1.5Io , = equivalent to 150% Io

At what distance will this intensity be located?

$I_1 = 1.5Io = Io / c^2$ Letting Io = 1, for a relative value, or use any specific value:

$1.5 = 1/c^2$ Solving for c:

$c = \sqrt{1/1.5} = \sqrt{0.667} = 0.8167$

(distance in question) = (c)(reference distance)
(distance in question) = 0.8167(reference distance)

If we let the reference distance = 1 or 100% of some distance, we have:

(distance in question) = 0.8167 of some unknown or any distance = 81.67% of the reference distance

If you are now closer at the 81.67% of the distance from the source, you went (100% - 81.67%) = 18.33% = 0.1833 of the way to the source. The remaining distance to the source can be found by knowing the distance you went closer to the source. If you went 3 miles closer:

3 miles is to 18.33% of the total distance, as x miles is to 81.67% of the total distance

$\frac{3 \text{ miles}}{0.1833} = \frac{x \text{ miles}}{0.8167}$, x miles = 13.367 miles , and the intensity at this distance from the source is 1.5Io

The total distance at the initial intensity (Io) known, measured and considered as 1=100%, to the source is:

$13.367\text{miles} + 3\text{miles} = 16.367\text{miles} = d_o$
This distance can also be checked from the fact of: c = distance in question / reference distance = d1 / do

do = d1 / c = 13.367 miles / 0.8167 = 16.367 miles

As shown above, the steps for this finding, by calculation, the distance to the source, you must have an initial intensity reading or measurement, the distance moved closer to the source, and then an intensity reading or measurement at that closer location.

Ex. What is the Intensity factor (n) when distance doubles (2)?

c = 2 = (distance in question) / (reference distance) = (2 reference distance) / (reference distance) = 2
$c^2 = 2^2 = 4$: this was squared since intensity is inversely related to the square of the distance

Note: since intensity decreases the farther from the source, this is a decreasing or dividing factor, and we will therefore take its reciprocal which is equivalent to dividing by that value:

$(n) = 1/c^2$ = a decreasing, intensity factor = (1/4) = 0.25
$I_1 = n\ Io = 0.25\ Io$: when the distance from the source is doubled, the intensity is only 25% of Io

Laser Light Laser means **L**ight **A**mplification by **S**timulated **E**mission Of **R**adiation, and other electromagnetic waves, such as radar, is not usually radiated outward in a sphere or wide angle pattern, but is emitted as a narrow angle, concentrated and powerful long distance energy (here, light) beam, and the inverse-square concept will not apply to narrow (low angle value, ideally 0°), low dispersion, transmission, or radiated "direct beam" energy transmissions. The dispersion or beam angle of this light will be a very low angle. When the diameter of the beam eventually doubles, its illumination area will increase by 4 units^2 more, but its intensity will decrease by 4 units per unit of area.

Units for the amount of visible light

To measure the amount of visible light, units of measurement or reference need to be developed and standardized. The amount of visible light transmitted from a light source is also called the intensity or brightness of that light. When light amount in a unit of area is concentrated to be in a smaller area, its intensity or brightness seen will increase. In terms of quantifying this mathematically, we have: **light intensity = light energy / area** , and this value could also be called the energy density and-or concentration of that light energy.

The amount of reflected light transmitted from an object and-or surface is called **Lamberts** , and the reflection of the light upon it is how we can see objects, and it is also very important in photography. For example, a light source that transmits much light energy does not always mean enough light is being transmitted to and-or upon an object, and-or reflected from it. The more photos emitted from a source, the more light energy is being emitted and the brighter it will be for that particular (mono-chromatic) light frequency or range of light frequencies it was composed of, and of which is seen as a certain color which can be a mix of light frequencies and their corresponding colors.

Before LED's and other modern form of lighting, the units for light intensity were based on something commonly available, and that is the light emitted from one candle as the light source, and it is roughly a point source that transmits light energy uniformly (ie., the same amount) in all directions Since it is very possible that different candles and their construction can emit various amounts of light, etc., the construction of the reference candle for the reference units of light intensity or brightness had to be defined also. For a general study, a commonly available wax candle can be used, and when also considering it as a fire hazard. A candle in a tall glass jar, perhaps having an inch of water in the bottom is one solution to prevent fires. A metal lid with holes in it, or a metal wire mesh (ie., an array, interwoven, and with air and heat holes) lid can prevent most material(s) from getting in this jar and possibly causing a fire.

Note that if a light reflector(s) is used behind the candle, much of its light energy can be reflected and directed forward, and this will effectively increase the light energy (light amount), intensity and-or brightness in that direction as if a candle itself was producing more (perhaps 3 time) light energy. Many modern LED's and-or flashlights use a reflector just for this purpose, but it is also possible to use a light reflector for many light sources. The amount of available light energy can also be increased by a factor of approximately (n) if (n) nearby candles are used, that is, the resulting sum of light energy transmitted is equal to the sum of the light energy transmitted from each candle. Likewise, the resulting sum of light energy through or upon an area or surface at a given distance away from the source is equal to the sum of the light energy from each source, and through or upon an area or surface at a given distance away from the source. In a reverse type of manner, if the distance between the light source and the object being illuminated by that light is decreased by a factor of (n), the light energy, intensity and-or brightness (here, in a basic or simple reasoning, and since our eyes and mind actually sense light levels in a logarithmic manner) will increase by the square of that factor, hence by (n^2). These values are due to what is called the inverse square (of the distance) relationship between the amount of radiated energy received from a point source and the distance from it. In short, the intensity of the energy will be reduced at an exponential (ie., increasing and non-constant) decline and-or rate (like the speed of the corresponding changes), and the decline and-or difference in value will therefore increase significantly for and between each unit of distance further away from that source.

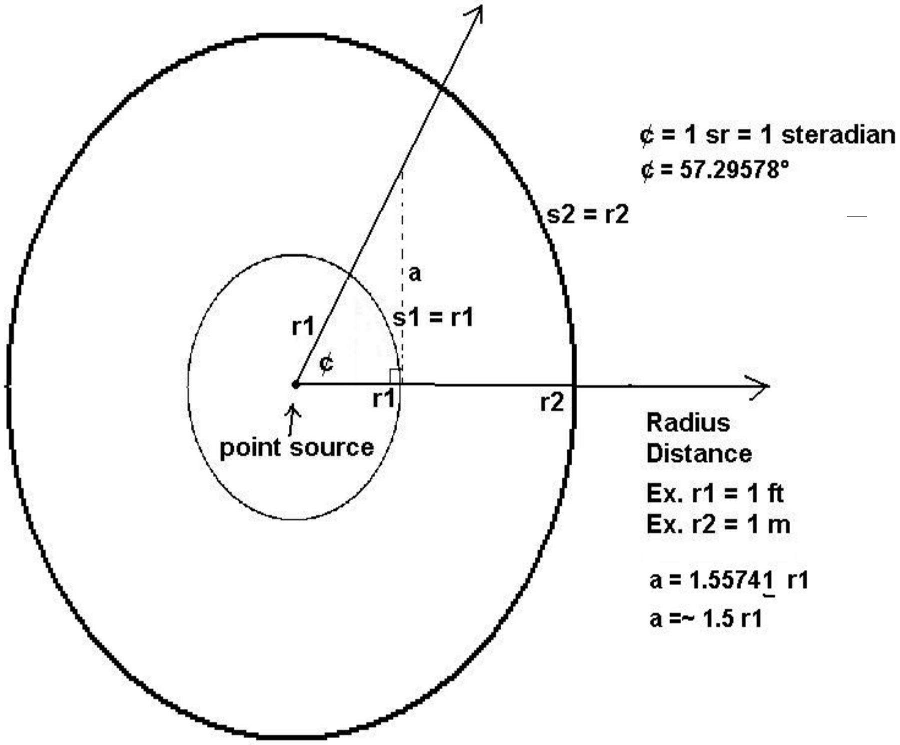
Because the candle is practically a point source of light, it radiates it outward like an expanding sphere of energy, and the amount of energy, intensity or brightness of the light available (to be seen and-or be upon a surface) per unit of area will decrease, and it is inversely proportional to the distance from the source. If the distance from the light source increases by a factor of (n), the light energy decreases by the square of that same factor, hence decrease by n^2, and which is the same as multiplying the intensity by (1/n^2) which equals reciprocal of n^2. Still, the total available energy has not changed, but it has dissipated or "spread out" and "got thinner (less of a concentration or intensity)" over a larger area, and the light energy, intensity or brightness per unit area has decreased. Due to this fact just mentioned, the energy, intensity or brightness seen, must then be measured at a particular amount and-or standardized unit of reference distance from the light source. Common units of reference distance when measuring light energy, intensity and-or brightness are the foot (ft) and the meter (m). 1 ft = 12 in. = 30.48 cm = 0.3048 m = about a third of a meter. 1 m = 100 cm =~ 3.28084 ft =~ 39.37 in =~ 39 in + (3/8) in =~ 3 ft + 3 in + (3/8) in and this value if very close, slightly less and in error or difference by only about: - 0.005 in

Light (or "light flux") energy, intensity or brightness transmitted from its source, such as the candle defined and standardized as the reference source of light energy is measured in units of **candlepower (cp)**, and more modernly as **candelas (cd)**. Note that the value of candlepower does not include its thermal energy created and-or its thermal power (energy/time), hence the word "candlepower" was not very descriptive of what this unit was intended to measure, which is light energy and not including the significantly more amount of heat energy produced at the candle's flame combustion of the hydrocarbon, wax fuel. Consider that a candle produces much more thermal energy than light energy. In the numeric sense of comparing output energy and power, a candle makes a much better room heater than a light source. The next and better choice of a word chosen to describe (visible) light energy, intensity and brightness is the word candela (cd), and of which is also like a redefined standard and-or recalculated value of a candlepower (cp):

0.981 candela (cd) = 1 candlepower (cp), and for most general purposes, they are considered equal in value and may be used interchangeably for non-critical calculations.

A candlepower was defined as the light intensity **[**and not a candle's thermal energy which is about 80W, and of which can heat a room if done safely in a large glass jar, however gas fumes and particles from a candle can become a health problem unless vented to the outside, and this is very possible to do and construct with a glass sided box and a pipe at the top that directs the particulates to the outside of the dwelling, meanwhile radiating as much heat as possible into the dwelling when needed**]** of a specifically constructed (formula and dimensions/design) reference candle with a burning flame radiating light or light energy onto a 1 square foot surface at a distance of 1 foot from just 1 candle. If this 1 foot value is considered as a radius value of a sphere, the sphere would have a surface area (= 4(pi)r^2 = 4(pi)(1^2)) of about 12.57 square feet, and if each square foot was illuminated by an energy or intensity of 1 candlepower of light, the source would then have a total of 12.57 candlepower of energy. Note that this does not mean a candle has a total light output of 12.57 candlepower, when in fact, the reference candle has a defined output of 1 candlepower =~ 1 candela. At
(1 candela / 12.57 steradians) of light energy transmitted about that light source at the center of an imaginary sphere, each steradian and-or square foot of that spheres surface at 1 foot away will have a light energy or intensity of the equivalent fraction of:: 0.079554494 candela / 1 steradian =~ 0.08 cd / sr

A figure to help understand a steradian and its size [Fig 252A]



As mentioned previously, the candlepower unit has been redefined to a better [than using various error prone candles and their expensive ingredients (materials, substances, wick dimensions, etc.) and constructions (design and manufacturing)] calibrated standard as the light energy from molten platinum at its minimal melting temperature, and shining (transmitting its light) through a 1 square centimeter opening. This amount of light energy or intensity is equal to 58.9cp =~ 60cp.

The candela was later redefined as the light energy and intensity having a single (mono-chromatic light) frequency of 540 Thz = 540 (10^12) hz, a green color, and having a power of (1/683)W =0.001464128 watts =~ 1.5 mw per steradian angle of transmission or radiation from the (point) source. This amount of energy can also be measured by the temperature increase of a (theoretical, ideal) **blackbody** or surface that absorbs (and does not reflect) all electromagnetic radiation, including infrared light (ie., heat energy and-or radiation and light), and its temperature will increase because the light energy absorbed was transformed into thermal energy and a corresponding increase in temperature of that surface and-or mass. A blackbody can also radiate the thermal energy and-or thermally caused electromagnetic frequencies of energy such as light. The frequencies of radiation it emits is determined by its temperature. If you interested further about this subject please research Planck's Law and "quanta" which is a discrete, minimal amount of energy for a photon of electromagnetic or radiation such as heat and light, and this was significant for the study of quantum theory. This minimal amount of energy per photon is a constant at any frequency, and is called **Planck's constant** = h.

As indicated above, the frequency of light and its specific corresponding energy used for the definition of a candela corresponds to a green color which is (supposedly) the color that the human eye is most sensitive to. Don't confuse the light energy or intensity of 1 candela with the emitted heat energy due to 1 burning candle, that is, a candela is a measure of the light energy only, and does not include a candles heat energy which is a much higher value than its light energy, and is actually about 80 watts (= 80 joules of energy / second) of heat energy.

Lumens (lm) are a defined measure of the energy or intensity of the light source (which has a particular value of energy called candlepower = candela, and if the light source is 1 candle, the total light energy from it is 1 candlepower) within a cross section of 1 radian angle which is also called a steradian (sr or sd) or square angle, and measured at a distance of (r) and upon a surface which will then have an area of r^2.

A **lumen** (lm), is a Latin word for light. Related word are: "illumination", "luminosity" = "brightness" or "light measure". A lumen is equal to the amount of visible light **energy** and-or power (ie., joules) emitted from a source of 1 candela (=~ 1 candlepower) for 1 second in a light beam having or limited to, or measured at, the dimensions of 1 **steradian** (sr or sd, "a solid or 3 dimensional [1 radian] angle"). A steradian is like a square shape and with opposite sides separated by an angle of 1 radian (= 57.2958°) from its vertex (at the center of an [imaginary] sphere or point source of the light), and extending outward from the center point of a sphere to a distance of (r), that (imaginary) solid angle will then intersect an area equal to [(r) (r)] = r^2 on the surface of the sphere.

Circumference of a circle = 2 (pi) (r). If the arc length on the sphere is equal to (r), then that length can divide into that circumference a total of : C / r = 2 (pi) (r) / (r) = 2 (pi) = 2 (3.14159265...) = ~ 6.28 . The corresponding angle that this arc suspends is called 1 radian angle. In brief, an angle of 1 radian is created when the distance to the arc on the surface of the circle or sphere, and the arc length are both equal to (r). Why all the fuss, why use radians and steradians? It is because these values are natural to all (and similar) circle, and 360° is a man-made value and-or angular system.

When the arc-length (s) = s = r: C / r = C / s = 2 (pi) (r) / (r) =~ 6.28 as mentioned above, hence a circumference has a total angle of (2 (pi) radians =~ 6.28 radians = 360° The length of the radius and-or arc-length of 1 radian angle angle can go about the circumference a total of 2 pi) =~ 6.28 times.

How many degrees is 1 radian angle? From the above equation, after dividing each side by 2 (pi) =~ 6.28 we have: 2 (pi) radians = 360° , as is to: 1 radian =~57.295778° . 360° / 57.295778° =~ 6.28

Extra: one-tenth of 1 radian angle and-or a tenth of its corresponding arc length would correspond to a tenth of 57.295778° = 57.29578° (1 / 10) = 5.729578° , and the arc-length also corresponds to an angle of 1 radian of a circle having a radius of 10 times less. Since these values are all linear and-or proportional, this concept can be used for any other value or factor than just 10, but using 10 as a divisor, it is easy to then find the quotient by moving the decimal point one digit leftward. Since the arc length is now 10 times less when the angle or radius is 10 times less, the corresponding area is (10^2) = 100 times less: (s/10)(s/10) = s^2 / 100 = A / 100, and the amount of light energy upon that amount of surface area will be 100 times less. If you were to use some lens and focus or concentrate (all that light energy at ro^2) = n lumens into an area that is a factor of (b) times smaller in units of area , (ie., Area / b), then the energy or lm per unit of area will increase by that same factor of (b), and here, the lm value measured for that area will be it will be a value of: (b) (n lm) .

At distances greater than (r) from a point source, the corresponding arc length (s) of 1 radian angle no longer equals the radius of the original circle, and the new arc length can be calculated from:

Since: C = 2 (pi) (r) = 2 (pi) (s for 1 radian angle) = 2 (pi) (distance from the center) , we have:

(s) corresponding to 1 radian angle = (distance from the center) = (new radius value = r)

Let It = total energy from the candle source = **1 candle power** = 12.57 lumens of light or illumination energy. Since 1 lumen is at (r) is dispersed over an area at r^2, and there are 12.57 sd sized areas about the point source or candle, therefore, there are (1 lumen / sd) (12.57 sd) = 12.57 lumens of light energy from 1 candle which has 1 candlepower of light energy.

2 candles flames side by side will then measure as 2 lm of candle light (illumination) on that same surface of 1 ft^2. Candle energy or candle power (candle energy / 1s = total candle energy/12.57 rads). For example 10W of light energy from the source will be (10W / 12.57 rads) at (r) and dispersed upon 1 square foot of area = 0.8W / ft^2 , but for an LED, this may be 1W / 12.57 = 0.08W / ft^2) at the reference measurement of 1 ft or 1m from the energy source, and here, the (number of candles at the source) = (number of lumens measured at ro or reference radius length) and the candlepower (cp) of the source = candela (cd) = (number of candles or=number of lumens at ro)(12.57).

Ex. 1 cp = 1cd = (1 lm / ro^2 at ro)(12.57 units^2 = 12.57 r^2) = 12.57 lumens *

We see that: cp = (lm at ro)/ 12.57 = number of candles used an emitting light power.
* We see that: lm at ro = (cp / 12.57) = light from a source of (cp) candles , upon ro^2 at ro

2 cp = light energy or power from 2 candles = 2 (12.57 lm) = 25.14 lm (= 2 lm at ro)

Ex. Two candles side by side will have a total of 2(12.57 lm) 25.14 lm of light energy, and will measure as 2 lm of light energy or power = (25.4 total lm / 12.57) at ro.
2 lm at ro will indicate the source of power is 2 candles = and producing a total of:
2 cp = (2 candles)(12.57lm/candles) = 24.4 lm

Ex. 100 lm total at a source is how many candle power at that source?
cp = lm / 12.57 = 100 lm / 12.57 =~ 8 cp : or use: cp = lm / (12.57 lm / cp)
lm = cp (1 cp / 12.57 lm) = 8 cp (12.57 lm / 1 cp) = 100 lm

Ex. If all the light energy = 1 candlepower = 1 candela from 1 candle is reflected and placed within a (concentrated) beam or beam angle of 1 steradian, that beam will have 12.57 lumens of energy at any cross sectional area at any distance from it. As the energy keeps getting farther and farther away from the source, the light energy or intensity (and measured in lumens) per unit area or energy per unit area will always be decreasing in value, and so the reference value of area is either 1 ft^2 (when r=1ft) or 1 meter^2 (when r=1m).

1 cp = 1 cd = 12.57 lm , multiplying each side by (n):
n cp = n cd = n (12.57 lm) = n (12.57) lm

1 cp = 1 cd = 12.57 lm , dividig each side by 12.57
1 lm = 1 cp /12.57 =~ 0.8 cp = 1 cd / 12.57 =~ 0.8 cd

lumens (lm) = energy intensity at a distance from the source, at 1 sd and on the standard reference amount of area, and of which is at r=1ft from the source, or r=1m from the source. If r = distance (d) from the source doubles, the light energy or intensity decreases by 4, and therefore, the lumens per unit area will decrease by 4.

$$(\text{ lumens per unit of area at d }) = \frac{\text{(lumens per unit area at do)}}{\left(\frac{d}{do}\right)^2}$$

: d = new distance from the source
do = a reference distance

1lm = 1cd within and onto a (ideally a black body; no reflection of energy received) (spherical) surface intersected by 1sd = 1 steradian or solid angle. If a point source of light radiates in all directions (ie., a spherical pattern), equal to: (surface area of a sphere) / (r^2) steradians = 4(pi)r^2 / r^2 = 4(pi) = 12.56637 steradians, the total amount of lumens emitted is: lumens = (candelas) (12.57).

1 lm = 0.08 cp : lm = lumens , cp = candlepower , 1 lm is a fraction of the light energy from a candle because 1 steradian angle is a fraction of the spherical transmission of the candle's light energy. mathematically:
1 cp = 12.57 lm

1 talbot = 1 lm of light energy / 1 s : this is like a light energy rate, hence (light joules / s) , however, there does not seem to be much use of this unit

Lux is the metric unit for light energy or light intensity, and it is very similar to lumen, except that the reference distance is 1 meter and the reference area is therefore 1 square meter. The reference angle is still 1 sd. and the total light energy or intensity at 1 ft from the source will equal the energy at 1 meter and so on from the source if the reference angle(such as 1 sd) is the same. In simple words, at 1 meter from the source, the light energy per same angle will equal that at 1 foot from the source, however, since the distance increased, the energy intensity or energy density = energy / (unit area) will decrease, but each area will still have 1 lumen of light energy upon its entire surface if the source is 1 candle transmitting or radiating 1 candle power of light (ie., photons, kinetic) energy.

Lux = light energy or intensity per or upon 1 square meter of surface at a distance of 1 meter from the source, hence lux = light energy / 1 sd. Since light energy can also be measured in lm / 1 sd , $\text{lux} = \text{lm} / \text{m}^2$, and $1 \text{ lux} = 1 (\text{lm} / \text{m}^2) = 1 \text{ lm} / \text{m}^2$

A good question is: Is 1 lux of light energy as bright as 1 lm of light energy, and the answer is no, and here is why:

First: $A = (r \text{ units})(r \text{ units}) = (r \text{ units})^2 = r^2 \text{ units}^2$

When $r = 1$, the corresponding area is: $1^2 \text{ units}^2 = 1 \text{ units}^2$
When r doubles from $r=1$to $r=2$, the corresponding area is: $2^2 \text{ units}^2 = 4 \text{ units}^2$, hence A increased by 4
$A1 = 4 (1 \text{ units}^2) = 4 Ao$

If the energy upon a surface of 1 units^2 is spread out over a larger area of perhaps 4 units^2, the total light energy there shining upon that larger surface is still the same value, however the energy density or light density, or energy intensity or light intensity has decreased by 4, and since brightness is proportional to light intensity, the apparent (and being more subjective at high levels) brightness will decrease by 4:

intensity (**I**) = energy density = energy / area. $Io = Eo / Ao$, and if area increase by 4, we have:the intensity decreasing by 4: $I1 = Io / 4 = (Eo / 4 Ao) = (Eo / Ao) / 4 = Io / 4$

Though at 1 sd, the total energy on the surface of 1 ft^2 will be equal to the total energy on the surface of 1 m^2 , we must consider energy density = energy intensity = energy concentration = energy / (unit area) , and this value of energy will actually decrease rapidly the farther from the source as it spreads out, and its value changes (here decreases) in a non-linear or non-proportional manner with respect to a change in distance, and is in fact it changes by an inverse square of the distance manner or factor of: $(1 / d^2)$. If d doubles (2) from 1 to 2, the value of this factor value decreases by 4. $1/d^2 = 1/2^2 = 1/4 = 0.25$. Energy density or brightness level at a distance (d) from the source = $E1 = Eo (1 / (d / do)^2$. Eo = energy density at (d). In a reverse or inverse manner, the closer you get to the source, the (measured) energy intensity will increase by a factor of 4 when the distance to it is halved, that is, it is essentially multiplied by a factor less than 1, and here it is a factor of $(1/2) = 0.5$ **Here is a general equation for the level of energy per unit area with respect to distance:**

$I1 = Io [1 / (d1 / do)^2] = Io (do / d1)^2$: do = original measurement reference distance to and-or from the source of energy , d1 is the new distance to and-or from the source of energy.

Ex: If the distance is now only half the distance to the source: $E1 = Eo (do / d1)^2 = Eo (10 / 5)^2 = Eo (2^2) = 4 Eo$.

Our eyes, if considered as light brightness sensors, do not sense light energy and-or its energy density in a linear type manner or relationship, but rather in a logarithmic type of manner or relationship. For example, when the light energy is increased many times, perhaps 10 times greater, we may only see that as 2 times greater and-or 2 times a brighter or intense, even though the energy density increased by the large factor of 10.

Note that theses have the same energy density::

1 lm / 1 ft^2 = 10.7639 lm / 10.7639 ft^2 due to equivalent fractions, and due to equivalent areas of 1m^2 = 10.7639 ft^2 (but having different units of measurement) :, this is also an equivalent measurement of the energy density of:
10.7639 lm / 1 m^2.

1 foot-candle (fc) = total energy or light upon a flat surface area of 1 ft^2 perpendicular to the source 1 foot away from 1 wax candle. If the surface was 1 meter away, and had an area of 1 m^2, the light intensity would be 1 lux, but since 1m is farther away than 1 foot, its energy density and-or intensity = brightness is less than the brightness of 1 defined foot-candle..

1 lm = 0.092937 ft-candles : about 9.3% of a ft-candle , mathematically:
1 ft-candle = 10.76 lm = 10.76 lux : and this is almost that of 1 cp = 12.57 lm

There are calibrated light meters called **photometers** that are available, and can measure light intensity using a small sensor and an electric circuit, and a reasonably well lighted room will have about 55 foot-candles measured near and upon its surface. It is possible to make a homemade light meter, but it will have to be calibrated, and it is very useful for experiments and data which can be used for various comparisons. There may be a certain distance used as a reference distance that is defined and-or necessary for some measurements and-or comparisons to other light sources. For example, the advertised light intensity of a certain brand of flashlight was measured by a photometer that was positioned one meter away.

Due to energy expanding outward from a point source in all directions, the greater the distance it is measured or seen at, the less intense the energy and-or light brightness becomes per square unit of area of surface according to the inverse square law (energy per distance) concept. Still, the total or collected energy emitted or measured per solid angle, such as 1 steradian angle from a source of energy is always the same no matter how far distant it was measured from the source.

From an ideal point source of energy which is being radiated in a sphere pattern, and since there are (4)(pi) ~= 12.57 steradians available in that sphere pattern, each steradian will contain:

total light energy available / 12.57 = 100% / 12.57 steradians = 1.0 / 12.57 sd =~ 0.07955 sd =
= ~ 8% of the total energy emitted from that source.

If you had an energy collector that collected the energy available in 1 steradian angle at any distance from the source, you can then calculate the total energy emitted from that source. For the previous values, if you collected 1w or 1 watt of energy per 1 steradian, and writing a proportion type of equation to solve for the total amount of energy emitted by the source.

If 1w is to 8% = 0.8, then Xw is to 100% = 1.0 as a relative and-or decimal value:

$$\frac{1w}{0.08} = \frac{Xw}{1.0} \quad \text{, after solving for Xw, we find:} \quad Xw = \frac{(1w)(1.0)}{0.08} = 12.57w$$

Although a steradian is formally defined as a cone shaped angle, if several of these cones were placed side by side, and just like two or more circles placed side by side, there would be gaps that need to be considered. To consider this fact, and the fact that any area can be represented as a square shape, the portion or area of a sphere intercepted by a radian angle can also be represented as a square shaped area of equal area so as to fill in the gaps. This square shape is called a square-steradian, and then the circular cone also becomes straight-sided, like a inverted pyramid shape. A smaller cone angle, but also in a square shape, is called a square-degree or **square-degee angle**, and this is based on a 1° solid angle from the point source. 1 sq. deg = 1deg^2 . Consider a circle or disk drawn on paper, and you draw lines outward from the center point and with each line separated by 1°, there will be 360 of these planar angles. If each 1 degree planar angle was rotated about the center, each and every one of the 360 planar angles could effectively create a total of 360 solid square angles. The total number of square-degree angles is therefore: (360)(360) = 129600 square-degree angles.

The light density from a source and-or upon an object (ex. 10 lumens/ ft^2) can be concentrated to be within a smaller (beam) angle and-or area on a surface being illuminated, and therefore it will have an increased intensity or energy density per unit area. In short, the light upon the subject (ex. wall, object being considered) will be brighter. This can be done by using lenses and-or curved (parabola-like) reflectors or mirrors.   Given (lm's / area), if the area decreases by a factor of (n), the corresponding lumen value will increase by that same factor of (n), and mathematically, both light energy density ratio's (lm/area) are equivalent fractions, and of which will resolve to the same number of lumens per unit of area. Since energy density is inversely related to area, it is not a proportional or linear mathematical relationship, but an inversely proportional type of relationship, hence an inverse mathematical type of relationship. When the area changes by a factor of (n), the, the corresponding number of lumens will change by a factor that is the reciprocal or inverse of (n), and that factor value is (1/n).

First:

Light Energy Created = Light Energy Out After Concentrating It   : ie., energy in = energy out   ,   power in = power out
candle power created = candle power out   : this value does not change by concentrating the available light

When the light energy density is concentrated to be a higher value, and the values of lm and A are inversely related:

Given, calculated or measured: When a variable changes by a factor of (n) in a inverse relationship:

$$\frac{lm}{A}$$

$$\frac{(1/n)\,(lm)}{(n)(A)} = \frac{(lm/n)}{(n)(A)} \quad \text{or} \quad \frac{(n)(lm)}{(1/n)(A)} = \frac{(n)(lm}{(A/n)}$$

Ex: (10 lm / 4 ft^2)   ,   (4)(10 lm) **/** (4 ft^2 / 4)   = (40 lm / 1 ft^2)   : area was decreased by 4 , and (lm/area) = light energy concentration or density , and brightness increased by 4

Note that due to the inverse physical and mathematical relationship , the fractions on the right and left are not equivalent fractions, and are not set as equal

A typical candle has a chemical (here, hydrocarbons in the wax) and-or combustion energy to light energy conversion efficiency of about 1%, and the remaining 99% is lost as heat energy. and the A typical candle will produce about 60W to 80W of heat energy.

A candle that has smoke or soot (mostly a carbon substance, mixed with some other substances) coming from it if the candle's heated and liquidfied wax not fully combusting properly (ie., incomplete combustion). The combustion process of candle wax will produce heat, light, carbon dioxide and some water molecules which are H_2O = hydrogen and oxygen. The lower part of a candle's combustion flame is higher in oxygen and has a blue color which is therefore high in (photon, light, rf, radiation) frequency and energy, and is the hottest section of that flame of which can be as high as about 1500°C for some candles.

The light output of 1 candle and-or its flame is said to be 1 candle or 1 candela = 12.57 lumens = 12.57 lm
This could be expressed as the light output from one lit (ie., ignighted) candle as being: 12.57 lumens / lit candle.
Having two lit candles side by side would produce about twice as much light and-or lumens (a measure or unit of light):
2 lit candles (12.57 lm of light / lit candle) = 25.14 lm of light

A candle can convert a (roughly) estimated value of 1%, or even less, of its combustion energy or input power to light energy and or power. The (energy and-or power) **conversion efficiency** = Pout / P in = P light / P in. The **efficacy** value is how much of that available output light energy and-or power is actually available upon the intended object and-or surface after various types of light production, transmission and-or interference losses such as a lamp shade which then effectively reduces the available light energy (lm / W).

Efficacy = (the output light energy and-or power applied to the intended object or surface) / (input energy and-or power). The units of measurement for the efficacy value are (lm / W). For a candle, and as a practical example, if 99% of the total energy is heat energy, then (100% - 99%) = 1% of the total energy is light energy. 99% of total energy is to 80W, as is 1% is to 0.8W, and from this, we can calculate: 12.57 lm / 0.8 W = 15.71 lm / 1 W = (1 lm / 0.063654 W) from a lit candle, and this can be used to measure the relative brightness and-or power of candles only since other light sources such as an LED typically have a higher value of:: (radiant light energy) / watt = for example: (100 lm / W) when it is concentrated into a narrow beam of light, perhaps within just 1 steradian (a 57.3° solid angle), or whatever the light (concentrated) beam angle is for that led flashlight due to it having a reflector and-or light concentrator, and then measured.

Note that the amount of input electric power does not mean there will be the same amount of light power emitted, and this is due to internal losses of that source lamp, and the major loss is due to heat energy also being produced.
If 1W (= 1J / 1s) of electricity or electric power is supplied and used for a lamp that has, say a 45% **conversion efficiency = (P light out produced / P electric energy in)** = 0.45 Mathematically: **P light = (P in) (conversion efficiency)**. For this example: P light = (1 W) (45%) = (1W) (0.45) = 0.45 W , hence (1W - 0.45 W) = 0.55 = 55% of the input electrical power was lost and-or wasted. It could also be said that the efficacy of the input electrical power was 45% when converted to and initially transmitted as light energy. Due to this reasoning, 1 watt of light energy does not mean or correspond to the same value, or, here for this specific example, 1 watt of electrical input power used to make it with that source having effective (light) energy losses due to it having a certain conversion efficiency that is less than 100%. How much electric power will then be needed for this particular source to produce 1 watt of light? From the above formula, we have: **P electric in = P light / (conversion efficiency)** = 1W / 0.45 = 2.222 W = 2.222 J / s . Note that this amount of input electrical power may not even be possible with some light sources such as 1 LED because it would get damaged and not function as a light source, and therefore multiple light sources will probably be needed to get a total sum of 1W of light energy. For example, if an LED had a conversion efficiency of 45%: let 2.222W = (electric power of 1 LED)(n LED's) = (0.06W)(n LED's). Solving for n, we have: n =~ 37 LED's . 37 LED's will be needed to produce 1W of light energy. Checking: (P / LED)(conversion to light efficiency)(37 LED's) = (0.6W / LED)(0.45)(37 LED's) = 1W. In general, most people want lumens or brightness from each light source instead of the total joules of light energy or watts (J/s) from several light sources.

In the Extras And Late Entries section of this book, there is an article on finding the conversion efficiency of an electric powered light such as an LED.

A more accurate analysis, measurement and calculation of a candle's light power = candlepower = candela was made by modern scientists, and the amount of usable or visible light energy from a candle is much less than 1% of the total energy of a typical candle that produces 80W, and of which is mostly heat energy.

1 cp =~ 1 cd = 12.57 lm = 0.0184 W of radiant light energy at 12.57 steradians = 0.001464128 W / steradian
: 0.0184W = 18.4 mW and 0.001464128 W =~ 1.47 mW =~ **1.5 mW**

$$\frac{12.57\ \text{lm}}{0.0184\text{W}} = \frac{683\ \text{lm}}{1\text{W}} = \frac{1\ \text{lm}}{0.001464\ \text{W}} \quad , \quad \frac{\text{light power}}{\text{heat power}} = \frac{0.0184\ \text{W}}{80\text{W}} = 0.00023 = 0.023\ \%$$

of a candles total energy is light energy,
= (a candle's electric to light conversion efficiency)

683 lm = 683 lm (1 cp / 12.57 lm) =~ 54 .34 cp = 1W of light energy = 1J / s

54.34 cp = 1W , and after dividing each side by 54.34: 1 cp = 0.0184026 W =~ 0.018W = 18mW of light energy
For comparison, an LED has about: (Pin)(conversion efficiency) = (Vin)(Iin)(%cf) = (3v)(0.020A)(45%) = (0.060W)(0.45) = 0.027W = 27 mW of light energy , this is a conversion efficiency of:
(light power out) / (light power in) = 0.027W / 0.060W = 0.45 = 45% . The LED has: 0.45 / 0.023 = 19.57 =~ 20 times more efficient than a candle. A typical 60W incandescent bulb can produce about 800 lm. 800lm/60W = 13.3 lm/W. A typical 0.06W LED produces 100 or more lm., and 100 lm / 0.6W = 1667 lm/W , and this is

(1667 lm/W) / (13.3 lm/W) = 125 times more lm per watt, and-or a conversion efficiency of 125 times more, and a therefore, 125 times less power is used. The LED's also produce much less heat per watt, and this is good in the summer time when trying to stay cool, and especially indoors where the temperature could be higher than outdoors.

It was mentioned that 55 lm per steradian is considered good lighting for a typical household room. A 60W, 800 lm incandescent lamp will therefore have: 800 lm total / 12.57 steradians = 63.64 lm / steradian, and this is slightly above the recommended 55 lm value for a well lit room. How many 100 lm LED's will create 800 lm? 800 lm / (100 lm / LED) = 8 LED's. How much electrical power will these 8 LED's use: (8 LED's)(0.06W / LED) = 0.48W , hence less than half a watt / steradian if the light from the LED is concentrated as such with its typical internal reflector. The incandescent lamp requires: 60W / 12.57 steradians = 4.77W / steradian, and this is (4.77W / 0.48W) = 99.44 times more energy or power, and which is nearly 100 times more energy and-or power required per steradian.

As of 2023, incandescent lamps are nearly all gone from common use, and for historical and-or reference purposes:

Incandescent Lamp And Rated Power	LM Rating	(LM / W)	(LM / steradian , rounded)	
40W	450	11.25	36	
60W	800	13.33	64	
75W	1100	14.67	88	
100W	1600	16	127	: about that of 1 white light LED

We see that the higher the input power of an incandescent lamp the higher the (LM / W) rating, hence the greater the conversion efficiency of electric power to light power. This is due to that when the filament in the bulb gets higher in temperature, it produces more light (photons) per watt and-or degree change in temperature.

Concentrating light energy

By using reflectors, the energy from a candle or light can be concentrated into a smaller angle or amount of steradians. If the number of steradians is reduced to a more narrow beam of light and having much more light energy can be created, and the number of lumens where it ismeasure will therefore increase. If the number of steradians is reduced by a factor of (n), the number of lumens per steradian will increase by that same factor in an inverse type of manner.

$$\frac{12.57 \text{ lm}}{12.57 \text{ sr}} = \frac{1 \text{ lm}}{1 \text{ sr}} = 1 \text{ cp} = 1 \text{ cd}$$

, and if we want to concentrate the light energy so to have (n) (1 lm) / sr = n lm / sr , we need to essentially concentrate the light's or source output transmission angle also hence reduce or concentrate it by that same value of (n). Generally, 1 cp is often noted as 1 cp = 12.57 lm of (total) light energy from 1 candle flame, as if all the light energy was concentrated or condensed into a smaller area such as 1 steradian and then this was the measured amount of all the lumens there, but the proper units for cp are actually:

1 cp = 12.57 lm / (sphere of outward transmission)

1 cp = 12.57 lm / 12.57 sr , and which is mathematically equal to the equivalent fraction of, and called: 1 candela = 1 cd :

1 cp = 12.57 lm / 12.57 sr = 1 lm / 1 sr = 1 cd

A reduction in the transmission angle (steradians) by a factor of (n) corresponds to an increase in the light energy concentration by that same factor of (n):

$$\frac{\text{steradians}}{n} \text{ corresponds to } n \left(\frac{\text{number of lumens}}{\text{steradian}}\right)$$

For conversions: 1 steradian = 57.3° solid angle
1° solid angle = 0.017452°

When highly reflective polished metal and-or mirrors were developed, it became relatively easy to concentrate the light energy from a candle (a wax fuel and-or powered lamp [light source]) or lantern (an oil powered or fueled lamp [light source]). by concentrating light, less energy and-or sources (such as multiple candles or other types of lamps) is needed, and this saves material, money and is just more practical to utilize. Some modern LED flashlights have a light concentration reflector and-or lens and possibly and adjustable beam angle which is done by effectively moving the lens closer to, or farther away from the LED(s). In general, low power red lasers can be purchased and used without a license, and for other colors, a license may be required for it to be used in some application.

A relatively recent (2015) development in LED flashlights is a **White Laser Flashlight**, and of which initially uses a blue laser light as the source which strikes a plate coated with yellow phosphor which will then create and re-radiate various other colors of light which will appear as white light. The result is a very narrow, concentrated or intense beam of white light which can travel several times farther than a standard high power LED flashlight which has a larger angle of dispersion or electromagnetic radiation. Their cost is still relatively high as of the year 2024. In general, many low power red lasers can be purchased and used without a license, but for other colors, a license and more precautions may be required due to the beam energy intensity, potential misuse and its dangers.

If you have a flashlight with a wide beam, say about 1 radian = 57.3° , you can experiment with concentrating the light from it by reducing the beam spread or transmission angle of that light by using a tube (ie., cylinder) shape of (shiny) aluminum foil. The shiny side of the aluminum foil should be on the inner surface of the cylinder shape. Make this tube shape about 12 inches long and having a diameter of about 3.25 inches. Perhaps you can also use a cardboard or plastic tube of similar dimensions. The size of the aluminum foil needed to make this shape will then need to be 12 inches long and having a width of (2 (pi)(r)) = (2 (pi) (d / 2)) = 2 (pi) (3.25 / 2) =10.21 inches, and this can be increased slightly to 11 inches so as to have some overlap of material to tape together more effectively. As you move the flashlight in and out of this tube, you will notice that the longer the tube reflector is, the more narrow (lower in angle) the beam of light.

Because aluminum foil is not a perfect reflector material that reflects 100% of the incident light upon it, some of the light energy transmitted from the flashlight will be absorbed, but still, the energy intensity or energy density (W/area = J-s / A), and light intensity (ie., brightness) of the beam will increase per unit area on a surface.

When a candlepower was redefined in terms of the most visible part of the light spectrum or range of colors.

The candlepower was effectively redefined by scientists to be in terms of just a portion of the light frequencies and their energy emitted, and this value was assigned as a new reference unit of light energy available. The new unit is called a **candela** and which is not exactly equal to 1 candlepower of light energy, and is rather a measure of light brightness or intensity, however, these values are very close in strict numeric or unit-less value and are usually considered as equal for most calculations. The light from a candle has various frequencies of red, green and blue light as its component frequencies and-or colors, and typically appears a faint or slight yellow color to our human eyes, and this could vary depending on the specific material construction of that candle being used, even though it was also defined to be made a certain standard way, but many people can not afford or have access these specially made (scientific, expensive) candles. The solution was to create a standard, reference frequency and-or color of light, and that is of a (single color and-or frequency, hence mono-chromatic) green color having a frequency of 540 (10^12) hz. This is basically the average frequency between the lower frequency red color, and the higher frequency blue color. Our eyes are also more sensitive to green colors than to red or blue colors. Due to this candela standard where some light energy (about 1.9%) was discarded from it, it is as if a candlepower of light energy was reduced in value and having a light energy value of (100% - 1.9%) = 98.1% of its defined value. If we then increase this candlepower value by 1.9% so as it to be 100% again, it will mathematically be equal to that of 1 candela:

1 cd = 1.0193$\underline{7}$ cp and mathematically: **1 cp = 0.981 cd** :(1.0 - 0.981 = 0.019 = 1.9%)

1 candela is still considered equal to 12.57 lumens.

Also, from the previous equations above, we have:
1 lm = (1 cp) (1 sr) = 1 cp-sr

1 lumen = 1 candle power of light = a total of 12.57 lumens transmitted spherically, at and-or through 1 steradian angle portion and-or its corresponding (imaginary surface) area which depends on the distance (r) from the source, however each area corresponding to 1 steradian angle, will have the same total amount of light energy passing through it, here measured in lumens (lm), passing (perpendicularly, at 90°) through it and-or upon its (imaginary) surface.

How much of the candles light energy and-or power is transmitted through, and-or corresponds to 1 steradian of the total of 12.47 steraidans of a sphere?

Let 1 cp = total light energy and-or power from 1 candle , and 1 sphere = 12.57 steradians , and mathematically: 12.57 sr = 1 sphere, and therefore, 1 sr = 1 sphere / 12.57 =~ 0.08 sphere = 8% of the sphere. If the source and-or transmission of energy was uniform and spherical, and due to that the total light energy level transmitted is proportional to the number of steradians involved, The lIght energy per steradian is about **8%** of the total amount of energy from the source.

Other units and facts pertaining to light:

foot-candles (fc) = lumens / 1 ft^2 : a type of light energy density measurement = light energy / area
A candle will transmit 1 foot-candle of light energy through 1 steradian through or onto a surface of 1 ft^2, and this area will be at a distance of 1 ft. from that candle. As the distance to the surface to be illuminated increases, the foot-candles or light energy density decreases rapidly due to the inverse square law relationship of (spherically transmitted) energy and distance. To achieve the same amount of brightness as that upon a surface that is a longer distance from the source, a light source transmitting more lumens is needed so as to have the same amount of foot-candle's of light energy. When the distance doubles from the light energy source that transmits spherically, say from 1 ft, to 2 ft, the energy density per unit area will decrease by 2^2 = 4. To regain the lost energy and-or restore the same amount of brightness, the amount of light energy from the source must then be 4 times greater, hence the lumen or candlepower value of it must be increased by 4, and so as to have the same intensity, brightness or energy density. An option to achieve this is to use 4 similar light sources or lamps.

1 fc = 1 lm / 1 ft^2 : as if the imaginary sphere had a radius of 1 foot, and each steradian had an area of 1 foot there.

lux = lumens / 1 m^2 : a type of light energy density measurement = light energy / area

1 m^2 = 10.7639 ft^2

1 lux = 1 lm / 1 m^2 = 1 lm / 10.7639 ft^2 , after dividing num. and den. by 10.7639, creating an equivalent fraction:
1 lux = 0.0929 lm / ft^2 = 0.0929 (lm / ft^2) = 0.0929 fc , and mathematically solving for (fc), we have:
Also: (1 lux is about 9.3% of 1 fc , or roughly 10% of 1 fc)
1 fc = 10.764 lux : =~ 10 lux

lamberts = is a measure of the light energy that was reflected from a surface. 1 lambert = 1 lumen / cm^2

> Named after the famous polymath (math and sciences, much like Newton) **Johann Heinrich Lambert**. Lambert is famous for his work in continued fractions, geometry, hyperbolic trigonometric functions, and optics. Lambert was the first to prove that (PI) was irrational. Lambert made progress in **photometry** which is the study and measurements of light, and progress in calculating orbits and locations of objects in space.

For an equation for the new energy level if the distance from the source changes: If the distance (d1) from the energy source changes by a factor of (n) so as to then be a distance of d2 = (n)(d1), the energy concentration, intensity or density (= energy / area) will change by the reciprocal or inverse of that factor, and this factor will then be: (1 / n^2):

(energy at d2) = (energy at d1) / n^2 : from n = (d2 / d1) , therefore mathematically: d2 = n d1
: relative energy values or use (energy / area) = intensity values

The color of a surface affects its reflectively of the light energy upon it.

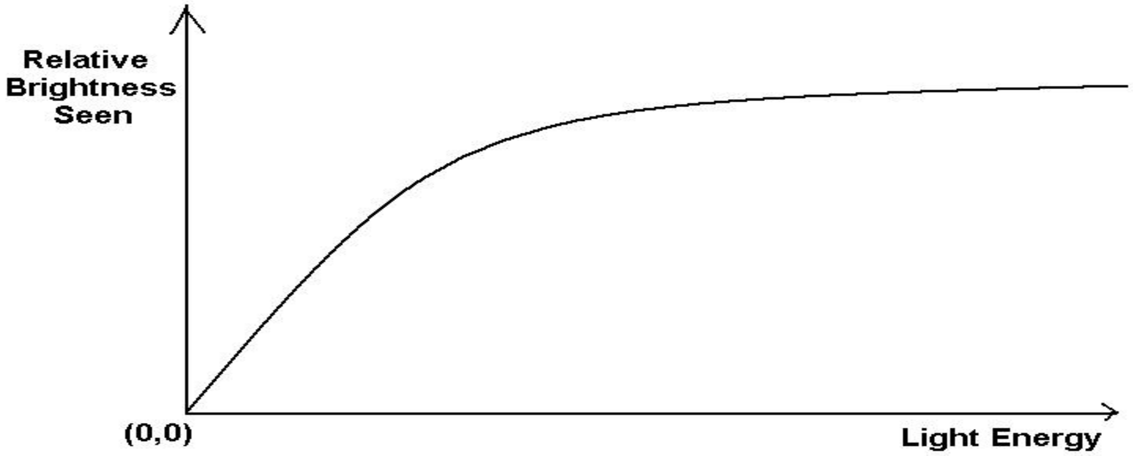
By increasing the amount of light energy upon a surface, the colors will appear brighter or more intense, and that is because the amount of energy of each reflected color has increased. Our eyes not only see or sense the colors associated with light frequencies, but also sense the amplitude or amount of energy of those light frequencies. When the reflected light energy is lower or "dimmer", the image detail or resolution is reduced, for example, making it more difficult to see clearly or read something, and the letters will appear as "fuzzy".

Most human eyes have special light sensing cells in them called cone cells, and there are three main types for sensing the three fundamental or **primary colors** or paint colors (ie., pigment small particles of matter having all the same color), as seen and not made by a direct (RGB) light source, but that of reflected light having a color or mix of : Red, Yellow and Blue (**RYB**) light, and each color has a different frequency and wavelength. The cone cells are of which is naturally tuned or resonant to a particular frequency and-or wavelength of these colors of light seen.

Our eyes are more sensitive to changes in dimmer or lower levels of light energy due to the logarithmic nature of our vision. As the light energy from a source or object increases to be a large amount, we generally can no longer sense small changes in "brightness" (ie., the energy level or amplitude of the light) when the amount of light energy changes..

The figure below shows how our eye respond to the energy levels of light, and it is in a logarithmic manner. When the energy levels or brightness is low, our eyes are more sensitive to the light energy changes, and when the energy levels or brightness is high, our eyes will usually not notice much change as the brightness or light energy level increases.

[FIG 252B]



The three **primary colors** of emitted white light are red, green, and blue, (RYB). Each has a particular electro-magnetic frequency. Red is lower in frequency than green, and blue has a higher frequency than green. When any two these colors combine, they make another colors called a **secondary color**. A primary color cannot be created by combining any other colors. If you take two colored lights and shine them both onto the same area of a white, reflective surface, you will see a different color created. Primary colors and-or their frequencies of light are then said as being **additive colors** because they can be combined to produce other colors. For example, a red and yellow light will combine to produce a orange light. A orange colored object will be sensed by our eyes as being a mix of red and yellow frequencies, and which results in a orange color being processed (ie., mixed) by our brain. The color perceived by additive color mixing is that the resulting color is apparently brighter than the other two. When all three primary colors are combined, they will produce white light if their intensities, energy, or wave amplitude levels are all the same amount. For example, if you keep reducing the intensity or energy of one primary color combined to make white light, you will eventually have a color created from only the other two primary colors combined. White light produced form red, yellow and blue is as if their frequencies were combined to produce a higher frequency that is the sum each color's frequency. Computer monitors use a **RGB** method via separate red, green and blue LEDs (Light Emitting Diodes) to produces an apparent resulting color of a pixel (picture element, part) on the viewing screen by varying the current to each of those three closely spaced LED's which then creates both the color and the shade, brightness or intensity of it.

Now lets consider the colors reflected from objects. We see a blue colored object, it is because it is absorbing all other colors such as red, green and any other possible color besides the color of blue actually seen. If we see a green surface, then the frequency of light corresponding to green was reflected off it. If we see an orange surface, then the frequency of light corresponding to red light and yellow light were reflected off of it. The same process can also be said for the other two primary colors. If you were to pass a beam of white light through a series of colored glass such as a red, blue, and color, and in that order, the red glass (or "red pass filter") will absorb all other colors and pass or transmit only the red color of light. When this red light is then passed through the blue glass, it will in theory, not be transmitted out at the opposite side of the blue glass since it has not received any blue color to pass through it, and the result is that it will absorb that incoming light, and no light will be transmitted, hence, as if it was black only. When paint and-or pigment colors are mixed this way, the resulting colors are called **subtractive colors.** Many printers use the cyan (green-blue), magenta (similar to: purple, violet, maroon) and yellow, (**CMY**) color system so as to produce a third color. CMY are not primary colors, but are secondary colors. The three colors (CMY) mentioned, when combined, they will create black, however, to reducing using these relatively expensive inks, black ink is rather used to produce the black color, and which is often needed for common, and-or large scale black text printing on white colored paper, such as for books. This color method just described is usually called the **CYMK** color system, and where **K** stands for blac**K** in order not to confuse it with blue. Red, Yellow and Blue (**RYB**) is also a subtractive color system or model, and so as to mix two colors to have a third color.

To create the 3 **secondary colors** of light (ie., and not paint reflecting those colors) from two of the 3 primary colors (**RGB**, red, green, and blue) of light, mix or combine (+) these two colors so as to produce a third and unique color:

red + green = yellow , red + **b**lue = violet (purple, magenta) , green + blue = teal (turquoise, aquamarine)

Dark colored surfaces absorb some amount of light energy and convert it to heat energy ,and therefore, they do not reflect as much light as a surface not a dark in color. The darker the surface, the more light energy absorbed, and therefore, they appear as darker in color. A perfect **black-body** is a surface of black color of which absorbs 100% of the light, heat or electromagnetic energy upon its surface and therefore it reflects no light energy, and it converts all the received light energy into thermal energy (ie., heat energy, and which has a lower frequency, and of which is actually leads to a kinetic energy of the matter (ie., atoms, molecules, electrons) rather than re-radiated or reflected as light energy. By measuring the increase in the temperature of a perfect black-body surface and its mass (perhaps being constructed of a square foot or square meter of thin aluminum plate), the amount of total light energy it received upon its entire surface can then be calculated. If the source of the received light was from a theoretical point source radiating or transmitting light outward or radially from it in a spherical manner, then the total amount of light energy transmitted and-or of that source would then be 12.57 times greater if the black body had a surface area which corresponds to a cross-sectional area of 1 steradian (solid angle). This factor value is so because a point or sphere has a total of 4 (pi) =12.57 steradians radially and-or spherically about it.

If some of the light energy reflected off an object or surface is at an angle and-or other direction away from your line of vision, then it will not appear as bright.

Sometimes very dense ice, such in the polar regions, appears to have a green-blue color, and this is due to all the other colors of the white light from the Sun being absorbed by that ice and-or now on it. The thicker the ice, the more blue (ie., a darker shade of blue) the output color will be. Snow appears as white because it is reflecting nearly all the white light that strikes it. The reflected light is also rather scattered at various angles, and therefore the snow is not like a polished mirror that reflects incoming light well, and at the same angle of incidence.

A simple light meter can be made using a circuit that contains a light sensor (perhaps a solarcell or a light sensitive resistor), and it measurements can be calibrated (adjusted or set so as to indicate accurate values of a measurement, here by using various light sources and an actual light meter, and recording those values for your circuit. The circuit could measure voltage, current or resistance when using a light sensor that ideally has a linear or proportional electrical output or response to the amount of light upon it's surface.

THE SPEED OF SOUND

For the value of the speed of sound (a compressed air wave traveling outward from its source) in the air, first consider that a person is repeatedly, visibly striking an object and periodically (same time) producing a sound every 1 second of time , and that it takes 1 second of time for that sound to reach the listener from a certain distance away. The speed of that sound can be calculated from knowing the distance equation: distance = (speed)(time), therefore mathematically: speed = (distance / time). If the distance was 1100 feet, and the time was 1 second, the speed of sound is about: **speed of sound ~= a distance of one-thousand and one-hundred feet per second = (1100ft / 1s). 1100 ft/s = 0.335.28 km/s =~ 335 m/s**

The common value for the speed of sound is the average value at sea level where there is an average air density and pressure of 14.7 psi.

How long will it take sound to travel 1 mile = 5280 feet? : 1 mile =~ 1.61 km

From d= v t , t = d / v , t = (5280 ft / 1) **/** (1100 ft / 1s) = 4.85 s = about 5 s

How long will it take sound to travel 1 kilometer = 1000 meters? : 1 km =~ 0.621 miles =~ 3281 feet

From d= v t , t = d / v , t = (1 km / 1) **/** (0.33528 km / 1s) = $2.98\underline{3}$ s =~ 3s ,

Many equations involving electromagnetic or sound energy radiation include an inverse-square relationship of the intensity and distance to or from a source. For example, for the forces of gravity, magnetism, electric force intensity, light intensity, and sound intensity. Sound is caused or created by a physical ("mechanical") force or pressure upon something. For example, an elector-mechanical vibrating speaker will giving air some kinetic energy, and then this compressed air or pressurized air that contains energy is transferred outwards and through a medium such as less pressurized air as a mechanical force, vibration or wave. A force applied to air will cause a compression and then a decompression ("rarefaction", reduction) of the small particles (atoms) of air matter as the energy is radiated or transmitted outward form that source, and this will then propagate (distribute, transmit, send) the sound energy (air compression, pressure, force) and affect other nearby particles of air or other matter like a "chain reaction" or "domino effect" of collisions, and at the same frequency as the source vibration and-or force.

Sound and vibrations travel faster in denser materials since the energy transfer is more efficient. There is more material or matter per unit volume to transfer sound or vibration energy, hence reducing the time of energy transfer from particles to particle.

A modern, closer and accepted estimate for the velocity or speed of sound in a 20°C = 68°F air temperature, about room temperature, is typically considered as 1125 feet per second = nearly 343 meters per second = 767 miles per hour, and this value increases fairly rapidly as the density of the material (transfer medium) that the waves pass will through increases.

The speed in sea-salt water is slightly less than 1 mile per second at about 4921 feet per second = 3355 miles per hour = 1500 meters per second. 4921feet ~= 5000 feet. 1 mile = 5280 feet. In dense metals such as aluminum and steel, the speed of sound has an average of about 16400 feet per second = about 3.1 miles per second = about 5000 meters per second.

The speed of sound also increases due to temperature since the molecules and-or atoms of the transmission medium (air, water, metal) are moving more quickly, and therefore can receive and transfer (or transmit) the sound energy in a shorter amount of time. Typically, the speed of sound changes by 0.6 m/s = 1.9685 ft/s = about 2 ft/s for each 1 degree increase (change) in Celsius temperature, or about 1.1024 ft/s for each degree change in Fahrenheit. A change of 1 degree Celsius temperature corresponds to a change of 1.8 degrees Fahrenheit.

The ratio of an objects velocity (ie., speed) to that of the speed of sound in air is called the Mach Number of that object. It was created by the Austrian physicist Ernst Mach in about 1887. It is basically how many time faster an object is going as compared to the speed of sound. Mach studied many things related to shock and (air) compression waves when an object travels faster than sound. The speed of sound can vary in air, depending on altitude and density of the air, but for general, practical or relative comparative reasoning, it can be considered as the speed of sound at sea level, and at 20°C = 68°F. Mach 1 = the speed of sound in air = 1125 ft/s = 767 miles/hour = 343m/s = 1235km/hour

$$\text{Mach number} = \text{Mach \#} = \frac{\text{object's velocity}}{\text{velocity of sound in air}} = \frac{\text{objects velocity meters per second}}{\text{Mach 1 = 343 meters per second}}$$

1 mile = 1mi = 5280ft = 1609.34 meters = 1.60934km

Ex. An airplane jet going Mach 2 is going: (2)(Mach 1) m/s = (2)(343 m/s) = 686 m/s = 1534.$\underline{5}$ mi/h

Underwater, where sound vibrations and can travel faster and therefore for a greater distance before it is naturally spreads or diverges out to a larger radius and its energy significantly decreases (ie., weaker, less intense per unit area) and becomes negligible, the water pressing against our eardrum sound sensors will greatly dampen (reduce, due to a resistance and loss of some free motion) it from vibrating efficiently, particularly with the high audio frequencies. Our denser head bone can effectively receive higher frequencies more efficiently, and even above the typical 10,000 to 20,000 hertz (ie., frequency) max. audio upper limit for most human (ear) hearing, and will effectively transmit the sound to the audio nerve sensors very well, and even to the point that some sounds, particularly the higher frequencies, are even louder or stronger than expected.

AIR PRESSURE AND ALTITUDE

Sound, essentially a transmission though air of energy in the form of pressure (compressed air) or force from the atoms of gasses in the air, is not transmitted through a vacuum such as in outer-space since there is no transmission medium or "conductor" like air or some other dense enough gas for the sound to travel through. Remember that on Earths surface, air is also pressurized (hence denser) from the column (height) of the amount and weight of the air above and around it. The air at high altitudes is said to be "thin" and having less oxygen, and this is due to the lack of air pressure concentrating (and increases density) or compacting it. This air pressure also helps our lungs inhale and inflate. The "standard air pressure" or "standard atmosphere" is about 14.7 pounds = 6.668kg per square inch, (psi), at sea level (0ft in altitude). This value will be less at higher altitudes, and for example, the air pressure at the top of Mt. Everest (29029 feet ~= 5.5 miles = 8848 meters high = ~= 30000ft) is only about (1/3) of that at sea level = (14.7psi)/3 = 4.9 psi, and this value is dangerously low and that oxygen tanks are usually needed, especially for highly active climbers. The air is not only at less pressure which is needed to inhale and expand the lungs, but the density of the air is also very low, and each breath does not have much necessary oxygen in it. At the top of Mt. Everest, the density of the air is about 33% as that at sea-level. Artificial (compressed, etc) air pressurization is used to maintain normal breathing and body pressure inside of a high altitude airplane, spacecraft, and spacesuits. The less dense, lower pressure air at high altitudes effectively makes sound production less efficient and therefore, the amplitude (volume, loudness) produced is lower, and even at the source and creation of the sound wave where not much vibration or force energy is transferred to the thin (ie., low density, low pressure) air to pressurize it into a transmitted wave. Air pressure can also change during a weather storm. At 62 miles = 100 km high is an imaginary boundary or line called the "**Karman line**" where the amount of air density (1 (10^-12) g/cc) and air pressure (1 (10^-11) psi =~ 0 psi, hence much like an empty vacuum) is considered negligible, and where airplanes cannot gain enough lift at that height and (orbit) velocity needed. Above this line toward higher altitudes is considered as the start of outer-space, but some others might consider that outer-space starts at the simple and "round-about" value of 100 miles high where there are even less atoms of air present. At these heights, it is still only a very small fraction (62 miles / 4000 miles = 0.0155 = 1.55%) of Earth's radius. Earth's atmosphere layer is much like that of the relatively thin layer of the shell of an egg.

Due to less air pressure at higher altitudes, heated water molecules in water have an easier time breaking free from the other water molecules, hence the water boils or "boils off" easier. At standard atmosphere pressure (1 atm = 14.7psi [pounds of pressure or force per square inch] at sea level. at 68°F air temperature), water boils at 100°C = 212°F. The hot water will evaporate (ie., vaporize) and become steam (a gas of water molecules) and rise upward since it is less dense than liquid water and the air. When this steam looses heat energy and cools to a lower temperature, it will condense (become more dense) together into larger and visible water drops like rain. The bubbles seen in boiling water are not uniquely hydrogen or oxygen gas bubbles, but are hot steam gas (water vapor) bubbles created by hot, high energy and moving molecules of the water vapor. The formula relating the boiling point of water to air pressure can be complex, and a simple, approximate formula can be created. The **boiling point of water** is directly related to pressure, but it is not a direct linear relationship, so here is a small chart for making a close estimate:

Pressure (psi)	**Typical Altitude (ft)**	**Boiling Point Of Water**	: Approximations for all values listed here.
4.3	30000	152°F = 67°C	: Mt. Everest 29032 = 5.4985 mi ~=5.5mi =~ 30000 ft , and
5.2	25000	163°F = 73°C	the total change in boiling temperature from sea level is
6.5	20000	174°F = 79°C	about 60°F. The corresponding change in air pressure is
8.2	15000	184°F = 84°C	10.34 psi. The mathematical relationship between the
10.0	10000	194°F = 90°C	boiling point and altitude is **nearly** linear at about:
11.6	6000	201°F = 94°C	Approximate rate of change = - (1° / 500 ft) = - (2° / 1000 ft)
12.6	4000	205°F = 96°C	
13.6	2000	208°F = 98°C	
14.7	0	212°F = 100°C	: sea level air pressure and boiling temperature point of water

A quick and simple estimate, especially for the lower altitudes, would be for every 2000ft higher in altitude, the air pressure will drop by 1psi., and the boiling temperature of water will decrease by about 4°F =~ 2°C . Another quick and simple estimate is that the boiling temperature of water will decrease by 1°F for every 500 feet higher in altitude, or 2°F =~

1°C for every 1000 feet higher in altitude. For conversions, 1foot = 0.3048 meters. Multiply feet by 0.3048 to convert to meters. Multiply meters by 3.2808398 to convert to feet. 1000ft = 304.8m, 1000m = 1km = 328$\underline{1}$ft. From this discussion, a reasonable estimation formula can be:

Boiling Point Of Water Estimation Formula (at altitude) = 212°F - 4°F (altitude feet /2000 feet)
Boiling Point Of Water Estimation Formula (at air pressure) = 212°F - 4°F (14.7psi - psi)

It will take less energy and time to boil water at a higher altitude where there is less pressure or compression applied to the water, and this will also decrease its density slightly. At low pressures upon water, it can actually expand and start (cold) boiling and off-gassing steam (water gas, water vapor, water molecules) since the molecules are not held together as much. Less energy is less joules of energy. Less energy per unit of time is less power (ie., rate of energy use or transfer) needed. P = energy / time = joules / second , and with units of watts. Now consider at sea level, for water to boil, the molecules of it must be given more kinetic energy by heating that water, and then when the water pressure becomes greater than the air pressure upon it, it will steam and boil into the lower pressure region of air that is at standard air pressure of 14.7psi.

The formula(s) for air pressure at an altitude can get complicated with many variables (height, temperature, air density, humidity=moisture in the air) to consider, and of which you may not have any data for their current values, but here is a simple estimation or approximation formula for the typical air pressure at a certain altitude less than 30000 feet, and the result differs by only about 5% or less than a real typical previously measured and-or current actual current value which can also vary due to many variables such as the air temperature:

$$\text{air pressure} = \frac{14.7\ \text{psi}}{3.2\text{^}(\text{height ft} / 30000\ \text{ft})}$$

: **A Simple Air Pressure Approximation Formula**
For Heights <= 30000 Feet
Height above the surface of the Earth = altitude.
A lookup table is also an acceptable alternative.

Or by converting 3.2 to an equivalent power of (e): e^1.163151... = 3.2 and e^(1.163151)^x = 3.2^x

$$\text{air pressure} = \frac{14.7\ \text{psi}}{(\text{e^}1.163151)\text{^}(\text{height ft} / 30000\ \text{ft})} = \frac{14.7\ \text{psi}}{\text{e^}(1.163151\ (\text{height ft} / 30000\ \text{ft}))} = \frac{14.7\ \text{psi}}{10\text{^}((0.50515)\ \text{H}/30000\ \text{ft})}$$

The value of **14.7 psi is also called 1 ATM** ("1 **ATM**osphere" of pressure) which is the normal, average and standard pressure of the atmosphere (air) at sea level. 14.7 psi = 1 ATM (typical "atmosphere (air) pressure" at sea level) = (nearly) 1 bar = 101 kPa , and Pa is a unit of pressure called a pascal, see the note below. The value of 3.2 is a compromise value so as to be more accurate near the ("center height" of this scale and formula) 15000ft height, and when not using other variables such as temperature. It was derived from that at 30000 feet, the known pressure is about 4 .36. 14.7 / 4.36 = an increase by a factor of about 3.37. Therefore, 4.26 is about 3.37 times lower than 14.7 4.26 / 14.7 =~ 0.304$\underline{3}$ =~ 30%. 4.36 psi is the air pressure at the top of Mt. Everest, and is about 30% of the air pressure at sea level. The values of the pressure data indicated a non-linear curve, hence some type of exponential equation could be used, and 3.2 was chosen as the (compromise value) base of this power value. You could simply use 3 for rough estimates. This equation for the air pressure at a certain height also requires a height variable. When the height or altitude is 0, the air pressure from this equation needs to be 14.7, and so it was included as the value to be worked with for all the other height and pressure values up to about 30000 feet. Note also that **14.7psi = 29.92 inches of mercury** = 29.92 inHg =~ 30 inHg, and other corresponding values can be found by setting up a proportion and-or equivalent fraction equation.

The air pressure at **0 ft elevation = 14.7 ps**i (pounds per square inch) at sea level = 1 atmosphere = 1 atm. Here are some typical values of air pressure (psi) and the altitudes. The higher the temperate,the greater the air pressure due to the movement of the air molecules having more kinetic energy. These values consider a constant temp. of about 57 °F :

1k ft. = 14.2 psi 2k ft = 13.7 , 5k ft = 12.2 , **10k ft = 10.1** , 15k ft. = 8.29 , 17878 ft = 7.35psi = 14.7psi/ 2
20k ft. = 6.75 , 25k ft. = 5.42 , 30k ft. = 4.36 , 35k ft. = 3.46 , 40k ft = 2.71 , 45k ft. = 2.10 , 50k ft. = 1.69
60k ft. = 1.05 , 70k ft. = 0.65 , 80k ft. = 0.406 90k ft. = 0.26 , 100k ft. = 0.16

If you are able, it would be a good exercise to plot the data points and draw an approximate graph and-or curve of the above data. This curve will rapidly slope downward from 14.7 psi at 0ft, and the slope will approach 0 at about 100k ft.

At 1k ft. below sea level, this can be expressed as: **-**1000 ft, such as in a deep cave or tunnel, the air pressure is actually higher than that at sea level, and is about 15.2 psi. The Dead Sea at -1370 ft has an air pressure of 15.5.psi. At 1 mile below sea level, the air pressure in a cave would be about 17.6 psi.

Extra: Although weather can affect several meters of surface temperature, the deeper you go below ground, the hotter it will be and the **geothermal gradient** or rate of temperature change with respect to land depth averages about +80°F increase per mile deep, and this is due to the heat from the hot core of the Earth being at high pressure and some radioactive decay of heavy radioactive metals producing heat

At less than 20000 ft, high, the air pressure drops in a fairly consistent, slightly linear way as the altitude increases, and for higher altitudes, the drop is more gradual and less, say per 1ft or 1000 feet. Let us find how much the pressure will drop with a simple or crude linear (line-like, constantly proportional) approximation:

$$\text{linear rate of change} = \text{slope} = m = \frac{\text{(change in y)}}{\text{(change in x)}} = \frac{\text{(change in PSI)}}{\text{(change in altitude)}} = \frac{(14.7 - 6.75)}{(20000 - 0)} = \frac{7.95}{20000} =\sim \frac{0.4 \text{ psi}}{1000 \text{ ft}}$$

(change in PSI) = (starting PSI) - (ending PSI) : with the right side terms reversed for a positive value
(ending PSI) = (starting PSI) - (change in PSI) = 14.7psi - (change in psi)

Change in PSI =~ (0.4 psi / 1000 ft) (ch. altitude in ft) =~ (0.0004 psi / 1 ft) (altitude in ft) : 0 ft to 20k ft
: Letting ch = change in altitude from the reference altitude
Solving for the altitude in ft, from this formula , we have:
altitude ft. = ((14.7 psi - PSI) / 0.0004 PSI)) (1 ft)

Ex. At 20000 ft, the PSI can be roughly estimated as:
PSI = 14.7psi - (0.0004 psi / ft) (20000 ft) = 14.7 psi - 8psi = 6.7 psi : 6.75 psi is listed above , likewise:

In terms of the common linear equation format:
y = mx + b ,
psi = m(altitude) + b =
psi = (slope)(altitude) + (y axis intercept when x=height =0)
psi = (-0.0004 psi / ft.)(altitude ft.) + 14.7 =
psi = 14.7psi - (0.0004psi / ft.)(altitude ft.)

Change in PSI =~ (0.2 psi / 1000 ft) (ch. altitude in ft) =~ (0.0002 psi / 1 ft) (altitude in ft) : 20k ft to 40k ft

Ex. At 30000 ft, the PSI can be roughly estimated as: (First, the psi at 20k ft is 6.75)
PSI = 6.75psi - (0.0002 psi / ft) (10000 ft) = 6.75 psi - 2psi = 4.75 psi : 4.36 psi is listed above

A more alternate value for CH. in PSI for altitudes from just 20k ft to 30k ft is: (0.00024 psi / 1 ft)

Using the main approximation formula equation for air pressure or PSI given previously, and solving for the height in altitude above sea level and up to 30000 ft, such as after measuring or given an air pressure (AP) psi value, we have:

Height in feet = 30000 ft [ln (14.7 psi / AP) / ln 3.2] = 30000 ft [ln (14.7 psi / AP) / 1.16315081]

Here are some other units used for pressure besides psi = pounds of weight force or pressure per square inch = pounds per square inch:

First note that: psi = pounds of force per square inch = pounds-force / 1 in sq.

Since 1 square foot has (12" x 12") = 144 in^2 of area,
1 psf = 1 pound of force / 1 ft^2 = 1 lbf / 144 in^2 = 0.0069444 lb / in^2 = 0.0069444 lb / in^2 = 0.0069444 psi

1 pascal = 1 Pa = (force/surface area) = pressure = 1n / m^2 = 1n / 10000cm^2 = 0.0001n / cm^2 : named after Blaise Pascal
: n = N = Newtons

1 pascal = 1N / m^2 = applying an energy of = F / A = Fd / Ad = F m^1 / m^3 = Work / Volume = Energy / Vol = **1J / m^3**

1 **ATM** =1 ATMosphere = ~ 29.92 inches in height of mercury (Hg) =~ 30 inches high of mercury =~ 76 cm = 760 mm Hg
1 ATM = 14.7psi = 101325 pascals ~= 100000 pascals : 1 ATM = the pressure of 1 atmosphere.

Ex. At tank of some gas is noted as having an internal pressure of 5 ATM = (5 ATM)(14.7 psi / 1 ATM) = 73.5 psi

1 **bar** of pressure = 100 Kpascals = 100KPa = 100000Pa = about 14.504psi , **and this is just slightly less than 1atm**

14.7psi = 14.7 pounds per square inch = 14.7 pounds per 6.4516 square centimeters, with about 2.2785 lbs / cm^2 which is equivalent to about: 1.034kg / cm^2 =~ 1kg / cm^2 =~ 9.8N / cm^2 =~ 10N / cm^2 =~ 100000 pascals

STP = **S**tandard **T**emperature and **P**ressure, and it depends on the substance, location, conditions, experiment, etc.
For the atmospheric (air) pressure at sea level: STP is: 0°C = 32°F and 14.7psi = 1 atm = 1 bar . To quickly convert from Kpa to psi, divide the Kpa by 6893 , (about 7000), for ex.: (100000Pa/1psi) / (7000Pa/psi) = 14.$\underline{3}$psi

As mentioned previously in this book, a device called a barometer can measure the local air pressure at various levels of height and weather conditions which can affect the local or nearby air pressure. By knowing the average air pressure at a certain height or altitude, a barometer can actually be used to also effectively measure (by calculation) altitudes after rearranging the air pressure formula to solve for height.

A note about barometric or atmospheric density, pressure and temperature changes.

In a car tire as it heats up from road friction, the air inside it will gain more kinetic energy and exert more force upon the inner wall of the tire, therefore, its internal air pressure will increase if its volume remains the same and does not expand too much from the pressure. When it is cold outside, the air within a cold tire will loose kinetic energy, and the air pressure in the tire will decrease and may even deflate it slightly which may cause a dangerous driving condition and-or increased tire wear due to the increased friction on the tire's outer surface.

Even apparently invisible gases like air have weight because it is composed of atoms of various gasses such as nitrogen and oxygen. The force of gravity will be upon this matter and they will tend to move downwards toward the ground unless the force of the wind is moving it in a different direction. The weight of air above the ground will cause a pressure (P = F / A = weight / area) on the ground or an instrument that measures the air pressure at any altitude. Due to the weight of the "air column" above the ground, the air at the ground will be at a higher density (d = m / V), weight (= force = ma = dmV, and pressure.

In the atmosphere, when temperature increases, the air pressure rather decreases due to that the atmosphere has room to expand in volume and therefore, it get less dense (= mass / volume). Less dense air means it will weigh less, and therefore exert less pressure upon an object. It is also of note that then warmer air, being less dense than the surrounding cooler air will then rise upward, and the cooler, more dense and pressurized air, and possibly more water moisture and a rain storm, will then move in towards a lower pressure region.

At higher altitudes where the air is less dense, a change in temperature will not affect the air pressure as much as it would at ground level where the air is more dense and the air pressure is 14.7 psi on average.

A formula for water pressure at a certain depth is:

$$\text{water pressure} = 14.7 \text{ psi} + \frac{(\text{depth ft})(14.7 \text{ psi})}{33\text{ft}}$$

: 33 ft ~= 10 meters, each extra 10 meters or 33ft of depth will add another 14.7 psi to the typical air pressure at the surface. With this equation, the values of pressure and depth have a linear mathematical relationship. For every 33 ft deeper, the water pressure increases by 14.7 psi =1atm. The pressure at 33 feet deep is: 1 ATM +14.7 psi = 29.4 psi = 2 ATM. Each foot adds: (14.7psi/33ft) = 0.4455psi per 1 foot. Note that 14.7lbs = 6.668kg. and 1 in^2 = 6.4516 cm^2

At 100ft , psi =~ 60 , At 200ft, psi =~ 104 , At 1000ft, psi =~ 460 , At 1km = 1000m = 3281ft, psi = 1476 ,
At 1mi = 5280ft, psi =~ 2367. To use the above formula for meters of depth, first divide the number of meters by 0.30479 to convert it to its equivalent feet units value. For every 100 feet more deeper, the water pressure will increase by about $44.5\underline{5}$ psi =~ 45 psi. For every 1000 feet more deeper, the water pressure will increase by about 450 psi.

The above formula is linear (ie., water pressure in a proportional mathematical relationship to depth) because water does not compress very much due to its fairly high density, molecule structure and could be considered as already being a highly compressed gas of steam [water vapor], but the formula for air pressure (ie., air pressure in relationship to height) is non-linear because air compresses much more than water because air is usually a gas that has a low density, except for liquefied air. The air pressure and-or density of the air at sea level is relatively high due to depth of it in the atmosphere region and-or column of the air and its mass above it, and the air pressure decreases relatively rapidly as the altitude increases.

For some basic verification that the pressure will increase by 14.7psi for every 33 feet deeper: 1 cubic inch of water weighs about 0.578 ounces. A foot or 12 inches high of this cubic inch will weigh: (0.578 oz/in) (12 in) = 6.936 oz/1ft .
(6.936 oz/ft)(33 ft) = 228.888 oz, and dividing this value by 16 oz/1lb, we find that 228.888 oz =~ 14.3lbs ,
33 feet = 10.0584 meters =~ 10m

When an object is subjected to a force and-or pressure (F/A), that object is said as being compressed or pressurized. If the force and-or pressure is reduced, then the object is said as being decompressed or depressurized.

The minimum depth that is considered as extremely dangerous for a diver to dive or go to without a special pressure protective diving depth suit or rigid surrounding safety vehicle, such as a "submarine" [below the water surface, a vehicle], is 130 feet. Here, the pressure is about 58 psi. This is about the pressure of a typical 26 inch bicycle (air pressure inflation) tube in the tire. At this depth, and for greater depths, a minimum amount of "decompression" time is needed so as to avoid decompression sickness (sometimes called "bends") as the diver travels upward to a lesser depth . Here, expanding in volume compressed nitrogen gas bubbles in the body, due to less pressure on them as the diver goes upward in the water to a less depth, are given some time to be safely eliminated from the body. The diver must ascend slowly at calculated amounts of time. People in high altitude, internally pressurized craft, where the air pressure is low at high altitude, may experience similar effects of a divers decompression sickness if the craft becomes unpressurized. Consider the fact that water boils faster or more easily in a lower pressure environment such as on a high mountain. There are fish and other life forms living their lives at crushing, high pressure depths of water. These forms of life have adapted their physical structure and internal life chemistry so as to prevent the high pressure from harming them. Most of these life forms have therefore adapted and are accustomed (used) to and need this high pressure environment so as to live and function properly. A balloon or air bubble would be squashed or compressed equally on all sides of it, and it will have a smaller diameter as the depth of it in the water is increased. In a reverse manner, the balloon or air bubble will expand as the depth and pressure decreases.

When an object goes in very deep water, the water pressure upon that object is not to be considered as just as a weight upon it forcing it downwards and the object weighing much more, but rather as a compression force acting upon all sides of the object equally, and not just downward upon it. The object will still have the same mass and weigh the same less the upward buoyant force upon it that is equal to the amount of water it displaces. In short, the buoyant force does not depend upon the depth of the object and the pressure upon it, but rather the weight of that amount of water being displaced.

An object, such as a fish or submarine, can remain at a fixed depth in water if the buoyant or upward force due to the displaced water equals the downward weight of that object.   Effective weight = (weight of the object - buoyant force)

A volume of air or space of 1 cubic foot will displace 1 cubic foot of water which weighs about 62.427lbs = 28.316kg. The total or resultant buoyant force of a lite weight container or balloon will be this amount. It could be said that the (upward) "lift force" (ie., buoyant force) of the displaced space or balloon under water is about 62.427lbs per cubic foot. If an object was suspended from this balloon and weighted less than 62.427lbs, it will still rise upwards in the water. If the object weighs 62.427lbs, it will remain at the same depth. Here, the total (downward) weight (force) of the object will then effectively cancel out the total (upward) buoyant force effectively pushing upon it from below due to the water pressure, and the object will have 0 net force acting upon it and it will remain (vertically) motionless at the same depth in the water.

Since water does not compress much, even at great pressure, its density at great depths is still about the same value as that at the surface, and which is: density of water = (mass / volume) = 1gram / 1cc.

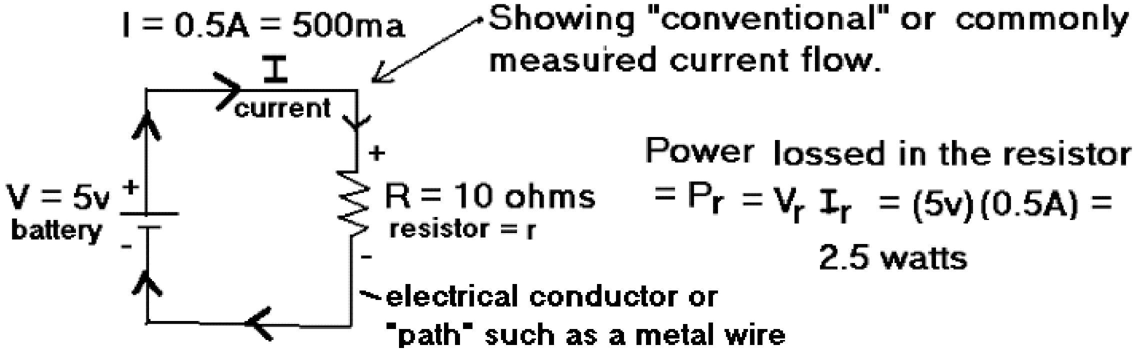
By rearranging the above formula for water pressure at a certain depth, we can solve for the depth given a certain value of water pressure:

$$\text{Water depth in feet} = \frac{(\text{PSI} - 14.7\text{psi})(33\text{ ft})}{14.7\text{ psi}}$$ **: depth of water in feet, given the pressure (psi) at that depth**

A SCHEMATIC DIAGRAM OF A BASIC ELECTRIC CIRCUIT

A schematic diagram or representative drawing of a simple electric (electrons with kinetic energy and electric charge) circuit (path, course) is shown below, and Ohm's Law is also used for some analysis and calculations of it. For electrons to flow, it needs a complete path to do so, and this is called a closed circuit, as opposed to an open circuit where a wire got cut open or something. A short circuit is where the electrical path has been shortened for some reason, perhaps by a piece of metal or wire. This short or shorted circuit is very dangerous for it can damage your device, drain the power supply, and-or cause smoke, and fire. Usually a circuit will cause some resistance in a circuit to be less or bypassed, and therefore , a high current will then be flowing in it.

Below, (I) is the identifier name of the variable for electric current and which is measured in units of amperes (A). Ir represents the amount of current through the resistor. V is used for voltage which electrically gives (electric) force and therefore transfers (kinetic, motional) energy form the battery (energy) supply to the electrons to flow. R is used for resistance which limits the flow of current and prevents the battery from draining of its energy too fast. Vr represents the voltage across the resistor. Conventional current flow is where the electrons are considered as coming out of the positive terminal of the battery, and this is also mathematically practical since the increase in energy of the electrons is then given a positive sign rather than a negative sign. [FIG 253]



I = voltage applied to total resistance / total resistance

I = 5v / 10 ohms = 0.5A

Vr = (Ir)(Rr) = (0.5amps)(10ohms) = 5 volts

The power lost in the resistor or resistance can also be considered as equal to the amount of power delivered to that resistor, and-or needed by that resistor to function properly, such as a heater, etc. Generally, in many circuits, heat is an indication of not only lost power, but of completely wasted power and energy. Because energy has a price and resources associated to it, a circuit and-or device should be designed so as to have a low waste of power, hence have a high energy efficiency or usage:

Power Input = Power Output + Power Losses : Also: = (min.) Power Input = (min.) Power Required

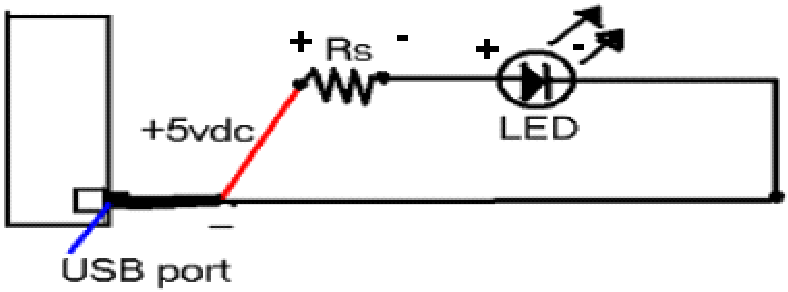
$$\text{Efficiency} = \frac{\text{Power Output}}{\text{Power Input}} = \frac{\text{(Power Input - Power Losses)}}{\text{Power Input}}$$

: divide Efficiency by 100 to find the equivalent percentage value

Since the efficiency value is a pure unit-less number, you can also use relative or percentage values so as to be non-specific in the actual detailed values, and have a useful generalization or basic value that easily express what is happening in a system(s) or device, and could even be used for comparing systems and costs. Here, power input would be expressed as: 100% = 1.0 as a relative or comparative value.

A simple LED (light emitting diode) circuit is a practical example of the above circuit when the resistance happens to be a LED. A white color LED initially requires [and actually tries to maintain it if slightly higher voltages of up to a maximum before **LED** "burnout" when greater than about **3.3v is applied]**, a 3v (such as from two 1.5v batteries in series) potential just to "turn on" and conduct current of about 20mA through it for its close to maximum brightness level. With a supply voltage of 5v, this will cause a higher than needed current to flow through the LED and it will be "burned-out" or ruined. This can be prevented by using a ("safety" or power reducing) resistor (R) directly in series or "in line" with that LED. The voltage across that resistor should be set to be equal to the supply voltage minus (less) what is needed for the LED to operate: Vr = 5v -3v = 2v. Since the resistor is in series (same line, circuit path or direct path) with that of the LED, 20mA of current will also be traveling through it. Since I = V/R : R for this particular circuit having a certain supply voltage, LED voltage and current: Safety resistor Rs = R ohms = Vr / Ir = (Vsupply - Vled) / (I led) = (5v - 3v) / 0.20A = 2v/0.020A = 100ohms : *

[FIG 254]



supply voltage = sum of voltage drops
+5vdc = Vr + Vled
+5 = Vr + 3v
+5 - 3V = Vr = 2v

Vr = Rs x I
2v = Rs x 0.020A : 20 milliAmps
Rs = 2v / 0.020 = 100ohms : design values, can adjust to your needs

If an LED is being powered by 3.3v across it and 0.020A through it, this is a power of: P=VI = (3.3v)(0.020) = 0.066w = 66mW. If more power is applied than this, the LED will most likely be quickly destroyed and not give off any light. **Also of note that the maximum reverse bias voltage that can be applied to an LED is also a low value of just about 5v on average**, if more than that is applied, it will usually be destroyed. To reduce the potential of a high valued reverse bias voltage being applied to a diode, a reverse biased diode in series with the reverse biased diode can effectively be used so as to act like an open circuit and-or very high resistance, thereby protecting the diode in reverse bias conditions, such as when it is being power4ed by an AC supply. As for reducing the forward bias current to an LED,,a resistor is used. An LED can also be powered by 120Vac household power, but a current reducing and safety resistor, and diode must then be used. A capacitor of say 100uF across (ie., in parallel to) the diode can store enough charge so as to help reduce any noticeable "flicker" of the light of the LED. To reduce wasted power by the resistor, rather use many more LED's to use that power.

Most of the basic USB ports of manufactured computers and other devices offer at least 1A of direct current before the circuit is shut off automatically before damage can occur. The power available from that USB port is: P = VI = (5v)(1A) = 5 watts. More advanced versions of USB ports offer 2 or slightly more amps. Only two of the 4 wires of a standard USB port and cord are needed, and those are the power supply wires which are the positive and negative wires so as deliver energy and power to an electronic circuit.

Standard USB 4-wire color codes:

RED = +5Vdc
BLACK = - , negative and-or battery-circuit ground
White = Data, positive signals (not needed in the above circuit, cut and insulate it)
Green = Data, negative signals (not needed in the above circuit, cut and insulate it)

* If your supply voltage is not the same as 5v, then you need to calculate a new value of the resistance needed.

A technical note about diodes and LED's which are a special type of diode: Before a diode "turns on" when the rated barrier voltage is finally reached, and its ("active" or controlled) resistance is very low and begins conducting like a short circuit (ie., a wire or closed switch), a diode will have a very high resistance of which is determined by the applied voltage (dropped or lost) across it. As the applied voltage across a diode increases from 0v, the diodes effective resistance will increase from infinity ohms, such as that of an open circuit and-or reverse (voltage) biased diode, and to a very low value of resistance, perhaps 0.1 ohms. In this manner, a diode can behave much like a voltage controlled adjustable or variable resistor and-or voltage divider. In place of a resistor, 3 plain (non-LED) diodes in series can drop about: 3 (0.65v) = 2v

If the power supply to this circuit was 3V, then no resistor would be needed and there would not be a power loss as heat energy in it. For the circuit shown above, here are some practical equations a circuit designer may consider:

Pin = (Vin)(It) = (5v)(0.020A) = 0.1W = 100mW = 0.1J/s
Pled = Pout = (3v)(0.020A) = 0.06W = 60mW
Presistor = (2v)(0.020A) = 0.04W = 40mW = Pwasted as heat energy = 0.04 J/s :100% of its power is wasted as heat
Ptotal = Pled + Presistor = 60mW + 40mW = 100 mW = Pin

circuit power efficiency = Pout / Pin = 60mW / 100mW = 0.60 = 60% efficient , however, the LED may only be 30%
efficient at converting the input energy and-or power to light. 60% of the power goes to the load
(an LED). The efficiency of an LED at producing light power is about 50%: Plight / Pled =~0.50

power loss = Pin - Pout = 100mW - 60mW = 40mW

% of input power wasted = Presistor / Pin = 40mW / 100mW = 0.40 = 40% : this is relatively, a high value

If the supply voltage was 3v, the power efficiency of this circuit would be 100% to the load (here an LED) because there would be no wasted power in a current limiting and-or safety resistor. 100% is a very high value.

By placing in more parallel and similar circuit branches, each consisting of the led and resistor, more light can be produced by that circuit, however, each branch will need 100mW of power more.

Total power and energy in or used will be:
(watts used) (time of use in seconds) = W-s = (joules/second) (seconds) = joules of energy or
(watts used) (time of use in hours) = W-h = (joules / second) (hours)(3600s) = joules of energy

With just 1 LED and resistor circuit branch being used, and considering 3600 seconds of use = 1hr of use, the total energy needed and used is:

(0.100W)(3600s) = 360 W-s = 0.1 W-hr : a 1W-hr rated battery, if fully charged, would last: 1W-hr / 0.1 W = 10 hours

If the supply voltage is 12V such as a typical automobile battery, 4 LED's in a series electrical connection can be placed in each parallel branch, and without any resistor needed. (4 LED's)(3V / 1 LED) = 12V. Each branch of 4 LED's will require:
(4 LED's) (60 mW / LED) = 240 mW = 0.240W : slightly less than a quarter of a watt = 0.250W = 0.250 J/s

LED's can be placed in series, and of which the same amount of current, say 0.020A, going through one will then be going through the other one. 2 LED's in series will require 3v per LED, hence (2 LED)(3v/LED) = 6v to be powered. The light energy output will be about the sum of the lumens (lm) of light energy and-or brightness from each LED, hence about: 2 LED (100 lm/LED) = 200 lm. Since our eyes see in a logarithmic manner, things illuminated by them both may not appear as twice as bright,, but nearly as so for low levels of lumens (ie., "illumination") or brightness seen. With a large array of LED's, adding or removing a single LED will not make much of a difference in the illumination or brightness of objects.

For LED's in parallel, they all can be powered by a 3v to 4v power source, but each will require about 20mA to 30mA of current, and the total current needed will be about: (number of LED's in parallel)(20mA).

The total light (or photon) output of any array of LED's will be the sum of the light output from each LED.

BASICS OF RADIO

A previous discussion about radio has been given in this book, and here is some more to consider on this topic:

In 1886, **Heinrich Hertz** (1857-1894), from Germany, studied electrically produced sparks, and also knew from Faraday's experiments and works that an electric field of the current (electrically charged electrons with kinetic energy) flowing through a wire could produce a magnetic field, and vice-versa. Static charges or charges at rest do not produce a magnetic field, but they will produce an electric field. Surely, the spark transmits some light, but does it then also transmit an electric or magnetic field and-or energy? Hertz made a metal loop with a small air gap (ie.,here, a "spark gap" or "spark-gap") in it and turned on his high voltage spark machine and noticed a corresponding spark being induced (hence being transmitted through the air, "wireless") or received in the distant metal loop, and which was seen as a small spark across the air gap. This was the first instance of deliberately transmitting and receiving electric energy or a radio wave. It began the huge field of (invisible and wireless) radio wave communication starting with Morse code that was already in use as for the telegraph (remote graphing communication via direct electrical wire connection) system of communication. Morse code is a standardized system of (usually) electromagnetic and-or heard pulses called dots (short pulses) and dashes (long pulses) that electronically represented information or data such as certain letters of the alphabet. For a radio or wireless system, theses pulses were also mechanically "tapped" (ie., so as to move a circuit breaking and making switch) by a trained operators hand, and which were then used to switch on or off so as to modify or modulate the transmitted radio wave. The spark-gap radio transmitter is capable of producing a very high frequency, but only for the duration of each and brief single pulse, that is, it did not create a continuous oscillation at a certain frequency. The tuning was limited by the inductors and capacitors used, and was "noisy" in terms of unwanted radio signals produced. Years later, an electronic tuned amplifier circuit was discovered by **Edwin Armstrong** in 1912 when using a vacuum tube amplifier modification that provided continuous oscillation at a (tune-able, adjustable) frequency. Hertz's discoveries also created a question of whether visible light was also a form of an electromagnetic energy and having a wave or frequency. Newton's previous experiments with light and (triangular, 3-sided) glass prisms showed that white light can be separated into various colors, and today we know that these colors of light have and are various electromagnetic or radio wave frequencies. It is of note that small drops of rain or some snow can also create a rainbow of colors when the incident Sun angle, and observer of the refraction are in a favorable position. In 1672, **Newton** then showed that the rainbow of colors can then be combined back together using a **glass prism** so as to form white light. It is of note that the glass prism was invented years before Newton was born, however Newton inspired a high interest in optics and the mysteries and possibilities of light. Generally, radio waves are considered as invisible electromagnetic waves containing energy, and which can go through non-metallic walls, whereas light does not unless it is a transparent wall that can essentially conduct light through it. Another common question was also created, and that is if invisible electromagnetic waves were also composed of and-or can be considered as (invisible) photons of light, and the answer is yes, and its that our eyes can only sense or see a small part of the electromagnetic energy spectrum or frequency range.

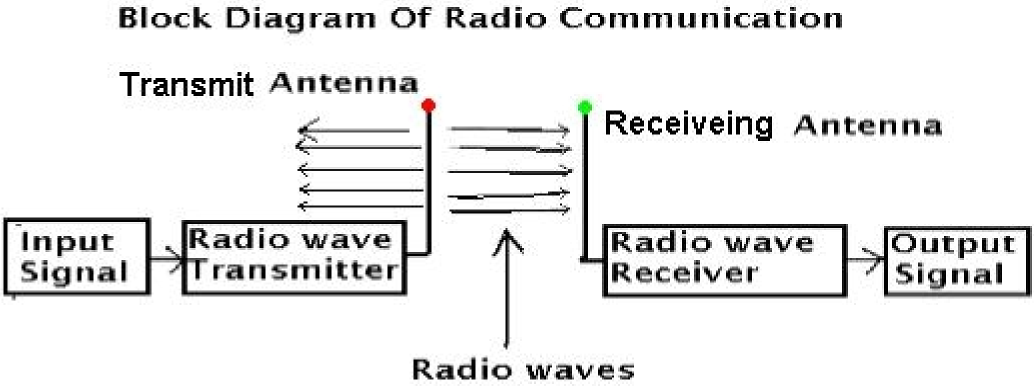
A transformer (essentially two coils side by side or intertwined so as to transfer electric energy from one coil to the other using magnetism or magnetic field transmission) is technically a short-range (ie., short distances) magnetic field transmitter and receiver device, hence a "transceiver", and since it also involves (input or created output) electricity or (electron) current, an electric transformer is more accurately and technically a (very) short-range electromagnetic transmitter and receiver. Note that a radio wave transmitter does not transmit electrons, but only the electric and magnetic fields and-or photons created in and upon the antenna surface by the flowing electron charge, current (ie., energized electrons) or movement. Metal such as iron can be used to help concentrate, direct or guide (ie., conduct magnetic "lines of force") a (transmitted) magnetic field or "current of magnetism (magnetic current)", typically in a transformer or choke.

A spark, such as lightening, actually produces many frequencies and it would be "heard" or converted to as unwanted noise on a modern AM radio, affecting many frequencies and causing circuit and-or audio interference. If the spark could be tuned to a certain frequency, receivers could also be tuned to a certain frequency so as to receive only a specific or desired transmitted radio wave and the information it contains being transmitted. This would allow the creation of many radio transmitting stations, each with their own legally permitted and assigned radio transmitting frequency (ie., for a radio "channel and-or station") and help eliminate signal interference problems in radio receivers that are tuned to select only one desired station with its assigned radio frequency and-or bandwidth (ie., frequency range or amount). The unit used to express a frequency value is called the hertz (hz) and it is named after **Heinrich Hertz**. 1hz = 1 cycle or complete periodic wave per 1 second of time, hence it is a wave or cycle rate. A wave can be thought of as a vibration. The hertz (hz) unit may sometimes be noted or indicated as cps (cycles per second) or the frequency per second.

In about 1884, **Guglielmo Marconi** (1874-1937), from Italy invented wireless (ie. radio wave) telegraphy that transmitted the communication of information using Morse code. This information was physically transmitted as long and short duration (pulsed) radio signals that were heard as tones and were received and recorded by the radio operator person using pen and paper so as to be decoded later. This was the first practical use of the newly discovered radio waves and inspired much further research till this day. He used the previously invented spark gap transmitter and improved it, but later, others would add better tuning controls (such as vacuum tube electric oscillators and tuners) of the transmitter and receivers so as to significantly reduce the bandwidth (ie., range) of frequencies transmitted for each transmitted pulse. With a capacitor and inductor, the radio waves could be tuned (adjusted) to be a certain frequency. The radio receiver would then also have to be "tuned" (for maximum signal strength reception) to this transmitted (limited) bandwidth or frequency which also electronically rejects or filters [remove out] all other frequencies. This "reduced (frequency) bandwidth" transmission and reception method allows many more radio-wave transmitting stations and their transmitted and received information from interfering with each other.

Even when tuned to a certain frequency so as to be efficient, much of a transmitted radio waves energy is often radiated or transmitted in a 360° directional pattern so as to be available to anyone nearby with a radio receiver, but then with much energy actually often wasted, but it will be available whenever needed as "wireless communication" without a direct wire connection. The amount of the transmitted radio wave energy received by a distant receiving antenna is very small and this amount is impractical to use directly to power most things, and it must then be greatly amplified so as to be useful such as in an amplified radio receiver. The exception to this is with crystal radios (receivers) which can function without any extra voltage source and-or signal amplification as long as very electrically efficient and sensitive (tuned) antenna, circuit, and earphones such as the (high-impedance, low voltage and power needed, very sensitive) carbon or (piezo) crystal types are used. As of the year 2021, carbon earphones are difficult to find and-or are expensive, but a piezo-crystal earphone can still be found and purchased for a few USD dollars each.

The assigned main frequency for a radio station is called the carrier frequency or wave since the input or information signal, such as an audio signal (ie., energy) is converted or transformed to a corresponding electric signal (ie., electric energy, voltage and current) equivalent in a microphone, and which will essentially be applied (ie., to modulate, change, adjust) to and "carried" on, integrated or combined with as part (such as the intensity or "amplitude" of) of that one main transmitted radio wave. At the receiver, the microphone signal will be removed out from the carrier wave signal in a process called demodulation, and it will eventually be heard through an output audio speaker. [FIG 255]



Though an antenna appears as an open circuit, the electric field (ie., voltage) created in the transmitter still causes electrons in the antenna wire to move by coulomb or electric forces, and a magnetic field at a right angle is then created with respect to the electric field created. A radio wave (an electro-magnetic [fields] wave and transmission of energy) is composed of an electric field and a magnetic field at a right angle to the electric field. The (RF or rf, radio frequency) signal received by the reception antenna is often very small and needs to be greatly amplified so as to be useful and-or heard. Often the input and output signals are ("low frequency", LF) audio signals such as speech and-or music. Today with modern "computer phones" with internet capability, transmission towers can send both audio and computer data using (digital, as in high power quick pulses, as opposed to an analog form) radio waves.

In a microwave oven, the (microwave) radio (rf) signal is relatively close to the food substance, and therefore is of a relatively high strength so as to apply much energy to the food molecules and-or water and cause them to gain kinetic energy and its associated heat due to collisions and infra-red (ie., heat) radiation. If you heat a piece of metal to being very hot, it will begin to emit thermal (ie., infra-red) radiation and it can be seen as a glowing red color. If heated further, it can emit a orange, yellow, and then white color like a electric light does. If the temperature of the metal changes by a factor of (n), the emitted radiation will change by a factor of (n^4), and this implies that to change the temperature of the metal by a factor of (n), that it will take (n^4) times more input energy applied to it since energy out = energy in. At a given temperature, if the surface area of that metal increases by (n), the total radiation will also increase by (n), hence it is proportional to the surface area. This forth power has been confirmed via tests and it is considered due to the metal's 3-dimensional crystal structure of nearby atoms. When heated, the atoms begin to vibrate and gain kinetic energy, releasing more free electrons and photons (ie, radiation energy) when they fall back into an orbit, and these photons also then cause a limited internal chain reaction of freeing more electrons and eventually increasing the thermal energy of the atoms and increasing the photon output at the surface of the metal. For more about this fourth power, research the Stephan-Boltzmann constant. **Extra**: A microwave oven's outer surface is generally of a continuous, flat sheet metal or metal screen construction so as to prevent the energy from leaking away or being radiated to where it is not desirable, and useless and dangerous for heating food, etc. This electromagnetic or RF shielding is often called a "**Faraday Cage**", and the glass on the viewing door has a metallic screen attached to it, and of which has viewing holes having a diameter of less than or equal to (1/10) of the wavelength of the microwave being used, and this effectively causes no radiation to be transmitted through and past them. It is also of interest, that this Faraday Cage system does not need to be electrically grounded as in a regular type of circuit, such as perhaps for a portable radio or communication device. RF shielding can both prevent the reception and transmission of RF energy and-or unwanted RF noise energy, and therefore an external, unshielded antenna will be needed to receive and-or transmit RF energy for communication.

Why radio communication? Consider for example that 1 watt of transmitted audio or sound energy may be discernible by the human ear at perhaps 300 feet away. Sound or air (physical) vibrational energy (ie., air compression [higher density] and rarefaction [lower density] transferring energy from air atoms to other nearby air atoms via collisions), like light and radio energy gets weaker and weak as it expands, or disperses outward from the source. 1 watt of radio wave energy which is essentially invisible (RF) light energy, and of which can be seen by the sensitive human eye much farther away, perhaps 1 mile away. Radio receivers and their circuitry can be much more sensitive than the human eye light sensor or receiver, and can even be very weak signals that were transmitted hundreds of miles away at high power, and can then amplify them to usable levels for hearing, etc.

It is interesting that radio wave energy is composed of an electro-magnetic (electric and magnetic) fields perpendicular (90°) to each other, and does not have electrons with kinetic energy to transfer energy from one location to the other as with common electric circuits. The electro-magnetic field is rather transmitted (ie., radiated, released outward) and-or received. (Usually invisible) photons can be considered as the energy carriers. The strengths or amplitudes of the electric and magnetic field waves vary periodically at the same frequency of that transmitted wave of energy.

In the above figure, the antenna is vertical and will transmit a radio signal outward in the horizontal direction all around it in a 360°, circular pattern. This pattern is therefore not a full spherical pattern in every possible direction including the vertical directions. For our radio system with vertical antennas, it is better to transmit the radio wave energy in just the horizontal directions. If the radio transmitter is 4 watts, the amount of energy transmitted horizontally per degree is:

$$\frac{\text{Total Transmitted Power}}{360°} = \frac{4 \text{ watts}}{360°} \overset{\sim}{=} \frac{0.011\text{w}}{1°} = \frac{11\text{mw}}{1°}$$

: a theoretical example, and the angle of signal reception may be much less when the receiver is farther away from the transmitter, and due to the inverse square law of energy transmission and reception, this value may be greatly reduced by the time it reaches the reception antenna.

If the receiving radio antenna is close enough to intercept 1° of the total transmitted signal, it will receive a theoretical maximum of 11mw of the transmitted radio wave energy. If the radio receiver has a minimum reception sensitivity of 11mw or less, then there will be enough radio signal received so as to be amplified and heard. A good radio receiver is said to have high **sensitivity** (ie., a low usable minimal voltage received ability, such as from a distance transmitter) and high **selectivity** (ie.,accepting only the desired frequency and-or range).

If the receiver is farther away and intercepts just 0.01° or one-hundredth of a degree of the transmitted signal., then the maximum radio energy power received is, and without considering the effective power loss or radio wave energy dispersion due to the distance between the transmitter and the receiver and the inverse square law of energy transmission and reception:

$$\frac{11mw}{1^\circ} = \frac{Xmw}{0.01^\circ}$$, after solving for Xmw:

$$Xmw = \frac{(0.01^\circ)(11mw)}{1^\circ} = 0.110\ mW = 110uW = 110\text{ micro-watts} = 0.000110\ W \quad : 11\ mW = 0.011\ W$$

Just like light, and of which is electromagnetic energy, the intensity, signal voltage and-or power level of a radio signal measured at a location from the transmission antenna (ie., energy radiator) is inversely related to the square of the distance from it: Just like light, most radio waves are also "line of sight" and need to be nearly free from obstruction. HF (High Frequency, "Shortwave Radio" - 3Mhz to 30 Mhz) radio has much more of an ability to reflect off the ionosphere and therefore travel a longer distance over terrain that is in the way of the line of sight and-or over the horizon or curve of the Earth. **Energy Received = Energy Transmitted / (distance^2)**. From this equation, if the distance doubles, the received signal is 4 times less, hence up to 4 times more energy may need to be transmitted if the initial received signal was very low. A certain radio receiver model usually has a general (minimum) input (RF, voltage or power), received sensitivity rating, and if the received radio signal is greater than this rating, that certain radio transmission should be able to be heard and-or understood clearly. Properly made and directed transmission and-or receiving antennas can improve the received signal strength and-or the transmission distance. If permitted, an "RF amplifier" can be used to transmit a stronger signal.

For best radio power transmission, it is best that the antenna be equal to the wavelength of the main carrier wave frequency being transmitted, and this seems to be particularly more critical as the frequency increases in value. A "tuned" or resonant antenna will give better transmission and-or reception. Certain fractions of a wavelength are also used so as a large impractical size antenna is not always needed. The antenna is then said to be properly tuned or set to a certain desired frequency. For these fractional sized antennas, their transmission efficiency is still near 90%. A "loading coil" is sometimes used in series with a short antenna so as it can appear electrically as a longer antenna with more impedance such as the "50 ohm" or "75 ohm" standard. An (automatic or manual) **antenna tuner**, which can be as simple as an inductor and-or capacitor which can help tune an (receiving and-or transmission) antenna so as its length does not need to be physically altered in length when the transmission frequency is changed. To help maximize a certain transmitted carrier or main signal, a **SWR** (**Standing Wave Ratio**, or Voltage Standing Wave Ratio (VSWR) and-or meter can be used. SWR has a technical derivation and formula beyond the scope of this book, but it can be roughly thought of as: SWR = (power out of the transmitter's electric circuit to the antenna / power actually transmitted from the antenna). This standing wave of radio power is not transmitted and is rather decaying back and forth through the resistance of the wire and-or transmitter circuit. Transmission loss is usually due to that an antenna mismatch (improper tuning [ie., length]), cable issues, or possibly some other problem such that some of the radio wave energy is not being transmitted outward ("forward") and away from the antenna and is rather then lost in other parts of the system and-or "reflected" (ie., essentially sent, transmitted) back into the transmitter circuit. A low SWR reading between 1 and no more than 2 is desirable. **The combination of a SWR meter and antenna tuner, both in a series connection to the antenna, can help optimize the transmitted radio signal and its energy.**

A transmitter can even perform tests and checks without using an antenna, but a "dummy load" resistor having an effective (LC, high frequency) impedance of often **50 ohms** (for HF), 75 ohms (for VHF), 300 ohms (low loss "twin lead" cable, for FM), and with an adequate power rating must then be used so as to not damage the transmitters electronics. Though a dummy load absorbs most of the transmitted power, a nearby radio may still be able to receiver enough energy so as to effectively hear a broadcast. Air is said to have an impedance of 377 ohms. The standard value of 50 ohms is an old and semi-arbitrary compromise value which considered the impedance matching for both lowest power losses (at 30ohms) and best signal transmission (at about 70 ohms) in (air-dielectric) coaxial cables, and then made into a default, recommended standard that all radio manufactures can then follow. Imagine the transmitted power not being transmitted or used such as it being lost in a resistor, but rather damaging the transmitter itself - usually the output transistor(s) of the signal amplifier. The combination of a monitoring, distant receiving station and-or a low power transmitted signal can also help with tuning an antenna system for maximum transmission. For maximum power transfer to the antenna, the impedance of the antenna should match that of the output impedance of the transmitter, and that is usually standardized at 50 ohms of impedance. Reactance is a function of the values of inductance (L) and-or capacitance (C), and signal frequency. If the frequency changes, the impedance of the antenna will then change and it will have to be reset to 50 ohms for maximum power transmission, such as by an antenna tuner, etc.

A **balun** is an electrical device, much like a small transformer, that allows the connection of **ba**lanced and **un**balanced transmission wires or lines for the best power transmission along a line. A balun can be used to reduce or eliminate the reactance of the antenna so as it appears as a 50 ohm load output or input to the 50 ohm radio. Some (balanced output) antennas have two out of phase signals at their output wires, and of which will then need to be connected to the single, signal input wire at the receiver, and a balun can do this. Balun's can prevent the outer shield metal of the cable from being a conductor for RF and-or noise. This helps with the transmission and-or reception of the desired radio signals. Balun's were once often seen on the ends of a TV aerial (ie., receiving antenna) or cable-tv coaxial line, and so that the TV's having a certain input impedance could efficiently display the received signal. One special balun is called a "**unun**" (ie., unbalanced to unbalanced) which can transfer power between two unbalanced (different impedance's) signal lines or cables. Some SWR meters may also include a power transmitted meter, and some radio receivers have a received, signal strength meter. **"Beam" or directional antennas** can focus the transmitted or received radio energy to or from a specific direction so as to aid communication and-or the need for more signal amplification.

What is polarized light and-or photons?

First, do not confuses the concept of polarized light with the concept of electrical polarity: (+) and (-).

Consider a vertical wire or pole, radio transmitting antenna. The electron moving in that antenna will travel along its physical direction. This will also create an electric field that will will oscillate (ie., like a repetative sine wave with its amplitude) along the length of that antenna (especially when properly tuned to the wavelength of that frequency), and here vertically (if placed vertically) in, and then transmitted away from that antenna in a vertical manner. This signal is then said to be vertically polarized. For best reception of the maximum power and-or signal strength, the receiving antenna should also be vertically polarized. For polarized light, the electric field oscillations are perpendicular to the direction of the travel of that light. Here in this case mentioned, the antenna signal is polarized vertically, and is traveling outward horizontally from that antenna.

Light from a household incandescent lamp bulb with its coiled metal milament is unpolarized, and is sending out its light waves and-or photons in various directions perpendicular to those directions of travel, but some of it can be made to be polarized by using say a vertical slot to behave like a polarization filter, and so as to pass only light having a vertical polarization in a certain direction.

COMMON CLASSIFICATIONS AND ALLOCATIONS OF VARIOUS FREQUENCY RANGES

The following is often called the (standardized) electromagnet frequency bands ("spectrum") or range classifications:

C = speed of light in a vacuum =~ 299,792,458 m/s =~ 300,000,000 m/s = 3(10^8)m/s =~ 186,000 mi/s = velocity = v

f=frequency or hz, = 1/ (time of each cycle) , w=wavelength distance in meters , hz = hertz = cycles per second = cps frequency (f) and wavelength (w) are mathematically, inversely related in value to each other, and are the factors of C or the speed of the wave.

C = fw , f = C / w , w = C / f , : Ex. If n is a multiplier or factor: At: nf, w = (1/n)w = w/n. At: nw, f = (1/n)f = f/n.
C is a constant, and (f) and (w) are inversely related to each other mathematically.

Wavelength is a real physical distance between each cycle (c , 1c = 1hz) or corresponding peaks of a wave of energy at a certain frequency and medium that the wave or energy is traveling through. **Wavelength = d / 1 c = d / c = d / 1 hz** . Wavelength is not the same meaning as the speed of sound or light transmission through a medium (material, such as air or a vacuum). A 1000hz sound pulse will travel faster through a medium of metal than in air, yet the frequency and its corresponding time (t=1/f) is always constant. Wavelength of a wave is determined by the frequency and speed of travel, hence wavelength is also determined by the medium it travels through.

Ex. For sound waves that travel at about 1125ft/s = 343 m/s in air, the wave length or wavelength of a 1000 hz sound wave is from:

distance = (speed)(time) = wavelength
wavelength = (speed)(time) = v t , and since frequency and time are reciprocals in value:

wavelength = (speed of sound)(1 / frequency) : For light, wavelength = (speed of light = c) (1 / frequency)
wavelength = v t = v / f = (d / t) / (1 / t) = (d / s) / (c / s) **= d / c** : For (rf), light or radio: w = vt = v / f = c / f =~
= (300 m/s) / (c/s) = 300 m/c

w = (speed of sound) / (frequency) = (speed of sound)(1/ frequency) = (speed of sound)(time) = (v)(t) = d
w = (1125ft/s)(1/1000c/s) = (1125 ft/s)(0.001s/c) = 1.125 ft / c : ft / c = feet per cycle, or simply feet

Note how the unites were expressed: (c/s) for frequency, and (s/c) for time , hence each has the mathematical inverse or reciprocal of the others units, and since frequency and time are inverse values of each other.

Sometimes a wavelength can be physically seen and-or demonstrated such as for sound waves in a pipe filled with a gas and a line of holes in it, and which will indicate where and the amount of sound or air pressure is in the pipe, and that distance or length between wave peaks or the troughs seen can be measured as the wavelength, and the corresponding frequency of it can be calculated using:

From: wavelength = distance = (speed)(time) = (speed)(1/frequency) = v / f , we have:

frequency = (speed / wavelength) = v / w : for radio waves , speed = v = C = 186000mi/s = 300000km/s

It is good to memorize that: wavelength = distance = (speed)(time) = w = v t = v / f , and that t = 1 / f and f = 1 / t

Extra: Ex. If the physical wavelength of a certain (rf) radio wave is say 0.1m , the number of these waves per unit distance or meter is simply found by division of this distance into the unit distance of 1 meter:

number of waves per meter = "wave number" = 1 meter / wavelength = 1 / 0.1 m = 10 : wpm

The following table is the frequency allocations for North America, and it may not apply to other locations, countries, and-or temporary allocations. These allocations were determined, set and governed by the FCC (USA, Federal Communications Committee) and in particular, so as to avoid a radio signal from interfering with any other radio frequency and signals.

Typical Frequencies (hz)	**Typical Wavelengths (m)**	**A Typical Frequency Group Name**
0 to 20Khz	3Km at 1000hz	Audio or radio (RF=radio frequency(s)), VLF (Very Low Frequency)
20Khz 525Khz	600m at 500Khz	**AM** (Amplitude Modulated or varied) radio LW (Long Wave) AM radio is 148.5Khz to 283.5Khz
525Khz to 1710Khz	300m at 1Mhz	**MW** (Medium Wave) **AM** radio, civilian, public 300Khz to 3Mhz is MF (Medium Frequency)
2.182Mhz	137.4m	Boating emergency communication, please monitor
3.2Mhz to 26Mhz	300m at 3Mhz 30m at 10Mhz Maritime Emergency (mhz):	**SW** (Short-Wave) AM, "**Am**ateur" and-or "H**am**", radio 3Mhz to 30Mhz is also called the **HF (High Frequency)** band. : Some: 4.125 Mhz , 6.215 , 8.291 , 12.290 , 16.420 156.8 (for short range areas and-or locations).
26.480 to 26.960 Mhz		U.S. Military
26.96Mhz to 27.41Mhz	11.1m at 27Mhz	CB (Citizens Band) **AM** radio, trucks, personal, 40 CHannels, **CH9 for emergency**, CH17 and CH19 are for trucks ("truckers"). While not transmitting, and if your radio(s) has enough supply power, **please monitor** CH1 and **CH9** for any possible emergency and-or assistance needed. A transmit (TX) license is not required for anyone to transmit on a CB radio. And: 4 watts is max. legal power transmission in the USA, and can generally reach 3 miles outward with direct line of sight travel. Some single side band (**SSB**) CB's can effectively transmit several times more power on **LSB (Lower Side Band) or USB (Upper Side Band)** near the main or fundamental (radio) carrier frequency assigned to a channel. People using CB, SSB mode are asked to use only CB channels 36 through 40. Any CB is very useful during an emergency, and since they are widely available and relatively inexpensive to purchase before they are even needed as part of communication for when telephone service is unavailable and-or not working. Each town should have at least one or two CB radios and operators. A CB or shortwave radio relay station/operator can be used at the top of a mountain to assist with any received communication, and to be relayed over. As of about the year 2021, some CB radios and-or transceivers (ie., can transmit and receive) also offer an RX/TX, **FM mode** for some clearer communication and very low AM noise interference. The max. output power is also limited to 4W.
27.450 to 28 Mhz		U.S. Federal Government
49 Mhz , **VHF** = 30MHz to 300 MHZ	10m to 1m	For (U.S.A., <= 100mW transmission) Walkie-Talkie radios

52.525 Mhz		A common **emergency** radio frequency, and uses a FM mode. Others in Mhz: 47.42 (Red Cross) , 39.46 (police)
54Mhz to 88Mhz	5.55m at 54Mhz	**VHF** TV CH 2 to 6, Very High Frequency
88Mhz to 108Mhz	3m at 100mHz	**FM** (Frequency Modulated or varied) civilian, public radio Many modern (as of 2022) phones include a FM radio app.
118Mhz to 137Mhz	2.47m at 121.5Mhz	VHF (Very High Frequency) Aircraft Band 121.5Mhz civilian **emergency**, 243.0 Mhz military emergency Please monitor these frequencies.
145.80Mhz (transmitted by the ISS, receives at 144.49Mhz and 145.20Mhz) 2.06m		**Shortwave**, space communications, such as for the ISS (International Space Station). Includes voice and data such as **SSTV (Slow Scan TV images)** of which images can be processed by a computer program (software) into an audio file. This file can then be sent using radio communication. The audio signal in the received radio signal (which has the image data coded into audio signals/tones being carried in/on as part of the high frequency radio signal) is sent to the computer mic input or other audio input of a computer of which an SSTV program can then process back into an image. Amateur ("HAM"), shortwave radio stations often experiment with using SSTV, and it is slow, but effective. There are many low-cost programs and standardized methods for data communication via radio that you should obtain and have available as a radio operator/communicator, and before any grid-down or emergency situation.
151.820 to 154.600		**MURS** radio, Multi-Use Radio Service, no license, usually for business communications, 2W max. transmission power. Has 5 FM communication channels and-or frequencies. Unlike FRS radio with a fixed antenna, MURS allows a removable or changeable antenna for better communication with the 2W.
156.8 Mhz		**A common emergency radio frequency**. VHF CH 16. Please monitor, listen and-or communicate with this frequency, and then alert authorities if needed.
174Mhz 220Mhz	1.5m at 200Mhz	VHF TV CH 7 to 13
300Mhz to 3000Mhz=3 Ghz	1m = 100cm at 300Mhz 0.5m = 50cm at 600Mhz 0.3m = 30cm at 1000Mhz 0.25m = 25cm at 1200Mhz	**UHF , Ultra High Frequency**. 406 to 406.1 Satellite emergency. Digital TV radio waves are 470Mhz to 850Mhz. **FRS** = Family Radio Service = 462Mhz to 467Mhz FM, 2W max FRS uses **FM** modulation which is less noisy than AM mod. **GMRS** = General Mobile Radio Service, is similar to FRS, but is more for business and requires a license to operate, and also can use "**repeaters**" (ie., transmission relays) to extend their range. FRS has 22 channels available as of the year 2021. GMRS includes the lower power FRS channels, and also includes 8 more high frequency channels, and generally GMRS allows up to 5W radio output power on channels 1 to 7, channels to 8 to14 are limited to 0.5W for both FRS and GMRS. GMRS channels 15-22 can have up to 50W. If the

GMRS power is lower than or equal to 2W max, such as like the FRS, then the license is waived or optional. **As with all radio communications, channels must be shared and not interfered with while being used by others, unless there is an emergency, and of which anyone without a license can and should legally use a nearby radio for transmission, and if they know how to use the radio on a minimal or greater level for emergencies.**

1Ghz to 100 Ghz 30 cm at 1 Ghz , Radar , 3 cm at 10 Ghz, **Microwaves** (microwave ovens)
10^9 hz and with 0.3 cm = 3mm at 100 Ghz = 100(10^9)hz. , 1(10^11)hz
1 Ghz = 1 billion hertz = 1,000,000,000 hertz = 1 (10^9) hz = 1 giga-hertz

2.4Ghz to 2.485Ghz 12.5cm at 2.4Ghz "**Bluetooth**", short-range radio communication system for data.
12.1cm at 2.485Ghz Devices with Bluetooth can share files and audio as digital data.
There is also offline WiFi and-or BlueTooth phone apps.

100Ghz to 1000Ghz = 1Thz 1mm=1000nm at 300Ghz Thermal **Infrared** (IR, heat electromagnetic radiation)
1mm = 1(10^-6)m 1 Thz = 1 trillion hertz = 1(10^12) hz. Thermal energy is due to the kinetic energy of particles, and this thermal energy also radiates some (infra-red) light or electro-magnetic energy.

10Thz to 100 Thz 30um to 3um **IR** (**infrared** to barely visible red color light)
= 10^13 hz to 10^14 hz 3um = 3 micrometers = 3(10^-6)m = 3000nm

1000Thz 580nm=Red, 545nm=Green, to 440nm=Blue **Visible Light** (**R**ed, **G**reen, **B**lue, "rainbow" spectrum =
= 10^15hz "**R**OY **G** **B**IV" = Red, Orange Green, Blue Indigo, Violet
= 300nm = 300(10^-9)m = 3(10^-7)m The color of an object is due to that all the other colors of the full spectrum or white colored light are absorbed by the atoms on its surface, and that light energy is converted mostly to thermal (heat, kinetic) energy such as for photosynthesis in plants. The reflected or emitted color of an object can help determine what element it is, such as silver, copper or gold. For example, copper has a red-brown color, but will emit a green color when heated by a flame.

10^16hz 10nm **UV** (Ultra Violet) light (**dangerous**, ex. sun burn, eye damage),
30nm = 30(10^-9)m = 3(10^-8)m such as some from the Sun, and much from an electric arc welder.

10^17hz
3 nm = 3(10^-9)m

10^18hz 1nm **X rays** (X ray machine in hospitals, radiation danger)
0.3nm = 0.3(10^-9)m = 3(10^-10)m 1nm = 1 nanometer = 1(10^-9)m
300pm = 3(10^-10)m

10^19hz 0.1nm **Gamma** rays (radiation danger)
= 0.03nm = 0.03 (10^-9)m = 3(10^-11)m 0.1nm = 0.1(10^-9)m = 1(10^-10)m
0.03nm = 30pm = 3(10^-11)m 1pm = 1 picometer = 1(10^-12)m

Ex. What is the approximate frequency of a radio signal that is said to be in or of the 40m "band" tuning range or wavelength?

$$f = \frac{v}{w} = \text{frequency} = \frac{\text{speed of light}}{\text{wavelength}} \approx \frac{300000 \text{ km/s}}{40\text{m}} = \frac{300000\text{km/s}}{0.040\text{km}} = 7500000 \text{ hz} = 7.5\,(10^6) \text{ hz} = 7.5 \text{ Mhz} \quad : 1\text{m} = 0.001\text{km}$$

The above assumed the signal is traveling in a vacuum such as space with a velocity of 299792458m/s =~ 300000m/s, but for a rf signal traveling through air at sea level where it is the most dense at 14.7psi, the signal's velocity is actually slower at about: 299702545m/s.

In general, a 1Mhz rf signal has a wavelength of 300m. If the frequency increases by (n), the wavelength will decrease by (n). Ex.: If the increase in frequency = n=10, a (1Mhz x 10) =10Mhz rf signal has a wavelength of (300m/10) = 30m. A 100Mhz rf signal has a wavelength of 3m, and this is within the common public radio FM band (ie., bandwidth or range of allocated frequencies). A 300Mhz rf signal has a wavelength of 1m.

Ex. What is the **wavelength** of a 100 Mhz signal: (300 M m/s / 100 M c/s) = 3 **m / 1c** = 3m , 3m =~ (3)(3.28)ft = 9.84 ft

Ex. An electron of a hydrogen atom that is struck by some energy, such as an electromagnetic wave or high speed particle of matter, will gain energy, and will eventually emit that extra energy as electromagnet energy (ie., photon, radio-wave) at a certain natural frequency due to the natural construction of the hydrogen atom. The frequency of the emitted wave of energy will have a wavelength of 21cm = 0.21m. The corresponding frequency of this wavelength is:

From: d = v t = w = v t , t = w / v , f = 1 / t = 1 / (w / v) = v / w
f =~ w / v = (3 (10^8) m/s / 0.21m) = 1428571429 hz =~ 1.4286 Ghz , often said as: 1.42 Ghz ,
1.42 (10^9)hz

A **spectroscope** (ie., a light-scope) is a device which has a glass prism, and which can essentially make a wide display (ie., a ("rainbow-like") spectrum) of all the component colors and-or frequencies contained in the received or incoming light energy, and which is then used to determine what element(s) is associated with that particular visible color(s) and-or corresponding frequency(s) of light. A **spectrometer** is like a spectroscope however it is rather used to determine the type and-or qualities of a sample of material such as an element when some type of energy, such as heat, is applied to it. After being heated, the element and-or its gases of, will have unique and identifying (black color, absorption) lines in its spectrum of colors (each having an associated frequency) seen on its spectrogram (ie., visible image). A few new elements were also discovered this way. A spectrometer can be used to determine what elements are in a substance. This discussion here is very short, and these topics, people and the history is beyond the scope of this practical math book.

When light beam enters a transparent medium such as a glass prism or lens, that light beam will enter that substance and then **refract** (ie., **bend**, change direction) and its direction of travel will be at a certain known angle (of refraction) with respect to the surface plane and-or a conceived "normal line" that is a perpendicular line with respect to the surface line of that material. A refraction effect is somewhat similar to the effect of having a mirror direct the light to another direction. When the light bean exits the transparent medium it will refract again at an angle that is determined by the new medium and its density. If a bean of light enters a medium straight-on or at no angle with respect to its flat surface, then it will not refract at an angle, but it will pass directly through it like a straight line passing through it, and without any refraction. The direction change can be estimated and-or thought of as being like that of having two wheels at the ends of an axel or left and right sides of line moving forward, and if one wheel slows down, there will be a direction change due to a rotation or torque effect about the slowed wheel acting as a type of pivot or rotation point.

Reflection and refraction are two different concepts. With reflection, the light does not enter the substance, but rather essentially "bounces" off its surface, and it could be at an angle. The angle of reflection (¢r) will be equal to the angle of incidence (¢i , incomming) light: ¢ reflection = ¢ incidence , hence their ratio is equal to 1. Now when light enters a medium such as water at an angle with respect to its surface, the angle of refraction or bending does not usually equal the angle of incidence, however it was found that::

$$\frac{\sin \text{¢ incidence}}{\sin \text{¢ refraction}} = \textbf{a constant}$$

value depending on the medium it will travel through : by **Willebrord Snellius** - - from the Netherlands , "**Snell's Law**" - independently in 1621, aka: "**law of refraction**". Also credited to **Ibn Sahl** of Iraq in 984.

See FIG 255A. You may also wish to also research the concept of the **refractive index** for a substance or material. In short: **refractive index = (RI) = velocity of light in a vacuum / speed of light in the medium material**
The wavelength of the light used will also determine the refractive index of the medium, and if the light used is white light with many frequencies, there will be a chromatic dispersion due to each color and-or frequency having a slightly different refractive index. Typically: for water, RI = 1.33 , glass, RI = 1.4 to 1.7 , quartz, RI = 1.54

It was also found in 1662 by **Pierre de Fermat** (1607-1665), from France, that the ratio of velocities of the light in the two mediums is also equal to this same constant of the law of refraction.

$$\frac{\text{v in medium 1}}{\text{v in medium 2}} = \frac{\sin \text{¢ incidence}}{\sin \text{¢ refraction}}$$

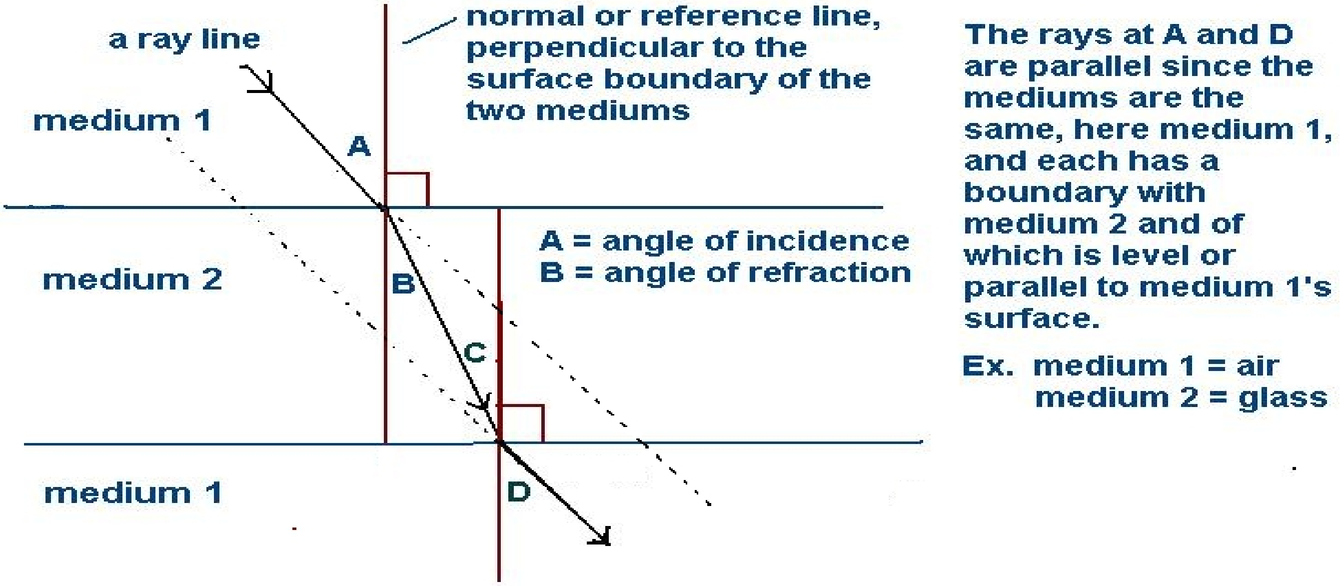
: Fermat's Principle (aka: "principle of least time of travel", and not necessarily the straight line path through a medium, such as for light travel)

The speed of light and-or an elector-magnetic wave will actually be reduced (ie., be slower) while going through a medium such as glass or water, and this is due to the electrons in the glass atoms gaining some kinetic energy and creating their own electro-magnetic field which will then interfere with electro-magnetic field and-or energy of the light wave. This physical interaction is what also causes refraction. Now since the frequency of the electromagnetic wave is still the same through the medium, but its propagation is now delayed or slower after all of the many electron and wave interactions involved over its length, the wavelength of the electromagnetic wave is actually shorter in the medium material it is passing through it.

The less clear and transparent the medium is, the more the light will not pass through it and it will be absorbed as heat in it. Some of the energy of the light passing through a medium will be converted to heat (ie., kinetic energy) in the medium, and therefore, its intensity will continue to lessen as it travels through that medium.

Beer's Equation or the (August - Johann) **Beer-Lambert Law** (1760 , 1852) of which is also based on the previous analysis of **Pierre Bouguer** in 1729, and which states the relationship of the absorption (A) of light or energy through a transparent medium, and it includes the length (L) of travel, concentration (C) of the medium and the molar absorptivity (e): **A = e L C** . A is a unitless number, usually called the Absorpton-coefficient of a material. (A) has a linear or direct relationship to (e), (L) and (C). The molar absorptivity coefficient value (e) depends on the specific substance or medium, and the wavelength of light used, and is measured and found by experiment with a unit length of a test sample, say 1cm: e = A / LC. The (molar) concentration, and which can also be thought of how much material is dissolved in a unit of a clear liquid, and this similar to a density value, but in terms of mol of mass, instead of mass, and with units of: mol / Liter = mol / (10 cm)^3 . Determining the amount of light passing through an object is a quick method to see if a standard(s) is being met in various chemistry, food and other industrial processes.

[FIG 255A]



In the above figure, let us consider medium 1 as air, and medium 2 as glass. Since glass is denser than air, the ray will bend toward the normal line. After traveling through the relatively dense glass, the ray will bend away from the normal line when it reaches the less dense air. It is of note that the frequency (ex. its color) and-or wavelength of the light ray will also help determine the refraction angle. Note that angle C becomes the next incident angle and of which by the geometry of this example is equal to angle B. and angle D = angle A due to parallel transversal lines, and is the next refraction angle as the beam or ray travels. Much of the last few paragraphs have dealt with optics, and the general subject is beyond the scope of this practical book.

CB RADIO (CB = Citizens Band Radio)

What is thought of as common radio frequencies, such as for CB, FRS, GMRS, AM public radio, FM public radio in one country, such as United States of America, may not necessarily be the same frequencies used in another country(s), You may need to purchase a radio intended for that country and its frequency allocations. Today as of the year 2024, many radios allow you to directly enter the numeric value of a radio frequency to listen (ie., receive) to and-or transmit on. You will also need to know if it is legal to transmit on a certain frequency in a country, and with or without a license to use a frequency(s). It may even be illegal in some countries to have a certain communication device without permission.
Some common radio frequencies allocations for the United States of America and its partner territories or locations when permitted:

CB radio CH1 has a frequency of: 26.965 Mhz , and each next higher CB CH number is 10 kHz higher in frequency, hence CH2 has a frequency of: 26.975 Mhz. A few channels are out of this step value of 10kHz, but are typically 20 kHz higher due to "backward compatibility" with older frequency allocations and equipment, etc. Further ahead in this book is a full listing of the North American CB radio frequencies, etc., but for now:

For CH1 through CH7 , CHn has a frequency of: 26.955 Mhz + (n 10 kHz) : **CH1 = 26.965 Mhz** = 26965000 hz
CH8 = 27.055 Mhz , and each higher channel trough CH15 is 10 kHz higher in frequency.
CH16 = 27.155 Mhz , and each higher channel through CH22 is 10 kHz higher in frequency.
CH23 = 27.255 Mhz
CH24 = 27.235 Mhz , and CH25 = 27.245 Mhz
CH26 = 27.265 Mhz , and each higher channel through **CH40 = 27.405 Mhz** is 10kHz higher in frequency.

CB Channel Use Considerations:

CH9 is for emergencies only **: if you are able, please monitor this channel, provide assistance and-or contact the necessary authorities such as the police or military assigned to that jurisdiction and-or location(s).**

CH1 is typically used by truckers
CH6 and CH11 are typically used by truckers for some longer distance communications, and after installing a RF amplifier to increase the signal strength, however, misuse may invite FCC law enforcement and a potential fine, and especially if it is causing interference on other radio frequencies, inappropriate language and-or used for illegal activities.
CH9 is for emergencies only and should be monitored when able, CH36 should also be monitored if able.
CH10 is typically for local roads truckers - perhaps letting a warehouse know that a shipment has arrived
CH17 is typically for highway northward or southward truckers , CH19 for eastward or westward truckers
CH36 through CH40 are typically used by **SSB** = **S**ingle **S**ide**B**and (USB or LSB) broadcasters only.
CH36-LSB is typically for calling (ie., requesting, finding) other SSB broadcasters
Each channel and-or frequency can essentially have 2 independent radio conversations at the same time since there is both an upper and lower sideband (USB and LSB) communication option for each SSB, CB channel. To make **SSB**, the main carrier frequency and either the adjacent upper or lower side band to it is first removed and that leaves just one side band, of which only it is then amplified, hence saving power which can then be applied to the sideband used. For example, given a 1 Mhz carrier wave and amplitude modulating or mixing it with a 1 Khz (audio, low frequency) wave, the output frequencies and-or bandwidth from that mixer in simple terms is: (1 Mhz + 1 Khz) = USB , and: (1 Mhz - 1Khz) = LSB. Most SSB radios come with what is called a "**clarifier**" or "fine tuner" control knob which allows a hand-operated (slight) frequency adjustment of the receivers desired or tuned frequency, and it is usually for the SSB mode of the radio. This can prevent a received voice from sounding too low or high, and distorted. The frequency bandwidth for AM, CB frequencies is 10 Khz, hence allowing a 5 Khz bandwidth or frequency range on both "sides" of the main carrier frequency. SSB CB has a bandwidth of 5 Khz, however it utilizes the same CB channel and its bandwidth assigned to a particular carrier frequency, however the SSB "tuning frequency" is slightly higher or lower than that standardized CB carrier frequency.

Technically, in the U.S.A., the maximum radiated RF power of a modulated (ie., a modified main RF carrier signal via a low frequency audio signal) or unmodulated CB carrier frequency is 4W and if this was given a 100%, full modulation by the input audio signal and its amplifier, it would be four times more at 16W of radiated RF power and would then be illegal. In short then, the output CB carrier frequency is generally less than 4W so as the (AM, typical, FM is now available [as of 2022] for CB as NFM = Narrow-Bandwidth FM) output modulated signal is <= 4W total. The average CB channel has a full wavelength of about **11m** =~ 36.1 feet = 36 ft + 1.2 in. CB = "11m band"

Canada and many other countries use similar CB channels and frequencies, and sometimes more, as the USA.

FRS radio has 22 FM channels available and are the modern equivalent of an improved "walkie-talkie" radio, but with a higher output power of **2W max. on most channels, but only 0.5W max. on channels 8 through 14**. These radios can transmit to about 1 mile in most cases, and generally need line of sight or a nearly uninterrupted radio signal path. Each FRS channel has a frequency separation of 25 khz and of which 2.5 khz is then the max. ("narrow band") FM frequency modulation centered on the carrier frequency of the channel being used. **CH 8** is considered as the FRS radio emergency channel, and it should be monitored if you are able. **CH 20** is considered the GMRS radio emergency channel. **GMRS is similar to FRS** and even includes the same FRS radio channels, but it allows **up to 50W** of RF signal being transmitted on channels 15 through 22. CH1 has a frequency of 462.5625 Mhz and with a 25khz skip up through CH7. CH8 has a frequency of 467.5625 Mhz , and with a 25 khz skip up through CH14. CH15 has a frequency of 462.550 Mhz , and with a 25 khz skip up through CH22 = 262.725 Mhz. This is how the newer frequency allocations eventually amounted to, and due to backwards compatibility, the old channel frequencies and identifying channel numbers were kept. Please follow the specific communication laws in your country.

A basic rule of thumb for transmission distance for a hand-held radio such as a CB, FRS, GMRS, etc. held at 5 or 6 feet above the ground is about 1 mile of transmission distance and-or range per watt of transmitted power. For a common 4 watt CB AM radio, you should expect about 4 miles of range if there are no obstructions to the signal. If you are up higher, such as in a building or on a mountain, the range will be much farther - at least double or triple since the horizon is farther and usually without obstructions such as hills and trees. If there are obstructions, you may get 1 mile or less of range. To maximize transmission distance, the antenna length should also be matched to the transmitted radio wavelength or some stated fraction of it, otherwise, there is a mismatch and lower power will be transmitted. If the mismatch is great and-or no antenna or high power 50 ohm load resistor is being used, the radio could be damaged.

If you are the radio operator of a group and-or family, it is wise to show each member the basics of using the communication system in the event of an emergency. Also write the instructions, radio frequencies, channels, time(s) of communication, etc. on a paper(s) that is kept near the system for access. During an electric grid-down and-or cell phone service not working, a radio (either a transmitter and-or receiver) becomes very wise and practical to have. Many people will have access to an FM and-or AM radio receiver, and local radio operators may then make local transmissions of information, etc, that can help with the needs of the local community. Many modern mobile-phones can even use an app (ie., program) that will let your phone activate an internal IC chip to receive FM radio broadcasts and play the sound through the phone speaker, etc, and often, an earphone or other wire in its place is needed so as to function as a needed, makeshift receiving antenna. Having some electric power backup like a solar electric generating system (solar panel(s), charger box, battery(s) and a power inverter) would help some electric devices function for extended time of use, particularly communication and light(s). Also remember the importance of having light in the dark, and the light from a modern phone can even be used as a practical emergency light. The camera on a modern phone can also be useful in grid down situations. A rocket stove can heat and cook things using dried scrap sticks of wood, and it also produces less smoke fumes.

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MAJOR OBJECTS IN OUR SOLAR SYSTEM

(Most of the values below should be considered as approximate and rounded.)

Object	**Diameter In Miles**	**Average Or "Mean" Distance From The Sun In Miles (perihelion + aphelion) / 2**	**Time Of 1 Orbit About The Sun**	**Eccentricity Of Orbit (elongation)**	**Time To Rotate Once, "Day Length" On Axis In relation to an Earth day**
Sun	865,000	0	0		30 days at its equator, due to sunspot observations
Planets:					
Mercury	3032	35 million = 0.37AU	88 days = 0.241yr	0.206 (very high)	59 days
Venus	7520	67 million = 0.72AU	225 days = 0.616yr	0.007 (very low)	243 days
Earth	7918	93 million = 1AU	365.256 days = 1 year	0.01671	~ 1 day = ~24h = 23h,56m,4s
Mars	4210	142 million = 1.53AU	687 days = 1.88 years 1 Mars-Year = 1M-Y	0.093	1.026 days = 24h , 37m , 26s

***** Largest "Minor" or "Dwarf" Planets, and with most within the main asteroid "belt", zone or region located between Mars and Jupiter: Much of this matter is thought to be from two or more planets or asteroids colliding and then fracturing apart.

***** Vesta	326	220 million = 2.37AU	3.6 years	0.0887	0.22 days = 5.28 hours
***** Ceres	588	257 million = 2.76AU	4.6 years	0.0758	0.38 days = 9.12 hours
***** Pallas	339	258.5 mill = 2.78AU	4.61 years	0.23	7.8 hours
***** Hygiea	276	292 million = 3.14AU	5.56 years	0.112	27.5 days
Jupiter	87000	484 million = 5.2AU	11.9 years	0.049	9.93 hours
Saturn	72000	887 million = 9.54AU	29.5 years	0.057	10.55 hours
Uranus	31000	1.784 billion = 19.2AU	84 years	0.046	17.23 hours
Neptune	30000	2.8 billion = 30.1AU	165 years	0.009 (low)	16.1 hours
Pluto	1478	3.67 billion = 39.5AU	248 years	0.248 (high)	6.39 days
* Eris	~ 1446	~ 9 billion =~ 96AU	~560 years	0.436 (very high)	~26 hours

The distance values given above are distances from the Sun, and since Earth is 93 million miles from the Sun on average, the closest or shortest distance from Earth to another object is less this distance. For example, the closest distance to or between Earth and Mars is calculated as a difference from the reference position of the Sun:

The difference between the distance to Mars from the Sun, and the distance to Earth from the Sun is:
142 million miles - 93 million miles = 49 million miles =~ 1.53AU = 1.0AU =~ 0.527AU . This is the (average or typical) distance between Earth and Mars.

Since Mars is a distance of 0.527AU from Earth, it will take a radio signal (8.333 min. / 1AU) (0.527 AU) = 4.39 minutes to reach Mars from Earth, or vice-versa, when this closest approach or minimum distance condition exists.

A spacecraft and-or deep space vehicle launched in 2006 and called "New Horizons" arrived at **Pluto** over 8 years later on July 14, 2015 and took close photos of Pluto as it quickly flew past Pluto, and without orbiting Pluto. These were the first close-up photos of Pluto for even the best telescopes could not resolve much detail of any other planet farther than Saturn. The photos revealed that Pluto is a red-pink and white colored solid (not gaseous) planet with a surface having much ice, high ice mountains, and contains some atmosphere. Because it is fairly small and not a gas covered giant planet like Jupiter, Saturn, Uranus and Neptune, and is in a relatively high tilted orbit about the ecliptic plane (solar plane of Earth's orbit about the Sun), some consider that Pluto, discovered in 1930, is not an actual planet, but rather a large asteroid or planetoid like Ceres that was discovered in 1801. Many people still firmly believe Pluto is a planet since it is round and even has a round moon. One current theory is that Pluto got placed into its current (inclined at a relatively large angle to the solar plane - common orbit of planets about the Sun) orbit by a large mass colliding with it.

Pluto has 5 moons, with the largest being called **Charon** (found in about 1978 from Earth based photography and some crude low resolution photos of it) which is spherical and amazingly half the diameter of Pluto, and such that they nearly orbit each other with a (bary-) center of orbit located outside of Pluto's diameter (1344 miles), and at a distance of about 2080 miles. The average orbit of radius of Charon about Pluto is about 12200 miles. The Pluto and Charon system also has 4 other smaller moons. Eris (discovered in 2005) has 1 known moon called Dysnomia. Before Pluto's official discovery and naming, Pluto was speculated by Percival Lowell (Lowell Observatory) ,Yerkes Observatory, and others to exist, for something was causing the orbit of Uranus to be altered slightly. In 1930, **Clyde Tombaugh** at **Lowell Observatory**, was the first to recognize tiny Pluto changing its position in two (previously made) photographs of the same section or celestial coordinates of the sky. Pluto's orbit has a high inclination to the Earth-Sun solar plane or ecliptic, and is about 17.2 degrees from it, and this is one of the main reasons that it was difficult to find, besides being relatively small and far from the Sun which is the light source of a planet's dim reflected light. Pluto receives only a small fraction of the Sun's light energy as compared to that of the Earth, and the reflected amount from Pluto then decreases significantly in intensity for us to then try to observe (ie., "see") it with telescopes on Earth. For comparison of Pluto's 17.2 degree orbit inclination, Mercury, the closets planet to the Sun, has the highest inclination of all the official or standard 8 or 9 planets (as of the year 2022) at 7°. Pluto

The spacecraft to Pluto's called "New Horizon" did not have enough fuel to slow down and go into orbit about Pluto, but rather the spacecraft aimed at Pluto so as to increase its speed due to the gravity of Pluto. Going too fast to orbit Pluto, the spacecraft got pulled slightly towards Pluto and this also changed the spacecrafts direction of travel. Its direction of travel was calculated and made so that it could photograph a large asteroid farther away than Pluto. This type of directional and-or speed change is known as a "(planetary) gravity assist".

Eris is a recent discovery in 2005 by **Mike Brown's team of astronomers at the Palomar Observatory**, California, United States. Eris is classified as a dwarf planet like Pluto and Ceres are. Eris and Pluto also have a moon which causes the status of a dwarf planet being in question, however some relatively small asteroids also have orbiting moons. The orbit of Eris has a large tilt or inclination from the Earth-Sun solar ("ecliptic") plane, and is an angle or tilt of 44° This large tilt from the standard ecliptic location of planets made the discovery of Eris as being unusual and unexpected, somewhat like Pluto's discovery. The orbit of Eris is also highly eccentric or elongated. As of about the year 2022, seven other, nearly spherical, dwarf planets smaller than Pluto have been discovered.

Current list of the known, large dwarf planets (or "protoplanets") and-or asteroids as of 2023: Pluto (m) , Ceres , Vesta , Quaoar (m), Sedna , Orcus (m) , Haumea (m) , Eris (m) , Makemake (m) , Gongong (m) , Pallas , Hygeia , and those with (**m**) have at least one moon. Pluto is the largest dwarf planet, and with Ceres being the second largest. Pluto, Ceres, and Vesta have been photographed very well by the DAWN spacecraft, and Ceres and Vesta have also been orbited by it for a time. Pluto and Ceres have a round shape, and Vesta has a round-like shape. Ceres was discovered by **Giuseppe Piazzi**, (1746-1826), from Italy, and it was found before the invention of photography to help sense the motion of objects orbiting the Sun over a relatively long period of time (days, months, years). Since its orbit was later found to be circular, it was deemed a planet-like object. Comets have orbits with a high value of eccentricity. At about the year 1800 many astronomers were making highly populated and detailed star maps. Pallas was discovered in 1802, soon after the discovery of Ceres, and they were both classified as **asteroids** ("star-like" as seen first by William Herschel and his telescope) which can be thought of as a "protoplanet", and because of their faintness, their shape could not be determined other than probably being small in diameter. Piazzi made several star catalogs of high quality, for stars visible to the eye and visible through a telescope of a certain size and magnification. Piazzi discovered Ceres using a 3" refactor (ie., glass lens) telescope, and he also used a large wheel-shaped measurement device with the telescope sightings so as to have a high degree of precision with the astronomical locations (ie., coordinates) of the objects seen.

In 1781, planet **Uranus** was discovered by **Frederick William Herschel** (1738-1822), from Germany and later England since 1757. The more distant planet **Neptune** was actually seen by several astronomers as more or less a mystery object for about 100 years and-or later calculated in advance to be in that area of the sky, and it was finally confirmed as a planet in 1846. Like Venus, Neptune has nearly a circular orbit, and yet it is very far form the Sun. Earth's orbit is usually considered circuilar. All of the other planets were known since antiquity: Mercury, Venus, Earth, Mars, Jupiter, and Saturn. Using various forms of light spectroscopy and temperature measurements, he was the first to noticed that (invisible) infrared light was present and had energy due since it caused an increase in temperature, and today, this is commonly known this as "heat rays" and-or "heat energy", but it is technically a form or light, much like how radio wave frequencies are not seen but are a form of light and-or RF (radio-frequency) energy. Some animals can see infrared light, and some can see ultraviolet light. In brief, all these forms of light are said as being composed of photons or particles of light.

Note that the Earth actually revolves on its axis in a slightly less amount of time than the accepted time standard of 24 hours per day. 23h, 56m, 4.1s is less than 24 hours by 3m, 55.9s =~ 3m, 56s, or nearly 4 minutes. Due to this, the position of the Sun at say solar Noon where it is at the highest angle above the horizon will be nearly 4 minutes faster or earlier than expected using the standard 24 hour per day clock. 4 minutes corresponds to about 1 degree of visual angle in the sky. It is possible to construct a clock to keep "solar noon day time" with 23h , 56m and 4.1s.

To find Earth's closest or perihelion distance to the Sun:

First , note that for a circle e=0, and with values of (e) near 0, it will indicate a slight oval or ellipse shape.

For a circular orbit: perihelion = aphelion = (average distance = radius)(1 - 0) = radius of orbit

average distance from Sun or focus of orbit = (aphelion + perihelion) / 2 : average orbital distance from the Sun

perihelion distance = (average distance)(1 - e) = (93 million miles)(1 - 0.017) = : here, e = eccentricity of Earth's
= (93)(10^6)mi (0.983) =~ 91.419 million miles elliptical orbit about the Sun
= a (1 - e)

To find Earth's farthest distance from the Sun, note that average distance is equal to the "center" of the orbit or ellipse:

aphelion distance = (average distance)(1 + e) = (av. distance) + (e)(av. distance) = (93 million miles)(1 + 0.017) =
= (93)(10^6)mi (1.017) =~ 94.581 million miles = a (1 + e)

From these equations, if you know average, and aphelion or perihelion distances, the eccentricity of the ellipse or orbit can be solved for from the above equation: (aphelion distance / average distance) = (1 + e)
e = (aphelion / average) - 1 , and:
(perihelion distance / average distance) = (1 - e) and e = 1 - (perihelion / average)

The eccentricity of the Moon's orbit about the Earth is about: e = 0.055, and its average distance from Earth, or the center of the Moon's elliptical orbit along its major axis distance (aphelion + perihelion) or plane is 240000 miles.

Some common astronomical terminology and facts for elliptical orbits:

AU = Astronomical (distance) Unit. This is the average distance that Earth is from the center of the Sun, and is used to simply indicate relative (ie., multiples of) or comparative distances that objects are from the Sun or other objects. Light or radio waves will take about 500 seconds = 8.333 minutes to travel 1AU = 93 million miles.

aphelion = An objects most distant location from the Sun. ("apo" means distant, and "helion" is from "helios" for Sun, A common use of "apo" or "ap" is "apart", meaning separated from). apogee = aphelion

The elongation or eccentricity (e) of Earth's orbit about the Sun is about 0.017. Earths average distance from the Sun is 93 (10^6) miles. The aphelion or farthest Earth is from the Sun is:
Earth's Aphelion = (Earth's average distance from the Sun) + (Earth's average distance from Sun)(eccentricity) =
(Earth's average distance) (1 + 0.017) =~ 94.6 million miles

Mar's Aphelion = (Mar's average distance) (1 + 0.093) = (142)(10^6) (1.093) ~= 155 million miles

perihelion = An objects closest location to the Sun. ("peri" means near, a common use of "peri" is "perimeter" and "periscope"). perigee = perihelion

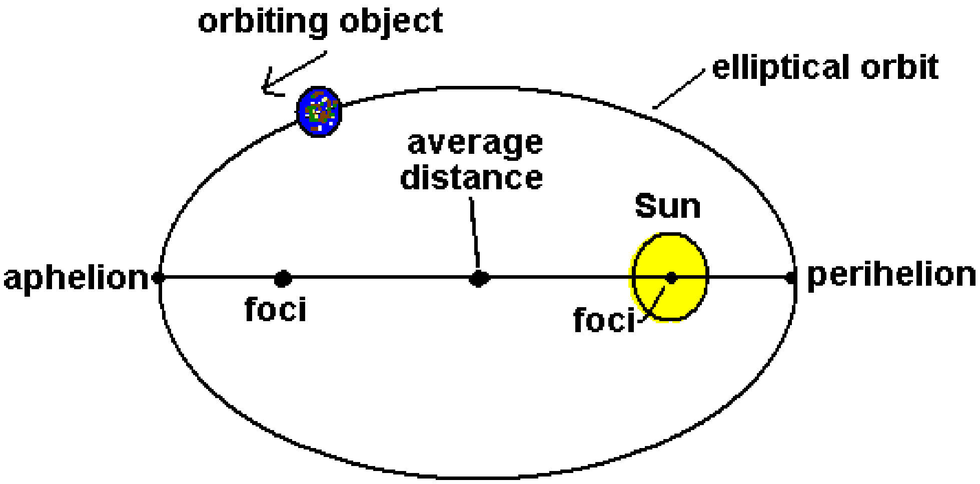
The elongation or eccentricity (e) of Earth's orbit about the Sun is about 0.017. Earths average distance from the Sun is 93 (10^6) miles = 93 million miles. The perihelion or closest Earth is from the Sun is:

Earth's Perihelion = (Earth's average distance) - (Earth's average distance from Sun)(eccentricity) =
(Earth's average distance) (1 - 0.01671) =~ 93 (10^6) (0.9833) = 91.45 million miles

Earth's aphelion - Earth's perihelion = a difference of only 3.2 million miles

Mar's Perihelion = (Mar's average distance from Sun)(1 - 0.093) = 142 (10^6) (0.907) = 128.8 million miles
Mar's Aphelion = (Mar's average distance from Sun)(1 + 0.093) = 142 (10^6) (1.093) = 155.2 million miles

Average Orbit Distance From Or To The Sun = (Aphelion + Perihelion) / 2 , [FIG 256]



This book contains a previous discussion and information about the ellipse curve, see: **ELLIPTICAL (ELLIPSE)**

The average speed of Earth going around the Sun is: speed = distance / time = circumference of orbit / 365.25 days =

(2 (pi) 93000000) mi / 365.25 days =~ 1599825 mi / day = 1599825 mi / 24 hr =~ 66600 mi / hr =~ 18.52 mi/s . The speed of the orbit will increase by (1+e) at perihelion, and decrease by (1-e) at aphelion, but the general equation for the instantaneous velocity is not a linear equation, and must take into the fact that the force of gravity is an inverse-square law type of result or value. Two common formulas for the velocity of a planet: (1): $v^2 = (GM / r)^2$, hence: $v = \sqrt{GM / r}$: G = universal gravitational constant, M = mass of the Sun , the planets mass is negligible , r = circular orbit radius
(2): $v^2 = (G(Ms + Me) (2/r - 1/a))^2$, hence **$v = \sqrt{GMs (2/r - 1/a)}$** : here Ms = mass of the Sun , a=semi-major axis of orbit : here, r=d is objects distance from M such as the Sun , Me is negligible when compared to Ms. To help verify this formula, consider when the orbiting object or planet is at a location equal to (a), then r=a, and the second factor becomes: $(2/r - 1/a) = (2/a - 1/a) = 1/a = 1/r$, and the formula becomes the familiar $v = \sqrt{GM / r}$. The primary formula given is often called the **vis-viva equation,** and which is a very old saying for "living force". Due to gravity and the concept of the inverse square law of energy and forces, the velocity or speed of an object will be determined by its current speed, and the force of gravity at that distance, and causing acceleration (ie., a change in velocity) of it. During an orbit of an object, though its velocity will change, the total energy of the object remains the same. The sum of its kinetic and potential energy is constant: Et = KE + PE. The angular momentum of the orbiting object will also be constant.

opposition = When a planet or other object is on the opposite (ie., 180° away) side of the Sun. In terms of a single planet's aphelion and perihelion at opposite sides of the Sun, these two points are both called apsis ("extreme side") points in its orbit. These points are actually the vertex, or "extreme points" of its elliptical shaped orbit about the Sun which is at one focus point of that ellipse. When Earth and Mars are on opposite sides of the Sun, this maximum distance between them is: 93 million + 142 million = 235 million miles = 2.57 AU . At this distance, it will take a light or radio signal (8.333 min / 1AU) (2.53 AU) =~ 21 minutes to go the maximum distance between Earth and Mars.

astronomical units = AU = a unit of distance or length that equals Earth's distance to the Sun = 93 million miles = 1AU
2AU = (2)(93 million miles) = 186 million miles
At 2 AU, the energy from the Sun decreases by 4 according to the inverse square law.
Earth receives about 1000 watts of solar energy / m^2 at its surface after a few hundred w/m^2 from the available energy from the Sun above the atmosphere and-or in space are reflected, absorbed and-or blocked by the atmosphere. At 2AU, the solar energy would be:
$1000 w/m^2 / 2^2 = 1000 w/m^2 / 4 = 250 w/m^2$

ecliptic = The imaginary ("solar") plane of Earth's orbit about the Sun. Earths polar (N, S) axis of rotation is also tilted about 23.4 degrees from being vertical (ie., 90 degrees, a right angle) to this plane, and essentially points to the star Polaris ("North Star"). All the other planets also reside close to this same plane, but their orbits and plane about the Sun are more or less slightly tilted by up or below by a small angle from the solar plane of Earths travel about the Sun, and they will also appear to travel across the night sky in the same general (ecliptic, east to west) location, path and direction as the apparent movement of the Sun, Moon and stars across the sky from east to west. This east to west apparent motion of the stars and planets is due to that the Earth is rotating to the eastward direction, and the generally fixed stars and planets then only appear to be traveling westward at night.

inclination = The angle between the actual orbit plane of a planet and Earth's solar plane (ecliptic).

magnitude = The apparent (logarithmic scale) or relative brightness of an object in space as seen or appears (only) from Earth and the distance to it. Brighter objects have a lower, possibly negative, value. An actual relative brightness value = $1 / (2.512^m)$: m = magnitude. For reference, the apparent brightness of the star called Vega is 0. The star Polaris, the "North" or "Pole (axis)" star has a apparent brightness of about: -3.6. Planet Venus has an apparent brightness of: -4.4. The brightest star in the night sky is Sirius with a apparent brightness of about -1.4. The apparent brightness of the Moon is -12.5. The apparent brightness of the Sun is -26. Surely this numbering system is not so intuitive, and seems backwards, and this is due to that the Greeks considered number 1 as the highest or most important for this system, and the multitude of less brighter objects would be assigned to the next integer values higher than 1 and up to include 6. On basic star charts, brighter stars are generally and simply indicated as being bigger even though size is a different concept than brightness. Generally, objects with a magnitude of 7 or more are not visible to the unaided eye, and a magnification system (binoculars, telescope, etc) is needed to gather a larger area of light, concentrate it and then focus a brighter image of it onto the eye or an image sensor such as that in a digital camera.

WARNING: Do not view the Sun with or without any type of lens, mirror and-or telescope, and instruct others not to do so because of the high possibility of permanent eye damage due to the intensity of the light and-or concentrated light. Telescopes use a thick light filter to view the Sun.

With this logarithmic apparent (visible by the unaided eye) brightness system, each next noted magnitude appears about 2.5 times dimmer. Where is 2.5 or 2.512 from? 2.512 is the 5th root of 100. 2.512^5 = about 100. The modern apparent magnitude formula was created by Norman Pogson in the mid 1850's. Note that Vega, in the original ancient system, had a magnitude of 1 meaning it is a star of most or significant importance. With the modern system, Vega is considered to have a magnitude of 0.

For an increase of 5 magnitudes, the apparent visible intensity or "flux" is estimated at about 100 times less. For a decrease of 5 magnitudes, the apparent visible intensity or "flux" is estimated at about 100 times more.

Are stars brighter for astronauts in space? The Earth's atmosphere reduces the intensity of received sunlight. Most estimates put the intensity of sunlight before reaching Earth's atmosphere is about 1370 watts/m^2, and will be 1000 watts / m^2 max. on Earth's surface. The difference in these values is 370 watts/m^2 and represents a loss of: 370w/1370w = 0.27 = 27% , hence the intensity of the sunlight and-or starlight in space is about 27% brighter. In terms of a change in apparent, visible magnitude, it is only a small change.

resolution = In terms of photographs or other images, resolution is a word describing the linear distance (such as a width) that the smallest discernible picture element (or "pixel") represents of that object being photographed. A more or "higher resolution" image means greater or "finer" detail is visible in that image, meaning a pixel corresponds to a smaller actual distance, and the image is a better representation of the true image or actual object. In terms of eyesight, resolution is rather an angle since an object, or even an image like a photograph, can be closer or farther away and another method to describe resolution is needed, and that is with an angle. Brighter objects tend to be able to be seen from great distances such as a candle or star in the darkness, but what is the minimal discernible distance or spaces between two of them in terms of angle? It depends on the person and how good their personal vision is, and an estimate is (1/60) of a degree = about 0.01$\underline{7}$° ("angular diameter" [ie., max. width]). This angle is also equal to 1 arc-minute in the DMS angular system. The "(lettered) eye charts" at eye doctors offices take many things such as this into account so as to help determine how well your vision is. "20/20" is often spoken as the eye or vision test result value for the best or accurate vision possible, with the "sharpest" (finest, smallest,minute) detail and resolution. Modern image sensors having millions of light senors in an array on a chip, have a very close "pixel" or sensor distance between each one, and can therefore have a resolution greater than our vision. An image sensor can be thought of as many tiny solarcells that convert light into electrical signals and data. A high quality (ie., high resolution), fine image sensor can effectively increase the power of an optical telescope by several times. For a hypothetical example, a 100x ("one-hundred power") optical (glass or mirror lens) telescope having a modern (electronic, digital) image sensor in place of the optical eyepiece, can approximate a 300x optical telescope and resolve (obtain) more detail on a fine (many viewing pixels [picture elements or parts]) viewing screen or printing (ie., a photograph) of the image. Ex. If 1° image of the sky is say 1000 pixels wide, then each pixel corresponds to how many degrees of the sky?:

$$\frac{\text{1000 pixels is to}}{1^\circ} \text{ as = } \frac{\text{1 pixel is to}}{x^\circ}$$, after solving for x°, we find that 1 pixel can resolve 0.001 degrees

The field of view one of the **Hubble's Space Telescope (HST)** system is about 200 by 200 arc-seconds of width and height. A total angle of 200 arc-seconds corresponds to a 0.056° angle. The received light is focused on to an image sensor with a pixel width and height count of about 4000 pixels respectively, hence a sensor with (4000 pixels) by (4000 pixels) = 16 Megapixels. If 4000 pixels corresponds to 0.056 degrees, 1 pixel corresponds to 0.056 degrees / 4000 = 0.000014 degrees. The image resolution can be said to be

0.000014 degrees of arc. (0.000014°)(3600 arc-seconds / 1°) =~ 0.05 arc-seconds. This telescope launched in 1990 was repaired by astronauts in1993 due to producing out of focus images that some might say look slightly "blurry", and has since made many important discoveries not yet practical with most telescopes on Earth. The diameter of the main mirror of the HST is about 7.9 feet wide and which can gather much light and therefore resolve faint (received low light) or "dim" objects of which may be due to their great distance away from Earth. The Webb telescope is composed of several mirrors with a common focus, and its total light gathering ability or area is about 5.4 times more than that of the Hubble telescope, and that Webb was made to be more sensitive to infra-red light. The maximum magnification power of the HST is 1440, but say 400 is often used for faint objects so as to obtain a brighter image, hence often no more power than that of a quality, consumer grade, land based telescope. **HST** has a low Earth orbit altitude of about 340 miles = 550 km. It has an orbit inclination with respect to the equator of about 28.5° Its focal length is about 190 ft.

Image resolution is much like or analogous to the precision or "fineness" of a number and-or measurement.

Image Resolution = Pixels / Image Size , ex. 200 pixels/inch , and Pixel Resolution can be defined as:
Pixel Resolution = Real Distance / Pixel , ex. 1 foot / pixel , and **Angular Resolution** = Angle / Pixel

* Of all the planets, Mars has the most elliptical (non, circular) orbit. Mars has a axial tilt similar to that of the Earth, and the value for Mars is about 25.$\underline{2}$° Mars also has a day or axis-rotation length just slightly longer than that of the Earth. This value for Mars is 1.026 days which is about a Earth day plus a half an hour.

* Jupiter, Saturn, Uranus, and Neptune are often called "gas giants" since their gas atmospheres are very thick.

* The distance a planet is from the Sun is noted above as an average distance which is the average of its closest and farthest orbital distances from the Sun. For example, Mars has a perihelion of about 128 million miles, and an aphelion of about 154 million miles from the Sun, and the average is about: (128Mm + 154Mm)/2 = 282Mm/2 =~ 141 million miles, and this is the value seen in the list above.

* At about 14 days past the winter **solstice** (ie. max. apparent tilt of Earth's pole axis, about 23.4° on about Dec 21, "the Sun (travel and-or height) stopping"), about Jan 4, Earth has an aphelion of about 94.5Mmi, and a perihelion of about 91.4Mmi at about 14 days past the summer Solstice (abut June 21) at about July 4. The average distance Earth is from the Sun is therefore usually noted as about 93Mmi. The perihelion and aphelion dates are also an average or typical value due to the small, but constant, gravitational (force) influences of other orbiting planets slowing down or speeding up the orbit of Earth around the Sun.

* Two planets are theoretically closest when they are both at their perihelion distance from the Sun, and at the same time and same side of the Sun in alignment like a line. For example, Earth's perihelion is about 91.4Mm, and Mar's perihelion is about 128Mm, and the difference between these two distance is about: (128Mmi - 91.4Mmi) = about 36.6Mmi

* The average distance between two stars is about 5 light-years. The average distance between two galaxies is about 1 million light-years.

* The distance to the closest star, and not including the Sun, Proxima Centauri is about: 4.24 light years. In theory, it will therefore take light and-or a radio signal from Earth a total of 4.24 years to reach Proxima Centauri, and vice-versa.

* The distance to AlphaCentauri is about: 4.37 light years.

* There is a rough estimated 10 stars within 10 light-years radius about Earth, 100 stars within 20 light-years radius about Earth, 1000 stars within a 100 light-year radius about Earth. The stars we see are usually withing the Milky-Way galaxy and less than 1000 light-years away. The density of stars in a galaxy is about 0.0035 stars / 1 cubic light-year, and with an equivalent fraction of about 1 star / 286 cubic light-years. By taking the cube root of 286, we have about 1 star at a average separation distance of abut 6.$\underline{6}$ light-years from the closest star to it. Stars can be gathered by together by gravity into groups called clusters which are then called a galaxy of stars. Galaxies can also be gathered together into super clusters.

* The distance to the center of the Milky Way Galaxy, from the position of Earth in it, is about 26000 light years.
* It has been estimated that it takes the Sun about 225 million years to orbit the Milky-Way galaxy.
* The diameter of the Milky Way Galaxy that out Sun is in, is estimated to be 75000 to 100000 light years across.
* The attractive force that holds the orbiting stars in a galaxy is often a **black-hole** which is a dense mass so large, its gravity force even stops light from going outward from it. The center of the Milky Way Galaxy is located in the direction of the constellation (ie., a unique pattern, group or arrangement of stars) called Sagittarius. This center is obscured by what can be called clouds of dust and debris, and can be viewed by some large radio and infrared (ie., heat) light capable telescopes using certain plastic and-or specialized types of glass since regular glass absorbs the infrared radiation and converts it to heat. This black-hole or theoretical dark, collapsed star which ran out of fuel that expanded it, and then having a very higher than average level of local amount of gravity (from a mass greater than an estimated 2.5 times that of the mass of our local Sun star) of which not even light has enough kinetic energy to be emitted from, and is called **Sagittarius A**. This black-hole has been imaged over several years of time using special infra-red sensitive telescopes. This black-hole currently has several high velocity stars in an elliptical orbit about it. This black-hole was discovered in 1974 by radio astronomers (**Dr. Kwok-Yung** aka. **Fred Lo**, **Bruce Balick**, and later, **Robert Brown**, and **Andrea Ghez**, the scientists for for the verification, and noticing a more than the average amount of star radio noise and-or or energy coming from this location.

* Radio astronomy was formally started in 1931 by **Carl Jansky** of the **Bell Labs** company, and who built a radio-telescope which was basically a large antenna construction on the ground that was tuned to receive a high frequency. This antenna (ie.,here, a radio energy receiver) could be rotated so as to be pointed in a desired directions in outer-space. He later noticed that there was always more radio noise in the direction of the Sagittarius constellation in our Milky-Way star galaxy. The ratio of black-holes to stars is estimated at about: 0.005 black-holes / 1 star =~ 0.5% , and which can also be stated by using division as about 1 black-hole among 200 stars. The distance to the center of the Milky-Way star galaxy and-or black hole is estimated at 25000 light years. In the northern hemisphere of Earth, and during the summer months, people can see the direction of Sagittarius-A in the in the southward direction in the constellation of Sagittarius (aka, The Archer). Sagittarius-A has galactic coordinates of about 17 hours and 45 min. and 40.041 sec. right ascension, and -29° , 0 arc-minutes and $28.1\underline{2}$ arc-seconds of declination. Sagittarius-A is about -5.6 degrees from the ecliptic, hence it is below it or at a lesser altitude in the sky.

* The latest estimated distance to the (current) "North Star" = "Pole Star" or **Polaris** is about 323 light years. Polaris is actually a 3 star system. Polaris actually differs by a small angle of 0.735° from the true location of Earth's north pole axis. The farther a person goes more south on the southern hemisphere of Earth, the less stars visible around the Polaris region since the Earth's bulge at the equator is essentially blocking the view. This book contains basic star maps. Throughout the year of Earths travel about the Sun, the stars will appear to have a counter-clockwise (ie., to the left from the top reference) circular orbit about Polaris which will still appear (nearly) fixed at its location, and this circular path is due to that Earth is rotating on its north to south, polar axis. As Earth orbits about the Sun during the yearly orbit, all the stars at day and-or night will shift in position along the solar plane (ie., the east to west, "ecliptic" line), and about Polaris by about (360°/365 days) =~ 1 degree per day.

* Due to Earth's 365 day, or yearly, orbit about the Sun, we will gradually see other stars in the Milky-Way galaxy at night.

* Most Comets have a highly eccentric (ie., "off from center". elongated, non-circle) orbit about the Sun. After studying the motion of comets, astronomer **Edmond Halley** (1656-1742), from England, predicted that they orbit the Sun like a planet. This was proven when (post named) **Halley's Comet** would return in 1758, and after previously been viewed by others every 76 years, hence Haley's comment was periodic, hence, it is in an (stable, elliptical) orbit about the Sun.

* It has been estimated that for a body to become **naturally round** due to its own gravity, it must have enough mass so as to be about 600 miles in diameter or greater.

* It has been estimated that for a body to have a hot, **molten liquid core** (usually of a metal(s), it must be at least 2000 miles in diameter or greater. This size is also the modern definition needed for a planet.

* The presence of large sized pieces of metal meteorites that landed on Earth suggest there was once another large body

in our solar system that probably collided with another and then fragmented into **asteroids** - perhaps as the "asteroid belt" or zone between Mars and Jupiter.

* The average distance between small pebble sized **asteroids** and large asteroids in the asteroid belt between Mars and Jupiter is about 500000 miles. As for smaller particles, this distance may be much less and since there are many more smaller particles, perhaps ten time more at least, hence having an average distance between these particles as being ten times less at 50000 miles.

* Using the distance to the Sun, and the **diameter of the Sun**, and using some right angle trigonometry, you can calculate the apparent width angle of the Sun as seen in the sky from Earth or any other planet. The Sun appears to be 0.5 degrees = half a degree, wide in the sky from Earth. For an example, knowing the distance from the Sun and the diameter of the Sun, you may solve this value for Mars. Another similar calculation is from the trigonometric concept that if you double (a factor or multiple of 2) the distance of the base of a right triangle, the height corresponding to that distance will likewise double (2). If the height is to remain constant, such as the diameter of the Sun, if the distance doubles (2), the angle will be divided by that same factor (2). For Mars, being a factor of 1.5AU from the Sun, the apparent angle of the Sun seen in the sky from the surface of Mars will be divided by that same factor. For Earth, the apparent solar angle or size is about 0.5°. For Mars, it is about 0.5° / 1.53 = $0.3\underline{3}$° on average since orbits are elliptical and the distance to the Sun changes varies.

From Earth's surface, the Sun appears to have a relative area of: (pi)(r^2) = 3.14 (0.5^2) units^2 = 0.785 degree units^2
From Mars's surface, the Sun appears to have a relative area of: (pi)(r^2)= 3.14 (0.327^2) units^2 = 0.336 degree units^2

Since Earth is at 1AU from the Sun, and Mars is at 1.53AU from the Sun, the average distance to Mars from Earth is: (1.53AU - 1.0AU) = 0.53AU. In terms of miles, this would be: (0.53)(AU distance) = (0.53)(93 million miles) = 49.3 million miles. This minimum distance is always changing to a larger value as the planets orbit about the Sun, especially when Mars is on the opposite side of the Sun and its distance from Earth is:
Earth's distance to Sun + Mars's distance to Sun = 1AU + 1.53AU = 2.53AU = 235.3 million miles.

The ratio of the relative apparent Solar areas of Mars to that of the Earth is: 0.336 units / 0.785 units = 0.43
The relative solar radiated energy (light, heat) reaching Mars will therefore be about 43% that of the Earth.
At 1AU in space from the Sun, and which is Earth's location or distance from the Sun, and when above the (energy absorbing) atmosphere, solar radiation is about **1370W/m^2** (yearly, orbit average). Due to Earth's atmosphere absorbing and-or reflecting some solar radiation, there is about **1000W (max.)** of solar radiation per square meter at Earth's surface at solar noon (considered 12 pm, and the start of the afternoon or evening) where the Sun is directly overhead. The ratio of: 1370w/m^2 / 1000w/m^2 = 1.37 and indicates there is 37% more solar energy per square meter above Earth's atmosphere and-or 37% less solar energy per square meter at Earths' surface at solar noon, and this equates to about 1000 w/m^2 / 1370 w/m^2 = 0.73 =~ 73% of the total available energy above Earth's surface. Due to the Moon having no atmosphere, the amount of solar radiation on the surface of the Moon is that reaching the upper atmosphere of Earth at 1 AU during its orbit, hence 1370 W/m^2 (yearly average). Due to the Moon's surface receiving much more ionizing radiation (including cosmic or galactic rays which are mostly the nucleus particle of ionized atoms, and free electrons which can break apart molecular bonds) from the Sun, other stars, nova, etc. than the surface of Earth, to protect an astronaut requires thicker and-or denser physical shielding so as to greatly reduce this type of harmful radiation.

Assuming low atmospheric losses on Mars, there will be about (1370W)(0.43) = 489W (max.) of solar radiation per square meter at the surface at Mars noon (ie., 12pm) when the Sun is directly overhead. Relatively, or comparatively, this is about only 49%, or about half, of that on Earth.

Another calculation for the solar or Sun radiation intensity per unit area at the surface of Mars is from the fact that it is at a distance of 1.53AU from the Sun. Since Earth is 1AU from Mars its relative solar radiation can be said to be a reference of 100% = 1.0. From the concepts of the inverse square law with energy and its intensity from the source, the intensity at Mars in relationship or comparison to that of the Earth can be calculated as:

Intensity at Mars = Intensity at Earth / 1.53^2 = 1.0 / 1.53^2 = 1 / 2.3409 = 0.427 =~ 43%

* The (Earth's) Moon orbital "tilt" (ie., inclination and-or declination) with respect to the solar or ecliptic plane as it orbits around Earth is a constant value of about 5.145° above or below that plane it during its orbit. The specific distance above the solar plane is sinusoidal in value and is constantly changing as the Moon orbits about Earth about every 30 days. The Moon's orbit about the Earth is nearly circular, and having only a very slight ellipse shape with a low eccentricity of e = 0.0549, however, if this is compared to other moons and planets, it is relatively large. Due to this fact, he Moon should and does appear as slightly larger when it is closer to Earth during its orbit about Earth which is at one focus of the Moon's orbit.

* If two ideal spherical objects have the same mass, they will each have the same amount of gravity attracting the other, and if given the proper circumstances (velocities, gravity, directions), they will orbit each other, and with the center of that orbit called the **barycenter** being located half-way between their center points, and as a point in empty space. If one mass is much larger, the barycenter of orbit will then be closer to that larger mass. The barycenter of the Earth and its distant Moon system is actually located in the Earth, and the Earth does not then appear much to be orbiting an empty point in space.

* When the Moon is said to be a "new moon" it is not illuminated. The Moon will afterwards be illuminated on its right or eastward side as viewed from the Earth. This begins the "phases of (illumination) of the Moon", and it is called "waxing". After a "full Moon" where it is fully lit or illuminated, the right or eastward side will then get dark throughout the next 14 days, and this is called "waning". The line between illuminated (lit) and not illuminated (dark) part of the Moon is called the "terminator line" and this appears to go from the north pole to the south pole of the Moon. When the Moon is highest in the sky as viewed by a person in the northern hemisphere on Earth, this north to south line will point to the southward direction on Earth, and this is another reference direction and way to navigate. Though we may see the Moon as being only partially lit, in reality, half the Moon which has a sphere or ball shape, is always illuminated by the Sun and its light, and half of the Moon is always in the dark or shadow of itself. As the Moon rotates on its own axis, a different portion of the Moon will be illuminated, and throughout its orbit about the Earth. When a new Moon or phase is viewed from Earth, the "far", "dark" or "back-side" of the Moon is completely illuminated.

Where did all the matter in the universe come from?

* The science of astronomy has shown that on average, galaxies are expanding outward from each other as if they were on the surface of an expanding balloon. The matter then has kinetic energy if it is moving. Where did all this matter and energy come from? Considering the **Big-Bang** (explosion) theory or not, it is thought by some that matter is natural and was always here in the universe, and it cycles through condensing, new big-bangs, and expansions. Many believe a "higher power", God and-or intelligence created the universe. That is, a God of the universe created the initial matter and-or energy. After all, consider that matter and-or energy just cannot create themselves from nothing, and even if the universe always existed, and that there has to be a source of them. Here, it is thought that a very large amount of condensed energy had a chain reaction of creating matter, heat, and the expansion (movement) of that matter away from that central location. Some say matter is a form of condensed and-or concentrated energy when we consider the equation E = mc^2. After this matter cooled, it began to collect together by the force of gravity. Large amounts of it eventually became dense enough so as to be ignited as stars, and with the left-over or remaining matter about that star collecting into orbiting planets, asteroids, comets etc. By gravity, stars eventually collected together into galaxies (groups of stars) that orbit a central point and-or core that is very large in mass and has a stronger gravitational field. This core is thought to be a black-hole which was created by the collision of many stars and the collection of that matter then having a very high gravitational field, and probably magnetic and electric fields, and that light cannot even escape from its strength. A black-hole has been found at the center of our Milky Way galaxy that the Sun is in and orbits about. We know its direction and-or location of it in space amongst the (night) sky and stars. This brings about a reasonable question as to whether we can find the direction and-or location in the (night) sky, stars and space as to where the Big-Bang originated at.

How the speed of light was first determined or estimated

The speed of light was once thought to be so high that it could not be known. In 1676, **Ole Romer**, (1644-1710), which is pronounced with a long-o sound such as used for "row", and may be spelled as Roemer. and he was an astronomer from Denmark, and who made the first close estimate as for the speed of light. With a telescope, he was studying the orbit of one of **Jupiter's moons called Io**, and noticed that it took about 1.7 days and would also periodically go behind Jupiter at this same time of 1.7 days, hence it was therefore a constant and could be seen to happen at a certain predictable time on Earth. The distance (93 million miles) to the Sun as known today, and which is also the radius of Earth's orbit about it was already previously estimated to be a lesser value. The diameter of Earth's orbit about the Sun is twice this distance, hence 186 Million miles. Roemer noticed that Io was "late" and that there was a several minute difference when Io was to go behind Jupiter. This happened when the Earth was on the opposite sides of the Sun every six months of time. Roemer understood that this time difference (16.7min, change in time, modern estimate) was most likely due to that light had a longer distance to travel and therefore took more time to reach Earth. Roemer's estimate for the speed of light was about 130000 miles per second, and this value is about (130000mi / 186000 mi) =~ 0.70 = 70% of the speed of light, and is fair or good estimation considering the (limited) knowledge available at those times.

In a more modern estimate and calculation of the speed of light, and using more accurate values, we have a closer estimate of the modernly accepted speed of light of: 186000 miles / second =~ 300,000,000 meters / second:

speed = distance / time = (change in distance) / (change in time) we have:

(2 x 93 million) / 16.7minutes = 186 million miles / 1002 seconds = 0.1856287 million miles / second
= 185,628.7 miles / second = (c) = light speed
= 299792458 meters / second = m/s

The above value for the speed of light is for when it is traveling through a vacuum medium that has no matter in it. Just like the speed of sound can change (increase) traveling through mediums (matter) such as air, water or metals, the speed of light can also be altered (slowed, decreased) as it travels through a medium such as glass. Though the frequency(s) and-or the colors seen of the light will remain the same, the wavelength distance of the light frequency(s) will be decreased in the glass and will resume light speed when it exist the glass material and into another material such as air.

Time	**Distance Light Will Travel (approximate) = (speed) (time)**
1 second	186,000 miles = 300,000 km = **"1 light-second" = distance light travels in 1 second**
1 minute	11,160,000 miles = 17,960,279 km =~ 18 million kilometers = **"1 light-minute"** = (c)(1 min)
1 hour	669,600,000 miles = 1,077,616,742 km =~ 1 billion kilometers = "**1 light-hour**" = (c)(1 hr)
1 day	16,070,400,000 miles = 25,862,801,808 km = "**1 light-day**" =~ 26 (10^9) km = 26 billion kilometers
1 year	5.88 trillion miles = 5.88 (10^12) miles = "**1 light-year**" ~= 9.5 trillion km =~ 9.5 (10^12) km

The distance to the Sun was first reasonably estimated by **Eratosthenes** by using observations, right-angle trigonometric calculations and geometry long before Roemer's speed of light discovery. Since ancient times, it was known that the Sun was farther than the Moon due to that during any eclipse seen, the Moon is always seen as covering the Sun. Since Roemer, the Earth to Sun distance had some better approximations. Once this distance was found, the diameter of the Sun could also be estimated using basic trigonometry using the Sun's observed angle of 0.5 degrees: Tan ¢ = Tan 0.5° = opposite side / adjacent side = Sun's diameter / distance to the Sun. Sun's diameter = Tan 0.5 (distance to the Sun) = 0.008726867 (93000000 mi) =~ 811600 mi , with a modern, actual value at: ~ 865000 mi

Roemers analysis for the speed of light was very creative and helpful, but it needed to be verified. **Armand Hippolyte Fizeau** (1819-1896), from France, made a novel, incredible and historic measurement method, and then a calculation with the measured value so as to estimate the speed of light in the year 1849. It is also of historic note that Hippolyte Fizeau and Leon Foucault took the first photograph of the Sun with sunspots on April 2, 1845. In brief, Fizeau's light speed measuring system used a high speed rotating disk that had equally spaced holes through it near its edge, much like a gear wheel with its protrusions or "teeth", and with each set being equidistant form the center of the disk. The spinning disk had to be large, and the holes farther from the center have a higher effective linear velocity than it would nearer to the center as it rotates. This device allowed for the observation and calculation of the time it took for a nearby bright light to travel to a distant location (here, about 5 miles away) and then be reflected back and seen directly through the next hole in the rotating disk. While the light is traveling, the disk continues to rotate, and that disk will only move a small angle of rotation and-or linear distance between each hole during the total time of that distance of light travel. This linear distance between each hole can be measured and-or calculated, and then the time it took for the hole to move that total distance can then be calculated.

For example: From: total distance = (speed)(total time) , we have: (total time) = (total distance) / speed :

rotation time = (total rotations) / (speed of 1 rotation) , Ex: (100 rotations or revolutions) **/** (5 rotations / 1 second) = 20s
Ex. 100 rps = 100 revolutions / 1 second , 1s / 100 revolutions = 0.01s / 1 revolution = 0.01s / 360°
= 0.0001s / 3.6° = $0.0000277\underline{8}$ s / 1°

equivalent linear time = Circumference / speed of linear travel , 1 rps = 360°/s = 2(pi) / s and 1 rps = 1C/s = 2(pi)r / s
From: time = distance / speed , solving for speed, we have for this example:
speed of light = (distance traveled) / time = is equivalent to this measurement :
= (distance of disk movement = arc length) / (calculated time of that disk or arc length movement)

Modern RADAR and-or an oscilloscope can be used to calculate the speed of radio wave (rf) and-or light energy. Fizeau measured the speed of light to be about only 4.3% greater than the modern accepted value of 186000 mi/s.

Taking Fizeau's and the modern accepted value of 186000 mi/s for the speed of light, we can then calculate and verify Earth's distance or radius of orbit from the Sun. First, as found by **Roemer**, the time difference across Earth's orbit distance is about 16.7minutes = (16.7 min)(60s/min) = 1002s , and:

Distance = (speed) (time) = (186000 mi/s)(1002s) = 186,372,000 mi : major axis length of Earth's Orbit
The above distance is like the diameter of Earth's nearly circular orbit about the Sun. Dividing this value by 2 to find the radius of orbit of Earth, we have: 186,372,000 mi / 2 = 93,186,000 mi =~ **93 million miles = 1 AU**

ORBITAL MOTION OF OBJECTS

Most orbits of the planets around are Sun are fairly circular with a low eccentricity (elongation [a greater distance than a circular radius] or shortening [a less distance than a circular radius] near the Sun). Still, all the orbits are periodic, and will predictably repeat every so many specific number of months or years. The position of a planet in its periodic orbit can still can speed up or slow down and this can make it difficult to predict an objects position. The reason for this is that the object was traveling (orbiting) around the Sun in a natural elliptical orbit with the Sun at one foci or focus point of that elliptical orbit. It is not difficult to understand that the closer the object is to the Sun, the faster it will be traveling. This is what happens when an object falls under the constant pulling influence of the force of gravity and it accelerates faster as it gets closer (perihelion) to the source mass of that gravity. Note that a circular orbit can be considered as a special instance of an elliptical orbit. It is not difficult to then understand that an object will then be traveling slower at the opposite end (aphelion) of this type of orbit, and again this is due to the constantly applied force of gravity constantly acting (pulling) upon the object and it decelerates. In 1609, **Johannes Kepler** (1571-1630), from Germany, published significant mathematical discoveries of orbital motion. Here are the three **Kepler's Laws** of orbital motion:

1. **Elliptical orbits**:
   Planets follow an elliptical orbit about the Sun at one foci point of that orbit. The speed of the object is not constant for objects in a natural gravitational induced (caused) orbit about the Sun. Speed increases near the Sun due to the increasing force of gravity of the Sun.

2. **Equal areas per same time**:
   From the orbiting object to the foci, equal orbital (planar, surface) areas will be covered in any specific amount of time.

   This coincides with a concept called **angular momentum** which essentially means rotational or orbiting momentum. Momentum = mv can be considered as a measurement of the amount of movement, and more technically, it is a measurement of the amount of inertia (resistance to a change in motion and direction) it has and the more energy required to change it's velocity and-or direction.. In terms of an orbiting body such as a planet about the Sun, the speed and kinetic energy (KE) of the object increases as it gets closer towards closer towards the Sun due to its gravity, and this "linear momentum" increases since the product of (mv) will now be larger, however, if the product now includes the radius distance from the Sun, this value is a constant (ie. "conserved" or "maintained") for the object. **angular momentum = mvr** : v and r are inversely related, and m is a constant. Note: mv = momentum . The product of (v) and (r) is a constant, and if the mass is constant, the angular momentum (mvr) value is constant for an orbiting object. If the mass of the object changes, the angular momentum will change.

3. In 1619, Kepler found that the orbital period (P) time of an object's orbit is proportional to, or directly (but not linearly) related to the objects farthest distance of the elliptical orbit. Basically, this states that more distant planets take longer to go around the Sun. For the specific and precise mathematical relationship: P^2 is directly related to the cube of (a) which is $(a)^3$. **For objects orbiting the same focus such as the Sun: P^2 / a^3 is a constant (ratio) value of 1** and regardless of its mass. **$P = \text{square-root}(a^3 \text{ au})$**. Kepler found this by applying Newtons law of gravitation, and it resulted in this equation below, and here using Earth's 1 year orbit time as a reference, and taking a ratio value:

$$\frac{(P = \text{orbital period yrs})^2}{(a = \text{semi-major axis in AU units})^3} = k = 1 \quad , \text{ or } = \frac{(\text{orbital years})^2}{(\text{average distance to the Sun in AU})^3} = \overset{\text{Ex: for Earth}}{\frac{(1\text{year})^2}{(1\text{au})^3}} = 1$$

(a) is essentially equal to the (average) radius (r) distance between the orbiting object and Sun.
Clearly, because of the definition, for Earth: $P^2 = k\,(1\text{ AU})^3. = 1$:k is a constant equal to 1 for Earth

Saturn which has an average AU of 9.537 , $P^2 = (a^3) =\tilde{}\ 9.537^3$, $P = \sqrt{9.537^3} = 29.45$ earth years (e.y.)

With Newton's motion and gravity laws in about year 1687, **Kepler's Third Law** can be shown as:

$P^2 = r^3 \dfrac{4(\pi)^2}{G(M1 + M2)}$: If M2 is a the mass of a planet or moon, its mass, maybe unknown, and is practically insignificant (as if 0kg) as compared to and-or combined (added) to the mass (Ms) of the Sun. We can let $Mt = (M1 + M2) = (Ms + 0) \cong Ms$. The multiplying factor to r^3 is a constant (k) for our solar system and Sun. Here (a) = semi-major axis of the elliptical orbit. a = r for a circular orbit, otherwise use the semi-major axis (and average orbit) distance with its regular units used, and not in (relative) AU units.

Solving for P, we have:

$$P = \text{time} = 2\,(\text{pi})\, r \sqrt{r} \;/\; \sqrt{GMs} = C \,/\, \sqrt{GMs / r} = \text{distance / velocity}$$

:use r = (a) for elliptical orbits
C = total distance traveled per orbit

From this equation, many calculations can be made about the planets and moons, such as their mass.
If needed, the mass of the Sun is: 1.989 (10^30) kg = about 2 (10^30) kg = 1 solar mass unit = 1 Msun = 1 Ms.
The mass of large astronomical objects is often measured (ie., compared to) and expressed with solar mass units.
Extra: Small, but dense stars can have as much or a greater amount of gravity than a large stare.

Kepler's Third Law is theoretically accurate, however it is only a very close approximation in actual reality due to that the gravity of planets slightly affect each other over long periods of time and so as to then be more noticeable. This is the case when trying to explain some orbit anomalies of some of the planets, and implied some unknown planets.

An object 2 orbiting about object 1 is in orbit due to the gravitational force of attraction. Object 2 is technically in a type of "free fall" about object 1 so as to remain in perpetual (non-ending) orbit, not escaping orbit, and not colliding into object 1. If you were to orbit a ball attached to the end of a string, wire or metal rod, the force of gravity between you and the object is insignificant, however, the there is a significant amount of **centripetal force** (ie., inward, centrally directed force, such as the gravity of a spherical planet or mass). This force is centrally directed about a point, and here, it is you. This force can be felt as the tension in the string pulling upon your hand, and it also pulls the same amount upon the ball in an equal and opposite manner. This force is perpendicular to the direction and motion of the ball and effectively produces a circular orbit or direction, and which can be thought of as like the average direction (like a 45° angle) due to the instantaneous forces upon it, which have a direction of straight downward (gravity) and straight horizontally (linear momentum). For a circular orbit due to a centralized force of gravity, this gravitational force can be considered as equivalent to a centripetal force:

When: centripetal force = gravitational force

$$Fc = Fg$$

, and:

$$\text{force} = (m)(a) = \frac{mv^2}{r} = mrw^2 = \frac{GM1M2}{r^2}$$

: where w is the angular or rotational velocity
v = wr , v^2 = (wr)^2 = w^2 r^2 , and
a = acceleration = v^2 / r
G is the universal gravitational constant.

$$\text{Note that angular velocity} = \frac{\text{angle traveled}}{\text{time}} = \frac{2(\text{pi})r}{\text{Period of rotation}} = \frac{6.28r}{P} = w$$

, therefore:
: t = P = T

$$\text{force} = mrw^2 = \frac{mr(2\ \text{pi})^2}{P^2} = \frac{4(\text{pi}^2)\ mr}{P^2}$$

: P is sometimes expressed as T since P is a time variable

$$\frac{4\ (\text{pi}^2)\ M1a}{P^2} = \frac{GM1M2}{a^2}$$

: after setting a=r , After solving for the P and (a) variables:
M1 = mass of the planet , M2 = Ms = mass of the Sun , and:

$$\frac{P^2}{a^3} = \frac{4(\text{pi}^2)M1}{GM1M2} = \frac{4(\text{pi}^2)}{GM2}$$

: the ratio value is a constant for our solar system with M2 = Msun , and all objects orbiting the Sun will have this same constant value for P^2 / a^3

For Earth , a = r = 1 AU , hence 1.0 = 100% of the relative distance to the Sun.

From the above equation for Fc = Fg, we have $\frac{mv^2}{r} = mrw^2$, and some variables can be solved for such as:
: r = orbit distance from the center of the Earth

$v = r w = \sqrt{GM / r}$: effective linear velocity for a object rotating or in circular orbit about another.
: This is also derived in the angles and rotation section in this book.
Also consider: $\mathbf{v = d / t = C / t = 2(pi)r / t}$: let t = Period of orbit or rotation = $T = P$
Mass of Earth = $Me = (5.9722)(10^{24})$kg , radius of Earth = $Re = r = 6378$km
Due to the inverse relationship, farther orbits with a larger radius will have a lower (ie., slower) effective linear velocity of travel, and a longer time period of orbit.

An orbiting object or mass held by gravity (ie, a centripetal force) obviously does travel in a straight line above the center point of Earth, but if the centripetal force was released, the object would leave or "escape" orbit and continue on in a straight line direction and which is perpendicular (ie., 90 degrees) or tangent to the orbit radius line at that instant.

Extra: From: $d = v t$, we have: $v = d / t = C / P$ =
= Average velocity of orbit = Circumference distance of orbit / Period time of orbit . $\mathbf{P = C / v = 2(pi) r / v}$

While orbiting, the object is always, slightly changing direction (ie., rather than going straight) in every instant of time, and for a change in direction, a force is needed such as the (constant) centripetal force due to gravity at that height or distance.

Force = (mass)(acceleration) = (mass)(change in velocity / change in time) = $(m)(v^2 / r)$:

a = (change in velocity) / (change in time)

Due to centripetal force, there is a change in direction toward the center, hence the radius (r) changes very slightly to a lesser value:

From: distance = (velocity)(time) or: (change in distance) = (velocity)(change in time) with substitution:
(change in r) = (v)(change in time) = (v) (change in velocity / a) solving for (a) , we have:

a = (change in velocity) (velocity) / (change in radius) , and for this situation with constant values for orbiting:

Since: (change in velocity) / (change in radius) is equivalent to the same ratio value of (velocity / radius) = v / r :

Then simplifying the equation for (a), it can then be shown that :

$\mathbf{a = v^2 / r}$ and that **Fc = centripetal force = ma = m v^2 / r** : $\mathbf{v = \sqrt{a r}}$ or $\mathbf{v = \sqrt{g r}}$
Such as for finding geostationary velocity when ($a=g$) is known at that altitude (r).

Centrifugal force is an outward directed force of a rotating object or material. It could be thought of as the equal and outward (from the center) opposite force of the centripetal force. It is sometimes used to separate materials that have different densities and-or to strain (ie., remove) liquids away and out from a more solid material. A motorized machine that is designed to do this is caused a centrifuge. When material is spinning like a flywheel, the outer rim portion is going at a higher velocity than the inner portion, and therefore it has more energy and pressure. This can sometimes be seen as an increase in the height of a fluid material near the outer rim, and therefore a decrease of the height of the fluid material near the center of rotation. Centrifugal force can be used to create simulated gravity for astronauts in a large rotating spacecraft.

Estimating the orbit time of a planet, and its distance from the Sun:

To make an equation for a planets orbit, the position of a planet(s) is studied and recorded for many years and the equation is refined. An estimate for the time of orbit of a planet can be made by noting how many degrees the planet has moved with respect to the background of seemingly fixed position of very distant stars during say one or several year of time, and where the Earth is at the same position every year with respect to the stars. Considering a linear or circular analysis, if the planet moved 10° degree in 1 year, how many years will it take the planet to move 360° for a complete circular orbit? We can solve this using a proportions type of equation so as to have an initial estimate of a planets orbit:

$$\frac{1 \text{ year}}{10°} \text{ is to} \quad \text{as} = \quad \frac{x \text{ years}}{360°} \text{ is to} \quad , \quad x \text{ years} = \frac{(360°)(1 \text{ yr})}{10°} = 36 \text{ years to orbit}$$

In general, the more distant the planet, the less the gravitational acceleration and-or pull force from the Sun, and the planet will take longer to orbit the Sun. Each orbiting planet around the Sun is also in a free-fall elliptical orbit about the Sun. If the planet has a greater angle of movement at each periodic time it is seen, it is in an elliptical orbit and is going closer to the Sun.

It is possible to find the current distance to a planet using trigonometry which includes two distant observers on Earth, each viewing it at a slightly different angle. For a star, visual parallax can be used, whereas the apparently changing position of a star is seen to have moved with respect to more distant, ("background") stars during, say after 6 months of Earth's orbit about the Sun, and where there is a maximum possible parallax or difference in the apparent visible angle.

Now that we can estimate the period of orbit of a planet, we can then use that value for estimating the distance to a planet with an assumed circular orbit like Earth, and this can be found using Kepler's Law as shown previously:

$$1 = \frac{\text{period\^{}2}}{\text{semi-maj. axis}} = \frac{P\text{\^{}}2}{a\text{\^{}}3} \quad : a = \text{semi-major axis of orbit, hence its average radius} = r \text{ in AU units}$$

$$a\text{\^{}}3 = P\text{\^{}}2 \quad : a = r = \text{distance from the Sun, and with AU units} \quad , (1 \text{ AU} = 93 \text{ million miles})$$

And:

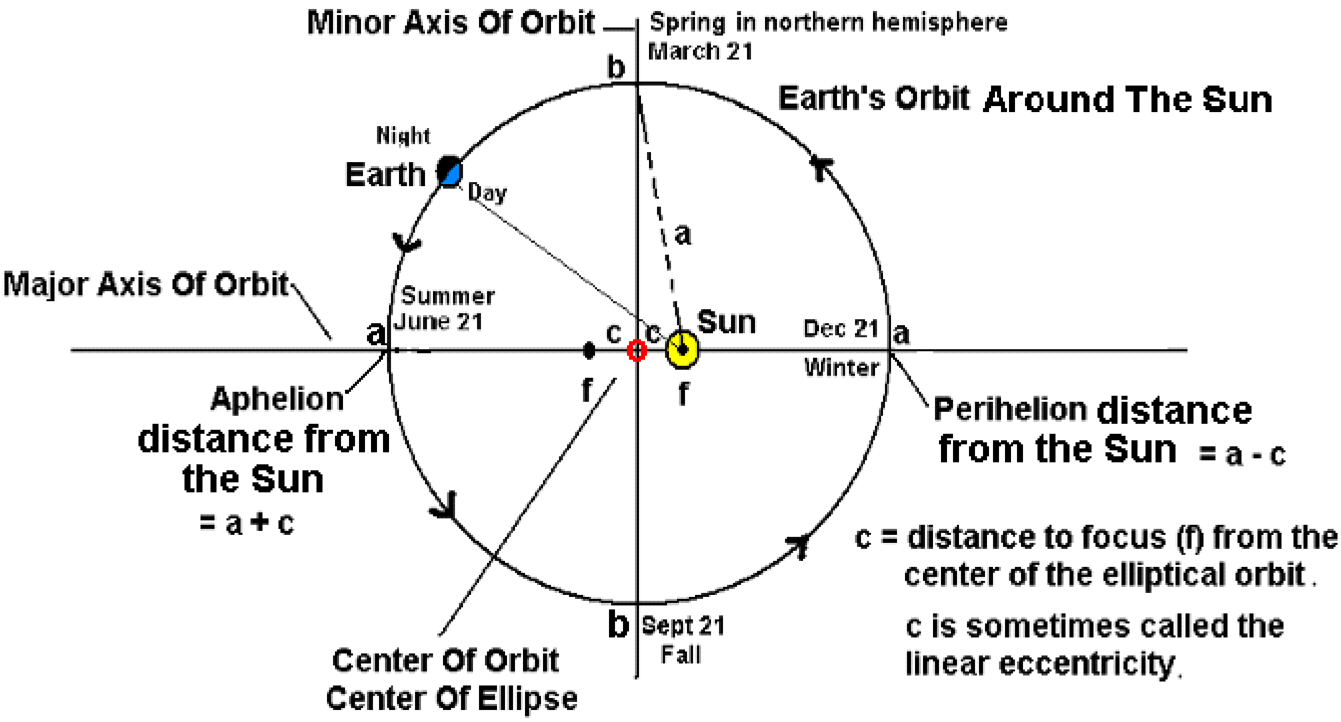
$$a = r = \text{distance from the Sun} = \sqrt[3]{P\text{\^{}}2} \text{ , and with AU units}$$

Extra: The Tides On Earth

As the Earth rotates on its axis once a day = 24 hours, the point on the Earth closest to the Moon, and on the opposite side of the Earth will experience a sea level or "tide" rise due to the gravity of the Moon attracting that water mass. The water at that location(s) will get deeper, and sometimes several feet deeper, and at several feet lower at other times of the day. The Moon also attracts the land mass of Earth just the same, but the liquid ocean and liquid interior of Earth are more easily moved because of their fluidity and-or its loose, less dense atomic structure. Due to what is mentioned, each point and-or longitude on the Earth facing the Moon will have 2 tides a day - a high tide, and a low tide. These tides actually help keep things in the ocean, and especially along the shores "churning" and "mixing", and this actually helps promote healthy life. Tides can also be dangerous if it is an "outgoing or receding" tide which can pull floating things farther out to sea. Tides can affect the bottom of a boat, essentially grounding it in shallow or no water. It is even possible to generate electricity by either water currents and-or tides. The Moon's gravity actually makes the orbit of Earth "wobble" periodically slightly nearer to and farther from the Sun, but in general, the path or orbit of the Earth around the Sun is nearly always considered as circular due to just the gravity of the Sun, and that the wobble motion is rather a negligible fraction of the distance to the Sun.

:

An illustration to describe many of the basic facts about the orbit of Earth: [FIG 257]



This view in the above image is as if looking down from far above the solar plane so as to view the entire ellipse of Earth's orbit. The image is a good and typical general approximation of Earth's orbit and the indicated days of the year.

The two foci on the major axis of an elliptical orbit are equidistant from the center of the ellipse where the major and minor axis intercept or cross. The center of the ellipse is also the center of orbit. Here (x) or the horizontal axis is the major axis, hence (a > b) and-or (b < a). Earth is closer to sun in the winter, but the northern hemisphere is effectively tilted away from the Sun. Using approximate values:

The eccentricity of Earth's orbit about the Sun is about: e = 0.01671
Earth's average orbit distance = a = (aphelion + perihelion) / 2 = 92.95M mi =~ 93M mi = major-axis / 2 = a
Perihelion distance, Earth's closest distance to the Sun is: 91.44 M mi = 93Mmi - (93Mmi * e)
Aphelion distance, Earth's farthest distance from the Sun is: 94.5 M mi = 93Mmi + (93Mmi * e)

c = Distance from the center of orbit to the focus = $\sqrt{a^2 - b^2}$ = (aphelion - perihelion) , and:

a^2 = b^2 + c^2 : c = foci distance is constant, and the length of the major and minor axis is constant, however, throughout the orbit, x or (current (a) value), and y or (current (b) value) are not constant.

$$a = \frac{\text{major axis}}{2} = \frac{(\text{aphelion} + \text{perihelion})}{2} = \sqrt{b^2 + c^2} = 92.95\text{M mi} =\sim 93\text{M mi}$$:average orbit distance from Sun

2a = total distance from any point on the ellipse to both foci = 2 (average distance) = 185.9M mi =~ 186M mi

$$\mathbf{e = eccentricity} = \frac{\mathbf{c}}{\mathbf{a}} = \frac{\sqrt{a^2 - b^2}}{a} = \frac{\sqrt{a^2 - b^2}}{\sqrt{\mathbf{a^2}}} = \sqrt{\frac{a^2 - b^2}{a^2}} = \sqrt{\frac{a^2}{a^2} - \frac{b^2}{a^2}} = \sqrt{1 - \frac{b^2}{a^2}}$$

$$\mathbf{e = \sqrt{1 - (b/a)^2}}$$

For the equivalence on the right side, since $b<a$, due to that the minor axis is always less than the major axis, their ratio will always be less than 1, and the difference from 1 of that ratio, will always be less than 1, and the square root of a value less than 1 will always be less than 1. (e) = eccentricity of an ellipse. ($e>0$) and ($e<=1$). (e) is between 0 and 1.

$e = c / a$: c = focus = (f) = $e a$, , and $a = c / e$

$$e = \frac{c}{a} = \frac{(\text{aphelion - perihelion}) / 2}{(\text{perihelion + aphelion}) / 2} = \frac{94.5\text{M mi} - 91.4\text{M mi}}{94.5\text{M mi} + 91.4\text{M mi}} = \frac{3.1\text{M mi}}{185.9\text{M mi}} = 0.01671$$

: c = center to foci = ea
a = semi-major axis
2a = major axis

aphelion + perihelion = 2a = major axis = 2(92.95M mi) = 185.9M mi
a = (aphelion + perihelion) / 2 = major axis / 2

aphelion = 2a - perihelion = a + c = a(1 + e)
perihelion = 2a - aphelion = a - c = a(1 - e)

aphelion + perihelion = (a + c) + (a - c) = a + c + a - c = **2a** : major axis
aphelion - perihelion = (a + c) - (a - c) = a + c - a + c = **2c** : distance between foci

c = aphelion - a
c = a - perihelion
c = a e = (92.95M mi) (0.01671) =~ 1.55M mi

a = c + perihelion
a = aphelion - c

aphelion distance = a + c = a + ae = **a(1 + e)** :a = semi-major axis = average or circular value, here extended by (ae)
perihelion distance = a - c = a - ae = **a(1 - e)** :a = semi-major axis, = average or circular value, here reduced by (ae)
:c = ae

2a = aphelion + perihelion = a(1 + e) + a(1-e) = a + ea + a -ea = 2a
aphelion - perihelion = a(1 + e) - a(1-e) = a + ae - a + ae = 2ae = 2c

$$e^2 = 1 - \frac{b^2}{a^2} = \frac{a^2 - b^2}{a^2} = \frac{c^2}{a^2} = \left(\frac{c}{a}\right)^2$$

: take the square root to find (e) , and mathematically:
: **e = c / a = f / a , c = e a , a = c / e**

$$b^2 = a^2 - e^2 a^2 = a^2 (1 - e^2) = a^2 - c^2$$

$$\mathbf{b} = \sqrt{a^2 (1 - e^2)} = \mathbf{a \sqrt{1 - e^2}}$$

: b = semi-minor axis , a = semi-major axis , the factor to a is < 1 , b < a
For Earth: e=0.01671 and e^2 = 0.0002792 , and:

b = 92.937M mi
Nearly Earth's average distance

$$\frac{b}{a} = \sqrt{1 - e^2} \qquad b = \frac{c\sqrt{1 - e^2}}{e}$$

(b) can also be shown to be the geometric mean of the aphelion and perihelion:

From the product of the aphelion and perihelion distances: (aphelion) (perihelion) =

$a(1 + e)\, a(1 - e) = (a + ae)(a - ae) = a^2 - a^2 e + a^2 e - a^2 e^2 = a^2 - a^2 e^2 = a^2 (1 - e^2)$

After taking the square root of the right side equivalence of their product, will result in the geometric mean type of expression of those two factors, and the result is (b) = semi-minor axis, as shown above, hence:

$$\mathbf{b = \sqrt{(aphelion)(perihelion)}}$$

For a practical verification of this, consider the special case of an ellipse where the aphelion = perihelion = radius = r, and this ellipse is actually a circle:

$$b = \sqrt{(r)(r)} = \sqrt{r^2} = r = a$$

In a similar manor to a previous derivation of (b) shown above: $\mathbf{a = \dfrac{b}{\sqrt{1 - e^2}}}$: the den. is < 1, therefore, a > b

If we let: $\mathbf{N} = \dfrac{\mathbf{aphelion}}{\mathbf{perihelion}} = \dfrac{a(1 + e)}{a(1 - e)} = \mathbf{\dfrac{(1 + e)}{(1 - e)}}$, also note here for example: aphelion = perihelion $\dfrac{(1 + e)}{(1 - e)}$

$N(1 - e) = (1 + e)$
$N - Ne = 1 + e$
$N - 1 = e + Ne$
$N - 1 = e(N + 1)$ and: $\mathbf{e = \dfrac{(N - 1)}{(N + 1)}}$: this value is always less than 1 as expected for (e)

The speed (Vp) of the orbit of Earth around the Sun will be the greatest when it is the Sun at perihelion, and the slowest speed (Va) when it is farthest away from the Sun at aphelion. The average or linear speed of Earth's orbit about the Sun can be calculated as:

$$Vo = \frac{\text{distance}}{\text{time}} = \frac{(2)(pi)(r\ mi)}{365.25\text{days}} = \frac{(2)(3.14159265)(92.9M\ mi)}{365.25\text{ days}} = \frac{583.708M\ mi}{365.25\text{ days}} = \frac{1.5981M\ mi}{1\text{ day} = 24hr} = \frac{66588\ mi}{1hr = 3600s}$$

$Vo = 18.497 mi/s \approx 18.\underline{5}\ mi/s$

Due to the force of gravity increasing nearer to the Sun, the average speed of orbit will increase by a factor of (e) at perihelion:

$Vp = Vo + \text{increase in } Vo$

$Vp = Vo + (Vo\, e) = Vo(1 + e) = 66588 mi/h + (1 + 0.01671) = 66588 mi/h\ (1.01671) = 67700 mi/h = 18.81 mi/s$

The average speed of orbit will decrease by a factor of (e) at aphelion:

$Va = Vo + \text{decrease in } Vo$

$Va = Vo - (Vo\, e) = Vo(1 - e) = 66588 mi/h + (1 - 0.01671) = 66588 mi/h\ (0.9833) = 65476 mi/h = 18.19 mi/s$

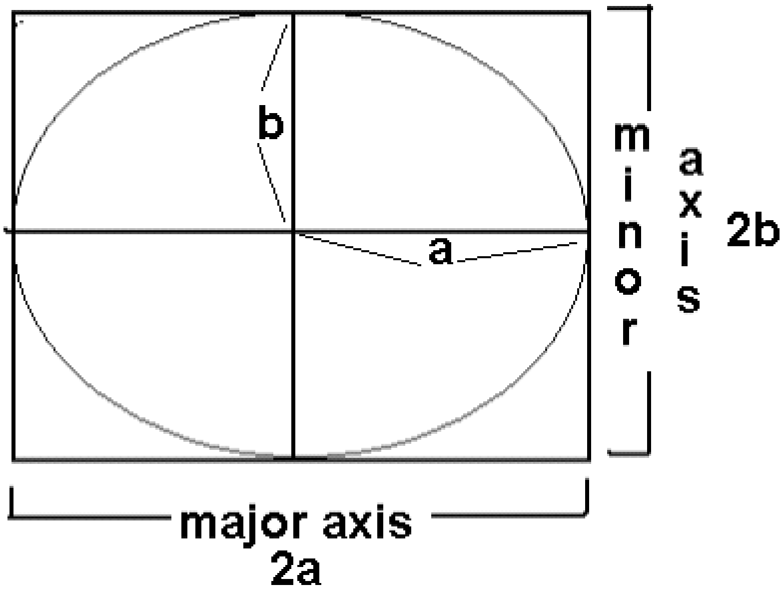
The distance between every planets can vary significantly throughout both of their elliptical orbits. For example, if a planet is on the same side of the Sun as Earth is, it will be closer to Earth, and it will be seen as bigger with a telescope, perhaps twice as big such as with planet Mars which is relatively close to Earth as compared to the more distant planets.

Most tables of information about the planets state the planets average orbit distance, and on occasion, you may need to know its actual position, and or its aphelion (farthest from Sun) and perihelion (closest to Sun) distances. As shown previously:

aphelion = 2a - perihelion = a + c = a(1 + e) : a = semi-major axis and average orbit distance
perihelion = 2a - aphelion = a - c = a(1 - e) c = distance from center of orbit to focus (Sun)
e = eccentricity of orbit value

THE PERIMETER AND AREA OF AN ELLIPSE

Although there are several successive approximation formulas, there is no simple and exact formula for the perimeter of an ellipse and-or the length of an elliptical orbit. The successive approximation formulas involve an infinite series of terms where the approximation gets ("converges", "approximates") closer and close to a specific value and true result.
[FIG 258]



$$\frac{\text{minor axis}}{\text{major axis}} = \frac{2b}{2a} = \frac{b}{a}$$

For a circle with e=0, the perimeter = circumference = 2(pi)r = 2(pi)a =~ 6.283$\underline{2}$a, where here (a)=the semi-major axis, and for a ellipse approaching a line shape with e=1, the perimeter approaches 4a. The ratio of these two values is a constant of about 0.66366$\underline{2}$ or its reciprocal of: ((2)(pi)(a)) / (4a) = **(pi) / 2** = (3.14159...) / 2 = 1.57$\underline{1}$. It is also of note that the ratio of the ellipse perimeter to that of corresponding rectangular perimeter will go from 1.57$\underline{1}$ at e=0 (ie., a circle) to 2 at e=1 (ie., a line). For a circle: a = b = radius = r , and (b / a) = 1 , but for an ellipse where b<a , then (b / a) <1 .

If the major and minor axis of an ellipse are changed by the same factor (n), their ratio will still be the same since it would create an equivalent fraction, and the perimeter of the ellipse will change by that same factor. Both of these ellipses will be similar ellipses that are simply like a magnified version of each other. The area of the ellipse will change by the square of this factor = n^2.

The author of this book has developed a crude formula for the perimeter length of an ellipse, but it is generally only close to the correct value when (b/a) <= 0.5, and the eccentricity (e) is then a fairly high value, say > 0.8. This formula is based on the above reasoning such as the difference between 6.2832a and 4a is 2.2832. This formula is initially based on the perimeter of the associated ("unsquashed", full amount) circle where r=a=semi-major axis, and it then includes a correction or adjustment. This formula is only a crude estimate, and should rather not be used:

An experimental formula for consideration and in general, DO NOT USE THIS FOR THERE ARE BETTER WAYS:
Rough Approximate Perimeter Of An Ellipse = $\frac{C}{e}$ - 2.2832a : typically where (b/a) <= 0.5, and where e >= 0.8,
, and-or its reciprocal where (a/b) >= 2 , and:
: C = circumference = perimeter = 2(pi)(a) , and

: (b) is not explicitly in this formula, but is in (e)

For lower eccentricities, where the ellipse is closer in shape to that of a circle, the above formula should rather consider Ce and not C/e as the basis of it, and due to the fact that when (e) is low, the value of (C/e) is very large. It is recommended to use the standardized formulas below for the perimeter or circumference of an ellipse:

Sir Ramanujan , is a historic mathematician from India and with a significant volume of work and with many being practical, created an amazing and very practical formula for the approximate perimeter length of an ellipse (such as of orbiting planets, etc.), and it has very good accuracy, and without using the somewhat complex mathematical summing series created with many divisions needed. It is also of note that this formula does not explicitly or directly use the variable (e) = eccentricity, however, the axis variables (a) and (b) are used which do actually define the value of (e):

$$Pe = (pi)(a + b)\left(1 + \frac{3h}{10 + \sqrt{4 - 3h}}\right)$$

: **Perimeter Of An Ellipse** , credited to **Ramanujan** from India
h = (a - b)^2 / (a + b)^2
Note here: a = semi-major axis , b = semi-minor axis ,
For a circle: a=b=r , h=0 , P = (pi)(r + r) = (pi)(2r) = 2(pi)(r)

It is conceivable that the distance between every two successive, very close points on the ellipse can be summed together so as to find the perimeter. This is can be done with a computer using the ellipse equation and the distance between two points equation. Since the ellipse is symmetrical about both axes, only one quarter of its perimeter, such as from x=0 to x=semi-major axis, needs to be found and then multiplied by 4 to find the entire perimeter value.

The author has found a very slight correction or adjustment value for (h), and which will improve the calculated perimeter result slightly, and particularly for when the eccentricity is high, and that is: h = h + (h)(0.0016523164869) =
= h (1 + 0.0016523164869) = h (1.0016523164869)

For the **area of an ellipse**, first consider the area of a circle being: A = (pi)r^2 = (pi)(r)(r) , and that we can find the area of an ellipse by letting one variable r=a=semi-major axis, and the other variable r=b = semi-minor axis::

The area of an ellipse is: **Ae = (pi) (semi-major axis)(semi-minor axis) = (pi) a b**

: **Area Of An Ellipse** , and here:
a = semi-major axis
b = semi-minor axis
For a circle: a=b=r , A = (pi) a b = (pi) r^2

There is a constant ratio value of the area of an ellipse or circle to that of the surrounding rectangular area with the same dimensions as the major and minor axes, and this value is: **ka = 0.785398163397448** , and this value happens to be the corresponding radian angle of a 45 degree angle, and it is also equal to: **(pi) / 4**. **Area of corresponding ellipse = (ka) (Area of corresponding rectangular area).** 1 / 0.785398163397448 = 1.273239545

The ratio of perimeters is not a constant but has a range from about 1.57$\underline{1}$ = (pi/2), to 2.0 as mentioned above, and the ratio of the two extreme values of this range is also k = 0.785398163397448. The perimeter of the ellipse within the rectangle will always be larger than the perimeter of the surrounding, corresponding rectangle. The area of the ellipse within the rectangle will always be less than that of the area of the rectangle.

One might ask, what are the dimensions of the axes and-or corresponding rectangle when **e=0.5** (ie., half of the maximum value of 1)? The answer to this is interesting, and it is when the ratio of **(b/a)** is nearly equal to: (epsilon / (pi)) = (2.718281828 / 3.141592654) = 0.865255979, and this is =~ **0.866025403** = sin 60°, and is nearly equal to the (square root of k) = 0.8662269254527578. Here, the ratio of the perimeter of the ellipse to that of the circumscribing or corresponding rectangle is about 1.5728246008$\underline{1}$ ~= (pi/2).

When the ratio of **(a/b =2)** and-or **(b/a = 0.5)**, e = 0.866025403 = sin 60° = square-root of (0.75). When **a/b = 2** and-or **b / a = 0.5** and-or **b/a=2,** that is, one axis is twice as long as the other, the ratio of the perimeters is a constant of: **1.61481450996017**. Another observation is that epsilon (e) is about equal to (pi) (sin 60°). When (b/a = 0.25) and-or (a / b = 4), the area is equal to (pi)(b^2).

When a=b, the ellipse is a circle and has the smallest perimeter for that value of: 2(pi)(r) = 2(pi)(a) = 6.283185307a. As (b) gets smaller for a given value of (a), the perimeter of the ellipse will decrease to its minimum value of 4a where (a) is the semi-major axis, and (b) will be equal to 0. When **a=b**, the ratio of the perimeters is a constant of: Pe/Pr = **1.570796325** = (pi)/2. The above ratio leads to the conclusion that when the ratio of (a/b) and-or (b/a) is a certain constant value, the ratio of the perimeters also has a specific constant ratio value.

When **a=b**, the ellipse will have an **e=0**, and it will be a circle shape and have the maximum area for a given value of (a). A = (pi)(r^2) = (pi) (a^2) where here, (a) = (radius = (r) of the circle) =semi-major axis. As (b) gets smaller, approaching 0, the area of the ellipse will also approach 0 and **e=1**, as for a line.

When **(b/a)** is about 0.66, and-or (a/b) is about 1.5, **(e) =** about **0.75**. A more precise ratio value is (b/a) = **0.661437827**.

When **(b/a)** = **($\sqrt{2}$ / 2) = 0.7071067811865475 , (e) = 0.7071067811865475 ,** and the **focus = b** , (a/b) = 1.414213562

Extra, from: Ae = (pi) (semi-major axis)(semi-minor axis) = (pi) a b , the ratio of the ellipse area Ae to that of the semi-minor circle area is: (pi) (semi-major axis) (semi-minor axis) / (pi) (semi-minor axis)^2 =
= (semi-major axis) / (semi-minor axis) , ex: a / b , 10 / 2 = 5 , hence the ellipse area is 5 times more.
Likewise: (semi-major axis circle area) / Ae = (pi) (semi-major axis)^2 / (pi) (semi-major axis) (semi-minor axis) =
(semi-major axis) / (semi-minor axis), ex: a / b , 10 / 2 = 5 , hence the circular area is 5 times more.
For circles, when r=radius changes by a factor of (n), the circumference (C) will also change by that same factor of (n), and the area (A) will change by the square of that factor of (n), and which is (n^2).

According to Kepler's Law about equal areas (and not equal angles) of an ellipse being traversed during equal time of an orbit about the focus (such as the Sun), this total area can be divided by the (average) time of orbit so as to find out the area segment of the ellipse traversed per unit of time:

$$\frac{\textbf{(ellipse segment area)}}{\textbf{time}} = \frac{\textbf{(total Area of elliptical orbit)}}{\textbf{(Period of orbit)}} = \frac{\textbf{ellipse segment area}}{\textbf{unit of time}} = \textbf{a constant for this ellipse}$$

As for finding the **average velocity** of an orbiting body such as a planet, and since the velocity is always changing due to gravity and especially when it is near the focus of the orbit such as the Sun where the gravity force is higher:

$$\text{From distance = velocity x time ,} \quad \textbf{v av.} = \frac{d}{t} = \textbf{v av.} = \frac{\textbf{Perimeter of ellipse orbit}}{\textbf{Period of elliptical orbit}} = \frac{\textbf{average distance traveled}}{\textbf{unit of time}}$$

Extra: Of possible use: Area / time = A / (d / v) = Av / d = d^2 v / d = dv = (vt)v = v^2 t

Extra: For some computer program which require quick graphics or image displays, there is fast algorithms (steps or sequence of computer instructions or program code) developed to display many common geometric shapes such as a line, circle and ellipse. These algorithms are based on integers values, rather than the slower floating point (true or "real" [in reality, possibility], decimal values with a fractional part) values. These algorithms work because the pixels or single dots on the screen have coordinates (ie., like address locations) that are numbered and accessed as successive integers. **Jack Bresenham** from the IBM (International Business Machines) company is credited in 1962 to the initial (Bresenham's) **line algorithm** for drawing-printers and-or for computer graphics today. This algorithm is "fast" so as to facility graphics (computer drawings, typically shown on a viewing screen or "TV") appearing more quickly and realistically when needed. It is of note that some computer languages, such as C, as standardized by an ANSI standard, do not include graphics functions (ie., to set pixels and-or colors, draw lines and curves such as circles, etc.), and it is up to the compiler to do so, such as by including functions to draw graphics on the computers viewing screen if the computer is capable of displaying graphics, such as when it has a graphics card (ie., circuit). An example of a function to set a single pixel on the screen would be something similar to: setpixel(horizontal position, vertical position, color)

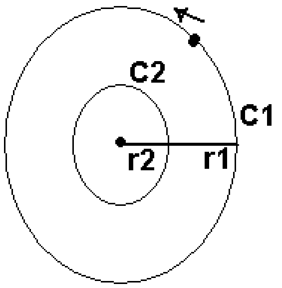
A note and example about angular momentum.

Angular momentum can be considered as a measure circular or orbital momentum, and-or circular or angular motion in general. Consider an object having a momentum of p = mv, and it is orbiting about a planet at a far away distance. If that object were to then be placed closer to the center of its orbit, its momentum is still the same if it has not gained any more energy or momentum along the way. Its momentum is still p = mv, and this is generally called an example of the concept of **the conservation of momentum**. Its velocity will remain the same, but its angular velocity will increase and it will make a complete orbit much faster now. An example of this would be where a ball is attached via a thin string to a rod or pole in the ground. The ball is then thrown so as to orbit about this pole. Meanwhile it is orbiting this pole, it is also getting closer to the pole because the string is getting shorter as it wraps around that pole. The ball will orbit much faster and give the appearance that it is actually going faster in terms of effective linear velocity = d/t, but it is rather the angular, circular or orbital velocity = (angle / second) and-or (rotations or orbits) / time that has increased, and it is still going the same linear velocity as that in its far away orbit.

For an orbiting object, if (r) decreases by (n), the circumference or distance of that orbit decreases by (n).

$$C = 2(pi)r \quad , \qquad \frac{C}{n} = \frac{2(pi)r}{n} = 2(pi)\left(\frac{r}{n}\right)$$

Given a constant velocity for an orbiting object with momentum = p=mv, if the orbit distance or radius of orbit is now (n) times less, the orbit distance (ie., C) is now (n) times less, and the orbiting time is also (n) times as less, and the angular velocity (ie., angle / time and-or rotations / time) is now (n) times more. If the orbiting object has not gained or lost any kinetic energy, its velocity will still be the same as that of the distant orbit, and its momentum will still be the same value. The following analysis may also be helpful to this subject: [FIG 259]



In the above figure, an object is in a circular orbit with a circumference of C1. It takes 1hour = 1h for the object to make an orbit. From d=vt ,its velocity is:

$$V1 = d / t - C1 / 1hr = 2(pi)(r1) / 1hr$$

In terms of rotations (rots), revolutions (revs) or full-orbit velocity (Vo):

$$Vo = 1 \text{ orbit} / hr = 1 \text{ rot} / 1 hr = 1 \text{ rev} / 1 hr = 1 \text{ cycles} / hr = 360° / 1 hr = 2(pi) \text{ radians} / 1 \text{ hour} = 1 \text{ wave} / 1 hr =$$
$$= 1 \text{ repetition} / 1 hr = w = \text{angular or circular velocity}$$

If (r) is changed by (n), either as a numerical co-efficient factor or divisor, C1 is then changed in the same way by this value of (n). If (r) is decreased by a factor of 2, hence (r/2), then C1 will also decrease by 2 to be a value of say C1/n = C1/2 = C2 in the above figure. Given the same velocity (V1) and time of 1 hour, the number of orbits with each being equal in length to C2 will also increase by the same value of (n):

$$\text{From } C1/n = C2 \quad , \quad C1 / C2 = n \quad , \quad \text{checking: } \frac{C1}{C2} = \frac{2(pi)(r)}{2(p)\left(\frac{r}{n}\right)} = n$$

$$C1 = n\,C2$$

same distance = same distance

And this implies that during the same time (t) necessary to orbit C1 , there would be many (ie., n) orbits of C2.

From $d = vt$ and $\frac{d}{t} = v$, Since a circumference is a distance: $C = vt$ and mathematically: $\frac{C}{t} = v$

From $C1 = n\,C2$ and dividing each side by t, we have:

$\frac{C1}{t} = \frac{n\,C2}{t} = v$: same velocity if no forces have act upon the object, and which would then change its speed or direction. This is mathematically equal to:

$V1 = V2$

same velocity = same velocity = v : linear or distance related velocity

Though the linear velocity is the same, the angular velocity = w/t is not the same. In the same amount of time, there will be (n) orbits associated with the object and C2, and its angular velocity will be (n) times greater:

$$w = \text{angular velocity} = \frac{\text{amount of rotation}}{\text{time}} = \frac{\text{orbits}}{t}$$

For the object in the orbit of C2, in the same time it takes to orbit at C1, it will make (n=2) complete orbits, hence it will go a total amount of rotation measured in degrees for (n) = 2 rotations.
Total Degrees = orbit1 + orbit 2 = 360° + 360° = (n orbits) 360° = (2)(360°) = 720°

$\frac{2 \text{ rotations}}{1 \text{ rotation}} = \frac{2 \text{ orbits}}{1 \text{ orbit}} = \frac{720°}{360°} = 2 = (n)$: after the same time duration or period
For the expressed ratio of angular velocities:

$$\frac{w2}{w1} = \frac{\frac{2 \text{ orbits}}{t}}{\frac{1 \text{ orbit}}{t}} \text{ or } = \frac{\frac{720°}{t}}{\frac{360°}{t}} = 2 = (n)$$: ratio of angular velocities

If (r) decreases by a factor of (n), the angular velocity increases by (n): $w2 = (n)\,w1$

A COMPUTER PROGRAM FOR THE PERIMETER AND AREA OF AN ELLIPSE

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/*----------------------------------------------------------------------------------------------------
EllipsePerimeterAndArea.c

        A program to input the major and minor axis lengths of an ellipse, and
        the perimeter of that ellipse will be calculated using Ramanujan's
        practical formula, that is perhaps easier than using the summation
        method (by hand and pen) with more mathematical divisions needed
        to make that method of calculation.  The area of the ellipse is also
        calculated.  Other values are also calculated.   You may wish to modify
        the program to reduce input errors, and also to display the perimeter
        values when the value of (h) in the perimeter algorithm is not used, and
        then when used.

        (c) JPA, Dec 3, 2021
------------------------------------------------------------------------------------------------------*/
#include "stdio.h"     /* for printf() */
#include "math.h"     /* for sqrt() */
#include "stdlib.h"   /* for system() */

double ellipseperimeter(double semimajoraxis, double semiminoraxis);
/* A function to calculate the perimeter of an ellipse.  The algorithm
   used is a practical equation by Ramanujan from India.  */
/*----------------------------------------------------------------------------------------------------*/
void main (void)             /* A sample program, and to test the function. */
{
double majoraxis=0.0;   /*  major axis ,  in most computer languages, the variable names
                             and-or identifiers cannot have spaces */
double minoraxis=0.0;   /*  minor axis  */
double a=0.0;   /* semi-major axis */
double b=0.0;   /* semi-minor axis */
double  perimeter=0.0;  /* perimeter of the ellipse */
double rectangleperimeter=0.0;
double e=0.0;               /* eccentricity */
double ellipsearea=0.0;
double rectanglearea=0.0;
unsigned char c=0;

for(;;){

system("cls");

printf("\nELLIPSE PERIMETER AND AREA CALCULATER PROGRAM.");
printf("\n(c) JPA 2021, From the Mathization ebook of math and science.");
printf("\n\n");

printf("Enter a positive length of the major-axis: ");
fflush(stdin); scanf("%lf",&majoraxis);    if(majoraxis < 0.0){ printf("\a");  exit(1); };

printf("Enter a positive length of the minor-axis: ");
fflush(stdin); scanf("%lf",&minoraxis);    if(minoraxis < 0.0){ printf("\a"); exit(1);  };

if(minoraxis > majoraxis){  printf("\a"); exit(1); }
```

```
a = majoraxis / 2.0;   /* adjust these to work with the minor axis values in the equations */
b = minoraxis / 2.0;

if(majoraxis >= minoraxis){     /*  :or use:   if(majoraxis a >= b){     */
    e = sqrt ( 1 - ( (minoraxis * minoraxis) / (majoraxis * majoraxis) ));
    };

if(minoraxis > majoraxis){
    e = sqrt ( 1 - ( (majoraxis * majoraxis) / (minoraxis * minoraxis)) );
    };

printf("\nEccentricity = %.15g",e);
printf("\nFocus = %.15g",e * a);

perimeter = ellipseperimeter(a,b);
printf("\n\nEllipse Perimeter = %.15g",perimeter);

rectangleperimeter = (2 * majoraxis) + (2 * minoraxis);
printf("\nCorresponding Rectangular Perimeter = %.15g",rectangleperimeter);

ellipsearea =  3.141592653589793 * (a) * (b);
printf("\n\nEllipse Area = %.15g",ellipsearea);

rectanglearea = majoraxis * minoraxis;     /* : or use:  rectanglearea = (a * b)   */
printf("\nCorresponding Rectangle Area = %.15g",rectanglearea);

printf("\n\nRatio Of Ellipse Area To The Rectangle Area = %.15g", ellipsearea/rectanglearea);

printf("\nRatio of Ellipse Perimeter To The Rectangle Perimeter = %.15g",perimeter / rectangleperimeter );

printf("\n\nPress A Key To Continue, or ESC to exit: ");
fflush(stdin); c=getch();  if(c==27){ break; };

majoraxis=0.0;   /* reset to 0,for the loop and program to be run again */
minoraxis=0.0;
perimeter=0.0;
};

};
/*-----------------------------------------------------------------------------------------------------------*/
double ellipseperimeter(double semimajoraxis, double semiminoraxis)
{
/*------------------------------------------------------------------------------------------------------------------------------------------------------
     Here is the  practical formula for a very close, approximate perimeter of an ellipse, created by Ramanujan from India:

    P  =  (pi) (a + b) (  1  +  ______3h______    )              : Perimeter Of An Ellipse , credited to Ramanujan from India
                           (       10 + \/ (4 - 3h)     )                h = (a - b)^2 / (a + b)^2 ,   here,  a=semi-major axis  ,
                                                                                                                 b=semi-minor axis.
                                                                         When (a) is high and (b) is low, (e) is high, and  the error is larger,
                                                                         at about  -0.016

     Area = (pi)(semimajor axis)(semiminor axis)      : Area Of An Ellipse
-------------------------------------------------------------------------------------------------------------------------------------------------------*/
```

```
double perimeter=0.0;

double a = semimajoraxis;
double b = semiminoraxis;

double h=0.0;   /* general use variables */
double x=0.0;
double y=0.0;

h = (a - b) * (a - b) / ( (a +b) * (a + b) );

h=h+(h * 0.00165231179593);       /* JPA's , (h) adjustment factor, gives an exact result on a=10, b=0.000000000000001,
                                        P = 40.0000000000000. This was found by trial and error adjustment  when
                                        observing the results for the perimeter.   It is reasonable on other values and is
                                      optional and-or experimental. Perhaps use it only when (a/b) >= 2 for better results.  */
x=4.0 - (3.0 * h);
y = sqrt(x);
x = 1 + ( (3*h) / ( 10 + y ) );

perimeter = (3.141592653589793) * ( a  +  b)  * x;

return perimeter;
}; /* end of program */
/*--------------------------------------------------------------------------------------------------*/
```

Here is an example of the screen output or display when using the above program:

```
ELLIPSE PERIMETER AND AREA CALCULATER PROGRAM.
(c) JPA 2021, From the Mathization ebook of math And science.

Enter a positive length of the major-axis: 2
Enter a positive length of the minor-axis: 1

Eccentricity = 0.866025403784439
Focus = 0.866025403784439

Ellipse Perimeter = 4.84444353479312
Corresponding Rectangular Perimeter = 6

Ellipse Area = 1.5707963267949
Corresponding Rectangle Area = 2

Ratio Of Ellipse Area To The Rectangle Area = 0.785398163397448
Ratio of Ellipse Perimeter To The Rectangle Perimeter = 0.807407255798853

Press A Key To Continue, or ESC to exit:
```

Extra: If both the axis (a and b) of the ellipse increase by the same factor value of (**n**), it essentially results in a magnification of that ellipse, and both the perimeter and focus (c) of it will be magnified or increased by that same value of (n). The area of the ellipse will increase by (n^2). **Both ellipses will have the same eccentricity = (e)** because the ratio of the axes (a/b) is still the same value. The outer rectangular perimeter will increase by (n) and its area will increase by (n^2).

ACID AND BASE

This discussion here about "acid and base" is only intended to be a simple and practical one for most people, yet it is still enough for most people to consider so as to have enough basic knowledge about it, and to then consider if they need to learn more about it elsewhere.

SEVER DANGER AND HEALTH WARNING: When working or using chemicals such as acids and bases, **wear proper and-or expendable clothing**, **gloves, breathing and eye protection**. Some chemical reactions or processes are very fast and can produce a lot of heat and gas when energy is released in a short amount of time, and hence they can be explosive (ie., rapidly expanding) and-or combustible, and cause injury, damage, and-or a fire. Rinse protective gear (clothing, eye protection, hand protection, etc.) them off thoroughly in water if it is permitted to reuse them again. Inspect them for tears, loose seams and holes, and are good working order. It is recommended to keep all chemicals and cleaning products hidden and out of reach of curious children and pets. If an accident happens, seek medical attention immediately; dial **911** (in U.S.A.) on a phone (unpaid phones still allow 911) for the emergency and so as to receive help. **A common and fast treatment for many chemical burns of the skin and-or eyes is to gently rinse (do not scrub) the area for up to 20 minutes using cool uncontaminated water poured over the area and-or shower in cool to warm water.** A fire extinguisher and-or a water-hose needs to be considered for some experiments involving heat and-or fire. Also, keep curious people and pets away from the experiment and-or work areas, have signs indicating the danger and to keep out, and properly clean up and dispose of any debris. **Properly label each chemical and-or mixture, usage, and potential dangers.**

The concepts of electric charges, acids and bases are widely used in the field or study of chemistry. An acid or base are general classifications given to substances, often in liquid (aqueous [aq], water) or dissolved form, which can help a chemical reaction take place due to a transport medium such as water. A common and rechargeable lead-acid (lead metal in an acid electrolyte) battery in an automobile has an electrolyte solution between its internal metal plates of each cell, and it is an acid solution, usually (dangerous) sulfuric acid (H2S1O4, hydrogen, sulfur and oxygen, avoid contact and-or inhalation, wear protection). The fluid substance or medium helps the distribution and mobility (ie., travel, movement) of all the atomic parts (molecules, protons, electrons) of the acid or base solution and substance(s) in it. The **pH scale** is a measurement of the strength (ie., "chemical power" (p) or "chemical potential ability", particularly of the hydrogen ions which are only protons, and without any [electrically balancing or neutralizing] electrons) of an acid or base aqueous substance. In short, the pH scale could be described is a measure how how acidic a substance is due to each substance having some level of hydrogen proton activity..

A neutral ("neutrally charged", or "balanced") atom has the same number of protons and electrons, and is said to be electrically neutral or chemically stable. When the number of electrons and protons are not the same in an atom, that atom is called an **ion**. Often an ion is positively charged since the atom is missing an electron in its orbit. An atom is said to be an **isotope** if the number of its neutrons has changed and this will also change the physical properties of that atom(s), material, substance, but the chemical and-or electrical properties of isotopes of the same atom are fairly similar.

An **acid** substance has or causes an abundance (ie., extra) of protons (ie., hydrogen ions) and can effectively release or "donate" them when that acid is put in a water solution, and so as to then have "balanced" or neutral atoms that have no net electrical charge. The abundance of protons is due to positive hydrogen ions (H+) that lack an electron because it has moved elsewhere, and what remains is a proton with a positive (+) charge. A free moving proton has much more mass than an electron and therefore, it has a higher kinetic energy than an electron, and can do more damage (via collisions, and freeing some electrons) and dissolve things. A proton also attracts electrons, and this can **dissolve** and-or break apart atomic structures, such as crystal bonds and-or molecular bonds at the boundary or surface layer of the acid and material, and then continue to dissolve the material. To make an acid solution, electrons need to be removed from it so as to have an abundance of protons. When an acid molecule disassociates (breaks apart) in water, it creates protons (+ , positive charge) particles, and anions (- , negative charge) particles from that acid molecule.

A **base** substance has a lack of protons and can "accept" them so as to become neutral atoms. A base substance has an abundance of electrons, and can release hydroxyle (hydroxide, OH^-) ions. As indicated by the sign, this molecule has a negative electrical charge, hence has more electrons than protons.

To determine how acidic or basic a substance is, a certain amount of it is dissolved in an aqueous (ie. aqua, "watery", liquid) solution such as a certain amount of pure water, and a pH measurement is then taken. This measurement is called a pH measurement and means: **pH** = **p**otential (strength) **H**ydrogen, and which for here, it is the hydrogen positive ion level in a substance, hence the proton and-or acid level. A positive hydrogen ion atom is indicated as: H^+, and which is actually just a single proton that has lost its electron, but can electrically attract one back into orbit around it. The pH scale of values is a "relative" (roughly, non-exact) type of logarithmic scale with a numeric base of 10. **Each integer increase or decrease in this scale means that the substance is now considered 10 times more or less of an acid or base than the previous value, hence the hydrogen ion concentration will change by a factor of 10.** This "small" logarithmic scale is needed due to the potentially high number (up to trillions) or concentration (ex. mol units) of atomic particles (H^+ protons from an acid, or OH^- from a base) involved. **OH^- is a hydroxyl ion** that consists of a bonded hydrogen and oxygen molecule, somewhat like a water molecule (H20) but having just one hydrogen atom, and is (net) negatively charged and is the main component of a base substance. A proton, being positively charged electrically, can attract electrons or molecules that are (net) negatively charged. These attractive or repulsive forces are very strong when the distances are small, and an acid can then (atomically, electrically) dissolve or separate objects such as molecules of a substance, and back into the atoms that composed it. A single proton when unbound to a neutron, nucleus, rigid crystal or molecule structure, is then able to easily travel about due to electric charges and-or forces influencing it to move.

The pH scale is currently defined as from 0 to 14 only. The pH scale was invented in 1909 by chemist **Soren Sorensen**. The formula for pH was made so that all the values (0 to 14) are simply positive in value and that water being nearly neutral (here, being a very weak acid, and very weak base) is in the middle of this range with a pH of 7, but it is still able to dissolve (ie., "corrosive") some materials over a length of time due to hydrogen ions (H^+, which are protons, and which have a positive electrical charged). Warmer water also has more hydrogen ions, and which will then also dissolve or loosen some materials faster. To help rationalize this strange valued scale, this scale can be somewhat thought of as the relative value or concentration of $[OH^-]$ in the substance. A pH of 0 means no concentration of $[OH^-]$ as expected for an strong acid substance which has plenty of protons. A pH of 14 means an incredibly high, maximum concentration or count of $[OH^-]$ as expect for a strong base substance that has plenty of electrons.

A pH value (the "**p**otential" of hydrogen, and-or its protons) to dissolve things for an acid is low because its ability or activeness to dissolve itself is low. A base substance, or even other acids rated with a higher pH, will have a higher pH rating because the ability of the (hydrogen, proton) acids ability to affect (chemically, electrically) it is higher.

Technically, a pH value is calculated as the negative of the logarithm (ie., an exponent value) of the hydrogen ion concentration amount. This concentration amount is measured in gram-moles per liter. The actual volume or amount of an acid substance in a solution will also determine the number of hydrogen ions (ie., protons) available and how active (dissolve-ability) that acidic solution is. In a way, the pH value indicated on the scale can be thought of as an inverse relative value or indication of the dissolving-ability of that specific acidic substance when it attracts electrons to it and pulls apart (ie., "dissolves") molecules. The pH scale is generally considered as an indication, ability or potential of a (aqueous, watery mix of) substance to attract protons and-or release electrons. Bases have a high pH, and the higher base pH, the more that the substance is affected by an acid and-or any substance with a lower pH including water with a pH of 7.

Hydrogen ion or proton concentration has units of: (moles of the mass of the ions in grams) / Liter

pH = - log (hydrogen ion or proton concentration in the solution) = $-\log[H^+] = -1\log[H^+] = \log[H^+]^{-1} = \log[1/H^+]$

Mathematically: (hydrogen proton or ion concentration in a liquid solution) = $10^{(-pH)} = \dfrac{1}{10^{pH}}$

> From the above formula , note that pH is a logarithmic value, and we see that the lower the pH, the higher the hydrogen ion or proton concentration.

Adding water to an acid will decrease the hydrogen ion concentration or amount of ions in the solution, and the pH of that solution will increase. As more and more water is added, the pH of that solution will become close in value to that of water which is 7.

A base has a pH greater than 7, ex: calcium carbonate, baking soda (sodium bicarbonate, pH=8), bleach, etc. Plain or pure water is said to have a pH of 7, since it does have some equal (neutral) amount of both hydrogen ions (ie., protons) and hydroxyl ions, and hence having a low dissolving ability with the relatively few positive hydrogen ions available, and the pH is considered as 7 (ie., in the middle) on the pH scale.

Pure water is said to have a "self ionization" process, and even has a very low electron (-) conductivity. In a volume of 1 liter of water, there is a concentration of about 10^-7 mols of hydrogen ions (H+):

pH water = - log 10^-7 = (-1)(-7) log 10 = +7 (1) = +7 : the log exponent rule was utilized here for simplification

An acid has a pH of 6 or lower. ex: vinegar and lemon juice. Vinegar, such as clear white "5%" acetic acid and the remainder being water. Common household food vinegar is sometimes called a weak acid and has a pH of about 2.4

min. pH				max. pH
0	**pH < 7**	**pH = 7**	**pH > 7**	**14**
	acid	**neutral**	**base**	

Pure hydrochloric acid (HCL, hydrogen and chlorine atoms combined into molecules, and creating a very strong or active acid) is rated and standardized as having the lowest pH of 0. Most soaps for household cleaning have a high pH, and are therefore a base substance.

If the hydrogen ion or proton activity in a solution is less than that of pure water, the solution is a base solution and its pH is actually greater than 7 according to the pH formula.

Acid substances can release or "donate" hydrogen ions (H+, protons) into a water solution. Base substances can release or "donate" hydroxide ions (OH-) into a solution. Acid solutions are highly chemically "active", and can atomically, electrically attract and break apart atomic bonds and therefore dissolve other substances such as molecules of compounds and crystallized elements (such has metals) into smaller particles and-or atoms or ions. Consider that a hydrogen ion (H+) is electrically positive and can attract loosely bound electrons (electrically negative, -e) of other substances, and this leads to molecules, and crystals of even a single elements having their atomic bonds broken apart, and that material or substance will then dissolve by and into in the acid solution.

Note that hydrogen atoms are also the smallest atoms and they can more easily "go through" other substances and materials than the atoms of any other type of element. This makes confining hydrogen gas a problem that needs to be resolved, such as for potential spaceflight use.

For some plants to grow well, they may need or favor a particular soil pH, climate, water availability, etc. Litmus paper "(pH) test strips" can measure, indicate and help determine the pH (via color of the red and blue dye-chemical coated wood cellulose papers) of a uniform or consistent (well mixed - "homogeneous", sometimes with a measured dilution of the solution, average) substance. A blue colored test strip will turn a shade of red in an acid solution. It is even possible to make these test strips at home with fairly common substances. Electronic pH meters are also available. Some of these have metal probe(s) made of non-reactive (ie., chemically stable) platinum metal, and less inexpensive pH meters may use some other inexpensive metal with similar characteristics. Acids and bases can dilute (ie., weaken by reducing the amount of a substance in a mixture) and even neutralize each other to a pH of 7 and actually produce some water (H20) if the cation (see the text below) is from an acid, and if their relative strengths are the same and-or the amount of each substance used is enough to essentially create the neutral state or a certain pH value desired. If the cation is from a base element such as a metal, a solution of an acid (such as having hydrogen protons) and that base metal will produce a (electrically neutral, "neutralization reaction") **salt which is formally defined in chemistry as a molecule composed**

of both a positive ion and a negative ion. This molecule is an ionic compound. Also produced is hydrogen gas, rather than water molecules. **Sodium-chloride (NaCl)** is a salt commonly known or called as table salt or food salt, and it is composed of both (metallic) sodium ions which are positive, and (non-metallic) chloride ions which are negative. Each sodium chloride molecule has one atom of sodium and one atom of chloride, hence a 1 to 1 ratio. Table salt is a hard and clear (stone, mineral) crystal if it is pure.

Cleaning **soap** is a (chemical, not specifically table-salt) salt substance made from sodium or potassium hydroxide as the (pH) base metal, and biological (organic, usually from plants, and therefore it containing some carbon) fats and-or oils which are also called "fatty (pH) acids". To clean things, a soap essentially weakens the bonds of oil molecules, and this will allows oil debris to be more easily removed from things, and then clean water is typically used to flush away the loosened oil and debris which also get partially dissolved by water itself. Dampening or soaking items in water for an amount of time is a helpful measure to soften debris before removal and-or cleaning.

An ion that is positive in charge is called a cation. An atom that looses an electron(s) is a positive ion, such as a hydrogen ion (H+) and of which is actually a single proton. Cation (ie., cathode metal ion) is a word similar to the word cathode which is an electric terminal point such as on a battery or wire. A cation can be created from a metal atom. **An ion that is negative in charge is called an anion** (ie., anode metal ion). An atom that has gained an extra electron(s) is called a negative ion. Anion is a word that is similar to the word anode which is an electric terminal point such as on a battery or wire. An anion can be created from a non-metal atom. Hydrogen is an atom that can be a cation or anion.

Both acids and bases are often dangerous because they can chemically "burn" and-or dissolve things and release large amounts of energy in a short amount of time (ie., high power, watts = joules/second), often in the form of heat, during the fast chemical reaction(s) taking place. This is also called a "chemical burn" or "caustic burn" such as when it injures someone.

The small amounts of acids and bases that humans do consume in foods is relatively weak and near the pH of water at 7. The pH of lemon juice that is generally known as a "weak acid" is about 2.5. A lemon or other citrus fruits have a "sharp", "tangy" or "sour" taste due to the citric acid in them, and the Latin word "acid" basically means "sharp" and-or "sour" tasting. Consuming some foods can cause and-or aggravate (painful, burning) heartburn and-or other digestive issues and teeth erosion. If you have this, a quick remedy is to drink something that does not have any acid in it and so as to reduce and-or "neutralize" the level of acidity in you. There are medications (prescribed by your doctor, and-or "over the counter" types without a doctors prescription needed) available so as to occasionally (when absolutely needed) prevent excessive acid from even being produced by the body for digestion, and so as to have much more control of the problem and possibly prevent it or reduce its severity. The pH of the human stomach, is about 2.5 on average, and it is a moderate level of acid so as to aid digestion by reducing all types of possible germs, and for pre-processing or preparing the nutrients in the food for absorption into the body through the walls of the intestines.

Bases, such as sodium hydroxide (ie. "lye" = "caustic soda"), can be used to break down fat molecules in a process called saponification **(**non-formally: that which produces a sap-like textured substance like feel of the semi-solid sap from a tree, and another similar definition is the "hardening" or "firming" of a substance**)** and is used to make soaps and with **glycerol** (C3O3H8, or "**glycerin**", a thick sugar-alcohol substance, and can be used as a moisturizer, used in some foods, etc.) as a byproduct. Hard **Soaps** are made of mostly oil and some sodium hydroxide placed into a water solution and heated or allowed to dry out or "cure" naturally. Soaps have a slippery feel to them. Soaps can mix into water, unlike insoluble oils. Base solutions are often called **alkaline** solutions and their pH is said to be of an alkaline value greater than 7.

Clean drinking water with a pH within the range of 6.0 to 8.5 is considered safe to drink. The pH of human blood is typically about 7.4, hence it is and-or should be a mild base solution. Blood is typically 9g of salt / liter = 9g/1000mL = 9g/1000g = 0.009 = 0.9% vital salt, and this is just slightly less than 1% salt. Salt is generally needed by the body for various electro-chemical reactions and processes such as for digestion which is the breakdown or dissolving of food so as to extract the nutrients (nutrition - minerals [elements] and vitamins [A, B, C D, etc]) to then be more available to and absorbed by the body.

Water is a mild solvent when the pH is less than 7. If you put a substance such as a hard candy, salt, sugar, or maybe a hard vitamin into a glass of water, it will slowly dissolve (break apart to smaller pieces), but for a given volume of water, there is a maximum amount for each specific substance that can be dissolved by and into that amount of water. Solubility is a measure of how well a certain substance or material can be dissolved (dissociates, "breaks off or apart from") in a specific substance. The units of solubility is grams of the substance per milliLiter = g/mL at room temperature. 1mL = 1cc. The maximum solubility of sugar at room temperature water is:

(180g sugar/100cc water) = (180g/100mL), and from this, dividing both the numerator and denominator by 100, we get: (1.8g sugar) / (1 cc water) = 1.8g sugar / 1 mL water = the solubility of table sugar in water

How much salt can dissolve in water? , How much salt is in sea water? , How much sugar can dissolve in water?

When a solution can no longer dissolve any more of a certain substance placed into it, that solution is called a **saturated** ("fully filled") solution, and this determines the (max.) solubility of that specific substance in a specific liquid. Solubility (ie., dissolvability, reduce to very small particle size) can also be expressed as the number of mols per liter (L). The **solubility of salt** (NaCl, sodium chloride, "table salt", crystals) in fresh, pure water is 35g/1kg = 35g/liter = 35g / 1000cc = = 35g/1000mL = 3.5g/100 mL = 0.35g / 10 mL = 0.35g / 1cc of pure water. 35g/1000g = 0.035 = is a relative value of **3.5%, by weight ratio, maximum** in any volume of water. For conversions: 35g salt / 1000g water = 1.235 oz salt / 35.274 oz water = 0.035 oz salt / 1 oz water. Sea or ocean water has this value on average, and is therefore already (salt) saturated. Adding more salt into this solution will not dissolve it, but rather the salt will be a **precipitate** (ie., "precipitation", "fall out from", "come out of solution") because it can not dissolve further, and any additional salt is then said as being insoluble. Rain coming from or falling out of the moist saturated clouds is also called precipitation. Precipitation is due to that there is not enough chemical energy left in the solution to dissolve any more of the substance such as salt. Perhaps this is how large salt deposits were made, and of which are sometimes mined for "table" or "food" salt and-or for industrial usage.

Granulated ("table") **sugar** has a solubility of about 1.8g / mL = 1.8g / cc =~ 54g sugar / 30 cc water =
54g sugar / 1 fl-oz water =~ 1.9 oz sugar / 1 fl-oz water. 1 fl. oz is actually defined as about =~ 29.5735$\underline{3}$ cc =~
30 cc. 30 grams of mass is often said as being a "food ounce".

Diluting and-or lessening an acid or base

If you add plain water into an acidic or base solution, you will dilute and weaken the concentration and ability of that acid or base solution. The pH of that solution will then change so that it is less of an acid or base and the pH of it will approach a value of 7 if enough water is added into the solution.

As the temperature of an acidic solution increases, it (ie., the atoms, ions, molecules) and any substance in that acid becomes more energized by the heat and become more chemically and-or electrically active. The protons and electrons become more mobile and potent with the higher kinetic energy, and can then dissolve things easier and faster as if more or a stronger acid was added into the solution. The heat energy can cause electrons to have enough kinetic energy to become free electrons, and or weaken the bonds of molecules so as the acid is more effective to dissolving them.

Adding **table salt** (Sodium-Chloride molecules, with the chemical name of NaCl) is a hard crystalline structure that is a compound or combination of sodium and chloride molecules) into a water solution is not known to change the pH of that water solution. Water will dissolve (here have enough electric forces to break apart the salt molecules ionic bonds) the salt into sodium ions (Na +) with a net positive electric charge, and chloride ions (Cl -) with a net negative electric charge. The (H2O) molecules of water will cause the sodium-chloride molecule to break apart (dissociate) into separate ions. The net positive hydrogen atoms or protons on the water molecule will be attracted to the net negative Cl ions. The net negative oxygen atoms on the water molecule will be attracted to the net positive Na ions.

A health tip about pH and acidic foods:

Some foods we intake (ie., eat) are slightly acidic (acid-like, having a low pH) foods, and in order to obtain the healthy nutrition in them, we need to eat those foods and-or use nutritional (vitamins, minerals = elements) supplements. In general, most naturally acidic foods usually have a pH just below that of water, which has a pH of 7. These acidic foods might have a pH rating between 5 to 7. Try not to eat and drink too much of foods that have high acidity. Some foods such as lemons, limes and many spices are known to be acidic, however, science has shown that some acidic foods effectively become as an alkaline foods in the bod after some digestion and-or processing by the body has taken place.

Bacteria are said to thrive (ie., grow, multiply) in an acidic environment, and also like to eat sugars of various types. High amounts of bacteria and sugar can be detrimental to health, such as causing tooth decay (ie., cavities or holes in our teeth). High amounts of sugar can also cause health problems due to an imbalance (often to much) of sugar in our bodies which is usually known as diabetes. The amount of sodium level in our blood can also become in balanced (often too much). We need salt to function properly, but either too little or too much can cause a health problems.

Acidic foods can also cause a chemical reaction on your teeth surface and cause them to decay and-or discolor.

Due to the above knowledge, a health tip or advice would be to rinse your mouth and teeth with plain, common water after eating a meal or snack. This will help remove debris or particles of food, and reduce the concentration and-or pH of any acids in your mouth which can damage the surface of your teeth and-or aggravate mouth ulcers. Brushing teeth and flossing (with a thin cord of material specifically made for that purpose) between them is also recommended. Many dentists also **recommend not rinsing your mouth out immediately after brushing with fluoride enhanced toothpaste** because it takes some amount of time for the fluoride substance to more fully interact with your teeth and obtain the most amount of health benefit. After brushing your teeth properly, let most of the matter in your mouth go down the drain, and then rinse your mouth and teeth after a few minutes, and then try not to eat or drink for 1 hour of elapsed time. There are various mouth and-or teeth rinses available at a relatively low cost, and only a small amount is needed at a time, and these are used as an additional supplement to brushing and cleaning your teeth.

Low pH mouth washes or rinses are also available, and if it not available, a small amount, perhaps a gram, of baking soda (sodium bicarbonate) in a cup of water will create a water solution with a high pH, hence a base solution. Do not drink this solution either because it does have some sodium in it due to the sodium bicarbonate in the baking soda. It is best not to use baking powder for this use purpose because it has some additional materials in its mixture, such as a trace amount of aluminum, and this type of powder is usually used to help bake some foods such as cakes, and if yeast was not being used, however, baking soda can be used, and of which already comes included with an acidic ingredient that will help release carbon dioxide bubbles from the baking soda (a base) when it is activated by the wet ingredients and acid. How much sodium is baking soda? Baking soda will have about 1280mg = 1.280g of sodium per 5cc or 1 teaspoon of volume. The RDA (Recommended Daily Allowance) of sodium or general guide is about **2300 mg = 2.3g total throughout the day** for an average working adult, and unless a doctor and-or nutritionist recommends otherwise. For some reference, a volume of about 5cc volume if water =~ **1 teaspoon of table salt** has about 2300 mg of sodium. 1280g / 2300g = 0.56, and this is over 50% of the RDA of sodium, and in just 1 teaspoon of baking soda. Because baking powder also includes a large amount of corn starch, the sodium content in it is 520 mg per teaspoon, and this amount is (1280mg - 520mg) = 760 mg less per teaspoon of volume. In terms of percentages, it could be said that the amount of sodium in baking powder is (520 / 1280) = 0.41 = 41% that of the sodium in baking soda, and-or that the amount of sodium in baking powder is (100% - 41%) = 59% less than the amount of sodium in baking soda. Be aware that other foods you eat may have sodium and you do not want to continue exceeding 2300 mg per day as an adult. People less than adult age, and smaller in size will obviously not usually need the mentioned adult RDA of sodium mentioned. 1 teaspoon of **table salt (NaCl)** will also contain the adult working male, RDA of **chlorine** which is about **2300 mg**. A level teaspoon of water weighs 5g, and a level teaspoon of fine salt weights about 6g. **With these numbers, a total of 4600 mg / 6000 mg = 0.77 = 77% =~ 75% or (3/4) of a teaspoon of level table salt will provide the necessary RDA of table salt for an adult working male.**

Some dentists can prescribe a tooth paste having a high amount of **fluoride** in it if your teeth could use an extra amount of it. They can also prescribe a special dental rinse and may contain **chlorhexidine** to help eliminated bacteria that can cause problems, either before or after a dental procedure. A dentist may also prescribe a 10 day round (ie., supply and procedure) of antibiotics of some type of Penicillin anti-bacterial such as Amoxicillin. It is difficult to avoid all cavities, but preventing most of them is possible. Try to have a regular dental checkup. Modern dental work is usually pain free with new forms of pain reliever available, and if requested by you, even your gums can be ("topical") numbed before any teeth "scraping" and-or flossing between teeth to remove plaque is performed.

Here are come common substances and their pH:

Substance pH (typical , average)

gastric acid , stomach acid , 2.0 on average
vinegar , white household , 5% acetic acid , 2.5 typical
lemon 2.5 : due to citric acid often found in citrus fruits
lime 2.5
soda 3.2 typ , the acidity is usually due to the added carbonic-acid to make the "bubbles", and the phosphoric acid. Your body will try to neutralize this with alkaline molecules if its able to, but otherwise it still is a significant acid, and it is recommended to not drink too much soda with a low pH which can cause **tooth erosion** and potential gastric issues like "acid reflux (gerd), "heartburn" or aggravate an ulcer. Regular Coca-Cola (R) soda has a low pH for soda, and is between 2.52. Pepsi (R) soda has a pH of about 2.53 Dr. Pepper (R) has a pH of about 2.89. A few of these sodas throughout a day of time is generally not a significant health issue.
tequila 3.2
white wine 3.3
grapefruit 3.4
apple 3.6
red wine 3.7 (typical)
grapes 4
beer 4.5 (typical) , (3 to 5) , hence acidic
coffee , no milk added 5
coffee , with milk added 6
tea , black no milk added 5.2
green tea , 7 before digestion, and 8.5 after digestion , and this is then said as being "alkalizing".
milk , 6.5 , slightly acidic
cheese 5 (typical)
white flour 5.5
various pepper plants are 5 to 6 before digesting, and can be 8 after digestion in the body
whole wheat flour 6
potatoes 5.5
garlic 6
carrots 6.2
cinnamon , before digestion 5

isopropyl alcohol 6 , "rubbing alcohol" , for household use such as a solvent and disinfectant , hence slightly acidic , and this is not for consumption like ethyl alcohol is. Avoid breathing its vapor, and contact with skin other than for short duration - war gloves if needed for longer duration of use. Alcohol can be flammable in high enough concentrations. It can be diluted with water, but this may reduce or negate its disinfectant ability. Chemical formula is: $C3H8O$, **Do not drink this type of alcohol**.

vodka , 6.5 (typical) when at 40% (edible in moderation or small amounts, **ethanol**) alcohol per volume, and with the remainder 60% usually being water. Do not drink this when concentration and focus is important, such as driving a car, operating machinery, taking care of others, etc, due to its sedative effect that can dull senses, lower awareness, and may produce a type of calming effect and-or bad behavior. In general, alcohol is not prescribed by health care as a medicine of any sort.

water 7 , distilled, pure water with no trace of other elements , makes a good pH reference calibration for a meter
When pH is 7 in a substance, the amount or concentration of hydrogen ions or protons (H+) is equal to the amount or concentration of hydroxide ions (-OH), and therefore, the number of these two different types of ions are equal and are then said to be in balance with no net amount of either type. The solution containing these ions is then said to be neutral, and the substance has no net or significant acid or base activity, and-or that their activities are equal in value. Consider that water can soften and-or dissolve many organic substances.
(number of hydrogen ions) - (number of hydroxide ions) = 0 in a neutral solution or substance.
Note that a neutral solution such as water still contains these two different types of ions.
Water is a common substance to dilute another substance so as to change the pH of that substance, either an acid or base substance, so as to have a pH closer to 7, but not necessarily 7, and this dilution by water process can not convert an acid into a base, or a base into acid.

table salt 7.0 , sodium-chloride molecules , a necessary nutrient , such as for digestion and body processes, however too much can cause or aggravate health issues such as hypertension, and some kidney problems

cocoa powder 7.3 , after it is reduced in acidity during its manufacture
human body internally , blood, 7.4 (common, average, typical) , hence **slightly alkaline**

cinnamon 5 ,after digestion 8.5
honey 5
spaghetti sauce or pizza sauce 5.75
salt 7
salt with iodine , manufactured to be about 7.5 before digestion, so as to stabilize the volatile iodine atoms from "gassing off" (evaporating).
salt water 7
table sugar 7
sugar water 7
rain 5.3 (typical, average)

soda pop soft drink 3 (typical, average) , it is **acidic** , many soda pop drinks contain phosphoric acid
apple cider vinegar 2.5 ,vinegars contain acetic acid that is made from a certain bacteria (acetic acid bacteria) added to alcohol during the fermentation process to make and-or convert it to the vinegar. Obviously then, this bacteria can thrive in the **acetic acid** of vinegar. "Mother of vinegar" is a safe cellulose substance often seen at the bottom of some unfiltered apple vinegar bottles, and is infused with the bacteria which if uncontaminated, it is often deemed as a healthy "gut bacteria".
hydrochloric acid 0

toothpaste 8 , typical, can be 7 or higher (less acidic), and this high pH means less acidic and so as to protect the (protective) tooth enamel coating from dissolving or eroding such as an acid would cause. The same can also be said for **mouthwash**. In general, avoid using toothpaste and-or mouthwash (or mouth-rinse) with a pH less than 7.0, hence avoid acids from damaging your teeth. If you do not have toothpaste available, you can rinse your mouth out with water so as to help neutralize any remaining acid or base in your mouth, especially after eating .

baking soda 8.5 , and this is measured when a specific amount of it is dissolved in a specific amount of water, and at a specific temperature of say 25°C for example. Temperature changes will affect the energy and-or chemical reactions, and the result of the pH measurement.

A pinch of baking soda added into a cup (8 fl-oz = 0.247L =~ 0.250 L = (1/4) L) of coffee is enough to neutralize (make it ineffective as an acid) the amount of acid in that amount of coffee so as it will have a pH of 7. This will also add a few (about 10) milligrams of sodium into your diet. If some extra water was then added into the coffee, it would also change its pH , but then it may not taste as good, but it is a solution to reduce the amount of acid in that cup of coffee. **Since the reference pH measurement for coffee was taken with these specific conditions (amounts, measurements, sizes, temperature, etc), pH measurements of cups of coffee should then be taken using those specific reference of measurement conditions, and this needs to be considered for other substances other than coffee.**

antacid , 9 typical, (8 to 9) , often used to neutralize excessive stomach gastric acid, but note that stomach acid is necessary for proper digestion of food. Some table salt (sodium-chloride) in the diet is also necessary so as to make that stomach acid and for other vital biological functions. There are some medicines which can also prevent excessive stomach acid production so as to then prevent excessive "hear-burn", "acid reflux" or "gurd". Some of these can be bought at a common store, and some others may require a prescription from a healthcare worker.

ammonia 12 , as a household cleaner

bleach 12 , chlorine bleach, stain dissolver/remover , common household cleaner and disinfectant that usually has a 6% to 8% concentration of sodium-hypochlorite in water. This high concentration of sodium-hypochlorite solution in "concentrated household bleach" is then often further diluted with water so as to make a less caustic and or corrosive cleaning and-or disinfecting ("germ fighting") solution, and a typical solution of this has 3% being bleach, and 97% being cool water. Do not add any other substances into a bleach solution. Use proper cleaning gloves and adequate ventilation when using bleach so as to help remove any dangerous chlorine gas it produced.

lime 12.5 , calcium carbonate , limestone , a strong base substance, can cause caustic (chemical reaction, releasing heat) **burns**

lye 13 , sodium hydroxide 14, **caustic** , a strong base substance , often used to make soap , also used as a drain pipe cleaner/unclogger/dissolver.

Although **soap** is often made using sodium hydroxide as one of its ingredients, the desired and safe pH of the hard soap is less than 10 and greater than 8, and with 9 as the average, and this is achieved by adding in the lesser pH of the oil and water ingredients when making that soap. Liquid soap usually has even a lesser pH at about 7.5 due to the higher amount of water in it. If you make soap, please obtain a pH meter and-or litmus paper strips to determine the pH of your soap so as to not be caustic to the skin. Ivory (R) soap has a pH of 9.5. Although the pH of a persons skin varies slightly from region to region, The average pH of human **skin** is 5.0, hence it is slightly acidic, and this helps reduce bacterial growth on it. When the pH of the skin is lower and-or more acidic, the skin can dry out when its natural oils are removed. A soap having a high pH can also dry the skin out, and hence it is recommended to use it quickly and-or use a soap with pH not much different than the pH of our skin. Certain skin diseases may require a special soap and-or pH so as to not irritate your skin.

pH and associated knowledge is very important in modern and-or advanced farming (many fruits and vegetables grow well in soil with an average pH of 6), chemistry, medicine, and many other sciences. pH measurements of each ingredient and result of a chemical reaction can help ensure it was done properly and-or obtained the desired and-or standard result expected.

Please read the following cautionary note about chemicals:

Many chemicals are potentially, in dangerous in some way, either by themselves and-or when mixed with another chemical, and they can very rapidly cause intense heat and-or an explosion during a chemical reaction of some sort: either expected or usually unsuspected. Please take the time to acquire and use eye protection and accurately follow the recipe and-or steps that scientists have previously developed so as to avoid injury and other problems. Scientists and chemists have much proper equipment available, knowledge and training so as

to make and do experiments more safely and avoid injuries and-or damages. Please label and store chemicals and other dangerous products in a safe and secure location. Do not touch, inhale, eat or breath unknown and-or potentially dangerous chemical substance, for even small amounts of some can cause health problems. Research each chemical and-or mixture to know what procedure(s) to do if an accident happens and-or how to dispose of any waste materials. If you have an issue, please seek medical help as soon as possible.

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SOME HELPFUL ELECTRONIC DEVICES AND FORMULAS

In 1799, at the dawn of the 19th century A.D., **Alessandro Volta** (1745-1827), from Italy, invented the first electrochemical device to produce electricity. He called this fundamental structure and unit of his device a **cell** (ie., container, compartment). A cell consisted of two different types (elements) of metals, such as copper and zinc discs, perhaps 1 inch (= 2.54 cm) or more in diameter, separated by a thin (liquid of which) **electrolyte** (such as for example: salt water, lemon juice, vinegar or other acid, for a liquid "wet cell") which is somewhat electrically conductive and allows the flow or transfer of charged particles (negative electrons, and positive ions) from and between the two pieces of metal. More technically, his cell is also known as "galvanic cell" found at about the same time by Luigi Galvani. This cell undergoes a process or chemical reaction called redox which is an oxidation (electron loss, oxidized, ex. iron oxide = "rust") and reduction (electron gain, or less oxidation, reduction in total positive charge) of the electrons in the two types of metals used, and of which are at two different electric states or potentials naturally due to the number of free electrons and-or weak electron bonds in their atoms. Even if an atom looses an electron, and not due to an oxygen atoms(s), it is still generally called **oxidation** so as to convey what has happened much like steel or iron rusting due to moisture (ie., water) and-or air (ie., oxygen).

By placing or stacking (piling) these cells in series or "pile" ("voltaic-pile") with each other, Volta invented the first **battery** (a multitude, or array of individual cells) that produced a higher voltage equal to the sum of voltages of each voltaic cell in the pile or series of cells. Many years after the discovery of the cell, it was found that electrons in the cell are essentially placed in and will pass through a higher electromotive field and will gain more kinetic energy called electric potential energy or "voltage". A battery also creates a more practical, controllable, reliable, and a reasonably constant amount of current ("electricity") which was very practical and useful for the many new discoveries and inventions that used electricity.

Unlike a "primary cell", a "secondary cell", such as a common lead-plate in acid ("lead-acid") battery in an automobile, can be recharged in a reverse type of manner to how it is allowed to generate electricity (electrons) and discharge it through an external electrical path or circuit. **In general, a battery is commonly known as a device to store electrical energy.**

A "chemical" battery, such as an alkaline battery uses the chemical energy of a paste placed between the electrodes so as to create current. An alkaline battery is a (sealed) "dry cell", and the electrolyte is a (chemical) paste consistency and where the electrolyte is not an acid pH and water mix, but a base pH chemical substance. One electrode in common alkaline batteries is zinc metal, such as used for the outer can or container structure.

In about 1827, after a few years of experimenting with the recently developed (voltaic) battery, **Georg Ohm,** in Germany**,** realized that when a voltage (V) was doubled, the resulting current (I) doubled, hence a direct and linear or proportional (ie., at the same rate of increase) relationship, which is sometimes called as having a "direct proportionality" between current and voltage. Ohm also noticed that when a resistance (R) doubled (by 2), the current was divided in half (by 2), hence there also a real physical and logical mathematical relationship between current and resistance, but it's an inverse linear relationship. As resistance increased, the current decreased by the same factor. These facts or relationships were summarized and expressed with a simple and most useful mathematical equation known as Ohm's Law:

$$\textbf{Current} = \frac{\textbf{Voltage}}{\textbf{Resistance}} \quad \text{or} = \quad \mathbf{I} = \frac{\mathbf{V}}{\mathbf{R}}$$

: Ohm's Law **Units: current = amps , voltage = volts , resistance = ohms**

Hence mathematically: **V = I R** **and** **R = V / I**

Resistance (R) is that which impedes (reduces) or prevents the flow of current. According to Ohm's law, current (I) is inversely related to resistance: I = V / R. For the total resistance of **resistors (R) in series**:

Two or more resistors in series essentially creates a greater resistance equivalent to a singer larger valued resistor:

Rt = R1 + R2 + **. . .** **:** series resistors or resistance

For the total resistance of **resistor in parallel**:

$1 / Rt = 1 / [(1/R1) + (1/R2) + . . .]$: parallel resistors : this was derived previously in this book

For just 2 resistors in parallel, the formula simplifies to:

$Rt = \dfrac{R1R2}{R1 + R2}$: Total resistance of 2 resistances in parallel. The net or total resistance is always lower than the least valued resistance. Two resistances in parallel reduces the total resistance of the circuit path because there is more conduction area or "room" for current to flow. There is a similar formula for two capacitors in series or two inductors in parallel.

A resistor can be thought of as having the qualities of an insulator and conductor, and which results in a low conductivity material that has a lower number of "free", "loose" or "unbound" electrons to help conduct current than a typical metal conductor has. Electrons passing through a resistor also have a higher chance of colliding with atoms and will then loose kinetic energy as heat energy. This wasted energy is therefore a power loss (joules/time = watts). In a certain way, a resistor is a type of semi-conductor. Given a resistance with a certain voltage across it and current through it, to effectively force the same amount of current though a higher valued resistance will take a higher amount of energy, and that is a higher electric potential energy, potential energy difference or electromotive force called voltage.

For the **capacitance of a capacitor (C)**

A **capacitor** is a relatively simple electronic device that can store an amount or capacity of electric charges, and that is why it is called a capacitor. Basically, the amount of charge it can store is mainly determined by the size of its two plates of metal or metallic-like substance. **DANGER**: A capacitor may already have energy stored in it which needs to be released, transferred or discharged through a relatively low resistance (ex. a 1ohm to 5 ohm power resistor) across its terminals before handling it.

A capacitor can store charge (ie., electrons and positive ions) on two parallel and separated metal plates (such as two pieces of aluminum foil separated by a sheet of thin paper, which could then be rolled up), much like a rechargeable battery. One plate will hold electrons, and the other plate will hold positive ions. The charges are held in position by the applied voltage across those plates and-or electrostatic forces when the voltage is removed. Since the charges are stored, they will have electric potential energy and have a voltage across the plates. Capacitors therefore can accumulate and store energy much like a battery stores energy.

Charge is essentially forced or driven onto the plates with a voltage (electric, electromotive force, emf, applied to the charges giving them kinetic energy to move or flow). The energy of the charge will be stored or maintained in position by an electric field potential (emf), and capacitance is a measure of its storage ability and will have units called Farads. The stored charge will then have a voltage (emf = electromotive force, potential ability) between them, and this internal voltage can then send those charges (ie. electrons) as an electric current through an electric circuit so as to balance those charges to 0.

As a capacitor is charging up (with current, electrons) due to a supplied voltage source, the internal emf across its two plates will be equal to that supply voltage and will essentially prevent any more charges from accumulating on its plates. This is sometimes called a "back-emf" or "(opposite measured polarity) reverse voltage". As this reverse voltage increases, it effectively decreases the net voltage applied to the capacitor, and therefore the current or charge to the capacitor will decrease. During charging, the repelling force due to the charge already on the capacitor is preventing more charge from being stored.

The time it takes a capacitor to charge up to the supply voltage is determined by the resistance in series with it, and which will essentially reduce the charging current or slow the rate of charge. As a "rule of thumb" or very close approximation, a capacitor is considered as charged or discharged in 5 "time constants", where 1 **time constant** is equal to: (Rohms series resistance between the charging voltage and the capacitor) (Cfarads) with units of seconds of time. Time of complete charge or discharge = **Ts = RoCf**

In about 1745 a crude type of capacitor with charge storage ability was discovered by **Ewald von Kleist** in Germany. Not long afterwards, capacitors were called "**Leyden Jars**" (named after the Leyden university where some initial experiments took place) and were very helpful as part of the initial discoveries about electric charges. Since the battery had a few years yet to be invented in about 1799, charges were usually created using various types of hand powered friction (of two surfaces) machines, and then those charges were stored in a Leyden Jar. These machines could produce a very high voltage, but little current, unlike a battery. In just a short time later, these jars were being connected in parallel so as to be able to hold more charge because the (foil) plate area was now essentially larger as the sum of the plate areas. In about 1750, the famous U.S. statesman **Benjamin Franklin** took interest in electricity and Leyden Jars. Franklin called the arrangement of parallel Leyden Jars a "battery" which could possibly be interpreted as: "an array with increased power". and soon created a type of relatively **simple electrostatic motor** which had only a low power ability, but it still rotated and showed that electricity could be useful and needs to be studied more. Franklin discovered that objects are always charged up or discharged with the same electric or charge "fluid" or substance (ie., called electrons today) transmitted through or along the wires to other objects. In short, it was this substance that gives charged objects their charge because that substance itself is the charge, or lack of it, and that the state of an object being charged is due to that (charged) substance. Franklin also expressed a controversial thought that charge is stored in the dielectric material of a capacitor. Since the plates of these Leyden Jars were relatively far apart, these capacitors could store a high voltage before dielectric breakdown and self-shorting out and-or discharging, but only a low amount of charge or current.

Since electrons have the same charge type, they repel each other, and it will then take a stronger force to essentially increase the electron particle density or number of them in the same given area and-or volume, and this will mean a stronger electric force, hence a higher voltage will be needed. An analogy would be that since a spring will have an equal and opposite force, it will take more force to squeeze that spring closer together, and in return, more energy is stored by that spring. As a capacitor charges, the capacitor charges up to the supply voltage, and then stops because there is no effective voltage difference for the current to flow, but is rather repelled due having the same electric polarity and net (emf) force.

Capacitance in terms of its electrical characteristics:

$Cf = Qc / Vv$: Q = charges measured in Coulomb (c) units, Qc = stored charge in the capacitor
V = voltage across the capacitor's two plates
f = Farads, the units of measurement for capacitance which means its capacity (ie., room, ability) to store charge or electrons per applied volt ($V = J / Qc$).

The unit of farad (f or F) is named after Michael Faraday who discovered that various dielectric materials between the plates of a capacitor increased or decreased its capacitance.

By observing the formula, capacitance can then be thought of as the amount of charge stored per applied volt across its place.

Extra: $Cf = Qc / Vv = Q / (J / Q) = Q^2 / J$, also :
$Vv = Qc / Cf = J / Qc$
$Qc = Vv\, Cf$

Capacitance in terms of its physical characteristics:

$Cf = k$ (plate area / distance between the plates) $= k (Am^2 / Dm)$ $:= Qc / Vv$: this fraction resolves or reduces to: $Cf =$ (charge / 1 volt)

A is the area of a plate in unitsof square meters.
. **D** is the distance in meters between the plates.
k = **e** = (electric) permittivity or= dielectric constant of the dielectric material placed between the plates of the capacitor.

permittivity is a measure of how much the dielectric material can permit or be electrically polarized by the presence of an electric (force) field such as that extending between the charged plates. For this formula, (e) is an absolute, true or specific value, and not any relative (r) or "comparison" (ratio, magnification) value to that of a vacuum (eo) that has a permittivity of about: $eo = k = (Qc/Vv)(d/d^2) = Cf / m =$ about $8.8541\underline{9} (10^{-12})$ F / m
If a dielectric material has an (e) value that is twice as much, C will be twice as much. $er =$ (e of material) / eo : er = relative permittivity

A **dielectric** is also an insulator material so as not to "short out" (ie., effectively, electrically connect, low resistance path) the plates, and allows charge flowing from plate to plate, and the capacitor no longer has charge storage ability. .

Some **r**elative permittivity (E**r**) values of some various dielectric materials in a capacitor that have a temperature of the "room" or "comfortable living" temperature of 72°F = 20°C :

Here is a table of relative perrmmittivity:

material	**relative permittivity = er = (permittivity of material) / (permittivity of vacuum) = e / eo**	
vacuum	1	
air	1.00054 = ~ 1	: at standard pressure of 14.7lbs/in^2, value is slightly greater than a vacuum. Though an air capacitor does not generally store much charge and has a low capacitance, it offers fine tuning of oscillators and radio receivers.
mineral oil	~2	
wax paper	2.5	
vegetable oil	2.5	
mylar	~3	
paper	~3.5 av	
nylon	4.5 av	
mica	4.75 av	: good
glass	7 av.	
diamond	8 av.	
aluminum oxide	~9	: very good
zinc oxide	~10	: high
tantalum-oxide	~26	: high
ceramic	30 av	: very high
niobium-oxide	45 av	: very high
glycerin	57 av.	: very high
water	85	: extremely high , this will get lower in value as the temperature lowers
titanium dioxide	115 av	: extremely high : sometimes used in the dielectric of "super capacitors". : some other alloys of titanium can have much higher values

The permittivity of water has a wide range and varies much with temperature, hence it is possible to use this type of capacitor as a thermometer sensor so as to measure a temperature such as air, and comparing its measured capacitance to that capacitors corresponding temperature scale that was pre-made, calibrated and known.

The energy stored in a capacitor and-or equivalent amount of work (W) needed to charge it up to that amount is:

W = (Cf)(V^2) / 2 : with units of joules : this is similar to the kinetic energy formula KE = mv^2 / 2 , but here mass has been replaced with capacitance and velocity (v) has been replaced with voltage (V). Note: joules of work = joules of energy

The Cf or farad rating of a capacitor is an indication of that capacitors charge holding ability or capacity, and not specifically the amount of charge on its plates at any moment. The energy stored within a capacitor also depends on the voltage applied across that capacitor. The maximum safe voltage rating of a capacitor is the dielectric breakdown rating of a capacitor, and it will therefore determine the maximum safe energy storage in that capacitor.

Some other formulas for the energy in a capacitor are:

Since: Capacitance = Cf = amount of charge held on the plates per volt (emf) across the plates = Q / V:

$$W = \frac{CV^2}{2} = \frac{(Q)}{(V)}\frac{V^2}{2} = \frac{QV}{2} \text{ , joules}$$

Since: V = Q/Cf ,

$$W = \frac{QV}{2} = \frac{Q\,(Q)}{2\,(Cf)} = \frac{Q^2}{2Cf} \text{ , joules}$$

For the total capacitance of **parallel capacitors**:

$Ct = C1 + C2 + . . .$ **:** parallel capacitors, this formula is similar to that for series resistors

Capacitors in parallel is essentially creating a larger capacitor with larger plates.

For the total capacitance of **capacitors in series**:

$1 / Ct = 1 / [(1/C1) + (1/C2) + . . .]$: this formula is similar to that for parallel resistors

When capacitors are placed in series, we see that their total capacitance is less than any one capacitor, and this is due to the effective larger amount of dielectric material involved, and when the dielectric material has a larger width between the plates, the plates are farther apart, and the resulting capacitance is less. If the capacitors are identical, the total capacitance will be half that of any one, and the breakdown voltage will be twice that of any one.

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Capacitors in series will create a voltage divider. For an AC voltage and-or signal applied to these capacitors, the voltage across each capacitor depends on the reactance (Xc) of each capacitor to that AC signal, and of which depends upon the frequency of the signal. Vc = Ic Xc Smaller capacitors will have a higher reactance (ie., impedance, , AC resistance) to the AC signal, and therefore a larger voltage across it.

First, a voltage source will cause current to flow into and be stored by electrostatic (or "Coulomb") force in an "empty" or uncharged capacitor. The rate of this charging (Coulombs per Second = Amps) depends on the circuit resistance to the flow of the charging current. The lower this rate, the longer it will take the capacitor to charge up. The capacitor will stop charging when the voltage of the capacitor equals the source or supply voltage because there is no difference in voltage for the charge (current, electrons) to be electromotively forced to due so. Vc grows while charging and is essentially a "back-emf" that will begin to limit the flow of current (as if it was repelling it due to electrostatic forces) because the effective supply voltage or emf is reduced even though Vs usually remains a constant voltage source.

When Vc = Vs, their difference or (electrical or voltage) potential difference is 0v: (Vs - Vc) = 0v, and there is no net voltage potential or available electromotive force condition to cause current to flow and the capacitor is done charging.

The effective (Ve) supply voltage = voltage potential difference, "emf" or "pressure" = (Vs - Vc).
Since Vc = 0v at t=0s of charging, Ve = (Vs - 0v) = Vs

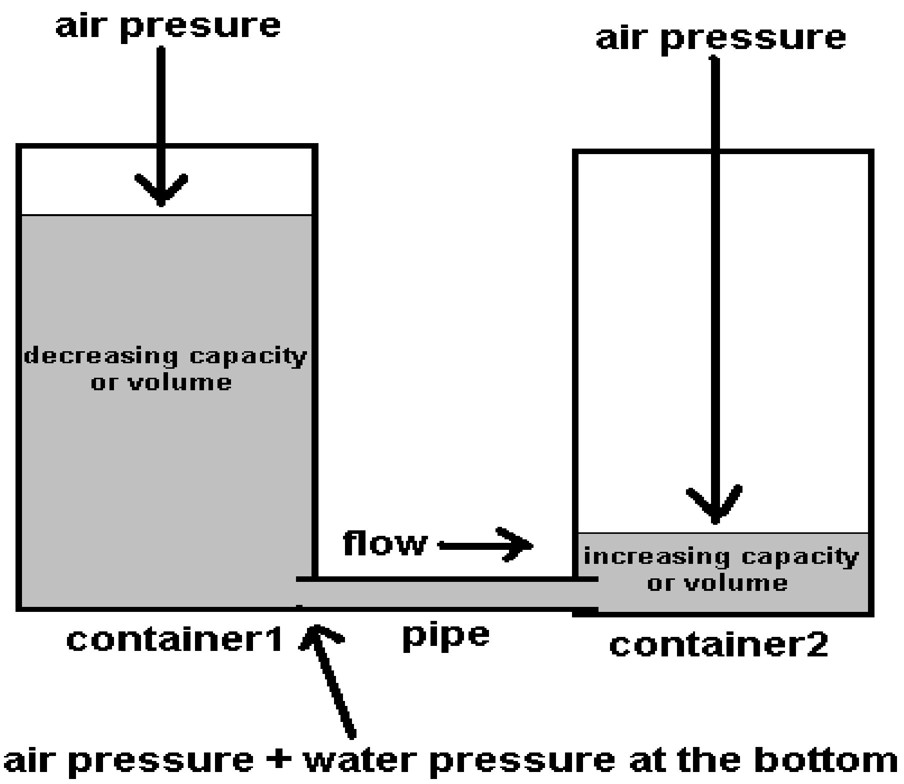
While the capacitor is charging, Vc will increase and Ve = (Vs - Vc) = effective potential difference, will decrease.

When Vc = Vs , Ve = (Vs - Vc) = (Vs - Vs) = 0v and current will cease to flow, and the capacitor is done charging.

The capacitance (Cf) rating of a capacitor is due to how much charge (Qc) it can hold on its plates due to the voltage across those plates. If the plates are small, then the amount of charge that it can hold will be reduced since like charges (ie., such as electrons) on the plate will repel each other, and it becomes difficult to place more charge onto smaller plates. Qf = Qc / V. Given an applied or supply voltage of Vs, the amount of stored charge in a capacitor is: Qc = Cf Vs . In the analogy below, an empty capacitor is charged up from a charged capacitor and its voltage will be half of that capacitors voltage, hence (Vs/2). The charge in that empty capacitor will become: Q = (Cf)(Vs/2). The maximum possible rated charge before a dangerous situation of the capacitor having "voltage breakdown" or conduction across its plates is: Qmax = (Cf)(Vmax rating).

Capacitor-water analogy for charging and discharging

Initial considerations for this analogy come from the water analogy of electric current flow being like that of water flowing like a current in a pipe. The water movement is influenced by a force and pressure at one end, and this is analogous to the voltage applied to a conductor (wire). The pipe diameter also determines the volume or the amount of the flowing water, and is analogous to the resistance between the two capacitors. This analysis below for a capacitor is mainly to understand the shape of the curve of how it charges, and the full analysis may not be perfect since the final water pressure and-or height of it is only half as that of the source. That is, when a capacitor charges, it charges up to that of the constant voltage source (emf, or pressure) applied to its input terminals, and not half of that voltage source. Still, the analogy described below is useful in several ways, and when considering two capacitors in parallel to each other, with one charged, and the other empty. [FIG 260]



In the above figure, each tank (with any maximum volume, size, shape or "capacity") represents a capacitor with a certain maximum capacity or volume. In this figure, container1 was filled up and container2 was initially empty. The amount or the volume of water present in each tank represent the total amount of electric charge in each capacitor. Note that a greater volume of water does not necessarily mean an increase in water pressure unless the height (ie., the depth of it) of that amount of water is also increased. Also, any surface of water on Earth will have the standard air pressure of 14.7psi applied to it, but if this is applied to both tanks of fluid in this system, these pressure values negate each other, adding to 0 net pressure to the actual fluid flow.

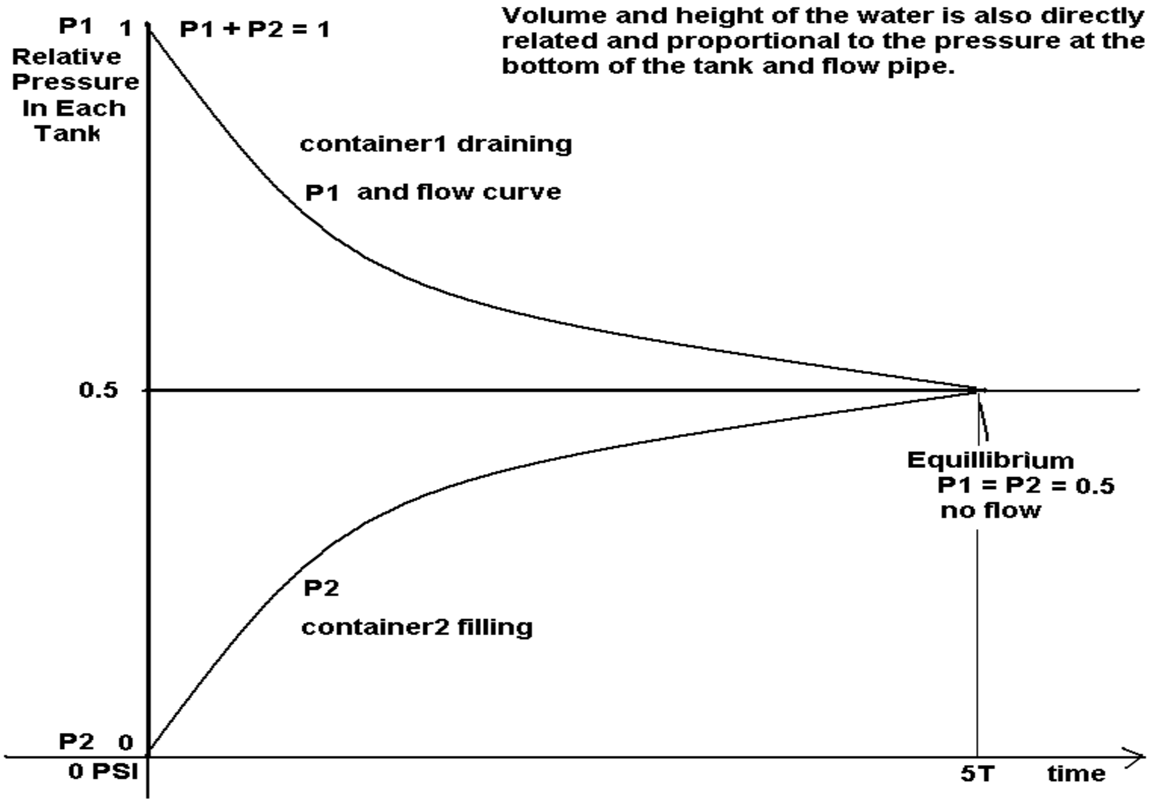
Initially, and when charging, at the bottom of container1, there is an higher pressure than at the bottom of container2 since the weight of the water also applies pressure to the bottom of the tank where the pipe is. There is a negligible difference in the water pressures of each tank due to the air pressure above, and it will not be considered further in this analysis, and in fact the two air pressures are of the same value and effectively cancel each other out mathematically in terms of net pressure and-or difference in pressure. Due to the volume and weight of the water in each tank, there is a difference in water pressure or force at each end and opening of the pipe. Here, the pressures corresponds to voltage and the difference in pressure or net effective pressure corresponds to a net or effective voltage difference. If there is a path to flow, the higher pressure (ie., a force) water will flow to a lower pressure region such as in container 2.

The volume of the tank represents the maximum amount of water that it can hold before overflowing and-or not allowing any more equal or lower pressure water into it if it was sealed with the filling pipe going into it, and this is analogous to the maximum amount of charge that a capacitor can hold or store before being damaged.

Water pressure and voltage pressure are analogous since they induce the flow of current, such as water and electrons.

During the discharging of container1 into container2, the water pressure is decreasing in container1 and increasing in container2. The pressure of the water in container2 is increasing and the difference in pressure is getting smaller and smaller till there is no pressure difference at the pipe openings, and the flow will cease. It could be said that the pressures are at equilibrium and have the same value.

Initially the flow is high but decreased to 0 due to the lack of a pressure difference. Initially the amount of water in container2 was 0, but increased till the pressure of the water in each tank was equal. It is not too difficult to imagine that when the pressure of container2 is half that of container1, that the flow is then half. Pressure against a flow is like a resistance to that flow, and the net effective or difference in pressure causing that flow is then reduced. [FIG 261]



When (e) is used for these equations, the formulas could probably be written to be somewhat similar to that of the catenary curve. The flow-rate (ex., gallons / hour) depends on the pressures involved which depends on the volume of water. The maximum flow-rate is at maximum effective pressure or "pressure difference" when t=0. The time till the pressures are the same or at equilibrium depends on the flow-rate and the specific volume of water available in container1 at t=0, and this is why non-specific or "relative values" (ie., percents, fractional values) were used in the above figure and analysis.

It was indicated that the pressure at the bottom of a tank is directly and proportionately related to the volume or height of the water in a container or tank. With two containers and a pipe system, the effective pressure at the bottom of the tank is due to what water remains in both tanks. Effective pressure is the water pressure difference of the two containers. **Because of P2 affecting and reducing the flow more and more as it grows, there is no linear-like relationship to the water height, volume and pressure of both P1 and P2.** In general if, container1 just drained out, there would be no back pressure and the flow would have a constant reduction with respect to time. In an electrical circuit and-or capacitor, this back pressure is analogous to a "back-emf" or emf of a reverse polarity, reducing the net voltage available to the sum of those two voltages.

The total time it takes till the flow stops could be set equal to 5 time constants = 5T. During the first time constant, the flow rate (= volume or amount **/** time) is relatively high, and then it decreases for each unit of time.
Ex. Below is a figure to verify this discussion. Two empty plastic bottles were hot-glued together along their height side and with some extra supporting pieces of wood. The actual volume does not matter for this initial experiment, but it can be used later so as to have some actual (rather than relative, portion or percent) values if needed. The bottles had a volume of about 2 liters each. Between the plastic bottles is a plastic tube with nearly 1/8" inside diameter. This tube was from a used spray bottle. The tube ends were place close to the center of the bottom of each bottle, and also hot-glued into position between the bottles and to prevent water leakage. A bottle cap is initially needed on the second bottle so as to help keep the water in the first bottle from entering into the second bottle. [FIG 262]

Initially, as a construction test, the cap was removed from the second bottle and it would fill up so as to be at pressure equilibrium (ie., the same pressure) with the first bottle. A line was drawn on each bottle at this height. With a ruler, also measure half this height and mark that line on each bottle. Use a timer and-or calculate the number of seconds until equilibrium is reach. Be patient and try to wait till they are exactly equal for it will help in the accuracy of the calculation. Due to a theorized natural flow, increment, decrement of the values in this type of system being analyzed, let's set this number of seconds till equilibrium is reached equal to 5T = five time constants, and after this amount of time and changes become negligible and are practically 0, and e^(-t/T) = 1/e^(t/T) = 1/e^(5T/1T) = 1/e^5) = 0.00674. Dividing this 5T value by 5 we will have the time value of one time constant (T) for this particular system of natural forces or pressure, and whose formula for analysis will include the natural number (e).

During this experiment, equilibrium took 11minutes = 660seconds = 5T.   Ts = 1Ts = (660s / 5) = 132s

For the volume (V2) of the second bottle being filled from the water (Vt=total volume) in the first bottle:

$$V2 = Vt\, e^{(-t/T)} = \frac{Vt}{e^{(t/T)}}$$

Using relative values, we can assign V2 = 100% = 1 at equilibrium and the maximum value of V2 has been reached. We can set V2 equal to half of 1 = 1/2 = 0.5 so as to find the time it will take to reach half of the equilibrium volume or final value of V2.   This value should then match that of the actual experiment, and it will also bring some verification to the formula used. Note that the actual physical volume of V2 at half its maximum value is only 1/4 of the total volume of V1=Vt at the start of this system.

$$0.5 = \frac{1}{e^{(t/132s)}}$$ solving for the indicated power of e:

: or = $0.5 = 1e^{(-t / 132\ s)}$, though this seems like a decaying function for V2, the reciprocal of it is indicated, and which therefore indicates a growing or increasing value.

$$e^{(1t/132s)} = 1/0.5 = 2$$

$$e^{(0.007575758t)} = 2$$ taking the ln of both sides:

$$\ln e^{(0.007575758t)} = \ln 2$$
$$0.007575758t = \ln 2$$ solving for (t):

$$t = \frac{\ln 2}{0b007575758} = \frac{0.693147181}{0.007575758} \approx 91.\underline{5}s$$: this value was verified when the experiment was performed again 91.5s ~= 1minute and 31 seconds = 00:01:31 of time

If 1 time constant = T = 132s, how many time constants is 91.5s where half of the final volume of V2 has been filled? Setting up a proportion or equivalent fraction:

$$\frac{1T}{132s} = \frac{xT}{91.5s}$$ after solving for xT , xT =~ 0.693T =~ 69.3% of one constant (T).

t/5T =   91.5s/660s = 0.1387 = 13.87% of the total time of filling, hence (1 - 0.1387) = 86.13 = 86.13% of the total time of (5T= 660s) remains until the equilibrium condition. 5Ts - 91.5s = 660s - 91.5s = 568.5s = 9m and 28s

It will take about (5T - 0.693T) =~ 4.30T more for the remaining half of V2 to be filled and the equilibrium condition exists, and this amount of time is:

4.30 (T) = 4.30(132s) = 568s = 9 minutes and 28 seconds , and this is (568 / 91.5) = 6.2 times longer than the first half of the volume of V2 being filled.

From the above analysis, if a container is filling under similar pressure conditions, and if it is halfway filled, then that amount of time is 69.32% of 1 time constant.   If the time to reach this half (0.5) amount of the final volume was 91.5s, then one time constant is then: 0.6932T = 91.5s , and after solving for T = 91.5s/0.6932 = 132s. This type of analysis can also be done for other fractions of the final volume, such as for 10%, 1% of the final volume.

For relative amounts in each container: Vt = 1 = V1 + V2   and   V1 = 1 - V2 = 1 - $e^{(-t/T)}$ and:

At (pressure, and volume) equilibrium: V1 = V2 , Vt = V1 + V2,   1 = V1 + V2,   1 = V1 + V1 = 2V1 ,   therefore mathematically, solving for V1, we find V1 = 0.5 = V2 at equilibrium.

Note that the amount or volume of water removed from one bottle will be added to the volume of the other bottle. Likewise, since the bottles are similar, the height decrease in one bottle will be a height increase by the other bottle. It could also be said that with this system, the sum of volumes is constant, and the sum of heights is constant.

If it is impractical to make the above device shown in the figure, a simpler device for experimentation with volume, pressure and time would be a plastic container with a 1/8 inch hole near the bottom of it. With this system, there would not be any effective "reverse" or "back" pressure from another container and therefore, the flow rate does not decline as much. With this system, the water analogy of the capacitor is not present, but the flow rate still decreases from the reduced water pressure of the reduced volume of water as time increases. The water volume, pressure and size of the hole determines the output flow rate, and therefore the total time until the container is empty. Since there is no "back" pressure reducing the flow, the volume will drain faster as compared to the system noted above. For a reasonable, but technically incorrect, linear-like estimate for the remaining volume left in the container, you can take the total time and divide it by 7 [6.6 to 7 is close] so as to have the time of the first quarter (1/4 = 0.25 relative value = 25%) of the volume drained, then double [2 is ok, and 1.9 is a better value] that time to find the time of the next quarter (50% of the volume drained off) and so on. For example: The first or upper (highest) quarter of the volume took 18s to drain out , the second quarter will drain off after about (2)(18s) = 36s. The third quarter will drain off at about (2)(36s) = 72s, and the remaining fourth quarter will drain out at about (2)(72s) = 144s. If the first quarter of the total volume took N seconds to drain out, the estimated total time till that total volume has drained out is: total time = (7)(Ns). A basic formula for an estimation of the percent of the volume left after time past is: % of volume remaining = 1 / 6.6^((1.2) time / total time). The formula is more accurate for the higher percentages of the volume remaining, but it can be adjusted after more experimentation. Solving for total time: (total time) = ($2.264483\underline{6}$) (time) / Ln (1 / percent remaining).

Another similar formula is: % of volume remaining = 1 / 9^(time / total time) , or:
%Vremaining = 1 / e^((2.4) time / total time) , or: %Vremaining = 1 - ((1.56) time / total time) which is a good approximation for up to 65% of the total time.

With the knowledge presented in this overall discussion, we can apply it to the flow of charge or current in a charging or discharging capacitor: With the above water analogy, volume two, (V2) will fill up to half of the initial value of the total volume (Vt). For a capacitor, the voltage across it, (Vc), after "filling up" or charging is equal to the supply voltage (Vt).

When an empty (of charge and voltage) capacitor is first charging, the flow rate (ie., amperes) of current is high, but then decreases due to that the effective voltage or potential difference (like a difference in pressure) causing that flow is decreasing. This constantly decreasing flow rate will then effectively increase the time it will take to charge up that capacitor to a certain amount of charge and-or voltage, hence charge up to an energy storage value.

The formula for the decreasing flow or charge rate (ie., current) to the capacitor will be one that expresses a natural-like decay with respect to time, and will cease at some certain amount of time, and this time is determined by the capacitor's capacity and the maximum charge rate initially to it. A higher capacity or capacitance means that at a certain initial or maximum charge (ie., current) rate value, it will take longer to charge that capacitor to a certain voltage. The formula for the increasing voltage of a capacitor will be one that expresses a natural-like increase in voltage over time and (practically, at 5T) ceases at some certain amount of time when the current ceases to a value of 0A due to a lack of voltage or pressure difference, and the capacitor has charged it up to the supply voltage.

Since (e) =~ 2.71828 is a natural-like value, it is typically used to express natural events such as growths. (e) is used to express the natural-like gravitational forces on a freely hanging chain from two supports at the same height above the ground, and which shape is called a catenary curve as discussed in this book.

$$V_c = e^{(-t/T)} (V_s) = \frac{V_s}{e^{(t/T)}}$$

: formula for the increasing voltage across a charging capacitor. Even though the voltage increases over time, the rate of this increase actually decreases over time (t). T is related to the total time or period it will take to charge and-or discharge that capacitor charged up to a specific amount of voltage. T is discussed further below. The negative exponent of (e) indicates there will be a natural decay over time.

$Vo = e^{(-t/T)}\, Vc$:formula for the decreasing voltage across a discharging capacitor

$Ic = e^{(-t/T)}\, It = \dfrac{It}{e^{(t/T)}} = \dfrac{Ve}{R}$:formula for the decreasing charging or discharging current of a capacitor
It or Imax is the maximum current flow which is determined by the series resistance (R) to and-or from that capacitor. The higher the resistance, the less the flow rate to and-or from that capacitor, and the longer the time of charging and-or discharging. Imax = Vmax / R = Vs / R.
The specific current flow is Veffective / R when charging, or Vc / R when discharging.
When charging, It = Imax = Vs/R , when discharging, It = Imax = Vc / R

A capacitor will take a certain amount of time to charge or discharge which depends on the rate (ie., amperes of current, coulombs per second) of charge flow and the maximum voltage to reach. R will limit current flow and increase the time of charge and or discharge. Resistance is a factor to the time to charge to a specific value. We can let time = (t) with units of seconds. The larger the capacitor, the longer it will take to charge up to a certain value. The capacitance of the capacitor (Cf) and R are factors of the (R)(C) time constant for a certain capacitor and resistance in series with it. It is of note that any resistance will reduce the power available at the output.

(t) = time in seconds, Ts = RoCf time constant or Period.

A capacitor is usually considered fully charged after 5 time constants = 5T = 5(RC)

When t = 5T, $e^{(-t/T)} = 1 / e^{(t/T)} = 1 / e^{(5T/T)} = 1/e^5 \approx 0.00674$, $Vc \approx Vs$

In a battery, the positive terminal is usually considered the higher potential where electrons are coming (repelled) out from, and they will travel through the circuit to the lower potential or pressure (attracted to the positive ions) in the battery.

When there is a thermal (ie., heat) energy difference between two items, say cubes of metal, the higher temperature metal will naturally transfer heat energy to the cooler metal and its temperature will rise. The temperature of the hot metal cube will decrease. This process will continue until both objects are at the same temperature, hence at thermal or temperature equilibrium. If the cubes are of the same metal they will have the same thermal conductivity, and if they are also of the same size they will have the same "**thermal mass**" (amount of stored thermal energy in its mass, some may might also call this "thermal inertia") and the concepts and curves will be similar to those shown above in this article for when there is a difference in pressure and one is decreasing as the other is increasing.

Also consider that since Cf = Q/V , and if two capacitors have the same farad or capacitance rating, and one is charged and the other is not, and if they are connected in parallel, we know that the total capacitance is twice that of a single capacitor, and this is so since the plate area is effectively doubled. The charged capacitor will charge up the empty capacitor. Each capacitor will receive half of the total charge (Q) in the first capacitor, and the voltage of each capacitor will be half of the total voltage on the first capacitor. A resistor can be used to increase the charging time and-or discharging time, and there will also be a power loss due to that resistance when current is flowing through it.

$$Ct = 2Cf = 2\,(Q/V) = [\,(Q/2) / (V/2) + (Q/2) / (V/2)\,] = 2\,[\,(Q/2) / (V/2)\,] = 2\,(Q/V) = 2Cf$$

In relationship to the water tank analogy, the water (ie., charge) level in each (similar) tank will be half as that of the full tank, and the water (ie., emf, voltage) pressure will also be half as that of the full tank. If a capacitor is being charged by a constant (non-reducing) voltage source, it will charge up to the same value of that voltage source. In the water analogy, a capacitor of a different value, say half = Cf/2, would have a lesser width, but still have the same physical height (as Cf, or container) so as to reach the same water pressure (ie., voltage) and then be at equilibrium. This smaller capacitor and-or tank will hold less water (ie., amount of charge) at equilibrium.

For the reactance (ie., impedance = Z, or effective resistance to ac signals) of a capacitor to AC signals:

As a reminder: **Resistance** will reduce the flow of current. **Impedance** (ie., to resist, to impede, an interruption, a blockage, a reduction) is the total effective resistance for AC current, and may include resistors and-or reactance. **Reactance** is how a device (such as an inductor and-or a capacitor) behaves or reacts when a particular frequency of AC is applied to it. It will behave as a certain value of equivalent resistance depending on both the device, its value, and the AC frequency applied. The units of resistance, impedance, and reactance are **ohms**. For a capacitor, the higher the applied AC frequency signal, the lower its effective resistance or impedance to current. A capacitor is said to more easily pass or allow higher frequency ac signals, hence it also rejects or does not easily pass lower frequency signals. For an inductor, the higher the applied AC frequency signal, the higher its effective resistance or impedance to current.

The **reactance of a capacitor** is how it reacts to an applied AC signal and is essentially a measure of its (perhaps unwanted, like a resistance) ability to impede (ie., resist) an AC (alternating current) signal. The specific value of its reactance is determined by both the capacity of the capacitor and the frequency of the applied AC signal. For example, consider that besides for an initial pulse of current to charge up a an empty (no charge on its plates) capacitor up to the input voltage, a capacitor will then block any direct current flow which has a frequency of 0hz. It becomes like an open circuit or very high resistance to low frequency AC signals trying to pass through it electrostaticly. This certain function of a capacitor is often used to isolate an AC input and signal from a connected devices DC (ie., a battery, etc.) power supply which may cause problems for both the input device and-or the signal efficiency. High frequency signals will pass through a capacitor with ease since it will have a low impedance to high frequency AC signals, and the capacitor will be practically like a low resistance piece of wire, hence like a short-circuit, or closed switch. Because of this frequency dependent value of reactance to ac signals, a capacitor can be used as a frequency dependent filter to block and-or allow desired and-or undesired frequencies. For example, unwanted high frequency signals can be "grounded" or sent to the ground connection and away from another circuit. Unwanted bass or low frequency signals can be greatly reduced and-or blocked from entering a circuit. Basically, reactance is how the device (a capacitor or inductor) reacts to an applied AC signal (here, in particular its frequency), and reduces that signal like a resistance would.

Xc = 1 / [2(pi)(Fhz)(Cf)] : **reactance of a capaacitor** in ohms units

Note for ex.: 1pF = 1 (10^-12)F , 120pF = 120 (10^-12)F =
120pF = 1.2 (10^2) (10^-12)F = 1.2 (10^-10)F

Cf = 1 / [2(pi)(Fhz)(Xc)]

As the input frequency to a capacitor increases, the charging and discharging time needed to reach or "follow" that input AC frequency and voltage signal is less, and the capacitor itself essentially reacts less with that signal and behaves more like a lower resistance to the signal. Remember that time=/frequency, so if the frequency is higher, the less time for each cycle of its wave. A smaller capacitor has a lower RC time constant and can quickly charge and discharge. A capacitor can be thought of as passing higher frequency signals easier with less reactance or impedance, and reducing lower frequency signals due to more reactance or impedance.

In general, the smaller the capacitor, the higher the frequency of the ac signals it will easily pass, and signals with frequencies below that high frequency will have more reactance and will be reduced as if a resistor was in series with that signal. To pass a low ac frequency signal with least impedance, a large capacitor (ie., a higher farad rating) should be used, and these types of capacitors are often found in audio equipment so as to pass low frequency ("bass") signals or audio. Large capacitors are often used to temporarily store energy (electrons and their voltage) so as to maintain a steady voltage value such as needed in a DC power supply to a circuit.

Using the above derived formula for Cf, the capacity or Farad value of a capacitor can be calculated after taking some measurements in a test circuit. This test circuit will consist of a voltage divider consisting of a known value series resistor (ex. r=1000 ohms) and the capacitor in question. An AC signal at a test frequency (ex. 2000 hz) is applied across the voltage divider circuit and the voltages (Vc and Vr) are measured with a (AC) volt meter. Xc = Vc / Ic. Note that: Ic = Ir = Vr / r. These calculated values can then be placed into the Cf formula. Except for the formula, a similar method can be used to calculate the value of an inductor. There are also inexpensive electronic meters available to measure resistors, capacitors and inductors. These meters are sometimes called LCR meters. Individual meters are also available for each electronic component mentioned, and some DMM (Digital Multi-Meters) can measure some of these components such as resistors and small valued capacitors.

Another way to find the value of a capacitance is through the RC time constant: $Ts = RC = RoCf$

$$Cf = \frac{Ts}{Ro}$$

After 1 time constant, 1t=T, the voltage across an empty charging capacitor will be about 63% of Vs:

$Vs = Vr + Vc$: Vr will go from Vs to 0v and Vc will go from 0v to Vs after ~5 time constants.

$Vs = \frac{Vs}{e^{(t/T)}} + Vs\, e^{(t/T)}$:If Vs = 1v , or as a relative value of 1, then: $1 = e^{(-t/T)} + e^{(t/T)}$
Extra: $Vs = Vs\,(e^{(-t/T)} + e^{(t/T)}) = Vs\,(1)$

$$Vc = Vs - Vr = Vs - \frac{Vs}{e^{(t/T)}} = Vs\,(1 - e^{(-t/T)})$$

First completely discharge the capacitor. Now with a resistor (R) in series with a charging capacitor, connect it to Vs and let the capacitor charge up to 63% of Vs while counting the number of seconds of time (t) it takes. This number of seconds is then equal to one RC time constant = Ts. For small capacitors, so as to be able to measure a reasonable amount of time greater than 1 second, use a very high value resistance for R. When smaller capacitors are used , they tend to require a high accuracy and low tolerance so as to be used in circuits, and because this method here can have timing error, this method is best suited for larger capacitors where a higher positive or negative (+,-) value (ie., farad units) tolerance is usually acceptable. Often a tolerance up to +,-10% is acceptable for most circuits unless otherwise stated.

$Cf = Ts / Ro$: calculating capacitance using the RC = Ts time constant

As a helpful example: Given $1uF = 1\,(10^{-6})F$ and $R = 1$million ohms $= 1\,(10^{+6})$: $Ts = RC = 1\,(10^{6})\,1\,(10^{-6}) = 1s$
This is generally not a practical value of time for most people to accurately measure, and it is basically too short or quick of a time to deal with, so it would therefore be better to increase R by 10 times, so as to have a value of 10 million ohms, and then Ts=10s

Because capacitors can store charge, they can maintain a voltage across their terminals of which can be used to help keep a fluctuating voltage more steady or constant, and this action is sometimes described as being an "electric flywheel" that accumulates and stores electric energy so as to be used later, and this is analogous or similar to a mechanical flywheel that has collected and stored the input kinetic or movement energy in its spinning or rotating mass with kinetic energy.

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For the inductance of an inductor

A magnet in motion, causing a changing magnetic (force) field intensity upon that wire, will affect the electrons in that wire and cause a voltage (ie., KE energy gain in the electrons, joules) and current to be induced (caused, generated) in a wire, and particularly in a coil (ie., having many loops or turns of wire) of wire where the effect is combined or summed to effectively be greater or amplified than if it was just a single length and-or coil of wire. This concept is therefore called induction (to induce, affect and cause), and the coil is called an inductor. If the relative motion between the magnet and coil is in one direction, the current will be in one direction, and if the relative motion is in the opposite direction, the current will be in the other direction. With an electric power supply, an electric current can be sent through an inductor. If the value of the electric current is constant, the voltage across it will be equal to the supply voltage, and the coil essentially becomes a (DC) electromagnet with a constant magnetic field about it. In this condition, it is useless as a transformer. Depending on the coils dimension, it will also have a (D) resistance value associated with it which does affect or limit the supply current. If AC current (particularly changing in magnitude, and direction) is flowing in an inductor, the created magnetic field will also be changing and the induce voltage and current in that coil will be such that it will be in the opposite direction or polarity to the supply or input voltage.

An inductor converts or transforms the (kinetic) energy of the input supplied current (ie., electrons) into a magnetic (flux) field (B) having potential energy, whereas a capacitor converts the energy of the input supplied current into an electric field and potential energy that can even be stored for a long duration of time without any more input energy applied.

The inductance (L) value of an inductor is a measure of its magnetic field creation and energy storage ability. Inductance is a measure of the magnetic field created by the applied current current. Inductance is a measure of the ability to create or induce a magnetic field. A transformer is a common example of an inductor(s), and the two coils, which are generally electrically unconnected are said to have mutual inductance because they will affect each other, initially via (varying in strength) magnetic fields. Michael Faraday is credited to this understanding.

Inductance = $L = B / I$ = (magnetic flux) / (current) : inductance has units of Henry's (H)

B has units of webbers (Wb) or the number of magnetic flux (force) lines, and this is determined by the applied current . Current = I has units of amperes. The units of inductance are: webbers / amp = Henry's

Ex.: 1H = 1 Wb per Amp.

1 Tesla = 1 webber per square meter = 1 (Wb / m^2) : magnetic field strength, concentration or intensity of the magnetic "flux lines".

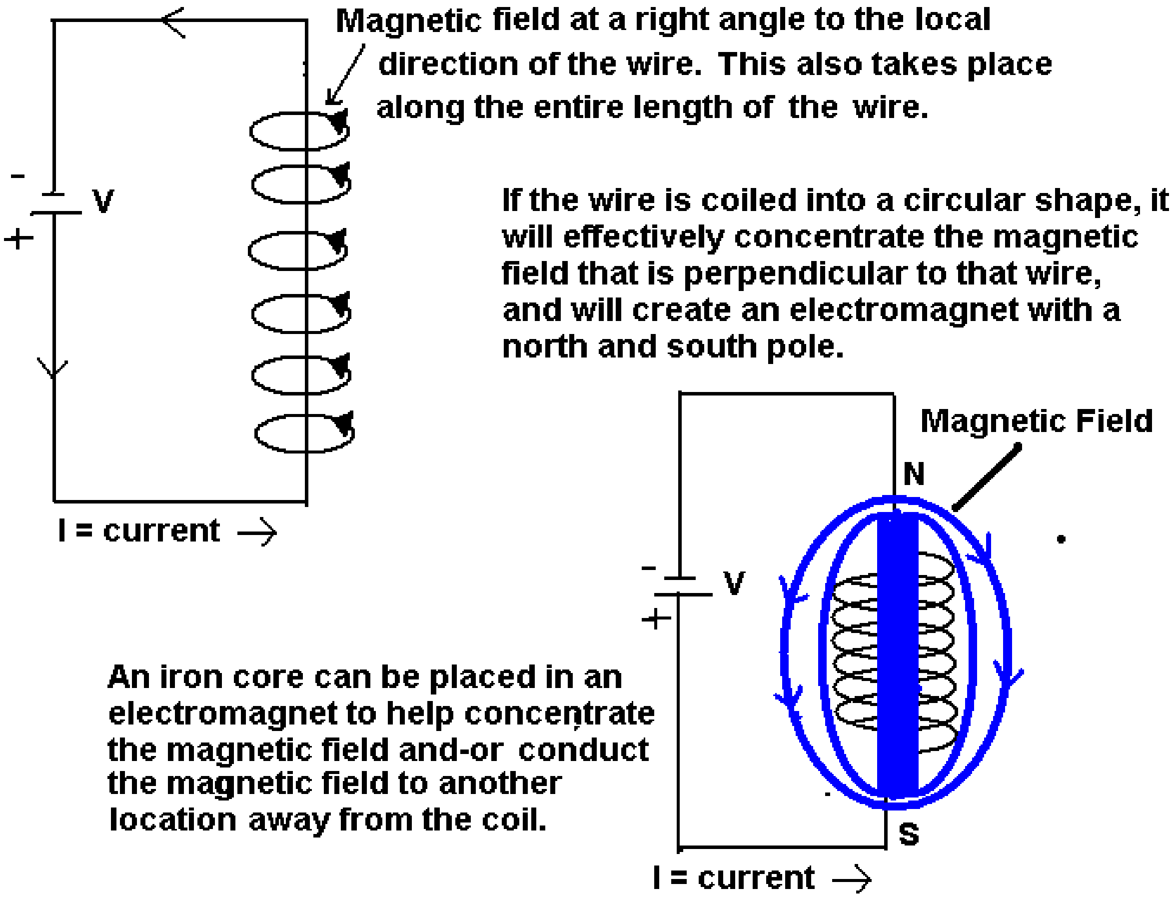
A coil is a good version example of a device with self-inductance When current passes through a coil, it will induce or create a magnetic field about it which will essentially "cut" (through) that coil and create an (opposing) current and ("back-emf" voltage, reverse polarity, opposing to current flow) voltage in the opposite direction to that voltage applied, and the result is that the current to that inductor will be reduced since there is less of a difference in voltage potential (ie., emf, the voltage) between the source voltage and the inductor. The inductance equation can be modified to include each wire turn (N) of a coil, which essentially sums up or effectively concentrates, adds to, and effectively magnifies the magnetic flux created or induced into a smaller area instead of a long length of wire, and therefor, the inductance of that coil will have a larger rating:

$L = N (B / I) = NB / I$: L = inductance i Henry's , N = number of turns, B = magnetic flux lines from one turn or loop, I = current in amperes (A)

Because an inductor reduces an alternating signal, it behaves much like a DC resistance that reduces or limits current, however, it will be called the reactance (Xl) of that inductor (and signal) since the effective AC resistance of inductor depends on the applied frequency of the AC or "(voltage) changing signal. Reactance has units of Ohms.

Passing a current through a length of wire produces a magnetic field that is curved perpendicularly around the center of that wire. A (opposite in polarity) voltage (emf, electric field) and corresponding current can then be induced (ie., like a simple transformer) in another or second wire that is parallel to that initial current carrying wire. The second wire, though parallel to the first wire, is actually then perpendicular to the direction of the magnetic field (flux or lines) direction of the first wire. **The magnetic field strength about a straight wire is: B = u I / (2 (pi) r)** . u is the permeabililty of free space = 4 (pi) (10^-74) , I is the current in amperes , (r) is the distance from the wire in meters. B is magnetic field strength in Teslas (T). The force on a length of parallel wire is: **F = L I B** , L is length in meters, I is current in amperes, and B is magnetic field strength, and F has units of Newtons.

With the previous understanding, given just one wire that is made into a loop or coil (ie., circular and spiral shape, like a spring, and usually coated or painted with a thin electrical insulation layer of paint) along its length. The (invisible) magnetic flux or field lines and the strength of the magnetism can "self induce" a voltage in the nearby (essentially parallel wire) loops or windings of the coil. This voltage will cause a current which in turn produces more magnetism. The magnetism or magnetic flux produced by each loop or turn of the wire in the coil is essentially summed together to produce a stronger magnetic field for a given amount of current. The resulting magnetic field of the entire coil will have a magnetic pole ("North" or "South", opposite poles) at each end. [FIG 263]



The more turns to make the coil, the greater the accumulated strength of the magnetic field produced. Each turn or coil will have a north pole in the center on one side of the loop or coil, and a south pole on the other side in the center. Placing magnets end to end or in series (North to South, and then North and South again) will create a stronger magnetic field and-or magnet. The energy of the electrons (ie., current) sent through that coil is converted to and stored as a magnetic (potential) field, and if the applied voltage and current to that coil then lessens or stops, the magnetic field decreases and essentially collapse down upon and passes through or "cuts" that coil again and induces (causes, creates) a voltage, much like a generator does, across that coil. Basically, a magnetic field will give electrons (ie., electric charges) enough energy to break free from their atoms and be free electrons (ie., current) in the wire. These free electrons now have a gained energy and can do useful work while releasing or using up their gained (kinetic) energy. The above figure of a coil is often called a **solenoid, and its magnetic field strength is B = u n I** , u is the permeability of free space, (n) is the number of turns per meter, and I is the current in amperes (A).

The first practical DC (direct current) generator was made by **Zenobe Gramme** (1826-1901), from Belgium, in 1869, and is type called a **dynamo** that has a commutator (ie., connection) ring that "(thin wire) or graphite brushes" would make contact with so as to deliver the power having more of a DC form with unaltering voltage polarity. In those times, there was no form of electrical rectification (such as with vacuum tube or solid-state diodes), and therefore mechanical rectification was the method used. Soon thereafter, Gramme also discovered that in a reverse manner, the dynamo was the basis of a DC powered motor, hence he invented the first practical DC motor to create rotational movement and-or power. An **alternator** is actually the basic form of an electric generator, and which makes AC electricity or power does not include a method to convert it to DC, and since the AC is useful in transformers and long distance wires so as to deliver energy. Essentially, a dynamo is a modified alternator. After Gramme made the DC motor, **Tesla** would invent the AC motor, and which also then showed people the usefulness of AC power. A **magneto** is an alternator that uses permanent magnets to create the magnetic field, rather than using electro-magnetic coils. A magneto system is often found in "(cord) pull start" petrol or gasoline engines where the force of the pull upon a cord by the user both moves and compresses the piston(s) and provides the energy to make the initial spark of electricity to cause the engine to run on its own power and generated electricity. In 1912 electric, high torque electric motors began to be used to start automobiles.

A basic form of one of the Maxwell equations is that EMF (electromagnetic force, voltage) induced is directly related to the rate of change (ie., speed) of the magnetic flux lines (B) with respect to time. This is also generally known as **Farday's Law of induction**.

voltage induced = emf = $-\frac{dB}{dt}$: emf = electromotive or electric force , B = magnetic flux , A stronger magnet will have a stronger magnetic field and-or more magnetic flux, and dB will then be higher.

: The faster a magnetic field cuts across a **wire**, or vice-versa, the higher the induced induced. Ex. The faster an electric generator spins, the higher the voltage produced. It is as if a higher force was in action to cause it. The higher the voltage (v = J / C) generated, the higher the current generated. dB / dt is the instantaneous change in magnetic flux with respect to an instantaneous (ie., momentary, brief, "quick", short, small, low) amount of time, hence the voltage is also an instantaneous value, and particularly with (rotating) electric power generators, and where the output waveform is a sine wave.

This induced voltage will then cause an induced (ie., generated) current to flow which will also create a magnetic field which actually opposes the original magnetic field applied that lead to its creation, and the voltage induced by that magnetic field is of opposite polarity to that which initially caused it. If the magnetic and-or flux field does not change, there will be no voltage induced in the other (output) coil of a transformer, hence transformers required AC electricity or pulsed electricity (ie., current) applied to the (input) coil for it to function as a (power) transformer and-or circuit isolation.

If there was no back-emf from the second wire, that second wire would then induce a voltage of the same polarity in the first wire and cause it to have a higher (ie., double) emf, and there would be a situation where the emf (v = J/Q) would keep increasing., however this cannot be the case when the input energy is limited to a certain value. There cannot be more energy out than that which as put in.

Inductance can be increased by using more side by side loops or turns of the wire so as to effectively increase the magnetic field strength, inductance and resulting generated voltage. The loops or **coil** of wire place a long wire into a smaller area that can be more affected by a relatively small magnet and its magnetic field.

Lenz's Law Formula: voltage induced = emf = - N dB / dt : very similar to Farday's Law of induction formula

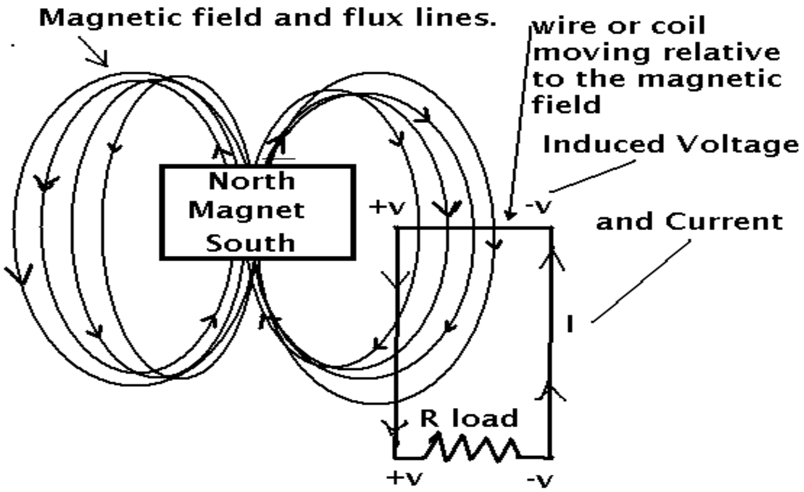
A common transformer which has two, non-connected, independent coils, functions by including another concept called mutual inductance. It may also be called as magnetic inductance. These coils are said as being only "magnetically (flux, energy) coupled", and that their (changing) magnetic fields (due to a changing or AC input signal) will influence each coil. The voltage produced in each turn and-or coil segment of wire is additive like small batteries in series, hence shown as a multiplication in the equation.

To create a simple **electricity generator** that creates a voltage and induced current, a single loop of wire is made to pass perpendicularly through a magnetic field, or vice-versa, and a small voltage and current will be generated. If there are N turns or loops in a coil of wire, the inductance and induced voltage will increase by that same factor of N and when the magnetic field effectively "cuts" or travels through only one half of the generators coil which has those turns of wire in the same direction, and so as the induced or generated voltage polarity and current will all have the same direction, and so as to not negate, reduce or cancel-out each other as a "back-emf" or voltage would. When the direction of relative movement of the magnet and coil changes to the opposite direction, the induced voltage changes polarity, and the induced charge or current changes direction, and this is the basis of AC electric power generation.

When a current is created or induced in a wire due to a magnetic field changing strength, the electron flow in the wire will cause the creation of a magnetic field about (ie., perpendicular to) that wire, and the polarity of this field will be opposite to that of the magnetic field which caused it. These opposite magnetic fields will then create a physical repulsion force applied to both the wire and magnet. In a motor that is to spin or rotate, this magnetic repulsion force is what causes the torque (ie., rotation force) which then rotates the rotor (ie., rotating) part of it.

To convert the mechanical-like input energy and-or force, to electrical energy, this force can be used to cause a rotation in both the turbine (ie., wind or water kinetic energy and-or force collector blades) and the electric generator which is mechanically connected via a shaft and-or gear(s) to the turbine. Remember, a non-changing magnetic filed just passing through a non-moving coil, or vice-versa, will not induce any change in voltage and-or current such as from being 0V and 0A. There would be no motion and-or kinetic energy available in this system so as to even be available to create, transfer to, and-or produce electrical energy. Gears can also be used to increase the speed of the rotation of the generator's rotor part , either to change torque (force) and-or its rotational speed (ie., "angular velocity") so as to have a maximum and safe energy output. Another method to alter the output electrical power after it is created is with a transformer so as to change the voltage and-or current available. Power = (Voltage)(Current) = VI watts = joules of energy / second. If the voltage changes by a factor of (n), the current will change by (1/n) since these two factors are inversely related if power is a constant value or product.

A basic representation of an AC electric power generator: [FIG 264]



Farday made some equations for the in a wire when it is moved past a magnet, or vice-versa. The equations are all somewhat similar. The faster the relative motion of a wire or coil, and the (perpendicular) magnetic field of an electrical generator, the faster the relative velocity of these two things and the greater the kinetic energy involved, and the faster the magnetic force is applied to the electrons in the wire and the greater the induced voltage will be. The stronger the magnetic field, the greater the number of electrons involved and the greater induced voltage (= J/Q) also. The length of wire perpendicular to and affected by the magnetic flux will cause a directly related increase in the generated voltage, and this is due to that there are more electrons being affected by the applied magnetic (force) field and given kinetic energy. A higher relative velocity between the wire and magnetic field, and a stronger magnetic field will effectively mean that the magnetic force will cause a higher amount of kinetic energy (Joules) in the electrons in that part of the wire. Remember that: voltage = (joules of energy / coulomb of charge) = V = J / Qc.

generated voltage = emf = - B v L : * , **B** = magnetic field strength or flux density perpendicular to the wire ,
v = motional velocity for a straight wire or angular velocity (¢ / t) for a rotating wire. **L** = length of wire = 1 turn , or set as N for number of turns of a rotating wire or coil. Due to Lenz's law, the negative or opposite value is indicated.
*** The above is a generalized equation and description of Faraday's Formulas.**
Extra: d = vt , v = d / t , dv = dd / dt = d¢ / dt

By using turns (ie., a coil) of wire, the voltage can be increased in an additive manner, and makes the size of the generator smaller and more practical, and since the addition is repetitive of the same value, this can be represented by multiplication by the number of turn: (total voltage) = (n turns) (the small voltage per single turn). This is much like placing batteries in series so as to increase the total voltage (or emf) available. The generated voltage with a generator is sinusoidal, hence the output is AC power. For the voltage and-or current at a particular instant of time, it depends on the angle of the magnets with respect to the coil. When the coil is perpendicular (ie., 90°) to the magnetic field, the maximum voltage is created. emf = (sine ¢) (B v L) . Generators and motors are nearly identical,except that a motor is often designed for high speed in many cases, and if this motor were to be used as a generator, a gearing system should be used so as to effectively increase and-or multiply the input speed (ie.,rmp) so as to have a usable and-or high enough voltage at the output of it.

The current (A or I) generated by a electricity generator depends on the value of the generated voltage and the resistance of the circuit. From V = J / Qc = I R we have: I = A = V / R .

Electrical Power = Pw = VA = V^2 / R = (emf)^2 / R = (B v L)^2 / R = B^2 v^2 L^2 / R = J / t = Fd / t = Fv

The **Lorentz Force** describes the forces upon an electric charge in an electro-magnetic field such as due to the presence of a magnetic and electric field acting upon it. There will be both an electric force and a magnetic force developed upon the charge. The Lorentz force is what causes a motor to rotate when electricity (ie., power = voltage and current [charges, electrons]) is passed through its coils, and of which are near a magnetic field. The interaction causes magnetic repulsion and the rotor (ie., spinning part) of the motor will then rotate. The direction of the electric and magnetic forces are at right angles to each other, much like the legs of a right triangle, and the resulting net force and direction is the vector sum of these two other component vectors.

$\overrightarrow{\textbf{Resultant Force}} = \overrightarrow{\textbf{Electric Force}} + \overrightarrow{\textbf{Magnetic Force}}$ **: the arrows indicate these are vectors which have a magnitude value and direction**

F = q E + q vB : q = charge E = electric field strength ,
v = velocity in (m/s) and is also a vector ,
B = magnetic field strength in teslas (T), here a vector
The value of vB is a vector product.

Protons and electrons will be directly attracted to the direction and source of an electric field. Protons will be attracted to the negative charges or source, and electrons will be attracted to the positive charges or source. It is of note that moving charges such as electrons in a wire will create a magnetic field perpendicular to and around that wire , and that a moving and changing magnetic field value will induce a (cause, produce, electric, free electrons, charges) current in a wire when they are moving relative to each other at right angle (90°, perpendicular to). The strength of the magnetic field created depends on the electric current in that wire, and the voltage induced in the wire due to the magnetic field depends on the strength of that magnetic field and how fast the strength of that field is changing - usually due to the relative motion of the wire and magnetic field. The higher the induced voltage (V = J/C = Energy / Coulomb of charges) created, the greater the current can be.

Neutrons are generally unaffected by an electric field since neutrons are considered as having no net electric charge. In a magnetic field, protons and electrons will experience a movement and change in the direction perpendicular to the magnetic field lines, each charged type will go in the opposite circular direction (CW or CCW). This will cause the electrically charged particle to go in a circular path.

Scientist, such as Einstein, have tried to consider and find if there is a unified or single source of the electric, magnetic and gravity (force) fields, and we know today there are other forces such as the nuclear weak and strong forces.

For creating a basic inductor component of an electric circuit, typically for frequency oscillators, tuners, and filters:

Lh = (permeability)(Nturns^2)(**A**rea of each turn) / (**l**ength of coil) : physical construction of an (coil) inductor

Lh = (u N^2 **A**m^2) / **l**m : u = permeability to magnetic fields, has units of (H / m) = (N / A^2)
Permeability is a measure of a materials ability to remain (permanently) magnetized.
It is also a measure of inductance per unit of length.
u = (magnetic field strength) / (nearby induced magnetic field strength) = B / H

We see that inductance is directly related to the number of turns of the coil, and the cross sectional area of the coil of which for a given current there is a greater magnetic flux and-or field created because the length of the wire and-or coil is longer and the resulting total magnetic field of the coil will then be greater. A basic formula for the minimum length of wire for a coil with thin wire is:approximately: **Coil Length** = (n turns) 2 (Pi) (radius of coil) = (n turns) (Pi) (diameter of coil), and you must remember to have some extra length for the straight leads and-or any connections to the coil. If the wire is thick and-or has thick insulation, and you want to be more precise, then the length of a single turn needs to be measured or calculated, and then multiply this by the number of turns of the coil.

For the total inductance of **inductors in series**:

Inductors in series essentially creates a longer inductor or "coil" with simply having more turns and length of wire.

Lt = L1 + L2 + . . . : this formula is similar to that for series resistors

For the total inductance of **parallel inductors**:

1 / Lt = 1 / [(1/L1) + (1/L2) + . . .] : this formula is similar to that for parallel resistors

For the reactance of an inductor

The reactance of an inductor (ie., inductive reactance) is essentially its ability to impede (ie., resist) an AC (alternating current) signal. The value of its reactance is also determined by the frequency of the applied AC signal to it. The impedance of an inductor is opposite in nature to that of a capacitor. For an inductor, its impedance increases as frequency increases, and at some point it will be like an open circuit or very high resistance to high frequency AC signals. Like capacitors, inductors can also be used as (frequency dependent) AC signal filters and in "tuned" (resonant frequency) oscillators. Reactance means how the inductor or capacitor reacts to the applied AC signal (ie, frequency).

$Xl = (2)(pi)(Fhz)(Lh)$: **reactance of an inductor**, **measured in ohm units**
$Lh = Xl / [(2)(pi)(Fhz)]$

The higher the input frequency, the higher the rate of change of the current going to it, and therefore, the higher the (ac) reactance induced because the rate of change of the (invisible) magnetic flux (lines) created will also be higher, and therefore the back-emf or reverse voltage induced is higher, essentially or effectively acting like a higher resistance impeding even more current flow. The equations for Xc and Xl are essentially, reciprocals of each other.

For a common example of an inductor, a typical audio speaker is an electro-mechanical transducer (ie., an energy or power transformer of sorts) that transforms or converts the input electrical signal energy to (vibrating, or pulsed) force and motion so as to apply pressure to the nearby air that acts as a sound conductor and transmits or radiates the air pressure or sound (a force due to compressed or decompressed, like a spring, air molecules). Our ears are essentially mechanical to electrical (nerve signals) transducers, much like how an electric microphone converts the received (sound, air) pressure or force into an electric signal. An audio speaker usually consists of an internal coil which is essentially an inductor or electro-magnet coil and which is standardized to be 8 ohms of impedance when a pure (AC) sine wave signal (voltage, current and frequency) having a frequency of 1000hz is applied through it. High power and-or high current speakers are often designed to have an impedance of 4 ohms.

For the wire length of an inductor: Given the circumference (C) of each turn, the basic length is simply the number of turns (N) times the Circumference length or distance. **Wire Length = Lw = NC** . You may then add on any additional length needed for the connection leads. From this equation,we have: N = Lw / C and C = Lw / N .

Some other common inductors are solenoids and relays which are essentially electromagnets to create a mechanical force or create an electrical operated switch..

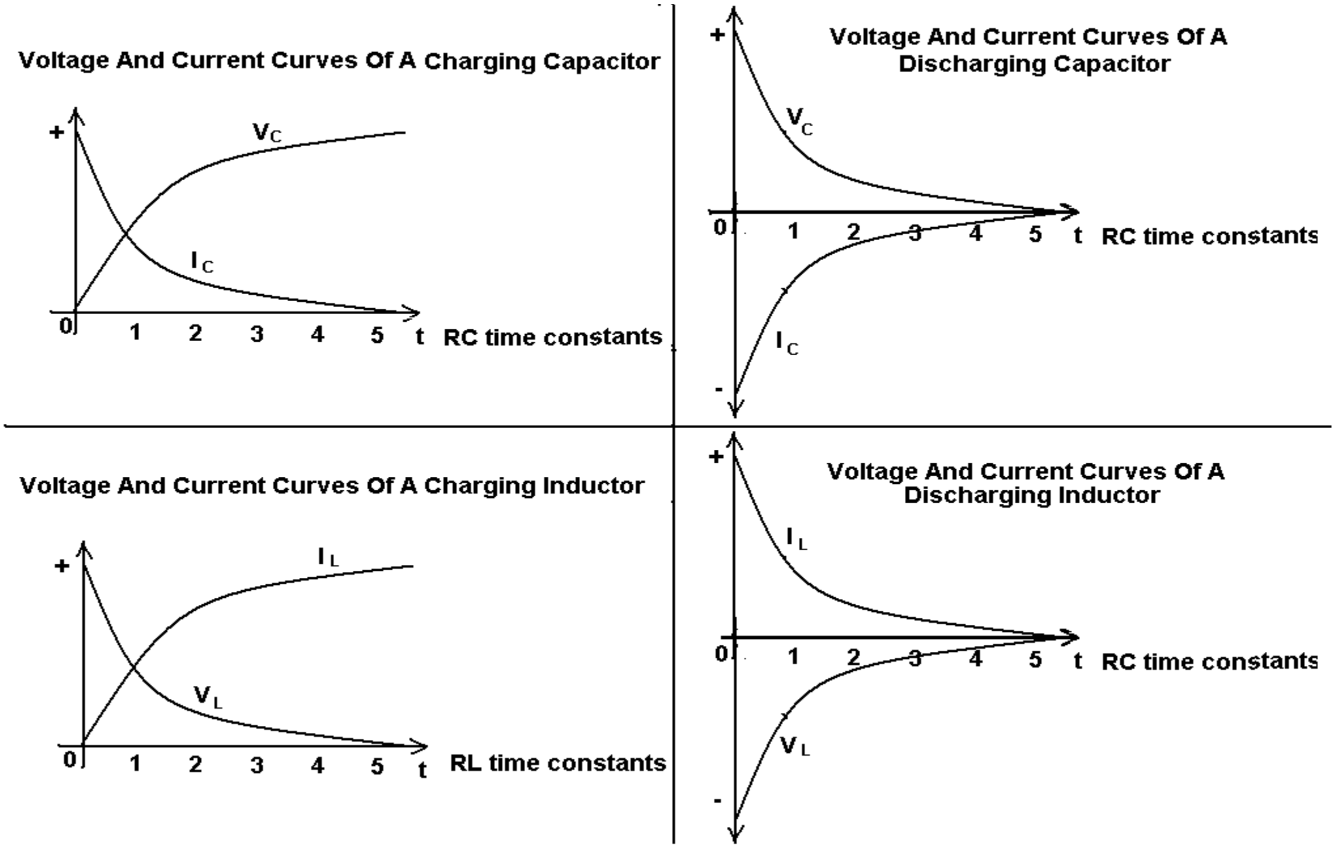
When a capacitor (C) and inductor (L) are in parallel to each other, such as in a (LC) tuner or "(LC) tank circuit", and each has its own reactance (X) to the applied input frequency, and the resulting or effective total reactance (Xt) and reactance formula is nearly identical to that of two resistors being in parallel ($R_{||}$) because reactance essentially limits current just like resistance does:

$Xt = \frac{Xc\ Xl}{Xc + Xl}$: **formula for two parallel reactances, such as in a LC tuning circuit**
At resonant frequency (Fr), Xc = Xl , and Xt or Zt is the highest possible.
(Xc + Xl) effectively becomes: (Xc + Xc) and-or = (Xl + Xl) = 2Xl = 2Xc ,and
Xt becomes: $Zt = Xt = Xc\ Xl / (Xc + Xl) = XcXc / 2Xc = (Xc)^2 / 2Xc =$
$= Xc / 2 = Xl / 2$

At what frequency (F) will Xc = Xl , We can solve this by setting Xc = Xl = (1 / 2(pi)FC) = 2 (pi)FL) , and we get the famous equation for the **resonant frequency of an inductor and capacitor:**

$F = 1 / [2(pi) \sqrt{LC}]$: Resonant Frequency Of An Inductor and Capacitor Circuit
$F = 0.159154943 / \sqrt{LC}$

Voltage And Current Curves Of A Charging And Discharging Capacitor And Inductor [FIG 265]



When done charging, the capacitor will store its energy in a electric field and charge. The (electric field). An inductor will store its energy in a magnetic field. When charging begins on a capacitor, the back-emf created at it 0, as like a short circuit, and the current will be at its maximum value.

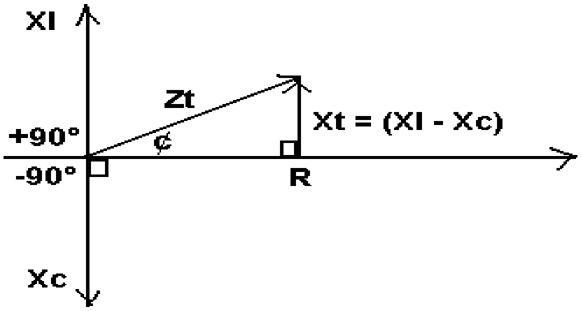
The discharging resistance or impedance and the corresponding time constant are generally not the same as the input or charging resistance and its corresponding time constant.

For a (magnetic field) charging of the inductor, the inductor behaves much like an open circuit with the full voltage available across its terminals. This initially fully impedes the current like a back-emf (reverse polarity voltage) at first and the current will then quickly rise in value to its maximum possible value, and the magnetic field and charge of the inductor will rise to its maximum possible value.

Note for example the change in the measured polarity or direction of the discharging current of the capacitor. This is so because the current is not entering the capacitor, but going out from the capacitor. If you do not reverse the leads of a voltage meter while measuring a voltage, you will also read a negative voltage during the negative half cycle of an AC wave, and in fact, you will also read a negative voltage across a component in a DC circuit if you switch the measuring leads around. An AC voltage or power source is much like taking the battery out of a DC circuit and switching the direction or polarity of it in the circuit, and very fast - either several to millions of times a second. This would be correctly called pulsed AC since the voltages are at maximum and not starting from 0 as in a typical AC signal such as a sine waveform.

The (dynamic) impedances or effective resistances of the charging and discharging capacitor and inductor are opposite or 180 degrees out of phase (ie., alignment, simultaneious) with each other and they can therefore effectively negate and-or nullify each other if they have the same magnitude or absolute value. These impedances are also both 90° out of phase with a resistor in series with it. It may be helpful to here to note a somewhat similar example that the common Sine and Cosine waves (and each instantaneous event or value) are very similar to each other but are not mathematical opposites or 180 out of phase with each other, but are 90 degrees out of phase with each other. Their "phase (or instance) shift" is said as being 90 degrees. Due to the above reasons, the total effective impedance to an AC input supply and-or signal is not simply R + Xl, or R + Xc, or R + Xl + Xc.. The total impedance is rather said to be a complex impedance that is composed of several parts and each being out of phase with the other two. The phase angle of the impedance due to a capacitance is -90 out of phase with that of the resistor, and this is said as being "leading" (already happened or earlier). The phase angle of the impedance due to an inductor is +90 out of phase with that of the resistor, and is this said as being "lagging" (or later in time). We can consider the value of the resistance (R) as a real value along the x-axis, and the other leg of this triangle would be the signed number sum of the inductive and capacitive reactances. If the inductive reactance is greater than the capacitive reactance, the circuit is said to be "inductive". If the capacitive reactance is greater than the inductive reactance, the circuit is said to be "capacitive". The effective resistance or total impedance (Zt) seen by the input signal is: [FIG 241B]

Total Impedance Of A Series RLC Circuit



When Xl = Xc, the phase angle is 0° and the circuit looks as "resistive" to the applied AC input signal.

From the Pythagorean Theorem:

$$Zt = \sqrt{R^2 + Xt^2}$$

Ex. An antenna for a certain radio receiver might be affected by some stray, nearby capacitance, and the antenna is then said as being "capacitive", and is then not being as efficient as it could be. This capacitive reactance can effectively be canceled or negated by using an inductor of a certain value which has the same reactance with the same applied input frequency.

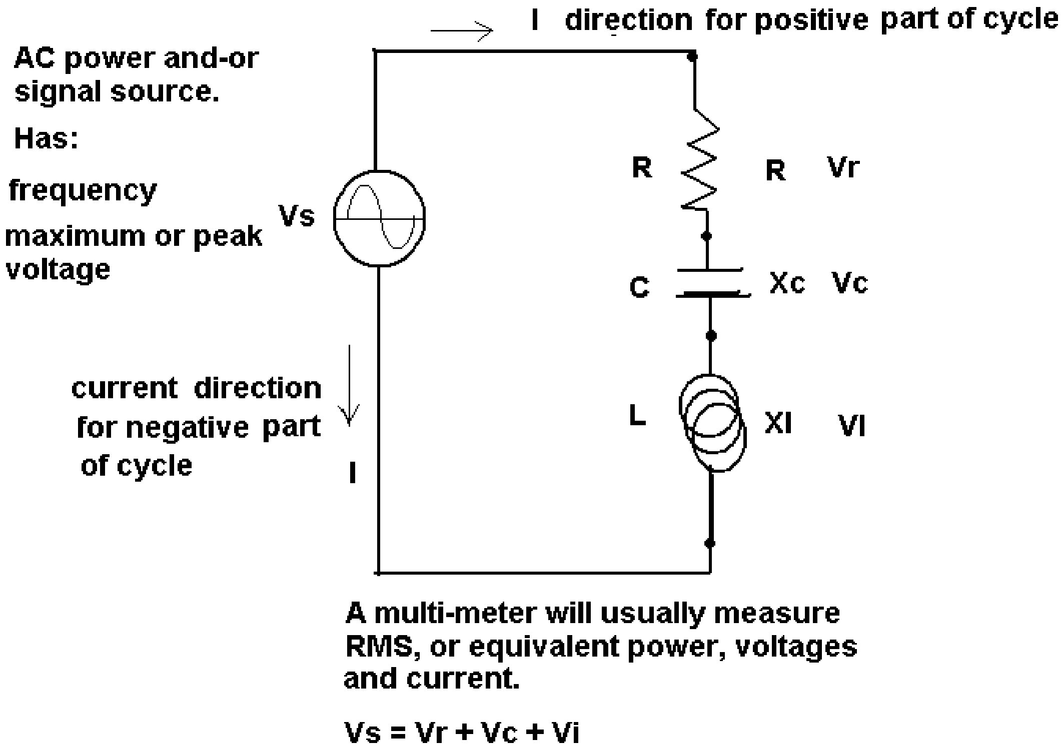
The concepts of Ohm's Law can be used for both DC and AC circuit analysis.

I peak = V peak / Impedance

It = Vt / Zt : AC , RMS, current through a RLC circuit and-or branch , **RMS** is the "root-mean-square" value of an AC voltage or current waveform. It is the effective or equivalent DC voltage or current.

Now for example, consider a certain given AC supply or signal frequency applied to a charging capacitor. If the charging resistance is very low, practically 0, the capacitor will quickly charge to that maximum AC supply or signal voltage which has its peak voltage at 90 degrees which is at one-fourth of its entire waveform period and-or cycle of time. A full cycle of a periodic waveform is a 360° waveform and it has a period of: P = Ts = 1 / Fhz. For a capacitor to charge up to the maximum input voltage during one 360° cycle of the input or applied waveform, the total time (5, RC time constants) of charging the capacitor must be less than or equal to one-fourth of the period time of the input frequency. In simple wording, the charging time of the capacitor, which is (5)(RC time constants), must be "fast" or short "enough" to properly handle one-fourth of the period time of the input frequency and signal, otherwise a portion of the input signal will get "clipped" (ie., lost, removed) or distorted.

An RLC (Resistor, Inductor and Capacitor) series circuit or current branch. [FIG 266]



Summarizing the basic electrical functioning of an inductor:

When a voltage supply sends a current to an inductor that has a low DC wire resistance, the current will be high. This high current will immediately create a high magnetic field about the coil and-or inductor. This growing or changing magnetic field will induce a voltage in the wire of that conductor and-or inductor, and this voltage will have a reverse polarity as that of the supplied voltage to that inductor. Due to this, this reverse polarity voltage is also called a back-emf or reverse-voltage emf. This voltage will reduce the effective applied voltage to that inductor, and the current to it will be very low at first since the back-emf is high, and then the current will increase as the magnetic field energy decreases over time and it is converted to electrical energy.

Voltage Divider Method To Find The Value Of An Inductor Or Capacitor

First, there are both capacitance and-or inductance meters available for purchasing, and as of the year (2022) they are widely available and relatively inexpensive. This note or discussion here has a forward reference to the general concept of the voltage divider that is shown after this one.

By having an AC voltage source and creating a voltage divider circuit with a resistor and either a capacitor or inductor in series with it, the value of that inductor or capacitor can be found from their reactance (ie., frequency related impedance or effective resistance) formulas:

From Ohms's Law: I = Vr / R amps , we have: R = Vr / I and Xl = Vl / I and Xc = Vc / I

From: Xl = 2 (pi) Fhz Lh , ohms , we have: Lh = Xl / [2 (pi) Fhz]

From: Xc = 1 / [2 (pi) Fhz Cf] , ohms , we have: Cf = 1 / [2 (pi) Fhz Xc]

For an example AC, series circuit, the AC voltage source might be 10 volts, and having an arbitrary sinusoidal frequency of say 1000hz. The resistance (R) also helps reduce the current to a safe level, perhaps 50mA. Hence R would then be calculated at: R = 10v/0.050A = 200ohms. Pr = (Ir)(R) = (0.50)^2 (200ohms) = 0.5Watts, hence use a resistor that is rated to handle at least a half-watt (1/2W = 0.5W = 500mW) of power (ie., loss in it as heat energy). For better accuracy, you can experiment with the value of the input power supply's or AC signal frequency. The voltage drop across a capacitor or inductor is equal to its reactance times the current through it, and this is Ohm's Law. Since a certain frequency will cause an associated value of reactance in an inductor or capacitor, by using a stable or constant voltage source, it is also quite possible to also measure or have a good estimate of the input frequency by measuring the voltage across the inductor or capacitor. A calibrated scale or chart can be constructed beforehand and checked via experimenting with a variable range of input frequencies.

Besides an initial pulse of current, DC current will not flow through any branch of a circuit containing a capacitor.

For an AC applied signal, although an inductor or capacitor have a reactance to the applied AC voltage and-or signal and will reduce the circuit and-or branch current and available power, there is no theoretical power loss in them like a resistor would have. The inductor and capacitor will eventually give any energy storage back to the circuit and-or load resistance.

An inductor and capacitor in series can create a filter, and in particular, one that easily passes a certain frequency better than both the higher and lower frequencies. At this (frequency band and-or range) pass frequency where the impedance is effectively 0 ohms, a safety resistor can be used in series with the inductor and capacitor so as to keep the current in that wire or circuit branch to a safe value. A resistor and-or wire resistance will also reduce the available power output, and increase the charging and discharging times of both the capacitor and inductor.

[This space for book edits.]

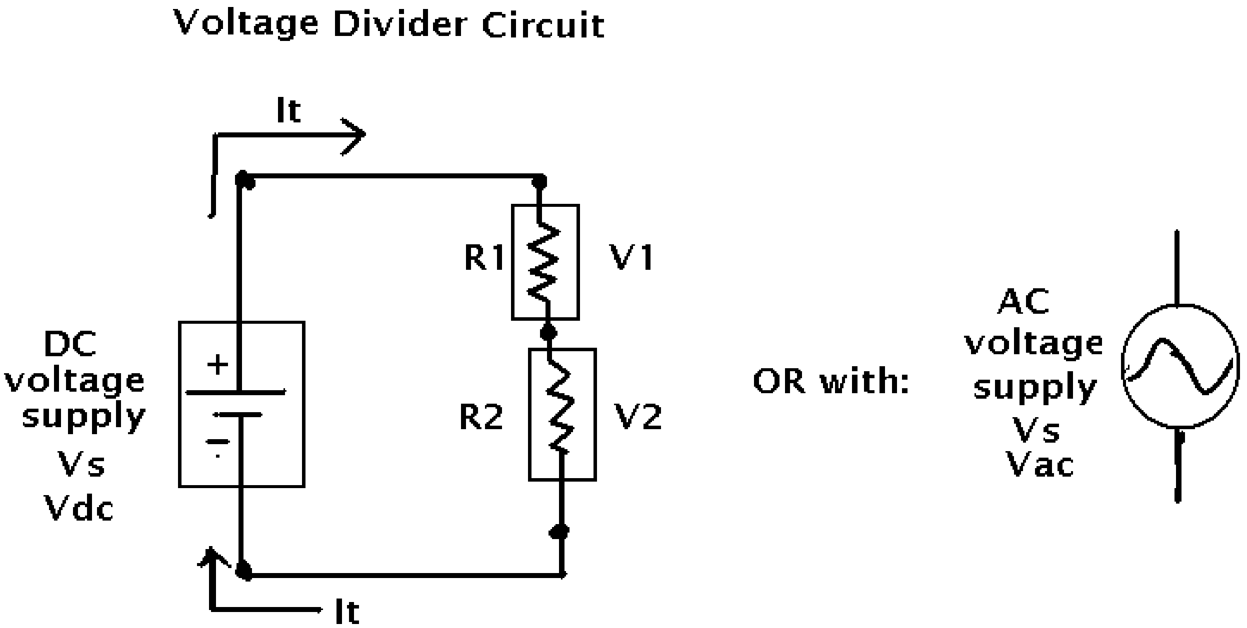
The Voltage Divider Concept

Whenever two or more resistances, and-or inductors, and-or capacitors are in series such as for AC circuits, each will have a certain value of resistance and-or impedance to the flow of AC electricity or current. There will essentially be a loss of energy trying to push that current though it and this results in voltage loss or drop through that component (a part or part of something larger, such an electric device and-or circuit). Because of this, there will be less voltage or (electrical, emf) potential energy available to the rest of the circuit. The higher the resistance to DC or AC, and-or impedance to AC signals (voltage, current,and frequency), the higher the corresponding voltage loss or drop in, through and-or "across it". There is a figure of an example voltage divider circuit shown below.

A common example of when a voltage divider circuit is needed is when a power supply being used is greater than 3Vdc and a voltage drop is then needed to reduce the voltage and current to a LED (that produces light) to its safe and recommended operating values. A voltage divider circuit will reduce the power to it and with a voltage of just 3Vdc is across that LED, and having about 20mA of (circuit, resistor limited) current through it. This circuit is also a form of voltage setting or regulation to the load, however a true **voltage regulator** can automatically keep the same or constant desired voltage to the load if the input supply voltage and-or load resistance changes. Most voltages regulators are not designed to increase the load's supply voltage if it is insufficient, but rater just reduce a larger supply voltage to the needed value.

A voltage drop across and-or through a component also means a loss of available energy or power in that component, and therefore a loss of power available to the rest of the circuit. For resistors, the loss of power is usually in the form of wasted heat energy, except when heat is actually desired such as for an oven and-or heater of some sort.

A voltage divider is a concept that the higher the impedance to (the same) current flow through two or more components in series, that the voltage drop or loss across and-or through it is directly proportional to its impedance. It will take a higher voltage to force a certain amount of current through a higher resistance to the flow of current. A circuit may be designed to include a voltage divider as a necessary part and-or function of the circuit, such as for example reducing the voltage to a certain usable and safe value or level; perhaps to control the volume in a radio by using a variable resistor controlled by the user via an external adjustable knob. The voltage divider concept is well known in the study of electricity and electronic circuits, study and applications of electricity. [FIG 267]



The voltage drop (V) across a component in a DC or AC circuit is equal to the current (I) through that component times the effective impedance (Z) to the flow of that current. If two of these values are known, the third value can be calculated by using Ohms Law ($I = V/R$):

$Vv = (Iamps)\ (Zohms)$: for two or more components are in series, the current is the same value through them

If the voltage is the same across each component in series, then their effective resistance is the same value.

Ex. Connected to an AC or DC supply voltage are two components (ex. resistor, capacitor, inductor) that are electrically connected in series with each other. It can be said that the (input) supply voltage is connected to and-or "across" those two components in series. Obviously, the sum of the voltages across each component cannot exceed the supply voltage, and in fact, it does equal the supply voltage. The actual voltage across each component will be less than the supply voltage:

$Vsupply = Vcomponent1 + Vcomponent2 + \dots$: voltages across series connected components

The current through each component connected in series is obviously the same for each, and that is the total (It) or supply current. The value of this current is determined by the supply voltage (Vt) and the total impedance (Z) to the flow of the generated or induced current.

$It = Vt / Rt$ or= Vs / Zt : It = Total current and $Zt = Z1 + Z2 + \dots$ = tatal (current) impedance

$I1 = I2 = \dots = It$ Using Ohms Law, substitution, and considering just two components in series:

$Vt = V1 + V2$

$Vt = (Z1\ It) + (Z2\ It)$ Since (It) is a factor to each term, and factoring it out, we have:

$Vt = It\ (Z1 + Z2) = It\ Zt$ solving for (It):

$It = \frac{Vt}{Zt} = \frac{V1}{Z1} = \frac{V2}{Z2}$: since the current (here, It) is the same value through series components

Indicated by the above equation for (It), and that each component has the same amount of current through it, when given just two components in series, the ratio of voltage across each component to its impedance is the same for each component, and this same ratio value equals the same current (here, It). Mathematically:

$\mathbf{\frac{V1}{V2} = \frac{Z1}{Z2}}$: **Equal ratio of voltages and impedances**. **Voltage Divider Equation** for two components in series, and if they are resistors: R1=Z1, and R2=Z2, and mathematically:

$$V1 = \frac{V2\ (R1)}{(R2)}$$

$$V2 = \frac{V1\ (R2)}{(R1)}$$

$Vt = Vs = Vin = V1 + V2 = (It\ R1) + (It\ R2) = It\ (R1 + R2) = (It)(Rt)$

If the output voltage (Vout) is the voltage across R1, then R1=Rout:

$Vout = V1 = V2\ (R1 / R2) = (It)(R1) = (Vin / (R1 + R2))\ R1 = (Vin / Rt)\ R1 = (Vin / Rt)\ (Rout)$

$Vout = Vin\ \frac{Rout}{Rt}$: A formula to calculate a series voltage across a component when the current is not known

$$\frac{V_{out}}{V_{in}} = \frac{R_{out}}{R_t}$$

: General equation for the ratio of corresponding voltages and impedances for series components. Vin = voltage across all the series components, Rt = total series resistance.

If you want a certain Vout, the proper value of Rout to achieve this must then be calculated from this formula:

$R_{out} = (V_{out} / V_{in})\, R_t$

In the above drawing, R1 and R2 can actually be one larger resistor with a lead output or terminal wire someplace between that larger resistor's end terminals.

If one, the other or both of the series resistors is made to be a variable resistor, the output voltage can likewise be made to be adjustable and stet to a needed value. This is an example of an **adjustable voltage regulator**. To regulate a voltage is to keep it at a steady or constant value for the circuit designed to work with that voltage. With a fixed value of resistance (ie., Rout) and the voltage across it is used as the output voltage, a problem can develop when a load (ie., that which draws the load of current and-or power such as another resistance, circuit application [ex. a radio, a speaker, motor, heater, a light, etc.]) is placed in parallel (ie., "next to", across its terminals or leads) to it so as to be powered by that output voltage. When two resistance are placed in parallel, their net or total resistance is actually less, and this will then affect the voltages of the voltage divider circuit, and since the resistance would be less, the voltage across it will also be less. A solution is to make a variable (user adjustable and set) voltage divider circuit or use a pre-made, adjustable voltage regulator circuit to supply power to experiments, projects and circuits.

Some basic ways to set an output or load voltage when there is a greater input or supply voltage are with resistors (Vr=IrR), diodes (Vd =~ 0.7), LED's (a red, amber or yellow led will drop about 1.8v to 2v, and a white led will drop about 3v to 3.5), and transistors being used as a variable resistance by varying the input base current, **Zener diodes** are a reverse voltage polarity (ie., setting, operation, bias) operated diode which "breaks down" and conducts at a specific manufactured and-or tested voltage, and so as to keep a constant voltage across it which is good for supply voltages that may vary by a few volts more than needed. Zener diodes behave like a common diode when forward biased. If must be noted that (common, single) LED's can only handle or pass relatively low currents of about 10ma to nearly 30ma, but (power) resistors and-or (power) transistors can handle more current and power dissipation, and of which is an undesirable waste of energy and a circuit should be designed so as to keep the power losses as low as possible, while maintaining circuit and-or load stability.

Today, pre-made voltage regulator circuits, sometimes called "modules", are relatively inexpensive to purchase, and most (but not all) convert a higher DC voltage to a lower DC voltage needed. Many modern voltage regulators are based on, or similar to, the common, inexpensive **LM317** IC (Integrated Circuit, "chip", general identification, however the **LM317T** is recommended), user adjustable (about 1.25vdc to about 37vdc, at 1.5A max limit before internal shutdown for safety, and typically used with a 5k ohms variable resistor) or voltage settable by using fixed resistors. The voltage dropout or loss across it to function is 3v minimum needed to function. This IC chip only has 3 terminal pins. Because these must drop a voltage so as to have a desired output voltage value, that voltage drop or loss within the voltage regulator passing the circuit current through it will also create an unavoidable, wasteful power loss in that IC, and a "heat sink" (and thermal conduction paste) should be considered to avoid damage to that chip. If the input voltage to the voltage regulator is close in value (say just 3 volts higher, as required for it to function properly) to the desired and set output voltage, then there is only a low voltage drop needed, and therefore a low power loss in that IC and a heatsink (or heat-sink) may not be needed. One nice feature of this particular IC is that the output voltage will remain the same, fixed at the desired value, even when the input supply voltage varies higher and-or the output load resistance varies. The LM317 chip manages to do this by sensing a small amount of output current that is fed back into a pin on the chip through a resistor, and this "feedback" or "sensing" current will actually be used to then automatically adjust, regulate and-or maintain the output voltage to the desired level. With a few extra parts, such as an AC voltage step-down (voltage reduction) transformer, 4 power diodes, and high capacity capacitor(s), especially at the input, and 4 power diodes, and two voltage setting resistors [note, the LM78XX fixed voltage regulators have these resistors built in], this voltage regulator IC circuit can also be made into an AC (such as from typical 120Vac household outlet voltages) to DC voltage regulator. The chip even has a short circuit protection at the output or load, and it will then essentially turn off the output current. The output current for this specific LM chip number is also limited to regulating a maximum of 1.5A of current. It is best then to use a transformer that can supply a minimum of 1.5A of current or greater. The diodes chosen must also be capable of handling a minimum of 1.5A of steady current, and the voltage ratings of the capacitors used need to be capable of safely handling the applied voltages so as to not be damaged. Capacitors with a voltage rating of 1.5 times higher or more than the max. input voltage expected to the LM317 chip are recommended. For a steady dc supply input to the LM317 type of voltage regulators, the input and output capacitors are optional, but small ones are often still recommended. Further ahead in this book, there is a circuit example of a variable voltage power supply based on the LM317. A premade circuit (called something such as a LM317 circuit board module or kit, voltage regulator) can also be purchased inexpensively (a few USD), such as from the **Ebay.com** website. Some of these kits include a LED digital display for the voltage and-or current, however, these small displays and-or meters are now relatively inexpensive to include in many projects. The transformer is usually sold separately, and is needed to convert AC household voltages to (usually lower) DC voltages. This is the most expensive part of a power supply, perhaps 20 USD new, and it is a good idea to save and-or obtain inexpensive and-or used or recycled ones if they are available. Due to this high cost, you may wish to place this into its own box and reuse it for various other projects and-or experiments. The LM317 has a maximum input voltage of about 40Vdc, but due to a voltage loss in it or "voltage dropout", 37Vdc is maximum. The common size and packaging of the LM317 is with TO-220 case (about 15mm by 15 mm), but the LM317L, LM317LZ and-or 78L05 version has the small plastic TO-92, three lead transistor-like case, and can handle 100mA of current, and is about half the price of the full size and power LM317. The **LM317HV** IC is capable of regulating 1.25vdc through 60vdc, 1.5A max., and with a voltage dropout of 4.2v. The **LM338** is a 5A version of the LM317. Similar to the LM317 are the preset (not generally adjustable, but one constant preset voltage by using internal resistors in the IC) voltage regulator IC chips designed for a specific output voltage such as the common **LM7805**, 5v voltage regulator (very good for USB projects) with a dropout voltage of about 2V, and a max. current of 1A to 1.5A - check data sheets with the L7805. Due to this voltage dropout, ,the input voltage will need to be at least 7V so as to have a 5V output. Another type of voltage regulator for power supply projects is called a **"buck converter", "boost converter", or "switching converter" (due to the [repetitive, oscillating] DC square wave pules used, typically having a frequency of several thousand hertz), "switching regulator"**, and these use a small transformer so as to either increase the output voltage and reduce the output current, or decrease the output voltage and increase the amount of current output, perhaps twice that of the input, and so these converters (or power inverters) will waste much less power than a series or inline voltage regulator which essentially operates as a active resistive voltage divider. An inexpensive DC-to-DC, step-down, buck-converter module is based on the **LM2596** IC chip. If you need a (voltage) step-up, buck-converter, check out the **MT3608 IC** and-or other circuit modules. **A (AC to DC, or DC to DC) switching regulator is usually between 90% and 95% efficient at converting the input power to output power.** A non-synchronous switching regulator with its AC to DC rectifying power diodes (like a bridge circuit) is very good, but is somewhat less efficient (maybe 10% less) than that of a synchronous regulator with its MOSFET's being used as fast (low turn-on and turn-off times, very low Vforward drop), low internal resistance, high current switches, and with lower internal power losses. These MOSFET's are in synchronous and have opposite (on/off) conduction states, and it improves switching time speed and therefore, output power efficiency. **Switching regulators are very helpful for charging batteries,** however they will mostly likely introduce **RFI (Radio Frequency Interference) noise** either due to the switching oscillator circuit itself and-or some noise as high frequency voltage spikes and-or "ringing" (some low-amplitude, higher frequency oscillations) in the output DC power even if the regulator has some basic noise filtering methods. This electrical signal noise might affect other sensitive electronic equipment - perhaps a sensitive radio and-or certain frequencies. A common switching or oscillating frequency for these devices is about 100Khz since the higher it is, the less power (Joules of energy / time) wasted and the more efficient it is. Common noise filtering methods for DC power are a low pass filter composed of a capacitor that will have a low reactance and impedance to high frequency signals and will then pass them to ground and pass the lower frequency DC signal to the circuit output. Another method is a ferrite [or iron?] (RF) **choke** - basically an inductor, either a bar or toroid [ring, "doughnut"] shape which has higher (ac) impedance and filter to higher frequency (ie., AC) signals and noise, therefore acting as an electronic signal filter to reduce the signal amplitude (ie., of the voltage, current and power) of higher frequency signals. This choke and the wire(s) wrapped on it are generally used in "common mode" filtering or rejection which means that a common (ac) voltage spike on the wire(s) or line(s) wrapped on this choke - much then like a transformer, but with the coils wrapped in opposite directions, will be reduced as the induced magnetic fields, voltage and current will effectively cancel each other out to an acceptable small or 0 level. The noise filtering methods, if any, on inexpensive devices are often not enough due to costs, but it is still possible to purchase noise filters and-or make your own after having a study on them and their design. A choke will pass DC current, but impede (changing) AC current. Even some "**pure sine wave**" power inverters (ie., 12vdc to 12vac) might still produce some RFI which may particularly affect some sensitive electronic devices and-or radio (RF) receivers and-or transmitters. "**Modified sine wave**" inverts are based on circuit shaping a square wave into a sine wave, and may produce more noise., but these inverters are less expensive, and are adequate for some devices.

Because resistors, capacitors, and-or inductors have an impedance (Zohms) to AC current or signal, these components can be used to make (frequency dependent) AC voltage divider circuits.

Current can be divided when it goes through a parallel branch or circuit. A parallel branch is essentially a **current divider** circuit or path that is part of a larger circuit. The sum of current through each parallel branch (b) will be equal the total supply current to all those branches. According to Ohm's Law, the specific current through a certain parallel branch is determined by the total supply voltage (Vb) that is applied to all the branches in parallel, and the total impedance (Zb) of that certain branch being considered. If a branch has several components in series, then each will also have that same branch current through it, much like when a steam that varies in depth, wide and shallow, or narrow and deep, will have the same total flow of liquid past each location along it if no other energy is input into that system at some location.

Total current available to parallel circuit branches = (current in branch 1) + (current in branch 2) + **. . .**

$It = Ib1 + Ib2 + \dots$ For just two parallel branches containing just resistors:

$It = \frac{Vb}{Rb1} + \frac{Vb}{Rb2}$ Combining fractions, and we also know that:

$It = Vt / Rt = Vb / Rt$ and for only two resistors in parallel, this becomes:

$It = Vb / [(R1\ R2) / (R1 + R2)]$, and:

$$It = \frac{(R2)(Vb) + (R1)(Vb)}{(R1)(R2)} = \frac{Vb\ (R1 + R2)}{(R1)(R2)}$$

$$\frac{Ib1}{Ib2} = \frac{Vb / Rb1}{Vb / Rb2}$$

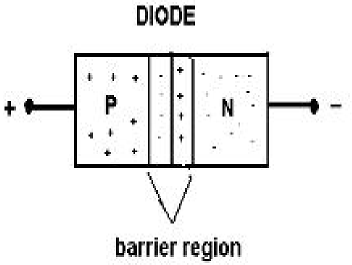
$\boldsymbol{\frac{Ib1}{Ib2} = \frac{Rb2}{Rb1}}$: **Ratio of currents and resistance of parallel circuit branches.**
This is an inverse type of ratio since current is inversely related to the resistance.
The lower the resistance of a branch, the higher the current through it.

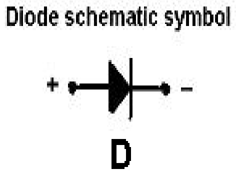
(Ib1)(Rb1) = (Ib2)(Rb2) = Vb = the same voltage across the two parallel branches

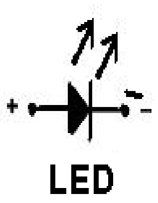
DIODE (An Electronic Current Flow, One-Way Or Direction Valve) : A passive device, but some power is lost in a diode

The word "diode" means "two-electrodes", but that is rather its physical description and not its electrical description. Its electrical description is much like an electronic one direction or way value or gate for current to flow.

An ideal or theoretical **diode** is an electronic device that conducts current in one direction, much like a mechanical valve that permits a fluid to flow in only one direction and is closed to the flow in the other direction. In actual practice and what can be manufactured at the present time, it is a device that conducts most current in one direction, and only a very small, usually negligible amount of current ("reverse current", or "reverse leakage current") in the opposite direction, hence it is not an ideal diode in terms of theoretical function, but it is still practical enough to be used in most circuits. An ideal diode also has 0 ohms of resistance like a short circuit or switch, wastes no power, turns on at any voltage above 0vdc, and has infinite impedance (ie., resistance) to the flow of current in the opposite direction, much like an open circuit. Most common (non-ideal) p-n silicon diodes, such as the very common, mass produced, and low cost (small signal power) **1n4148** "fast switching" diode in a small glass cylindrical case have a rated turn-on (forward or conduction) voltage of about 0.7Vdc, and the diode has a low ON or (forward) conduction resistance, usually less than 1ohm. It was first produced as the (very similar) **1n914** in 1960 by the Texas Instruments Co. It's maximum continuous rated current is 300mA. and has a maximum reverse voltage rating of 75v. Some 1n4148 diodes may have a continuous current rating of only 150mA to 200 mA. Its maximum on-off switching frequency is rated at 100mHz. General purpose low power (<=2A) silicon diodes are very common and inexpensive. [FIG 268]







A solid-state diode is composted of two pure crystalline metal pieces, often silicon, but could be germanium as used initially, and that were made to have an abundance of charge carriers of either free electrons (as for N or negative type material) or free "holes" (as for P or positive type material) so as electrons and holes can essentially move or be relocated through the crystal lattice structure of the materials. When the two semiconductor metals are joined together, the charge carriers of each type of semiconductor material will essentially cross at the thin junction region and create a (electrostatic, charges) conduction or voltage barrier (that must be first overcome by an applied external electric potential so as the diode can conduct electricity) across that effective "depletion [low conduction] region" or junction which can be thought of as being void of any charge carriers (ie., hence, a very high resistance to current, until it is overcome by a "voltage pressure or breakdown") which would allow easy conduction. The voltage to overcome a diode's junction or barrier region is called the barrier or "turn on" (conduction) voltage. When the **barrier region** is naturally created with a sufficient, limited number of charges (electrons and holes) near the junction, it will also prevent more charges from continuously neutralizing each other across the junction or barrier region and then that region growing larger in size. From this discussion, if someone was to call the barrier region a electrically neutral or neutralized region, I would agree with them.

For current to flow through the diode, it must overcome (be greater) or "breakdown" the current flow barrier at the semiconductor junction. The barrier (breakdown) voltage is about 0.7v for a silicon diode, and 0.3v for a germanium diode. At this voltage the diode is said to be fully on or conducting current, and having a very low resistance. Because this must be overcome to conduct electricity. The applied breakdown voltage will essentially be the force needed to push and conduct charge through the barrier region as if eliminating it. Because there is a voltage loss across a diode, there is also a power (P) loss through a diode (d) that is equal to:

Pd = (current through the diode)(voltage drop across the diode)

Pd = (current through the diode)(barrier voltage) , and:
Pd = (Id)(Vd)

The diode symbol on the right is used in electronic circuit schematics (ie., representative plans, diagrams, drawings), and it is symbolic of one of the first diodes used and that is the point-contact diode, and where a pointed tip of a wire was placed on the surface material of the diode so as to find a good location where (diode) rectification happened. The banded indicated side or end of a diode is the negative or cathode end.

Each diode will have a maximum voltage, current and power rating. Each diode also has a maximum reverse or "inverse voltage" that can be applied before a diodes barrier electrically breaks down and starts to conduct current in the opposite direction. This is a diodes breakdown voltage, and is usually much higher (maybe 25v to 1000v, check the data or information sheet or literature) than the "forward" or "turn-n" (conduction) voltage of the diode, such as 0.7v. These diode parameters, like any other circuit component or device, must be taken into consideration when making a circuit that has certain parameters (ie., values) and requirements. Some galena, germanium, and Schottky diodes have a low forward voltage (Vf or Vd) of about 0.25V, but common silicon general purpose diodes, they have about a 0.6 to 0.7v drop across them needed to conduct well.

At lower applied voltage levels before a diodes barrier voltage (about 0.7v) is reached, a diode behaves like an open circuit or high resistance to the flow of electricity. An ideal diode would turn on and conduct electricity at exactly 0.7v and always behave like either an open or closed circuit depending on the applied voltage. In practice however, diodes are not ideal, and some forward current will flow depending on the applied voltage that is below the "turn on" voltage. With applied voltages of less than 0.7v, the diode will essentially behave like a voltage controlled resistor or resistance. As a thought experiment, it will change from say 100kohms at very low applied voltages to, about 1 ohm at 0.7v when the diodes barrier or junction voltage is reached and it will essentially "turn on" or "switch on" to conduct (forward, normal) current like a low value resistor, and ideally like a short circuit or switch.

The **effective resistance of a diode** is: (voltage across the diode) / (current through it) = Vd / Id

Diodes are often used in or for: radios, removing voltage transients ("spikes", quick and large voltage electric pulses, such as from static electricity, lightening and other (high voltage, high power) electronic noise such as from inductors and motors), voltage references, and in some voltage regulator circuits such as for a full wave AC rectifier (ie., wave inverter) or "bridge rectifier" in electronic power supplies or signal conditioning and processing so as to create a near DC signal. Many semiconductor devices, such as (BJT) transistors, essentially have diode junctions within them such as between the base and emitter (pn) semiconductor materials, and it is important to first have a basic understanding of diodes so as to help understand transistors and other "active (electronically controlled, operated) devices". Sometimes diodes are used to protect electronic devices or components such as transistors (BJT's and FET's). For example, two reversed Zener diodes in series can prevent high voltage spikes from damaging the thin insulating gate of a FET by then conducting and shunting the energy to the ground and away from the FET transistor. The conduction voltage would be the (specific rated breakdown) voltage (Vz) of the Zener diode, plus the normal conduction voltage of a Zener diode (ie., Vd=0.7v). The "turn on" voltage of this arrangement would be: (Vz + Vd)

Because a radio wave (RF energy) received by an antenna is a very small amount, both in voltage and current, there is not enough voltage available to over come the conduction or forward voltage of a (0.7v) silicon diode, and rather a diode such a germanium with a forward voltage of about 0.25v is used. Still, even with a germanium diode, there will be signal power loss through the diode and this will reduce the audio output level. Some people have even applied a low ("bias") voltage across (ie., parallel to) a diode so as to help it conduct, and so that the diode is not damaged by a high current,, a very high resistance (such as a fixed safety resistor, say 10k ohms in series with a variable 1Mohm variable resistor) is in series with the battery of say having 1.5v or possibly 3 small solar cells if there is enough sunlight. It is also possible to amplify a weak radio signal before or after the diode by using a transistor, and in fact, a single transistor can be used as both a diode due to its base-emitter pn junction, and it can also be used to provide some signal amplification. In this circuit, a BJT transistor and-or amplifier is biased by a supply voltage, but is biased to be barely in the ON or conduction state due to the very low DC current allowed into the base by using a fairly high resistance and-or impedance to the AC signal. The base-emitter diode junction will then have a high resistance also. The base bias resistance is say 100Kohms to 1M ohms when the voltage source is about 5v to 12v, and this also prevents "loading" or "dampening" of the oscillations in the LC "tank" or tuning circuit, and it helps ensure that a highest voltage of the received signal is applied to the input of the transistor. An FET transistor can be used also, and it was designed to already have a high input impedance. **As a reminder, when two resistances are in parallel, the effective or combined resistance is less than or equal to half of the largest resistance, and is slightly less in value than the smaller resistance.** A signal will see or be affected by the combined resistance and-or "the load". If the load resistance is low, more current can flow. If the load resistance is high, less current will flow. With impedance matching, the most electric power (= V **I**) is transferred.

Power diodes can safely pass or handle currents >= 1A. The **1n4000** or= **1n400X** series (1n4001 to 1n4007) are low cost silicon diodes in a black cylindrical case, and are only capable of low frequency (a few khz maximum) switching applications such as for power rectification, reverse polarity protection and safety, and for circuit applications and-or experiments. The 1n400X series can safely pass 1A. The **1n4001** can handle a reverse voltage of 50V. **The 1n4007** can handle a reverse voltage of 1000 volts. The **1n540X** series are 3A versions of the 1n400X series.

A **thyristor** is like diode and a transistor combination, and having a third semiconductor layer. It has a third terminal between the other two layers, and it is called a gate which can turn the thyristor's conduction and diode action on or off, however it cannot amplify or modulate the current through the device like a transistor can. A thyristor is essentially a non-mechanical, electronic (semiconductor) relay. The **SCR** (silicon controlled rectifier) and **Triacs** are based on the thyristor. An SCR is generally for DC (ie., one direction) current and-or power (VI) control to a device, such as motors and dimmer circuits. A Triac is similar to an SCR,however it is made to also allow AC current flow, and is essentially two parallel SCR's in reverse connection. Triacs typically have a lower current ability than that of an SCR.

TRANSFORMER

Today, the common meaning of the word (electronic or electricity power) **transformer** is an electronic device to transfer power from one circuit to another, such as converting 120v AC signal (voltage, current and frequency) house-hold power to a safer, lower DC signal and power. A transformer can be designed to alter or change how that power (P = voltage x current, watts) will be transformed and output as either a maximum voltage or current. A transformer will have a maximum input voltage and current rating, and each transformer will have a maximum output voltage and current rating, and generally due to the wire gauge used. Transformers can also be used to electrically isolate two parts of a circuit since the power is transferred magnetically with a magnetic field extending from the input coil to influence or induce a voltage and cause an electric current in the output coil and circuit. The amount of current in the output circuit depends on the resistance of the transformer output coil wire and the external circuit attached to it.

The main construction of a transformer is essentially that of two (magnetic field) inductor coils placed near each other. The input power is transferred magnetically from the input or primary coil to the output or secondary coil. Many transformers have both coils or "windings" interwoven (crossing, among), next to (ie., parallel to) each other so as to be more efficient at transferring all their input power available to the output. The magnetic field of a wire created by current has a circular in shape around the length direction which will affect (ie., "cut", influence) another wire in parallel to it. This, along with using metal (magnetic conductors) to direct the current created or induced magnetic fields can increase the efficiency of that transformer. A power transfer efficiency greater than 90% is satisfactory and typical of a transformer. The ("dc", constant, natural, "ohmic") resistance of a transformer's internal coils (ie., inductors) of wire will cause an energy (joules) or power loss. Power = joules/second. Most of the energy loss is converted to and appears as heat energy.

For a magnetic field to induce a voltage and current in a coil, that magnetic field must be changing (increasing or decreasing) in intensity, growing or reducing in strength, so as the "magnetic (field) lines" are increasing and extending (ie., moving) outward and essentially "cut" (through or "across") those windings of the coil and induce a voltage in that coil. Either the wire (with electrons) and-or the magnet field must be moving with respect to each other. For maximum induced power, each should also be perpendicular to the other. A constant or steady magnetic field will not induce a voltage in a nearby stationary coil. Because of this, the input signal to a transformer must be a form of AC (or pulsed DC) power having an AC voltage and corresponding induced AC current. The initial pulse of DC current to an inductor or transformer will induce a temporary changing strength magnetic field and then it will remain at a steady level until the current is lessened or removed.

The coils of a transformer are magnetically coupled so as to transfer electric power magnetically. If the input and output coils of a transformer have the same wire thickness size and number of windings, the output voltage will be (ideally) the same as the input voltage. In a (voltage) step-up (ie., increase) transformer, the output coil has more windings or turns - often of a thinner gauge wire. In a (ie., voltage) step-down transformer, the output winding has less turns - often of a thicker gauge wire. Due to energy conservation, the output power will remain the same (constant) value as the input power. Since power = (voltage)(current), and is a constant, if voltage increases at the output, current will decrease. If voltage decreases at the output, current will increase, and this is an inverse physical relationship, and a mathematical relationship to express it:

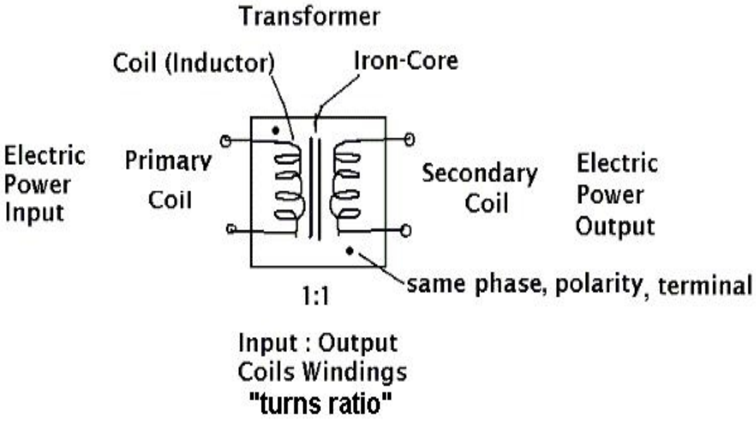
Power Out = Power In : due to "conservation" (no actual loss, only a transformation of) of energy or power
$Po = Pi$
$(Vo)(Io) = (Vi)(Ii)$ mathematically:

$\frac{Vo}{Vi} = \frac{Ii}{Io}$: the ratio of output voltage to input voltage is equal to the reverse (ie., inverse) ratio of the output current to the input current. These are "reverse or non-corresponding ratios".

Each transformer will have a maximum input and output power (voltage and current) rating. Even if the maximum output power drawn is not exceeded, if more current is drawn by the load device or circuit (ie., the resistance of the load circuit) than what that transformer is rated at, the coil can get hot and damaged. Fuses and-or circuit breakers with same current rating as that of the transformer can be used to protect the transformer and-or load circuit.

Can a transformer be used in reverse of that which it was designed and-or rated for? In theory, a "**reverse fed**" transformer could work, however we must remember the maximum current, voltage and power ratings of each coil (input and output coils) of the transformer. so as to not cause a thermal problem which will most likely damage the transformer and-or cause a fire.

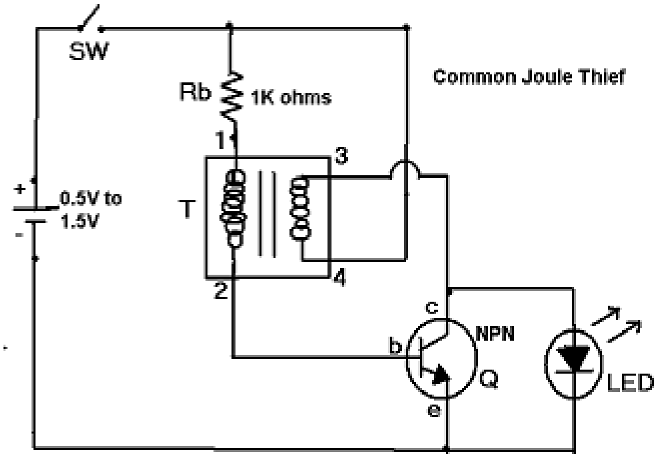
Electric power companies decided to transfer power to distant customers in the form of a very high voltage (thousands of volts) and low current. Placing a higher voltage on the wires effectively reduces power losses due to the high current resistance of very long transmission wires which would significantly reduce the available current and power (voltage x current) from the generator. This is because power available and-or power loss is related to the square of the current:
P = I^2R. Because this high voltage is impractical and dangerous to use for households (homes and small electrical devices), it must be reduced to a practical and safe level, and a voltage step-down transformer is usually used for that. By using a step-down transformer, household voltages supplied by the electric utility company and its "grid" (wires) system are reduced from a high voltage to be lower voltage - typically 240vac and 120vac so as to sufficient enough to be able to run household machinery and incandescent lamps. Still, this voltage is even too high for many low power devices such as radios and TV's which require far less power, and the voltage to them is reduced by using another transformer - often placed directly inside the device, or an external (120vac to DC) plug in transformer. [FIG 269]



In the above drawing of a transformer, you can imagine a pulsating AC (ex., from a 120v, 60hz, household power outlet) input voltage and current signal applied to the primary coil and then creating a corresponding (ex., 60hz) pulsating magnetic field which will extend outwards around it and interact with the secondary coil and induce an (AC) voltage across it. If the transformer is a 120vac to 24vac voltage step-down transformer , that turns ratio and-or voltage ratio is 120vac / 24vac or= 5 / 1 or= 5 to 1, and should be indicated as 5:1 on the packaging and-or data sheet. The iron metal core helps conduct and transfer the magnetic field and its energy from one coil to the other, and it improves the efficiency of the transformer so that the power output from the transformer is nearly equal to the input power to that transformer.

A very useful, small and inexpensive circuit with a transistor (small NPN plastic type), a resistor (~1K ohms), a small homemade transformer using two (usually in parallel) strands of thin wire (about 30 gauge, such as from a junked speaker coil) wrapped (about 22 or slightly more turns) around a small (iron particles) ferrite ring or core, a 1.5V penlight battery, and a white light LED that you may wish to explore is more popularly called a "**Joule Saver"** circuit, and which is sometimes incorrectly called a "Joule Thief". A more appropriate name and-or description is the joule saver and-or energy retriever circuit since most of the energy stored within a battery can be retrieved out from it. This circuit oscillates at several thousand hertz (cycles per second, here, pulsed DC) and boosts (ie., increases) a low voltage output of a 1.5v (or less) battery to about 3 or more volts necessary to make a LED give off a decent amount of light. Though voltages higher than about 3.5Vdc are not recommended for a white LED, but if those voltages are only briefly or temporarily applied as a pulse, there won't be as much of damage issue, and may actually improve the usefulness and-or amount of light (ie., photons, radiated RF energy) output from the LED. You may also look up the topic of pulsed led circuits for improved brightness and-or high brief flashes of light. The circuit is somewhat like a solar powered, low powered (small internal battery) garden and-or decoration light, and that charges up during the daytime light and automatically "turns-on" at night and produces some light, however those circuits seems to stop working when the battery has drained to about 1V. The Joule Saver circuit will continue reasonably working and providing some light to as lows as about 0.5 volts remaining in the battery. This circuit will help get (ie., not actually steal) or retrieve the remaining energy (ie., Joules) out of the battery so as to be useful and not otherwise wasted energy and resources. It is particularly helpful for making a small size flashlight and-or when only one, 1.5v standard or 1.2v (typically rechargeable) battery is available. Note that it is still generally always possible to connect batteries in series so to get create higher voltages necessary for devices. For example, connecting two 0.5v (ie., low voltage and-or drained) batteries in series will produce (0.5v + 0.5v) = 1v. Some experimenters have even powered a joule saver using a "voltaic pile" battery made from copper and zinc metal for each voltaic cell, and having just a few voltaic cells stacked in series so as to obtain a larger voltage. Between each piece of metal was a piece of paper soaked with an electrolyte of either lemon juice and-or vinegar, possibly with a small amount of dissolved salt. Possibly, a switch could be used to temporarily remove any battery in series so as to save some energy.

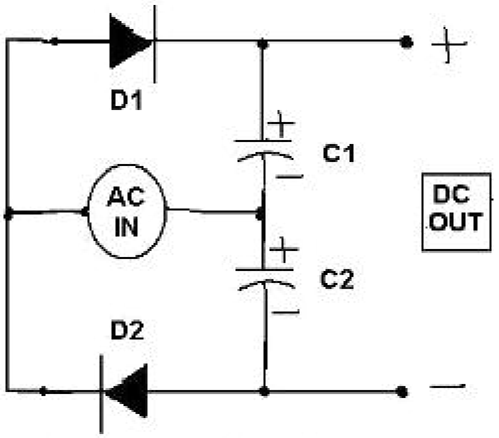
The circuit given below is the commonly available, standard and popular circuit of a Joule Saver circuit, and of which some people still experiment with and improve upon such as for the maximum brightness, efficiency and the longest "life" (time of use) of the battery(s) possible. For example, a small ceramic capacitor (perhaps 1000pf) can be placed across the base resistor (Rb). If an oscilloscope is available, you can see the waveform(s) produced on its display screen, and so as to help perform experiments and "fine-tune" (adjust) the device for maximum efficiency. The circuit is based on an electronic oscillator that oscillates at say at about 20Khz to 40Khz frequency depending on the circuit and its construction, and particularly the construction of the (transformer) coils. The waveform applied across the LED is similar to a pulsed square wave. The transformer output or secondary wires are reversed so as to apply a positive voltage to the collector of the transistor. The voltage waveform applied to the transistor is generally a short duration spike for each cycle of the waveform and-or oscillator. If possible, a circuit builder should obtain some low cost "alligator" clamp-ended leads (lead-in and-or lead-out wires) or "hookup-up wires" so as to more easily build and or-test circuits without the initial need for soldering and-or de-soldering. [FIG 270]



The led is a common white light output, 5mm or 3mm diameter LED, and requiring a DC current of about 20ma for full, and device-safe brightness. The transformer (T) is relatively small with thin gauge wire being 30 (typical) to 22 American Wire Gauge (AWG) of enamel coated wire, and with about 22 turns of each wires being in parallel ("**bi-filer**"), albeit not always indicated by the schematic diagram) to each other. This wire pair should be wrapped with a single layer around a circular shaped **ferrite** ("bead", torroid "doughnut ring" core) **core** that is about a 1/4 inch to 3/4 inch in diameter. The ferrite increases the inductance of the coils and permits the use of less turns of wire needed, and it is fast and has low magnetic energy losses of the electrical power. The transistor (Q) such as the **2n3904 NPN** is a low power, inexpensive, small black plastic npn silicon transistor. It has a current amplification or gain of about 100 to 300 depending on the current (200 mA max, 40 Vce, 625mW total power handling, with a cut off frequency [no more gain = 1] at 300 mHz.). A pnp type of transistor such as the common **2n3906** can be used, but you must then reverse the polarity of the battery so as to have a forward or conduction bias applied to that type of transistor. The battery should drain down to about 0.5v, but if a germanium transistor is used, the battery energy (ie., joules) can drain even lower because the turn on voltage of a germanium transistor is about half that of a silicon transistor or diode, hence about only 0.3v. Some common germanium transistors are the 2n1309 (pnp, gain of 90), 2n1307 (pnp, gain of 60). Because of the limited desire and availability of germanium transistors today (2022), they are currently several times higher in price than the mass produced silicon types. Germanium transistors generally have about half the power handling capability of similar sized silicon transistors, and a higher reverse bias "leakage current". You can experiment if the value of Rb should be slightly higher or lower when a germanium transistor is used, and so as to help conserve some energy of the battery. Two 1.5 batteries in series can give a long time of use for this circuit, and effectively allowing each cell to drain to 0.25v so as to have the minimum sum and voltage of about 0.5v needed for the circuit to produce some light.

If you have an oscilloscope, you can visually observe the oscillations of this circuit when placing the oscilloscope's input or sensor probes across one or more of this circuit's components such as the LED.

Here is a simple passive AC to DC voltage booster circuit of which for here is actually a voltage doubler circuit. There are also similar passive circuits such as to triple, and quadruple the output voltage. [FIG 270B]

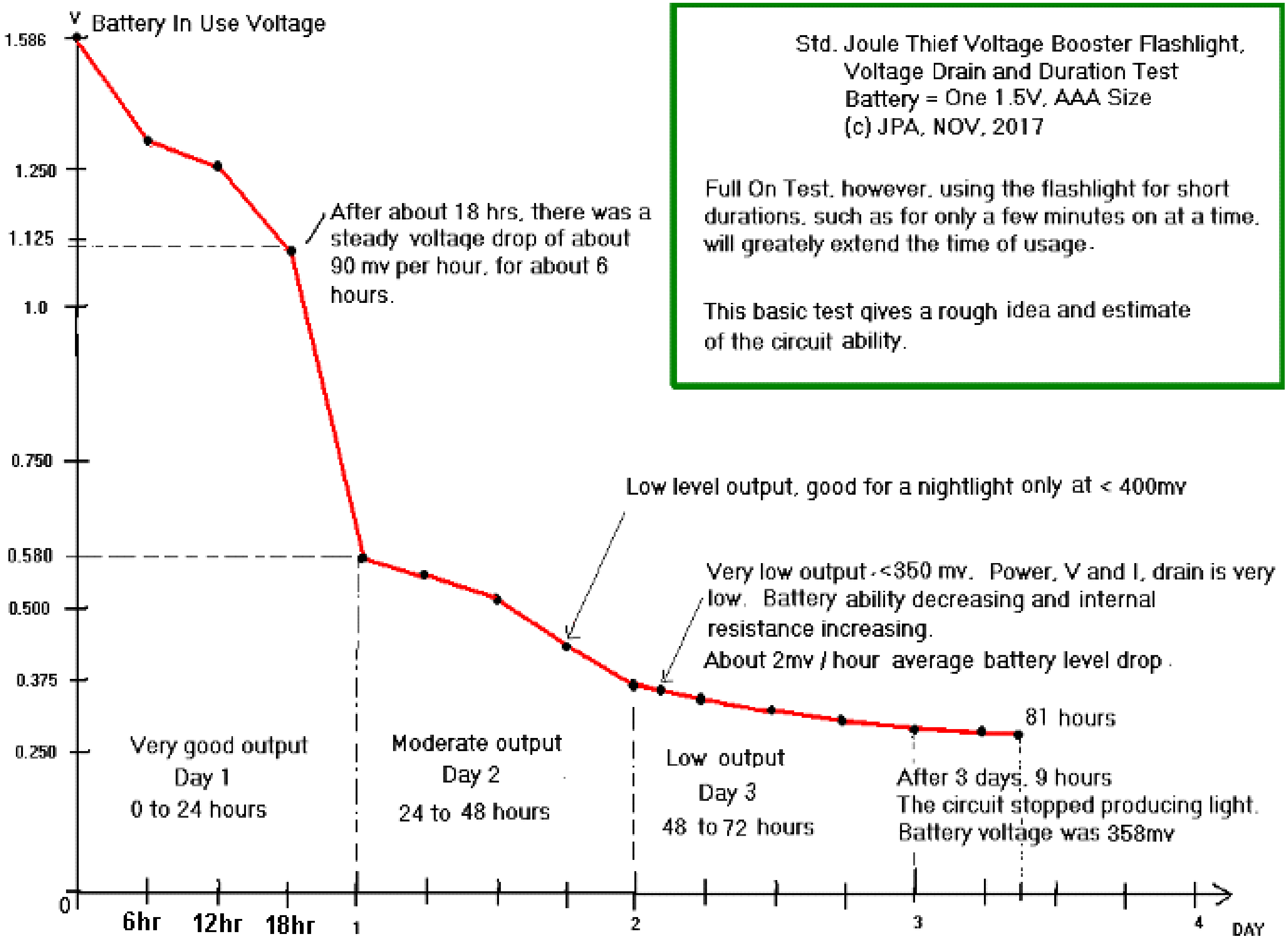


Vdc out =~ Vc1 + Vc2 + Vd = Vac rms + Vac rms + Vd ,
hence about twice the AC rms voltage plus
the voltage across the one conducting diode.
during each half-wave or half-cycle of the
input AC voltage waveform.

The values of C1 and C2 are identical, and will need to be found by experimentation with the load and-or current needed, but for the sake of simplicity, the larger they, are the better in terms of a higher output current. During each half-cycle of the AC input voltage, one capacitor will charge, and since both are in series, the total voltage available will be the sum that is on each capacitor, hence doubling and-or much like two batteries in series. The diodes direct the charging current, and they also prevent the capacitors from discharging back through the generator and-or the other capacitor.

An (active) voltage booster circuit was created and actually patented in about 1930, and this used a vacuum tube based amplifier and oscillator, and later, similar concept and patented versions by the early 1950's used the more fast and efficient - lower power loss, transistor. It should be of no surprise that if a transformer can boost an input oscillating (with a frequency, and changing in value) voltage, then an oscillator circuit could be used if the supply voltage is a DC supply voltage such as that from a portable battery, and so as to have a varying signal for the transformer to function. These concepts can be found in power inverters where a ("automobile") 12 volt or 24 volt, high load (ie., current) DC battery is converted to ("household") 120 Vac using an oscillator and transformer. Though the voltage is boosted, say 10 times, the maximum current is relatively low, being reduced say 10 times by the transformer. A 120W load on this type of inverter will require (from P=V/I, then: I = P/V = 120W/120V) = 1A of current. A 1200W load will require 10A, and very thick wire must be used for safety. Batteries save and-or create charge or current, and a high circuit current will deplete a battery relatively fast, and "deep cycle" batteries are preferred where they can be significantly depleted or drained without much negative effects, and are said to have a "longer life" or number of charging and discharging cycles.

The figure below shows the battery voltage of the "joule saver" circuit shown previously in [FIG 270] when working with a load (here, a mainly the LED) with respect to time. [FIG 271]



Std. Joule Thief Voltage Booster Flashlight,
Voltage Drain and Duration Test
Battery = One 1.5V, AAA Size
(c) JPA, NOV, 2017

Full On Test, however, using the flashlight for short durations, such as for only a few minutes on at a time, will greately extend the time of usage.

This basic test gives a rough idea and estimate of the circuit ability.

After about 18 hrs, there was a steady voltage drop of about 90 mv per hour, for about 6 hours.

Low level output, good for a nightlight only at < 400mv

Very low output -<350 mv. Power, V and I, drain is very low. Battery ability decreasing and internal resistance increasing.
About 2mv / hour average battery level drop.

After 3 days, 9 hours
The circuit stopped producing light.
Battery voltage was 358mv

Many small sized, low power solar powered "garden lights" have a small internal IC chip, a small rechargeable battery, and a few other components such as a small inductor (ie., no quartz version yet) working together so as to boost voltages between 0.9v and 1.5v to a 3v value for a white LED to function. This 4-pined low-cost, transistor-like chip is part of an oscillator circuit that forms a voltage "boost converter" and-or a "switching regulator". The most common IC for this is the 5252F, but there are some similar ones such as: CL0116 and XY8018 as of the year 2024.

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Wire Gauges (AWG)

AWG = America Wire Gauge. This system is sometimes called the Brown And Sharp (B&S) gauge system. The thinner the diameter of a wire, the less effective "room", area and-or cross sectional diameter it has for passing current. There will be less free electrons to assist with conduction, and therefore, that thin wire resistance will be higher than that of a thicker wire of the same length. Just like a resistor component in a circuit, a higher resistance for a length of wire,the higher the voltage drop (loss) and power loss for a given current through it. A wire thickness and length combination must chosen so as the current though it does not cause it to overheat and melt its protective insulation and then be a danger to people and-or circuitry. With some high current devices, such as various toasters and heaters, it is recommended to not to use any extension cord wire since the total power lost through the wire may cause it to overheat due to the high amount of current (ex., 10A, for (120v)(10A) = 1200w) used in those devices. Some wire is also made and intended to be hot, such as in various heaters and toasters. Like the internal wire ("filament") in most incandescent light-bulbs, these wires may be a low resistance when power is not applied to them and they are not hot yet, but will have a much higher resistance and power loss through them when hot. Due to this initial cold, low resistance value, there is a high valued, temporary "inrush" of current which will stabilize to the normal operating current and its corresponding power rating. This inrush of current can cause a worn-out light-bulb filament to "blow-out" (melt to an open circuit, no circuit, no current, no light produced), when it is first turned on, unless some current control to it is used.

The longer a wire is, the more total resistance between its endpoints. There will be an increased energy loss or voltage drop across it for a given current, and therefore there will be an increase in power lost through that wire which will then not be available to the rest of the circuit. In most circuits for home use, this is usually not an issue. For high power or wattage devices, a long cord or wire can effect how that device operates, such as reducing available power significantly, such as needed for the speed and-or power of a motor.

Obviously, the thinner a wire is, more parallel wires or coil turns of it could be placed side by side per unit of length measurement of that wire. This is basically what the Brown And Sharpe (B&S), or (North) American Standard Wire Gauge is based upon and became the commonly used **AWG = American Wire Gauge.** Thinner wires will therefore have a gauge (ga.) number that is higher since more of them can be placed in parallel per unit of distance. For example 30 gauge is thinner than 20 gauge. With this system, the actual gauge number is an (stepped) integer value corresponding to a certain logarithmic (ie., an exponent) value. For this system, the wires are also assumed to be round and having a solid core, however multi-core or stranded core wire that can carry the same amount of current as a solid core wire, will be slightly thicker, but will have the same rated gauge. Basically, the AWG system is based on a wires internal cross sectional area and therefore, its conductance ability for current or electricity, rather than just its diameter dimension.

The reasoning and actual formula used to calculate a gauge for the AWG system is a overly complicated for the average person to always consider, but for simplicity, the gauges are based on cross sectional area and current ability, and some simpler formulas derived from some results of the AWG formula, are given below. Here are some key points and relationships of this AWG system:

1. When the diameter of a wire is half (ie., 0.5) of that of another, the gauge will increase by 6.

   . When the diameter of a wire is half of another, its radius is also half, and the cross sectional area is decreased by 4, and therefore, the conductance decreases by 4, and this is similar to saying that the resistance increases by 4.

   A1 = (pi)r^2 and A2 = (pi)(r/2)^2 = (pi)(r^2 / 4) , therefore: A1 / A2 = 4 and A2 / A1 = 1/4 = 0.25

2. When the gauge increases by 3, its cross sectional area is half, its (safe) conductance ability is half, and that wire can only carry half (ie., 0.5) the current, and this is equivalent to the resistance increasing by 2.

4. When the gauge increases by 10, the cross-sectional area is reduced by about 10, and the conductance is 10 times less, and this is equivalent to the resistance being 10 times more.

5. When the gauge increases by 20, the diameter decreases by 10. This is equivalent to saying that when the

diameter increases by 10, the gauge decreases by 20.

Wire diameter or "thickness" sizes can be expressed in mils (thousandths of an inch), or millimeters (mm, metric):

1 mil = 0.001 inch = a thousandth of an inch = 0.00254cm = 0.0254mm , mathematically, solving for 1mm:
1 mm = 39.37007874 mils = about 40 mils
1 cm = 393.7 mils =~ 394 mils
1/16 inch = (1/16)" = 0.0625 in = 62.5 mils
1 square-inch area = (1in)(1in) = 1in^2 = (1000mils)(1000mils) = (1000)(1000) mils^2 = 1,000,000 square mils
1 square mil = (1mil)^2 = (0.001 in)^2 = (0.001)^2 (in^2) = 0.000001 in^2 = a millionth of a square inch
= (0.0254mm)^2 = (0.0254^2)(1 mm)^2 = 0.00064516 mm^2
1 circular mil (CM) area has a diameter of 1 mil = 0.0254mm, and its area is: A =(pi)(r^2)= (3.14159265)(0.0254mm/2)^2 =
= 0.000506671mm^2 = (5.0671)(10^-4) mm^2

For the ratio of a square area to that of a circular area (Ac) having the same width or diameter, is to first consider or draw a circle in a square such that its diameter (d) of that circle equals the side (s) length of the square, say 1 unit long. Therefore s = d = 2r , where r=radius of the circle. The area of the circle will clearly be less than that of the surrounding square. The area of a circular mill (a thousandth of an inch) is less than that of a square mill, but by what factor?:

Ac / As = (pi) r^2 / (2r)^2 = (pi)r^2 / 4r^2 = **(pi) / 4** = 3.14159265 / 4 = **0.7853981** (ie., roughly about 79% or 80%)
We find that a circular mil is about (100% - 80%) = 20% less than that of the area of a square mil. The perimeter of the circle and square mentioned above also have this same ratio: **Cc / Ps** = 2(pi)(r) / 4 (2r) = (pi) / 4 = **0.875398163**

A wire diameter or gauge might be expressed in an area value such as square millimeters: mm^2, and to find the value of its diameter, you can use the area of a circle formula: A = (pi) r^2 , r = sqrt (A / (pi) , d = 2r. 6 AWG gauges lower is 2 times the diameter and 4 times the area.

For a stranded wire containing several smaller strands of wire, the notation of its structure is usually expressed as:
AWG # "Number of strands / AWG gauge"

Some Common AWG and their corresponding approximate diameters, thickness or width. Resistance at ~ 72 °F :

Gauge Number	**Inches**	**Millimeters**	(Solid copper wire is considered for the other parameters). 1in.= 2.54cm =25.4mm
	Diameter		1mm = 0.03937 in. =~ 0.04 in =~ four-hundredths of an inch
#0	0.325	8.26	Area = A = (pi)(r^2) = (pi) (d/2)^2 =~ 55.6 mm^2 **:cross sectional, circular area**
#2	0.258	6.55	=~ (1/4) in = 0.25 in = a quarter of an inch wide , A =~ 33.7 mm^2
#4	0.2	5.2	A =~ 21.2 mm^2
#6	0.162	4.11	=~ 4mm , A =~ 13.3 mm^2
#8	0.1285	3.26	=~ (1/8)inch = 0.125in , A =~ 8.3 mm^2
#10	**0.102**	**2.6**	**=~ (1/10)inch = 0.1 in. , resistance = 1ohm/1000ft = 0.001/ft , A =~ 5.3 mm^2**
#12	0.081	2.05	=~ 2mm , A = 3.3 mm^2 , Note that this can carry four times that of 18AWG
#14	0.064	1.63	=~ **(1/16) inch** = 0.0625in , A =~ 2.1 mm^2
#16	0.051	1.29	A =~ 1.3 mm^2
#18	0.0403	1.024	=~ **1mm** , A =~ 0.82 mm^2
#20	0.0311	0.789	=~ (1/32)in = 0.03125in , resistance = 10ohm/1000ft = 0.010 ohm/ft , A =~ 1cir.mil
#22	0.0253	0.6436	
#24	0.0201	0.510	=~ 0.5mm = 1mm / 2 = (1/2) mm = half a millimeter
#26	0.01594	0.4049	=~ (1/64)inch = 0.015625in
#28	0.0126	0.321	
#30	**0.01**	**0.254**	**= (1/100)in =~ 0.25mm , resistance = 100ohm / 1000ft = 1ohm / 10ft = 0.1 / ft**
#32	0.008	0.202	, resistance =~ 164 ohms / 1000 ft =~ 0.164 ohms / 1 ft

#34	0.0063	0.160		, resistance =~ 0.264 ohms / 1ft
#36	0.005	0.127		, resistance =~ 0.418 ohms / 1 ft
#38	0.004	0.1		, resistance =~ 0.670 ohms / 1 ft
#40	0.0032	0.081		, resistance = 1000ohms/1000ft = 1ohm / 1ft
#50	**0.001**	**0.0254**	**= (1/1000)in = 1mil**	**,** resistance = 10 ohms / 1 ft

As you can see in the above data, it is easier for most wire installers and users to use simple gauge numbers than trying to remember the very small differences between the diameters of each gauge of wire. There is a pattern that can be had from this data, and a basic formula was derived out from it so as to fit that data. The following formula is based on the fact that when the gauge decreases by 20, the thickness should increase by a factor of 10 for each instance of this, and resulting in a more general formula that will include a variable for each instance of this process where the gauge decreased by 20. We now have this factor to multiply to the diameter: 10^instances or= 10^x. We will use AWG #50 and its corresponding diameter as the starting reference value for this particular formula, and so as to have a formula with only one unknown variable, and that is the gauge number of which you are trying to find the thickness or diameter of:

Considering gauge #50 which is 0.001" = 1mil : this should be memorized if using the following formula:

Thickness of Gauge: = (0.001") 10^((50 - Gauge#) / 20) : **SIMPLIFIED WIRE THICKNESS FORMULA BASED ON 50AWG = 0.001"**

Ex. What is the thickness or diameter of 28 AWG?

Thickness of 28 AWG = (0.001") 10^((50 - 28)/20)
= (0.001") 10^(22/20)
= (0.001") (10^1.1)
= (0.001") (12.59)
Thickness of 28 AWG = 0.0126 inches

Thickness of Gauge = (0.1") 10^((10 - Gauge#) / 20) : **ALTERNATE SIMPLIFIED WIRE THICKESS FORMULA BASED ON 10AWG = 0.1"**

Extra: After solving for Gauge #:

Gauge # = 50 - 20 log ((1000)(thickness in.) : **AWG #** : **1 inch = 2.54 cm = 25.4 mm = 1000 mils**
Gauge # = 10 + 20 log (0.1 in / thickness in) : **Alternate AWG #** **and 1 mm = 0.03937 in = 39.37 mils**

Resistnace Estimation: For each increment in AWG gauge, the resistance is about n =~ 1.25 times greater
Ten guages higher is ten times more resistnace.

This value of : n=~1.25 can be found from the geometric series: R , Rn , (Rn)n , ((Rn)n)n .... : R = resistance
R , Rn^1 , Rn^2 , Rn^3.......
Letting: Rn^10 = 10R : at ten AWG gauges higher

After solving for (n) by using logarithms, we find **n = ~ 1.258925** : This is also the 10th root of 10

Two AWG gauges will result in a factor of: (1.258925)(1.258925) = 1.5848931925 =~ 1.585 =~ 1.6

Some people in the United Kingdom (UK, England area) may still use the (outdated) **Standard Wire Gauge (SWG)** which is also called the British or Imperial wire gauge. For some basic reference, here are some SWG gauges and their approximate diameters: 33 SWG = 0.25mm, 25 SWG = 0.5mm, 19 SWG = 1mm. 14 SWG = 2mm. 11 SWG = 3mm , 8 SWG = 4mm. 6 SWG =~ 5mm. 4 SWG =~ 6mm, 2 SWG = 7mm **SAE gauge is the Standard American English gauge** or Standard Automotive Engineer Gauge for automobile wires and for screw diameter and pitch (ie., threads or turns per inch). An SAE Gauge wire size for an equivalent number AWG wire gauge will be slightly less or thinner in value, hence its maximum current will also be slightly less. These references to other and-or outdated guages is still useful for older devices.

Ampacity (Maximum Rated Current)

Ampacity is a word formed from the words "amps" and "capacity", and this word may sometimes be discontinued and superseded by other words such as "safe capacity" or "maximum amps" to express the maximum safe current in a wire. As for now, we will consider the word "ampacity" to be the recommended, safe heat and maximum rated current passing ability or capacity of a certain size [thickness] and-or type (element, copper, aluminum, etc.) of wire, and it is usually specified for a certain air temperature such as 20°C. The maximum temperature rating (typically some tested value between 60 °C =~ 120 °F and 90 °C =~ 180 °F) for a wire is usually determined by its insulation material, and clearly, if the insulation is hot and-or melting, a thicker wire is then needed and-or a lower amount of current used. The maximum current value rating is lower for thinner wires since they have a lower conductance per cross sectional area and a higher resistance per unit of distance. If the diameter (or radius) of a wire doubles, the cross sectional area (A) increases by 4:

A2 / A1 = (pi)(2r)^2 / (pi)r^2 = 4(pi)r^2 / (pi)r^2 = 4

Even though a wire has a recommended maximum rating, a further safety margin has been set by the National Electrical Code (NEC) of the USA, and most likely for any country. In general, it states that in a circuit branch and its given wire, the maximum current of that wire should be no more than 80% of the wires ampacity rating.

BELOW IS A LIST OF SOME TYPICAL AMPACITY RATINGS OF COMMON **SOLID CORE COPPER WIRE** AT ROOM TEMPERATURE, AND SHORT DISTANCES - NOT FOR LONG DISTANCE POWER TRANSMISSION, AND NOT FOR HIGH FREQUENCIES WHERE THE CURRENT WILL TEND TO MOVE ALONG THE SURFACE ("SKIN" OR WIRE SURFACE EFFECT), AND THE EFFECTIVE WIRES RESISTANCE, HEAT AND POWER LOSSES EFFECTIVELY INCREASE. TECHNICALLY, IF A WIRE OF A FIXED LENGTH AND WHICH THEN USUALLY HAS A FIXED RESISTANCE AND-OR AC IMPEDANCE THEN CARRIES MORE CURRENT THAN A PREVIOUS AMOUNT, IT WILL THEN CAUSE A HIGHER VOLTAGE LOSS OR DROP IN AND-OR ACROSS IT DUE TO OHMS LAW (V = I R = A R). THIS WILL HAVE MORE OF AN AFFECT, AS A PERCENTAGE, ON THE VOLTAGE OUTPUT ON LOW DC VOLTAGE LINES THAN ON THE TYPICAL HIGHER AC VOLTAGE POWER LINES WHERE TYPICALLY LESS CURRENT IS USED SO AS TO PROVIDE THE SAME AMOUNT OF AVAILABLE POWER (ie., energy/time).

AWG#	**MAX. RECOMMENDED CONTINUOUS (DC) CURRENT = AMPACITY RATING BASED ON DIAMETER**
0	A safe recommended, maximum value is 125A , : this gauge is also identified as AWG 1/0
2	A safe recommended, maximum value is 115A.
4	A safe recommended, maximum value is 85A.
6	A safe recommended, maximum value is 60A.
8	47.6 A =~ 50 A
10	30 A : needed for some high current household devices and-or some 12v vehicle systems
12	18.8 A. : the modern, common wire gauge required for many household (homes) and its electrical devices -
14	11.8 A =~ 12A - 18.8A =~ 20A
16	7.4 A
18	4.6 A =~ 5 A
20	2.96 A =~ 3 A
22	1.87A :this and AWG 24 are common ("hookup") wire gauges for many homemade electronic projects, ~ 2 A
24	1.18 A =~ 1A
26	0.74 A =~ 0.75A = 750mA
28	0.46 A =~ 0.5A
30	0.294 A =~0.3A : this is a common, fine wire gauge for making homemade coils such as for a voltage booster
32	0.186 A

The formula used for the above data is based on AWG 10 of solid copper wire with a ampacity rating of 30A, and the fact that for every 3 gauge increase, the cross-sectional area and conductance is halved (1/2 = 0.5). It must be noted that area is not linear or proportional to the radius or diameter of a wire, but rather it increases exponentially by the square of the radius. If the diameter or radius doubles, the area and conductance increases by 4, the resistance decreases by 4, and max. current can increase by 4, and this is the case when the AWG number is 6 less.

ampacity = (30A) 0.5^((Gauge# - 10) / 3) : **A basic ampacity formula for AWG solid copper wires**

This book does not have a table of powers of 0.5^x, but this can be used instead if needed:

From: ln 0.5 = -0.693147181 , we have: e^(-0.693147181) = 5 , and raising each side to the (x) power, we have:

0.5^x = e^(-0.693147181)^x = e^(-0.693147181 x) = 10^(-0.301029996 x) : a form of: n^x = (e^a)^x = e^ax

ampacity = (30A) 0.25^((Gauge# - 10) / 6) : **Alternate basic ampacity formula for AWG solid copper wires**

and : 0.25^x = e^(-1.386294361)^x = e^(-1.386294361x) = 10^(-0.602059991)^x

Because a wire has a maximum current rating, we can find the maximum voltage that can be applied to that wire:

Vmax = (Ampacity Rating) (Resistance of the wire) : has the form of: Vmax = Imax R or= V = I R

Because a wire has a maximum safe current or ampacity, and a voltage (Vmax) rating, there will be a maximum safe power loss in that wire = Pmax. The formula is based on: Power = Voltage x Current = V I = I^2 R = V^2 / R

Pmax = V I = (Vmax) (Ampacity Rating) or:

Pmax = I^2 R = (Ampacity Rating)^2 (Resistance of the wire) or:

Pmax = V^2 / R = (Vmax)^2 / (Resistance of the wire)

For increased safety and-or reduction of power losses in a wire, use a wire that has a greater current ability than what is actually needed for the total amount of amps needed and-or used for all the devices in series that are powered by the same power supply. It is possible to have other electrical devices connected in parallel (ie., another current branch) to the battery, and of which thinner wires can be used if they are rated for the current needed or used by the load device. Be sure to consider if the total current drawn from the battery from all the powered devices is below the maximum amount that the battery is rated for. For safety, fuses and-or circuit breakers should be used on each current branch from the battery. To reduce power losses in a wire due to increased length and its increased resistance and voltage drop (ie., energy loss) across it, use a thicker gauge of wire.

When current is drained from a battery, it will also reduce in voltage. Generally, most 12v lead-acid batteries are considered "dead" or drained at about 10 or 11 volts, and this is because it is below the rated voltage of the battery, and that there is no guarantee that it is sufficient for the devices being powered. At 11 volts or less, there is still useful power in a battery for devices designed to operate at lower voltages, and you should then consider having a voltage regulator so as to have a lower, but stable and-or adjustable voltage supply. Deep-cycle batteries can hold their rated voltage much longer than non-deep-cycle batteries, therefore they can deliver much more of their internal energy storage as useful power during a time period of usage. Some rechargeable lead-acid batteries area rated as deep-cycle, and some rechargeable batteries are also known as automatically having a deep-cycle like quality, such as with the lithium types.

Batteries are charged so as to store those charges for later use, and high current devices will drain those charges faster, and this is why some battery systems use batteries connected in parallel so as to have more (energy and-or power) capacity and ability, and at the same supply voltage. This greater energy supply can be used to transfer or deliver more total energy or power (VI) for a longer time. Total power = (total energy) / (total time) = (energy / time unit) = = (joules / time) = (VI) / time , such as watts / hour. A device may use and-or be rated in (watts / second), and to find how many (watts / hour) it uses or used per hour, multiply the number of (watts / second) by 3600s since an hour is 3600 seconds of time (T):

$$\text{total power} = \text{total watts} = \frac{\text{watts}}{\text{second}}\ (\text{number of seconds}) = \left(\frac{\text{watts}}{\text{second}}\right) T_s \quad : \text{for 1hr of time, use } T_s = 3600$$

$$\text{total power} = \text{total watts} = \left(\frac{\text{watts}}{\text{1 hour}}\right)(T_h) = \left(\frac{\text{watts}}{\text{3600s}}\right)(T_s) \quad : 1000\text{w} = 1\text{kw} \ , \ \text{ex: } 500\text{w} = 0.5\text{kw}$$

Thick wires are often made by stranding many thinner diameter wires together, and this helps to keep a thick wire more flexible so it can be bent around turns and-or rolled up on a reel. .

Though not mentioned much, and typically discouraged for good reason, in an emergency or difficult situation, it is possible to use two identical lengths of identical wire connected in parallel, so as to effectively create a thicker wire to double the current ability. If the wires are not identical in diameter and ampacity (max. ampere rating), the maximum possible current should only be considered as twice the maximum current or ampacity of the thinner diameter wire.

A wire has some resistance (Rohms) to current, and it will have a voltage and power loss in and through it. A wire could get hot and be a fire hazard. If the power loss is significant and cannot power a device safely and-or properly, then a thicker wire should be used since they have less resistance, and then there will be less of a voltage loss across it, and more available to the device being powered.

Wires of long length will have more resistance, and therefore will have a higher voltage drop or loss across and-or through them, and therefore a higher power loss through them: P=VI. If needed, it is sometimes possible to raise (increase, "step-up") an (AC) voltage or current with the aid of a transformer, but the total **power** available cannot be increased. Because power loss is related to the square of the current through a wire or resistor: P = I^2 R, which is sometimes called the **Joule-Lenz Law** of ohmic or resistive heating or loss (of power) and which is exactly compatible with Ohms' law (I = V/R), power companies usually generate and distribute their output power in the form of a very high voltage so as to reduce power losses due the otherwise large amount of current that would need to pass through that resistance of a very long wire. This will also prevent the wires from getting too hot and melting as it transfers all that power from one location to another. With this system using a relatively low amount of current, the magnetic field around the wire will be kept relatively small, but the electric field about the wire will be relatively high. These wires from the power company are often called "power" or "transmission" lines, and because of their high voltage, there is an increased chance of "arcing" (ie., sparking, short circuiting) to nearby objects of lower resistance, and therefore being a safety and fire hazard. These high voltages will be stepped down to lower voltages for cities and homes by using a local (nearby) transformer so as to have a more manageable, practical and safe voltage level. High towers hold the high voltage wires up off the ground, and large insulators prevent various arcing problems due to the metal tower itself, and also to insulate the wires from rain, moisture and snow which may promote arcing. Typical rain water does not conduct much electricity at low voltages, but at high voltages, it can "break down", where the atoms are "ionized" (separated into atomic parts) and can then conduct charges (electricity, current) fairly well, and therefore may cause arcing and-or (a low resistance path) "shorting out" into and through nearby objects (particularly metal, trees and people) and into the ground. This amount of power is very dangerous and can pass through and cause serious life-threatening injury to people and-or fires.

Metric Wire Gauge (MWG)

MWG Gauge = (10) (wire diameter in mm) , mathematically:

(wire diameter in mm) = (MWG) / 10

Extra: For the rated conductance and-or resistance of a specific metal see the topic in this book of:
A List Of Common Metals And Their Relative Electricity Conductance

POWER

Energy is the ability to do work such as applying a force to an object for a certain amount of time. Force is the application of energy, and then all that energy can be transferred to that object if there is no energy losses during the transfer of that energy. This transferred and gained energy in the object is usually transformed or converted to kinetic or motional energy of the object and giving it a faster motion, speed or velocity. The basic unit of energy is the joule (J). Power is a (average) measure of how much energy is available or used (ie.,transferred) during a certain amount of time or time period.

Watts are the units of power. Power = (energy / time) watts = (joules / time) watts.

1 Watt of power = 1 Joule of energy used per or during 1 second of time = 1 Joule-second = 1W

1 W = 1 J / s . The watt is named after James Watt who invented a newer, more efficient and more powerful
steam engine for boats and other machinery.

For an example of electrical power: A 1 ohm resistor having a total (electric, electron) current flow or conduction of 1 Ampere = 1A constantly through it for 1 second, will gain a total amount of energy of 1watt, mostly in the form of (often wasted) heat energy. A formula is shown below. The source that supplied that energy will have correspondingly used and-or lost 1 watt of energy or power. Usually for resistors, the gain in energy is seen as a gain in heat or temperature, and the power it gained is usually called a power loss or power drop in or through it because its no longer available to the rest of the electrical circuit. Power losses should be avoided since it is usually wasted energy, time, money and resources.

For electricity and circuits: P = V I = (IR)I = I^2R = V (V/R) = V^2 / R : with units of watts (W)

From Ohm's Law of: current flow = voltage (ie., emf) applied / resistance = I = V / R = voltage / resistance.

A voltage (emf, electromotive, electrical or electronic force due to electrical charges attracting or repelling) of 1 volt applied to a circuit having 1 ohm of resistance will induce or create 1ampere of current to flow through that circuit.

Mathematically, from Ohm's Law:

V = I R : Voltage can be thought of as the force or pressure to the available current and giving it kinetic energy. Voltage is also called the electromotive force = "emf", and voltage is also called the potential difference between two points since it will take a difference in this force or pressure so as to have any current to flow, and at a value due to that net voltage value of the difference. For example, If one point in a circuit is measured to be 302v and another point is at 300v, the potential difference between those two points is (302v - 300v) is only 2 volts, and this 2 volts may be across a resistor, and the current through that resistor is determined by that voltage, energy or (emf) difference, loss or drop of and across it: I resistor = Ir = Vr / R ohms .

.

1V = (1A)(1ohm) The greater the current (I=V / R), the more electrons (ie., matter, mass) with kinetic energy, and the greater the total energy being transferred. Power = (voltage)(current) = VI

When 1 joule (J) of energy is applied to 1 coulomb (c) of charge (Q, such as electrons), that charge will gain 1 joule of energy, and the energy density and-or electrical potential (ie., voltage, emf) ability of that stored (in electrons, as motion) energy is called 1 volt:

1J / 1c = 1 volt. :This energy is often the (kinetic, moving) energy or power gain of the electrons moving through a wire or circuit. This energy stored in the electrons as kinetic energy ca be used to do useful things.
Ex. 2J / 0.5c = 1V , Ex. 2J / 2c = 1V , Ex. 4J / 2C = 2V

Mathematically rearranging: 1V = 1J / 1c we have:

1J = 1V 1Qc : or 1 coulomb of charge at 1 volt of potential energy contains 1 joule of energy
For another ex.: 2 coulomb of charge at 0.5 volts will contain 1 joule of energy

The product of the voltage potential and the amount of charge having it, is equal to the total amount of energy in that charge. Another description is: When 1 volt of electromotive force (emf) is applied to 1 coulomb (c) of charge (Q, such as electrons) for 1 second, that charge will gain 1 joule of energy. This may sometime be expressed as: J = V C

1J = 1V 1Qc : Extra: The units of force are commonly newtons (N), 1N = (1 kg)(1 m/s^2). 1J of energy was initially defined as the energy used for and transferred by a force of 1N constantly applied to a 1kg mass for a distance of 1meter, and this will accelerate (change of, increase its speed) it to, or by 1 more meter per second if there is no losses (undesired transfers or wastes of energy) of that energy.

1J = 1N-m = V C = energy = work = (force)(distance) = (mass)(acceleration)(distance) =
= (kg)(m/s^2)(m) = kg m^2 / s^2 = kg-m-a

Also, mathematically: **1V = 1J / Qc , Voltage is the joules of (electric potential) energy per coulomb of charge. V = J / Q = (I)(R)**

Considering and dividing each side of the above equation by the time used so as to find energy used per second or unit of time, will result in the following definition of the watt unit of electrical power:

Since 1A = 1 coulomb (c or C) of charge (Q) particles (ie., electrons, current) passing a point per second = 1C / 1s:

1J / s = 1V (Qc / s) = 1V 1A = 1W : 1 watt of power or energy used

Mathematically, 1 joule is also called a "watt-second" = 1 watt of energy applied for 1 second = **1Ws = 1J of energy**

In relationship to Ohm's Law we find that power is:

Pw = V I = (I R) I = (V) (V / R) = I^2 R = V^2 / R

Also, from the above equation, we can derive:

V = J / Q : voltage (emf) is equivalent to the energy (joules) per unit (coulombs) of charge. Voltage can be thought of as a measure of the (potential) energy available that can create the electromotive force (emf) needed to move (attract and-or repel) charges such as an electric current. More voltage means more energy and electric force or ability to move charges.

Mathematically from this, we can derive:

J = QV :If the charge particle or mass was a single electron, there is a measure of the (kinetic) energy that 1 electron will gain or loose going through a 1volt potential difference (emf = electromotive force). This amount of energy is often called an electron volt: **1 eV = 1.602 (10^-19) J .** Besides this point, protons and especially electrons are traveling very fast, and even though an electron has a very small mass, its kinetic energy (KE) is significant and is verified, for example, when resistors get warm, and in electrolysis (ie., electro-chemistry). An electron being repelled by other electrons reduces its kineitic energy. This is often spoken as simply electrons strking an atom, and-or a collision of sorts.

When an electron breaks free from its atom, the remaining atom is said to be ionized. Electrons farther out in orbit around the nucleus of the atom are easier to break free because the electric force (ie., Coulomb force) that holds them in that orbit position (ie.,radius, distance) is less. It will take less applied energy or force to break them free from their atom. This amount of energy is sometimes called that electron's ionization energy or ionization potential. Dmitri Mendeleev who is generally credited to making the first practical periodic table of elements discovered the concept of ionization energy. When an atom has a larger number of protons, its nucleus is larger and it also has more electrons and with some at a farther distances from the nucleus, and the ionization energy required to free more distant electrons is then even less. A hydrogen atom which has only 1 electron relatively close to its nucleus will require a relatively high amount of electron volts to break that electron free from its close orbit with a high electric attracting force.

For hydrogen, the ionization energy is often noted at about: 13.6 eV or = 1312 kJ / mol of hydrogen atoms.

An electron can also go to a higher orbit or "(electron) shell" after gaining a certain amount of energy that is less than the ionization energy for it to break free from the orbit of that atoms nucleus, and if this, or possibly another electron falls back to a lower orbit, its gained energy is then released as some form of electromagnetic radiation (light, xrays, etc) energy.

The minimal ionization energy required to split a water molecule into its hydrogen and oxygen gas elemental components of each water molecule, is said to be about 1.23V. This electro-chemical process is usually called **electrolysis**. The hotter the water, it and its electrons have more gained (thermal and kinetic) energy and the ionization energy will be less. Using a voltage higher than 1.23V is often said to be unnecessary and a waste of power, particularly with current passing through the water having a small amount of an electrolyte substance to allow the movement or conduction of current (ions and electrons, in opposite directions) between the two electrodes (+ pos ,and - neg). The higher the temperature of the water solution is also an indication of wasted energy in the form of thermal energy, and is usually caused by applying too much power (voltage and-or current) and-or that the effective resistance of the water solution is high and power is being wasted through that resistance.

Extra, from: 1 eV = 1.602 (10^-19) J , solving for 1J by dividing both sides by 1.602 (10^-19), we have:

1J =~ 6.2415 (10^18) eV : = 1J = (1Qc)(1V) = (1 coulomb of electrons)(1 volt) electron volts ,

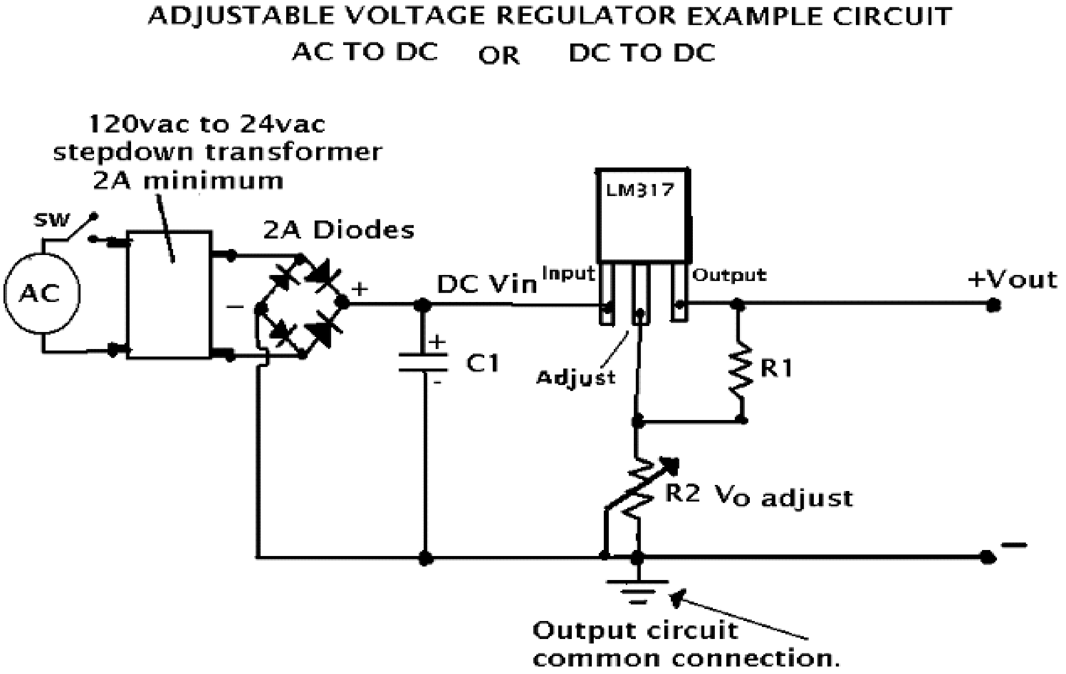
As a reminder and comparison to the above values, this was stated previously in this book:

1C of charged particles or electrons = (6.24)(10^18) charges or electrons,
and if each side is divided by the factor of (6.24)(10^18), we find the
charge (Qc) of 1 single electron to be: 1.602564 (10^-19) C

From: V = J / Q , we have:
J = Qc V , and if the amount of charge is 1 electron, and V=1 volt:
(1.602564)(10^-19)C (1V) J = (charge of an electron)(1V) J = 1ev

Below is an example circuit so as to have a relatively inexpensive, adjustable or variable **electric power supply source** for electronic devices, circuits and experiments. In particular here, only the voltage is adjustable. You can make or purchase such products already built.

Adjustable Voltage Regulator Power Supply Example Circuit This one uses the LM317 or LM317T IC . [FIG 272]



A note about the LM317 linear voltage regulator IC was previously mentioned in this book. There are many similar IC's available, in particular, they are not meant to be adjustable, but rather have a fixed, predetermined output voltage such as 3V, 5V which is the common USB devices voltage, 6V, 9V, 12V, and 15V. You can look-up or research the LM317 IC **data sheet** (device information and-or specifications) and-or packaging notes for the LM317 specifications, notes and example circuits of which the above circuit is actually based upon. You do not need a LM317 integrated circuit to make a stable or constant voltage supply, but it is inexpensive (about $1.00 to $3.00 USD, depending on the quantity purchased) to obtain and implement (use) and simple to use in a circuit. The LM317 type of "series" voltage regulators have an adjustable or variable output. A premade LM317 voltage regulator circuit can also be purchased inexpensively, such as from the Ebay.com website. The transformer is usually sold separately, and is needed as part of the entire circuit to convert AC household voltages to DC voltages. The circuit indicates a transformer with an output of 2A minimum, and this will ensure that the full rated 1.5A output current of the LM317 can be achieved if so needed, otherwise a lower output current from a smaller transformer can be used, however if too much current (perhaps from a shorted output circuit) is drawn from it, it could damage that transformer when the LM317 cannot shut-down its output at currents less than 1.5A. An "inline" (ie., series) fuse (of a necessary value, for example 0.25A, 0.5A, etc) should be used to prevent circuit damage issues for either this power supply and-or equipment such as an expensive and-or vital radio. Here are some notes for the above **LM317 IC, Adjustable Voltage Regulator Circuit:**

Vin = The input voltage applied to the LM317 input pin should be about 3vdc min. to about 39vdc max. There will be a voltage drop (ie., a "dropout voltage") across this IC due to its internal circuitry, and will therefore, it will not be available at the output. The power loss in this IC is the voltage drop across or through it times the current through it. P=VI To reduce power losses in the chip, use a minimum input voltage to it, say only 3 or 4 volts higher than the output voltage. Once the minimum input voltage is reached, any higher input voltage can be regulated (a steady v.).

Vo = 1.25vcd to about 37vdc , The max. output voltage depends on the input voltage. This voltage is adjustable via a variable resistor (R2), but can be set or fixed at a desired voltage using a fixed resistor for R2.

Io = 1.5Amax. Higher output currents will be sensed by LM317 IC and it will stop the flow of current.

The transformer, diodes and in general, the capacitors, are not needed in the input power supply is a DC voltage such as from a battery.

Transformer and Diodes = Common AC household power can also be used if it is converted to a form of DC with by using a capacitor "ripple filter" and-or temporary energy storage circuit. Usually this circuit consists of a transformer, and four, 2 Amp (or more) power diodes (or a premade full-"bridge" rectifier) so as to make a full ac or sine wave rectification (to having one (curent flow) direction and-or voltage polarity, hence [unregulated] dc), and eventually to have a fairly steady [ie., regulated, low fluctuations, changes, low "ripple voltage"] dc voltage by using a large capacitor.

The output voltage of the transformer should be in the range of about 3vac to 30vac that the IC can regulate, and be able to deliver the amount of current needed (up to 1.5A max.). The transformer should be rated at about 2A or higher for safety if 1.5A will be needed. The maximum output voltage of this regulator circuit also depends on the maximum available voltage of the transformer, or DC input voltage such as from a battery. If possible, the circuit can be constructed to allow both house power and also have terminals for a DC input voltage supply. **If the voltage regulator circuit is DC input powered only, then a transformer and diode(s) are not needed -** using the transformer with a steady DC current is also not even possible, and diodes will cause a waste of DC power due to the voltage drop across them. It is possible to use a power diode(s) before the load circuit so as to obtain very low voltages if necessary.

IC Pins of the LM317 = External electrical contacts: Looking at the actual front or "top view" side of the 317IC chip, from left to right are the internal-external terminals or pins: 1=adjust , 2=output on center pin , 3=input.
The pins shown in the figure or schematic are not numbered, but are just indicated as for what it is for, and to make the rest of the circuit drawing easier to understand. Look at the IC packaging for the pin numbers and-or corresponding function of it. **These pins and or the IC chip itself can be thought of as equivalent to that of a high current or power transistor** with its collector pin being the input pin, the emitter pin being the output pin, and the base pin being the adjust pin for the amount of current passing through the transistor from the emitter to the collector. This concept is essentially varies the collector to emitter (CE or C-E) resistance of a transistor, and therefore it can vary the voltage drop across it and any other output resistors which for this IC are generally not in the output current circuit, but are relatively high in value and are rather used to provide a current reference, control or bias (electronic circuit settings) amount to the M317 IC so as to automatically adjust its output voltage to be a stable or constant desired value. There are also low current LM317 available that look like small plastic transistors.

R1 = a value of 220 or 240 ohms is typical

R2 = User selectable for a fixed output voltage such as by using 1 fixed resistor, or a selectable resistor by using a user selectable or operated rotary switch (ex. single pole to 6 throw = 1 input pole or terminal, to 6 possible output poles or terminals), or use a 5k ohm linear variable resistor so as to have a variable output voltage. Here, the center terminal (pin 2) of the variable resistor should be connected to pin 1 or pin 3 of the variable resistor in such a manner that the output voltage increases as the variable resistor (potentiometer = "pot") is turned clockwise. This output voltage setting of the power supply can be indicated at the adjustment knob to this variable resistor, and can be done with simple pen lines on paper and set affixed with tape or glue during the calibration of those lines. An analog voltmeter is also typical on power supplies so as to quickly set a good rough value or approximation of a voltage. There are also inexpensive,small LED voltmeters that will normally display 3v or higher voltages.

The higher R2 is, the lower the "(IC chip) sensing current" and the lower the output voltage of the LM317 IC circuit. Though not often needed, an optional capacitor, say 10uF, across R2 can help improve the output voltage stability by helping to maintain a steady "sensing current" and therefore helping to reduce any output ripple voltage due to input and-or load impedance and current fluctuations.

C1: 0.1uF, usually for AC to DC conversion, and is optional if the input is from a DC source which may briefly vary in output power. Also use another capacitor of higher value, perhaps 1000uF or higher, if the input is from an AC source, such as from the output of a transformer and rectifier diodes, so as to have a more constant valued, consistent, "steady" or "smoother" DC voltage without much, or possibly no variation or "ripple" (rises and-or declines) in the value of the input and output voltage of the IC. Make sure the voltage rating of this capacitor is higher by at least 1.5 times the maximum voltage across it. Another optional high capacity capacitor (ex. 1000uF) is often used at the voltage output of the IC, but a smaller 10uF capacitor is typical and optional.

SW: is the main voltage regulator circuit power on-off switch. This should be used between the mains house power and the transformer. Another switch can be used at the output terminals, or better yet, at the input pin to the LM317 IC chip, and so as to be able to immediately turn on or shut off any power to the load that is still available in high storage value capacitors if used. To prevent fire and other damage, do not connect the household power to the transformer until the circuit is built and ready to use, however the transformer can be initially checked before connecting any circuit to it, and so as to determine and-or check its output voltage(s). In general, unplug or switch off the transformer and-or power supply from the household power when it is not in use, and or when checking it. Before examining a circuit and to avoid a potential electric shock, it is a good idea to first discharge any large capacitor in a circuit by connecting a low resistance conductor across its two terminals.

Fuse: A slow-to-blow ("slowblow") or melt to be an open circuit, 0.5A safety fuse is also recommended in series with the primary or input coil of the transformer. The LM317 IC has a max amperage ability of 1.5A, however there may be a short in another part of the circuit, and which can cause circuit damage, and or a fire due to high heat. Due to this it is best to consider a 2A fuse in series with the output transformer. There may be some low current, resetable circuit breakers which can be used in place of a typical (thin metal, one time use) fuse.

Power, On-Off Light: A LED can be used as a power on-off indicator to the LM317. If it is a red, yellow or green LED, it will require about 15ma of current to be bright enough, and about 2V to turn on. With a 24V input to the LM317, the safety resistor (Rs) to the LED will need to drop (24V - 2V) = 22V. The resistance of this resistor will be about: Rs = Vr / Ir = 22v / 0.015A = 1466 ohms. This light and series resistor parallel (to the LM317 input) branch circuit will be placed from the input of the LM317, and will need to be connected to the resistor and LED in series, and then to the common circuit ground so as to have a complete circuit or path for the current to flow through the LED. A white or blue LED will require about 3vdc at 20mA to 30mA of current. Typically, LED's are sold with the positive (+) or "anode" lead being the longer of the two LED terminals or leads. The shorter lead is the negative (-) or "cathode" lead and is usually attached to the small reflective cup that holds the LED's semiconductor diode

Voltage And Or Current Meter - This is optional. A voltage meter (digital or analog) will be able to indicate the actual voltage at the output terminals of the voltage regulator. It's leads which may include a series, current limiting resistor(s) and possibly a switch to set the range. The meter and its safety circuit will be placed across the output terminals of the voltage regulator where the available output power of the voltage regulator circuit is accessed.

There are some similar IC voltage regulators available with a lower voltage dropout (ie., or loss, necessary for the regulator to function, the **LM317 needs about 3V** to operate, hence the input voltage will need to be 3v extra than the desired output voltage), a LDO voltage regulator, of about 1.5V or even less, and these are ideal when the supply voltage is just enough or close in value to the output voltage needed, however, if the input voltage needs to be reduced significantly, there will still be a significant voltage and power loss in the IC. Because of the low operating voltage or dropout voltage, these regulators will not waste as much internal power as standard voltage regulators, and could save several watts of power. The power loss in these IC's may be only 50% as that of a standard 3 volt dropout regulator.

Voltage regulators, even of the same identification number can come in various cases or packaging and ratings. The TO-92 is the typical small plastic case often used for transistors, and it is about 5mm wide by 5mm high Multi-pin chips or DIP (Dual Inline Packages) have pins on both (L and R) sides of its plastic case and can be placed into sockets so as to be removable. The LM317 for the above circuit used the TO-220 case which is about 10mm wide by 15mm wide . There are other voltage regulator IC's similar to the LM317 which can pass more current and handle more internal power loss, and they should then also have a **heat sink** (or heatsink, heat-sink, a thermal conductor of heat, "to draw away heat", the LML317 , of the 220 case size or larger, typically has a max. power of 15W or possibly more) and thermal conductive paste applied to them so as improve heat transfer and-or removal, and so as to prolong its useful "life" or time of functioning:

IC and Max. Current Rating : LM317 1.5A , LM350 3A , LM338 5A : all generally require heat sinks

Using the LM317 as a constant amount of current power supply

Due to a fixed or set constant or regulated output voltage, if the value of the resistance of the load device being powered by the LM317 changes in resistance, the amount of current going to it will then also change. This can be problematic and can cause the load device to not perform as expected, and or get damaged by excessive current. For example, an LED that has a maximum load current of say 30 mA. To obtain a stable or constant current from the LM317, the resistors R1 and R2 shown in the above circuit are to be removed, and replaced by a (calculated) resistor connected to the output pin of the LM317 IC. The load device will be connected to the other end or terminal of this resistor, and also connected to this terminal of this load resistor is a wire going directly to the adjust pin of the LM317. The LM317 is designed to have a stable or constant output voltage by having the voltage on the adjust pin to always be regulated (ie., automatically adjusted to internally the IC) so as to always have 1.25V regardless of the load's current drawn by the device being powered.

With this minimal circuit mentioned, the resulting output voltage to the load device will then be determined by the input voltage as usual, less the ("dropout") voltage lost through the LML317 to function as a voltage regulator, and less the 1.25V across this current regulating resistor between the LM317 and the load device. To calculate the value of this current regulating resistor (Rcr) use:

Rcr = Vcrs / Icrs = 1.25V / (constant load current desired) , ohms : cr = current regulating resistor

An example of where this circuit can be used is to add or remove a LED diode(s) that are in series electrical connection, and so as to still have the desired current going through those LED's, and without the need to redesign the circuit to increase or decrease the current through those series LED's. In this example the modification would most likely be a series (safety) resistor being changed in value.

Due to the constant current through the resistance of the load device, it will then, according to Ohms's Law (V = IR), also have a constant voltage drop across it if its resistance does not change.

If needed, another LM317 can be used in series as a voltage regulator before the LM317 being used as a current regulator. This would then make a good power supply for experimental purposes and for collecting performance data of the load device.

The maximum current output of the LM317 is limited to 1.5A, however, there are external circuits available which can increase this value, and of which is usually involves using a high current, power transistor capable of handling or safely passing several amps, say 5A of current through it, and yet the LM317 maintains it set voltage output to the load. There are other IC's that are similar to the LM317, and of which have higher current such as 3A and 5A. A heat sink is always recommended with any voltage regulator and-or transistor when passing current of more than 1A through it.

A constant current regulator has good possibilities for charging up batteries at a constant amount of current, and so as to reduce the total charging time needed. The typical recommended amount of current to charge a battery is 10% or less of that batteries maximum rated output (ie., the load current = current to the load) of current.

Here are some very similar 3 pin, leads or terminals IC's that are similar to the LM317, however these are more modern ones that are becoming available and of which have a **low voltage dropout** or minimal voltage loss across them - typically 1.5v max at higher regulated voltages and this value will be less at lower voltages, perhaps 1v. This low voltage dropout also means there is less power loss in these IC's. These IC's are about twice the cost of the LM317.

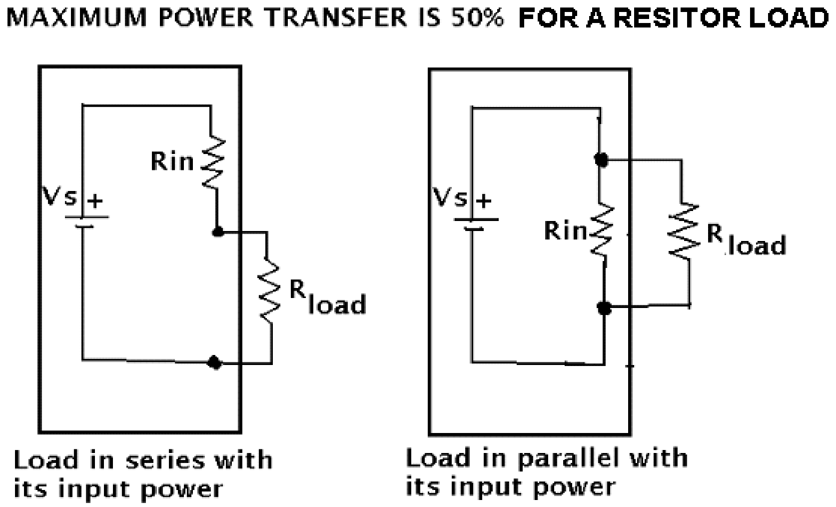
IC and Max. Current and Power Rating : LM1086 1.5A , 15W : for these, LM may sometimes be noted as LT .
and Max Power Dissipation Rating: LM1085 3A , 30W These voltage regulators are also noted as
Please use a heat-sink LM1084 5A , 45W requiring a capacitor at the output to ground -
(a thermal absorber and radiator LM1083 7.5A , 60W for stability, about 10uF. An optional resistor,

with a large surface area)

say 4.7K, across a capacitor can be used as a safety resistor to slowly drain its charge.

MAXIMUM POWER THAT CAN BE TRANSFERRED TO A RESISTIVE LOAD

[FIG 273]



Given a power source or supply as indicated in the box in above drawing, the **maximum power that can be transferred to a resistive load** is 50% of the supply power. This may bring to mind the concept of an equal and opposite force. In the above figure, Rload is the load or external device being powered by the power supply. This also implies that 50% of the supply power to the load is lost and-or not available to the load. Rin represents all the internal resistance of the circuit of the source of energy. So as to have the maximum power condition to the load, the load impedance is the same (ie., "matches", "impedance matching") as the source impedance. Due to this power transfer concept, given a certain value load or impedance, a circuit is often designed with an internal or source impedance equal to this load impedance value so as to give or transfer maximum power to it. It is of important note that if the power supply is a battery, that its internal resistance will increase as it drains of stored current and-or energy, and therefore there will be more internal power loss to that batter and it will also increase in temperature which can physically degrade the battery.

In the left figure, the total internal resistance (Rin) is in series with the voltage source and Rload. This is essentially a voltage divider circuit. In the right figure, the total internal resistance is in parallel to the voltage source and internal resistance. This is essentially a current divider circuit. Note that a battery also has an internal resistance, for example 1 ohm, and this will limit current and cause a voltage drop and power loss internally of the battery. In general, the more a battery is drained of charge, the less voltage is will have, and the greater its internal resistance such as due to its chemical composition of the electrolytes.

In the above figure, the impedances shown are represented as resistances, but it could actually be composed of various resistors, capacitors and inductors which have a net impedance to the flow of AC signals (voltage, current, and frequency). Capacitors and inductors have a (AC, frequency dependent) reactive impedance which limit current much like a resistance would, but the electrical formula for maximum power transfer is more involved than that for resistors, and is beyond the scope of this book. Note that even a power source has a small internal (series) resistance which may be about only 1 ohm or less, but it could be significant for some circuits, especially if a battery's voltage is getting lower and its internal resistance is increasing and the available output power from that battery power source is greatly reduced by a growing internal resistance. Also:

At maximum power transfer to the load:

Pload = 50% Psource and here:

Pload = (load voltage)(load current) = (load current)^2 (load resistance) = (load voltage)^2 / (load resistance)

Why is it limited at only 50% maximum of the source power?

For the load in series with the power source, if you increase its resistance, you will increase the voltage across it due to the voltage divider concept, but then reduce the current through it since the total resistance is now larger. If you decrease the load resistance, you will decrease the voltage across it.

For the load in parallel with the power source, if you decrease its resistance so as to increase the current through it, there will actually be a larger voltage drop across the resistance of the battery or power source because that resistance will become more significant and a larger power will be lost through it and it will be unavailable to the load resistance. When a battery is shorted (ie., no load, 0 ohms external resistance), all of its power is lost internally through its internal resistance and it will get hot (due to electric energy converted to thermal energy) and possibly damaged, and-or it could vent gas or explode due to the force or internal pressure of the hot gasses created.

Even for a capacitor as a load (like a [charge storage] battery) and being charged with energy, a maximum of half of the supplied energy is wasted just to charge it. It is usually wasted due to the internal resistances of the battery and-or wires.

From V = J / Qc , we have: J = VQc ,

Due to the maximum power transfer theorem, the maximum energy delivered to the capacitor and stored is equal to only half (1/2 = 0.5 = 50%) of the supply energy used to charge it up, therefore, the above equation needs to be modified for use with a capacitor:

Dividing both sides by 2: J / 2 = (VQc) / 2 : (1/2) = 0.50 = 50%

If you were to lift an object to a height, due to equal and opposite forces of which acts like a resistance to movement that you must then use energy to overcome, half of the total energy you used will be wasted as heat by your muscles, etc, and the object will gain the other half of the total energy in the system. This energy will be stored as gravitational potential energy (GPE) until it is released to fall and to where the GPE will be turned into more and more kinetic energy (KE) as it falls and-or is being pulled by the gravitational force of Earth and the object gets faster and faster due to the constant force and resulting acceleration in velocity of 9.8m/s^2. In short, only half of the total energy involved with this system or instance got transferred to the object.

Energy Storage In A Capacitor

Maximum energy stored in a capacitor = Jmax = (1/2) (supply energy joules) = (VsQc / 2) joules

For a relationship of energy to just a capacitors voltage and a capacitance, a charged capacitor can store this much energy:

From: $Cf = Qc / Vc$ mathematically:

$$Vc = \frac{Qc}{Cf} \quad \text{squaring both sides we have:} \quad (Vc)^2 = \frac{(Qc)^2}{(Cf)^2} \quad \text{therefore:} \quad (Qc^2) = (Vc)^2 (Cf)^2$$

From: $Jc = \frac{Vc\ Qc}{2}$:Joules of energy in a capacitor. , Using substitution for the above expression for Vc:

$$Jc = \frac{(Qc)\ Qc}{(Cf)\ 2} = \frac{(Qc)^2}{2\ Cf} = \frac{(Vc)^2\ (Cf)^2}{2\ (Cf)^1} = \frac{(Vc^2)\ (Cf)}{2}$$

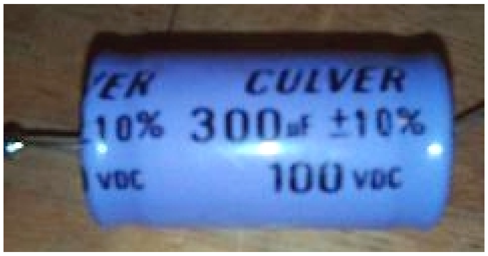
: Energy available in a capacitor, and it's exponentially related to the voltage across it.

Consider a 1F capacitor charged up to 1 volt across it, its energy is 0.5J
Consider a 1F capacitor charged up to 10 volts across it, its energy is: 50J , which is 10^2 = 100 times more than when it was charged to 1 volt.

Here in the image below is a fairly large capacity (300uF) non-polarized (NP, having no specific + or - terminals) capacitor that is typical of being used in audio circuits, power supplies, etc. and where relatively high amounts of current and-or power are involved. The capacitor is indicated as being made by the Culver company. The size of this capacitor in the image is about 4cm long, by 2cm wide or in diameter. It is rated at 300uF, with a tolerance of +,- 10%, and having a maximum voltage applied rating of 100Vdc. Due to the tolerance rating, the actual capacitance may be between 270uF and 330uF:

300 uF - (10%) of (300 uf) = 300 uF - (10%)(300 uF) = 300 uF - 30 uF = 270 uF and:
300 uF + (10%) of (300 uf) = 300 uF + (10%)(300 uF) = 300uF + 30 uF = 330 uF : a total range of: 330uF - 270uF) = 60uF

[FIG 274]



The "**energy density**" (EDc, or energy storage capacity) of a capacitor (c) is the amount of energy that can be stored in the volume or weight of the capacitor. This value is sometimes used for comparing various energy storage devices. A high EDc is usually desired, and at a low cost.

EDc = energy per unit of weight = Jc / weight

As an indication of the amount of energy or power in a battery, a battery is given an **amp-hour (Ah) rating,** which is how much current (A) can be safely drawn per second for 1 hour. A battery rated at 10Ah, can supply 10 amps (=10 Coulombs/s) of current for 1 hour before its considered drained or impractical to use for the rated voltage. That same battery can also have less of a current drawn over a longer period of time such as for example. 5 amps for 2 hours is equal to the same amount of energy as 10 amps for 1 hour = 10Ah. 10Ah = (10amps)(1hour) = (5amps)(2hours). = (5)(2)(amp)(hours) = 10Ah. There will be some energy left in the battery, but its voltage will be less than its rated voltage. Deep cycle (charge and discharge ability) batteries have a relatively high amp-hour rating. We see that an amp-hour rating is really an indication of the amount of current that can be drawn and the total expected time of that. Note that since a battery has a small internal resistance, and if more current is drawn, the larger the voltage drop across its internal resistance, and the lower the available output voltage. Batteries are often tested using an expected load of current such as for the headlights (having a resistance) of an automobile so as to determine the actual available output voltage and status of that battery.

The energy in a battery may also be rated in terms of milli-amp hours (mAh). 1000mAh = (1000)(10^-3)A = 1.0Ah

Ex. 500mAh = (500)(10^-3)A = 0.5A for 1 hour

Since 1A = 1 coulomb of charge per 1second = 1Qc/1s
1Ah = 1coulomb of charge per 1second, for 1 hour = 1 amp for 1 hour , since 1 hr = 3600 seconds:
(1coulomb / 1s) x 3600 seconds = 3600 coulombs of charge in 1 hour

Since J = (Qc)(V) , and if this specific battery is rated at 1.2V :

J = (3600coulombs)(1.2v)
J = 4320J of energy

From: Watts = Vv Ia

The total energy or power available in a battery can also be calculated as:

Wh = (average usable voltage)(amp hour rating) = (Vav Ia)(h) = Wh = (W)(h) = **watt-hours** = total watts used for that many total hours

The average usable voltage corresponds to the battery's (usable, and safe for the battery) amp hour rating.

Ex. A battery rated at 12v and having an amp-hour rating of 50ah, can supply a rated maximum of (12v))(50a) = = 600 wh or= 600 watts for one hour. If this energy was applied to a 100w motor, then 600wh/100w = 6h of total running time of that motor can be expected. This also does not mean that the battery is completely drained of energy and having a very low voltage, but its output-load voltage may be 11V of which is less than its rated 12V working-usable, expected or rated value.

watts = Joules of energy per second = J/s ,

watt-hours = Wh = (J/s)(1hr) = (J/s)(3600s) Joules of energy = W-h : Wh is essentially joules of energy per hour

Electric power from the electric power generation plant is usually measured with an outside household meter and sold in units of kilowatt hours: KWh units. At April 2020, the average residential (home, household) cost of 1KWh of electricity in the United States of America (U.S.A) is 13.31 cents =~ $0.1331USD

1KWh = 1000Wh = 1(10^3)Wh :1 kilo-watt-hour = 1000 watt-hours
This is equivalent to a total of 1000W of power delivered and-or used for a total of 1 hour of time and not necessarily between 1hour of continuous time, but can be spread out over various amounts, times and durations of energy usage. Once 3600J of energy have been delivered and used, that is equivalent as equal to 1 watt-hour = 1Wh of energy. 3600J/1hr = 3600J/3600s = 1J/s = 1 watt .

1KWh = (1000)Wh = (1000)(Wh) = (1000)(3600J) = 3,600,000 total joules of energy.

To convert a value of watts into equivalent kilowatts, divide that value by 1000: x watts / (1000w/kw) = z kw
To convert seconds to hours, divide the amount of seconds by 3600: x seconds / (3600s/h) = z hours
To convert minutes to hours, divide the amount of minutes by 60: x minutes / (60m/h) = z hours

Ex. A 100W = (100J/s) appliance that is ran (ie., turned on) for 10 hours, used a total amount of energy of:
(100W) (10h) = (100)(10)(W)(h) = 1000 Wh = (1000Wh / 1000) KWh = 1 KWh.
The total cost is: (Total KWh used) (cost per 1 KWh)

Total cost = (1KWh) (13.31 cents / KWh) = (1)(13.31) cents = 13.31 cents
Note also that 100W is equivalent to: 100 Joules of energy / 1 second and if this continued for a total of 1 hour = 3600s: 100J/s x 3600s/h = 360000J/h

Ex. Household or residential electricity is set at about 120vac in the U.S.A. How much current does a 100w appliance draw or use each second:

From: Power = (volts)(current) Watts = (V)(I) W we have:

Ia = W / V = 100w / 120v = 0.833A = 833mA : It could be also said that for every 100w, 0.833A is drawn. And:
The cost of 100watts per hour or 100Wh is from:
(cost / 1000w) = (cost / 1Kw),
Since 100w is only a tenth of 1000w, its cost is also a tenth:
cost per 100Wh = ((cost per 1KWh) / 10) ,
and for the above example: (13.31 cents) / 10 = 1.331 cents.
To run a 100W appliance for 1 hour will cost 1.331 cents

The wire to this device or appliance must be rated to safely handle a current of 0.833A or more. It must have an ampacity (or maximum continuous current rating) rating of this value or greater.

The effective current resistance or impedance of this 100W device or appliance is:

Ro = Vv / Ia = 120v / 0.833A = $144.\underline{1}$ ohms

Another useful equation is:

$$\text{Total Energy} = \text{E joules} = (\text{Power})(\text{time}) = \left(\frac{J}{s}\right)(s) = (V\,I)(s) \text{ joules} \quad \text{or=} \quad \text{VAs}$$

Ex. A 12v , 50 aH rated battery can supply a rated and safe maximum power of:

P = (V)(I) = (12v)(50A) = 600W of power , and since this is rated for 1 hour, hence 600 wH =
600 watt-hours of total energy

Pin = Pout
wH in = wH out : note that the number of watts and hours need not be the same, just as long as their product is the same. Considering a 300w device being powered by a 120Vac inverter:

First: From P=VI , I = P/V = 300w/120v = 2.5A

From wH = (watts)(hours) , hours = time = wH / watts = 600wH / 300w = 2h , max.
For the battery and its current out, another formula is: max. time = aH / A
And:

energy in = energy out : Joules
600 wH in = 600 wH out
(12v)(50A)(1H) = (120v)(2.5A)(2H) , note also that at 120v, the max safe, available current is: 600w/120v = 5A.
The voltage increased by 10, and the max. amperage decreased by 10.

600 watts for 1 hour = 300 watts for 2 hours : the lower usage rate (energy/time) of the available energy, the longer the available time of usage of the available energy.

Ex. For a 48v , 50aH dc battery system, such as by connecting 4, 12V batteries in series connection:

For this battery, the maximum rated amount of power usage is: P = VI = (48v)(50A) = 2000w for 1 hour , and this corresponds to 2000wH of energy , Joules

P in = P out considering the maximum rated value for this battery and-or system:
2000w = 2000w
(V)(I) = (V)(I)
(48v)(50A) = (120v)(16.67A) : notice the high amperage from the battery into the 120Vac power inverter, and this requires very thick and-or short cables, typically 3 feet. Copper pipes flattened at each end with drilled holes can be used. Note that the voltage was increased by 4, and the current decreased by 4, still, to get that 16.67A for the device being powered, 50A is actually drawn from the battery.
And:

energy in = energy out : Joules
wH in = wH out : Joules , wH = The product of: (watts)(time) = (watts)(hours) , with units of Joules
2000 wH in = 2000 wH out : considering the max. rated and safe values for this example
(48v)(50A)(1H) = (120V)(A)(H) , now considering if a 300w device is being used, I = W / V = 300 / 120 = 2.5A
(48v)(50A)(1H) = (120V)(2.5A)(H) , after solving for H we have: H = time = 2000wH / 300w = 6.67H
(48v)(50A)(1H) = (120V)(2.5A)(6.67H) : The 300w device can be ran for up to 6.75H

USING A MULTI-METER TO MEASURE ELECTRICAL VALUES

A multi-meter which is now often a digital multi-meter (DMM) with a digital numeric display rather than a (relatively expensive) analog meter display with a dial or scale and pointer hand, can be used to directly measure voltage, resistance, and current if it has an ammeter or current function. It is still possible to use an individual meter for each measurement. Some of these meters include a (electrical, conductance) continuity function so as to check if a wire is functioning and not broken and having a very high resistance of an open circuit, but rather having a very low resistance. Sometimes a value cannot be measured directly, and a calculation, usually using Ohm's Law, can then be made, especially when finding a value of current and trying to avoid placing the meter in series with a wire and-or component.

Before proceeding, a multi-meter is calibrated to display the effective DC value of the AC voltage or current when the meter setting is for AC waveform, and this value is the RMS value of that AC waveform and-or signal. Vpeak = Vp =~ (1.414) Vrms and mathematically: Vrms = Vpeak / 1.414 =~ (0.707) Vp. Since power = (V)(I), the product of the RMS values of each will give the equivalent DC power. Note also that 0.707 is one-half of 1.414 = sqrt(2). Here are some generalities and considerations of measuring and the measured or measurement values when using a Multi-Meter:

TO MEASURE THIS VALUE , YOU CAN DO THIS

Voltage , Set the meter on a high voltage scale and-or range. Place the measuring terminals or probes across the circuit element or component. Reduce the scale and-or range so as to find the voltage scale and-or range having a good precision and voltage display.

An ideal voltage meter has a very high to infinite internal resistance so as to not draw any current and alter the device or circuit being measured. Many common volt meters have about 1M ohm resistance.

Although a crude estimation of a batteries current state of charge can be determined by simply using just a volt meter, it is better to place the battery "under load", that is, some current is being drawn by the load or device being powered. For this, you can connect and turn on the device to be powered and then measure the battery voltage. An alternative to using the specific devices to be actually used and powered, is to use a resistance such as from a resistor of perhaps in the range of 5 ohms to 10 ohms. An alternative load for a car would be to temporarily turn on its headlights or some other lights. When a battery is under load, its internal resistance can then be accounted for by being in the actual circuit, and this resistance will have a voltage drop or energy loss across and-or through it, and will result in the actual available and lower voltage and measured value or "reading". A battery may have a measured voltage of 12 volts without a load,and 10 volts with a load being powered, and this may be insufficient for it to function properly. An alternative to this problem is to use a voltage regulator designed for a device that was designed for and requires a lower input voltage, say 9v.

Resistance , Set the meter on a high resistance scale and-or range. Do not place the terminals across an element having current through it, hence also having a voltage reading across it. This could damage the meter. Place the terminals across the (unpowered) resistance. Reduce the scale and-or range so as to find the resistance scale and-or range having a good precision and resistance display. Be wary of any other possible resistance in parallel and-or series with the resistance you are trying to measure, for it will then give an unintended measurement reading or displayed value on the meter. In some instances, you may need to unpower the circuit and cut or remove one of the two resistor leads so as to measure its resistance value. Resistors are usually color coded so as to have color bands on them to indicate its rated value of resistance: Black (0), Brown (1), Red (2), Orange (3), Yellow (4), Green (5), Blue (6), Violet (7), Grey (8), White (9), The third band is the power of 10 multiplier to the previous two bands. The fourth band is the "tolerance" code, but it is actually the possible, percent difference in error from the rated color coded value and actual true or measured value: Brown (+,- 1%) , Red (+,- 2%) , Gold (+,- 5%) , Silver (+, - 10%).

An ideal resistance meter has a high internal resistance, and since this resistance will be in parallel to the resistance being measured, it will have a very low (parallel resistance) effect on the displayed value of resistance.

It is very possible to measure the value of a resistance indirectly: From $V = I R$, $R = V / I$

If the current is not known or can not be measured, a form of the voltage divider formula for two resistors in series can be used to determine the value of a resistor, here R2:

$\frac{V1}{V2} = \frac{R1}{R2}$, $R2 = R1\, V2 / V1$, also: $I = V1 / R1 = V2 / R2$: I = same series current

A resistance meter can be used to determine the **continuity** of a wire. If it has a low resistance,the wire is still conducting. If the resistance of a wire is very high, that wire is probably broken, and is non-conducting.

Amps = I = **Electric Current** , Set the meter on a high current scale and-or range. Place the probes in series with any circuit component or element in that circuit branch of which you want to directly find the current for. Reduce the scale and-or range so as to find the current reading having a good precision and display.

So as to not impede the circuit current being measured, a current meter will have a very low internal resistance, perhaps a tenth of an ohm. This resistance is actually a parallel resistance to the coil or sensor within the meter, and is often called a "shunt resistor", and is practically a short circuit to it. Shunt or a shunting resistor diverts ("shunts)" the high current away from the sensitive current sensor so as to not damage it. An amp meter is rated at a maximum amount of current to measure, and if it is exceeded, it may blow an internal fuse, or cause damage to the meter, etc. Some amp meters, sometimes called "(current) clamp or loop meters", have magnetic field sensors that can indirectly measure the current flowing through a nearby wire, and without the need to put the meter in series with the circuit.

Use Ohm's law for any calculations and-or checking needed: $I = V / R$, $V = IR$, $R = V / I$

P = electrical power = $V I$ watts = joules / second

$P = I^2 R = V^2 / R$ = watts = J / s

An indirect way of measuring current is by the voltage across a resistance and then using Ohms law to calculate the current: $I = V / R$. This works will for a pre-made circuit that has a known resistance value, however, we do not want to lower the actual current to the load resistance by introducing a resistor into the circuit to measure its current. A compromise is by using only a very low, calibrated and known "precision resistance value" value such as 0.1 ohms or near that value, and typically a power resistor type of say 10W to 100W power handling capability. The calculated value will be only slightly less than the actual current value without the introduced resistance for the measurement. Please read the following note:

Due to the resistance of the wire probes of the meter, this value will be added to the resistance being measured Typically for values of say 1 ohms or greater, it is a relatively insignificant value, however, when measuring a very low resistance, the value of resistance of the probe is relatively significant and must then be deducted.

To measure the value of the resistance of the probes, simply connect the end points of the probes together, and then simply observe how much resistance is then measured by the meter. Since most inexpensive amp or current meters can only measure about 10A maximum, and only for a few seconds, the above method is a good option when trying to measure current greater than say 10A. For relatively high currents, there are also current measuring devices which simply encircle a wire and measure its flowing current indirectly via the magnetic field that current creates.

Some meters and-or multi-meters can measure other electrical values such as: continuity or low resistance check, diode check, transistor check and-or gain (amplification), capacitance, inductance, temperature, frequency, and battery test.

Another type of meter is a **power meter** that can sense and measure both current and voltage in a circuit. The power value is internally calculated as: Power = (voltage)(current), and this can also be done manually if the current and voltage is known. This device is generally plugged into a home's electric power socket ("outlet"), and then the device being measured is then plugged into the power measuring device or meter. With this meter, you can determine how much power is actually being drawn by a device, perhaps when checking if the device is using a greater amount of power than its rated value. You can also check the power used by each device in a home or building and so as to sum, determine or estimate the total kilowatt-hours (kWH) being used, and to then estimate the corresponding electric bill (fee, price, cost) that is based on the amount of kilowatt-hours (kWh) used. Some electric power utility companies may sometimes estimate the amount of power a home used, and it could be based on the average monthly usage and-or a previous amount for a certain month of the year. A bill or financial charge from the electric company an-or the supplier (ie., [wire, maintenance, service and repair] distribution company) will usually have the number of KWh used, and the cost per KWh of electricity.

To measure capacitors and inductors, you will need a capacitor and-or inductor meter. These measurements are usually performed on the circuit element (here a capacitor or inductor) when it is not in a circuit. Before measuring the capacitance of a capacitor, be sure that it is discharged such as by connecting a low value resistance across its leads. If a capacitor or inductor is in a circuit, please remove all power to that circuit, discharge the capacitor, and take one lead of it out of the circuit so as to the measure that capacitor and-or inductor. Some mult-meters may include a capacitor measuring function and possibly an inductor function. It is possible to measure the value of a capacitor or inductor by using a voltage divider circuit that includes a series resistor, and with AC signal applied across just those two circuit elements or components. For example, solve for the reactance of the capacitor (Xc), and then use the reactance of the capacitor formula and solve for the capacitance (Cf).

Some multi-meters can also measure the **temperature** of something via a temperature sensor probe that plugs into that multi-meter. Some multi-meters can also measure **frequency** within a certain range. As of the year 2010 or so, there are many inexpensive digital **oscilloscopes** available, and which can view the waveform(s) of an electronic signal with respect to time. It is recommended that the measurement sample rate or frequency of a digital oscilloscope be at least 10 times the frequency of the wave being measured so as to construct it reasonably accurate on the display screen.

A COMPUTER PROGRAM TO FIND THE AVERAGE AND RMS VALUE OF A SINE WAVEFORM

An **RMS** value is the Root Mean Square value of a waveform. RMS means the square root of the average (ie., mean) of all the squared instantaneous values of the wave form. RMS is considered because the average of a sine waveform above and below the 0 value has a resultant value of 0, and by squaring the curve values, the result is always positive in value. These positive values are then averaged, and the square root of that average must then be taken or undone so as the resultant value is not left as excessive in value. RMS is generally considered as the equivalent DC (both the voltage and current sine wave forms) power available to a resistive (ie. resistor) load. Power is also the product of voltage and current, hence this is essentially a squared value in terms of the voltage and corresponding current sine waveforms which both have a maximum relative value 1.0 Because of the above facts, the RMS value is slightly higher than the average value of the voltage and-or current sine waveforms, and the RMS value should be considered as the power value.

The two programs below are very similar in construction and with slight mathematical adjustments.

```
/* Sine Average Value.c

   A experiment to find the average sine value, of a sine curve.  I take many "samples" values as the angle
   increases slightly in value up to 90 degrees which is about 1.57 radians, and then divide the growing sum
  of the values by the number of values or samples taken.

   (c) JPA June 8, 2018

-----------------------------------------------------------------------------------------------------*/

#include "stdio.h"
#include "math.h"

/*----------------------------------------------------------------------------------------------------*/
void main(void)
{
double increment=0.0000001;   /* increment a "millionth" = (one-millionth) = (one/millionth) = (1/million) = (1/1,000,000)   */
double angle=0.0;
double sine=0.0;
double sum=0.0;
unsigned long int n=0;

for(;;){

        sine=sin(angle);
        sum=sum+sine;      /* NOTE , If this statement was made to be:  sum = sum + sine * sine;   such as when
                              considering the product of the voltage and current at each instant, the result will be 0.5, and
                              then doubling this for the other half of the wave (180° to 360°) will mean (ie., average) result
                              in 1.  Since squared values were used to create the result, we can take the square-root of
                              this 0.5 value, we have about  0.707  as the equivalent DC value for the voltage and-or
                              current, and this is called the RMS value of the AC voltage.  See the RMS program below. */

        angle=angle+increment;
        n=n+1;
        if(angle>=1.570796326){  break; };     /* 90 degrees ,  if this is doubled for 180 degrees, the average is still the
                                                  same over 180 degrees since a sine waveform is symmetrical about 90°
                                                  and will yield the same average value. */
    };
```

```
printf("\nThe average value of the sine waveform  is:  %lf",sum/n);    /*  Displayed is:  0.636620  , and if it may matter, this
                                                                                     value is close in value to, but slightly greater than:  1 - (1/e)

printf("\n\nPress a key."); fflush(stdin);  getch();

};
/*-----------------------------------------------------------------------------------------------------------*/
```

If the peak voltage of a sine waveform was 9 volts, the average value of that waveform and effective or equivalent dc voltage is then: (9v)(0.636620) = 5.73v

```
/*------------------------------------------------------------------------------------------------------------------------------------------------*/
/* Sine RMS Value.c

   RMS = Root Mean Square  =  Square-Root of the Average of the squared values.

   This program is based on sineaveragevalue.c that does not consider RMS, but the basic values of
    the curve such as for up to 90 degrees, and that average value is also the same value even if 180
   degrees is considered.  To consider the equivalent dc power of the entire wave, and so that the
    positive and negative halves do not cancel or negate each other to 0, the RMS method is done
    where squared values are considered.  Nonetheless, the RMS value of 0.707 is close in value to
    0.637 = 2 / (pi) , as found in the above program to find the average value of a sine waveform.
    1 / 0.637 = 1.57 = (pi) / 2

    If given an RMS voltage of Vrms, the peak (Vp) value of that waveform is (Vrms)(sqrt 2) =~ 1.414 Vrms = Vp
    mathematically then:   Vrms  =~  Vp / 1.414  =  0.707 Vp .   Also note that 1.414^2 = 2   Most AC volt and current
    meters measure the RMS value of that AC waveform and which is the "effective (as like, equivalent)" value in
    in terms of the effective power measurement of that signal, and is therefore not the peak values which would be seen
    on an oscilloscope.  Consider that Pw=(Vv)(Ia) for DC values, now with an AC waveform:
    Pw = (Vrms)(Irms), and so if Vac has a peak of 1Vp  then  Vrms = 0.707 Vp,  and Irms = 0.707 Ip   , and the equivalent
    DC power is:  Pw = (0.707Vrms)(0.707 Irms)  =  0.5W and not 1W as would be with DC.  If the RMS voltage was
    1 Vrms, and the RMS current was 1Arms such as through a 1 ohm resistor, Vp = 1.414 V  and  Ip = 1.414 A , and
    Prms = (Vrms) (Irms) = (1Vrms)(1Arms) = 1W of equivalent DC power.

    Extra considerations:  1 / 0.707 = 1.414 = sqrt(2)      and      0.707 / 0.5 =  1.414  =  sqrt(2)   ,  sqrt(0.5) = 0.707
                           0.5 / 0 .707 = 0.707      and     0.707^2  =  0.5       ,     0.707 (2) = 1.414
                           (0.707)(1.414) = 1
                           (0.637)(0.637) = 0.637^2 = 0.4058      and      0.637 / 0.4058  =  1.57  =  (pi) / 2

   (c) JPA Jan, 2021
 -------------------------------------------------------------------------------------------------------------*/

#include "stdio.h"
#include "math.h"

/*-----------------------------------------------------------------------------------------------------------*/
void main(void)
{
double increment=0.0000001;
double angle=0.0;
double sine=0.0;
double sum=0.0;
```

```
unsigned long int n=0;

for(;;){

          sine = sin(angle);    /* :using the compilers built in ANSI C, sin() function.  There is an extra NOTE below. */
          sum = sum + (sine * sine);  /* squaring each sine value */
          angle = angle + increment;
          n = n+1;
          if(angle>=1.570796326){  break;  };    /* if greater than 90 degrees  = (pi)/2  =~  3.14159265 / 2 = 1.570796325 */
      };                                                   /* 1.570796326 rads = 90 degrees */

printf("\n%lf",sqrt(sum/n));     /* taking the square root of the average value  ,  the result is 0.707107  */

printf("\n\nPress a key."); fflush(stdin);  getch();

};
/*----------------------------------------------------------------------------------------------------*/
```

If the peak voltage of a sine waveform is 9 volts, the RMS value of that waveform is then:
(9Vp)(0.707107) = 6.36Vrms

When a DC signal (direct [non-alternating direction] voltage, current and frequency) such as a power supply is constant, the output power is:

> Po = Pmax = (voltage max.)(current max.). If that power signal is on for a certain amount of time and then off for that same amount of the time, then the power available is clearly half of maximum power possible = Pmax. These facts are part of the concepts known as "**duty cycle**" and (equivalent resistor load) "effective power". On an oscilloscope instrument, the waveform of this would looked like square waves and is usually called "**pulsed DC**".

$$\textbf{duty cycle} = \frac{\text{dc power on time}}{\text{dc power off time}} \quad , \quad \textbf{effective power} = (\text{Max. DC power at 100\% duty cycle })(\text{duty cycle})$$

A power inverter that converts 12vdc to a 120vac, 60hz pulsed dc signal for common household devices that connect to the electric utility power ("grid") supply. The inverter device simply pulses the 12vdc input supply by 60 hz with a 50% duty cycle, and then raises that signal by 10 times with a voltage step-up transformer so as to create 120v pulsed ac. The output of this type of inverter is not very close to being an ideal sine waveform, but is useful for some non-sensitive electronic devices such as basic lights and resistive heaters, etc. Square waves such as pulsed DC can contain and-or cause electronic noise and interference, and can potentially damage an electronic device. For input power signal sensitive devices, a "pure sine wave" type of inverter should be used, and which shapes the output power waveform so as to be nearly sinusoidal (ie. having a smooth curve, sine waveform shape). If a 12Vdc voltage is boosted to 120Vac by using a transformer or inverter, to power a 120W device will require only 1 amp of current. Pw = (V)(I) = (120Vac) (1A).

In brief, a 12vdc to 120vac will boost the voltage by 10 and decrease the current by 10. Due to this, the current needed from the 12vdc battery will need to be 10 times more to result in the same available power. In theory, this will drain that battery 10 times faster, and charging it will take ten longer. A 24vdc, 36vdc, and 48vdc battery system will therefore use less current, last longer per charge than a 12vdc system. The tradeoff is the higher costs.

$$\underset{\text{Pin}}{(12\text{vdc})(\text{A})} = \overset{\text{Pout}}{\frac{(\text{n})(12\text{dc})\ (\text{A})}{\text{n}}} = \overset{\text{Pout}}{(120\text{vac rms})\ \frac{(\text{A})}{\text{n}}} \quad : \text{ here the factor of n=10 for the same amount of power}$$

Ex: (12vcd)(10A) = (120vac)(1A) = 120w

[This space for book edits.]

A general example of a car or automobile battery and (electrical power) inverter system.

The mechanical analogy of an electrical inverter system is a (mechanical) gear system that can alter and-or transfer mechanical (ie., direct, physical contact forces) power. Given a constant value of power = P = (voltage)(current) = VA or VI, the two variables: V and A will be altered by the **inverter**. The physical and mathematical relationship of those two factors of P is that they are **inversely** related. If one increases by (n), the other decreases by that same factor of (n) when the power value is a constant value.

A common 12V, non-deep cycle or drain, car or automobile battery has about a 50 amp-hour rating = 50 aH. This means it can supply 50A maximum for 1 hour, or mathematically 25A for 2h, and an equation for this is:

Rated Discharge Time = aH / a : expected discharge time until the car battery needs to be recharged.
aH = amp hour rating, a = A amps = current.

Rated Discharge Time = wH / w : **wH = watt-Hour rating = (V)(A)(H) = (V)(aH rating)**

After this rated discharge time, a car battery will usually have power still in it, but it has fallen below its rated 12V and is perhaps now just 10v or 11v, and which is enough to power circuits for emergency LED lights and-or other devices that can operate at lower voltages. You may which to then consider some type of voltage regulator and-or circuit for these. It is also of not that the current drawn from a battery during engine ignition is often higher than its aH rating, but its for just a brief amount of time.

Ex: If a device that uses 5A of current is connected directly to a fully charged 12V, 50aH car battery, the expected time of it to function using this battery is: time = 50 aH / 5a = 10H = 10 hours. For this to happen, the device also needs to operate at 12V.

Ex.: While using an inverter: Power in from the battery = Power out to a 20W device
20W = 20W , and:
(12v)(1.67A) = (120vac)(0.167A) : P=VI , I = P/V = 20 / 120v = 0.167A

In general, regardless if the system uses an inverter or not: A 12V, 50aH battery can supply a maximum power of:
Pmax = (V)(Imax) = (12v)(50A) = 600W, and this was tested and rated to last for 1 hour = 60 minutes of time. This can also be expressed as a total safe amount available as: (Power)(Time) = (watts)(Hours) = (V)(I)(H) = 600wH.

For a non-inverter system, using just the battery: A 20W device can be powered for 600w / 20w = 30 times longer than 1 hour, hence for 30 hours = (30h)(60min/h) = 1800 minutes of continuous or intermittent use. If the device is used only for a few minutes at a time, say for 5 minutes, the battery can be expected to provide this 20w of energy for: 1800min / 5min = 360 times. A battery array, either series and-or parallel, will increase the stored energy available and the amount of time a fixed or certain amount of power can be drawn. A 12v, 50aH rated battery would have a corresponding (maximum) watt-hour rating of: (12v)(50A) = 600wH, and therefore it can supply a maximum of 600w of power for 1 hour of time.
If the battery (mass-weighed) 18.1kg = 40lbs, its corresponding **energy density** or **specific energy** would be rated as: (wH / mass) = (600wH / 18.1 Kg) = $33.\underline{2}$ wH / 1 Kg . It is generally easier to weigh a battery than to calculate its volume, but an energy density could possibly consider the volume of the battery as: (wH / volume). The energy density rating of a battery is sometimes expressed in Joules of energy per mass. It was shown in this book that:

Energy = J = VAs = Ws = (Power)(time)

1J = 1 Ws , if the amount of seconds was 3600s = 1h , we have after multiplying each side by 3600:

(3600) J = (1 ws) (3600)
3600 J = 3600 ws
3600 J = 1 w (3600s)
3600 J = 1 w (1h) = 1 wH , or by switching sides:

1 wH = 3600J

Considering the above mentioned battery rating of 33.2 wH / Kg , a proportion type of equation can be created, or by simple multiplication, and so as to find the safe and expected amount of energy in that battery, we have:

(33.2 wH / Kg) (1 wH)
(33.2 wH / Kg) (3600 J / 1wH)
119520 J / Kg : expected or rated energy, although the battery also has more in it

For two batteries in parallel, the amount of time available to draw a certain amount of current will increase by the same factor of two. This is much like two capacitors in parallel that will double the total capacitance and-or maximum charge storage.

To effectively increase the total energy storage for a system, a larger constructed battery of the same voltage can be used, and this can be effectively achieved by placing similar (ie., same voltage, aH, etc) batteries in parallel (P) connection: positive to positive, and negative to negative. Similar batteries in a series (S) connection: negative to the positive terminal of the other battery, will effectively increase the available voltage by the value of each battery. The current value of charge level or state, and voltage level of each battery should be very close to avoid electrical damage such as an inrush and-or drain of a high value of current of which can damage electronics and can be a fire hazard, To help prevent this, charge the batteries so as to have the same voltage or within say 0.1V from a reference or goal voltage. Some systems have what is called "balancing resistors" so as to help maintain identical voltage levels for all the batteries once they are charged up.

A 48vdc system will require a 48v battery or series array of smaller voltage batteries, and a 48v (battery) charge controller, and it helps to keep the input voltage to the charge controller a few volts higher, say by 5v higher, due to the energy requirements and-or voltage drop or loss due to the charge controller circuit. A charge controller prevents overcharging a battery to a higher voltage than its safe recommended voltage, and which can then damage a battery. In general, the larger the battery in terms of its size or volume, the higher its aH rating. When single batteries are connected together, the aH rating of each battery still remains the same, and it is recommended to not exceed the maximum amps of its aH rating because it is the safe rated value of current passing in, out, or through that battery, and the aH rating does consider any other batteries in a system and their energy storage and possible extended time available. The aH rating is both a measure or indication of the amount of charge storage, time available and its safe maximum, continuous current.

For a 12v system, 50A from the battery is required to have 600W of power. For a 48v system how much current will be needed so as to have 600W of power?

From: Pw = V **I** = 600w = (48v)(Ia) and **I** = 600w/48v = 12.5A of current drain from the battery. This is the maximum safe current, and it should not be exceeded. Fuses and-or circuit breakers can prevent this.

Note also that 48v/12v = 4 times increase in system voltage , corresponding to a 50 A/12.5A = 4 times decrease in system current. When the current between the battery and the inverter is lower, thinner and-or longer wires can be considered. If 4 similar , 12v, 50aH batteries are connected in series so to make a 48v battery, the total amount of energy stored will technically be 4 times higher than any one battery, likewise it will also take 4 times more energy and-or power to charge to its full voltage. As mentioned previously, though the stored energy may be higher, the aH rating of a battery cannot be exceeded.

You can always use fuses and-or circuit breakers to prevent high current and-or short circuit damage to your power system, including the energy input and charging system. For example if a battery is rated at 50aH, then use a 50aH fuse - probably a slow-blow fuse, or circuit breaker rated at 50aH or less that is needed for all devices being powered. For extra safety, each device being powered can also have its own individual fuse or circuit breaker.

Typically, inverters are made for 12V, 24V, 36V and 48V battery systems. The 48v inverters are generally cheaper, perhaps half the price than the 12v inverters because in the 48v inverters a smaller step-up transformer (**1:2.5**) is needed, that is, the input of 48dc voltage needs to be boosted by only 2.5 times so as to produce 120vac. 12v systems, such as in a RV or "camper vehicle", will require much more current so as to provide power (P=VI). This can drain the charge in a battery fairly quick. An option to is to then use several, identical 12V rated batteries in parallel. A 12vdc system can also be converted to a 120vac system by using a 12vdc to 120vac inverter, however the current from the battery will still be high, and the parallel battery method will still be encouraged.

The maximum safe and-or available current for a 12V battery having a 50aH or (12v)(50aH) = 600w maximum rated power output is: (Note, the actually result is 600 watt-hours = 600wH.)

P max. in = P max. out : Power = energy used and-or supplied / time , P in = power from the battery
600w in = 600w out
(Vin)(I in) = (Vout)(I out)
(12v)(50A) = (120V)(I out) , after solving for I out:

I out = 5A : this value is one-tenth (1/10) the current drawn from the battery by the inverter. The ratio of voltages equals the reverse ratio of currents: (120 Vout / 12 Vin) = (50 Ain / 5 Aout) = 10

Note that the time of this power usage, either input at the inverter and-or output on the AC line is still the same value for a certain max. safe wattage available. It is incorrect to think that the time of use will be: time = aH / A = 50 aH / 5 = 10h , but the max. time value is rather one-tenth of this value, and is 10h / 10 = 1h, and this corresponds to that from the battery to the inverter as used for this 600W of continuous use and where Pin = Pout: time = 50aH / 50 = 1h . To help overcome the maximum amount of time available at a give wattage, the current drawn can be significantly reduced if the battery system has a higher voltage. Its aH rating will also increase if the batteries are in parallel, and then the time of use of the available power will also increase. If the output device used only half of the 600w max. available or 300w , and-or this could be calculated as 600w/2 = 300w, and-or 600w/300w = 2, and then the time of use would be twice as much or 2 (1h) = 2h. Also, 300w/600w = 0.5 and the time value will increase by the reciprocal of this ratio: 1/0.5 = 2 , As a check:

Energy in = Energy out **:** (Power)(time) = total energy = total Joules
(Power)(Time) = watt-hours in = watt-hours out : Power = joules / s = watts
: **(Power)(time)** = (J / s) (s) = (w)(s) = **J = Energy J**

$\frac{3600J}{1s} = \frac{1J}{s}$ 3600s = 1 w for 1 hour = **1wH** = 1w (3600s) = 3600 ws = **3600J**
1wH (1000) = 1000 wH = **1kwH** = 3600J (1000) = **3600000 J**

600wH in = (600w)(1hr) = (12V)(50A)(1hr) = (300w)(2 hr) = (120v)(2.5A)(2h) = 600wH out

To effectively double the amp hour rating of a battery system, you can connect two similar batteries in parallel. For two similar 50aH batteries connected in parallel, the total aH rating of this battery array and-or system is: (number of batteries in parallel) (aH rating) = (2 batteries)(50 aH / battery) = 100aH. Similar voltage batteries connected in parallel will have the same output voltage as any one battery. To be safe and not permit high currents to flow amongst batteries when they are connected together, please make sure they are at or near the same voltage level. Remember, it is a potential or voltage difference that will cause a current to flow

For **charging a battery**, many consider a fair and safe value of **one-tenth the amp value in the aH rating**, hence 1/10 the maximum input or output rating of current, or less. This reduces internal heat damage in the battery and gives more time for the chemical reactions to take place, and simply requires a relatively low power supply to charge that battery. Ex.: For a 50aH battery, a max. safe value for charging is: 50A / 10 = 5A of current. The theoretical amount of time to charge the battery when considered below its rated value and is "drained", and at this amount of current is: time = (aH rating / A) = 50 aH / 5 A = 10H = 10 hours . The ratio of this charging or discharging current to the rated capacity in amp-hours is sometimes called the **C-rate** or Charging and Discharging rating. This value is essentially a percentage of the aH rating:

C-Rate = current drawn in amps / max current in the aH rating :It is recommended that a C-Rate be less than

1 = 100% of the aH rating of that battery

A typical 100W, 12V rated solar panel can deliver about: from P = VI , we have I = P/V = (100w / 18max open circuit voltage) = about 5.5A maximum of current. The time it takes to charge a battery to its full rated capacity is determined by both the safe amount of recharging current, and how drained of energy the battery is and-or its current amount or capacity of energy. When charging a battery, it is best to monitor its voltage if it is not performed by a charger or voltage regulator device. To do this, temporarily stop charging the battery, and before and while measuring its voltage, temporarily draw an amp or so of current from it, such as for a light and-or other devices, and so as to have its true available output voltage value while being used. A battery itself has an internal resistance of which there will be a voltage drop or loss across and due to it, and this will not take place until it is in a circuit with current being drawn.

As an extra note for a more completeness of the energy conversions mentioned above:

To convert aH to J , it depends on the voltage used because it is the voltage that will give and increase the kinetic energy (J) of each electron in the moving current. More voltage means more (electric, electromotive) force applied to the electrons, and they will then have more kinetic energy and are therefore transferring more energy (Joules).

From: Power = (energy) / (time) = Pw = W = J / s

1 J = 1 Ws = 1 VAs , since 1h = 3600 seconds, and multiplying each side by 3600:

3600 J = 3600 Ws = 3600 VAs = VAh = W (3600s) = Wh : **VAh = (VA)(h) = (V)(Ah) = Wh**

(1V)(1aH) = 3600 J : 1 aH could be for example: 1A drawn for 1h or 0.5A drawn for 2h , etc. , **aH = (A)(h)**
J = (V)(aH) / 3600

Some data about a common example of a disposable or non-rechargable battery, here an AA electro-chemical cell.

These single cell, but often simply called batteries, have a typical voltage of 1.5V, whereas most rechargeable batteries of the same size are usually rated as 1.2V . These batteries are often tested using a 0.050A =50mA drain of current. For example it might take 12h for the battery to not be able to deliver 50ma. The rating of this battery would be:

(Power)(Time) = (watts)(time) = (1.5V)(0.05A)(12 h) = (1.5)(0.05)(12)VAh = 0.9 aH = 900maH

Internal Chemistry	maH rating max, expected, safe rated values at 1A load drain (Typical)	
zinc-carbon	600 maH = 0.6 aH	: "standard duty". Ex. If a device uses 25ma of current, it can function for: time = aH / A = 0.6aH / 0.025a = 24H = 24h
zinc-chloride	1200 maH = 1.2 aH	: "medium duty"
alkaline	2200 maH = 2.2 aH	: "high duty" (high current ability)

Though most rechargeable batteries, such as Lithium-Ion, are more expensive than non-rechargeable batteries, they can be recharged repeatedly and this results in a comparable and significant cost savings - perhaps by 50 times the purchase cost, thereby creating a way for an incredible financial savings over time. Many lithium batteries are rated at 1000 to 2000 full charging cycles, and some as high as 5000 cycles, and if the battery is not drained much, it can be recharged perhaps 10 times more. The energy density of lithium batteries is greater than that of standard disposable or lead-acid batteries, and has an average energy density of about 5 time more by volume.

[This space for book edits.]

Lithium Batteries During Cold Temperatures:

A current (as of 2024) problem with lithium batteries is that their current will decrease as temperature decreases. This is due to the electrolyte becoming not as conductive, and then potentially some lithium being plated on the anodes which will permanently degrade the performance of the battery. Lithium batteries are best used when the temperature is between (15°C = 59 °F) to (45 °C.= 113°F). A lithium battery can be permanently damaged if it is used when its internal temperature is less than 0°C = 32°F. At cold temperatures the electrolyte can expand and damage the internal structure of the battery. Special blankets can help keep a battery warm, and-or a special electric heater for batteries. Some lithium batteries may even have an internal electric heater. Even with all these facts, it is also said that lithium batteries still perform better in the cold air temperatures as compared to lead-acid batteries, but then lead-acid batteries do not have the risk of being damaged so easily by cold weather and this invites the possibility of also including a lead-acid in an automobile so as to start it, and warm the lithium battery. **It is recommended to only charge a lithium battery when the temperature is greater than or equal to 32°F = 0 °C** Some lithium batteries include a battery management system (BMS) which will help keep the battery safe, such as from discharging too much, and then the battery will appear as if it is drained and not working until it is recharged. The BMS also helps from over charging a battery with too much current and-or too high a voltage. If cold weather is approaching, the lithium battery can be removed and stored in a warmer climate, and-or be sure it is charged up. Batteries can "self discharge" slowly, and if they are left sitting too long, their voltage will go so low that the BMS will refuse to begin charging them, and so it is best to maintain the lithium batteries at no less than 50% of its total charge state or greater. Fully charge the batteries when possible when they are in use.

Note that an automobile and-or some electric motors require a high amount of power and-or torque to start the engine, and this requires a high amount of current so as to produce a high magnetic field having much power to rotate that high power electric motor which will "turn-over" or start a gasoline engine or electric motors for an electric car. These **starter batteries** must have what is called as a high amount of **"cold cranking amps" (CCA)**, and can start a petrol (ie., oil or gasoline) engine in cold weather, say at -20°C, and even so, this battery should only be charged when the temperature is above freezing. A typical, average sized car engine will require about 350 CCA (ie., 350A) to start, and larger engines will require about 1 amp more per cubic inch more in engine volume. Note that the rated volume of an engine is usually only for the volume of the empty cylinders in that engine, hence combustion related volume only, and not the engines total volume. These batteries have large internal plates and can then supply a large amount of that current and-or power very quickly, however, they cannot maintain that amount of current or charge being drawn for too long, and the maximum time rating of that current is for about 30 seconds. Then if the vehicle (car, etc) combustion engine has started, that battery can then be automatically recharged by the electricity generator ("alternator") that gets mechanically powered by the engine.

```
/*-----------------------------------------------------------------------------------------------------------------------------------------------------------------*/
```

A COMPUTER PROGRAM TO FIND THE AVERAGE VALUE OF ANY CURVE

```
/* AverageValueOfAnEquation.c

   The example equation shown below is "hard coded" (ie., explicitly, constant), here as a parabolic or
   second-degree quadratic equation, and can be changed for any equation you need.

   This method will find the average of y=x^2,  between the user settable upper and lower "bounds" or
   limits of x1 and x2.

   N sample values will be taken and summed together, and then divided by N.

   When the limits are between x=0 and x=1, the average value of y = x^2 is 0.33333...
   This is the case whether the curve is increasing from x=0 to x=1 or decreasing from x=1 to x=0.

   (c) JPA ,  April 16, 2021
 -----------------------------------------------------------------------------------------------------------*/

#include "stdio.h"
#include "math.h"

/*-----------------------------------------------------------------------------------------------------------*/
void main(void)
{
double y=0.0;
double xlower=0.0;       /*  settable lower limit or value of the integrating or summing */
double xupper=0.0;       /* settable upper limit or value of the integrating or summing */
double x=0.0;
double increment=0.0000001; /* essentially how the value of x will change so as to calculate each new term of the sum */
double sum=0.0;
unsigned long int n=0; /* this is a like a loop counter, but is essentially the number of "samples" or "data values" counter */
double average=0.0;
char ch=0;

start: ;        /* : This is  a label which is essentially a location and-or memory address in a program that can then be
                    quickly called with the goto command.  The goto command is often seen in the BASIC Language.  It may
                    be spoken that the program will "jump" (ie., the jmp command or instruction in the more fundamental
                    assembly or (strict, pure, "low-level" , actual) machine code language) to these locations in the program
                    and continue running or executing the program code or instructions there. Programming languages such
                    as BASIC and C are said as being "high level" (and much easier to program and edit) languages. */

/* system("cls");  optional, to clear the screen, this function is part of the stdlib.h to then include */

printf("\nThe equation is:  y=x^2");
printf("\n\n");
printf("\nInput lower value of x: ");  scanf("%lf",&xlower);
x = xlower;
printf("\nInput upper value of x: ");  scanf("%lf",&xupper);

if(xlower>=xupper){               /* : some user input error checking */
       printf("\nPlease use a lesser value for the lower value of x.");
       printf("\nPress a key to continue, or ESC to exit the program: ");
```

```
        fflush(stdin);  ch=(char)getch(); if(ch==27){ exit(0); };
        goto start;
    };

for(;;){
          y=x*x;       /*   <-----  is C's expression for the equation in question, and to find the average value of:   y = x^2   */
          sum=sum+y;
          x=x+increment;
          n=n+1;
          if(x>xupper){  break; };
        };

average = sum/n;

printf("\n%lf",average);          /* from 0 to 1, the average value is 0.33333......   = 1/3  */

printf("\n\nPress a key to continue or ESC to exit the program"); fflush(stdin);
ch=(char)getch();  if(ch==27){ exit(0); };

/* clear values of the variables to 0, reinitialize after each loop */
y=0.0;   xlower=0.0;  xupper=0.0;  x=0.0; sum=0.0; y=0.0;  n=0;  ch=0;

goto start;
};
/*--------------------------------------------------------------------------------------------------------------*/
```

BASIC CONCEPTS OF INTEGRATION

In the topic of calculus, there is a concept called integration. The basic meaning of integration is the summing of all the small parts so as to find the whole thing of which it came from. In calculus, the small parts are actually infinitesimally (as small as possible) small parts. This discussion and notation below is meant to be very basic so as the reader can gain at least a minimal understanding and to research the topic further if needed. If needed, you can review the topic of Understanding Basic Calculus shown previously in this book.

If given variable x, all the small bits of x are theoretically the infinitely small change or "differentials" (ie., a difference) of x = (dx), and will sum to that variable of x, regardless of any actually value of x. We can consider dx as like a very small first dimensional length amount and is equivalent to a very small change, piece or bit of the entire x or length that is infinitely long, and if each is identical and having a theoretical infinitely small size:

Sum of all the small bits of x = x1 + x2 + x3 + **. . .** = dx + dx + dx + **. . .** = S dx = S 1 dx = x :S means summation
an alt. expression: S 1x^0 dx = x or integration

When the summation symbol does not have any indication of the bounds (boundary) or limits of the summation, it is considered an indefinite, undefined or a general summation and which basically means a complete, full, or infinite summation of all the small parts. When the limits of the summation are indicated, the integral is called a limited, definite or defined integral. Here is a helpful pseudo-example of both integration and the notation used:

1 brick = Sum of all of its grains of sand : for practical purposes, any grain cement substance is not considered here

weight of 1 brick = Sum of the weight of all its grains of sand

weight/brick = S of weight of each piece of sand , If the weight of each same piece of sand is or averages about 0**.**001g and if the sum weight of each brick is 2kg:

$\frac{2kg}{brick}$ = (**n**umber of grains of sand) (0**.**001g) , mathematically , the **n**umber of grans of sand in 1 brick is (2kg / **b**rick) / 0.001 = (2000g/**b**) / 0**.**001g = (20)(10^5) grains / **b**
Expressing the indicated product with summation notation:

weight g = $\underset{x=1}{\overset{x=n}{S}}$ (0.001g) : Here, (0**.**001g) is a constant for the weight of any amount of bricks considered.
: x = term and-or grain number, and (n) is the number of grains of sand considered

If y = a function of (x) = f(x), its derivative with respect to the (x) variable is noted as: y ' = f ' (x) = (dy / dx) = d f(x) / dx .

(dy / dx) represents the (infinitely small) instantaneous rate of change of (y) with respect to (x), and when (x) changes by an infinitely small value. (dy / dx) is mathematically either a constant such as for a linear equation or a derived equation from the given function. In short, a derivative actually represents the slope value of the curve or function at a specific value of the independent variable, location or point, and is therefore is the instantaneous rate of changes in the variables or can be thought of as the equal the instantaneous slope (m), of a very small line segment, at one point on the curve..

$\frac{\text{(instantaneous change in y)}}{\text{(instantaneous change in x)}} = \frac{dy}{dx}$ = derivative = f'(x) = instantaneous slope , mathematically:

(dy) = (derivative) (dx) : though (dx) is a theoretical constant equal to an infinitely small value, we see that (dy) is not and (dy) depends on the product indicated

dy is an infinitely small part of (y) = f(x). The sum or integral of all these small parts of (y) will therefore equal (y) = f(x):

y = f(x) = S dy = S (derivative) dx = S f'(x) dx

Area Under The Curve Of A Function And Between The X-Axis

Given a function and its curve, integration can be used to find the area beneath that curve (See the figure below.) which is bounded by that curve and the (x) or independent variable axis. Since we are not finding an infinite area, but rather a much smaller, defined, definite, limited, or bounded portion along the (x) axis, we will indicate (ie., notation, express) this with "boundary" or "limit" numbers next to the summation symbol. The upper or higher value or limit of (x) will be placed higher near the S symbol, and the lower or smaller value of (x) will be placed lower near the S symbol. Here is a notation for the rectangular Area (theoretically, length times width) that is defined or bounded by the curve and the (x) axis between the values of x=a, and x=b, and it is an example of a "definite integral" with a specific result or sum rather than a generalized integral, sum or result:

First: $y = f(x) = A$, and which could be expressed as A(x) since A is a function which depends on (x). Ex. A=x^2

$$\frac{dy}{dx} = \frac{d(\,f(x)\,)}{dx} = f'(x) = \text{derivative} = \frac{dA}{dx} \quad , \quad \text{mathematically:}$$

$$d(\,f(x)) = f'(x)\,dx \quad , \quad \text{and since } A = y = f(x):$$

$$dA = f'(x)\,dx \qquad \text{summing (S), all these infinitely, and infinitesimally small bits:}$$

$$y = f(x) = A = S\,dA = S_{x=a}^{x=b}\ f'(x)\,dx = \int_{x=a}^{x=b} f'(x)\,dx$$

: The area will have the same units as those of (x) or some other variable being integrated. The term(s) being summed is formally called the integrand.

In the above notation, each bit of Area is an infinitesimally small rectangular area (dA), and each with a base or width of (dx) and a corresponding height of y(x) = f(x).

Area = A = (base)(height) = Sum of each small rectangular, differential area = S dA

For each (dx) or small base length of the small portion of the Area, hence dA and which is not a constant value, (dx) is defined as an infinitesimally small constant value. but the specific, actual or corresponding value of (y) is not defined as a constant value, and is actually variable in value, and essentially is the corresponding value of the function using a specific value of (x), hence the (y) value of the function at that specific value of (x), hence y at f(x) = y(x) = f(x), and the result is the (variable) height of the curve above the (x) axis at that point.

The derivative of a function is the rate of change of (y) with respect to a change in (x) when the change in (x) is infinitesimally small in value, and therefore, the corresponding change in (y) will also be infinitesimally small and it is generally and symbolically not equal to the same value as the change in (x), and this is due to the specific functional equation and relationship of (x) and (y).

Given y=f(x) = "y equals a function of x" (ie., y, the output or result, depends directly upon the value of x and the equation it is in) below is some notation for: "the derivative of y with respect to x", and since the changes are infinitesimally small, unlike that when finding a general slope value, it is more properly spoken as: "the differential of y with respect to the differential of x":

$$\frac{d\,f(x)}{dx} = \frac{dy}{dx} = \frac{\text{change in y --> 0}}{\text{change in x --> 0}} = y\,' = f\,'(x)$$

: notation for the derivative of the function of (x) with respect to x, the independent variable. It is a constant or an expression for the point to point (on the curve of the specific equation, or instantaneous rate of change of (y)=f(x) with respect to (x). Mathematically:

$d\,f(x) = \frac{dy\,dx}{dx} = dy = f'(x)\,dx$: the product of the derivative of the function and the infinitesimally small change in (x) or dx is equal to a differential of the function. The sum of differentials, dy, of a function is equal to that entire function: f(x) If this sum is limited or bounded, the result is not equal to that function, but rather is essentially a defined, limited or bounded area.

Notice that: $dy = f'(x)\,dx$ has the form of a linear or line equation:
$y = m\ x$, and $m = y / x$ = the slope or rate of change of y with respect to a change in x, and:

The sum of all the small bits of y will equal that entire function:

$S\,dy = y = f(x)$, and $S\,dy = y = f(x) = S\,(\,f'(x)\,dx\,)$: as shown above.

$dy = f'(x)\,dx$ = (derivative of the function) dx Note : $\frac{dy}{dx} = \frac{d\,(f(x))}{dx} = f'(x)$ = derivative

Given that: $S\,dy = y = f(x) = S\,(\,f'(x)\,dx\,)$, this clearly shows that to solve for the sum of all the differential bits of the function, we need to take the anti-derivative of the derivative, so as to have the original function = f(x) or= which may be expressed with the notation of F(x) = anti-derivative. Also:

First remember that : d(value) = a very small or instantaneous amount, bit or "differential (amount or value)" of the stated value, and do not consider this as the derivative of a value or function which is actually the value of: dy / dx = d(function) / dx . A differential is actually a very small instantaneous value and should be considered as a constant value for a given function, and not variable in value.

$y = S\,dy = f(x) = S\,(f'(x)\,dx) = S\,(\,(\text{derivative})\,(dx)) = S\,\frac{dy}{dx}\,(dx) = S\ dy = S\,f'(x)$

Also consider: $\frac{dy}{dx} = \frac{d\,(f(x))}{dx}$ = (derivative) , then $dy = \frac{d\,(f(x))\,(dx)}{dx} = d\,(f(x))$

$dy = d(f(x))\,dx = (\text{derivative})(dx) = \frac{d(f(x))\,(dx)}{dx} = d(f(x))$, by summing both sides:

$S\,dy = S\,d(f(x)) = S\,(\text{derivative})(dx)$
$y = f(x)) = S\,(\,(f'(x))\,(dx)\,)$: Shows f(x)) is both the result of the summation and the anti-derivative of the function: f'(x).

$S_{a}^{b}\ f\,'(x)\ dx$ = Area between x=a and x=b = F(b) - F(a). If a=0, F(x) = F(0) = no Area, and Area = F(b)

It could also be said that this bounded or defined area is the difference in areas from or bounded by: ((x=0) to (x=b)) and is the area between or bounded by: ((x=0) and (x=a)).

Consider a line, y=x , with an infinite number of infinitesimally small line sentiments, piratically points side by side. Each segment is a "differential" of that line with a value of dx. The sum of all these segments is that entire line: $x = S\,dx$. If we want to just consider the length of only a portion of that line, hence the length of a line segment (with a length that could be thought of as a multiple of dx, and yet not the full line x), we need to also limit the sum between two selected or boundary values of x, such as x=a and x=b:

length of line from points x=a to x=b = $x_{x=a}^{x=b}$ = x_{a}^{b} = length of bound portion of that line = a line segment

$x_{x=a}^{x=b}$ = (limited multiple of)(dx) = xb - xa : the length of the line segment is the difference between (b) and (a) which is the difference between the upper and lower limiting values.
From: **(a) + distance = (b)** , **distance = (b) - (a)**

x_{a}^{b} = (limited multiple of)(dx) = S_{a}^{b} 1 (dx) = S_{a}^{b} x^0 (dx) = $x\hat{\ }1_{a}^{b}$ = (x^1=b) - (x^1=a) = $x_b - x_a$

= (x at b) - (x at a) = (x=b) - (x=a) = x(b) - x(a) = y(x=b) - y(x=a) = y(b) - y(a)

Also note that: y = f(x) = x = S dx = x^1 = S x^0 dx : x^0 is a constant value of 1, and is the derivative of x^1, and x^1 is the antiderivative of x^0. (dx) becomes automatically included into the sum, and here the sum is x^1

Given a derivative of a function, this is noted as: y ' = f '(x) . When given a derivative, noted here as y ' or= f '(x) , we can find the original function (F) from which this derivative was derived out from.

Consider that F represents a function and its derivative is the function: y=f(x). F is the notation for the anti-derivative of y=f(x). In more symbolic form: F is the anti-derivative of its, or a given, derivative.

The derivative of F is therefore equal to:

d F / dx = F '(x)

In short, the derivative of the anti-derivative of a given function is equal to that same given function.

The derivative of y=f(x) with respect to x is: y' = f'(x) = dy/dx . Given the derivative equation = dy/dx, the antiderivative of this is y=f(x) which is the original given function.

In the notation shown being summed or integrated, all the bits of the derivative of a function which is f '(x) = y '(x) will sum to Y = F(x) which is the "whole thing", the "whole function" and not just a "bit of the function", and it is also the anti-derivative function of the function being integrated.

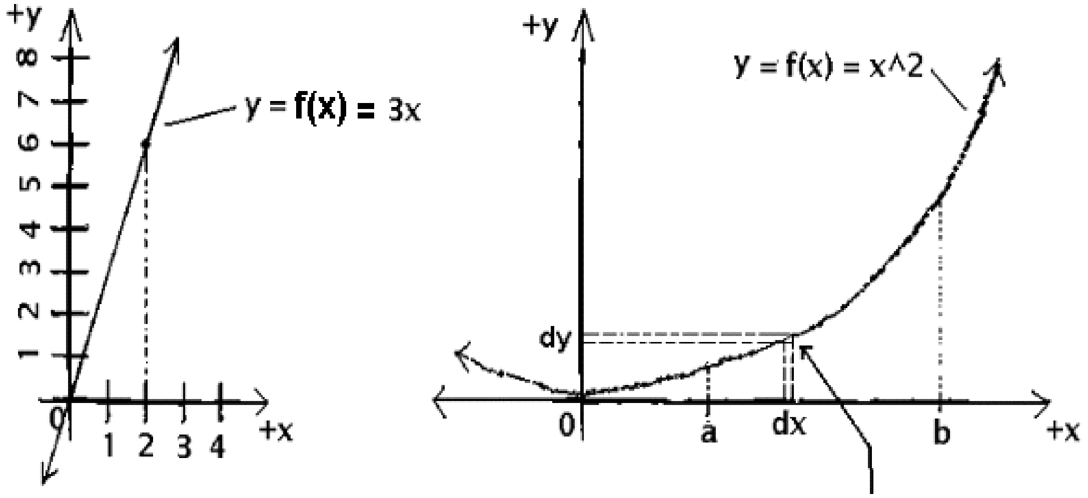
The area (A) beneath both the bounded curves of a line and parabola will be developed below.

In the graphing of the parabola curve on the right, the area (A) to be found is indicated as defined, limited or bounded by the lower bound value of (x=a) and the upper bound value of (x=b). An infinitesimally small "slice" or "differential area" (dA) bounded by dx which is a very small change in (x), and the height=y at that value of (x) is indicated within the total area being found.

Now considering the line graph. The area being found is triangular and is easy to see that it is bounded by (x=0) and (x=2) as the triangles base, and having a height that is equal to: y(2) = f(2) = 3(x) = 3(2) = 6. The area of this triangle is:

At = $\frac{\text{(base)(height)}}{2} = \frac{(2)(6)}{2} = \frac{12}{2}$ = 6 square units

[FIG 275]



Essentially a very small right triangle wth sides (dx) and (dy), and -or a very small (hypotenuse) line with the slope of = (dy) / (dx) at that point, and which is the derivative of the given function, here: x^2

Before proceeding, first notice that for the line, that the slope value = m = dy/dx is a constant value anyplace on that curve (here a line), and that for the parabola curve, dy/dx is not a constant value and is actually an equation. dy/dx is called the derivative when the changes are infinitesimally small and considered at a single point of the curve.

Given this equation of this function (f) = y = f(x) = 3x, and if the derivative of some other function is 3x, that function (F) is:

$$F = \frac{3x^{(1+1)}}{(1+1)} = \frac{3x^2}{2} \quad \text{or=} \quad 1.5x^2 \quad : F(x) \text{ is the anti-derivative of } y = f(x) = 3x$$

We can take the derivative of this function = F as a check:

$$F'(x) = \frac{dF}{dx} = \frac{d}{dx}\frac{3x^2}{2} = \frac{(2)\,3\,x^{(2-1)}}{2} = \frac{6x^1}{2} = 3x \quad : \text{checks}$$

Just for some extra completeness, the derivative of the line equation of: y = f(x) = 3x is:

$$y' = f'(x) = \frac{dy}{dx} = \frac{d\,f(x)}{dx} = \frac{(1)\,3x^{(1-1)}}{1} = 3x^0 = 3(1) = 3$$

: the derivative of a line is a constant, and is equal to the slope=m of that line.

Now consider:

$$f(x) = 3(x) \quad \text{and} \quad F(x) = \frac{3x^2}{2}$$

The derivative of: $F(x) = \frac{3x^2}{2} = 3x = f(x)$: $F'(x) = d(F(x)) / dx = f(x)$: F(x) is the antiderivative of f(x)
f(x) is the derivative of F(x)

A(a to b) = A(x=a, to x=b) = A(x=b) - A(x=a) = Ab - Aa : The bounded area is equal to the difference of two areas

Area of a thin section or slice of the bounded region = A slice = (base of slice)(height of slice) = (dx) (y value at that location of x), however, since y=f(x)=A itself, we would then be trying to multiply (dx) by the entire area (A), and we rather just need an infinitesimal amount of y=f(x) = A at a time, and this value is dA, then sum all these "differential areas" bits:

A slice = dA = (dx)(dy) , dy = dA / dx = A'(x) = f'(x) , The antiderivative of A'(x) = A = y = f(x)

A = Area of all the "differential areas" = dA, and if A found by summing (S) all these (dA):
A = S (area of each slice or differential area) = S dA

y = f(x) = A(x) = S dA = S (dy)(dx) = S f'(x) (dx) = F(x) = Y or = y = f(x) : F(x) is the antiderivative of f'(x) = A'(x)
F(x) = A

Let a limited value of being determined A = the summed area of all the area slices from x=(0 to b) =

$$A_a^b = S_a^b\ dA = S_a^b f'(x)\,(dx) = A_b - A_a = F(b) - F(a)$$

: the Area is limited so as to have a definite sum or integral, rather than an indefinite result of just the function itself. (a) and (b) are the limits or bounds of the summation of the differential areas where each is dA. This general expression is sometimes called the **Fundamental Theorem Of Calculus**.

Another way to consider or look at the derivation and meaning of all this:

$$y = f(x) = \text{function1} \quad , \quad \frac{d\,f(x)}{dx} = \frac{dy}{dx} = (\text{derivative of function1}) = \text{function2}$$

: function1 is the anti-derivative of of function2

y = f(x) , and dy / dx = d (f(x)) / dx = f'(x) , and therefore, mathematically:

dy = d(f(x)) = f '(x) dx : f(x) is the anti-derivative of f '(x) , after integrating (ie., S) each term, we find:

S dy = S d(f(x)) = S f '(x) (dx) , and these terms sum to:

y = f(x) = f(x)

$$\text{Area} = F(x) = Y = S\,dy = S_0^x\ (dx)\,(\text{derivative of function1}) = S_0^x\ f'(x)\ (dx)$$

Note that the infinite sum of all the (bits of x) = S (dx) = (x). To help understand this, consider all the infinite, infinitesimally small line segment lengths or bits (or infinitely small line segments) of a line, and which when put, added, joined or combined together, they will sum up to or create that whole (infinite) line. This was shown previously in this topic.

If we consider the function = f =3x as a derivative of another function = F, then F is the anti-derivative of 3x which is equal to: F = (3x^2) / 2

$$S_{0}^{b}\ f'(x)\,(dx) = S_{0}^{b}\ 3x\ dx = D_{0}^{b}\ \frac{(3x)(x)}{2} = D_{x=0}^{x=b}\ \frac{3x^2}{2} = \frac{3(b^2)}{2} - \frac{3(0^2)}{2} = \frac{3(b^2)}{2} \quad \text{or}= 1.5\,b^2 \quad : D = \text{difference}$$

For the example above when b=2, the area beneath the curve of $y=3x$, from $x=0$ to $x=b=2$ is:

$$\text{Area} = \frac{3(2^2)}{2} = \frac{12}{2} = 6 \text{ square units}$$

For another example, the area beneath the curve of $y=3x$, from $x=a=1$ to $x=b=2$ is:

$$\text{Area} = \frac{3(2^2)}{2} - \frac{3(1^2)}{2} = 6 - 1.5 = 4.5 \text{ square units}$$

For the parabola curve of $y=f(x) = x^2$, Its anti-derivative is: $F(x) = \frac{x^3}{2}$, and for the area beneath this curve curve, bounded by (b) and (a) and the (x) axis:

$$\text{Area} = S_{a}^{b}\ (dA) = S_{a}^{b}\ f'(x)\,dx = F_{a}^{b} = \left.\frac{x^3}{2}\right|_{x=a}^{x=b} = F(b) - F(a) = \frac{b^3}{2} - \frac{a^3}{2} = \frac{b^3 - a^3}{2} \quad \text{: with square units of the (x) axis.}$$

If the limits were: $x=a=0$ and $x=b=1$, the area beneath the function or curve of $y=x^2$ is:

$$\text{Area} = \frac{b^3}{2} = \frac{2^3}{2} = \frac{8}{2} = 4 \quad \text{: with square units of the (x) axis}$$

Ex. You are filling a 100 gallon container with a watering hose. The flow rate (gallons per hour = gal/hr) is constant and it fills the 100 gallon container in 5 hours. What is the flow rate:

$$\text{flow rate} = \frac{\text{volume of water}}{\text{time to move and-or fill that volume}} = \frac{V}{T} = \frac{100 \text{ gallons}}{5 \text{ hours}} = \frac{20 \text{ gal}}{1 \text{ hour}} \quad \text{: rate of filling}$$

From this data, we can mathematically derive a formula for the volume (V) of water in the container with respect to the time (T) variable:

Volume of water = volume = V = (rate) (Time) = $R\,T$: a linear equation with the rate = slope = m = R

$V_{gal} = (R)(T) = (\text{gal / hour})\ (T \text{ hours})$ and for the specific example above:

$100\text{gal} = (20\text{gal / hour})\ (5 \text{ hours})$

From the above equation, the derivative of Volume with respect to Time is:

First: V is a function of both (flow) rate and time. Here, the rate is a constant value of 20gal/hour, and then V becomes a function (relationship) of just the Time variable, and with the rate being a multiplier. Since the rate of change of one variable with respect to another is constant, the equation for V is a linear equation. Expressing V as a function of Time:

$V= f(T) = R\,T^1 = (20\text{gal/hour})\ T^1$

By knowing how to take derivatives as shown previously in this book:

$\frac{dV}{dT}$ = R = (1) (20gal/hour) T^(1-1) = (20gal/hour) T^0 = (20gal/hour) : a constant

If just given the above derivative equation, or constant flow rate with respect to time, we can integrate that constant or equation so as to find the volume equation of which it is from.

First note mathematically from the above expression:

dV = (rate) dT = R dT : extra, since dV and dT are instantaneous values, so is the (rate) or derivative.

S (dV) = S (rate) (dT)

V = (rate) (T)

If the container had some initial volume; (+) of water or deficit; (-) of water, that can be included into the formula as an addition of this constant (C). This will only shift the curve of the volume equation in the vertical direction, and will not change the general shape (ie.. slopes and-or the rate of changes) of the curve This is also a more formal or complete way to write an equation. This initial constant has no affect on the flow rate, and only has an affect on the total volume at negative time vales before the start of time when it is considered as Th=0h:

V = (rate) (T) + C : checking, at T=0h, V = 0 + C = C, the initial or starting amount of water volume

The formula for distance is very similar to the above formula for volume.

distance = (rate) (time) = (speed) (time) = (velocity) (time) = D = V T

In the situation here, V = velocity, and not volume.

$\frac{d\,D}{d\,T} = \frac{V\,T\text{^}(1-1)}{1}$ = V T^0 = V(1) = V mathematically:

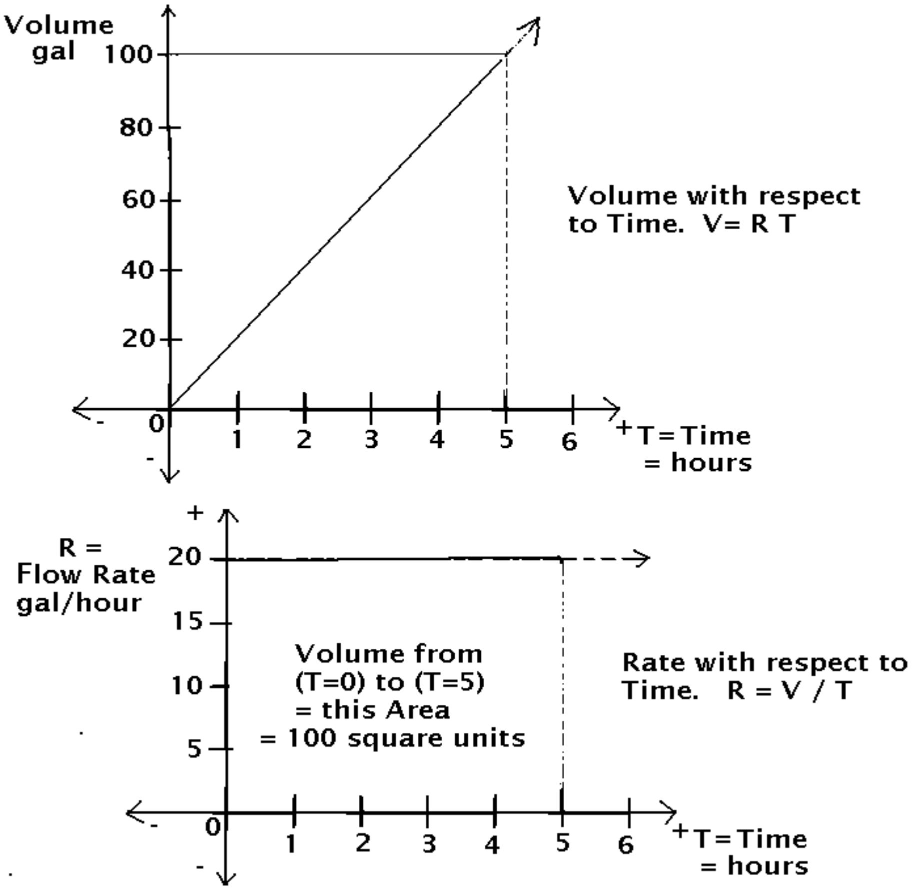
dD = (V) (dT)

S (d D) = S (V)(dT)

D = V T

See FIG 276 below.

[FIG 276]]



Area on the above graph = Volume of water = RT = (base)(height) = (5hr)(20gal / hr) = 100 gal

Above in the first image, consider R = V / T = slope of the line

A COMPUTER PROGRAM TO FIND THE AREA BENEATH A CURVE SEGMENT

```
/*******************************************************************************************************************
  AreaUnderACurve.c

   (c) JPA , April 2020

   This is a program to find the area beneath the curve of a function y=f(x) and the x-axis, and has a width
   that is bounded by a lower limit of (x=a), and an upper limit of (x=b).  This program uses a successive
   approximation method of summing up many smaller areas or thin, narrow "slices" or segments of area,
   and so as to find the total area by summing smaller areas.  This is a nice program to understand, and is
   very useful for when standard methods of calculus are not being used for some reason.

   Adjust the program as needed.  For example, since the function is hard coded or set fixed as:
   y = f(x) = x^2,  and this is essentially a simple parabola curve and you will have to change it if you want
   to use other functions and their curves.

*************************************************************************************************************/

double findarea(double lowerlimit_a, double upperlimit_b);

/************************************************************************************/
void main(void)
{

                  /* findarea(), function use example program:  */

double area=0.0;
double a=0.0;     /* lower limit of integration */
double b=0.0;     /* upper limit of integration  */

printf("\n\nThe function currently used in this program is:  y = f(x) = x^2 ");
printf("\n");
printf("\nInput a = lower x limit of integration: ");  scanf("%lf",&a);
printf("\nInput b = upper x limit of integration: "); scanf("%lf",&b);

if(a>b){
          printf("\n\aPlease use an upper limit that is greater than the lower limit.");
          printf("\nPress A Key To End The Program");  getch();  exit(0);
       };

area=findarea(a,b);

printf("\n\nThe approximate summed area is:  %lf square units.",area);
printf("\n\nPress a key");  getch();

exit(0);
};
/*****************************************************************************************/
```

```
/****************************************************************************************/
double  findarea(double lowerlimit_a, double upperlimit_b)
{
double area=0.0;
double a=0.0;
double b=0.0;

double dx=0.0000001;    /* Increment in the x variable , equivalent to the width of a
                                    slice of area .  Here, dx is hard coded as a constant, and
                                    you can use a smaller value for increased precision in the
                                    result.  This value will give about 6 digits of precision in the
                                    result.   Use a smaller value for more precision.

                               An option is to let the user enter this dx value, OR: modify the
                               program to let the user enter the number of slices, steps, or
                               increments, and then dx can be found from that as for example:
                               Instead of the user entering 0.001 for dx, the increment in x, the
                               user can enter 1000 for the number  of steps or slices of area,
                               and steps would also essentially be a loop counter variable.

                              dx = (upperlimit_b  -  lowerlimit_a) / steps
                              steps  =  (upperlimit_b  -  lowerlimit_a) / dx                              */

double y=0.0;            /* y = f(x) = equivalent to the height of a slice of area */
double x=0.0;           /* current position along the x axis during the summation process */

x=lowerlimit_a;      /* let's start x and the summing at the lower limit */

for(;;){                 /* successive approximation, summing loop        */

             y = x * x;         /* the hard coded function of the curve of which the
                                    area beneath it ant the x-axis is being found.
                                    here: y = f(x) = x^2  and is C coded as:   x * x    ,  if something like
                                    y = ax^2 = a * (x * x) is used, the result or area will be a factor of (a) times more. */

            if(y<0){ y=y*(-1.0); };  /* make y always positive, since area is always a positive value: */

            area = area + (dx * y);    /*  Summing each slice of area to the total accumulated area.
                                                  (dx * y) is analogous to:  Area slice = (width) x (height) */

            x=x+dx;         /* incrementing (x) by the value of dx , and so as to have a new value of (x) for the loop */

           if(x>=upperlimit_b){ break; };    /* break from the summing loop */
        };

return area;
};
/******************************************************************************************/
```

Use ex.: If 0 and 1.0 are input, the result is 0.3333...   : square units = units^2
Use ex.: If -1 and 1.0 are input, the result is 0.6666... : hence twice as much due to the symmetry of the parabolic curve
Use ex: If 0 and 2.0 are input, the result is 2.67

A COMPUTER PROGRAM TO FIND THE ARC LENGTH OF A CURVE SEGMENT

```
/**********************************************************************************************
  ArcLengthOfACurve.c

   This coding is mostly based on the algorithm found in the program
   AreaUnderACurve.c program that is shown in this book.

   (c) JPA,  Dec 11, 2021,

  This is a nice program to understand, and it is very useful for when standard
  methods  of calculus are not being used and-or not known such as for
  when a difficult equation(s) is involved.      This is  very handy when the
  anti-derivative is difficult to find when using a standard calculus method.

   Basically, this is a form of integration by summing up many small distances
   along the entire (arc) length of the curve that was bound or delimited by limits.
   In the sample use program below, the function or curve is y=f(x) = mx, a linear
   or line equation,   and the limits or boundary section or definition, are from x=a
   to x=b.

*********************************************************************************************/
#include "stdio.h"
#include "math.h"    /* : if needed, such as for sqrt(number) and pow(base, exponent */
#include "stdlib.h"   /* : for system() */

double findarclength(double lowerlimit_a, double upperlimit_b, double a);
/*  This program currently uses a hard coded, or constantly defined
     function, y = f(x), and of which needs to be changed for other equations.
     The specific function used is for a liner or line function. */

/*********************************************************************************************/
void main(void)
{
/* Function use example program:  */

double a=0.0;   /* here the slope (m) of the line function:  y = f(x) =ax = mx   */

double L=0.0;      /* lower limit of integration */
double U=0.0;      /* upper limit of integration  */

double arclength=0.0;

char c=0;

for(;;){   /* main loop */

system("cls");

printf("\nA Program For:  Arc Length or Length Of A Curve Segment");

printf("\n\nThe function currently used in this program is the linear function:");
printf("\ny = f(x) = mx");
printf("\n");
```

```
printf("\nInput the slope (m) or steepness value: "); fflush(stdin); scanf("%lf",&a);
printf("\nInput the lower x limit of integration: ");  fflush(stdin); scanf("%lf",&L);
printf("\nInput the upper x limit of integration: ");  fflush(stdin); scanf("%lf",&U);

/* NOTE:  %f did not work to input a double,  so use:  %lf,  , otherwise it gets 0  */
/*  %lf  is the (data type, size) format specifier for a double float, or "long float"   */
/*  The word double here means double precision, such as twice the number of decimal places possible. */

if(L>U){
            printf("\n\aPlease use a upper limit that is greater than the lower limit.");
            printf("\nPress A Key To End The Program");  getch();  exit(0);
          };

arclength=findarclength(L,U,a);

printf("\n\nThe approximate arclength is:  %.7g units.",arclength);
                                                     /*  %lf , can also use %f */

printf("\n\nPress a key to continue, or ESC to exit");
fflush(stdin); c=getch();  if(c==27){  exit(0); };   /* : user pressed the escape key */

};   /* end of main loop */

};
/*****************************************************************************************************************************/
double  findarclength(double lowerlimit_a, double upperlimit_b, double a)
{

double dx=0.0000001;
  /* increment in the x variable , equivalent to the width of a slice of area .  dx is a constant,
     use a smaller value for increased precision in the result.
     This value will give about 6 digits of precision for the fractional value after the decimal point.
     You can use a smaller value for more precision, but it will take more loops and time to calculate
     the result. an option is to let the user enter this value, OR: enter the number of slices, steps, or
     increments, and  then dx can be found from that as:  Ex, instead of the user entering 0.001 for dx,
     the increment in x, the user can enter 1000 for the number of steps or slices of area
      dx = (upperlimit_b  -  lowerlimit_a) / steps   */

double y=0.0;       /* : y = f(x) , equivalent to the corresponding height at value used for (x)  */
double x=0.0;

double x1=0.0;      /* : for point1 on the curve, coordinates = p1(x1,y1);  */
double y1=0.0;
double x2=0.0;      /* : for point2 on the curve, coordinates = p2(x2,y2);  */
double y2=0.0;

double distance=0.0;  /* distance between two points on the curve */
double arclength=0.0;      /* total sum of distances = arc length */
int n=0;

x=lowerlimit_a;    /* : let's start the value of x, and the summing, at the lower limit */
```

```
for(;;){
        x1=x;

        y1 = (a * x1);    /* : 1:  The hard coded function of the curve of which the
                                   area beneath it ant the (x) axis is being found.
                                   here: y = f(x) = ax + b ,   and is C language coded as:
                                   (a)( x) + 3   , for a linear equation, b shifts the entire curve
                                   vertically, and the arc length still remains the same, hence
                                   its not used here.  */

        /* : Now we have the first point's coordinates , p(x1,y1) */

                         /*  Three other example functions to consider, and the arclength() should be rewritten as needed,
                             in particular, for any arguments needed to be sent to this function:  */

                        /* 2:   y1 = a * (pow(b,x1));     : such as for an exponential function:  y = (a)(b^x),
                                   and the user could first input the base value (b) and multiplier (a).  */

                        /* 3:   y1 =  sqrt((a * a) - (x1 * x1));     : such as for a circle function:  r = x^2 + y^2 ,
                                   and the user could first input the radius value (r).   */

                        /* 4:  For an ellipse:  This will give 1/4 of the perimeter depending on the limits, hence multiply by 4:

                                    y  = +-  b  \/  1  -  _x^2_          : Note that when x=0, y= +-b as should be expected, and
                                                           a^2

                                    y1 =  minoraxis * sqrt( 1 -  ((x1 * x1) / (majoraxis * majoraxis)) );   */

       x2 = x1 + dx;    /* : increment x so as to find the next close point and its coordinates */

       y2 = (a * x2);     /* 1: Now we have a nearby, second point's coordinates  p(x2,y2)  */

                             /*  2:   y2 = a * (pow(b,x2)); */
                             /*  3:   y2 = sqrt( (a * a) - (x2 * x2));  */
                             /*  4:    y2 = minoraxis * sqrt( 1 -  ((x2 * x1) / (majoraxis * majoraxis)) );   */

        /* For the distance between two close, successive points on the curve:
           This uses the standard distance equation, for the distance between two points on a plane:
            distance = sqrt ( (change in x)^2 + (change in y)^2 )
            In theory, for two very close points, this is an infinitely small distance value
            for that segment or portion of a curve between x1 and x2:                              */

        distance = sqrt( (x2 - x1) * (x2 - x1) + (y2 - y1) * (y2 - y1) );   /* : this is similar to:   hypotenuse = sqrt (x^2 + y^2)  */

        arclength =arclength + distance;  /* : add each distance value to the total distance called arclength */

         x = x + dx;    /* : incrementing x by the value of dx, so as to get the coordinates of the "next point" on the curve */

        if(x>=((upperlimit_b - dx))){ break; };
    };

return arclength;
};
/*****************************************************************************************************************************/
```

[This space for book edits.]

BASIC CONCEPTS OF VECTORS AND RESULTANT FORCES AND MOTION

Consider an airplane or other object ascending or descending in the sky. For detailed scientific analysis, the airplane can be considered as having two basic types of motion; a horizontal motion and a vertical motion. Each motion has a velocity (ie., speed) and direction. More specifically, it will have both a horizontal (ie., "x") velocity or speed, and a vertical (ie., "y") velocity or speed. These two types of motions can be considered separately over time so as to find a change in either the horizontal or vertical distances traveled. These two motions can also be combined or summed into one resultant (ie., true, actual, apparent) motion that is often called the resultant or effective velocity and direction. These two basic or component motions with both having a magnitude (ie., size) and direction are formally called **vectors**, and their combined or summed vector is called a **resultant vector**. A resultant vector can be considered as the effective vector of several other vectors. Since a plane generally does not just go vertically and-or horizontally upon landing or taking-off, the resultant direction or angle is someplace in-between the horizontal (0°) and vertical (90°) dimensions, and the plane will have a small angle or direction of ascending or descending between those two extreme values.

If an object such as a plane or ball has a horizontal velocity and direction, and a vertical velocity and direction, the resultant velocity and direction is composed (ie., a combination) or determined by those two values, and the actual effective or resultant velocity and direction of that object will be between those two values, hence like an upward rise and-or downward decline.

If two or more forces are acting upon an object at the same time, there is a single, net, effective (as if, or equivalent to a single force or vector) or resultant force that would be equivalent to those two component or vector forces.

Ex. An object is being forced both rightward, and also leftward with the same value of force. The net or resultant force is obviously 0. We can assign direction to the forces using positive and negative signs. For rightward, we will use the positive sign, and for leftward, the opposite direction, we will use the negative sign which indicates a 180° rotation in the opposite direction. Expressing the combining of forces or vectors (have both a magnitude or value and direction) algebraically:

$$F1 + F2 = Fr$$

: General equation for the sum of forces. The forces can have any sign.
Fr = resultant force.

$$+F1 + (-F2) = Fr$$

Since the magnitude of F1 = F2, and using substitution:

$$+F1 + (-F1) = Fr$$
$$+F1 - F1 = Fr$$
$$0 = Fr$$

Since the resultant force is 0, the object will not have any motion due to F1 and-or F2. Even though there is no effective or resultant force that causes a motion for the object, the object can also be considered as being (somewhat, depending on the material) compressed (like a spring) by those forces acting upon or applied to it.

If F1 was 3, and F2 was -5, the resultant force is:

$$F1 + F2 = Fr$$
$$3 + (-5) = Fr$$
$$3 - 5 = Fr$$
$$-2 = Fr$$

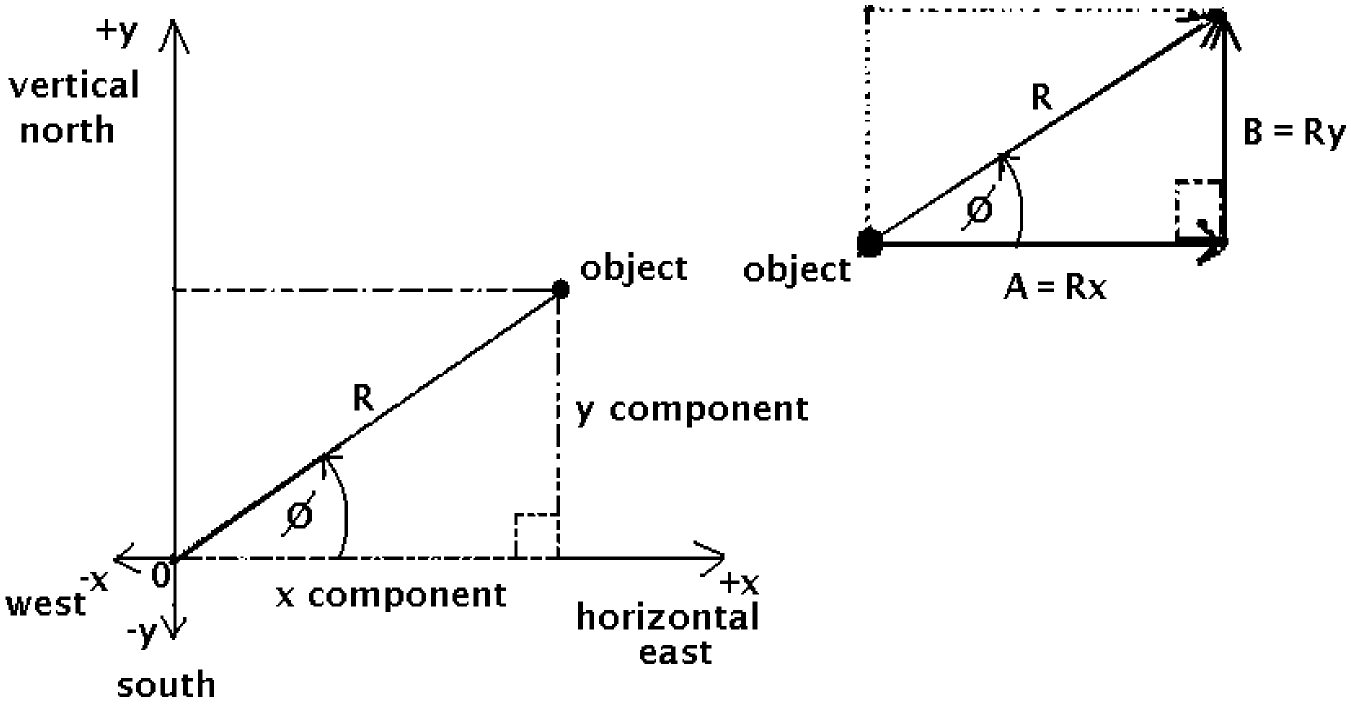
: the object will resultantly or effectively have a force of 2 applied to the object into the leftward direction.

Vectors may be noted (notation, indicated, expressed) as the variable identifier with an arrow "directional" line over it and which does not necessarily indicate the direction of that vector:

Ex.: $\overrightarrow{F1} + \overrightarrow{F2} = \overrightarrow{Fr}$

The following drawing shows two common and fairly similar methods to visualize several component (or "dimensional") vectors and their corresponding effective or resultant vector. The one on the right is generally the methods used to

visualize force vectors acting upon an object, and the resultant (R) force and direction is shown. The length of the vectors indicated by lines and directional arrows is proportional to the magnitude (size, value, amount) of that vector. This method of showing the resultant is often called the "head to tail" method, where one vector starts at the head or arrow of the previous one. The method on the left is very similar and is typically called the rectangular method and it produces the same resultant. Another vector summing method is called the "parallelogram" or "tail to tail" method which starts two vectors at the same position and considers these 2 sides of a parallelogram, and the complete 4 sided parallelogram which often looks like a slanted rectangle, is then drawn. The diagonal in the parallelogram from the starting point is the **resultant vector** that represents the resulting or effective magnitude and direction of something. [FIG 277]



R=real or resultant vector

If A and B are vectors, the resultant (R) vector is the sum of those two vectors. In vector notation form this is expressed as:

$R = A + B$: a special vector expression or notation form for the sum of two vectors. It is incorrect to sum their magnitudes for the resultant magnitude of the resultant "summed", combined, output vector or result.

From the study of trigonometry:

$\sin \text{¢} = \frac{y}{R}$ and $\cos \text{¢} = \frac{x}{R}$

$y = R \sin \text{¢}$ $x = R \cos \text{¢}$

If x and y are two component or vector forces acting upon the object:

$Fy = R \sin ¢$ and $Fx = R \cos ¢$

In a reverse type of manner, if you were given the two component vectors, you can derive the resultant vector:

$\tan ¢ = \frac{\text{opposite}}{\text{adjacent}} = \frac{y}{x}$ and $¢ = \arctan(\tan ¢)$

$R = \frac{y}{\sin ¢} = \frac{x}{\cos ¢}$

$R = \sqrt{x^2 + y^2}$: this has the form of the Pythagorean Theorem

Just as a point can be identified by two values: an x coordinate and a y coordinate, P(x,y), a vector or vector line on a graph is also identified by two values: a magnitude and direction. V(magnitude, direction), where direction could be an indicated angle, often with respect to the horizontal axis. When the direction of the vector is not explicitly given, it can be found from its horizontal (x) component and vertical (y) component vector as indicated above with the trigonometric, triangle or Pythagorean analysis.

Since each vector has a horizontal (x) component, the resultant vector's horizontal component will simply be the sum of those two horizontal components. Since each vector has a vertical (y) component, the resultant vector's vertical component will simply be the sum of those two vertical components. Expressing the notation of the combining or summing of vectors so as to find the resultant vector, and optionally including notation for when there are more vectors to sum or combine so as to find the resultant vector:

R = effective or resultant vector

Rx = effective horizontal component or vector of the resultant vector
Ry = effective vertical component or vector of the resultant vector

A and B are vectors

Ax = effective horizontal component or vector of vector A
Ay = effective vertical component or vector of vector A

Bx = effective horizontal component or vector of vector B
By = effective vertical component or vector of vector AB

$R = (\bar{A} + \bar{B}) = (Rx, Ry) = (Ax + Bx + \ldots, \; Ay + By + \ldots)$

In the above drawing, since Ay=0 and Bx=0 we then have:

Rx = Ax and Ry = By, and that A and B vectors are as legs or sides of a right triangle, and that the Pythagorean Theorem can be used to solve for the value of the resultant vector.

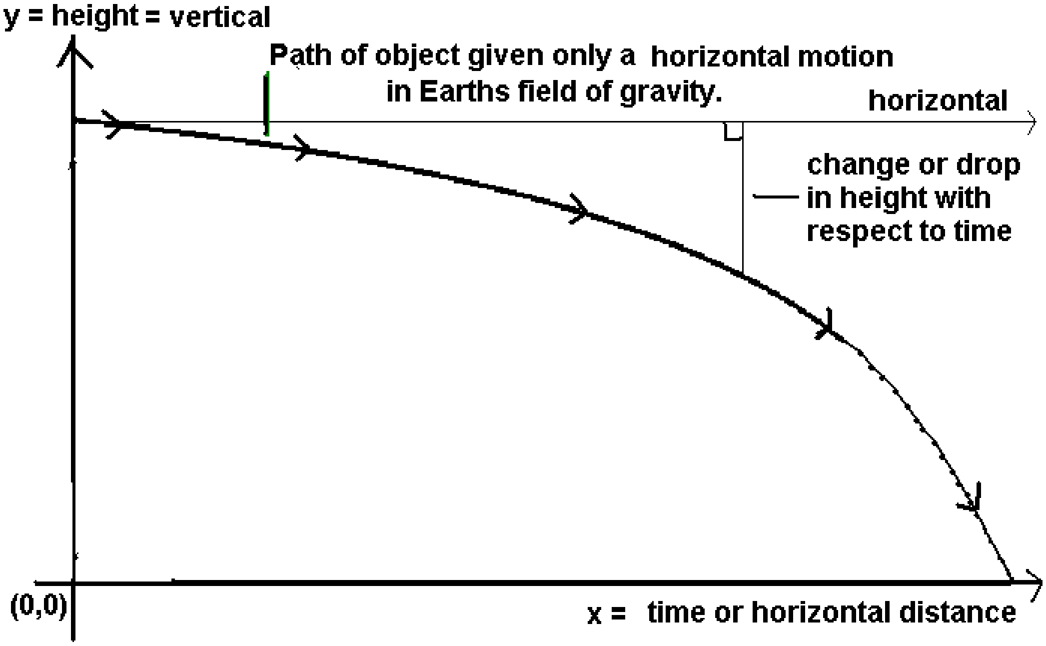
$R = \sqrt{(Rx)^2 + (Ry)^2} = \sqrt{(Ax)^2 + (By)^2}$

With this previous equation, we see that the (right-triangle) legs or component vectors of the resultant vector are the (vector) sum or resultant of all the horizontal component vectors, and then all the vertical component vectors of each individual vector.

A 3 dimensional or spherical reference and measurement system may use coordinates that are something like this:

point = P(radius distance to the point from the origin (0,0,0), horizontal or rotational angle on the x horizontal plane
from the x axis reference line, vertical angle above the horizontal plane from the origin vertex).
Ex. point(r=distance= 5m , 30° h , 45° v)
A linear or non-angular version for a 3 dimensional system was shown previously as: P(x, y, z)

Ex. A ball that is thrown across a field will have both a horizontal component of motion or travel due to the input energy applied to it, and a vertical component of motion or travel due to the force of gravity constantly acting upon it. Its path, locus or curve will be like that of a upside-down or inverted parabola due to that as time increases, its vertical velocity, here downward, is increasing with respect to time, hence it rapidly approaches the surface with respect to time. This increase in vertical velocity is due to that its vertical velocity (v = d / t = at) and acceleration (v / t) is related to a squared variable, here (s): a = g = 9.81 meters / s^2 = 9.81 m/s^2. In short, as time increases, its vertical velocity rapidly increases exponentially with respect to it, that is, the vertical velocity and time are not linear or proportional as would be for a linear relationship and-or equation. The horizontal velocity will remain the same, but it will get less or decrease at a very slow rate with respect to time due to the ball colliding with air and creating a drag force in the opposite direction to its travel where some of the kinetic energy of the ball will be transferred and-or lost to many air particles of which each then gains a small amount of that kinetic energy transferred to it. [FIG 277A]



A COMPUTER PROGRAM TO CALCULATE SOME STATISTICS OF DATA

This computer program will calculate some common statistics of numeric data stored as ASCII text strings in a text file (has the .txt filename extension, and contains readable ASCII text characters. If possible you may even create a program so as to enter numeric values and store, edit and-or append them in a text file. The text file can also be easily edited by nearly anyone, just like any text file. The data can always be edited and-or quickly reused without having to find, calculate, and-or reenter the many values again. In the following computer program, the statistics calculated for the floating point numeric data (ie., signed decimal values with or without a fractional portion) are:

1: Number of data values, elements, instances or samples
2: Average of the data values
3: Average Deviation of the data values
4. Standard Deviation of the data values

```
/*-------------------------------------------------------------------------------------------------------------------------------
DataStatistics.c

(c) JPA,  Aug. 2021

           This will read a text file for text segments that are strings of ASCII numeric values.  The program
            will calculate the average of those data values as:  (sum of values) / (number of values).

           The format of the input data file was kept as simple as could be so as to have a simple program,
           however you can always improve upon it, such as for allowing comments bounded by a delimiter
           character such as *, etc.

           The data is decimal values, separated by a space(s) or new line. The program is written so allow
           up to 10000 data values.

--------------------------------------------------------------------------------------------------------------------------------*/
#include "stdio.h"
#include "stdlib.h"
#include "math.h"     /* : for sqrt() */

/*-------------------------------------------------------------------------------------------------------------------------------*/
void main(int argc,  char * argv[] )
{
unsigned char c;
int n=0;
int t=0;
int e=0;
FILE *file1=NULL;
unsigned char filename1[1000]={ "\0" };      /* the file to either code or decode, the input file   */
double datavalues[10000];
double averagevalue=0.0;
double deviation=0.0;
double deviationsum=0.0;
double averagedeviation=0.0;
double standarddeviation=0.0;
double temp=0.0;
float  x=0.0;

system("cls");
```

```
fflush(stdin);

if(argc==2){ strcpy(filename1,argv[1]);      /*  arg[1] is the second real argument, and is the "input"
                                                   dragged and dropped (with the mouse,) file. */

                printf("\n\nINPUT FILE:  %s",filename1);   /* fflush(stdin);  getch();    test ok    ex.  mydatafile.txt   */
           };

printf("\n");

if(argc == 1){         /* : if no file entered on the command-line argument, and only the program name is an argument  */
             printf("\nPlease enter the text filename with the data values expressed as an ASCII text strings on each line: ");
             fflush(stdin);
            gets(filename1);
            };

/* This will  check if the file even exists */
file1 = fopen(filename1, "rt");      /*  : open the file for access, here reading in text mode */
                                      /* text mode is similar to a viewable screen output mode, where numbers and text are
                                         read and-or written to as readable strings of text characters, and not generally as
                                         binary numeric values such as for numeric data.    Also consider for example that
                                         in text mode, a n indicated (as a string of ASCII text characters) value of 100 will take
                                         three bytes: "1 =49,  "0"=48, and "0"= 48 , however in byte mode it will require only 1
                                         byte and it will be: 100 expressed in binary numeric form as 8 binary bits:  01100100 */
        if(file1==NULL){
              printf("\n\aREAD ERROR --- OR FILE DOES NOT EXIST.");
              printf("\n\nPress A Key To Exit This Program >");  fflush(stdin); getch();  printf("\n"); exit(1);
             };

            for(;;){       /* : a loop to get each number (here, actually a string of ASCII text characters) or data element
                                that is stored in the file  */

                    e = fscanf(file1,"%f",&x);

                     /* fscanf() is similar to scanf() to get keyboard read text input, but is for reading computer files,
                        reads formatted text data. This is a simpler alternative to using fgets() and then having to
                        use a function to convert the text string to a floating point numeric value.  If values are
                        not stored in text or string character, ASCII form in the file, but binary numeric values, then
                        the file should be opened for accessing in binary (pure byte) mode, instead of text mode.
                        Here is a similar example of how to access coordinate values of points stored in a text file
                        and having the format:  x-value , y-value    , and for example:   5 , - 4.2
                        The coding for this would be:        e = fscanf(file1,"%g ,%g",&x,&y);
                        The comma in the format specifier will cause a scan for a comma in between the text values.     */

                    if(e==(-1)){  break; };      /* : The eof = end of file, is a C language defined constant, and is often -1 */

                    printf("\n%g",x);             /* : opt. to display all the input data on the screen */

                    datavalues[n]=(double)x;     /* : store the read values from the file into the numeric
                                                      floating point values array */

                   n=n+1;      /* : increment the index and-or counter to store the data into the next
                                    array element and-or memory location  */
                 };
```

```
fclose(file1);   /* : the file data is no longer needed for the rest of the program, so here, it is closed, no assigned access */

if(n==0){   printf("\nNo Data Found, Press A Key To Exit The Program");  fflush(stdin); getch(); return(1);  };

printf("\n\nNumber of data values = %d",n);

for(t=0;;){      /* : Loop to find the AVERAGE of the data values. */

                    averagevalue = datavalues[t] + averagevalue;     /* : calculate the sum of the data values */

                     /*Calculate the average of the data values:  */
                     t=t+1;  if(t==n){  averagevalue = averagevalue / (double) n;   break;   };
                };

printf("\nThe average value is:  %g",averagevalue);


for(t=0;;){    /* : Loop to find the AVERAGE DEVIATION from the average. */

                  temp = averagevalue - datavalues[t];     /* : a temporary deviation value */
                  if(temp<0){ temp = temp * (-1);  };          /* : make pos. if it was a neg. value */
                  deviationsum = deviationsum + temp;    /* : calculate the sum of the deviations from the average */
                  t=t+1;   if(t==n){  averagedeviation = deviationsum / ((double)(n-1));   break; };
           };

printf("\nThe average deviation is:  %g",averagedeviation);


for(t=0,deviationsum=0.0;;){      /* : Loop to find theh STANDARD DEVIATION  from the average. */

                   temp = averagevalue - datavalues[t];      /* : a temporary deviation value */
                   temp = temp * temp;         /* : squaring the deviation */
                   deviationsum = deviationsum + temp;
                   t=t+1;  if(t==n){    averagedeviation = deviationsum / (double)(n-1);
                                        standarddeviation=sqrt(averagedeviation);
                                        break; };
          };

printf("\nThe standard deviation is: %g",standarddeviation);

printf("\n\nPress A Key To Exit Program > ");  fflush(stdin);  getch(); printf("\n");

return(0);
};

/*---------------------------------------------------------------------------------------------------------------------------------*/

/* Use example:

   Here, the example or test input (plain, ASCII, with no formatting symbols or commands from a word processor) text file,
   as shown below is called (ie., its filename) NumericData1.txt, which contains the string or ASCII text form of the
   numeric data to use in the program.  In this way, a text file is being used as data storage that a program can use.
   A program can be made so as to input and-or create data files, and the data can be stored as numeric a values or
```

readable (and-or easily editable or creatable by anyone, even without the main program) ASCII text values which must then be converted to numeric values so as to be used by arithmetic in the (data) statistic program above.

If you want to make various notes or comments within the data file, you can delimit it or set it apart by using a special character such as * and then keep checking for this character and-or byte so as to find the start of the note or comment. You can then access each byte of the note or comment and build a text string with it if you need to display it, or to simply skip over it until the next * character is found, and then continue accessing the file data as normal.

```
-3
7.5
8.5
9.5
```

Program Output displayed on the screen:

```
INPUT FILE:  NumericData1.txt

Number of data values = 4
The average value is:  5.625
The average deviation is:  5.75
The standard deviation is: 5.80768       : according to the standardized or accepted formula for standard deviation

Press A Key To Exit Program >
```

```
------------------------------------------------------------------------------------------------------------------------------------------*/
```

A COMPUTER PROGRAM TO CREATE, SAVE AND LOAD RANDOM VALUES

```
/* -------------------------------------------------------------------------------------------------------------------------------------------------------
randomdata.c

A computer program to generate or make, view, save and load (store, write)
random byte, 8-bit  (0 to 255 decimal, a total of 256 values) or single byte values
to a file.  For this program,  each random byte value is only used once in the array
or list of values.

This minimal or basic C-program was created as an example of several useful
programming concepts which may be adopted, modified and used in other
programs.  If interested about computer programming, such as C-programming,
there are many book, articles, and even some courses available elsewhere.
Depending on what kind of program you are making, knowledge of some math
is a good application of, and can help with computer programming. Seek
assistance from others where necessary.  With just the basics of C-programming,
or any other computer programming language, many useful programs can be
created.  Just like any other topic, computer programming is fun, but can become
an endless cycle of learning and experimentation without a useful result, and
therefore, it is good to put what you know into useful computer program(s) or
ideas so as to have something actually useful to you and-or others.

The compiler used for this C source code program was:  TinyC Compiler

(c) JPA 2018
-------------------------------------------------------------------------------------------------------------------------------------------------------*/

#include "stdio.h"        /* for printf(), etc. */
#include "stdlib.h"       /* for exit(), system(), etc. */

/* This array is declared here as "global" or outside of all functions so as to be accessible
to any and all functions without  the need of  sending a "(memory) pointer" (essentially a
computer memory address value assigned to the start of a memory location and/or data)
to a function: */

unsigned char bytevaluesarray[256];


/* function "prototypes" (forward, in advance, declarations) and which help explain the
(any) arguments required, and the function usage, etc, to the user and compiler: */

void makedata(void);
void viewdata(void);
void savedata(void);
void loaddata(void);
void searchdata(void);

/*------------------------------------------------------------------------------------------------------------------------------------------------*/
void main(void)          /* the main() function is the required, formal entry or start of any C program */
{

char c;
```

```
for(;;){   system("cls");

            printf("\nRANDOM VALUES MENU");
            printf("\n");
            printf("\n1 Make New Data");
            printf("\n2 View Data");
            printf("\n3 Save Data");
            printf("\n4 Load Data");
            printf("\n5 Search Data");
            printf("\n0 Exit");

            printf("\n\nEnter Selection: ");
            fflush(stdin);
            scanf("%d",&c);

            if(c==1){ makedata(); };
            if(c==2){ viewdata();   };
            if(c==3){ savedata();  };
            if(c==4){ loaddata();   }
            if(c==5){ searchdata(); };

            if(c==0){ exit(0); };

           /*    if((c<0) || (c>5)){  continue; };  optional, since will continue looping anyway, for other values entered */
        };
};
/*------------------------------------------------------------------------------------------------------------------------------------------------------*/
void savedata(void)
{
FILE * file1;       /* declare a file pointer variable, so as to point to a specific  file and its data so as  to access it */

unsigned char filename1[256]={ "bytes.dat\0" };       /* the filename string or array or characters */
unsigned char c;
int n=0;
int e=0;   /* for file access (during reading or writing) error checking */

file1 = fopen(filename1, "rb");        /* open the file for reading for reading mode, so as to check to see if it already exists */

                        /*------------------------------------------------------------------------------------------------------*/

          if(file1==NULL){                      /* if the file does not yet exist */

                      file1 = fopen(filename1, "wb");        /* open the file for (non-appending)  writing mode, filesize will be
                                                                  cleared to 0 bytes.   File data will essentially be overwritten */

                      if(file1==NULL){         /* if there was  a problem opening or creating the file */
                                        printf("\n\aFILE CREATION, WRITE, OR ACCESS  ERROR 1.");
                                        printf("\n\nPress A Key To Continue");   getch(); return;
                                     };

                     rewind(file1);        /* set the file pointer to the start of the file, where it should be already */
                     goto WRITEDATA;
                };
                          /*---------------------------------------------------------------------------------------------------*/
```

```
                if(file1 != NULL){                          /* if the file already exists */
                          fclose(file1);          /* close the file if it exists, so it could be opened in write access mode */


                          printf("\n\n\aThe byte.dat file already exists, do you want to overwrite it? 1=Yes, 0=No : ");
                          fflush(stdin); scanf("%c",&c);
                          c=c-48;  /* convert asciii code to a strict numeric value */
                          if((c<0) || (c>1)){   return; };
                          if(c==0){   return;  };

                          file1=fopen(filename1,"wb");          /* when the choice was c=1 */
                          rewind(file1);

                          if(file1==NULL){                                /* if there was  a problem opening or creating the file */
                                      printf("\n\aFILE CREATION, WRITE, OR ACCESS  ERROR 2.");
                                      printf("\n\nPress A Key To Continue");   getch(); return;
                                   };

                          rewind(file1);
                         };
            /*-----------------------------------------------------------------------------------------------------------*/
WRITEDATA:                                          /* write the bytes to the file */

            for(n=0;n<=255;n++){
                               c=bytevaluesarray[n];
                               e=(char)fwrite(&c,1,1,file1);   /* check the usage note below for fwrite()  */
                               if(e==0){  printf("\n\aWRITE ERROR FOR OUTPUT FILE, 3\n");
                                      printf("\nPress a key"); fflush(stdin); getch();  return;
                                     };
                            };

             fclose(file1);       /* formally, close the file */

printf("\n\nCompleted writing the data to the:  byte.dat  file.  Press a key.");  fflush(stdin); getch();

return;
};
/*----------------------------------------------------------------------------------------------------------------------------------------*/

/* The function prototype, of the fwrite() ANSI standard C-function is:

size_t fwrite(const void * ptr, size_t size, size_t nobj, FILE * stream);

fwrite() returns the number of objects or (array) elements written, or 0 on error. * ptr is the pointer (address) of the data
to be copied and written to the file.  size is the bytes of each element, nobj is the number of objects or elements to write.
size_t is a data type (essentially a byte size) that is similar to the integer data type of 2 bytes.  * stream is a pointer to the
destination location where the data is to be written or sent to.  fread() is the corresponding similar function to fwrite()  */
/*----------------------------------------------------------------------------------------------------------------------------------------*/
```

```
/*------------------------------------------------------------------------------------------------------------------------------------------*/
void makedata(void)
{
int n=0;
int m=0;
int z=0;
unsigned int r=0;                    /* the returned random value from the random number generator */
int t=0;
unsigned char c=0;                /* will hold the random byte */
unsigned char found=0;        /* used as a "toggle" value, to determine if the byte value is already in the array */
unsigned char zerofound=0;  /* a toggle, used since all the bytes are set to 0 initially */

unsigned long M=0;             /* total number of random values generated, used or picked, even if it was already
genereated  previously */
unsigned long SUM=0;        /* for the sum of all the random values picked */

                                  /*-------------------------------------------------------------------------------------------------*/

srand(time(NULL));     /* seed, initialize or set  the computers random number generator with a starting value so as further
                                random numbers can be chosen.  The seed value can be any value in general, and not
                                necessarily the computers current time value in seconds as returned by time(NULL) which is
                                pseudo-random or near random in value enough to be used since it changes every second.
                                You can also use:  time(0) for the argument to srand(), instead of time(NULL).
                                The "random number" generator will be seeded (ie., initialized with a starting value) using srand()
                                = "seed random" pseudo-random numbers using the current time as a pseudo-random value.
                                The values returned by rand() will be based on this seed value and a built in algorithm to choose
                                the next pseudo-random number.  srand() returns a two byte  integer value (0 to 65535).*/

system("cls");             /* the system() function can send commands to the computer system.  Here the command is: cls
                                which is the PC command to clear the standard output display screen.  */

/* Lets optionally clear (set all the byte values to 0) the array just for some extra safety. */

for(n=0;n<=255;n++){
                                bytevaluesarray[n]=0;
                           };
                        /* ------------------ Fill the char or byte array with random byte values -----------------------*/

/* Lets fill each byte of the array with a random value, and where each byte is only  used once.  To accomplish this,
the array must be checked if each new random value was already used or not.  If it was already entered, the loop
will just continue without taking any further action, and it was not entered, it will be placed at the current position in
the loop, and the loop will then continue.  */

n=0;
for(;;){
         r=rand();                         /* get the next available generate random number, rand() returns an
                                              integer number data type, typically 2 bytes */

          c=(unsigned char)r;    /*  we only need a 1 byte value to place into each element or position of the array */
          if(c==0){
                       if(zerofound==1){    continue;  };
                       if(zerofound==0){  zerofound=1;   };
                    };
```

```
          M=M+1;   SUM=SUM+c;

      /* Let's check if this random value was already placed in the array, if so, then just continue, if not, then place it in: */

        found=0; z=0;
        for(;;){      /* for loop 2 */
                    if((bytevaluesarray[z])==c){                              /* if that byte is already in the array */

                                                            found=1;    if(c==0){ n=n+1; };
                                                            break;
                                                         };

                    /* if(z==n){  break; };  */     /* optional, if actually at the next "empty" byte position, no need to go further */
                     z=z+1;  if(z>=256){  break; };         /* finished checking through the entire array */

              }; /* end of for loop 2 */

        if(found==0){  (bytevaluesarray[n])=c;  n=n+1;  if(n>=256){ break; };  };             /* the byte was  not found in the
                                                                                                array, so  place it in the array at the
                                                                                                current position */
      if(n>=256){ break; };
    };  /* end of for loop 1 */

printf("\n\nTo fill the 256 byte array with unique values required %d random values",M);
printf("\nto be generated.  The average value of those %d values is %d.",M,SUM/M);

printf("\n\nCompleted making the random values data, press a key."); fflush(stdin); getch();

return;
}
/*------------------------------------------------------------------------------------------------------------------------------------------------------------*/
void loaddata(void)
{
FILE * file1;        /* declare a file pointer variable, so as to point to a specific  file and its data so as to access it */

unsigned char filename1[256]={ "bytes.dat\0" };        /* the filename string or array or characters */
unsigned char c;
int n=0;
int e=0;   /* for file access (during reading or writing) error checking */

file1 = fopen(filename1, "rb");        /* open the file for reading for reading mode, so as to check to see if it already exists */
                    /*--------------------------------------------------------------------------------------------------------------------*/

        if(file1==NULL){                      /* If the file is not there yet, or is a problem accessing it  */
                              printf("\n\aFILE READ OR ACCESS ERROR.");
                              printf("\nPress a key to continue. "); fflush(stdin); getch(); return;         };
                                     /*-----------------------------------------------------------------------------------------------*/

            /* read each byte of the file and load or place it into the bytevaluesarray[]  */
          for(n=0;n<=255;n++){
                                        e=(char)fread(&c,1,1,file1);       /* put next byte read from the file into variable c */
                                        if(e==0){  printf("\n\aREAD ERROR FROM INPUT FILE\n");  exit(1);  };
                                        bytevaluesarray[n]=c;
                               };
```

```
                fclose(file1);      /* formally, close the file */

printf("\n\nCompleted reading the data from the:  byte.dat  file.  Press a key.");  fflush(stdin); getch();

return;
};
/*---------------------------------------------------------------------------------------------------------------------------------------------------------*/
void viewdata(void)
{
int n=0;
int t=0;
unsigned long M=256L;
unsigned long SUM=0L;
                                        /*--------------  Display each random byte value in the array. -------------------*/
system("cls");
                                        /* Here,the first value is at offset 0 in the array which is no offset into it, and
                                           is indicated as byte number 0, but it is actually the first (1) byte in the
                                           array.  You may modify the code as to how you want to display these. */

printf("\nHere are the array offsets of the array elements, and the corresponding values");
printf("\nthey are now set to:");
printf("\n\n");

for(n=0,t=1;n<=255;n++,t++){
                                    printf("%3d %3d    ",n,bytevaluesarray[n]);
                                    SUM=SUM+bytevaluesarray[n];
                                    if(t==5){  printf("\n");  t=0;  };
                               };

printf("\n\nThe average value of all the 256 data values shown above is: %lu",SUM/M);

printf("\n\nPress a key."); fflush(stdin); getch();

return;
};
/*-------------------------------------------------------------------------------------------------------------------------------------------------------:*/
void searchdata(void)
{
int n=0;
unsigned char c=0;
int m=0;
int b=0;

for(;;){
            system("cls");
            printf("\n\n1 Find the logical (0 to 255) position, offset or address of a byte value.");
            printf("\n2 Find the byte value at a logical (0 to 255) byte number, position,");
            printf("\n  offset or address.");
            printf("\n0 Finished searching.");
            printf("\n\nInput Selection Number: ");
            fflush(stdin); scanf("%c",&c);    c=c-48;    if((c<0) || (c>2)){  printf("\a");  continue;  };

            if(c==0){   return;  };
```

```
        if(c==1){          /* ----------- Find logical (0 to 255) position or address of a byte value. ----------- */

                     printf("\n\nInput the byte value (0 to 255) to find: ");
                     fflush(stdin);  scanf("%d",&b);

                     for(n=0;n<=255;n++){
                                         m=bytevaluesarray[n];
                                         if(m==b){
                                                  printf("\nByte value %d was found at offset %d",b,n);
                                                  printf("\n\nPress a key");  fflush(stdin); getch();
                                                  break;
                                                };
                                       };

               };

        if(c==2){          /* ----------  Find byte value at the logical (0 to 255) byte number address. ---------- */

                      printf("\n\nInput the offset (0 to 255) of the byte: ");
                      fflush(stdin);  scanf("%d",&m);

                      b=bytevaluesarray[m];

                      printf("\nThe byte value of %d is at address or offset %d",b,m);
                      printf("\n\nPress a key");  fflush(stdin); getch();
               };

    };

return;
};
/*-----------------------------------------------------------------------------------------------------------------------------------------------------*/
```

Here is an example of the screen output of the "view data" menu option in the above computer program, it shows the logical (ie., offset from 0) byte number, and the corresponding value (between 0 to 255) stored there:

```
  0  88    1  94    2  72    3 138    4 249
  5 124    6  49    7  71    8 147    9 117
 10  43   11 228   12  92   13 123   14  55
 15 173   16 126   17 223   18 235   19  44
 20 107   21  28   22  76   23 113   24  23
 25 148   26 190   27 142   28 165   29 122
 30 156   31 133   32  78   33  21   34  81
 35 168   36  26   37 231   38 193   39  91
 40  70   41 204   42  82   43 252   44 209
 45 197   46 186   47 180   48   3   49 215
 50 224   51  53   52  25   53  74   54 205
 55   9   56 109   57 174   58 206   59  59
 60  61   61  33   62  24   63  93   64   4
 65 221   66  85   67 145   68 226   69  15
 70 153   71 118   72  96   73  77   74 242
 75 159   76 229   77 167   78 169   79 149
 80  11   81 106   82 185   83  66   84 112
 85 143   86 101   87 202   88  29   89 239
 90  63   91 199   92 203   93 102   94 189
```

```
 95 103    96  41     97 125    98  58     99 179
100  95    101 208    102 111    103 233    104 104
105 212    106  67    107   7    108 181    109 244
110 255    111 116    112 157    113 161    114  89
115 176    116   5    117  42    118 184    119 201
120 135    121 120    122  68   123 139     124 195
125 130    126 151    127 121    128  90    129 177
130  40    131  62    132 152    133  37    134   8
135 232    136 100    137  46    138  34    139  65
140  80    141 160    142 140    143  14    144 146
145  18    146 219    147  83    148  10    149 207
150  45    151 119    152  20    153 200    154 234
155 245    156  86    157 251    158 238    159 216
160  13    161  12    162 247    163  97    164  35
165   6    166 183    167 211    168 210    169 128
170 192    171  36    172 178    173 129    174 241
175 214    176  57    177  87    178  22    179 137
180  69    181 191    182 227    183 254    184 222
185 110    186  17    187  75    188  48    189  30
190  64    191   0    192 163    193 162    194  73
195   2    196 237    197  19    198 108    199  60
200 136    201 155    202   1    203 144    204 171
205  56    206  16    207 187    208  32    209 114
210 158    211 134    212 182    213 248    214 218
215 166    216 175    217 253    218  98    219 196
220 141    221  99    222 127    223 164    224 246
225 150    226  51    227 225    228 230    229  38
230 213    231 154    232 236    233 198    234 132
235  47    236 105    237  52    238 115    239 188
240 240    241 131    242  50    243  84    244 194
245  27    246  31    247 220    248 217    249 170
250  39    251 243    252 172    253 250    254  79
255  54
```

The average value of all the 256 data values shown above is: 127 , and this is approimately 256 / 2 =~ half of 256

The above data is what is printed or displayed on the viewing screen of the computer used. The binary data was converted to ASCII text characters to be displayed on the screen, but in the computer file or memory, the data is stored as binary values. For example, a (text) screen display of the number 1 is the ASCII text character coded as a binary value of 49d = 0011001b, however the (numeric) binary value for 1 is 00000001b and this is what is saved on disk or memory if it was stored as a binary value and not ASCII or "text values". A file can be opened (ie., for read and write access) in binary mode or text mode. When a data file contains readable characters it may be best to access the file in text mode, but this mode does some extra things such as converting a carriage return (ASCII 13) character to two bytes of data: a carriage return (ASCII 13) and line-feed (ASCII 10) characters, hence it automatically adds a byte to the stream of data. If this is not desired, then open the file in (pure, strict) binary access mode so as to access only the (exact) saved bytes of the file. fprintf() is the usually function for "text printing" to a file. For example a text string (ie. with the %s format specifier) can be printed to a file with fprintf(). If you want to first "build" a string, such as containing multiple variables, etc, one function you can use for this is the **sprintf()** function which will catenate (ie., string catenation, appending or joining) text strings together rather than use the strcat() function and other numeric to text conversion functions, such as itoa() to build a string. Once a string is built, it can then be displayed and-or saved to an opened text file, etc. This function essentially prints or sends the resulting output to a string instead of the screen as printf() would.

Ex. sprintf(mystring,"The Filename is: %s, and the number is %d",string1,n); printf("%s",mystring);

A COMPUTER PROGRAM FOR A COUNT-DOWN AND COUNT-UP TIMER

```
/* -----------------------------------------------------------------------------------------------------------------------
       timer.c

        Some measurements require the time or duration of an event or process.
      . Below is a program that has both a "count-down" and "count-up" timing
        options.  Several instances of these programs can usually be ran
        simultaneously on the same computer.  The precision of this timer is 1s,
        and accuracy is +,- 1s and it is good for many practical applications such
        as for cooking recipes, procedures and processes which require timing.

(c) JPA Feb 2018
------------------------------------------------------------------------------------------------------------------------*/

#include "stdio.h"
#include "conio.h"    /* : for kbhit() which checks if any key is currently being pressed while this statement is processed.  */

unsigned long  secondstimer(void);
/* This is a function for a "count-up" timer. */

int  countdowntimer(unsigned long seconds);
/* This is a function for a "count-down timer, returns 1 when finished, 0 when not. */

/*----------------------------------------------------------------------------------------------------------------------*/
void main(void)
{
unsigned long  n;
int a=0;
unsigned long s=0;

unsigned long hours=0;
unsigned long minutes=0;
unsigned long secs=0;
char c;

int beeps=0;
unsigned long b=0;

                         /* -------------sample program to demonstrate the secondstimer()  -----------  */
start: ;

printf("\nTimer Program.");

printf("\n\nThe current computer system time is:  "); system("time < nul");    /* optional, a method to easily display the
                                                                                    computers system's current  time setting */
printf("\n");                                                                  /* the nul input prevents any needed user input  */
printf("\Multiple instances of this program can be ran.");

for(;;){
  printf("\n\n1 Seconds Timer");
  printf("\n2 Countdown Seconds Timer");
  printf("\n0 Exit Program");
  printf("\n");
```

```
   printf("\nEnter Selection: \n:");
   fflush(stdin);  c=getch();  printf("%c ",c);
            if(c=='1'){ goto secondstimer;  };
            if(c=='2'){ goto countdowntimer;   };
            if(c=='0'){  exit(0); };
 };
                                                  /*------------------------------------------------------------------------------------------*/
secondstimer: ;

fflush(stdin);  printf("\n\nPress a key when you want to start the timing. "); getch();
printf("\nPress The ESC  Key When Finished Timing: ");
n=secondstimer();

printf("\nThe number of seconds is: %ld",n);

hours=n/3600;                              /* optional, display the number of hours,minutes, and seconds that have passed */
minutes=(n-(hours*3600))/60;       /* converts the number of seconds to its equivalent:  H:M:S    */
secs=n%60;
printf("\n   %ldh:%ldm:%lds", hours, minutes, secs);
printf("\nPress A Key"); getch();  goto start;

                                        /*------------------------------------------------------------------------------------------------*/
countdowntimer: ;                          /* sample program call, start countdowntimer()  */

printf("\n\nEnter the number of seconds to count down from to 0, for example, for 5 minutes");
printf("\nenter 300 since 60s/m x 5m = 300s > ");

fflush(stdin);  scanf("\n%ld",&s);  printf("Press a key when ready to begin: "); fflush(stdin); getch();

if(a==0){    printf("\nUser Terminated The Countdown Timer, Press A Key. "); getch(); goto start;   };
if(a==1){               /* beep, ring alarm bell , till user presses a key */

                fflush(stdin);

                for(;;){
                            if(beeps==0 ){
                                printf("\a");       /* bell, ascii, beep sound */
                                printf("\n*********************** COMPLETED *************************\n");
                                printf("\nPress The ESC Key");
                                beeps=1;
                                };

                           b=b+1;
                           if(b==20000L){  beeps=0;  b=0L;   };

                          if(kbhit()){  c=getch(); if(c==27){   goto start;    };    };   /* esc key was pressed */
                     };     /* kbhit() is non-ANSI C, however it common enough to be  considered as in the ANSI  C standard. */
            };               /* Many compilers include kbhit() due to its ability, however it is possible to create a similar function. */

return;
}
/*-----------------------------------------------------------------------------------------------------------------------------------------*/
```

```
unsigned long  secondstimer(void)
{
unsigned long start=0;    unsigned long end=0;   char c=0;    unsigned long temp=0;   unsigned long seconds=0;
unsigned long hours=0;   unsigned long minutes=0;    unsigned long secs=0;

start=time(0);     /* A prewritten, standardizd C-language function to get a computer systems current seconds count. */
temp=start;

fflush(stdin);

for(;;){

        if(kbhit()){    c=getch();  if(c==27){ break; /* esc key */};  };

              /* optional, display the seconds that have passed */
          if( temp != time(0)){  seconds=seconds+1;  temp=time(0);
              printf("\n%ld",seconds);

                /* optional display if desired :   hrs:min:sec   */
                  hours=seconds/3600;   minutes=(seconds-(hours*3600))/60;   secs=seconds%60;
                  printf("   %ldh:%ldm:%lds",hours,minutes,secs);
            };  /* end of if  */

      };  /* end of loop */

 end=time(0);              /* get the computer systems current seconds count */
 return (end-start);      /* could also use:   return seconds;  */
}
/*-------------------------------------------------------------------------------------------------------------------------------*/
int  countdowntimer(unsigned long seconds)
{
 int result=0;    /* return result=1 when finished counting down (or up to the seconds argument */
int c;
unsigned long start=0;;
unsigned long temp=0;

printf("\nPress the ESC key to terminate early or when finished using the seconds timer. ");
printf("\n%ld",seconds);

start=time(0);    /* get the computer system's current seconds count */
temp=start;

          for(;;){
                       if(kbhit()){   c=getch();   if(c==27){   return 0;  };   };    /*  so user can press ESC key to exit early */

                       if( temp != time(0)){   seconds=seconds-1;
                                                temp=time(0);
                                                printf("\b\b\b\b\b\b\b\b\b\b\b\b\b%ld        ",seconds);    /* /b = backspace esc. char */
                                                if(seconds==0){  return 1; };
                                                };
                  };
};
/*-------------------------------------------------------------------------------------------------------------------------------*/
```

A COMPUTER PROGRAM FOR A STOPWATCH TIMER

This is somewhat similar to a count-up seconds timer as shown in the last program: timer.c, but with this program shown below, it allows the user to pause and continue the seconds timer.

```
/*------------------------------------------------------------------------------------------------
stopwatch.c

   A timer with a user pause ability that is started at the press of a key,
   and ended at the press of a key, and then displays the elapsed time.

   (c) JPA September, 2021


  Extra:   A pseudo-type of code and-or steps so as to time a process
           such as a function, etc. is:

           1.  starttime=time(0);

           2.  Some process to be completed.

           3.  endtime=time(0);

           4.  total_seconds = endtime - starttime;

           5.  Display the number of seconds.

------------------------------------------------------------------------------------------------*/

#include "stdio.h"
#include "stdlib.h"

/*----------------------------------------------------------------------------------------------*/
void main(void)
{
unsigned long starttime=0L;
unsigned long endtime=0L;
unsigned long seconds=0L;
int n=0;
int c=0;
int pausemode=0;
unsigned long pausetimestart=0L;
unsigned long pausetimeend=0L;
unsigned long pausedseconds=0L;

/*--------------------------------------------------------------------------*/

system("cls");

printf("\nSECONDS STOPWATCH PROGRAM by JA, (c) 2021, From The Math eBook: Mathization.");

printf("\n\nThe seconds count displayed are approximate,");
printf("\nand within about 1 second of error.");
printf("\n\nPress A Key To Begin The Seconds Counter: ");fflush(stdin);  getch();
```

```
start: ;

/* Let's reinitialize the variables here such as for when the program is restarted in the program by the user: */

starttime=0L; endtime=0L;  seconds=0L;  n=0;  c=0;  pausemode=0;  pausedseconds=0L;

starttime=time(0);   /* : get the current value of elapsed seconds
                          from the computers internal clock  */

system("cls");

printf("\nSECONDS STOPWATCH PROGRAM by JA, (c) 2021, From The Math eBook: Mathization.");

printf("\n\nPress A Key To End The Seconds Counter, Or");
printf("\nPress Space To Pause or Continue The Counter: ");
/*------------------------------- main timer loop ----------------------------------------------------*/

for(;;){

        endtime=time(0);    /* : essentially, the current computer time in seconds */

        seconds = endtime - starttime;
        seconds = seconds - pausedseconds;

        if(pausemode==0){  printf("%c%d       ",13,seconds);  }; /* ASCII 13 = CR = Carriage Return to the start of the line*/

        if(kbhit() == 1){
                      c=getch();
                      if(c==32){  if(pausemode==0){ pausemode=1;  /* The space key for pause or continue was pressed */
                                                    pausetimestart=time(0);

                                                  }else{  pausemode=0;
                                                          pausetimeend = time(0);
                                                          pausedseconds =
                                                          pausedseconds + (pausetimeend - pausetimestart);
                                                                                  };

                               /* printf("\n                    %d  ",pausemode);  for testing, shows 0 or 1 toggled */

                               continue;

                               };

                      break;   /* : when any key except the space key was pressed */

                      };

       };   /* : end of main timer loop */

/*---------------------------------- display the results ----------------------------------------------*/

printf("\a");
```

```
/* printf("\nTotal Seconds = %ld",endtime - starttime);  */  /* optional */

printf("\n\nPress A key To restart the Program, or ESC to end the program: ");

fflush(stdin); c=getch();

if(c==27){  return;  } else{ goto start; };


};
/*-----------------------------------------------------------------------------------------------*/
```

A COMPUTER PROGRAM TO COUNT THE NUMBER OF KEY PRESSES

```
/*--------------------------------------------------------------------------------------------------------------------------
ButtonPressCounter.c

    This program will keep a count of how many times the UP Arrow button was pressed.
    This is useful for some people at certain times where many things need to be counted
    and they may not have any other device to do so.

    (c) JPA.  Nov 21, 2021

-------------------------------------------------------------------------------------------------------------------------*/

#include "stdio.h"

/*-------------------------------------------------------------------------------------------------------------------------*/

void main(void)
{
int n=0;                        /* : a counter, increment  */
unsigned char c=0;         /* : assigned to each new character entered */

long int delay=0;
long int d=0;

printf("\n\n");

printf("Button Counter Program");
printf("\n\n");
printf("Press the UP-Arrow key to increment the counter value by 1.");
printf("\nPress the DOWN-Arrow key to decrement the counter value by 1.");
printf("\nPress ESC to quit the program.");

printf("\n\nEnter a time delay value (1 to 100) between key presses: ");
fflush(stdin);
scanf("\n%d",&delay);
if( (delay>100) || (delay<1) ){  printf("\a");  exit(0);  };


printf("\n\nBEGIN");
printf("\n");
printf("\n");

                    /* --------------  main loop to sense and count the key presses ------------------------- */

    for(;;){
                fflush(stdin);
                c=(unsigned char) getch();
               /* To see each charcter value for testing then use:   printf("\n%d ",c);*/

               if(c==224){    c=(unsigned char)getch();     /* pressed a special key, such as the arrow keys */

                               if(c==72){  n=n+1; };      /* UP-ARROW key     */
                               if(c==80){  n=n-1;  };      /* DOWN- arrow key  */
```

```
                         if(c==75){  ;   };              /* left arrow ke y         */
                         if(c==77){  ;   };              /* right arrow key        */

                        printf("%c%d        ",13,n);     /* 13 = Carriage Return */     /* 8 = Backspace  */
                                                        /* : display the current count of key presses */
                        /* delay loop */
                        d = delay *  6000000;
                        for(;;){
                                 d=d-1;
                                 if(d==0){   break;  }
                                };

                        continue;
                     };

           if(c==27){   printf("\a");  break;  };    /* user pressed the ESC key */

      };   /*  ----------------------  end of main for loop --------------------------------*/

printf("\n\nPress A Key To Exit The Program:  ");  fflush(stdin);  getch();
return;
};
/*------------------------------------------------------------------------------------------------------------------------------------------------*/
```

A COMPUTER PROGRAM TO ENTER AND DISPLAY A TEXT STRING

Below is a simple computer program (for a PC computer that runs .exe (executable, "runable", working, operational, ready, already compiled to the machine or internal processor code of the computer) data files which are computer programs for it. You can study it, and-or improve upon it, perhaps even writing your own string handling functions. There are computer programs in this book which show how to enter (ie., input, type in) and mathematically work with and-or display numeric values.

A "text string" is also known as a "string of text", repetition or array of text characters, one after the other in succession, and it essentially has the same structure as that of a sentence in a book. On a technical or advanced level of knowledge with computers, when entering and-or displaying numeric values, it is done by visually entering and-or displaying ASCII text characters (ie., a graphic or symbol for each) from the keyboard (the standard input) and screen (the standard output), and of which can then be converted to or from actual binary numeric values. Consider that letter or text character A has a (ASCII) code value of numeric 65. B is 66, and so one. These are small values and can be contained within a single byte (8 binary bits or digits in length) of memory which can hold equivalent decimal system numeric values from 0 to 255. Just the same, there are (ASCII) text characters codes for the text or graphic numbers of 0 through 9. Text character 0 is assigned the (ASCII) code number of numeric 48, and text character 1 is assigned (ASCII) code of numeric 49, and so on. If text character 0 was assigned an (ASCII) code number of numeric 0, this would interfere with the null character of 0 that is often used to signal the end of a string. Also the ASCII codes from numeric 0 to 32 are often used for "control codes" for the display and-or printer system. A sentence is an example of a character or text string. When you enter numeric value from the keyboard, it is entered and-or displayed as a series of individual ASCII text characters, such as keyboard or text characters 1, and then 5 and then 9, and these can be stored as a text string having the ASCII numeric codes of: 49, 53, and 57 respectively, and each can be converted by a premade, standard or built-in computer function so as to be a real binary coded number value that can be used mathematically such as an operand for a mathematical operations in your program.

```
/* -------------------------------------------------------------------------------------------------------------------------------
    StringExample.C

     This example program shows how enter and display a string (ie., several) of
     (ASCII encoded) text characters on the standard output or display (ie., the
     computer screen).  For example, a string could be a persons name, or a sentence
     to display.

    To get a single text character, the "get character" function:  getch() is one of the
    methods that be used.  A single text character is equivalent to a string length of 1.

   (C). J.P.A.  July 2021
---------------------------------------------------------------------------------------------------------------------------------*/

#include "stdio.h"      /* : where the functions used in this program are located in or at */
#include "string.h"
/*------------------------------------------------------------------------------------------------------------------------------*/
void main(void)
{
int string1length = 0;
char c = 0;
unsigned char string1[129];        /* declaring an array to hold many values, here, the values of
                                      displayable, or entered text characters */

                       /* : above, declares an array (ie., multiple) of char (ie., 8 bit, byte) data type,
                          and the computer will allocate and set aside the amount of memory
                          necessary for it, which can be calculated as:
```

```
                    total memory needed =
                       =  (data size in bytes per element)(number of elements)

                    Here, the memory allocated is 129 bytes, with 128 being
                    used to hold a string, and 1 byte for numeric 0 which is
                    a byte value set to numeric 0,and not ASCII text character 0
                    which is assigned a code value of 48.

                    An unsigned numeric data type indicates that numeric values are to be
                    stored there as an unsigned numeric value, hence the sign is to be
                    discarded when storing a value there, and this allows twice as much
                    positive or sign-less values since the first or most significant bit
                    is essentially not set as the numeric sign indicator for the numeric value.

                    Be sure to always allow 1 extra byte or character of memory in this
                    character or  byte array that will hold, store or contain a text string.
                    This extra byte will ensure there is a null or numeric 0 character to
                    be placed at the end of a (here) 128 byte text string.  The scanf() will
                    automatically do this for you when you press the Enter key.  The 0
                    or "null" byte will be placed in the next byte position past  the last
                    text character of the string.  Using scanf(), once the ENTER key has
                    been pressed, it essentially signals to the scanf function that you have
                    input the completed string of text characters or bytes.  Scanf will then
                    "null terminate" that string, and so as to know when the end of the array
                    and-or string has been reached when displaying and-or processing that
                    string.  There could be a system error if the user enters a string that
                    exceeds the total allocated size of the array.

                    When an array is declared, given a identifier or name, it is also possible
                    to also assign values to its  elements.                              */

printf("\nEnter A String: ");
gets(string1);                    /* : A C function to input a string, and allows space characters.
                                     scanf() will terminate a string when the first space is entered.
                                     Ex.  scanf("%s",string1);
                                     scanf() does allow special modifications to its basic format
                                     specifiers, such as %s, so as to then allow certain characters
                                     such as spaces to be input.  */
 ;
printf("\n\nThe string you entered is:  %s", string1);     /* display the string  */

string1length=strlen(string1);      /* : A standard C function to calculate the length of a string up to, but  not including
                                       the null terminator character of 0 in this character count or length */

printf("\nThe length of the entered string is %d characters or bytes", string1length);

printf("\n\nPRESS THE ESC KEY or CTRL Break,TO EXIT THIS PROGRAM: ");


   for(;;){
         c=getch();                 /* : A standard C function to input a single character from the keyboard. */
                                    /*   Using getch() is also a way to pause any program, such as to read the screen .*/
         if(c==27){   exit(0);   };  /* : Exit the program when the user enters ASCII 27, the ESC key.  */
      };
```

```
return;
};
/*-------------------------------------------------------------------------------------------------------------------------------------------------------------*/
```

How are the text characters displayed on the screen? The ASCII character codes of the desired text characters are first placed into memory for the general program to access. When a text character is to be displayed on the screen, such as with the printf(), it is essentially converted to and placed into the video (display) memory area as an array of bits (ex., an 8 x 8 grid, array ["dot matrix"] of bits, hence 8 bytes per character) that will get displayed on the viewing screen, bit by bit, pixel by pixel, line by line, row by row, across the screen, and then displaying the next row of the text characters, and so on at high speed. This entire process is done repeatedly so as to keep the image available on the screen so the viewer has time to see it.

The computer programs contained within this book can also inspire some mathematicians to be able to find some quick solutions with a computer programs(s), and-or for computer programmers to consider more mathematics in their programs. Both mathematicians and computer programmers can even consult and collaborate each other so as to more knowledgeable and productive, and make something useful.

There is often several ways methods to do the same thing in a computer program, and some are more efficient (quicker). For example, here are some other methods to display this text string are as follows:

1. Indexing the string array or "array indexing" so as to have a desired element or piece of data in the entire array. This is also called "array offsetting", where the array offset is the index or element number into that array. The first element starts at the first byte of the array in memory, and therefore, it has no offset, hence 0 offset.

2. Using a **pointer**, where the pointer variable or identifier contains the address of a memory location being accessed.

   A: "Pointer indexing" by incrementing the pointer to the next memory address so as to access the data there.
   B: "Pointer Arithmetic" where the initial pointer's memory address is held the same.

First, a basic example of a pointer that can be in a program:

```
unsigned char c=65;     /*  :declaring and initializing a char (ie., a byte) data type and variable, here c  */
unsigned char * p;       /*  : declaring a pointer, or declaring a variable (here, p) that has a pointer data type, here
                                 the data type is:  char *  ,  and  which can be read as "a char or character pointer" or
                                 "a pointer to a char data type"  */

p = (unsigned char *) &c;       /* :assigning the memory location or address of variable (c) to the pointer p */
                                /* : & retrieves the address of (c), and (unsigned char *) is a (data) "typecast", where
                                   here the data (here &c) is an address and is being typecast or altered so as to a
                                   pointer data type because it will be assigned to a pointer data type.

                                    &  can be thought of as the "address retrieval operator", and
                                    *   can be thought of as the "data access operator" for a memory location, such as
                                        assigned to a pointer.   Here the access is read (a copy, retrieval) or
                                       write (set, assign value or new value)  */

printf("\n%d",c);     /* : 65 will be displayed */
```

Now that pointer (p) has been assigned the address of (c), you can even access (read, write) the data of (c), for ex:

```
*p = 66;       /* This sets the data that the pointer is pointing to, hence the data at the address it is assigned to access.
                  This will not change the address or memory location that (p) points to.  If you  wanted to change the
```

```
                  address assigned to that pointer, you can increment the pointer, and here fore example to the next
                  char or byte since that is the data type or memory size of this pointer :  p = (p+1);                    */

printf("\n%d",c);     /* : 66 will be displayed  */
printf("\n%d,*p);     /* : 66 will be displayed since the pointer (p) is assigned to the computer memory address of variable
                           (c).  It could be said that (p) points to (c).  *p is used to access the data that (p) is pointing to, hence
                           it is accessing the data held by, at or assigned to variable (c)   */
```

Why use pointers if we can already use variables? When the amount of data and-or possible variables and their identifier names become large, it is then difficult to assign, remember and use so may variable names and identifiers, and it is easier to access data by using an assigned corresponding number or memory address, especially if that data is held in successive memory and is repetitive in its data type and-or structure, such as having any data types and or elements.

A pointer with a FILE data type has been shown previously in this book. This type of pointer is used to access (read, and-or write) computer (binary) files that were stored byte by byte on disk drives. For example: FILE * myfile; When a FILE pointer is first associated to a file and-or filename, that pointer is set to access the first byte of that file that is located on the disk and its memory array of (digital, binary, 8 bit) bytes. When a FILE pointer is incremented by 1 in the program, it will then be able to access the next byte of that file. The C-language also has (ANSI) standard, common and-or required functions for accessing files such as fread() and fwrite(), and these are analogous to those of accessing program, data input from the keyboard such as with scanf(), and accessing program, data output to the display screen (computer "monitor"). More technically, a FILE pointer points to a file data structure (a struct data type in the C language) assigned to a file being accessed and that structure holds information about a desired file on data storage, such as its current access or offset (from its start) location within it. This file data structure is usually "transparent" to the programmer since the FILE pointer itself is used by the file functions that (transparently) access that file data structure, and the functions will appear to be accessing the file directly.

The following methods are shown as portions of a larger program of which you can test and use them within, such as the program shown previously to display a text string:

/* **Method 1**: Displaying the string using string or **array indexing** (ie., numerical amount or offsetting from the start) each element of the array: */

```
/* Some initial program preparation before the loop, hence "loop preparation": */
printf("\n");  n=0;

for(;;){                              /* : a  loop to display the input string, (;;) is used to indicate an
                                           "infinite loop"  that will keep running or executing until
                                            the loop is somehow terminated, and then the next
                                            statements in the program will be run.

                                            Can also use any loop preparation and or termination condition in the for-loop
                                            declaration:

                                                          Format:  for(initialization , do while or until , adjustments)
                                                          The statements are separated by a semi-colon.
                                                           ex:  for(n=0;c!=0;n=n+1){                                    */

          c=string1[n];               /* (n) is used as the index or offset from the start of the array
                                           and-or its starting memory location. */

          if(c==0){   break;  };      /* at the null terminator of the string,  break or terminate the
                                           loop (program repetition),  otherwise:  */
          printf("%c",c);             /* %c is the printf() format specifier to display a single
```

```
                                         character on the screen */
        n=n+1;                  /* increment n by 1so as to have the next array element, here a  char value  */
    };

/* Method 2:   Displaying the string using  pointer incrementing it to point to the next memory location or text
               character in that string: */

/* Some initial preparation before the loop:      */
printf("\n");
n=0;
p = string1;      /* : setting the pointer to point to the first byte of the string1character array  */

  for(;;){
        c =(unsigned char) *p;          /*  :is used to access the data at a memory location or pointer.
                                            Can also use:     c = (unsigned char)  *(p+n); ,orr increment the pointer by 1
                                             below with p=p+1, and use c=*p;                                             */
        if(c==0){   break;  };
        printf("%c",c);
        p=p+1;    /* increment the pointer to hold the next memory value and-or point to the next memory location.
                    Incrementing a pointer that has a certain data type will increment it by the byte or memory size of
                     and allocated for that data type.  The data type of a pointer is that of the type of data it that it points
                     to in memory, and here, it is a char data type which is 1 byte of memory.  Incrementing this pointer
                     by 1 will increment its reference address by 1 byte.  */

      };  /* end of the for-loop */

/* Method 3:   Displaying the string using pointer arithmetic applied to a pointer, where the initial value of the
              pointer remains unchanged:   */

/* Some initial preparation before the loop:      */
printf("\n");
n=0;
p = string1;     /* : setting the pointer to point to the first byte of the string1 character array  */

  for(;;){
        c =(unsigned char) *(p+n);      /* (p+n) resolves or sums to the address of the next byte,
                                           hence this C language expression is a temporary pointer.  */
        if(c==0){   break;  };
        printf("%c",c);
        n=n+1;
      };

/* Method 4:  Displaying the string using an array name or identifier as a pointer to the start or first or starting
               memory location of that array:   */

/* Some initial preparation before the loop:      */
printf("\n");  n=0;


  for(;;){
        c =*(string1 + n);          /* : when n increments by 1, it will increment the string1 pointer by 1 so as
                                       to point to the address of the next byte or character in that array */
        if(c==0){   break;  };
        printf("%c",c);
```

```
            n=n+1;
        };
```

Pointers can even be assigned as having the same address that another pointer is assigned or pointing to. For example, if p1 is a char pointer assigned to a memory location, and say p2 is NULL or 0, hence not assigned to a memory location yet, p2 can later be assigned to a memory address by setting it equal to another pointer. An older saying or equivalent word for the word "NULL" is "NIL"..

Ex.

```
/* initial declarations: */

char c=65;
char * p1 = NULL;   /* NULL means an unassigned pointer, and or 0, or an address of 0 */
char * p2 = NULL;

/* an example of using the variables and or identifiers in the program: */

p1=(unsigned char *) &c;      /* : a (data) type cast of the address of a variable, here (c), is considered here,
                                   so as it can be properly assigned to an actual pointer variable. */

p2 = p1;          /* : setting pointer p2 equal to the same address that p1 is pointing to, or in reference to */

printf("\n%d",*p2);    /* the result displayed is 65  */

/*-----------------------------------------------------------------------------------------------*/
```

In the **BASIC** programming language, the default keywords to access a memory location are: PEEK and POKE. PEEK is used to read or get a byte stored in a memory location, and the general syntax format is:

```
    variable = PEEK (memory_address);
```

POKE is used to set (ie., write) a value into a memory location, and the general syntax format is:

```
    POKE (memory_address, value);
```

The above syntax looks similar to a function call in the C programming language.
For comparison:

```
BASIC language:     value  =  PEEK (memory_address);       : PEEK is analogous to READ
      C language:     value = *pointer;

BASIC language:      POKE (memory_address, value);         : POKE is analogous to WRITE
      C language:      *pointer  =  value
```

The BASIC language was one of the first "user friendly" ways to create computer programs. Programs became fairly or relatively easy to understand, write and edit. There are many **BASIC** language interpreter programs and "dialects" of BASIC - often suited for a particular computer and-or graphics abilities of it. BASIC is called an interpreted language because the program is not compiled into a standalone executive program, but rather each line is executed one at a time. Most of the BASIC interpreters use or require a similar syntax or format of the source program or input language, and there has been an attempt to standardize BASIC and you can research this if you want. There are also some BASIC compilers now available so as to generate a standalone executive program for a specific computer system.

A Computer Program For A Homemade Text String Input Function

```
/*-----------------------------------------------------------------------------------------------------------------------------------------------------
   mygetstring.c

   As a programmer learns more about the C language and-or computer programming, they will eventually consider a
   need for having their own homemade functions such as string function(s).  A text string is a series or string of readable
   text characters.  ANSI C, has several string input functions, and the main ones are scanf() and gets().  The source code
   for the homemade  function show below should also give a programmer some useful ideas to consider when they
   write their own functions.

   (c). JPA 2021
------------------------------------------------------------------------------------------------------------------------------------------------------*/

#include "stdio.h"

/* Here is the function prototype for mygetstring()              */

void mygetstring(unsigned char * s, int maxchars, int display);   /* a homemade getstring() */

    /* 1.  s is a pointer to the string and-or its memory, and where you want to access (read, write)  the input string.
           maxchars is the total number of characters to accept into this string.
           display is 1 to display the input string, and 0 to not display the input string.

       2.  A potential issue is that when backspacing, the cursor won't go back to the previous line on the display,
           hence a max. of 80 characters can be seen so as to be  edited if the string input starts at the beginning
           of a line of the display.

       3.   Allows use of the BACKSPACE  key or ESC key so as to terminate the input early with no valid input.

       4.   Shows a way to determine if a special key such as a function or arrow keys has been pressed.
   */
/*----------------------------------------------------------------------------------------------------------------------------------------------------*/
void main(void)
{
unsigned char c;
unsigned char string1[129];        /* declaring an array to hold many values, here, the values of
                                      displayable, or entered text characters */

printf("\nPlease enter some text up to 128 maximum characters. ");
printf("\n:");
mygetstring(string1,128,1);      /* call mygetstring() with two arguments,
                                    the string indentifier, and the max. number of input characters */
if(string1[0] != 0){  printf("\n:%s",string1);  };   /* or use NULL for 0, see if an input string exists  */

printf("\n\nPRESS THE ESC KEY or CTRL Break,TO EXIT THIS PROGRAM: ");

    for(;;){
            c=getch();                   /* : A standard C function to input a single character from the keyboard. */
            if(c==27){   exit(0);    };   /* : Exit the program when the user enters ASCII 27, the ESC key.  */
         };
```

```
return;
};
/*--------------------------------------------------------------------------------------------------------------------------------------------------------*/
void mygetstring(unsigned char * s, int maxchars, int display)          /* : a homemade getstring()  */
{
int n=0;                          /* : a counter, increment  */
unsigned char c=0;        /* : assigned to each new character entered */
unsigned char * p=s;     /* : a pointer to the start of the string sent to the function */
unsigned char * p1=p;   /* : to store the starting address of the string sent to the function */

/* A potential issue is that when backspacing, the cursor won't go back to the previous line on the display. */

     for(;;){
                 fflush(stdin);
                 c=(unsigned char) getch();
                /* To see each charcter value for testing then use:   printf("\n%d ",c);*/


                if(c==224){    c=(unsigned char)getch();     /* pressed a special key, such as the arrow keys */

                                 if(c==72){  /* printf("\nUP  ");  */    };         /* UP ARROW key was pressed */
                                 if(c==80){  ;   };        /* down arrow */
                                 if(c==75){  ;   };        /* left arrow */
                                 if(c==77){  ;   };        /* right arrow */

                                continue;
                            };

                if(c==0){    c=(unsigned char)getch();     /* pressed a special key, such as the function keys */

                                 if(c==59){  /* printf("\nF1 ") ;  */   };         /* F1 key was pressed, then do something.  */
                                 if(c==60){  ;   };        /* F2 */
                                 if(c==61){  ;   };        /* F3 */
                                 if(c==62){  ;   };        /* F4  */
                                 if(c==63){  ;   };        /* F5  */
                                 if(c==64){  ;   };        /* F6  */
                                 if(c==65){  ;   };        /* F7  */
                                 if(c==66){  ;   };        /* F8  */
                                 if(c==67){  ;   };        /* F9  */
                                 if(c==68){  ;   };        /* F10 */
                                 if(c==133){  ; };        /* F11 */
                                 if(c==134){  ; };        /* F12  */

                                continue;
                            };


                 if(c==27){    *p1=0;  break;  };    /* : user pressed the ESC key, user quit this step, null terminate the
                                                           string at the beginning of it   */
                 if(c==13){    *p=0;   break; };     /*  : user pressed the ENTER key null terminate the string */

                 if(c=='\b'){                              /* : user pressed the BACKSPACE key */

                            if( (n>0) && (n<=128) ){           /* : if not at the start of the input string's first memory location */
```

```
                              printf("\b"); printf(" ");  printf("\b");   /* : backspace, then overwrite with a space, then
                                                                              backspace again to re-position the text cursor */
                              p=p-1;  n=n-1;   /* : decrement these values, so as to essentially
                                                  let the user redo the input at that text position */
                              continue;

                            }   else {   if(n==0){            /* : already at the start of the string, don't backspace */
                                              printf("\a");     /* : alert the user with the ASCII bell character and sound  */
                                              continue;
                                              };
                                     };

                     };   /* end of user pressed backspace */


                           /* User text characters for the string have offsets: 0 to 127 = 128
                              total text characters,  offset 128 should then contain a null, 0, character:  */
               if(n==128){  printf("\a");     /* : the user has entered the max. number of text characters */
                           continue;       /* : don't display any more characters */
                         };

               *p=c;  /* place, write, store the input , displayable, character into the string[] sent to this function */

               if(display==1){
                              printf("%c",c);       /* : optional, to display the currently entered character   */
                            };

               p = p+1;       /* : point to the next memory location in the string array */
               n = n+1;       /* : incrementing the current number of displayable characters in the string */


          };   /* end of main for loop  */


return;
};
/*-------------------------------------------------------------------------------------------------------------------------------------------------*/
```

[This space for edits.]

LIST OF ASCII TEXT CHARACTER CODES

ASCII is the American Standard Code for Information Interchange (communication), and is used as a standard for machine and computer programming and display screens. These are essentially 1 byte values, hence 8 bits of which can be sent over 8 parallel wires, all at the same time. ASCII was standardized in the early 1960's by what would later become the ANSI which is the American National Standards Institute.

7 Bit ASCII Standard Text Character Codes: (0 to 127), and 8 Bit ASCII Extended Text Character Codes (128 to 255):

ASCII codes 0 through 31 are usually non-text display codes that are used to control (ie., send basic commands for various functions of) a machine such as a printer or computer display screen. The table below was printed using a C language computer program.

```
 32       33 !     34 "     35 #     36 $     37 %     38 &     39 '     40 (     41 )
 42 *     43 +     44 ,     45 -     46 .     47 /     48 0     49 1     50 2     51 3
 52 4     53 5     54 6     55 7     56 8     57 9     58 :     59 ;     60 <     61 =
 62 >     63 ?     64 @     65 A     66 B     67 C     68 D     69 E     70 F     71 G
 72 H     73 I     74 J     75 K     76 L     77 M     78 N     79 O     80 P     81 Q
 82 R     83 S     84 T     85 U     86 V     87 W     88 X     89 Y     90 Z     91 [
 92 \     93 ]     94 ^     95 _     96 `     97 a     98 b     99 c    100 d    101 e
102 f    103 g    104 h    105 i    106 j    107 k    108 l    109 m    110 n    111 o
112 p    113 q    114 r    115 s    116 t    117 u    118 v    119 w    120 x    121 y
122 z    123 {    124 |    125 }    126 ~    127 ¦    128 Ç    129 ü    130 é    131 â
132 ä    133 à    134 å    135 ç    136 ê    137 ë    138 è    139 ï    140 î    141 ì
142 Ä    143 Å    144 É    145 æ    146 Æ    147 ô    148 ö    149 ò    150 û    151 ù
152 ÿ    153 Ö    154 Ü    155 ¢    156 £    157 ¥    158 P    159 ƒ    160 á    161 í
162 ó    163 ú    164 ñ    165 Ñ    166 ª    167 º    168 ¿    169 ¬    170 ¬    171 ½
172 ¼    173 ¡    174 «    175 »    176 ¦    177 ¦    178 ¦    179 ¦    180 ¦    181 ¦
182 ¦    183 +    184 +    185 ¦    186 ¦    187 +    188 +    189 +    190 +    191 +
192 +    193 -    194 -    195 +    196 -    197 +    198 ¦    199 ¦    200 +    201 +
202 -    203 -    204 ¦    205 -    206 +    207 -    208 -    209 -    210 -    211 +
212 +    213 +    214 +    215 +    216 +    217 +    218 +    219 ¦    220 _    221 ¦
222 ¦    223 ¯    224 a    225 ß    226 G    227 p    228 S    229 s    230 µ    231 t
232 F    233 T    234 O    235 d    236 8    237 f    238 e    239 n    240 =    241 ±
242 =    243 =    244 (    245 )    246 ÷    247 ˜    248 °    249 ·    250 ·    251 v
252 n    253 ²    254 ¦    255
```

Common ASCII machine control and-or formatting control characters and-or values:

0 = NULL
7 = A = Alert and-or a bell sound. To consider people with hearing problems, you can do something such as displaying the word "alert!" or "error!" on the screen.

In C, you can use the escape (from the normal text displaying, display and-or machine system control)character symbol followed by letter b so as to have access to, and signal the system to use the bell character, and from within a printf() statement. These control characters are also called an "escape sequence (of characters)" The backslash character is used first, so as to signal an escape sequence. For example:

```
printf("\a Did you hear a bell sound?");  This is equivalent to:
printf("%c Did you hear a bell sound?,7);
```

Note, to display a backslash character and-or the quote character, use for ex.

```
                 printf("   \\    \"   ");
```

8 = BS = Backspace = Go one text position "back" or leftward on the current line.
10 = LF = Line Feed = New Line = Go to the next display and-or text line.
13 = CR = Carriage Return = Return to the start of the current line of text.
27 = ESC = **Esc**ape Character = Typically used to start a special process and-or hault in the program, or to control the display.
32 = SP = Space, "blank" Character, "word separator", the absence of any visible char.

Pressing the Enter key usually generates both a CR=13 and LF=10 characters or commands.

The ASCII characters can optionally be entered by pressing and holding the ALT(Alternate) key and then entering the text character code on the numeric keypad, and then releasing the ALT key. The Windows ((t) Microsoft Co.) computer operating system may redefine the look of the extended characters and-or depending on the font being used. Note for example that given a capital or upper-case letter such as character A = 65, the corresponding lower-case version is 32 higher, hence character (a) = (65 + 32) = 97.

ASCII Images or Art is term used to describe images composed of ASCII text characters, either for graphics or photos. This is an actual method that can be used for communication systems without dedicated graphics ability, but still having text ability to work with. The result is a mediocre image if viewed from a few feet away or more, but it is much better than nothing. Today, with modern computer systems, ASCII art is generally unnecessary, however computer programs are available to convert (.bmp, .jpg, etc.) images to ASCII art and with color ability, and which can then make something such as interesting and artful posters. Many of the first electronic computer games before programmable, pixel graphics and colors were developed used ASCII text characters to display things such as values, locations, and (movable) objects on the screen. It was common before about 1985 to purchase a book containing BASIC language (text) source code programs that you could type in yourself and run on a personal computer(PC).

More ASCII Control Characters   (generally for communication to send ASCII text files)

1 = SOH = Start Of Heading
2 = STX = Start Of Text
3 = ETX = End Of Text
4 = EOT = End Of Transmission
5 = ENQ = Enquirer (ie., a question, find)
6 = ACK = Acknowledge
9 = HT = Horizontal Tab (ie., several spaces such as for making columns, indentation)
11 = VT = Vertical Tab   (ie., insert blank lines)
12 = FF = Form Feed   (ie., eject current page, prepare for next page to be printed)
14 = SO = Shift Out   (typically for printer color selection)
15 = SI = Shift In
16 = DLE = Data Link Escape
17 = DC1 = Device Control 1 (the control depends on the device and how it was programmed)
18 = DC2 = Device Control 2
19 = DC3 = Device Control 3
20 = DC4 = Device Control 4
21 = NAK = Negative Acknowledge
22 = SYN = Synchronous Idle
23 = ETB = End Of Transmission Block
24 = CAN = Cancel
25 = EOM = End Of Medium   ,   26 = SUB = Substitute

28 = FS = File Separator   ,   29 = GS = Group Separator
30 = RS = Record Separator   ,   31 = US = Unit Separator

Extra:

On a Window's © operating system computer ALT 0169 is the copyright symbol: © and(c) can be used.

On a Window's ® operating system computer ALT 0174 is the registered symbol: ® and (R) can be used.

Why the ASCII standard uses 7 bits instead of a complete 8 bit byte.

The basic answer to this is that the 8 bit was used for error checking. For example, if the 8th bit was set it could mean there was either an even number of set bits (ie. "1's") transmitted. If the receiver counted that an odd number of bits were received after counting them, then there was an error in the transmission of the data, and a request to resend that data again could be sent to the transmitter device. Though this will not find all the possible errors, it is still practical. It is even possible to send the received data bact to the sender so as to compare the data and-or compare the unique checksum value of all the data send and received, and which surely will indicate if there was a data error. A slower method to find transmission errors would be to send each byte twice and compare those, and even send it back to the sender for comparring. If there was an error, the two same bytes could be sent again and comparred again.

Besides encoding the control characters into the standard set of ASCII printable characters, it is possible to make a special byte value (say 1, 127, etc.)to signal or indicate that a control value or control character (here, still a byte value) follows.

USING COMMAND LINE ARGUMENTS AND-OR DATA WITH A COMPUTER PROGRAM

```
/*-------------------------------------------------------------------------------------------------------------------------------------------
    CommandLineArguments.c

    This program shows an example of how to send optional and-or various argument data (such as text and-or numbers)
     to a program when it is first ran on a personal computer (PC) having a  DOS (Disk Operating System) command line,
    and then how to access those arguments from within the program so as to be used within the program.  The
    command-line is sometimes called the computer's operating system input and display (ie., typing, basic input and
    screen output) or (access or system) "terminal", and or the "DOS (Disk Operating System, of commands and access)
    command-line".

    Format example:   myprogram.exe  argument1  argument2 . . .

                             Examples of the arguments:   a word, sentences (ie., words with space between them)
                                                           bounded by quotes (ie. "word  word  word"), numbers, a
                                                           filename, etc., and these are input as ASCII text characters

   Each argument should be separated by a space character.  Extra spaces will be ignored unless they are
   within quotes such as for a string of text characters.

   Using (mouse, left-click) "drag and drop" of an icon or filename onto the programs icon or filename, is also
   a way to send a filename and-or the file as an argument.  The file can then be accessed within the program
   without asking the user to type in or select what file to use.   With the Windows operating system (technically
   a program itself), you can place a program in the SENDTO folder so as when you mouse-right-click on a
   filename, one of the menu options is "send to" so as to directly send that file to your program(s), and without
   needing to find and enter long directories, etc. To open the SENDTO folder or directory, press the START
   button the screen, select RUN, and then enter Sendto, and then drag and drop, or copy, your program to be
   in there.   To RUN or execute an .EXE format computer program, simply double click on it, or enter its name
   on the computer system's (DOS) command-line.  To see a list of files in a folder or directory, enter DIR on the
   command-line.  To see a list of DOS commands available for that computer system, enter:  HELP, and for
   help with a specific command, enter:  HELP command.   If possible, try to always make backup copies of
   your programs and-or data, for these devices tend to fail, can get damaged, hacked, lost or stolen.  Put these
   in a safe and protected place.

  The arguments are considered as null terminated (0), text strings, and you may then need to, for example,
  convert a ASCII text string to a binary numeric value that can be assigned to a variable, etc.  One common
  C function for this is the atoi( ) which is the "ASCII text string to numeric integer" conversion function, and for
  converting an ASCII text string to a floating point value, use the atof( ) function.  The return value is a double
  precision floating point number.  Leading spaces will be ignored, a sign and decimal point can be used, and
  the last character accepted will be either a space or a text character other than a number.  strtod( ) is a similar
  function.  To convert a string to a long double floating point number, use strtold( ).

  Since main( ) function is the default or required entry point of a C program, the arguments are passed
  or sent to argument holders described in the main().  Just like for any function with arguments, each
  argument to a function has a data type and an identifier, and here they are also specific C keywords and-or
  variables and their identifiers (identifying names) that need to be used so as to access that data.  Still,
  you can always assign these values to your own variables used in the program.

  The first argument is always the program name itself, and is identified with an offset of 0 as:
  argv[0].   argv[ ] means "argument vectors" array  which essentially means text strings for the arguments.
  argc = argument count, will hold the number of total arguments sent to the program on the command line.

   (c) JPA, Aug. 2021
```

```
-------------------------------------------------------------------------------------------------------------------------------------*/

#include "stdio.h"
#include "stdlib.h"    /* : for the exit(), although just a return statement would be alright in this program
                            exit() is often used to terminate a program early after perform some cleanup
                            such as closing open files, etc  */
#include "string.h"   /* : so as to include the strcpy() and its header, which includes the function
                            prototype or example format, and which can help determine if the arguments
                            sent to it are proper. */

/*-----------------------------------------------------------------------------------------------------------------------------------*/
void main(int argc,  char * argv[] )
{
int n=0;
unsigned char mystring[1000] = { "\0" };   /* : optional, a way to null terminate the string, here the first byte and-or
                                                initialize the string with some data.  This here is initializing mystring[ ]
                                                array with a string that is just a null terminator (0 , byte value) character,
                                                but it could also be initialized with a complete string within quotes.    */

printf("\nTotal Number Of Command Line Arguments = %d",argc);
printf("\n\n");

            for(;;){
                      strcpy(mystring,argv[n]);      /* : using the string copy function, here argv[n] is copied to mystring */
                      printf("\nArgument %d = %s",n,mystring);     /* : display each command line argument */
                      n=n+1;
                      if(n>=argc){  break;  /* break or exit from this loop or repetition */  };
                   };


printf("\n\nPress A Key To Exit This Program: ");  fflush(stdin);  getch();  return(0);

exit(0);    /* : can also use:   return;  or something like return or return(0);
               0 is usually used to indicate everything was ok, no known errors or issues   */
};
/*---------------------------------------------------------------------------------------------------------------------------------*/
```

FOR PROGRAM USAGE SECURITY, YOU CAN USE A PROGRAM ACCESS CODE

Perhaps you want to use a **program access code** to be entered on the command line or at some point during the program once started to run. This can be used so as to help limit the access to that program you are making. It is possible to do this in a varitey of simple ways, such as with the code used in the above program to display the command-line arguments to the computer system and program to be run or executed (.exe). For exampe, you can use the string compare function strcmp(), and compare say the first word, or the program access text (including numbers) or words surrounded by quotation marks, and which follow the program name on the comand line. For example:

```
myprogram.exe  1397abz              : there the program access code entered on the command line is 1397abc
```

To Learn To Program A Computer Device

A computer is a machine that can be programmed to have certain steps, and modernly it is a electronic machine that is often based on digital or binary data rather than some physical switch, cog or voltage level. It is of note that binary data can also be used for switching something on or off, and or to store data such as a number or letter. A computer that can be programmed (ie., "told what steps to do") is often used to then control another machine of which may be called as a "computer controlled machine".

To learn C programming, you can take a course or two and-or obtain some books about it. The computer programming books by **Herbert Schildt** are a good purchase to make. Some of these books also include the basics of the C programming language and with many programming examples to consider. Some of these books also include some general computer science topics which are a great aid for any programmer. Schildt's books have made formal computer science and programming into a practical reality for the average person, and so as to consider making some useful programs without a costly formal education in computer science. Another very good learning tool is the **Learn C Now book** by **August Hansen** which includes examples programs and a fast C editor and compiler to test your programs, however. note that it does not create the .EXE programs for standalone operation without the compiler, and is rather used for quickly testing and finding errors in programs you make. The author of Mathization has bought and used this specific book required a local college course about C programming, however, it is also good for any person interested in the fundamentals of the C language, and without the need to take the course, but it does help. People new to a programming language need to "keep it simple" and focus on the keywords and-or commands and syntax used for that language, and what they mean and how to use them. Learn by writing simple test programs that use a specific keyword(s). This book here, **Mathization**, tries to give the programmer a general or practical math education, and with C programs related to math that are generally not found in most computer programming language books and-or courses. These programs are also a good example to reference so as to create your own programs.

Some may desire to also take a course in computer science. For learning C, you will need a C source code compiler such as TinyC . This book considers the TinyC compiler which is generally an ANSI standard, C compiler with no specific computer system (such as Windows (R)) and-or its own specific programming functions to then consider, and which then tends to complicate learning the basics of any computer language. Other compilers may include many other specialized functions to be used for a certain or unique computer system, and such as for basic graphics to put a pixel, line or circle on the screen, or to make some sounds. These compilers must also be compatible with the specific computer system being programmed, but the ANSI C standard is generally compatible with any computer, and you generally will not have to learn or relearn C programming for general programming (creating computer programs) on any ANSI C compatible compiler for any computer system.

With programming, and much like math, just the basics of it can go a long way and be most useful or practical. Rather than make exquisite and specialized programs that take much knowledge and years to develop, it is sometimes better or more practical to make many relatively simple, useful and practical programs or even functions( ) that can calculate some numeric values of which others, such as at a factory, group or business, can utilize. Writing your own homemade functions is a good learning experience. **Writing functions** or the programming code or steps to do a certain task(s) in a program, and of which can be reused in many of your programs and possibly by other programmers, is a very useful way to become an experienced computer programmer. Strive to learn the ANSI C standard keywords and functions first, and before trying to write programs for a specific computer system that requires a specific and capable C compiler which also has some non-ANSI C standard functions for that computer system. Learning how to program or work witih a specific computer operating system with its own non-ASCII requirements with a given computer language such as C is another programming issue and challenge that you may or may not encouonter.

As a computer programmer, you are generally not expected to be initially proficient in the science and-or math necessary for any certain program or company, however you may need to research what is necessary and-or collaborate with other employees and those knowledgeable, such as other programmers, so as to make a specific and customized computer program. Please keep a written record so as you and-or others may need this data later. You may even be designated to do other things such as making the morning coffee and-or sweep the floors so as to be a multi-tasking employee with several useful and valued abilities. This book contains many programs for the common math functions found on a scientific calculator, and these are generally already part of the C language standard and available as prewritten math functions, and hence you do not need to be a "math programmer" to do much general programming in terms of those, but it would benefit any programmer if they know some math so as to make more advanced programs and-or to understand or edit programs written by others.

Many of the computer programs and-or functions shown in this book are relatively uncomplicated, and they may even be overlooked by many when scientific calculators and-or prewritten math computer functions are available in the C compiler. The set of ("scientific") fundamental math functions shown in this book, such as for those found on a scientific calculator or a program compiler, seem to be very rare, and should therefore be of general interest to computer programmers who are oriented to writing mathematical calculations. What is also important and shown in these functions is how to create some specialized loops, such as for performing successive approximation to "zero in on" or to get as close as possible to a numeric result in terms of a certain precision needed.

It is best to make a study of the available C keywords, syntax and functions available in the plain ANSI C current standard. You will find that it is not too difficult, and due to the fact that certain functions like printf() and scanf() are used rather extensively, and that test programs and-or functions to see the results can be made relatively short. You can always make programs as complex as you want from just using the basics, interest, practice, program testing and editing.

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A COMPUTER PROGRAM TO DISPLAY THE BITS OF A BYTE

```
/*----------------------------------------------------------------------------------------------------------------------
    BinaryToDecimal.c

    There are many ways to access (read, write or set) the bits of a byte, below is
    one example of doing so in a functional form or coding.  The C programming
    language has several "bitwise" operators that work with the bits of bytes, and
    of which you may wish to reasearch and learn.  For example the leftshift (<<) and
    rightshift (>>) operators:

    Format example:  bytevalue1 >> number of bits to shift rightward, to produce bytevalue2
                             Ex.  10000010  >> 1 would become 01000001

    Note, for example, that the rightshift operator will actually shift the least significant
    bit off the byte and it is lost unless it was first saved before that operation, and perhaps
    then placed after the operation into the most significant bit, and the operation essentially
    becomes a rotate_right operation that ASNSI C does not actually have.   Note that (>)
    and (<) are conditional operators.

    To invert each bit in a byte from 0 to 1, or from 1 to 0, the complement operator (~) can be
    used.  This is also called inverting the bit value or state.  This requires only one operand:

    Format example:   bytevalue2 = ~bytevalue1     : ex. ~10000001  would become 01111110

    The C programming language has many bitwise logical operators such as:

    & for AND        : Note that && is for logical (decision making) AND for two conditions.
                     : example format:  bytevalue1 & bytevalue2 to produce bytevalue3
                       This "ANDing" works on corresponding bits of the two bytes.  If both
                       bits are set to 1, the result is 1, otherwise the result is 0.

    | for OR         : Note that || is for logical (decision making) OR for two conditions.
                     : example format:  bytevalue1 | bytevalue2 to produce bytevalue3
                       This "ORing" works on corresponding bits of the two bytes.  If either
                       bit is set to 1, the result is 1.

    ^ for XOR        : example format:   bytevalue1^bytevalue2  to produce bytevalue3
                       This is "eXclusveOR", it is similar to the binary OR function, however it excludes one extra truth,
                       and that is if both bits are set to 1, the result is not 1, but 0, since this condition is to be "excluded".

   When working with numeric values, a programmer generally does not need to convert a
   decimal value to its binary equivalent, and vice versa,  since the compiler will do this so
   as the computer can work with binary values behind the scenes or transparently.  For
   example, when you enter a decimal value of 5, the compiler will convert this to its
   corresponding binary value of 00000101 so as the computer electronic circuitry can
   work with it such as in mathematical operations.

    There may be times when you explicitly want to enter and convert and-or display these
    binary values.  Some calculators have similar conversion functions available.  In the
    example program below, error or input checking is kept to a minimum,and may be
    improved upon.  One example of error checking is to determine if any other characters
    other than 0 or 1 have been placed in the string, and then signal an error.  It may be
    best to put any error checking in the function itself, since any programmer who desires
```

```
    to use the function does not need to worry about the program coding to do so, and
    can simply concentrate on what the function is supposed to do.

    This program example below is to demonstrate using the following homemade functions:

    btod()   : binary to decimal conversion of an ASCII text string containing 1's and 0's
               ASCII text characters, hence a string of text, and the return values is a
               numeric integer.  Programmed to work with up to an 8 bit unsigned byte value
               only,  however, you may wish to include a function parameter for the sign, say
               0 for unsigned, and 1 for signed.

               unsigned char btod(char * binary_text_string);

    dtob()  : decimal to binary ASCII text string
               Programmed to work with 8 bit byte values only.

               void dtob(int number, char * binary_text_string);

Three extra homemade bit functions are included:

   rotateright()    : rotates the bits of a byte left, and does not loose any bits or data
   rotateleft()     : rotates the bits of a byte right, and does not loose any bits or data
   complement()  : C already has a built in complement function, for example: ~byte , and
                     this will invert or change the state of each bit from 0 to 1, or 1 to 0.
                     Ex:  Using common decimal values :   254 = complement(1)    ,  1d = complement(254)  , and:
                     Binary equivalents are:  11111110 = ~00000001   ,   00000001 = ~11111110

   The C programming language has standard ANSI, C functions to shift the bits of a byte,
   but either the LSB (Least Significant Bit)  or  MSB (Most Significant Bit) is lost unless its value is saved first.
   (c) JPA 2021
------------------------------------------------------------------------------------------------------------------------*/

#include "stdio.h"
#include "string.h"

          /* Function (compiler) prototypes for error checking,and usage:  */

unsigned char btod(char * binary_text_string);         /* binary to decimal */
void  dtob(int number, char * binary_text_string);     /* decimal to binary */

unsigned char rotateright(unsigned char byte);        /* rotate the bits of a byte, righward */
unsigned char rotateleft(unsigned char byte);         /* rotate the bits of a byte, leftward */
unsigned char complement(unsigned char byte);   /* change the state of each bit of a byte */

/*----------------------------------------------------------------------------------------------------------------------*/
void main(void)
{
unsigned char string1[9];
unsigned char b=0;
unsigned int d1=0;
unsigned char d=0;
int L=0;
int a=0;
```

```
system("cls");
fflush(stdin);

                                   /*************   btod()  use example    *****************/
for(;;){
          printf("\n\nEnter A Binary Value Up To 8 Bits of 0's and 1's, or press ENTER: ");
          fflush(stdin);
          gets(string1);
          if(string1[0]==0){  break; };                          /* : user pressed the Enter key */

          L=(int) strlen(string1);
          if(L>8){  printf("\n\aEnter Only Up To 8 Binary Digits or Bits!");
                     printf("\nPress A Key");
                     fflush(stdin); getch();  continue;
                   };

           b=btod(string1);
           printf("\n%s = %d",string1,b);
           string1[0]=0;    /* a simple way to null terminate a string at the first character */
          };

                        /*****************   dtob() use example    ***************/

for(;;){
          printf("\n\nEnter A Decimal Value (1 to 255) or enter 0 to exit loop: ");
          fflush(stdin);
          scanf("%d",&d1);
          if(d1==0){  break;   };                          /* : user pressed the Enter key */

           if((d1<0) || (d1>255) ){  printf("\n\aEnter A Value From 0 to 255 only!");
                     printf("\nPress A Key\n");
                     fflush(stdin); getch();  continue;
                   };

           d=d1;

           dtob(d,string1);
           printf("\n%d = %s",d,string1);

        };   /* end of for() loop */

for(;;){                    /***********   example of the rotate byte functions ****************/

       printf("\n\nEnter A Decimal Value 1 to 255, or 0 to Exit Loop:  ");
       fflush(stdin);
       scanf("%d",&d);
       if(d==0){ break; };

      dtob(d,string1);
      printf("\nThe corresponding binary value is:  %s",string1);

      b=d;
      b=rotateleft(b);
      dtob(b,string1);
```

```
       printf("\nThe rotate-left value of this is: %d = %s",b,string1);

       b=d;
       b=rotateright(b);
       dtob(b,string1);
       printf("\nThe rotate-right value of this is: %d = %s",b,string1);
   };

printf("\nPress A Key To Exit The Program: "); fflush(stdin); getch();
return;
};
/*-----------------------------------------------------------------------------------------------------------------------*/
unsigned char btod(char * binary_text_string)
{
unsigned char b=0;
int n=0;
unsigned char c=0;
unsigned char binaryweight=1;

/* Let's find the end of the binary string so as to begin the conversion at the
   least significant bit = LSB: */

for(;;){
            c=binary_text_string[n];
            if(c != 0){ n=n+1;  };
            if(c==0){  break;  };
        };

/* A debugging test:   printf("\nbits = %d",n);  fflush(stdin); getch();  */

n=n-1;   /* rather than use [n-1] below, and since the first data element is at offset 0 */
for(;;){    /* Loop to convert the binary value to its equivalent decimal value. */

            if(binary_text_string[n] == '1'){   b=b+binaryweight; };    /*  '1' = ASCII 50  */
            binaryweight = binaryweight * 2;
            n=n-1;
            if(n<0){ break; };
       };
 return b;
};
/*-----------------------------------------------------------------------------------------------------------------------*/
void dtob(int number, char * binary_text_string)
{
int n=0;
unsigned char binaryweight=128;

strcpy(binary_text_string,"00000000\0");   /* : ensures a starting value with a zero value */

                  for(;;){
                           if(number>=binaryweight){
                                                binary_text_string[n]='1';
                                                number = number - binaryweight;
                                               };
```

```
                    binaryweight = binaryweight / 2;
                    n=n+1;
                    if(n==8){ break; }
                  };

return;
};
/*-----------------------------------------------------------------------------------------------------------------------*/
unsigned char rotateright(unsigned char byte)
{
unsigned char b;

b=byte&1;          /* store the value of the LSB, bit 1 = logical bit 0 */
byte=byte>>1;    /* perform the right shift of the byte */
if(b==1){ byte=byte|128; };     /* set the MSB equal to the previously stored LSB */

return byte;
};
/*----------------------------------------------------------------------------------------------------------------------*/
unsigned char rotateleft(unsigned char byte)
{
unsigned char b;

b=byte&128;
byte=byte<<1;
if(b==128){ byte=byte|1; };

return byte;
};
/*----------------------------------------------------------------------------------------------------------------------*/
unsigned char complement(unsigned char byte)       /* Invert each bit of the binary value. */
{
 /* This is also called the 1's or binary complement. and adding 1 to  this value gives the 2's complement */

byte = byte^255;    /*  You can also use C's special, one operand, binary complement operator:  byte = ~byte; */
                    /*  You can also use:  byte = 255 - (byte)  ,  and this is equal to 11111111b - (byte) ;  */
                    /* byte^255  is equivalent to C's syntax for:  complement = (byte) XOR (255)  =  (byte)^11111111
return byte;
};
/*-----------------------------------------------------------------------------------------------------------------------*/
```

Example of program use and screen output:

```
Enter A Binary Value Up To 8 Bits of 0's and 1's, or press ENTER: 10000001

10000001 = 129     : or=  10000001b = 129d

Enter A Decimal Value (1 to 255) or enter 0 to exit loop: 130
130 = 10000010

Enter A Decimal Value 1 to 255, or 0 to Exit Loop:  3

The corresponding binary value is:  3 = 00000011
The rotate-left value of this is: 6 = 00000110
```

```
The rotate-right value of this is: 129 = 10000001    : rotate is like a byte or bit location shift, but does not loose any digits

Press A Key To Exit The Program:

--------------------------------------------------------------------------
```

A homemade way to left shift a binary value is to multiply it by 2, and for a right shift, divide that binary value by 2. When this is done the most significant bit or the least significant bit will be lost if its value (ie., 1 to 0) or some indication such as a toggle or flag variable (ie., set to 1 or 0) was not stored before the mathematical operation.

Ex. A computer byte or char value of 145d = 10010001b , multiplied by 2 is: 00100010b = 34d
The MSB was lost after the left shift operation.

In a similar manner to the above reasoning, for example, when working with single byte or characters(char) values only, they will "roll over, to zero" like an automobiles mile counter when the highest count is reached, and that value is 11111111b = 255d. When 1 is added to this value, the leading carry is lost unless it was somehow saved first, and that binary value will then become 0000000b = 0d. So that this does not happen when you need to consider higher values, you can first define the variables as integers which are often 2 bytes ("wide" or "width") and have a maximum value before rollover of: 11111111 11111111b = 65535d. By having various data types or byte sizes available in a compiler and computer, it can help increase speed and saves memory for other things.

USING THE #DEFINE , C LANGUAGE COMPILER DIRECTIVE FOR TEXT SUBSTITUTION

```
/* ---------------------------------------------------------------------------------------------------------
define1.c

    A basic test of using the #define , compiler preprocessing directive or for text substitution.
    This can be used to redefine keywords, mathematical operation symbols, constants , etc.
    # indicates a certain compiler directive or instruction follows, much like a command or
     option for the compiler program, and usually for how to process the source code.
    The #define data_used   is commonly called a macro.  They are much like program
    "global constants".

Ex.  #define ROWS 50
      #define COLUMNS 80             /* : can change the values as needed for the program, etc */
      #define PI 3.14159265
      #define cube(x) (x * x * x)         /* : this is a inline, function-like macro  , here x can be a integer or float  */

   (c) JPA , June 4, 2025
------------------------------------------------------------------------------------------------------------*/
#include "stdio.h"

#define  rightshift  >>      /*    perhaps make:   leftshift <<   */
#define RS  >>               /*   : an alternate example,  and   perhaps make:   LS <<     ,  ffor left shit  */

#define cube(x) (x * x * x)

/*--------------------------------------------------------------------------------------------------------*/
void main(void)
{
unsigned char c = 128;      /* 128d  =  10000000b  */

printf("\n%d",c);               /* displayed is:   128     */
```

```
c = c rightshift 1;               /* : the compile will convert this to:  c  = c  >> 1   , and 1 is the number of shifts */
printf("\n%d",c);                  /* 64d  =  01000000b  */

c = c RS 1;                      /*  : the compiler will convert this to:  c = c  >> 1  */
printf("\n%d",c);                  /* 32d  =  00100000b  */

printf("\n\n"The cube of 2.5 is %.15g",cube(2));  /* : 15.625 */    /* Can use %lu for unsigned long values only */

printf("\n\nPress A Key :"); fflush(stdin); getch();

};/*-----------------------------------------------------------------------------------------------------------*/
```

A COMPUTER PROGRAM TO CONVERT A DISTANCE BETWEEN THE METRIC AND ENGLISH MEASURING SYSTEMS

```
/*-----------------------------------------------------------------------------------------------------------------
   conversions.c

   A C language computer example program showing several functions to convert
   from the English system to the Metric system.  Functions such as this are
   very helpful and convenient to computer programmers either good with math or
   not so good, and-or don't want to be bothered creating such functions.  These
   functions  are somewhat unique and can be used as an example to create many
   other functions and-or programs for conversions.

   (c) JPA Aug. 2021
------------------------------------------------------------------------------------------------------------*/

#include "stdio.h"
#include "stdlib.h"   /* for exit()  */

double FeetToMeters(double feet, double inches, double eighths, double sixteenths, double thirtytwoseconds );
/* If a argument is not needed, use 0.0 for it.                                                  */
/* 1ft = 12in =  0.3048m = 30.48cm   ,  1inch = 2.54cm =  25.4mm                                 */
/* The return value is the equivalent length or distance in meter units.                         */

double MetersToFeet(double meters, double centimeters, double millimeters);
/* a modified  function might contain a kilometer argument  */

/*------------------------------------------------------------------------------------------------------------------------------------------*/
void main(void)
{
double  result=0.0;
double feet;
double inches;
double eighths;
double sixteenths;
double thirtytwoseconds;

double meters;
double centimeters;
double millimeters;
```

```
unsigned char ch=0;

long int f=0L;   /* whole portion of the feet */
double f2=0.0;  /* fractional portion of the feet */

double ninches=0.0;
double totalinches=0.0;
double totalmeters=0.0;

double totalcentimeters=0L;
double mm=0.0;


selection0: ;  /*----------------------------  PROGRAM MENU   ---------------------------------------------------------*/

for(;;){
          system("cls");
             fflush(stdin);

          printf("\nTHIS PROGRAM HAS TWO FUNCTIONS:");
          printf("\n\n1. Feet To Meters Conversion");
          printf("\n2. Meters to Feet Conversion");
             printf("\n\nESC To Exit Program");

          printf("\n\nEnter Your Selection To Begin: ");  fflush(stdin);  ch=getch();
          if(ch=='1'){  goto selection1;  };   /* : Using the goto command is discouraged and is mainly
                                                        used as a quick and easy method to change program
                                                        exection flow within a C function.  Using goto is popular
                                                        in the BASIC language  */
          if(ch=='2'){  goto selection2;  };
             if(ch==27){  exit(0);  };      /* : user pressed the ESC key */
      };

selection1:  ;

for(;;){                         /*---------------- main loop to use FeetToMeters()  --------------------*/
             system("cls");
             fflush(stdin);

          printf("FEET TO METERS CONVERSION PROGRAM.  (c) 2021, JPA");

             result=0.0;  feet=0.0;  inches=0.0; eighths = 0.0, sixteenths = 0.0, thirtytwoseconds = 0.0;
             totalmeters=0.0; totalcentimeters=0L;  mm=0.0;

          printf("\n");

          printf("\nWith this program up to 5 units can be used: \nfeet, inches, eighths of 1in, sixteenths of 1in, and
thirtytwoseconds of 1in");
             printf("\n\nEnter a value of 0 if it is not used.");
             printf("\nUse a negative value such as -1  to return to the MENU. ");

             printf("\n\nFor some reference: 1 ft = 0.3048m , 1in = 2.54cm , 1mile = 5280ft = 1609.34m");
             printf("\n1m = 39.37in = 3.28084ft  ,  1/8in = 0.125in  ,  1/16in = 0.0625in");
             printf("\n1/32in = 0.03125in  , quarter-inch = 1/4 (1in) = 0.25in ");
```

```
            printf("\n1 half-inch = 1/2 (1in) = 0.5in ,  three-fourths inch = 3/4 (1in) = 0.75in");
            printf("\n1.234 meters = 1 meters + 23 centimeters + 4 millimeters");

            printf("\n");
            printf("\nEnter feet = ");   fflush(stdin);   scanf("%lf",&feet);
           if(feet < 0){  goto selection0; };  /* ie., if a negative value was entered */

         printf("Enter inches = ");   fflush(stdin);   scanf("%lf",&inches);  if(inches<0){ goto selection0; };
         printf("Enter eighths = ");    fflush(stdin);    scanf("%lf",&eighths);  if(eighths<0){ goto selection0;  };
         printf("Enter sixteenths = ");   fflush(stdin);   scanf("%lf",&sixteenths);  if(sixteenths<0){  goto selection0;  };
         printf("Enter thirtytwoseconds = ");   fflush(stdin);   scanf("%lf",&thirtytwoseconds);
                 if(thirtytwoseconds<0){ goto selection0;  };

         result = FeetToMeters(feet,inches,eighths,sixteenths,thirtytwoseconds);
            /* first,let's display the number of feet in strict decimal form:  */
         printf("\n%.14g feet = ",feet + (inches/12) + (eighths/96) + (sixteenths/192) + (thirtytwoseconds/384));
            printf("%.14g meters",result);

            printf("\n\nPress A Key To Continue With This Function or Press ESC for the MENU: ");
            fflush(stdin); ch=getch();  if(ch==27){ goto selection0;  };
      };


selection2:  ;

for(;;){                              /*---------------- main loop to use  MetersToFeet()  --------------------*/
            system("cls");
            fflush(stdin);

         printf("METERS TO FEET CONVERSION PROGRAM.  (c) 2021, JPA");

            result=0.0;  meters=0.0; centimeters=0.0; millimeters=0.0;
            f=0L; f2=0.0; inches=0.0;  ninches=0.0;  totalinches=0.0;

         printf("\n");

         printf("\nWith this program up to 3 units can be used: \nmeters, centimeters = (1/100)m = 0.01m , millimeters =
 (1/1000)m = 0.001m");
            printf("\n\nEnter a value of 0 if it is not used.");
            printf("\nUse a negative value such as -1 to return to the MENU.");

            printf("\n\nFor some reference: 1 ft = 0.3048m , 1in = 2.54cm , 1mile = 5280ft = 1609.34m = 1.60924 km");
            printf("\n1m = 39.37in = 3.28084ft  ,  1/8in = 0.125in  ,  1/16in = 0.0625in");
            printf("\n1/32in = 0.03125in  , quarter-inch = 1/4 (1in) = 0.25in ");
            printf("\n1 half-inch = 1/2 (1in) = 0.5in ,  three-fourths inch = 3/4 (1in) = 0.75in");
            printf("\n1.234 meters = 1 meters + 23 centimeters + 4 millimeters   or=   1 meter + 234 mm");

            printf("\n");
         printf("\nEnter meters = ");   fflush(stdin);   scanf("%lf",&meters);  if(meters < 0){  goto selection0; };
         printf("Enter centimeters = ");   fflush(stdin);   scanf("%lf",&centimeters);  if(centimeters<0){ goto selection0; };
         printf("Enter millimeters = ");   fflush(stdin);   scanf("%lf",&millimeters);    if(millimeters<0){ goto selection0; };

         result = MetersToFeet(meters, centimeters, millimeters);
            /* first,let's display the number of feet in strict decimal form:  */
```

```
            totalmeters = meters + (centimeters/100.0) + (millimeters/1000.0);    /* essentially converted each terms to
                                                                                     meter units */

            printf("\n%.14g meters = %.14g feet",totalmeters,result );

             f=(long int) result;   /* get the whole portion of the decimal value of feet  */
             f2 = result - f ;         /* get the fractional portion of the decimal value of feet */

            printf("\n\n%ld Feet + ",f);
           /*  printf("\nFraction of 1 foot = %.9g = : ",f2);  test ok */

            totalinches = f2 * 12.0;         /*  (1/12) = 0.0833333333333...   ,  note.  0.5ft x 12in/ft  = 6 inches   */
            ninches =  totalinches;
            printf("%.14g inches = ",totalinches);

            printf("\n%ld Feet + %ld inches + %.14g 16ths of an inch = ",
                f,(unsigned long)ninches,((double)ninches - ((long int)ninches)) / (1.0/16.0) );

           printf("\n\nPress A Key To Continue With This Function or ESC For The MENU: ");
           fflush(stdin); ch=getch();  if(ch==27){  goto selection0;  };
      };

return;
};
/*-------------------------------------------------------------------------------------------------------------------------------------*/
double FeetToMeters(double feet, double inches, double eighths, double sixteenths, double thirtytwoseconds )
{
/*  1ft = 12in =  0.3048m = 30.48cm   ,  1inch = 2.54cm =  25.4mm                                   */
/*  1in = 1ft / 12  =  0.083333333 ft                                                                */
/*  1eighth = 1in/8  =  0.125in   ,  1ft = (8/in x 12) = 96 eighths                                  */
/*  1sixteenth = 1in/16  =  0.0625in  =  (1in/8)/2   ,  1 ft = 192 sixteenths                        */
/*  1thirtytwoseconds = 1in/32  =  0.03125   ,  1ft =   384 thirtytwoseconds                        */

double result=0.0;

/* For simplicity, let's convert each argument to feet, and then multiply once by 0.3048 to
   convert the number of feet units to its equivalent value of  meter units.  */

result = 0.3048 * ( feet + (inches * 0.083333333333333) + (eighths/96.0) + (sixteenths/192.0) +
(thirtytwoseconds/384.0) ) ;

return result;
};
/*-------------------------------------------------------------------------------------------------------------------------------------*/
double MetersToFeet(double meters, double centimeters, double millimeters)
{
double result=0.0;

/* For simplicitry, let's convert each argument to meters, and then multiply by 3.28084 to
   convert the number of meter units to its equivalent value of feet units. */

result = (meters + (centimeters / 100.0) + (millimeters / 1000.0)) * 3.28084;
```

```
return result;
};
/*-----------------------------------------------------------------------------------------------------------------------------------------*/
```

Example Of Program Output:

```
THIS PROGRAM HAS TWO FUNCTIONS:

1. Feet To Meters Conversion
2. Meters to Feet Conversion

ESC To Exit Program

Enter Your Selection To Begin:


FEET TO METERS CONVERSION PROGRAM.  (c) 2021, JPA

With this program up to 5 units can be used:
feet, inches, eighths of 1in, sixteenths of 1in, and thirtytwoseconds of 1in

Enter a value of 0 if it is not used.
Use a negative value such as -1  to return to the MENU.

For some reference: 1 ft = 0.3048m , 1in = 2.54cm , 1mile = 5280ft = 1609.34m = 1.60924 km
1m = 39.37in = 3.28084ft  ,  1/8in = 0.125in  ,  1/16in = 0.0625in
1/32in = 0.03125in  , quarter-inch = 1/4 (1in) = 0.25in
1 half-inch = 1/2 (1in) = 0.5in ,  three-fourths inch = 3/4 (1in) = 0.75in
1.234 meters = 1 meters + 23 centimeters + 4 millimeters   or=   1 meter + 234 mm

Enter feet = 1
Enter inches = 1
Enter eighths = 1
Enter sixteenths = 1
Enter thirtytwoseconds = 1

1.1015625 feet = 0.33575625 meters

Press A Key To Continue With This Function or Press ESC for the MENU:


METERS TO FEET CONVERSION PROGRAM.  (c) 2021, JPA

With this program up to 3 units can be used:
meters, centimeters = (1/100)m = 0.01m , millimeters = (1/1000)m = 0.001m

Enter a value of 0 if it is not used.
Use a negative value such as -1 to return to the MENU.

For some reference: 1 ft = 0.3048m , 1in = 2.54cm , 1mile = 5280ft = 1609.34m
1m = 39.37in = 3.28084ft  ,  1/8in = 0.125in  ,  1/16in = 0.0625in
1/32in = 0.03125in  , quarter-inch = 1/4 (1in) = 0.25in
1 half-inch = 1/2 (1in) = 0.5in ,  three-fourths inch = 3/4 (1in) = 0.75in
1.234 meters = 1 meters + 23 centimeters + 4 millimeters
```

```
Enter meters = 1
Enter centimeters = 1
Enter millimeters = 1

1.011 meters = 3.31692924 feet

3 Feet + 3.80315088 inches =
3 Feet + 3 inches + 12.85041408  16ths of an inch =
3 Feet + 3 inches + 25.70082816  32nds of an inch

Press A Key To Continue With This Function or ESC For The MENU:
```

Also of note, for example, in the above program to convert feet to meters, you may also initially ask the user if they want to convert a decimal value of feet to meters, and-or if they want to convert the number of feet and-or fractional parts of feet to meters.

[This space for book edits.]

A COMPUTER PROGRAM WITH SOME HOMEMADE MATH FUNCTIONS

This computer program shows some examples of homemade math functions that can be used within any C program that needs them, and-or rewritten for other computer programming languages.   In particular, here they are some trigonometric functions. Also shown after this program, is how to create the header (.h) file with these functions so that it can later be included in any C program that needs them; such as the computer program shown here.   This header file is called HomemadeMathFunctions.h. To reduce program sizes, you may also place each pre-made function in its own header file, and you can then easily include that function(s) in your program(s) so as to make a shorter source code for editing.

```
/* ---------------------------------------------------------------------------------------------------
    myprogram.c

    This program is to test including a header file called:
     HomemadeMathFunctions.h that contains some homemade
     function prototypes and functions.  To reduce program size,
     it is a good idea to perhaps make each function into its own
     header file and include only that when it is needed by the
     program.

    A programming topic you may wish to research, is that of
    creating object files, (dynamic and static) linking, and libraries.

(c) JPA  ,  2021

---------------------------------------------------------------------------------------------------*/

#include "stdio.h"
#include "HomemadeMathFunctions.h"  /* : this header file is shown below  */

/*---------------------------------------------------------------------------------------------------*/
void main(void)
{
double angle=0.0;
double trigvalue=0.0;
unsigned char ch;

system("cls");

printf("\n1 rad is %.14g degrees.",RadiansToDegrees(1.0));
printf("\nThe arcsine of 0.7 is %.14g degrees.",dasin(0.7));
printf("\nThe tangent 45 degrees is %.14g.",dtan(45.0));
printf("\n45 degrees is %.14g radians",DegreesToRadians(45));
printf("\nThe arcsin of 0.5 is: %.14g degrees",dasin(0.5));
printf("\nThe arccosine of 0.6 is: %.14g degrees",dacos(0.6));
printf("\nThe arctangent of 1 is: %.14g degrees",datan(1.0));
printf("\n");
printf("\nPress A Key To Continue: "); fflush(stdin);  getch();

for(;;){
            system("cls");
            printf("\nEnter A Degrees Angle or 0 to exit:  ");
            scanf("%lf",&angle);
            if(angle==0){  break; };
            if( (angle <0) || (angle > 90)){
```

```
                         printf("\n\aPlease Use An Angle Between 0 and 90 degrees");
                         printf("\nPress A Key To Continue or ESC to exit");
                         fflush(stdin); ch=getch(); if(ch==27){ break; };
                         continue;
                        };

            printf("\nThe sine of %.14g degrees is %.14g",angle,dsin(angle));
            printf("\nThe cosine of %.14g degrees is %.14g",angle,dcos(angle));
            printf("\nThe tangent of %.14g degrees is %.14g",angle,dtan(angle));

           printf("\n");
           printf("\nPress A Key To Continue or ESC to exit: ");
           fflush(stdin);  ch=getch();  if(ch==27){  break; };
        };

return;
};
/*---------------------------------------------------------------------------------*/
```

/* Output of the above example function use program:

```
1 rad is 57.295779513082 degrees.
The arcsine of 0.7 is 44.427004000806 degrees.
The tangent 45 degrees is 1

Press A Key
---------------------------------------------------------------------------------*/
```

```
/* ------------------------------------------------------------------------------------------------------------------
   HomemadeMathFunctions.h

    Some examples of homemade math functions which can be used in any program that
    needs them.   You may also explore the programming topic of object files, libraries,
    and (static and dynamic) linking files.  This file is mostly about the code for some
    functions, and does not contain a main( ) function since it is already within the program
    you are creating.  This header file will be saved as a plain ASCII text file with a  .h filename
    extension so that you know what it is for.

   (c) JPA 2021
--------------------------------------------------------------------------------------------------------------------------*/

#include "math.h"  /* : included here incase it was not included elsewhere, and since these
                          homemade functions will (function) call and use the standard C math functions.  */

/* These conversions can also be done "inline" or in a function.  The ANSI C, standard math functions
   use and return radian angles, and these conversion functions are useful for easily using degree
   values which are often used and needed more commonly.
*/
                         /*  Function Prototypes, and often found in a header file, and they help describe to
                            the compiler and-or programmer what to expect and use for the argument data.  */

double RadiansToDegrees(double radians);       /* : converts a radian angle to a degrees angle */
double DegreesToRadians(double degrees);     /* : converts a degrees angle to a radian angle */
```

```
double dsin(double degreesangle);
                        /* : this function returns the sin of a degrees angle, rather than
                             the sin of a radian angle that is required for the standard
                             ANSI, C sin() */

double dcos(double degreesangle);  /* : returns the cos of a degrees angle */
double dtan(double degreesangle);  /* : returns the tan of a degrees angle  */


double dasin(double sin)    ;  /* : returns the degrees angle = arcsin (sin angle) for
                                    the sin of the angle */
double dacos(double cos);    /* : returns the degrees angle = arccos(cos angle)  */
double datan(double tan);     /* : returns the degrees angle = artcan(tan angle)    */

/*-----------------------------------------------------------------------------------------------------------------------------*/
double RadiansToDegrees(double radians)    /* convert radians to degrees , angle */
{

return radians *  57.29577951308231; /* constant is: 180 degrees divided by  (PI =~ 3.141592654)  */
};
/*-----------------------------------------------------------------------------------------------------------------------------*/
double DegreesToRadians(double degrees)    /* convert degrees to radians, angle */
{

return degrees *  0.0174532925199433;   /* constant is (pi)  divided by 180   */

};
/*-----------------------------------------------------------------------------------------------------------------------------*/
double dsin(double degreesangle)   /* returns the sine of a degrees angle value */
{
 /* Lets first convert the degrees angle argument to it's equivalent radian value that
    the sin() requires as an argument */

return sin(degreesangle * 0.0174532925199433);  /* 0.0174532925199433 =  (PI) / 180.0 */
                                                                       /* PI =~ 3.141592653589793  */
};
/*-----------------------------------------------------------------------------------------------------------------------------*/
double dasin(double sin)  /* returns the degrees angle = arcsin (sin angle) for the sin of the angle */
{
double angle=0.0;

angle=asin(sin);       /* C's asin() returns a radian angle */

/* lets convert this radian angle value returned by the std. math function, to it's corresponding
    degrees angle  */

return (angle * 57.29577951308231);                  /* about 57.29577951308232 degrees,
                                                                        per 1 rad.  */
                                                                     /* = 180.0 / (pi)= 3.141592653589793  */
};
/*-----------------------------------------------------------------------------------------------------------------------------*/
double dcos(double degreesangle)   /* returns the cosine of a degrees angle value */
{
 /* Lets first convert the degrees angle argument to it's equivalent radian value that
```

```
   the sin() requires as an argument */

return cos(degreesangle * 0.0174532925199433);  /* 0.0174532925199433 =  (PI) / 180.0) */
                                                 /* PI ~ 3.141592653589793  */
};
/*-------------------------------------------------------------------------------------------------------------------------*/
double dacos(double cos)  /* returns the degrees angle = arccos (cos angle) for the cos of the angle */
{
double angle=0.0;

angle=acos(cos);    /* C's acos() returns a radian angle */

/* lets convert this radian angle value returned by the std. math function, to it's corresponding
   degrees angle  */

return (angle * 57.29577951308231);                  /* about 57.29577951308232 degrees,
                                                         per 1 rad.  */
                                                      /* = 180.0 / (pi = 3.141592653589793)  */
};
/*-------------------------------------------------------------------------------------------------------------------------------*/
double dtan(double degreesangle)   /* returns the tangent of a degrees angle value */
{
 /* Lets first convert the degrees angle argument to it's equivalent radian value that
   the sin() requires as an argument */

return tan(degreesangle * 0.0174532925199433);  /* 0.0174532925199433 =  PI/180.0) */
                                                 /* PI ~ 3.141592653589793  */
};
/*-------------------------------------------------------------------------------------------------------------------------*/
double datan(double tan)  /* returns the degrees angle = arctan (tan angle) for the sin of the angle */
{
double angle=0.0;

angle=atan(tan);      /* C's atan() returns a radian angle */

/* lets convert this radian angle value returned by the std. math function, to it's corresponding
   degrees angle  */

return (angle * 57.29577951308231);                  /* about 57.29577951308232 degrees,
                                                         per 1 rad.  */
                                                      /* = 180.0 / (pi = 3.141592653589793)  */
};
/*-------------------------------------------------------------------------------------------------------------------------------*/
```

A COMPUTER PROGRAM TO CONVERT A TEXT STRING TO A LONG INTEGER

Below is a homemade string function that will convert a numeric value input as a readable text string of ASCII text characters and convert it to a numeric data type that can be used for a mathematical operation. The standard or an ANSI compatible C language already has some standard functions to do such conversions, but knowing how to make your own functions is very useful for a programmer.

```
/*----------------------------------------------------------------
stringtolongint.c

stringtolongint()  converts a text string to a
signed long integer numeric value. This function
is similar to the standard C function of atoi() and atol().
There are several ways a user could make such a
function, and the one shown below is one example.

A string null value or decimal point will signal the end
of the conversion.  Leading spaces are allowed in the
text string using scanf(), but other spaces signal
the end of the string input using scanf(). You can also
try gets() to allow spaces if needed.  You can also
use commas in the number, for example, -2,000

A more genereal text to numeric function would be
to write one that accepts a double float text input
and converts it to a double float, numeric data type.

So far, the program can display about:
+/- 2,000,000,000,  about  +/- 2 billion
The actual values are:
-2,147,483,648 to 2,147,483,647 for signed values and
unsigned long int, max.  is  4,294,967,295

(c) JA Oct. 2013, updated for the Mathization ebook on
                       Nov. 24, 2022
----------------------------------------------------------------*/

#define start main     /* an example of a compiler directive or
                          command, define is used to define a text
                          alia or substitution in the program */

#include "stdio.h"
#include "math.h"   /* used for sqrt() in the example usage program */

long int stringtolongint(char * string);
```

```
/* It is quite possible to use this function twice so as to construct a
    double float numeric data type.  Enter the string and break it into
    two parts, the whole part and the fractional part.  Send each string
    part to this similar function, say from within a function called
    stringtodouble(char * string) ,  but for the fractional part , the
    weights or powers of 10 will be divided by 10.  A double float
    variable being created will be the sum of both of the whole or
    integer part, and the fractional part of 1.
*/
/*-------------------------------------------------------------------------------------*/
void start(void)   /* example program to use the function */
{
char string[20];
long int n;

printf("\nEnter an integer value: ");
scanf("%s",string);

n=stringtolongint(string);
printf("\nThe integer part of the value you entered is: %ld",n);

printf("\nThe square root of %ld is %.15g",n,sqrt(n));

/* printf("\nOn this computer the size of a long integer data value is:  \n%d bytes",sizeof(long int)); /*
          /* usually 4 bytes , for testing */

printf("\n\nPress A Key");  fflush(stdin);  getch(); exit(0);

return;
};
/*----------------------------------------------------------------------------------------------*/
long int stringtolongint(char * string)
{
char ch;
long int n=0;
int sign=0;
char digits[10]={ 0,0,0,0,0,0,0,0,0,0,0 };  /* :This is actually a string, and with each byte initialized to 0 */
unsigned long int powerof10=1;
int l=0;

while(1)              /* Lets get each numeric digit into an array for processing. */
{
    ch=*string;     /* : Read the char or byte value at the address that string contains and points to. */
```

```
    if(ch==0){  break; };     /* null terminator */
    if(ch=='.'){  break; };     /* decimal point, or to skip over use:  string=string+1; continue  */
    if(ch=='-'){  sign=-1; };  /* negative sign */

    if( (ch>47) && (ch<58) ){    /*  If the text character is a numeric digit:
                                       0=ascii 48,  9=ascii 57  */
       digits[l]=ch-48;              /*  convert ascii letter to a number */
       l=l+1;                        /*  essentially the number of digits also */
      };

    if(l>9){  break; };     /* : for if there are 10 or more digits, and could also use return -1 here
                                 for an error signal indicating the entered value is to big. */
    string=string+1;      /* point to next character in the text string  or char = byte array */
};

    /* Create the decimal number value from the array of numeric digit values. */

   while(1)      /* : Any value other than 0 is considered a truth, and 1 is used here and is constant, and
                      will create an infinite loop until it is broken out of, such as with a break statement.  */
          {
            n=n+(digits[l-1] * powerof10);  /* create the positional sum ,
                                               used [l-1] since first digit is at offset 0 */
            l=l-1;   if(l==0){ break; };
            powerof10=powerof10*10;      /* create the decimal weight of the next digit leftward. */
          };                              /* In a float, the digits rightward of the decimal point
                                               have a digit weight value that is divided by 10. */

if(sign==-1){  n=-n; };    /* :For if the input was to be a negative number.  Can also use:   n  * (-1)   */

return n;;
}
/*-----------------------------------------------------------------------------------------*/
```

Here is an example of using the above program and-or function:

```
Enter an integer value: 100.234

The integer part of the value you entered is: 100
The square root of 100 is 10

Press A Key
```

COMPILING A C PROGRAM WITH A C-COMPILER APP AND-OR MAKING A PHONE APP

The following are some basic and helpful notes about compiling a C program on a phone system and-or making a phone app (ie., application, a [computer, phone program] as of March 2022. The operating system of the phone must be a version of the Android Operating System, and does not consider iOS (ie., the iphone operating system) and phones developed by the Apple Co. There are several C and-or C++ compilers available and they usually can be found on the Google Play ("App Store") website. The author will only discuss two such apps available that he has tried in a minimal manner, and for the following minimal discussion. Be sure to use an APK extractor so as to save any apps that you may need to have again if the phone is damaged and-or needs to be reset or "cleared" via a factory reset or software re-installation selection in the phones settings, and if so, any data stored on it will be erased, and therefore, you will first need to consider saving that data onto a computer, memory card, etc. As for the apps mentioned below, they may or may not still be available, but there is usually some similar apps available.

In your internet (ie., "web") browser such as the Chrome browser, search for Google Play and go to that website.

1. **For the Mobile-C Compiler App**

   This is a easy to use basic C compiler, and is similar to the TinyC compiler in terms of simplicity and speed, however Mobile-C does not make Apps. It will run a C source code program, and is great for learning and-or editing programs. There is a free version and a pay version available. There are other compilers available in the app menu, such as: C++, Javascript, Python and Lua.

   On the Google Play search bar or input, enter: Mobile-C
   Choose to install the Mobile-C App.
   Open the Mobile-C App.

   In the settings menu, there are several default compilers to use. New programmers should select the (basic) C compiler. This compiler seems to be a version of the TinyC compiler (TCC).

   Your C source files can be accessed from the Mobile-C folder (ie., directory) or the external storage directory which is often where most of the user files for the phone are placed. You can press on the filename to load (ie., move, place) that source file into the C program editor, or press Run so as to compile and run (ie, execute or perform the program code or instructions) the file without editing.
   If you choose to edit the file, there is a Run button in the editor.

   To allow the Mobile C app to access to the External Storage to the app, and which is the phones main storage location, directory or folder, select your phone settings, and then select the Mobile-C app, and in its permissions, allow access to the phones storage.

   This app can also edit a .txt (ie. a text) file much like editing a .c file. This app can also view .jpg images when you press on its filename.

   The reader may also read the C4droid section below for some relevant information.

2. **For the C4droid Compiler and App maker App**

   This is an easy to use App that can edit a C source code files and also make an APK file which can then be installed as an runable or "executable" App on your phone.

   Press the Google Play Store App/Link on phone. On the Google Play Store search bar or input, enter: C4droid (ie., C For the Android operating system on the phone). Choose to install the C4droid App. The price of it was about $4USD as of March 2022, and is a good deal. This can be paid for using a Google Pay account having your bank account and debit card assigned to it. Open the C4droid App. You may also want to download and-or install from the Google Play Store (ie. Google's certified apps website location) the free GCC, and SDL (Simple Direct media

Layer Software , and is sometimes called a Software or Graphics Defined Library). SDL contains premade graphics functions for a certain electronic device such as a computer or smartphone. SDL is used as a "plugin" or addition to the C4droid compiler, and so your programs can call these functions(). It is possible that OpenGL might be a fancier option to the more basic SDL, but you will have to check with this.

With this app, there are several compilers available in the Menu. Select Compiler option, such as: TCC (TinyC compiler), or GCC (such as for some graphics with the SDL plugin, and for C++), or the G++ compiler to compile C++ source code programs. For some of these to function, you may need to install GCC and SDL form the Google Play Store if they were not already included with the C4droid App. In the Preference menu menu, you may also select the "Close shell after program exit" option. Shell is essentially the command-line and program input/output environment, and is sometimes called the "terminal". If you are programming using only the ASCI C language, then select the TCC compiler in the compiler preferences.

When you compile and run the Your C source files or accessing other files with C4Droid, you make will usually be found and accessed in the device (ie., such as a phone) memory storage which for an android phone is usually the /storage/emulated/0/ folder or directory. This directory may be sometimes noted as being "external storage" on some apps. A more complete file path of your user stored area is: file:///storage/emulated/0/Download/ of which can even be entered on your internet browser (such as Google Chrome) so as to view and-or access the files in your memory. On some Android system phones using Google Chrome: file:///sdcard/ can be used to view the files in the phone storage. When accessing a file using a C4droid program, note that scanf() will stop the string input at the first space, and so you may rather need to use gets().

The Android system comes with an app to display the files in memory storage: Settings App ---> Storage ---> Files . Google Files is another app made by the Google company, and which can be used to view what files are in memory. There are various ways to save your files on your phone to an external device(s) for file safety, backup and storage: file share via the internal Bluetooth (radio, communication, transmitter) system, USB cable to a computer, between two phones, phone to USB data storage. Some may require specialized cables for data transfer, such as for computer to computer and you will have to research this. One such cable is called an On-The-Go (OTG) cable such as for saving files directly to a USB data storage device or another phone. To save a file on your phone to a computer drive (including any attached USB drive) plug the phones USB charging cable into the computer USB port. In the phones settings/notifications select FILE TRANSFER mode. On your computer, you should now see the phone installed as like a storage drive to access, then locate the file on your phone and copy it and then paste it where you want it to go.

I set the "Buttons" location to "move all to the menu" (in the upper right of the editing screen). Select the Open menu button to access the files and to click on it to load it into the C editor/compiler. When done entering a C source file, it is good to Save it, and then you can then compiler it to check for errors. If there are no errors to fix, then you can Run and test it.

If you are satisfied with the C program, you can then choose Export so as to make an **APK** file from your compiled C program. Most of the default data there can be used, but it is best to enter a application title (ie., name) so as you know what it is about and does for when the memory about the specifics of and making it are slowly forgotten, and-or you share your app with others to use. The name of the APK file will still be that of the C source file, except that it will have the .APK filename extension. The APK file created will be placed in:

Ex: Filename: /storage/emulated/0/myprogram.apk

Note that on an "Android phone": /storage/emulated/0/ can be thought of as your phone's main directory or folder of the phones internal storage. A file may be for ex. located at: /storage/emulated/0/myfolder/myfile

Package name: com.myprogram : This associated, app data folder will be located in the directory: /storage/emulated/0/android/data

Application title: MyProgram

The APK file created will be placed in the same directory mentioned. You can press or click on this APK file so as to install it. You will get a warning about installing unknown (here, "homemade") Apps, and not from the popular App sites such as Google Play Store which tries to verify all apps that they have available as being safe for the device and free of bad things like malware. You may also be asked about the App not requiring or not-requiring any special (phone, system) access such as file access permission. Install the App and, if no special device access such as files for media and-or photos is needed, then select "Deny" as a safety measure. The created app will usually be found in your main App folder, and you can then press and hold on its icon and then "drag and drop" it to one of your phone's screen pages for easy access. The APK file that was made by the C4Droid compiler can be shared, traded and-or possibly sold. C4Droid will access files in the main internal storage memory > android > .com.app_name > files directory. You can select and copy a file into this directory.

Here is a basic example of how to access a file called myfile in the main Internal Storage directory or folder of the phone:

```
FILE *fp;       /* Creating a pointer to a file */
.
.
fp=fopen("/storage/emulated/0/myfile.txt,"rb");      /* Setting the file pointer identified as fp to a certain file. */
if(fp==NULL){  printf("\nFile Error , Press A Key");  fflush(stdout);
                fflush(stdin);  getch();  exit(0);
               }
```

Often by pressing on a file and selecting "File Info" from the menu, you can see what directory it is in, for example:

The phones main storage folder is:	/storage/emulated/0/
Camera photos are often in this folder:	/storage/emulated/0/DCIM/Camera/
Screenshots are often in this folder:	/storage/emulated/0/Pictures/Screenshots/
Downloads are often in this folder:	/storage/emulated/0/download/

Often a phone will come with a Google App called Files By Google, and this can be used to view the files in your phones internal storage. Most phones using the Android operating system also have a method to access your phones internal files, and that is to select: Settings menu or button > Storage > Files , and you can select to display it with the Android's Files App. Note that a "folder" is analogous to the older terminology of a "file directory" or location.

When running your app program, you can select Menu , Preferences, and (Shell) Terminal type. For simplicity, select either the "screen" or "screen-256color". The default is usually Linux. You may also wish to select the Close Window On Exit, and this will fully terminate the app.

There could be issues with some functions, etc, such as with the system() function. For example to clear the screen: system("cls") might not work in the resulting exportedl app. An alternative is to try using just this: clrscr(). Check the menu options and-or uses GCC instead of TCC. You could always print several blank lines to effectually or crudely clear the screen. Be sure to include all the header files needed by the functions in the program. Sometimes there are known solutions to try for certain issues, such as using fflush(0), for to process and clear all buffers, or fflush(1) instead of fflush(stdin) of which can be used before a user input such as: printf("\nPress A Key"); fflush(0); getch(). Also consider using fflush(stdout) so as to be sure text is displayed on the screen. Avoid using the ASCII bell sound character, such as in printf("\a"); ,because it may not work and-or cause issues with the Android phone operating system and cause the program to crash. Instead of using something like #include "stdio.h", rather use: #include <stdio.h> . Seek assistance from people and-or websites more knowledgeable if you need more help. Often, a solution is relatively simple.

The c4droid compiler wants the main() to have an integer return value, rather than void. So use something such as: int main(void) , and at the end of the program use something like: return 0 , or some other integer value.

There are special apps to create APK files from a list of the applications already installed on your device such as a phone. These files can be used for backup reinstalling, and-or for sharing when needed. This type of special app is called an APK extractor, and there are several available from the Google Play App Store. When you share an app, or need to reinstall it, you will need the APK file for that defines a specific App, and then you can install that App by pressing on the APK filename. You phone will process the .apk file in a program called a process installer.

Note that an .exe formatted program for a pc (personal computer, having the Intel Co. type of processor) will not work on another system such as the Android operating system for some phones. It is possible to obtain a third-party app that can simulate a pc, DOS system and run some .exe programs. Look up or search for DOS simulator apps. Likewise, some pc programs for the Windows operating system may be able to simulate the Android phone system and so as to create/edit phone apps. I, the author of this book have recognized C++ as being a superior programming language to (plan, basic) the C language in terms of capability, however, its syntax seems awkward, but there are also many similarities also. If you are learning and-or building simple programs, then knowing C++ is generally not needed, and may in fact complicate the issue of learning a practical programming language by its greater capabilities and-or complexities. **I recommend a new programmer begin with the C language.**

First, learn a computer language such as C and-or C++ before trying to make programs for the "Windows" or Android (ie, phone) operating system which have an increased complexity to learn and use for making even simple programs and-or apps, however, those do make a fuller use of those more robust, modern systems and to have the benefits and practicality of them such as graphics, mouse, sound and on-screen buttons ability for your programs and apps, and lets the users interact (ie., inputting, controlling) with them. For most people, just knowing C programming (ie., its syntax and structures) fairly well and being able to create some simple computer programs and-or phone apps is practical and enough for them.

The Android operating system began its creation in about 2003. It is based on the Linux kernal (core, main) operating system that was designed in about 1983 to be a new alternative to the Unix operating system, and so as to be used for personal computers (PC). By 2007, Android was being used in ("smart", ie., computer, mobile, "cell") phones, and these versions are said as being owned by Google Inc. of which currently owns the YouTube .com website. As of 2021, Android 12 is the latest version of the Android operating system. Some non-proprietary portions of the Linux and Android operating systems software are said as being "free, and open source (ie., publicly accessible)" software, and which can potentially be used and-or modified as needed.

Because programming a machine such as a computer to do useful and-or fun things is imaginative and-or beneficial, it can become not only a habit, but an addiction of which is not needed , and it may interfere with your health, your well-being, your friends, family and relationships. Much thinking can be stressful, and you need to be aware so as to be in a preventative state. Many get addicted to endless cycles of learning things and without actually having a goal of doing something with what they learned, and that is to create some useful programs for others, and rather than "figure-out" every possible detail of a computer system(s) and which is generally unrelated to most programs. An example of this would be a carpenter or auto mechanic who endlessly studies tools, parts and their design, practical tricks and how be a master at multiple choice exams, and yet never fixes or makes anything useful for others. Consider how much a person studies to learn, and then doesn't or can't even remember much of it, and to then apply it well. "Keep things simple" is a helpful concept wanted by most users of programs who just want them to be simple to learn and use, of which is often spoken as being "user friendly" and rather than full of a programmer's most complex and artistic programming skills possible.

Many computers, even the ones that have the Windows operating system, still have a "DOS mode" of programming where it may be called a "shell mode" and-or a "terminal (input, output, system accessing) mode" where commands such as: DIR and COPY, are typed and entered on the "command line" shown on the screen. There are several compilers still available so as to make computer programs on and for the older "DOS system" [technically a program in itself, like the Windows system is] computers. These compilers might be for example, C language compilers, and which the non-ANSI C ones have more functions to access the more advanced, modern computer systems. These functions can be used to access many of its system hardware and-or functions such as graphics, video, mouse input, and sound output, and without any Windows programming involved.   These programs you create may be called "DOS" or "terminal" programs.   Many of these older types of computers and-or systems also came with the BASIC language installed and-or on disk.   Today, we can download a BASIC language compiler so as to still use BASIC.   I believe the C computer programming language was created to overcome some limitations of the BASIC computer language which itself was a giant leap from the more difficult to read, edit, and-or program machine or assembly programming languages.   The DOS (Disk Operating [ie., accessing and control] System is user accessed by typing in special DOS commands which are technically a type of (system command and control) programming language or tool.   A BATCHFILE is simply a plain text file, much like a C language program, but with the .bat filename extension, and is essentially a list of DOS commands which may include some decision ability such as the Basic-like IF and GOTO commands. The commands are generally processed from the top of the file to the bottom, one by one just like regular computer programs. Modern forms of DOS seem to have less commands available to the user.   A batchfile is a program of DOS or system commands. To see a list of the DOS commands, enter: HELP on the command line.   To see the specific HELP data about a particular command, enter the commands name followed by a space character and-or then: /?

For much information about programming a device with the Android operating system, please visit: https://developer.android.com   There is also a link to this website and it is given on a page that is called something like: "Browse All Of Googles Produces And Services", and it was currently: https://about.google/products   , and then select the menu selection called: Android OS (ie., Android Operating System). There are also many websites and-or webpages throughout the internet that offer many tips and-or answered questions about programming any well known language and-or program.   Some other search words for Android programming are: Android App Development Kit, Android SDK (Software Development Kit) and Android Studio. There are also various websites that have associated programmers who can make apps for you for a fee, and just let them know what you are considering and negotiate a fee. You may need to first copyright your idea(s) or be able to somehow verify you were its creator, perhaps in a previous email to yourself and-or to a friend to save, or perhaps a published (ie., publicized, offered to the general public) video, and before letting others utilize them for their own control and benefit.   Even if you let others use your ideas or app and-or a version of it freely, you still need to have some type of copyright or verification so as others will not claim and-or independently create the same general type of idea, and then they may say that you and-others using it can no longer use their idea or need permission and-or a fee to use it.

After making the C source code for a program with C4Droid, I have noticed that this file was not displayed right in Windows, Notepad program. In particular, it was missing some new lines and was seen as jumbled together.   **Windows Notepad** program is supposed to be a pure/only ASCII text editor. I put this .c source code into the DOS EDIT.com text editor and it was displayed right, and could then be saved and then displayed in Windows Notepad. An option to correct the .c source code for proper display in Notepad is to make a copy of the file using a homemade program as shown in this book, and when a byte value of 10 (ASCII for LF = Line Feed) is read, then first write a byte value of 13 (ASCII for CR = Carriage Return), and then write the byte value of 10. In short byte 10 will then effectively become a byte 13 and then followed by 10. The overall resulting sequence then becomes CR, LF.

[This space for edits.]

TABLE OF TRIGONOMETRIC VALUES FOR ANGLES FROM 0 TO 90 DEGREES

Deg	Rad	Sin	Cos	Tan
0.000	**0**	**0**	**1**	**0**
0.001	1.745329251994e-005	1.745329251906e-005	0.9999999998477	1.745329252172e-005
0.002	3.490658503989e-005	3.49065850328e-005	0.9999999993908	3.490658505406e-005
0.003	5.235987755983e-005	5.235987753591e-005	0.9999999986292	5.235987760768e-005
0.004	6.981317007977e-005	6.981317002306e-005	0.9999999975631	6.981317019319e-005
0.005	8.726646259972e-005	8.726646248895e-005	0.9999999961923	8.726646282124e-005
0.006	0.0001047197551197	0.0001047197549283	0.9999999945169	0.0001047197555025
0.007	0.0001221730476396	0.0001221730473357	0.9999999925369	0.0001221730482475
0.008	0.0001396263401595	0.0001396263397059	0.9999999902522	0.0001396263410669
0.009	0.0001570796326795	0.0001570796320335	0.999999987663	0.0001570796339714
0.010	0.0001745329251994	0.0001745329243133	0.9999999847691	0.0001745329269716
0.011	0.0001919862177194	0.00019198621654	0.9999999815706	0.0001919862200782
0.012	0.0002094395102393	0.0002094395087081	0.9999999780675	0.0002094395133017
0.013	0.0002268928027593	0.0002268928008125	0.9999999742598	0.0002268928066528
0.014	0.0002443460952792	0.0002443460928478	0.9999999701475	0.0002443461001421
0.015	0.0002617993877991	0.0002617993848086	0.9999999657305	0.0002617993937803
0.016	0.0002792526803191	0.0002792526766896	0.999999961009	0.000279252687578
0.017	0.000296705972839	0.0002967059684856	0.9999999559828	0.0002967059815458
0.018	0.000314159265359	0.0003141592601913	0.999999950652	0.0003141592756944
0.019	0.0003316125578789	0.0003316125518012	0.9999999450166	0.0003316125700344
0.020	0.0003490658503989	0.0003490658433101	0.9999999390765	0.0003490658645764
0.021	0.0003665191429188	0.0003665191347127	0.9999999328319	0.0003665191593311
0.022	0.0003839724354388	0.0003839724260036	0.9999999262826	0.0003839724543091
0.023	0.0004014257279587	0.0004014257171776	0.9999999194287	0.000401425749521
0.024	0.0004188790204786	0.0004188790082292	0.9999999122702	0.0004188790449774
0.025	0.0004363323129986	0.0004363322991533	0.9999999048071	0.0004363323406891
0.026	0.0004537856055185	0.0004537855899445	0.9999998970393	0.0004537856366666
0.027	0.0004712388980385	0.0004712388805974	0.999999888967	0.0004712389329205
0.028	0.0004886921905584	0.0004886921711068	0.99999988059	0.0004886922294616
0.029	0.0005061454830784	0.0005061454614674	0.9999998719084	0.0005061455263004
0.030	0.0005235987755983	0.0005235987516737	0.9999998629222	0.0005235988234475
0.031	0.0005410520681182	0.0005410520417206	0.9999998536313	0.0005410521209136
0.032	0.0005585053606382	0.0005585053316026	0.9999998440359	0.0005585054187094
0.033	0.0005759586531581	0.0005759586213145	0.9999998341358	0.0005759587168454
0.034	0.0005934119456781	0.0005934119108509	0.9999998239311	0.0005934120153323
0.035	0.000610865238198	0.0006108652002066	0.9999998134218	0.0006108653141808
0.036	0.000628318530718	0.0006283184893763	0.9999998026079	0.0006283186134014
0.037	0.0006457718232379	0.0006457717783545	0.9999997914894	0.0006457719130048
0.038	0.0006632251157578	0.000663225067136	0.9999997800662	0.0006632252130016
0.039	0.0006806784082778	0.0006806783557155	0.9999997683385	0.0006806785134025
0.040	0.0006981317007977	0.0006981316440876	0.9999997563061	0.0006981318142181
0.041	0.0007155849933177	0.000715584932247	0.9999997439691	0.000715585115459
0.042	0.0007330382858376	0.0007330382201885	0.9999997313274	0.0007330384171358
0.043	0.0007504915783576	0.0007504915079067	0.9999997183812	0.0007504917192593
0.044	0.0007679448708775	0.0007679447953963	0.9999997051304	0.00076794502184
0.045	0.0007853981633974	0.0007853980826519	0.9999996915749	0.0007853983248885
0.046	0.0008028514559174	0.0008028513696683	0.9999996777148	0.0008028516284155
0.047	0.0008203047484373	0.0008203046564402	0.9999996635501	0.0008203049324317
0.048	0.0008377580409573	0.0008377579429621	0.9999996490808	0.0008377582369476

0.049	0.0008552113334772	0.0008552112292289	0.9999996343068	0.0008552115419739
0.050	0.0008726646259972	0.0008726645152352	0.9999996192282	0.0008726648475213
0.051	0.0008901179185171	0.0008901178009756	0.9999996038451	0.0008901181536003
0.052	0.0009075712110371	0.0009075710864448	0.9999995881573	0.0009075714602216
0.053	0.000925024503557	0.0009250243716377	0.9999995721649	0.0009250247673958
0.054	0.0009424777960769	0.0009424776565487	0.9999995558678	0.0009424780751335
0.055	0.0009599310885969	0.0009599309411726	0.9999995392662	0.0009599313834455
0.056	0.0009773843811168	0.0009773842255042	0.9999995223599	0.0009773846923423
0.057	0.0009948376736368	0.000994837509538	0.999999505149	0.0009948380018345
0.058	0.001012290966157	0.001012290793269	0.9999994876335	0.001012291311933
0.059	0.001029744258677	0.001029744076691	0.9999994698134	0.001029744622648
0.060	0.001047197551197	0.0010471973598	0.9999994516887	0.00104719793399
0.061	0.001064650843717	0.00106465064259	0.9999994332593	0.001064651245971
0.062	0.001082104136236	0.001082103925055	0.9999994145254	0.0010821045586
0.063	0.001099557428756	0.001099557207191	0.9999993954868	0.001099557871888
0.064	0.001117010721276	0.001117010488992	0.9999993761436	0.001117011185846
0.065	0.001134464013796	0.001134463770452	0.9999993564958	0.001134464500485
0.066	0.001151917306316	0.001151917051567	0.9999993365433	0.001151917815815
0.067	0.001169370598836	0.001169370332331	0.9999993162863	0.001169371131846
0.068	0.001186823891356	0.001186823612739	0.9999992957246	0.00118682444859
0.069	0.001204277183876	0.001204276892786	0.9999992748583	0.001204277766058
0.070	0.001221730476396	0.001221730172465	0.9999992536874	0.001221731084258
0.071	0.001239183768916	0.001239183451772	0.9999992322119	0.001239184403203
0.072	0.001256637061436	0.001256636730702	0.9999992104318	0.001256637722904
0.073	0.001274090353956	0.001274090009249	0.999999188347	0.001274091043369
0.074	0.001291543646476	0.001291543287408	0.9999991659576	0.001291544364611
0.075	0.001308996938996	0.001308996565174	0.9999991432636	0.00130899768664
0.076	0.001326450231516	0.001326449842541	0.999999120265	0.001326451009466
0.077	0.001343903524036	0.001343903119504	0.9999990969618	0.0013439043331
0.078	0.001361356816556	0.001361356396057	0.999999073354	0.001361357657554
0.079	0.001378810109076	0.001378809672196	0.9999990494415	0.001378810982836
0.080	0.001396263401595	0.001396262947914	0.9999990252244	0.001396264308959
0.081	0.001413716694115	0.001413716223208	0.9999990007027	0.001413717635932
0.082	0.001431169986635	0.00143116949807	0.9999989758764	0.001431170963766
0.083	0.001448623279155	0.001448622772497	0.9999989507455	0.001448624292473
0.084	0.001466076571675	0.001466076046483	0.9999989253099	0.001466077622062
0.085	0.001483529864195	0.001483529320021	0.9999988995698	0.001483530952544
0.086	0.001500983156715	0.001500982593108	0.999998873525	0.00150098428393
0.087	0.001518436449235	0.001518435865738	0.9999988471756	0.00151843761623
0.088	0.001535889741755	0.001535889137905	0.9999988205216	0.001535890949456
0.089	0.001553343034275	0.001553342409605	0.999998793563	0.001553344283617
0.090	0.001570796326795	0.001570795680831	0.9999987662997	0.001570797618724
0.091	0.001588249619315	0.001588248951579	0.9999987387318	0.001588250954789
0.092	0.001605702911835	0.001605702221842	0.9999987108594	0.001605704291821
0.093	0.001623156204355	0.001623155491617	0.9999986826823	0.001623157629832
0.094	0.001640609496875	0.001640608760897	0.9999986542005	0.001640610968831
0.095	0.001658062789395	0.001658062029678	0.9999986254142	0.00165806430883
0.096	0.001675516081915	0.001675515297953	0.9999985963233	0.001675517649839
0.097	0.001692969374435	0.001692968565719	0.9999985669277	0.001692970991868
0.098	0.001710422666954	0.001710421832968	0.9999985372275	0.00171042433493
0.099	0.001727875959474	0.001727875099696	0.9999985072227	0.001727877679033

0.10	0.001745329251994	0.001745328365898	0.9999984769133	0.001745331024189
0.11	0.001919862177194	0.0019198609978	0.9999981570652	0.001919864535985
0.12	0.002094395102393	0.002094393571219	0.9999978067554	0.002094398164747
0.13	0.002268928027593	0.00226892608084	0.9999974259839	0.002268931921107
0.14	0.002443460952792	0.002443458521345	0.9999970147508	0.002443465815699
0.15	0.002617993877991	0.002617990887418	0.999996573056	0.002617999859157
0.16	0.002792526803191	0.002792523173742	0.9999961008996	0.002792534062113
0.17	0.00296705972839	0.002967055375002	0.9999955982815	0.002967068435202
0.18	0.00314159265359	0.00314158748588	0.9999950652019	0.003141602989056
0.19	0.003316125578789	0.003316119501059	0.9999945016606	0.00331613773431
0.20	0.003490658503989	0.003490651415224	0.9999939076578	0.003490672681596
0.21	0.003665191429188	0.003665183223057	0.9999932831934	0.003665207841549
0.22	0.003839724354388	0.003839714919243	0.9999926282675	0.003839743224803
0.23	0.004014257279587	0.004014246498464	0.9999919428801	0.00401427884199
0.24	0.004188790204786	0.004188777955404	0.9999912270311	0.004188814703745
0.25	0.004363323129986	0.004363309284747	0.9999904807207	0.004363350820702
0.26	0.004537856055185	0.004537840481175	0.9999897039489	0.004537887203494
0.27	0.004712388980385	0.004712371539373	0.9999888967156	0.004712423862756
0.28	0.004886921905584	0.004886902454025	0.9999880590209	0.004886960809121
0.29	0.005061454830784	0.005061433219812	0.9999871908648	0.005061498053225
0.30	0.005235987755983	0.00523596383142	0.9999862922474	0.0052360356057
0.31	0.005410520681182	0.00541049428353	0.9999853631687	0.005410573477182
0.32	0.005585053606382	0.005585024570828	0.9999844036286	0.005585111678304
0.33	0.005759586531581	0.005759554687997	0.9999834136273	0.005759650219701
0.34	0.005934119456781	0.005934084629719	0.9999823931648	0.005934189112008
0.35	0.00610865238198	0.006108614390678	0.9999813422411	0.00610872836586
0.36	0.00628318530718	0.006283143965559	0.9999802608561	0.00628326799189
0.37	0.006457718232379	0.006457673349044	0.9999791490101	0.006457808000734
0.38	0.006632251157578	0.006632202535817	0.9999780067029	0.006632348403026
0.39	0.006806784082778	0.006806731520562	0.9999768339347	0.006806889209402
0.40	0.006981317007977	0.006981260297962	0.9999756307054	0.006981430430496
0.41	0.007155849933177	0.0071557888627	0.9999743970151	0.007155972076945
0.42	0.007330382858376	0.007330317209461	0.9999731328639	0.007330514159382
0.43	0.007504915783576	0.007504845332927	0.9999718382517	0.007505056688444
0.44	0.007679448708775	0.007679373227783	0.9999705131787	0.007679599674766
0.45	0.007853981633974	0.007853900888711	0.9999691576448	0.007854143128984
0.46	0.008028514559174	0.008028428310396	0.9999677716501	0.008028687061732
0.47	0.008203047484373	0.008202955487522	0.9999663551946	0.008203231483648
0.48	0.008377580409573	0.008377482414771	0.9999649082785	0.008377776405367
0.49	0.008552113334772	0.008552009086827	0.9999634309016	0.008552321837525
0.50	0.008726646259972	0.008726535498374	0.9999619230642	0.008726867790759
0.51	0.008901179185171	0.008901061644096	0.9999603847661	0.008901414275704
0.52	0.009075712110371	0.009075587518675	0.9999588160075	0.009075961302997
0.53	0.00925024503557	0.009250113116797	0.9999572167885	0.009250508883275
0.54	0.009424777960769	0.009424638433144	0.9999555871089	0.009425057027175
0.55	0.009599310885969	0.0095991634624	0.999953926969	0.009599605745333
0.56	0.009773843811168	0.009773688199249	0.9999522363688	0.009774155048386
0.57	0.009948376736368	0.009948212638374	0.9999505153083	0.009948704946972
0.58	0.01012290966157	0.01012273677446	0.9999487637875	0.01012325545173
0.59	0.01029744258677	0.01029726060219	0.9999469818066	0.01029780657329
0.60	0.01047197551197	0.01047178411625	0.9999451693655	0.0104723583223
0.61	0.01064650843717	0.01064630731131	0.9999433264644	0.01064691070939

0.62	0.01082104136236	0.01082083018208	0.9999414531032	0.0108214637452
0.63	0.01099557428756	0.01099535272322	0.9999395492821	0.01099601744037
0.64	0.01117010721276	0.01116987492942	0.9999376150011	0.01117057180553
0.65	0.01134464013796	0.01134439679537	0.9999356502602	0.01134512685133
0.66	0.01151917306316	0.01151891831575	0.9999336550596	0.01151968258841
0.67	0.01169370598836	0.01169343948525	0.9999316293992	0.01169423902739
0.68	0.01186823891356	0.01186796029854	0.9999295732792	0.01186879617893
0.69	0.01204277183876	0.01204248075031	0.9999274866996	0.01204335405366
0.70	0.01221730476396	0.01221700083525	0.9999253696605	0.01221791266222
0.71	0.01239183768916	0.01239152054803	0.9999232221618	0.01239247201524
0.72	0.01256637061436	0.01256603988335	0.9999210442038	0.01256703212338
0.73	0.01274090353956	0.01274055883589	0.9999188357865	0.01274159299726
0.74	0.01291543646476	0.01291507740032	0.9999165969098	0.01291615464753
0.75	0.01308996938996	0.01308959557134	0.999914327574	0.01309071708484
0.76	0.01326450231516	0.01326411334363	0.999912027779	0.0132652803198
0.77	0.01343903524036	0.01343863071187	0.999909697525	0.01343984436308
0.78	0.01361356816556	0.01361314767075	0.999907336812	0.01361440922531
0.79	0.01378810109076	0.01378766421495	0.9999049456401	0.01378897491713
0.80	0.01396263401595	0.01396218033915	0.9999025240093	0.01396354144918
0.81	0.01413716694115	0.01413669603803	0.9999000719198	0.01413810883211
0.82	0.01431169986635	0.01431121130629	0.9998975893715	0.01431267707655
0.83	0.01448623279155	0.01448572613861	0.9998950763646	0.01448724619314
0.84	0.01466076571675	0.01466024052966	0.9998925328992	0.01466181619254
0.85	0.01483529864195	0.01483475447414	0.9998899589753	0.01483638708538
0.86	0.01500983156715	0.01500926796672	0.9998873545931	0.0150109588823
0.87	0.01518436449235	0.0151837810021	0.9998847197525	0.01518553159394
0.88	0.01535889741755	0.01535829357495	0.9998820544537	0.01536010523096
0.89	0.01553343034275	0.01553280567996	0.9998793586967	0.01553467980398
0.90	0.01570796326795	0.01570731731182	0.9998766324817	0.01570925532366
0.91	0.01588249619315	0.0158818284652	0.9998738758086	0.01588383180065
0.92	0.01605702911835	0.0160563391348	0.9998710886777	0.01605840924557
0.93	0.01623156204355	0.01623084931529	0.999868271089	0.01623298766908
0.94	0.01640609496875	0.01640535900136	0.9998654230425	0.01640756708182
0.95	0.01658062789395	0.0165798681877	0.9998625445384	0.01658214749444
0.96	0.01675516081915	0.01675437686898	0.9998596355768	0.01675672891757
0.97	0.01692969374435	0.0169288850399	0.9998566961577	0.01693131136187
0.98	0.01710422666954	0.01710339269513	0.9998537262812	0.01710589483798
0.99	0.01727875959474	0.01727789982936	0.9998507259474	0.01728047935655
1.0	0.01745329251994	0.01745240643728	0.9998476951564	0.01745506492822
1.1	0.01919862177194	0.01919744239969	0.9998157121216	0.01920098090772
1.2	0.02094395102393	0.02094241988336	0.9997806834748	0.02094701390966
1.3	0.02268928027593	0.02268733357278	0.9997426093227	0.02269317458436
1.4	0.02443460952792	0.02443217815265	0.9997014897812	0.02443947358526
1.5	0.02617993877991	0.02617694830787	0.9996573249756	0.02618592156919
1.6	0.02792526803191	0.02792163872357	0.9996101150404	0.02793252919659
1.7	0.0296705972839	0.02966624408511	0.9995598601194	0.02967930713181
1.8	0.0314159265359	0.03141075907813	0.9995065603657	0.03142626604335
1.9	0.03316125578789	0.03315517838853	0.9994502159418	0.03317341660413
2.0	0.03490658503989	0.0348994967025	0.9993908270191	0.03492076949175
2.1	0.03665191429188	0.03664370870656	0.9993283937787	0.03666833538873
2.2	0.03839724354388	0.03838780908752	0.9992629164106	0.03841612498281
2.3	0.04014257279587	0.04013179253256	0.9991943951144	0.04016414896719

2.4	0.04188790204786	0.0418756537292	0.9991228300989	0.04191241804079
2.5	0.04363323129986	0.04361938736534	0.9990482215819	0.04366094290851
2.6	0.04537856055185	0.04536298812925	0.9989705697907	0.04540973428152
2.7	0.04712388980385	0.04710645070964	0.998889874962	0.04715880287748
2.8	0.04886921905584	0.04884976979561	0.9988061373414	0.04890815942085
2.9	0.05061454830784	0.05059294007671	0.9987193571842	0.0506578146431
3.0	0.05235987755983	0.05233595624294	0.9986295347546	0.05240777928304
3.1	0.05410520681182	0.05407881298478	0.9985366703262	0.05415806408703
3.2	0.05585053606382	0.05582150499316	0.998440764182	0.05590867980926
3.3	0.05759586531581	0.05756402695957	0.998341816614	0.05765963721203
3.4	0.05934119456781	0.05930637357596	0.9982398279238	0.05941094706601
3.5	0.0610865238198	0.06104853953486	0.9981347984219	0.06116262015048
3.6	0.0628318530718	0.06279051952931	0.9980267284283	0.06291466725365
3.7	0.06457718232379	0.06453230825296	0.9979156182722	0.06466709917287
3.8	0.06632251157578	0.0662739004	0.997801468292	0.06641992671492
3.9	0.06806784082778	0.06801529066525	0.9976842788356	0.06817316069632
4.0	0.06981317007977	0.06975647374413	0.9975640502598	0.06992681194351
4.1	0.07155849933177	0.07149744433269	0.9974407829309	0.07168089129321
4.2	0.07330382858376	0.07323819712763	0.9973144772245	0.07343540959262
4.3	0.07504915783576	0.07497872682633	0.9971851335251	0.07519037769975
4.4	0.07679448708775	0.07671902812682	0.9970527522269	0.07694580648363
4.5	0.07853981633974	0.07845909572784	0.9969173337331	0.07870170682462
4.6	0.08028514559174	0.08019892432886	0.9967788784562	0.08045808961469
4.7	0.08203047484373	0.08193850863004	0.996637386818	0.08221496575765
4.8	0.08377580409573	0.08367784333232	0.9964928592495	0.08397234616948
4.9	0.08552113334772	0.08541692313737	0.9963452961909	0.08573024177855
5.0	0.08726646259972	0.08715574274766	0.9961946980917	0.08748866352592
5.1	0.08901179185171	0.08889429686644	0.9960410654108	0.08924762236563
5.2	0.09075712110371	0.09063258019778	0.995884398616	0.09100712926494
5.3	0.0925024503557	0.09237058744656	0.9957246981846	0.09276719520463
5.4	0.09424777960769	0.09410831331851	0.9955619646031	0.09452783117928
5.5	0.09599310885969	0.09584575252022	0.9953961983672	0.09628904819754
5.6	0.09773843811168	0.09758289975915	0.9952273999818	0.0980508572824
5.7	0.09948376736368	0.09931974974364	0.9950555699612	0.09981326947148
5.8	0.1012290966157	0.1010562971829	0.9948807088288	0.1015762958173
5.9	0.1029744258677	0.1027925367872	0.9947028171172	0.1033399473877
6.0	0.1047197551197	0.1045284632677	0.9945218953683	0.1051042352657
6.1	0.1064650843717	0.1062640713362	0.9943379441332	0.1068691705503
6.2	0.1082104136236	0.107999355706	0.9941509639723	0.1086347643566
6.3	0.1099557428756	0.109734311091	0.9939609554552	0.1104010278158
6.4	0.1117010721276	0.1114689322063	0.9937679191606	0.1121679720759
6.5	0.1134464013796	0.1132032137679	0.9935718556766	0.1139356083016
6.6	0.1151917306316	0.1149371504929	0.9933727656004	0.1157039476751
6.7	0.1169370598836	0.1166707370993	0.9931706495385	0.1174730013956
6.8	0.1186823891356	0.1184039683065	0.9929655081065	0.1192427806806
6.9	0.1204277183876	0.1201368388346	0.9927573419294	0.1210132967651
7.0	0.1221730476396	0.1218693434051	0.9925461516413	0.1227845609029
7.1	0.1239183768916	0.1236014767405	0.9923319378855	0.1245565843662
7.2	0.1256637061436	0.1253332335643	0.9921147013145	0.1263293784461
7.3	0.1274090353956	0.1270646086014	0.99189444259	0.1281029544531
7.4	0.1291543646476	0.1287955965776	0.9916711623831	0.1298773237169
7.5	0.1308996938996	0.1305261922201	0.9914448613738	0.1316524975874
7.6	0.1326450231516	0.1322563902571	0.9912155402515	0.1334284874343

7.7	0.1343903524036	0.1339861854183	0.9909831997148	0.1352053046478
7.8	0.1361356816556	0.1357155724343	0.9907478404714	0.1369829606388
7.9	0.1378810109076	0.1374445460371	0.9905094632383	0.1387614668393
8.0	0.1396263401595	0.1391731009601	0.9902680687416	0.1405408347024
8.1	0.1413716694115	0.1409012319376	0.9900236577166	0.1423210757029
8.2	0.1431169986635	0.1426289337055	0.9897762309078	0.1441022013377
8.3	0.1448623279155	0.144356201001	0.989525789069	0.1458842231255
8.4	0.1466076571675	0.1460830285624	0.989272332963	0.1476671526079
8.5	0.1483529864195	0.1478094111296	0.9890158633619	0.1494510013491
8.6	0.1500983156715	0.1495353434437	0.988756381047	0.1512357809366
8.7	0.1518436449235	0.1512608202472	0.9884938868087	0.1530215029812
8.8	0.1535889741755	0.152985836284	0.9882283814466	0.1548081791176
8.9	0.1553343034275	0.1547103862995	0.9879598657694	0.1565958210043
9.0	0.1570796326795	0.1564344650402	0.9876883405951	0.1583844403245
9.1	0.1588249619315	0.1581580672545	0.9874138067509	0.160174048786
9.2	0.1605702911835	0.1598811876918	0.987136265073	0.1619646581215
9.3	0.1623156204355	0.1616038211034	0.9868557164068	0.163756280089
9.4	0.1640609496875	0.1633259622416	0.986572161607	0.1655489264724
9.5	0.1658062789395	0.1650476058607	0.9862856015372	0.1673426090814
9.6	0.1675516081915	0.1667687467161	0.9859960370705	0.169137339752
9.7	0.1692969374434	0.168489379565	0.9857034690889	0.1709331303467
9.8	0.1710422666954	0.170209499166	0.9854078984835	0.1727299927553
9.9	0.1727875959474	0.1719291002794	0.9851093261548	0.1745279388944
10.0	0.1745329251994	0.1736481776669	0.9848077530122	0.1763269807085
10.1	0.1762782544514	0.175366726092	0.9845031799744	0.1781271301699
10.2	0.1780235837034	0.1770847403196	0.9841956079692	0.1799283992793
10.3	0.1797689129554	0.1788022151163	0.9838850379335	0.1817308000657
10.4	0.1815142422074	0.1805191452506	0.9835714708134	0.1835343445874
10.5	0.1832595714594	0.1822355254921	0.983254907564	0.1853390449315
10.6	0.1850049007114	0.1839513506127	0.9829353491496	0.1871449132152
10.7	0.1867502299634	0.1856666153856	0.9826127965436	0.1889519615851
10.8	0.1884955592154	0.1873813145857	0.9822872507287	0.1907602022186
10.9	0.1902408884674	0.1890954429899	0.9819587126964	0.1925696473232
11.0	0.1919862177194	0.1908089953765	0.9816271834477	0.1943803091377
11.1	0.1937315469714	0.1925219665259	0.9812926639922	0.1961921999321
11.2	0.1954768762234	0.19423435122	0.9809551553492	0.1980053320081
11.3	0.1972222054754	0.1959461442425	0.9806146585466	0.1998197176992
11.4	0.1989675347274	0.1976573403791	0.9802711746217	0.2016353693715
11.5	0.2007128639793	0.1993679344172	0.9799247046208	0.2034522994237
11.6	0.2024581932313	0.201077921146	0.9795752495993	0.2052705202874
11.7	0.2042035224833	0.2027872953565	0.9792228106218	0.2070900444279
11.8	0.2059488517353	0.2044960518418	0.9788673887617	0.2089108843441
11.9	0.2076941809873	0.2062041853966	0.9785089851018	0.2107330525689
12.0	0.2094395102393	0.2079116908178	0.9781476007338	0.21255656167
12.1	0.2111848394913	0.2096185629038	0.9777832367586	0.2143814242497
12.2	0.2129301687433	0.2113247964554	0.9774158942861	0.2162076529457
12.3	0.2146754979953	0.213030386275	0.9770455744353	0.218035260431
12.4	0.2164208272473	0.2147353271671	0.9766722783342	0.219864259415
12.5	0.2181661564993	0.2164396139381	0.9762960071199	0.2216946626429
12.6	0.2199114857513	0.2181432413965	0.9759167619387	0.2235264828971
12.7	0.2216568150033	0.2198462043528	0.9755345439459	0.2253597329968
12.8	0.2234021442553	0.2215484976195	0.9751493543056	0.2271944257987

12.9	0.2251474735073	0.223250116011	0.9747611941912	0.2290305741974
13.0	0.2268928027593	0.2249510543439	0.9743700647852	0.2308681911256
13.1	0.2286381320113	0.2266513074369	0.9739759672791	0.2327072895546
13.2	0.2303834612633	0.2283508701107	0.9735789028732	0.2345478824949
13.3	0.2321287905152	0.2300497371881	0.9731788727771	0.2363899829963
13.4	0.2338741197672	0.2317479034942	0.9727758782094	0.2382336041481
13.5	0.2356194490192	0.2334453638559	0.9723699203977	0.2400787590801
13.6	0.2373647782712	0.2351421131026	0.9719610005785	0.2419254609626
13.7	0.2391101075232	0.2368381460656	0.9715491199976	0.2437737230066
13.8	0.2408554367752	0.2385334575786	0.9711342799096	0.2456235584648
13.9	0.2426007660272	0.2402280424773	0.9707164815782	0.2474749806315
14.0	0.2443460952792	0.2419218955997	0.970295726276	0.2493280028432
14.1	0.2460914245312	0.243615011786	0.9698720152847	0.2511826384788
14.2	0.2478367537832	0.2453073858788	0.9694453498951	0.2530389009606
14.3	0.2495820830352	0.2469990127227	0.9690157314069	0.2548968037538
14.4	0.2513274122872	0.2486898871649	0.9685831611286	0.2567563603677
14.5	0.2530727415392	0.2503800040544	0.9681476403781	0.2586175843559
14.6	0.2548180707912	0.2520693582431	0.967709170482	0.2604804893164
14.7	0.2565634000432	0.2537579445848	0.9672677527759	0.2623450888925
14.8	0.2583087292952	0.2554457579358	0.9668233886045	0.2642113967728
14.9	0.2600540585472	0.2571327931547	0.9663760793213	0.266079426692
15.0	0.2617993877991	0.2588190451025	0.9659258262891	0.2679491924311
15.1	0.2635447170511	0.2605045086426	0.9654726308792	0.2698207078179
15.2	0.2652900463031	0.2621891786409	0.9650164944723	0.2716939867274
15.3	0.2670353755551	0.2638730499654	0.9645574184578	0.2735690430822
15.4	0.2687807048071	0.2655561174868	0.9640954042341	0.2754458908533
15.5	0.2705260340591	0.2672383760783	0.9636304532086	0.2773245440598
15.6	0.2722713633111	0.2689198206153	0.9631625667977	0.2792050167703
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15.8	0.2757620218151	0.2722802470406	0.9622179935293	0.2829714772241
15.9	0.2775073510671	0.2739592186924	0.9617413095492	0.2848574933532
16.0	0.2792526803191	0.275637355817	0.9612616959383	0.2867453857588
16.1	0.2809980095711	0.2773146533024	0.9607791541576	0.2886351687611
16.2	0.2827433388231	0.2789911060392	0.9602936856769	0.2905268567319
16.3	0.2844886680751	0.2806667089208	0.9598052919752	0.2924204640956
16.4	0.2862339973271	0.2823414568429	0.9593139745401	0.294316005329
16.5	0.2879793265791	0.2840153447039	0.9588197348682	0.2962134949621
16.6	0.2897246558311	0.285688367405	0.9583225744651	0.2981129475787
16.7	0.2914699850831	0.2873605198497	0.9578224948453	0.3000143778165
16.8	0.293215314335	0.2890317969445	0.9573194975321	0.3019178003682
16.9	0.294960643587	0.2907021935983	0.9568135840576	0.3038232299812
17.0	0.296705972839	0.2923717047227	0.956304755963	0.3057306814587
17.1	0.298451302091	0.2940403252323	0.9557930147983	0.3076401696599
17.2	0.300196631343	0.295708050044	0.9552783621223	0.3095517095007
17.3	0.301941960595	0.2973748740778	0.9547607995028	0.3114653159542
17.4	0.303687289847	0.2990407922561	0.9542403285163	0.3133810040507
17.5	0.305432619099	0.3007057995043	0.9537169507482	0.315298788879
17.6	0.307177948351	0.3023698907504	0.9531906677929	0.3172186855863
17.7	0.308923277603	0.3040330609255	0.9526614812536	0.3191407093792
17.8	0.310668606855	0.3056953049631	0.9521293927421	0.3210648755236
17.9	0.312413936107	0.3073566177998	0.9515944038794	0.3229911993458
18.0	0.314159265359	0.3090169943749	0.9510565162952	0.3249196962329
18.1	0.315904594611	0.3106764296307	0.9505157316278	0.326850381633

18.2	0.317649923863	0.3123349185122	0.9499720515247	0.3287832710562
18.3	0.319395253115	0.3139924559674	0.9494254776419	0.3307183800747
18.4	0.321140582367	0.3156490369471	0.9488760116445	0.3326557243238
18.5	0.322885911619	0.3173046564051	0.9483236552062	0.3345953195021
18.6	0.3246312408709	0.3189593092981	0.9477684100096	0.336537181372
18.7	0.3263765701229	0.3206129905857	0.947210277746	0.3384813257607
18.8	0.3281218993749	0.3222656952305	0.9466492601157	0.3404277685604
18.9	0.3298672286269	0.3239174181981	0.9460853588275	0.3423765257287
19.0	0.3316125578789	0.3255681544572	0.9455185755993	0.3443276132897
19.1	0.3333578871309	0.3272178989791	0.9449489121575	0.3462810473341
19.2	0.3351032163829	0.3288666467386	0.9443763702375	0.34823684402
19.3	0.3368485456349	0.3305143927132	0.9438009515832	0.3501950195735
19.4	0.3385938748869	0.3321611318837	0.9432226579476	0.3521555902892
19.5	0.3403392041389	0.3338068592338	0.9426414910922	0.3541185725307
19.6	0.3420845333909	0.3354515697503	0.9420574527873	0.3560839827314
19.7	0.3438298626429	0.3370952584231	0.941470544812	0.3580518373949
19.8	0.3455751918949	0.3387379202453	0.9408807689542	0.3600221530958
19.9	0.3473205211469	0.3403795502131	0.9402881270104	0.3619949464801
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20.4	0.3560471674068	0.3485720473218	0.9372819894919	0.3718966663499
20.5	0.3577924966588	0.3502073812595	0.9366721892484	0.3738846794848
20.6	0.3595378259108	0.3518416484047	0.936059535739	0.3758752888799
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20.8	0.3630284844148	0.3551069624081	0.934825676396	0.3798643654902
20.9	0.3647738136668	0.3567379993196	0.934204474321	0.3818628674187
21.0	0.3665191429188	0.3583679495453	0.9335804264972	0.3838640350354
21.1	0.3682644721708	0.3599968081201	0.9329535348255	0.3858678858936
21.2	0.3700098014228	0.3616245700821	0.9323238012155	0.3878744376263
21.3	0.3717551306748	0.363251230473	0.9316912275855	0.3898837079472
21.4	0.3735004599268	0.3648767843376	0.9310558158625	0.3918957146512
21.5	0.3752457891788	0.3665012267243	0.930417567982	0.3939104756149
21.6	0.3769911184308	0.3681245526847	0.9297764858883	0.3959280087977
21.7	0.3787364476828	0.3697467572738	0.9291325715341	0.3979483322421
21.8	0.3804817769348	0.3713678355502	0.9284858268809	0.3999714640748
21.9	0.3822271061868	0.3729877825758	0.9278362538989	0.4019974225069
22.0	0.3839724354388	0.3746065934159	0.9271838545668	0.4040262258352
22.1	0.3857177646907	0.3762242631394	0.9265286308718	0.4060578924424
22.2	0.3874630939427	0.3778407868185	0.92587058481	0.4080924407983
22.3	0.3892084231947	0.379456159529	0.9252097183858	0.4101298894601
22.4	0.3909537524467	0.3810703763503	0.9245460336123	0.4121702570735
22.5	0.3926990816987	0.3826834323651	0.9238795325113	0.4142135623731
22.6	0.3944444109507	0.3842953226598	0.923210217113	0.4162598241834
22.7	0.3961897402027	0.3859060423243	0.9225380894562	0.4183090614197
22.8	0.3979350694547	0.3875155864521	0.9218631515885	0.4203612930882
22.9	0.3996803987067	0.3891239501402	0.9211854055657	0.4224165382877
23.0	0.4014257279587	0.3907311284893	0.9205048534524	0.4244748162096
23.1	0.4031710572107	0.3923371166036	0.9198214973217	0.4265361461392
23.2	0.4049163864627	0.393941909591	0.9191353392552	0.428600547456
23.3	0.4066617157147	0.395545502563	0.9184463813431	0.4306680396351
23.4	0.4084070449667	0.3971478906348	0.917754625684	0.4327386422474

23.5	0.4101523742187	0.3987490689252	0.9170600743851	0.4348123749609
23.6	0.4118977034707	0.4003490325569	0.9163627295622	0.4368892575412
23.7	0.4136430327227	0.401947776656	0.9156625933396	0.4389693098524
23.8	0.4153883619747	0.4035452963524	0.9149596678498	0.4410525518581
23.9	0.4171336912266	0.4051415867799	0.9142539552343	0.4431390036219
24.0	0.4188790204786	0.4067366430758	0.9135454576426	0.4452286853085
24.1	0.4206243497306	0.4083304603814	0.912834177233	0.4473216171847
24.2	0.4223696789826	0.4099230338416	0.9121201161723	0.4494178196199
24.3	0.4241150082346	0.4115143586051	0.9114032766354	0.451517313087
24.4	0.4258603374866	0.4131044298245	0.9106836608062	0.4536201181636
24.5	0.4276056667386	0.4146932426562	0.9099612708765	0.4557262555326
24.6	0.4293509959906	0.4162807922604	0.9092361090471	0.4578357459832
24.7	0.4310963252426	0.4178670738011	0.9085081775267	0.4599486104117
24.8	0.4328416544946	0.4194520824462	0.9077774785329	0.4620648698226
24.9	0.4345869837466	0.4210358133675	0.9070440142915	0.4641845453292
25.0	0.4363323129986	0.4226182617407	0.9063077870366	0.466307658155
25.1	0.4380776422506	0.4241994227454	0.9055687990111	0.468434229634
25.2	0.4398229715026	0.4257792915651	0.904827052466	0.4705642812123
25.3	0.4415683007546	0.4273578633872	0.9040825496608	0.4726978344484
25.4	0.4433136300066	0.4289351334031	0.9033352928633	0.4748349110147
25.5	0.4450589592586	0.4305110968083	0.9025852843499	0.4769755326982
25.6	0.4468042885105	0.432085748802	0.9018325264051	0.4791197214015
25.7	0.4485496177625	0.4336590845875	0.9010770213221	0.4812674991437
25.8	0.4502949470145	0.4352310993723	0.9003187714022	0.4834188880617
25.9	0.4520402762665	0.4368017883677	0.8995577789552	0.4855739104108
26.0	0.4537856055185	0.4383711467891	0.8987940462992	0.4877325885659
26.1	0.4555309347705	0.4399391698559	0.8980275757606	0.4898949450225
26.2	0.4572762640225	0.4415058527917	0.8972583696743	0.4920610023978
26.3	0.4590215932745	0.4430711908242	0.8964864303834	0.4942307834316
26.4	0.4607669225265	0.4446351791849	0.8957117602394	0.4964043109874
26.5	0.4625122517785	0.4461978131098	0.894934361602	0.4985816080534
26.6	0.4642575810305	0.4477590878388	0.8941542368394	0.5007626977438
26.7	0.4660029102825	0.4493189986159	0.8933713883278	0.5029476032996
26.8	0.4677482395345	0.4508775406894	0.8925858184521	0.5051363480896
26.9	0.4694935687865	0.4524347093118	0.8917975296052	0.5073289556118
27.0	0.4712388980385	0.4539904997395	0.8910065241884	0.5095254494944
27.1	0.4729842272905	0.4555449072335	0.8902128046111	0.5117258534969
27.2	0.4747295565425	0.4570979270587	0.8894163732913	0.5139301915111
27.3	0.4764748857945	0.4586495544843	0.8886172326549	0.5161384875623
27.4	0.4782202150464	0.4601997847839	0.8878153851364	0.5183507658106
27.5	0.4799655442984	0.461748613235	0.8870108331782	0.5205670505517
27.6	0.4817108735504	0.4632960351199	0.8862035792312	0.5227873662187
27.7	0.4834562028024	0.4648420457246	0.8853936257544	0.5250117373824
27.8	0.4852015320544	0.4663866403399	0.8845809752151	0.5272401887532
27.9	0.4869468613064	0.4679298142606	0.8837656300887	0.529472745182
28.0	0.4886921905584	0.4694715627859	0.8829475928589	0.5317094316615
28.1	0.4904375198104	0.4710118812194	0.8821268660177	0.5339502733272
28.2	0.4921828490624	0.4725507648691	0.881303452065	0.536195295459
28.3	0.4939281783144	0.4740882090471	0.8804773535092	0.5384445234822
28.4	0.4956735075664	0.4756242090703	0.8796485728666	0.5406979829687
28.5	0.4974188368184	0.4771587602596	0.878817112662	0.5429556996384
28.6	0.4991641660704	0.4786918579406	0.877982975428	0.5452176993606
28.7	0.5009094953224	0.4802234974432	0.8771461637056	0.5474840081549

28.8	0.5026548245744	0.4817536741017	0.8763066800439	0.5497546521928
28.9	0.5044001538264	0.483282383255	0.875464527	0.552029657799
29.0	0.5061454830784	0.4848096202463	0.8746197071394	0.5543090514528
29.1	0.5078908123304	0.4863353804235	0.8737722230355	0.556592859789
29.2	0.5096361415823	0.4878596591387	0.8729220772698	0.5588811095998
29.3	0.5113814708343	0.4893824517488	0.8720692724321	0.561173827836
29.4	0.5131268000863	0.4909037536151	0.8712138111202	0.563471041608
29.5	0.5148721293383	0.4924235601035	0.8703556959399	0.5657727781878
29.6	0.5166174585903	0.4939418665842	0.8694949295052	0.56807906501
29.7	0.5183627878423	0.4954586684324	0.8686315144382	0.5703899296733
29.8	0.5201081170943	0.4969739610276	0.8677654533689	0.5727053999421
29.9	0.5218534463463	0.4984877397538	0.8668967489356	0.5750255037476
30.0	**0.5235987755983**	**0.5**	**0.8660254037844**	**0.5773502691896**
30.1	0.5253441048503	0.5015107371595	0.8651514205697	0.5796797245379
30.2	0.5270894341023	0.5030199466302	0.8642748019537	0.5820138982337
30.3	0.5288347633543	0.504527623815	0.8633955506068	0.584352818891
30.4	0.5305800926063	0.5060337641212	0.8625136692073	0.5866965152984
30.5	0.5323254218583	0.5075383629607	0.8616291604415	0.5890450164206
30.6	0.5340707511103	0.5090414157504	0.8607420270039	0.5913983513995
30.7	0.5358160803623	0.5105429179116	0.8598522715969	0.5937565495564
30.8	0.5375614096143	0.5120428648706	0.8589598969307	0.5961196403933
30.9	0.5393067388663	0.5135412520582	0.8580649057236	0.5984876535943
31.0	0.5410520681182	0.5150380749101	0.8571673007021	0.6008606190276
31.1	0.5427973973702	0.5165333288666	0.8562670846003	0.6032385667467
31.2	0.5445427266222	0.5180270093731	0.8553642601605	0.6056215269924
31.3	0.5462880558742	0.5195191118795	0.8544588301328	0.6080095301944
31.4	0.5480333851262	0.5210096318406	0.8535507972753	0.6104026069728
31.5	0.5497787143782	0.522498564716	0.8526401643541	0.6128007881399
31.6	0.5515240436302	0.5239859059701	0.851726934143	0.615204104702
31.7	0.5532693728822	0.5254716510723	0.850811109424	0.617612587861
31.8	0.5550147021342	0.5269557954967	0.8498926929869	0.6200262690161
31.9	0.5567600313862	0.5284383347223	0.8489716876291	0.6224451797657
32.0	0.5585053606382	0.5299192642332	0.8480480961564	0.6248693519093
32.1	0.5602506898902	0.5313985795181	0.8471219213821	0.6272988174489
32.2	0.5619960191422	0.5328762760707	0.8461931661276	0.6297336085912
32.3	0.5637413483942	0.5343523493898	0.8452618332219	0.6321737577492
32.4	0.5654866776462	0.535826794979	0.844327925502	0.6346192975442
32.5	0.5672320068982	0.5372996083468	0.8433914458129	0.6370702608075
32.6	0.5689773361502	0.5387707850069	0.8424523970071	0.6395266805826
32.7	0.5707226654021	0.5402403204777	0.8415107819453	0.6419885901269
32.8	0.5724679946541	0.5417082102827	0.8405666034957	0.6444560229135
32.9	0.5742133239061	0.5431744499507	0.8396198645344	0.6469290126334
33.0	0.5759586531581	0.544639035015	0.8386705679454	0.6494075931975
33.1	0.5777039824101	0.5461019610144	0.8377187166204	0.6518917987383
33.2	0.5794493116621	0.5475632234926	0.836764313459	0.6543816636122
33.3	0.5811946409141	0.5490228179981	0.8358073613683	0.6568772224013
33.4	0.5829399701661	0.550480740085	0.8348478632634	0.6593785099158
33.5	0.5846852994181	0.5519369853121	0.8338858220672	0.6618855611957
33.6	0.5864306286701	0.5533915492433	0.8329212407101	0.6643984115131
33.7	0.5881759579221	0.554844427448	0.8319541221305	0.6669170963744
33.8	0.5899212871741	0.5562956155003	0.8309844692743	0.6694416515222
33.9	0.5916666164261	0.5577451089797	0.8300122850954	0.6719721129376
34.0	0.5934119456781	0.5591929034707	0.829037572555	0.6745085168424

34.1	0.5951572749301	0.5606389945632	0.8280603346225	0.6770508997015
34.2	0.5969026041821	0.5620833778521	0.8270805742746	0.6795992982245
34.3	0.5986479334341	0.5635260489376	0.8260982944958	0.6821537493689
34.4	0.6003932626861	0.5649670034249	0.8251134982783	0.6847142903417
34.5	0.602138591938	0.5664062369248	0.824126188622	0.6872809586016
34.6	0.60388392119	0.5678437450531	0.8231363685344	0.6898537918621
34.7	0.605629250442	0.5692795234308	0.8221440410307	0.6924328280932
34.8	0.607374579694	0.5707135676844	0.8211492091337	0.6950181055238
34.9	0.609119908946	0.5721458734455	0.8201518758738	0.6976096626445
35.0	0.610865238198	0.573576436351	0.819152044289	0.7002075382097
35.1	0.61261056745	0.5750052520433	0.818149717425	0.7028117712404
35.2	0.614355896702	0.5764323161698	0.8171448983351	0.7054224010261
35.3	0.616101225954	0.5778576243835	0.8161375900802	0.708039467128
35.4	0.617846555206	0.5792811723427	0.8151277957286	0.7106630093812
35.5	0.619591884458	0.5807029557109	0.8141155183563	0.713293067897
35.6	0.62133721371	0.5821229701573	0.813100761047	0.7159296830662
35.7	0.623082542962	0.5835412113561	0.8120835268918	0.7185728955611
35.8	0.624827872214	0.5849576749872	0.8110638189893	0.7212227463384
35.9	0.626573201466	0.5863723567358	0.8100416404458	0.723879276642
36.0	0.628318530718	0.5877852522925	0.8090169943749	0.7265425280054
36.1	0.63006385997	0.5891963573533	0.807989883898	0.7292125422547
36.2	0.631809189222	0.5906056676199	0.8069603121438	0.7318893615114
36.3	0.6335545184739	0.5920131787992	0.8059282822485	0.7345730281949
36.4	0.6352998477259	0.5934188866037	0.8048937973559	0.7372635850259
36.5	0.6370451769779	0.5948227867513	0.8038568606172	0.7399610750285
36.6	0.6387905062299	0.5962248749656	0.8028174751911	0.7426655415338
36.7	0.6405358354819	0.5976251469755	0.8017756442438	0.7453770281825
36.8	0.6422811647339	0.5990235985156	0.8007313709487	0.7480955789279
36.9	0.6440264939859	0.6004202253259	0.7996846584871	0.7508212380388
37.0	0.6457718232379	0.6018150231521	0.7986355100473	0.7535540501028
37.1	0.6475171524899	0.6032079877453	0.7975839288252	0.7562940600292
37.2	0.6492624817419	0.6045991148624	0.7965299180242	0.7590413130521
37.3	0.6510078109939	0.6059884002657	0.7954734808549	0.7617958547335
37.4	0.6527531402459	0.6073758397233	0.7944146205354	0.7645577309667
37.5	0.6544984694979	0.6087614290087	0.7933533402912	0.767326987979
37.6	0.6562437987499	0.6101451639013	0.7922896433552	0.7701036723356
37.7	0.6579891280019	0.6115270401858	0.7912235329675	0.7728878309424
37.8	0.6597344572539	0.612907053653	0.7901550123757	0.7756795110496
37.9	0.6614797865059	0.6142852000989	0.7890840848347	0.7784787602549
38.0	0.6632251157578	0.6156614753257	0.7880107536067	0.7812856265067
38.1	0.6649704450098	0.6170358751408	0.7869350219613	0.7841001581082
38.2	0.6667157742618	0.6184083953576	0.7858568931754	0.7869224037201
38.3	0.6684611035138	0.6197790317951	0.7847763705331	0.7897524123645
38.4	0.6702064327658	0.6211477802783	0.7836934573258	0.7925902334286
38.5	0.6719517620178	0.6225146366376	0.7826081568524	0.7954359166678
38.6	0.6736970912698	0.6238795967094	0.7815204724188	0.7982895122101
38.7	0.6754424205218	0.6252426563357	0.7804304073383	0.8011510705588
38.8	0.6771877497738	0.6266038113645	0.7793379649315	0.804020642597
38.9	0.6789330790258	0.6279630576493	0.778243148526	0.8068982795913
39.0	0.6806784082778	0.6293203910498	0.777145961457	0.809784033195
39.1	0.6824237375298	0.6306758074313	0.7760464070665	0.8126779554527
39.2	0.6841690667818	0.6320293026649	0.7749444887042	0.8155800988039
39.3	0.6859143960338	0.6333808726276	0.7738402097265	0.8184905160865

39.4	0.6876597252858	0.6347305132023	0.7727335734973	0.8214092605418
39.5	0.6894050545378	0.6360782202778	0.7716245833877	0.8243363858175
39.6	0.6911503837898	0.6374239897487	0.7705132427758	0.8272719459725
39.7	0.6928957130418	0.6387678175156	0.7693995550469	0.8302159954806
39.8	0.6946410422937	0.640109699485	0.7682835235935	0.8331685892351
39.9	0.6963863715457	0.6414496315692	0.7671651518153	0.8361297825525
40.0	0.6981317007977	0.6427876096865	0.766044443119	0.8390996311773
40.1	0.6998770300497	0.6441236297614	0.7649214009184	0.8420781912861
40.2	0.7016223593017	0.645457687724	0.7637960286346	0.8450655194919
40.3	0.7033676885537	0.6467897795105	0.7626683296957	0.8480616728487
40.4	0.7051130178057	0.6481199010631	0.7615383075367	0.8510667088561
40.5	0.7068583470577	0.6494480483302	0.7604059656	0.8540806854635
40.6	0.7086036763097	0.6507742172659	0.7592713073349	0.8571036610749
40.7	0.7103490055617	0.6520984038304	0.7581343361976	0.8601356945537
40.8	0.7120943348137	0.6534206039901	0.7569950556518	0.8631768452273
40.9	0.7138396640657	0.6547408137173	0.7558534691676	0.8662271728915
41.0	0.7155849933177	0.6560590289905	0.7547095802228	0.8692867378162
41.1	0.7173303225697	0.6573752457941	0.7535633923016	0.8723556007495
41.2	0.7190756518217	0.6586894601187	0.7524149088957	0.8754338229228
41.3	0.7208209810737	0.6600016679609	0.7512641335035	0.8785214660562
41.4	0.7225663103257	0.6613118653237	0.7501110696305	0.8816185923632
41.5	0.7243116395777	0.6626200482157	0.748955720789	0.884725264556
41.6	0.7260569688296	0.6639262126522	0.7477980904985	0.8878415458505
41.7	0.7278022980816	0.6652303546544	0.7466381822854	0.8909674999719
41.8	0.7295476273336	0.6665324702495	0.7454759996829	0.8941031911598
41.9	0.7312929565856	0.6678325554711	0.7443115462312	0.8972486841735
42.0	0.7330382858376	0.6691306063589	0.7431448254774	0.9004040442979
42.1	0.7347836150896	0.6704266189588	0.7419758409756	0.9035693373483
42.2	0.7365289443416	0.671720589323	0.7408045962867	0.9067446296769
42.3	0.7382742735936	0.6730125135098	0.7396310949786	0.9099299881777
42.4	0.7400196028456	0.6743023875837	0.7384553406259	0.9131254802927
42.5	0.7417649320976	0.6755902076157	0.7372773368101	0.9163311740174
42.6	0.7435102613496	0.6768759696827	0.7360970871197	0.9195471379071
42.7	0.7452555906016	0.6781596698681	0.73491459515	0.9227734410822
42.8	0.7470009198536	0.6794413042615	0.7337298645029	0.9260101532352
42.9	0.7487462491056	0.6807208689589	0.7325428987874	0.9292573446357
43.0	0.7504915783576	0.6819983600625	0.7313537016192	0.9325150861377
43.1	0.7522369076096	0.6832737736808	0.7301622766207	0.9357834491848
43.2	0.7539822368616	0.6845471059287	0.7289686274214	0.9390625058175
43.3	0.7557275661136	0.6858183529274	0.7277727576572	0.942352328679
43.4	0.7574728953655	0.6870875108044	0.726574670971	0.9456529910219
43.5	0.7592182246175	0.6883545756938	0.7253743710123	0.9489645667149
43.6	0.7609635538695	0.6896195437357	0.7241718614375	0.9522871302494
43.7	0.7627088831215	0.6908824110769	0.7229671459096	0.9556207567464
43.8	0.7644542123735	0.6921431738704	0.7217602280984	0.9589655219629
43.9	0.7661995416255	0.6934018282758	0.7205511116803	0.9623215022995
44.0	0.7679448708775	0.694658370459	0.7193398003386	0.9656887748071
44.1	0.7696902001295	0.6959127965923	0.7181262977632	0.9690674171938
44.2	0.7714355293815	0.6971651028546	0.7169106076505	0.9724575078326
44.3	0.7731808586335	0.698415285431	0.7156927337037	0.9758591257685
44.4	0.7749261878855	0.6996633405134	0.7144726796328	0.9792723507258
44.5	0.7766715171375	0.7009092642999	0.7132504491542	0.9826972631157
44.6	0.7784168463895	0.7021530529952	0.712026045991	0.986133944044

44.7	0.7801621756415	0.7033947028105	0.710799473873	0.9895824753188
44.8	0.7819075048935	0.7046342099636	0.7095707365365	0.993042939458
44.9	0.7836528341455	0.7058715706787	0.7083398377245	0.9965154196977
45.0	**0.7853981633975**	**0.7071067811866**	**0.7071067811865**	**1**
45.1	0.7871434926494	0.7083398377245	0.7058715706787	1.003496765061
45.2	0.7888888219014	0.7095707365365	0.7046342099636	1.007005800319
45.3	0.7906341511534	0.710799473873	0.7033947028105	1.010527191963
45.4	0.7923794804054	0.712026045991	0.7021530529952	1.014061026942
45.5	0.7941248096574	0.7132504491542	0.7009092642998	1.017607392972
45.6	0.7958701389094	0.7144726796328	0.6996633405134	1.021166378545
45.7	0.7976154681614	0.7156927337037	0.698415285431	1.024738072939
45.8	0.7993607974134	0.7169106076505	0.6971651028546	1.028322566226
45.9	0.8011061266654	0.7181262977632	0.6959127965923	1.031919949281
46.0	0.8028514559174	0.7193398003387	0.694658370459	1.035530313791
46.1	0.8045967851694	0.7205511116803	0.6934018282758	1.039153752265
46.2	0.8063421144214	0.7217602280984	0.6921431738704	1.042790358044
46.3	0.8080874436734	0.7229671459096	0.6908824110769	1.046440225309
46.4	0.8098327729254	0.7241718614375	0.6896195437357	1.050103449091
46.5	0.8115781021774	0.7253743710123	0.6883545756937	1.053780125281
46.6	0.8133234314294	0.726574670971	0.6870875108044	1.05747035064
46.7	0.8150687606814	0.7277727576572	0.6858183529274	1.061174222811
46.8	0.8168140899334	0.7289686274214	0.6845471059287	1.064891840325
46.9	0.8185594191853	0.7301622766208	0.6832737736808	1.068623302615
47.0	0.8203047484373	0.7313537016192	0.6819983600625	1.072368710025
47.1	0.8220500776893	0.7325428987874	0.6807208689589	1.076128163821
47.2	0.8237954069413	0.7337298645029	0.6794413042615	1.079901766202
47.3	0.8255407361933	0.73491459515	0.6781596698681	1.083689620312
47.4	0.8272860654453	0.7360970871197	0.6768759696827	1.087491830246
47.5	0.8290313946973	0.7372773368101	0.6755902076157	1.091308501069
47.6	0.8307767239493	0.7384553406259	0.6743023875837	1.095139738823
47.7	0.8325220532013	0.7396310949786	0.6730125135098	1.098985650536
47.8	0.8342673824533	0.7408045962868	0.671720589323	1.102846344242
47.9	0.8360127117053	0.7419758409756	0.6704266189588	1.106721928983
48.0	0.8377580409573	0.7431448254774	0.6691306063589	1.110612514829
48.1	0.8395033702093	0.7443115462312	0.667832555471	1.114518212887
48.2	0.8412486994613	0.7454759996829	0.6665324702494	1.118439135312
48.3	0.8429940287133	0.7466381822854	0.6652303546544	1.122375395322
48.4	0.8447393579653	0.7477980904985	0.6639262126522	1.126327107212
48.5	0.8464846872173	0.748955720789	0.6626200482157	1.130294386362
48.6	0.8482300164693	0.7501110696305	0.6613118653236	1.134277349255
48.7	0.8499753457212	0.7512641335035	0.6600016679609	1.13827611349
48.8	0.8517206749732	0.7524149088957	0.6586894601187	1.142290797791
48.9	0.8534660042252	0.7535633923016	0.6573752457941	1.146321522027
49.0	0.8552113334772	0.7547095802228	0.6560590289905	1.150368407221
49.1	0.8569566627292	0.7558534691676	0.6547408137173	1.154431575567
49.2	0.8587019919812	0.7569950556518	0.6534206039901	1.158511150443
49.3	0.8604473212332	0.7581343361977	0.6520984038304	1.162607256427
49.4	0.8621926504852	0.7592713073349	0.6507742172658	1.16672001931
49.5	0.8639379797372	0.7604059656	0.6494480483302	1.170849566113
49.6	0.8656833089892	0.7615383075367	0.6481199010631	1.174996025099
49.7	0.8674286382412	0.7626683296957	0.6467897795105	1.179159525794
49.8	0.8691739674932	0.7637960286346	0.6454576877239	1.183340198996
49.9	0.8709192967452	0.7649214009184	0.6441236297614	1.187538176797

50.0	0.8726646259972	0.766044443119	0.6427876096865	1.191753592594
50.1	0.8744099552492	0.7671651518153	0.6414496315692	1.195986581111
50.2	0.8761552845012	0.7682835235935	0.6401096994849	1.20023727841
50.3	0.8779006137532	0.7693995550469	0.6387678175156	1.204505821911
50.4	0.8796459430051	0.7705132427758	0.6374239897487	1.20879235041
50.5	0.8813912722571	0.7716245833877	0.6360782202778	1.213097004093
50.6	0.8831366015091	0.7727335734974	0.6347305132023	1.217419924558
50.7	0.8848819307611	0.7738402097265	0.6333808726275	1.22176125483
50.8	0.8866272600131	0.7749444887042	0.6320293026648	1.226121139379
50.9	0.8883725892651	0.7760464070666	0.6306758074313	1.230499724141
51.0	0.8901179185171	0.777145961457	0.6293203910498	1.234897156535
51.1	0.8918632477691	0.778243148526	0.6279630576493	1.239313585483
51.2	0.8936085770211	0.7793379649315	0.6266038113645	1.243749161427
51.3	0.8953539062731	0.7804304073383	0.6252426563357	1.248204036353
51.4	0.8970992355251	0.7815204724188	0.6238795967094	1.252678363808
51.5	0.8988445647771	0.7826081568524	0.6225146366376	1.257172298919
51.6	0.9005898940291	0.7836934573258	0.6211477802783	1.261685998418
51.7	0.9023352232811	0.7847763705331	0.6197790317951	1.266219620661
51.8	0.9040805525331	0.7858568931754	0.6184083953575	1.270773325645
51.9	0.9058258817851	0.7869350219613	0.6170358751407	1.275347275038
52.0	0.9075712110371	0.7880107536067	0.6156614753257	1.279941632193
52.1	0.9093165402891	0.7890840848347	0.6142852000989	1.284556562176
52.2	0.911061869541	0.7901550123757	0.612907053653	1.289192231785
52.3	0.912807198793	0.7912235329675	0.6115270401858	1.293848809575
52.4	0.914552528045	0.7922896433552	0.6101451639013	1.298526465881
52.5	0.916297857297	0.7933533402912	0.6087614290087	1.303225372841
52.6	0.918043186549	0.7944146205354	0.6073758397233	1.307945704421
52.7	0.919788515801	0.7954734808549	0.6059884002657	1.31268763644
52.8	0.921533845053	0.7965299180242	0.6045991148624	1.317451346593
52.9	0.923279174305	0.7975839288252	0.6032079877453	1.322237014477
53.0	0.925024503557	0.7986355100473	0.601815023152	1.32704482162
53.1	0.926769832809	0.7996846584871	0.6004202253259	1.331874951503
53.2	0.928515162061	0.8007313709487	0.5990235985156	1.336727589586
53.3	0.930260491313	0.8017756442438	0.5976251469755	1.34160292334
53.4	0.932005820565	0.8028174751911	0.5962248749656	1.34650114227
53.5	0.933751149817	0.8038568606172	0.5948227867513	1.351422437946
53.6	0.935496479069	0.8048937973559	0.5934188866037	1.356367004027
53.7	0.937241808321	0.8059282822485	0.5920131787992	1.361335036296
53.8	0.938987137573	0.8069603121438	0.5906056676199	1.366326732684
53.9	0.940732466825	0.807989883898	0.5891963573533	1.371342293302
54.0	0.9424777960769	0.809016994375	0.5877852522925	1.376381920471
54.1	0.9442231253289	0.8100416404458	0.5863723567358	1.381445818754
54.2	0.9459684545809	0.8110638189893	0.5849576749872	1.386534194986
54.3	0.9477137838329	0.8120835268918	0.5835412113561	1.391647258305
54.4	0.9494591130849	0.813100761047	0.5821229701573	1.396785220187
54.5	0.9512044423369	0.8141155183563	0.5807029557109	1.401948294476
54.6	0.9529497715889	0.8151277957286	0.5792811723427	1.407136697421
54.7	0.9546951008409	0.8161375900802	0.5778576243835	1.412350647706
54.8	0.9564404300929	0.8171448983351	0.5764323161698	1.417590366489
54.9	0.9581857593449	0.818149717425	0.5750052520433	1.422856077432
55.0	0.9599310885969	0.819152044289	0.573576436351	1.428148006742
55.1	0.9616764178489	0.8201518758738	0.5721458734455	1.433466383205
55.2	0.9634217471009	0.8211492091337	0.5707135676844	1.438811438224

55.3	0.9651670763529	0.8221440410307	0.5692795234308	1.444183405853
55.4	0.9669124056049	0.8231363685344	0.5678437450531	1.449582522843
55.5	0.9686577348569	0.824126188622	0.5664062369248	1.455009028672
55.6	0.9704030641089	0.8251134982783	0.5649670034249	1.460463165594
55.7	0.9721483933609	0.8260982944958	0.5635260489376	1.465945178671
55.8	0.9738937226128	0.8270805742746	0.5620833778521	1.47145531582
55.9	0.9756390518648	0.8280603346225	0.5606389945632	1.476993827851
56.0	0.9773843811168	0.829037572555	0.5591929034707	1.482560968513
56.1	0.9791297103688	0.8300122850954	0.5577451089797	1.488156994534
56.2	0.9808750396208	0.8309844692743	0.5562956155003	1.493782165669
56.3	0.9826203688728	0.8319541221305	0.554844427448	1.499436744741
56.4	0.9843656981248	0.8329212407101	0.5533915492433	1.50512099769
56.5	0.9861110273768	0.8338858220672	0.5519369853121	1.510835193615
56.6	0.9878563566288	0.8348478632634	0.550480740085	1.516579604828
56.7	0.9896016858808	0.8358073613683	0.5490228179981	1.522354506896
56.8	0.9913470151328	0.836764313459	0.5475632234925	1.528160178695
56.9	0.9930923443848	0.8377187166204	0.5461019610144	1.533996902454
57.0	0.9948376736368	0.8386705679454	0.544639035015	1.539864963815
57.1	0.9965830028888	0.8396198645344	0.5431744499507	1.545764651873
57.2	0.9983283321408	0.8405666034957	0.5417082102827	1.551696259241
57.3	1.000073661393	0.8415107819453	0.5402403204776	1.557660082093
57.4	1.001818990645	0.8424523970072	0.5387707850069	1.563656420228
57.5	1.003564319897	0.8433914458129	0.5372996083468	1.569685577118
57.6	1.005309649149	0.844327925502	0.535826794979	1.575747859969
57.7	1.007054978401	0.8452618332219	0.5343523493898	1.581843579779
57.8	1.008800307653	0.8461931661276	0.5328762760707	1.587973051394
57.9	1.010545636905	0.8471219213821	0.5313985795181	1.594136593572
58.0	1.012290966157	0.8480480961564	0.5299192642332	1.600334529041
58.1	1.014036295409	0.8489716876291	0.5284383347223	1.606567184561
58.2	1.015781624661	0.8498926929869	0.5269557954967	1.612834890991
58.3	1.017526953913	0.8508111094241	0.5254716510723	1.619137983349
58.4	1.019272283165	0.8517269341431	0.5239859059701	1.625476800881
58.5	1.021017612417	0.8526401643541	0.5224985647159	1.631851687129
58.6	1.022762941669	0.8535507972753	0.5210096318406	1.638262989995
58.7	1.024508270921	0.8544588301328	0.5195191118795	1.644711061816
58.8	1.026253600173	0.8553642601605	0.5180270093731	1.65119625943
58.9	1.027998929425	0.8562670846003	0.5165333288666	1.657718944253
59.0	1.029744258677	0.8571673007021	0.51503807491	1.664279482351
59.1	1.031489587929	0.8580649057236	0.5135412520582	1.670878244512
59.2	1.033234917181	0.8589598969307	0.5120428648706	1.677515606331
59.3	1.034980246433	0.8598522715969	0.5105429179116	1.684191948278
59.4	1.036725575685	0.8607420270039	0.5090414157504	1.690907655785
59.5	1.038470904937	0.8616291604415	0.5075383629607	1.697663119326
59.6	1.040216234189	0.8625136692073	0.5060337641212	1.704458734498
59.7	1.041961563441	0.8633955506068	0.504527623815	1.711294902107
59.8	1.043706892693	0.8642748019537	0.5030199466302	1.718172028254
59.9	1.045452221945	0.8651514205697	0.5015107371594	1.725090524422
60.0	**1.047197551197**	**0.8660254037844**	**0.5**	**1.732050807569**
60.1	1.048942880449	0.8668967489356	0.4984877397538	1.739053300215
60.2	1.050688209701	0.8677654533689	0.4969739610275	1.746098430539
60.3	1.052433538953	0.8686315144382	0.4954586684324	1.753186632472
60.4	1.054178868205	0.8694949295052	0.4939418665842	1.760318345797
60.5	1.055924197457	0.8703556959399	0.4924235601035	1.767494016243

60.6	1.057669526709	0.8712138111202	0.4909037536151	1.774714095593
60.7	1.059414855961	0.8720692724321	0.4893824517488	1.781979041782
60.8	1.061160185213	0.8729220772698	0.4878596591387	1.789289319004
60.9	1.062905514465	0.8737722230355	0.4863353804235	1.796645397821
61.0	1.064650843717	0.8746197071394	0.4848096202463	1.804047755271
61.1	1.066396172969	0.875464527	0.483282383255	1.811496874981
61.2	1.068141502221	0.8763066800439	0.4817536741017	1.818993247281
61.3	1.069886831473	0.8771461637056	0.4802234974432	1.82653736932
61.4	1.071632160725	0.877982975428	0.4786918579406	1.834129745188
61.5	1.073377489977	0.878817112662	0.4771587602596	1.841770886034
61.6	1.075122819229	0.8796485728666	0.4756242090703	1.849461310193
61.7	1.076868148481	0.8804773535092	0.4740882090471	1.857201543314
61.8	1.078613477733	0.881303452065	0.472550764869	1.864992118485
61.9	1.080358806985	0.8821268660177	0.4710118812194	1.872833576372
62.0	1.082104136236	0.8829475928589	0.4694715627859	1.880726465346
62.1	1.083849465488	0.8837656300887	0.4679298142606	1.888671341631
62.2	1.08559479474	0.8845809752151	0.4663866403399	1.896668769437
62.3	1.087340123992	0.8853936257544	0.4648420457246	1.904719321107
62.4	1.089085453244	0.8862035792312	0.4632960351199	1.912823577266
62.5	1.090830782496	0.8870108331782	0.461748613235	1.920982126971
62.6	1.092576111748	0.8878153851364	0.4601997847838	1.929195567863
62.7	1.094321441	0.888617232655	0.4586495544843	1.937464506325
62.8	1.096066770252	0.8894163732913	0.4570979270587	1.945789557644
62.9	1.097812099504	0.8902128046111	0.4555449072335	1.954171346174
63.0	1.099557428756	0.8910065241884	0.4539904997395	1.962610505505
63.1	1.101302758008	0.8917975296052	0.4524347093118	1.971107678635
63.2	1.10304808726	0.8925858184521	0.4508775406894	1.979663518141
63.3	1.104793416512	0.8933713883278	0.4493189986159	1.988278686367
63.4	1.106538745764	0.8941542368394	0.4477590878388	1.9969538556
63.5	1.108284075016	0.894934361602	0.4461978131098	2.005689708259
63.6	1.110029404268	0.8957117602394	0.4446351791849	2.014486937092
63.7	1.11177473352	0.8964864303834	0.4430711908242	2.023346245365
63.8	1.113520062772	0.8972583696743	0.4415058527917	2.032268347069
63.9	1.115265392024	0.8980275757606	0.4399391698559	2.041253967122
64.0	1.117010721276	0.8987940462992	0.4383711467891	2.050303841579
64.1	1.118756050528	0.8995577789552	0.4368017883677	2.059418717851
64.2	1.12050137978	0.9003187714022	0.4352310993723	2.068599354919
64.3	1.122246709032	0.9010770213221	0.4336590845875	2.077846523564
64.4	1.123992038284	0.9018325264051	0.432085748802	2.087161006596
64.5	1.125737367536	0.9025852843499	0.4305110968083	2.096543599088
64.6	1.127482696788	0.9033352928633	0.4289351334031	2.105995108622
64.7	1.12922802604	0.9040825496608	0.4273578633872	2.115516355532
64.8	1.130973355292	0.904827052466	0.4257792915651	2.125108173157
64.9	1.132718684544	0.9055687990111	0.4241994227454	2.134771408104
65.0	1.134464013796	0.9063077870367	0.4226182617407	2.14450692051
65.1	1.136209343048	0.9070440142915	0.4210358133675	2.154315584313
65.2	1.1379546723	0.9077774785329	0.4194520824462	2.164198287535
65.3	1.139700001552	0.9085081775267	0.4178670738011	2.174155932562
65.4	1.141445330804	0.9092361090471	0.4162807922604	2.184189436438
65.5	1.143190660056	0.9099612708765	0.4146932426562	2.194299731165
65.6	1.144935989308	0.9106836608062	0.4131044298245	2.204487764009
65.7	1.14668131856	0.9114032766354	0.4115143586051	2.214754497813
65.8	1.148426647812	0.9121201161723	0.4099230338416	2.225100911321

65.9	1.150171977064	0.912834177233	0.4083304603814	2.235527999504
66.0	1.151917306316	0.9135454576426	0.4067366430758	2.246036773904
66.1	1.153662635568	0.9142539552343	0.4051415867799	2.256628262976
66.2	1.15540796482	0.9149596678498	0.4035452963524	2.267303512444
66.3	1.157153294072	0.9156625933396	0.401947776656	2.278063585667
66.4	1.158898623324	0.9163627295622	0.4003490325569	2.288909564012
66.5	1.160643952576	0.9170600743851	0.3987490689252	2.299842547236
66.6	1.162389281828	0.917754625684	0.3971478906348	2.310863653882
66.7	1.16413461108	0.9184463813431	0.395545502563	2.321974021679
66.8	1.165879940332	0.9191353392552	0.3939419095909	2.333174807955
66.9	1.167625269584	0.9198214973217	0.3923371166036	2.344467190065
67.0	1.169370598836	0.9205048534524	0.3907311284893	2.355852365824
67.1	1.171115928088	0.9211854055657	0.3891239501402	2.367331553953
67.2	1.17286125734	0.9218631515885	0.3875155864521	2.37890599454
67.3	1.174606586592	0.9225380894562	0.3859060423243	2.390576949508
67.4	1.176351915844	0.923210217113	0.3842953226598	2.402345703099
67.5	1.178097245096	0.9238795325113	0.3826834323651	2.414213562373
67.6	1.179842574348	0.9245460336123	0.3810703763503	2.426181857712
67.7	1.1815879036	0.9252097183858	0.379456159529	2.438251943345
67.8	1.183333232852	0.92587058481	0.3778407868185	2.450425197889
67.9	1.185078562104	0.9265286308718	0.3762242631394	2.462703024894
68.0	1.186823891356	0.9271838545668	0.3746065934159	2.475086853416
68.1	1.188569220608	0.9278362538989	0.3729877825758	2.487578138596
68.2	1.19031454986	0.9284858268809	0.3713678355502	2.500178362257
68.3	1.192059879112	0.9291325715341	0.3697467572738	2.512889033523
68.4	1.193805208364	0.9297764858883	0.3681245526847	2.525711689447
68.5	1.195550537616	0.930417567982	0.3665012267243	2.538647895664
68.6	1.197295866868	0.9310558158625	0.3648767843376	2.551699247056
68.7	1.19904119612	0.9316912275856	0.363251230473	2.564867368439
68.8	1.200786525372	0.9323238012155	0.3616245700821	2.578153915272
68.9	1.202531854624	0.9329535348255	0.35999680812	2.591560574377
69.0	1.204277183876	0.9335804264972	0.3583679495453	2.605089064694
69.1	1.206022513128	0.934204474321	0.3567379993196	2.618741138042
69.2	1.20776784238	0.934825676396	0.3551069624081	2.632518579913
69.3	1.209513171632	0.9354440308299	0.3534748437793	2.646423210287
69.4	1.211258500884	0.936059535739	0.3518416484047	2.660456884463
69.5	1.213003830136	0.9366721892484	0.3502073812595	2.674621493927
69.6	1.214749159388	0.9372819894919	0.3485720473218	2.688918967236
69.7	1.21649448864	0.9378889346119	0.3469356515733	2.703351270931
69.8	1.218239817892	0.9384930227596	0.3452981989985	2.717920410478
69.9	1.219985147144	0.9390942520947	0.3436596945856	2.732628431237
70.0	1.221730476396	0.9396926207859	0.3420201433257	2.747477419455
70.1	1.223475805648	0.9402881270104	0.340379550213	2.7624695033
70.2	1.2252211349	0.9408807689542	0.3387379202453	2.777606853915
70.3	1.226966464152	0.941470544812	0.3370952584231	2.792891686511
70.4	1.228711793404	0.9420574527873	0.3354515697503	2.808326261489
70.5	1.230457122656	0.9426414910922	0.3338068592338	2.823912885601
70.6	1.232202451908	0.9432226579476	0.3321611318837	2.839653913143
70.7	1.23394778116	0.9438009515832	0.3305143927132	2.855551747189
70.8	1.235693110412	0.9443763702375	0.3288666467386	2.871608840857
70.9	1.237438439664	0.9449489121575	0.3272178989791	2.887827698624
71.0	1.239183768916	0.9455185755993	0.3255681544572	2.904210877676
71.1	1.240929098168	0.9460853588275	0.3239174181981	2.920760989299

71.2	1.24267442742	0.9466492601157	0.3222656952305	2.937480700323
71.3	1.244419756672	0.947210277746	0.3206129905857	2.954372734604
71.4	1.246165085924	0.9477684100096	0.3189593092981	2.971439874557
71.5	1.247910415176	0.9483236552062	0.3173046564051	2.988684962743
71.6	1.249655744428	0.9488760116445	0.3156490369471	3.006110903495
71.7	1.25140107368	0.9494254776419	0.3139924559674	3.023720664615
71.8	1.253146402932	0.9499720515247	0.3123349185122	3.041517279111
71.9	1.254891732184	0.9505157316278	0.3106764296307	3.059503847001
72.0	1.256637061436	0.9510565162952	0.3090169943749	3.077683537175
72.1	1.258382390688	0.9515944038794	0.3073566177998	3.096059589318
72.2	1.26012771994	0.9521293927421	0.3056953049631	3.114635315898
72.3	1.261873049192	0.9526614812536	0.3040330609255	3.133414104222
72.4	1.263618378444	0.9531906677929	0.3023698907504	3.152399418564
72.5	1.265363707696	0.9537169507482	0.3007057995043	3.171594802363
72.6	1.267109036948	0.9542403285163	0.2990407922561	3.191003880498
72.7	1.2688543662	0.9547607995028	0.2973748740778	3.210630361639
72.8	1.270599695452	0.9552783621223	0.295708050044	3.230478040689
72.9	1.272345024704	0.9557930147983	0.2940403252323	3.2505508013
73.0	1.274090353956	0.956304755963	0.2923717047227	3.270852618484
73.1	1.275835683208	0.9568135840576	0.2907021935983	3.29138756132
73.2	1.27758101246	0.9573194975321	0.2890317969445	3.312159795747
73.3	1.279326341712	0.9578224948453	0.2873605198497	3.333173587472
73.4	1.281071670964	0.9583225744651	0.285688367405	3.354433304968
73.5	1.282817000216	0.9588197348682	0.2840153447039	3.375943422591
73.6	1.284562329468	0.9593139745401	0.2823414568429	3.39770852381
73.7	1.28630765872	0.9598052919752	0.2806667089208	3.419733304551
73.8	1.288052987972	0.9602936856769	0.2789911060392	3.442022576669
73.9	1.289798317224	0.9607791541576	0.2773146533024	3.464581271549
74.0	1.291543646476	0.9612616959383	0.275637355817	3.487414443841
74.1	1.293288975728	0.9617413095492	0.2739592186924	3.510527275335
74.2	1.29503430498	0.9622179935293	0.2722802470406	3.533925078986
74.3	1.296779634232	0.9626917464265	0.2706004459759	3.557613303092
74.4	1.298524963484	0.9631625667977	0.2689198206153	3.58159753563
74.5	1.300270292736	0.9636304532086	0.2672383760783	3.605883508761
74.6	1.302015621988	0.9640954042341	0.2655561174868	3.630477103515
74.7	1.30376095124	0.9645574184578	0.2638730499654	3.655384354652
74.8	1.305506280492	0.9650164944723	0.2621891786409	3.680611455724
74.9	1.307251609744	0.9654726308792	0.2605045086426	3.706164764325
75.0	1.308996938996	0.9659258262891	0.2588190451025	3.732050807569
75.1	1.310742268248	0.9663760793213	0.2571327931547	3.75827628777
75.2	1.3124875975	0.9668233886045	0.2554457579358	3.784848088366
75.3	1.314232926752	0.9672677527759	0.2537579445848	3.811773280078
75.4	1.315978256004	0.967709170482	0.2520693582431	3.839059127324
75.5	1.317723585256	0.9681476403781	0.2503800040544	3.866713094899
75.6	1.319468914508	0.9685831611286	0.2486898871649	3.89474285493
75.7	1.32121424376	0.9690157314069	0.2469990127227	3.923156294129
75.8	1.322959573012	0.9694453498951	0.2453073858788	3.951961521347
75.9	1.324704902264	0.9698720152847	0.243615011786	3.981166875449
76.0	1.326450231516	0.970295726276	0.2419218955997	4.010780933536
76.1	1.328195560768	0.9707164815782	0.2402280424773	4.040812519505
76.2	1.32994089002	0.9711342799096	0.2385334575786	4.071270712997
76.3	1.331686219272	0.9715491199976	0.2368381460656	4.102164858732
76.4	1.333431548524	0.9719610005785	0.2351421131026	4.13350457625

76.5	1.335176877776	0.9723699203977	0.2334453638559	4.16529977009
76.6	1.336922207028	0.9727758782094	0.2317479034942	4.197560640431
76.7	1.33866753628	0.9731788727771	0.2300497371881	4.230297694195
76.8	1.340412865532	0.9735789028732	0.2283508701107	4.263521756678
76.9	1.342158194784	0.9739759672791	0.2266513074369	4.297243983692
77.0	1.343903524036	0.9743700647852	0.2249510543439	4.331475874284
77.1	1.345648853288	0.9747611941912	0.223250116011	4.366229284035
77.2	1.34739418254	0.9751493543056	0.2215484976195	4.40151643899
77.3	1.349139511792	0.9755345439459	0.2198462043528	4.437349950242
77.4	1.350884841044	0.9759167619387	0.2181432413965	4.473742829212
77.5	1.352630170296	0.9762960071199	0.2164396139381	4.510708503662
77.6	1.354375499548	0.9766722783342	0.2147353271671	4.548260834484
77.7	1.3561208288	0.9770455744353	0.213030386275	4.586414133306
77.8	1.357866158052	0.9774158942861	0.2113247964554	4.625183180964
77.9	1.359611487304	0.9777832367586	0.2096185629038	4.664583246891
78.0	1.361356816556	0.9781476007338	0.2079116908178	4.704630109478
78.1	1.363102145808	0.9785089851018	0.2062041853966	4.745340077456
78.2	1.36484747506	0.9788673887617	0.2044960518418	4.786730012367
78.3	1.366592804312	0.9792228106218	0.2027872953565	4.828817352193
78.4	1.368338133564	0.9795752495993	0.201077921146	4.871620136197
78.5	1.370083462816	0.9799247046208	0.1993679344172	4.915157031071
78.6	1.371828792068	0.9802711746217	0.1976573403791	4.959447358451
78.7	1.37357412132	0.9806146585466	0.1959461442425	5.004511123898
78.8	1.375319450572	0.9809551553492	0.19423435122	5.050369047431
78.9	1.377064779824	0.9812926639922	0.1925219665259	5.097042595709
79.0	1.378810109076	0.9816271834477	0.1908089953765	5.14455401597
79.1	1.380555438328	0.9819587126964	0.1890954429899	5.192926371837
79.2	1.38230076758	0.9822872507287	0.1873813145857	5.242183581113
79.3	1.384046096831	0.9826127965436	0.1856666153856	5.292350455697
79.4	1.385791426083	0.9829353491496	0.1839513506127	5.343452743758
79.5	1.387536755335	0.983254907564	0.1822355254922	5.395517174319
79.6	1.389282084587	0.9835714708134	0.1805191452506	5.448571504414
79.7	1.391027413839	0.9838850379335	0.1788022151164	5.502644568991
79.8	1.392772743091	0.9841956079692	0.1770847403196	5.557766333751
79.9	1.394518072343	0.9845031799744	0.175366726092	5.613967951127
80.0	1.396263401595	0.9848077530122	0.1736481776669	5.671281819618
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80.3	1.401499389351	0.9857034690889	0.168489379565	5.850240956633
80.4	1.403244718603	0.9859960370705	0.1667687467161	5.912355021466
80.5	1.404990047855	0.9862856015372	0.1650476058607	5.975764364433
80.6	1.406735377107	0.986572161607	0.1633259622416	6.040510327118
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80.9	1.411971364863	0.9874138067509	0.1581580672545	6.243208607008
81.0	1.413716694115	0.9876883405951	0.1564344650402	6.313751514675
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81.9	1.429424657383	0.9900236577166	0.1409012319376	7.026366229041
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83.0	1.448623279155	0.9925461516413	0.1218693434052	8.144346427974
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83.5	1.457349925415	0.9935718556766	0.1132032137679	8.776887356869
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83.7	1.460840583919	0.9939609554552	0.1097343110911	9.057886686238
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83.9	1.464331242423	0.9943379441332	0.1062640713362	9.357235532477
84.0	1.466076571675	0.9945218953683	0.1045284632677	9.514364454222
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84.8	1.480039205691	0.995884398616	0.09063258019779	10.98815013809
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85.6	1.494001839707	0.9970527522269	0.07671902812683	12.99615983897
85.7	1.495747168959	0.9971851335251	0.07497872682634	13.29957410233
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86.3	1.506219144471	0.9979156182722	0.06453230825297	15.46381410007
86.4	1.507964473723	0.9980267284283	0.06279051952932	15.89454484386
86.5	1.509709802975	0.9981347984219	0.06104853953487	16.3498554761
86.6	1.511455132227	0.9982398279238	0.05930637357597	16.83191481342
86.7	1.513200461479	0.998341816614	0.05756402695958	17.34315455927
86.8	1.514945790731	0.998440764182	0.05582150499318	17.88631037992
86.9	1.516691119983	0.9985366703262	0.05407881298479	18.46447093073

87.0	1.518436449235	0.9986295347546	0.05233595624296	19.08113668772
87.1	1.520181778487	0.9987193571842	0.05059294007673	19.74029095106
87.2	1.521927107739	0.9988061373414	0.04884976979563	20.4464860637
87.3	1.523672436991	0.998889874962	0.04710645070965	21.20494878968
87.4	1.525417766243	0.9989705697907	0.04536298812927	22.02171001046
87.5	1.527163095495	0.9990482215819	0.04361938736535	22.90376554842
87.6	1.528908424747	0.9991228300989	0.04187565372921	23.85927719624
87.7	1.530653753999	0.9991943951144	0.04013179253257	24.89782618864
87.8	1.532399083251	0.9992629164106	0.03838780908753	26.03073580292
87.9	1.534144412503	0.9993283937787	0.03664370870657	27.27148613097
88.0	1.535889741755	0.9993908270191	0.03489949670251	28.63625328291
88.1	1.537635071007	0.9994502159418	0.03315517838854	30.14461886555
88.2	1.539380400259	0.9995065603657	0.03141075907814	31.82051595376
88.3	1.541125729511	0.9995598601194	0.02966624408512	33.69350893397
88.4	1.542871058763	0.9996101150404	0.02792163872358	35.80055328902
88.5	1.544616388015	0.9996573249756	0.02617694830789	38.18845929701
88.6	1.546361717267	0.9997014897812	0.02443217815267	40.91741160098
88.7	1.548107046519	0.9997426093227	0.02268733357279	44.06611319549
88.8	1.549852375771	0.9997806834748	0.02094241988337	47.73950140636
88.9	1.551597705023	0.9998157121216	0.0191974423997	52.08067258673
89.0	1.553343034275	0.9998476951564	0.0174524064373	57.28996163071
89.1	1.555088363527	0.9998766324817	0.01570731731183	63.65674116282
89.2	1.556833692779	0.9999025240093	0.01396218033916	71.61507011945
89.3	1.558579022031	0.9999253696605	0.01221700083526	81.84704111458
89.4	1.560324351283	0.9999451693655	0.01047178411626	95.48947517099
89.5	1.562069680535	0.9999619230642	0.008726535498388	114.5886501291
89.6	1.563815009787	0.9999756307054	0.006981260297976	143.2371216692
89.7	1.565560339039	0.9999862922474	0.005235963831434	190.9841863773
89.8	1.567305668291	0.9999939076578	0.003490651415238	286.4777340104
89.9	1.569050997543	0.9999984769133	0.001745328365913	572.9572133495
90.0	**1.570796326795**	**1**	**0**	**[infinite, undefined]**

Note that for example: 1.745329251994e-005 = 1.745329251994 (10^-5) = 0.00001745329251994

To find the trigonometric values of angles greater than 90°, convert the angle to its corresponding trigonometrically equivalent acute reference or corresponding angle that is less than or equal to 90°.
Ex. Sin 100° = Sin (180° - 100°) = Sin 80°

Here is an example of how to find the trigonometric values of angles that are between the table-list entry values.
A non-proportion method of interpolation is used.

Ex. Find SIN 37.5°

Here is a method that uses a common type of linear (line-like) interpolation using proportions:

$$\frac{\text{SIN } 37^\circ}{37^\circ} = \frac{\text{SIN } 37.5^\circ}{37.5^\circ} \qquad \text{after using substitution and solving for SIN } 37.5^\circ$$

$$\text{SIN } 37.5^\circ = \frac{37.5^\circ \ \text{SIN } 37.5^\circ}{37^\circ} = \frac{37.5^\circ\,(0.60181502315205)}{37^\circ} = 0.6099476586000507$$

For the angles in the method above, you can use it's degrees or radian equivalent and get the same result.

The true value of SIN 37.5° is 0.6087614290087208. The calculated value has a difference of: +0.001186229591329968 from this true value, or about: +0.001

Here is a slightly better linear interpolation method that uses the "point slope" form of a line which will consider a more correct slope value for those close points or area of the (sin) curve, here, points close to the known sin 37° Since the slope value of a line is a strict (ratio) numeric value, it is unitless, therefore, use radian (unitless, strict numeric) angle values.

$m = \dfrac{(y2 - y1)}{(x2 - x1)}$: slope of a line, between two points (locations) on it

$(y2 - y1) = m\,(x2 - x1)$: "points and slope" for m of a linear (line) equation

The slope of a tangent line (ie. between two close points on the curve) at a point on the SIN curve is not a constant value as for lines, since the SIN curve is non-algebraic and non-linear. The slope value on a line or curve is equal to the derivative of the curve at that point or location. The derivative of the SIN curve at any point, hence also the entire curve, is equal to the COS curve.

Here, the (x) values are the angle values, and the (y) values are the corresponding SIN values:

$(y2 - y1) = m \quad (x2 - x1)$

(SIN 37.5° - SIN 37°) = COS 37° (37.5° - 37°) After some simplification and transposing:

SIN 37.5° = COS 37° (0.5°) + SIN 37° 0.5° = 0.00872664625997165 (radians):

SIN 37.5° = 0.79863551004729 (0.00872664625997165) + 0.6018150231520484

SIN 37.5° = 0.6087844327388832 : a calculated ("close" mathematical estimate-approximation) interpolated value

This calculated value differs from the true value by only: 0.000230037301623964, or about:: +0.0002

This above table, and many others in the book, were made with the assistance of a computer which is essentially a fancy automatic calculator machine with a typing (letters and numbers) keyboard input screen. This electronic machine can process what are called programs which are composed of smaller steps of the desired calculation, and for its input and output such as on the screen and-or printer. In the old days, tables such as this were calculated by hand and pen, usually with many people involved, and some to compare or check the work of the others. But once it was done, it was treasured and distributed in books to be used for many years and even till this day.

TABLE OF e^x

0.000	1	0.001	1.0010005001667	0.002	1.002002001334	0.003	1.0030045045034
0.004	1.0040080106773	0.005	1.0050125208594	0.006	1.0060180360541	0.007	1.0070245572668
0.008	1.0080320855043	0.009	1.0090406217739	0.010	1.0100501670842	0.011	1.0110607224447
0.012	1.0120722888661	0.013	1.0130848673598	0.014	1.0140984589385	0.015	1.0151130646157
0.016	1.0161286854061	0.017	1.0171453223252	0.018	1.0181629763898	0.019	1.0191816486174
0.020	1.0202013400268	0.021	1.0212220516375	0.022	1.0222437844704	0.023	1.0232665395472
0.024	1.0242903178906	0.025	1.0253151205244	0.026	1.0263409484734	0.027	1.0273678027635
0.028	1.0283956844214	0.029	1.0294245944751	0.030	1.0304545339535	0.031	1.0314855038865
0.032	1.0325175053051	0.033	1.0335505392413	0.034	1.0345846067281	0.035	1.0356197087996
0.036	1.0366558464909	0.037	1.0376930208382	0.038	1.0387312328785	0.039	1.0397704836502
0.040	1.0408107741924	0.041	1.0418521055455	0.042	1.0428944787508	0.043	1.0439378948506
0.044	1.0449823548884	0.045	1.0460278599087	0.046	1.0470744109569	0.047	1.0481220090797
0.048	1.0491706553245	0.049	1.05022035074	0.050	1.051271096376	0.051	1.0523228932832
0.052	1.0533757425134	0.053	1.0544296451194	0.054	1.0554846021551	0.055	1.0565406146755
0.056	1.0575976837366	0.057	1.0586558103955	0.058	1.0597149957103	0.059	1.0607752407402
0.060	1.0618365465454	0.061	1.0628989141872	0.062	1.063962344728	0.063	1.0650268392313
0.064	1.0660923987615	0.065	1.0671590243842	0.066	1.068226717166	0.067	1.0692954781746
0.068	1.0703653084788	0.069	1.0714362091483	0.070	1.0725081812542	0.071	1.0735812258684
0.072	1.0746553440638	0.073	1.0757305369147	0.074	1.0768068054962	0.075	1.0778841508846
0.076	1.0789625741573	0.077	1.0800420763926	0.078	1.0811226586701	0.079	1.0822043220703
0.080	1.083287067675	0.081	1.0843708965668	0.082	1.0854558098295	0.083	1.0865418085482
0.084	1.0876288938088	0.085	1.0887170666984	0.086	1.0898063283051	0.087	1.0908966797183
0.088	1.0919881220282	0.089	1.0930806563263	0.090	1.0941742837052	0.091	1.0952690052585
0.092	1.0963648220808	0.093	1.0974617352681	0.094	1.0985597459172	0.095	1.0996588551261
0.096	1.100759063994	0.097	1.101860373621	0.098	1.1029627851085	0.099	1.1040662995589
0.100	1.1051709180756	0.101	1.1062766417634	0.102	1.1073834717279	0.103	1.108491409076
0.104	1.1096004549156	0.105	1.1107106103557	0.106	1.1118218765065	0.107	1.1129342544793
0.108	1.1140477453865	0.109	1.1151623503414	0.110	1.1162780704589	0.111	1.1173949068545
0.112	1.118512860645	0.113	1.1196319329486	0.114	1.1207521248842	0.115	1.1218734375719
0.116	1.1229958721333	0.117	1.1241194296905	0.118	1.1252441113673	0.119	1.1263699182884
0.120	1.1274968515794	0.121	1.1286249123673	0.122	1.1297541017803	0.123	1.1308844209475
0.124	1.1320158709992	0.125	1.1331484530668	0.126	1.134282168283	0.127	1.1354170177815
0.128	1.1365530026971	0.129	1.1376901241657	0.130	1.1388283833246	0.131	1.139967781312
0.132	1.1411083192672	0.133	1.1422499983309	0.134	1.1433928196446	0.135	1.1445367843513
0.136	1.1456818935949	0.137	1.1468281485204	0.138	1.1479755502742	0.139	1.1491241000036
0.140	1.1502737988572	0.141	1.1514246479847	0.142	1.152576648537	0.143	1.153729801666
0.144	1.1548841085249	0.145	1.156039570268	0.146	1.1571961880508	0.147	1.1583539630299
0.148	1.159512896363	0.149	1.1606729892091	0.150	1.1618342427283	0.151	1.1629966580818
0.152	1.1641602364321	0.153	1.1653249789427	0.154	1.1664908867784	0.155	1.1676579611051
0.156	1.1688262030899	0.157	1.1699956139009	0.158	1.1711661947077	0.159	1.1723379466807
0.160	1.1735108709918	0.161	1.1746849688139	0.162	1.175860241321	0.163	1.1770366896885
0.164	1.1782143150927	0.165	1.1793931187114	0.166	1.1805731017233	0.167	1.1817542653084
0.168	1.1829366106478	0.169	1.184120138924	0.170	1.1853048513204	0.171	1.1864907490217
0.172	1.1876778332139	0.173	1.188866105084	0.174	1.1900555658204	0.175	1.1912462166124
0.176	1.1924380586507	0.177	1.1936310931271	0.178	1.1948253212348	0.179	1.1960207441679
0.180	1.1972173631218	0.181	1.1984151792932	0.182	1.1996141938799	0.183	1.2008144080808
0.184	1.2020158230963	0.185	1.2032184401277	0.186	1.2044222603776	0.187	1.2056272850499
0.188	1.2068335153496	0.189	1.2080409524829	0.190	1.2092495976573	0.191	1.2104594520813
0.192	1.2116705169649	0.193	1.2128827935191	0.194	1.2140962829562	0.195	1.2153109864897
0.196	1.2165269053343	0.197	1.2177440407059	0.198	1.2189623938216	0.199	1.2201819658999

0.200	1.2214027581602	0.201	1.2226247718233	0.202	1.2238480081114	0.203	1.2250724682475
0.204	1.2262981534562	0.205	1.2275250649632	0.206	1.2287532039953	0.207	1.2299825717808
0.208	1.2312131695489	0.209	1.2324449985303	0.210	1.2336780599567	0.211	1.2349123550614
0.212	1.2361478850785	0.213	1.2373846512436	0.214	1.2386226547935	0.215	1.2398618969661
0.216	1.2411023790007	0.217	1.2423441021378	0.218	1.2435870676191	0.219	1.2448312766875
0.220	1.2460767305874	0.221	1.2473234305641	0.222	1.2485713778643	0.223	1.249820573736
0.224	1.2510710194284	0.225	1.2523227161919	0.226	1.2535756652782	0.227	1.2548298679403
0.228	1.2560853254323	0.229	1.2573420390098	0.230	1.2586000099295	0.231	1.2598592394492
0.232	1.2611197288283	0.233	1.2623814793273	0.234	1.2636444922078	0.235	1.2649087687329
0.236	1.2661743101669	0.237	1.2674411177753	0.238	1.2687091928249	0.239	1.2699785365838
0.240	1.2712491503214	0.241	1.2725210353082	0.242	1.2737941928162	0.243	1.2750686241185
0.244	1.2763443304895	0.245	1.2776213132049	0.246	1.2788995735417	0.247	1.2801791127783
0.248	1.281459932194	0.249	1.2827420330698	0.250	1.2840254166877	0.251	1.2853100843312
0.252	1.2865960372848	0.253	1.2878832768346	0.254	1.2891718042678	0.255	1.2904616208729
0.256	1.2917527279397	0.257	1.2930451267594	0.258	1.2943388186242	0.259	1.295633804828
0.260	1.2969300866658	0.261	1.2982276654337	0.262	1.2995265424294	0.263	1.3008267189517
0.264	1.3021281963009	0.265	1.3034309757784	0.266	1.3047350586869	0.267	1.3060404463307
0.268	1.3073471400149	0.269	1.3086551410465	0.270	1.3099644507332	0.271	1.3112750703846
0.272	1.3125870013111	0.273	1.3139002448247	0.274	1.3152148022387	0.275	1.3165306748676
0.276	1.3178478640273	0.277	1.319166371035	0.278	1.3204861972091	0.279	1.3218073438695
0.280	1.3231298123374	0.281	1.3244536039353	0.282	1.3257787199868	0.283	1.3271051618172
0.284	1.3284329307528	0.285	1.3297620281215	0.286	1.3310924552523	0.287	1.3324242134757
0.288	1.3337573041234	0.289	1.3350917285285	0.290	1.3364274880255	0.291	1.33776458395
0.292	1.3391030176393	0.293	1.3404427904317	0.294	1.341783903667	0.295	1.3431263586863
0.296	1.3444701568321	0.297	1.3458152994481	0.298	1.3471617878796	0.299	1.3485096234729
0.300	1.349858807576	0.301	1.351209341538	0.302	1.3525612267095	0.303	1.3539144644423
0.304	1.3552690560897	0.305	1.3566250030062	0.306	1.3579823065479	0.307	1.359340968072
0.308	1.3607009889372	0.309	1.3620623705034	0.310	1.3634251141322	0.311	1.3647892211862
0.312	1.3661546930295	0.313	1.3675215310276	0.314	1.3688897365474	0.315	1.370259310957
0.316	1.371630255626	0.317	1.3730025719255	0.318	1.3743762612276	0.319	1.375751324906
0.320	1.377127764336	0.321	1.3785055808938	0.322	1.3798847759572	0.323	1.3812653509056
0.324	1.3826473071195	0.325	1.3840306459808	0.326	1.3854153688728	0.327	1.3868014771803
0.328	1.3881889722894	0.329	1.3895778555876	0.330	1.3909681284638	0.331	1.3923597923082
0.332	1.3937528485125	0.333	1.3951472984698	0.334	1.3965431435745	0.335	1.3979403852225
0.336	1.3993390248109	0.337	1.4007390637385	0.338	1.4021405034053	0.339	1.4035433452127
0.340	1.4049475905636	0.341	1.4063532408622	0.342	1.4077602975141	0.343	1.4091687619265
0.344	1.4105786355077	0.345	1.4119899196677	0.346	1.4134026158177	0.347	1.4148167253704
0.348	1.41623224974	0.349	1.417649190342	0.350	1.4190675485933	0.351	1.4204873259122
0.352	1.4219085237186	0.353	1.4233311434336	0.354	1.4247551864799	0.355	1.4261806542815
0.356	1.4276075482638	0.357	1.4290358698539	0.358	1.4304656204799	0.359	1.4318968015717
0.360	1.4333294145603	0.361	1.4347634608786	0.362	1.4361989419604	0.363	1.4376358592412
0.364	1.439074214158	0.365	1.4405140081492	0.366	1.4419552426545	0.367	1.4433979191152
0.368	1.4448420389739	0.369	1.4462876036747	0.370	1.4477346146633	0.371	1.4491830733866
0.372	1.4506329812932	0.373	1.4520843398328	0.374	1.4535371504569	0.375	1.4549914146182
0.376	1.4564471337711	0.377	1.4579043093712	0.378	1.4593629428758	0.379	1.4608230357434
0.380	1.4622845894342	0.381	1.4637476054097	0.382	1.465212085133	0.383	1.4666780300684
0.384	1.468145441682	0.385	1.4696143214411	0.386	1.4710846708147	0.387	1.4725564912731
0.388	1.4740297842881	0.389	1.4755045513331	0.390	1.4769807938826	0.391	1.4784585134131
0.392	1.4799377114023	0.393	1.4814183893293	0.394	1.4829005486748	0.395	1.4843841909209
0.396	1.4858693175514	0.397	1.4873559300513	0.398	1.4888440299073	0.399	1.4903336186074
0.400	1.4918246976413	0.401	1.4933172685	0.402	1.494811332676	0.403	1.4963068916636
0.404	1.4978039469581	0.405	1.4993025000568	0.406	1.500802552458	0.407	1.502304105662
0.408	1.5038071611701	0.409	1.5053117204856	0.410	1.5068177851129	0.411	1.5083253565581

0.412	1.5098344363287	0.413	1.511345025934	0.414	1.5128571268844	0.415	1.514370740692
0.416	1.5158858688706	0.417	1.5174025129351	0.418	1.5189206744022	0.419	1.5204403547902
0.420	1.5219615556186	0.421	1.5234842784088	0.422	1.5250085246833	0.423	1.5265342959665
0.424	1.5280615937841	0.425	1.5295904196634	0.426	1.5311207751332	0.427	1.532652661724
0.428	1.5341860809676	0.429	1.5357210343973	0.430	1.5372575235483	0.431	1.5387955499569
0.432	1.5403351151611	0.433	1.5418762207006	0.434	1.5434188681165	0.435	1.5449630589513
0.436	1.5465087947494	0.437	1.5480560770563	0.438	1.5496049074195	0.439	1.5511552873877
0.440	1.5527072185113	0.441	1.5542607023423	0.442	1.5558157404341	0.443	1.5573723343418
0.444	1.5589304856219	0.445	1.5604901958327	0.446	1.5620514665337	0.447	1.5636142992864
0.448	1.5651786956535	0.449	1.5667446571995	0.450	1.5683121854902	0.451	1.5698812820932
0.452	1.5714519485776	0.453	1.5730241865142	0.454	1.574597997475	0.455	1.576173383034
0.456	1.5777503447665	0.457	1.5793288842494	0.458	1.5809090030614	0.459	1.5824907027825
0.460	1.5840739849945	0.461	1.5856588512805	0.462	1.5872453032256	0.463	1.5888333424161
0.464	1.59042297044	0.465	1.5920141888871	0.466	1.5936069993485	0.467	1.595201403417
0.468	1.5967974026871	0.469	1.5983949987546	0.470	1.5999941932174	0.471	1.6015949876744
0.472	1.6031973837266	0.473	1.6048013829763	0.474	1.6064069870275	0.475	1.6080141974858
0.476	1.6096230159584	0.477	1.6112334440542	0.478	1.6128454833836	0.479	1.6144591355586
0.480	1.6160744021929	0.481	1.6176912849017	0.482	1.6193097853019	0.483	1.6209299050121
0.484	1.6225516456523	0.485	1.6241750088442	0.486	1.6257999962113	0.487	1.6274266093786
0.488	1.6290548499726	0.489	1.6306847196215	0.490	1.6323162199554	0.491	1.6339493526056
0.492	1.6355841192052	0.493	1.6372205213892	0.494	1.6388585607938	0.495	1.640498239057
0.496	1.6421395578187	0.497	1.6437825187201	0.498	1.6454271234041	0.499	1.6470733735153
0.500	1.6487212707001	0.501	1.6503708166063	0.502	1.6520220128835	0.503	1.6536748611828
0.504	1.6553293631571	0.505	1.6569855204609	0.506	1.6586433347503	0.507	1.6603028076832
0.508	1.6619639409191	0.509	1.6636267361191	0.510	1.6652911949459	0.511	1.666957319064
0.512	1.6686251101397	0.513	1.6702945698405	0.514	1.6719656998361	0.515	1.6736385017975
0.516	1.6753129773976	0.517	1.6769891283108	0.518	1.6786669562132	0.519	1.6803464627827
0.520	1.6820276496989	0.521	1.6837105186428	0.522	1.6853950712974	0.523	1.6870813093472
0.524	1.6887692344785	0.525	1.6904588483791	0.526	1.6921501527387	0.527	1.6938431492486
0.528	1.6955378396018	0.529	1.697234225493	0.530	1.6989323086186	0.531	1.7006320906766
0.532	1.7023335733668	0.533	1.7040367583907	0.534	1.7057416474516	0.535	1.7074482422542
0.536	1.7091565445052	0.537	1.7108665559129	0.538	1.7125782781873	0.539	1.7142917130402
0.540	1.7160068621849	0.541	1.7177237273365	0.542	1.7194423102121	0.543	1.7211626125301
0.544	1.7228846360109	0.545	1.7246083823764	0.546	1.7263338533505	0.547	1.7280610506586
0.548	1.7297899760278	0.549	1.7315206311872	0.550	1.7332530178674	0.551	1.7349871378007
0.552	1.7367229927213	0.553	1.7384605843651	0.554	1.7401999144695	0.555	1.7419409847741
0.556	1.7436837970197	0.557	1.7454283529493	0.558	1.7471746543074	0.559	1.7489227028404
0.560	1.7506725002961	0.561	1.7524240484245	0.562	1.7541773489771	0.563	1.7559324037072
0.564	1.7576892143698	0.565	1.7594477827218	0.566	1.7612081105217	0.567	1.7629701995299
0.568	1.7647340515085	0.569	1.7664996682212	0.570	1.7682670514337	0.571	1.7700362029135
0.572	1.7718071244296	0.573	1.7735798177529	0.574	1.7753542846563	0.575	1.777130526914
0.576	1.7789085463025	0.577	1.7806883445996	0.578	1.7824699235852	0.579	1.7842532850409
0.580	1.7860384307501	0.581	1.7878253624978	0.582	1.789614082071	0.583	1.7914045912585
0.584	1.7931968918507	0.585	1.7949909856399	0.586	1.7967868744203	0.587	1.7985845599877
0.588	1.8003840441398	0.589	1.8021853286761	0.590	1.8039884153979	0.591	1.8057933061082
0.592	1.807600002612	0.593	1.809408506716	0.594	1.8112188202286	0.595	1.8130309449602
0.596	1.8148448827228	0.597	1.8166606353306	0.598	1.8184782045991	0.599	1.8202975923459
0.600	1.8221188003905	0.601	1.8239418305541	0.602	1.8257666846596	0.603	1.827593364532
0.604	1.8294218719979	0.605	1.8312522088858	0.606	1.833084377026	0.607	1.8349183782509
0.608	1.8367542143942	0.609	1.8385918872919	0.610	1.8404313987816	0.611	1.8422727507029
0.612	1.8441159448971	0.613	1.8459609832074	0.614	1.8478078674789	0.615	1.8496565995583
0.616	1.8515071812945	0.617	1.8533596145381	0.618	1.8552139011414	0.619	1.8570700429588
0.620	1.8589280418463	0.621	1.8607878996621	0.622	1.8626496182659	0.623	1.8645131995195

0.624	1.8663786452865	0.625	1.8682459574322	0.626	1.8701151378241	0.627	1.8719861883312
0.628	1.8738591108247	0.629	1.8757339071775	0.630	1.8776105792643	0.631	1.8794891289619
0.632	1.8813695581488	0.633	1.8832518687053	0.634	1.8851360625139	0.635	1.8870221414587
0.636	1.8889101074259	0.637	1.8907999623032	0.638	1.8926917079807	0.639	1.8945853463501
0.640	1.896480879305	0.641	1.8983783087409	0.642	1.9002776365552	0.643	1.9021788646474
0.644	1.9040819949186	0.645	1.9059870292719	0.646	1.9078939696125	0.647	1.9098028178471
0.648	1.9117135758847	0.649	1.9136262456361	0.650	1.9155408290139	0.651	1.9174573279327
0.652	1.9193757443089	0.653	1.9212960800611	0.654	1.9232183371095	0.655	1.9251425173764
0.656	1.9270686227859	0.657	1.9289966552643	0.658	1.9309266167395	0.659	1.9328585091414
0.660	1.934792334402	0.661	1.9367280944551	0.662	1.9386657912365	0.663	1.9406054266838
0.664	1.9425470027368	0.665	1.9444905213368	0.666	1.9464359844276	0.667	1.9483833939545
0.668	1.950332751865	0.669	1.9522840601083	0.670	1.9542373206359	0.671	1.956192535401
0.672	1.9581497063588	0.673	1.9601088354665	0.674	1.9620699246831	0.675	1.9640329759698
0.676	1.9659979912897	0.677	1.9679649726078	0.678	1.9699339218909	0.679	1.9719048411082
0.680	1.9738777322304	0.681	1.9758525972306	0.682	1.9778294380835	0.683	1.9798082567661
0.684	1.981789055257	0.685	1.9837718355372	0.686	1.9857565995893	0.687	1.9877433493983
0.688	1.9897320869507	0.689	1.9917228142354	0.690	1.9937155332431	0.691	1.9957102459665
0.692	1.9977069544003	0.693	1.9997056605412	0.694	2.0017063663879	0.695	2.0037090739412
0.696	2.0057137852037	0.697	2.0077205021802	0.698	2.0097292268773	0.699	2.0117399613038
0.700	2.0137527074705	0.701	2.01576746739	0.702	2.0177842430772	0.703	2.0198030365488
0.704	2.0218238498235	0.705	2.0238466849223	0.706	2.025871543868	0.707	2.0278984286854
0.708	2.0299273414013	0.709	2.0319582840448	0.710	2.0339912586468	0.711	2.0360262672401
0.712	2.0380633118599	0.713	2.0401023945432	0.714	2.042143517329	0.715	2.0441866822586
0.716	2.0462318913749	0.717	2.0482791467234	0.718	2.0503284503511	0.719	2.0523798043075
0.720	2.0544332106439	0.721	2.0564886714136	0.722	2.0585461886722	0.723	2.0606057644772
0.724	2.062667400888	0.725	2.0647310999665	0.726	2.0667968637762	0.727	2.068864694383
0.728	2.0709345938546	0.729	2.073006564261	0.730	2.0750806076741	0.731	2.077156726168
0.732	2.0792349218188	0.733	2.0813151967048	0.734	2.083397552906	0.735	2.085481992505
0.736	2.0875685175862	0.737	2.0896571302361	0.738	2.0917478325432	0.739	2.0938406265984
0.740	2.0959355144944	0.741	2.098032498326	0.742	2.1001315801904	0.743	2.1022327621865
0.744	2.1043360464155	0.745	2.1064414349807	0.746	2.1085489299876	0.747	2.1106585335436
0.748	2.1127702477582	0.749	2.1148840747433	0.750	2.1170000166127	0.751	2.1191180754822
0.752	2.12123825347	0.753	2.1233605526962	0.754	2.1254849752832	0.755	2.1276115233553
0.756	2.1297401990391	0.757	2.1318710044633	0.758	2.1340039417587	0.759	2.1361390130581
0.760	2.1382762204968	0.761	2.1404155662119	0.762	2.1425570523427	0.763	2.1447006810308
0.764	2.1468464544197	0.765	2.1489943746552	0.766	2.1511444438853	0.767	2.15329666426
0.768	2.1554510379316	0.769	2.1576075670544	0.770	2.1597662537849	0.771	2.1619271002819
0.772	2.1640901087061	0.773	2.1662552812207	0.774	2.1684226199906	0.775	2.1705921271834
0.776	2.1727638049685	0.777	2.1749376555176	0.778	2.1771136810046	0.779	2.1792918836053
0.780	2.1814722654982	0.781	2.1836548288635	0.782	2.1858395758838	0.783	2.1880265087439
0.784	2.1902156296306	0.785	2.1924069407332	0.786	2.1946004442429	0.787	2.1967961423532
0.788	2.1989940372599	0.789	2.2011941311608	0.790	2.2033964262559	0.791	2.2056009247477
0.792	2.2078076288406	0.793	2.2100165407413	0.794	2.2122276626588	0.795	2.2144409968041
0.796	2.2166565453905	0.797	2.2188743106337	0.798	2.2210942947514	0.799	2.2233164999636
0.800	2.2255409284925	0.801	2.2277675825624	0.802	2.2299964644002	0.803	2.2322275762346
0.804	2.2344609202967	0.805	2.23669649882	0.806	2.2389343140399	0.807	2.2411743681944
0.808	2.2434166635234	0.809	2.2456612022692	0.810	2.2479079866765	0.811	2.2501570189919
0.812	2.2524083014645	0.813	2.2546618363456	0.814	2.2569176258888	0.815	2.2591756723497
0.816	2.2614359779865	0.817	2.2636985450595	0.818	2.2659633758312	0.819	2.2682304725665
0.820	2.2704998375324	0.821	2.2727714729984	0.822	2.275045381236	0.823	2.2773215645192
0.824	2.2796000251241	0.825	2.2818807653293	0.826	2.2841637874154	0.827	2.2864490936655
0.828	2.2887366863649	0.829	2.2910265678012	0.830	2.2933187402642	0.831	2.2956132060461
0.832	2.2979099674415	0.833	2.300209026747	0.834	2.3025103862617	0.835	2.304814048287

0.836	2.3071200151266	0.837	2.3094282890863	0.838	2.3117388724745	0.839	2.3140517676018
0.840	2.3163669767811	0.841	2.3186845023275	0.842	2.3210043465586	0.843	2.3233265117943
0.844	2.3256510003567	0.845	2.3279778145702	0.846	2.3303069567618	0.847	2.3326384292605
0.848	2.3349722343979	0.849	2.3373083745076	0.850	2.339646851926	0.851	2.3419876689914
0.852	2.3443308280446	0.853	2.3466763314289	0.854	2.3490241814897	0.855	2.3513743805749
0.856	2.3537269310347	0.857	2.3560818352215	0.858	2.3584390954905	0.859	2.3607987141987
0.860	2.3631606937058	0.861	2.3655250363738	0.862	2.3678917445671	0.863	2.3702608206522
0.864	2.3726322669984	0.865	2.3750060859771	0.866	2.3773822799621	0.867	2.3797608513295
0.868	2.382141802458	0.869	2.3845251357285	0.870	2.3869108535243	0.871	2.3892989582311
0.872	2.3916894522372	0.873	2.3940823379329	0.874	2.3964776177111	0.875	2.3988752939671
0.876	2.4012753690986	0.877	2.4036778455057	0.878	2.4060827255909	0.879	2.4084900117589
0.880	2.4108997064172	0.881	2.4133118119754	0.882	2.4157263308456	0.883	2.4181432654423
0.884	2.4205626181825	0.885	2.4229843914856	0.886	2.4254085877732	0.887	2.4278352094696
0.888	2.4302642590014	0.889	2.4326957387977	0.890	2.4351296512899	0.891	2.4375659989119
0.892	2.4400047841002	0.893	2.4424460092935	0.894	2.444889676933	0.895	2.4473357894623
0.896	2.4497843493277	0.897	2.4522353589776	0.898	2.454688820863	0.899	2.4571447374375
0.900	2.459603111157	0.901	2.4620639444797	0.902	2.4645272398666	0.903	2.4669929997809
0.904	2.4694612266885	0.905	2.4719319230575	0.906	2.4744050913586	0.907	2.476880734065
0.908	2.4793588536523	0.909	2.4818394525987	0.910	2.4843225333848	0.911	2.4868080984936
0.912	2.4892961504107	0.913	2.4917866916242	0.914	2.4942797246246	0.915	2.4967752519049
0.916	2.4992732759607	0.917	2.5017737992899	0.918	2.5042768243932	0.919	2.5067823537734
0.920	2.5092903899363	0.921	2.5118009353898	0.922	2.5143139926443	0.923	2.5168295642131
0.924	2.5193476526117	0.925	2.5218682603581	0.926	2.5243913899731	0.927	2.5269170439796
0.928	2.5294452249033	0.929	2.5319759352725	0.930	2.5345091776179	0.931	2.5370449544726
0.932	2.5395832683725	0.933	2.5421241218559	0.934	2.5446675174636	0.935	2.547213457739
0.936	2.5497619452281	0.937	2.5523129824794	0.938	2.5548665720438	0.939	2.5574227164751
0.940	2.5599814183293	0.941	2.5625426801651	0.942	2.5651065045438	0.943	2.5676728940292
0.944	2.5702418511877	0.945	2.5728133785883	0.946	2.5753874788025	0.947	2.5779641544044
0.948	2.5805434079706	0.949	2.5831252420805	0.950	2.5857096593158	0.951	2.5882966622611
0.952	2.5908862535031	0.953	2.5934784356317	0.954	2.5960732112389	0.955	2.5986705829195
0.956	2.601270553271	0.957	2.6038731248932	0.958	2.6064783003887	0.959	2.6090860823628
0.960	2.6116964734231	0.961	2.6143094761802	0.962	2.6169250932469	0.963	2.619543327239
0.964	2.6221641807746	0.965	2.6247876564746	0.966	2.6274137569625	0.967	2.6300424848643
0.968	2.6326738428089	0.969	2.6353078334275	0.970	2.6379444593542	0.971	2.6405837232255
0.972	2.6432256276808	0.973	2.6458701753619	0.974	2.6485173689135	0.975	2.6511672109826
0.976	2.6538197042192	0.977	2.6564748512757	0.978	2.6591326548072	0.979	2.6617931174716
0.980	2.6644562419294	0.981	2.6671220308437	0.982	2.6697904868801	0.983	2.6724616127073
0.984	2.6751354109964	0.985	2.6778118844211	0.986	2.6804910356578	0.987	2.6831728673859
0.988	2.685857382287	0.989	2.6885445830457	0.990	2.6912344723493	0.991	2.6939270528875
0.992	2.696622327353	0.993	2.6993202984411	0.994	2.7020209688497	0.995	2.7047243412795
0.996	2.7074304184338	0.997	2.7101392030188	0.998	2.7128506977432	0.999	2.7155649053186

1.0	**2.718281828459**	1.1	3.0041660239464	1.2	3.3201169227365	1.3	3.6692966676192
1.4	4.0551999668447	1.5	4.4816890703381	1.6	4.9530324243951	1.7	5.4739473917272
1.8	6.049647464413	1.9	6.6858944422793	2.0	7.3890560989307	2.1	8.1661699125677
2.2	9.0250134994341	2.3	9.9741824548147	2.4	11.023176380642	2.5	12.182493960703
2.6	13.463738035002	2.7	14.879731724873	2.8	16.444646771097	2.9	18.174145369443
3.0	20.085536923188	3.1	22.197951281442	3.2	24.532530197109	3.3	27.112638920658
3.4	29.964100047397	3.5	33.115451958692	3.6	36.598234443678	3.7	40.447304360067
3.8	44.701184493301	3.9	49.40244910553	4.0	54.598150033144	4.1	60.340287597362
4.2	66.686331040925	4.3	73.699793699596	4.4	81.450868664968	4.5	90.017131300522
4.6	99.484315641934	4.7	109.94717245212	4.8	121.51041751873	4.9	134.28977968494

5.0	148.41315910258	5.1	164.0219072999	5.2	181.27224187515	5.3	200.33680997479
5.4	221.40641620419	5.5	244.69193226422	5.6	270.42640742615	5.7	298.86740096706
5.8	330.29955990965	5.9	365.03746786533	6.0	403.42879349273	6.1	445.85777008251
6.2	492.74904109325	6.3	544.57191012593	6.4	601.84503787208	6.5	665.14163304436
6.6	735.09518924197	6.7	812.40582516754	6.8	897.84729165041	6.9	992.27471560502
7.0	1096.6331584284	7.1	1211.9670744926	7.2	1339.4307643944	7.3	1480.2999275845
7.4	1635.9844299959	7.5	1808.042414456	7.6	1998.1958951041	7.7	2208.3479918872
7.8	2440.6019776245	7.9	2697.2823282685	8.0	2980.9579870417	8.1	3294.4680752838
8.2	3640.9503073323	8.3	4023.8723938223	8.4	4447.0667476998	8.5	4914.7688402991
8.6	5431.6595913629	8.7	6002.9122172609	8.8	6634.2440062778	8.9	7331.9735391559
9.0	8103.0839275753	9.1	8955.2927034824	9.2	9897.1290587438	9.3	10938.019208165
9.4	12088.380730217	9.5	13359.726829662	9.6	14764.781565577	9.7	16317.607198015
9.8	18033.744927828	9.9	19930.37043823	10.0	22026.465794806		

To calculate a value that is not listed in the table above, we can use a method that is similar to that previously described above for calculating a value that is not listed in the SIN x table.

Since the slope of the e^x curve is equal to e^x we will use:

(y2 - y1) = m (x2 - x1)

(y2 - y1) = e^x (x2 - x1)

Here are two entries from the table, and we need to find the value of e^0.5405:

e^0.540 = 1.7160068621849 and e^0.541 = 1.7177237273365

The result should be a value that is someplace between these to values, of about 1.7165.

(e^0.5405 - e^0.540) = e^x (0.5405 - 0.540) After some simplification and transposing:

e^0.5405 = e^0.540 (0.0005) + e^0.540

e^0.5405 = 0.000858003 + 1.7160068621849

e^0.5405 = 1.7168648651849 : a calculated ("close" mathematical estimate-approximation) interpolated value

This calculated value differs from the true value by -0.00085822, or about:: -0.0008, which is less than a thousandth.

TABLE OF ln x

0.000	(undefined, infinite)	0.001	-6.9077552789821	0.002	-6.2146080984222	0.003	-5.809142990314
0.004	-5.5214609178622	0.005	-5.298317366548	0.006	-5.1159958097541	0.007	-4.9618451299268
0.008	-4.8283137373023	0.009	-4.7105307016459	0.010	-4.6051701859881	0.011	-4.5098600061838
0.012	-4.4228486291941	0.013	-4.3428059215206	0.014	-4.2686979493669	0.015	-4.1997050778799
0.016	-4.1351665567424	0.017	-4.0745419349259	0.018	-4.017383521086	0.019	-3.9633162998157
0.020	-3.9120230054281	0.021	-3.8632328412587	0.022	-3.8167128256238	0.023	-3.772261063053
0.024	-3.7297014486342	0.025	-3.6888794541139	0.026	-3.6496587409607	0.027	-3.6119184129778
0.028	-3.5755507688069	0.029	-3.5404594489957	0.030	-3.50655789732	0.031	-3.473768074497
0.032	-3.4420193761824	0.033	-3.4112477175157	0.034	-3.381394754366	0.035	-3.3524072174927
0.036	-3.324236340526	0.037	-3.2968373663379	0.038	-3.2701691192558	0.039	-3.2441936328525
0.040	-3.2188758248682	0.041	-3.1941832122778	0.042	-3.1700856606988	0.043	-3.1465551632886
0.044	-3.1235656450639	0.045	-3.1010927892118	0.046	-3.079113882493	0.047	-3.0576076772721
0.048	-3.0365542680742	0.049	-3.0159349808715	0.050	-2.995732273554	0.051	-2.9759296462578
0.052	-2.9565115604007	0.053	-2.93746336543	0.054	-2.9187712324179	0.055	-2.9004220937497
0.056	-2.882403588247	0.057	-2.8647040111476	0.058	-2.8473122684357	0.059	-2.8302178350764
0.060	-2.81341071676	0.061	-2.7968814148088	0.062	-2.780620893937	0.063	-2.7646205525906
0.064	-2.7488721956225	0.065	-2.7333680090865	0.066	-2.7181005369557	0.067	-2.7030626595912
0.068	-2.688247573806	0.069	-2.6736487743849	0.070	-2.6592600369328	0.071	-2.6450754019408
0.072	-2.6310891599661	0.073	-2.6172958378337	0.074	-2.603690185778	0.075	-2.5902671654458
0.076	-2.5770219386958	0.077	-2.5639498571285	0.078	-2.5510464522925	0.079	-2.5383074265151
0.080	-2.5257286443083	0.081	-2.5133061243097	0.082	-2.5010360317179	0.083	-2.4889146711855
0.084	-2.4769384801388	0.085	-2.4651040224918	0.086	-2.4534079827286	0.087	-2.4418471603276
0.088	-2.4304184645039	0.089	-2.41911890925	0.090	-2.4079456086519	0.091	-2.3968957724653
0.092	-2.3859667019331	0.093	-2.3751557858289	0.094	-2.3644604967121	0.095	-2.3538783873816
0.096	-2.3434070875143	0.097	-2.3330443004788	0.098	-2.3227878003116	0.099	-2.3126354288475
0.100	-2.302585092994	0.101	-2.2926347621409	0.102	-2.2827824656979	0.103	-2.2730262907525
0.104	-2.2633643798408	0.105	-2.2537949288246	0.106	-2.2443161848701	0.107	-2.2349264445202
0.108	-2.2256240518579	0.109	-2.216407396753	0.110	-2.2072749131897	0.111	-2.1982250776698
0.112	-2.189256407687	0.113	-2.1803674602698	0.114	-2.1715568305876	0.115	-2.1628231506189
0.116	-2.1541650878758	0.117	-2.1455813441844	0.118	-2.1370706545165	0.119	-2.1286317858706
0.120	-2.1202635362001	0.121	-2.1119647333854	0.122	-2.1037342342489	0.123	-2.0955709236097
0.124	-2.0874737133771	0.125	-2.0794415416798	0.126	-2.0714733720307	0.127	-2.0635681925235
0.128	-2.0557250150625	0.129	-2.0479428746205	0.130	-2.0402208285266	0.131	-2.032557955781
0.132	-2.0249533563958	0.133	-2.0174061507604	0.134	-2.0099154790312	0.135	-2.0024805005437
0.136	-1.9951003932461	0.137	-1.987774353154	0.138	-1.9805015938249	0.139	-1.9732813458514
0.140	-1.9661128563728	0.141	-1.958995388604	0.142	-1.9519282213809	0.143	-1.9449106487222
0.144	-1.9379419794061	0.145	-1.9310215365616	0.146	-1.9241486572738	0.147	-1.9173226922034
0.148	-1.910543005218	0.149	-1.9038089730367	0.150	-1.8971199848859	0.151	-1.8904754421672
0.152	-1.8838747581359	0.153	-1.8773173575897	0.154	-1.8708026765685	0.155	-1.8643301620629
0.156	-1.8578992717326	0.157	-1.8515094736338	0.158	-1.8451602459552	0.159	-1.8388510767619
0.160	-1.8325814637483	0.161	-1.8263509139977	0.162	-1.8201589437498	0.163	-1.8140050781754
0.164	-1.8078888511579	0.165	-1.8018098050816	0.166	-1.7957674906256	0.167	-1.7897614665654
0.168	-1.7837912995789	0.169	-1.7778565640591	0.170	-1.7719568419319	0.171	-1.7660917224795
0.172	-1.7602608021687	0.173	-1.7544636844844	0.174	-1.7486999797676	0.175	-1.7429693050586
0.176	-1.737271283944	0.177	-1.7316055464083	0.178	-1.7259717286901	0.179	-1.7203694731414
0.180	-1.7147984280919	0.181	-1.7092582477163	0.182	-1.7037485919053	0.183	-1.6982691261407
0.184	-1.6928195213732	0.185	-1.6873994539038	0.186	-1.6820086052689	0.187	-1.6766466621275
0.188	-1.6713133161522	0.189	-1.6660082639225	0.190	-1.6607312068217	0.191	-1.6554818509355
0.192	-1.6502599069544	0.193	-1.6450650900773	0.194	-1.6398971199188	0.195	-1.6347557204184
0.196	-1.6296406197516	0.197	-1.6245515502441	0.198	-1.6194882482876	0.199	-1.6144504542576
0.200	-1.6094379124341	0.201	-1.6044503709231	0.202	-1.5994875815809	0.203	-1.5945492999403

0.204	-1.5896352851379	0.205	-1.5847452998437	0.206	-1.5798791101926	0.207	-1.5750364857168
0.208	-1.5702171992808	0.209	-1.5654210270173	0.210	-1.5606477482647	0.211	-1.5558971455061
0.212	-1.5511690043101	0.213	-1.5464631132727	0.214	-1.5417792639603	0.215	-1.5371172508545
0.216	-1.532476871298	0.217	-1.5278579254417	0.218	-1.523260216193	0.219	-1.5186835491656
0.220	-1.5141277326298	0.221	-1.5095925774644	0.222	-1.5050778971099	0.223	-1.500583507522
0.224	-1.4961092271271	0.225	-1.4916548767777	0.226	-1.4872202797099	0.227	-1.4828052615007
0.228	-1.4784096500277	0.229	-1.4740332754279	0.230	-1.4696759700589	0.231	-1.4653375684603
0.232	-1.4610179073158	0.233	-1.4567168254164	0.234	-1.4524341636244	0.235	-1.448169764838
0.236	-1.4439234739565	0.237	-1.439695137847	0.238	-1.4354846053107	0.239	-1.4312917270506
0.240	-1.4271163556401	0.241	-1.4229583454915	0.242	-1.4188175528254	0.243	-1.4146938356416
0.244	-1.4105870536889	0.245	-1.4064970684374	0.246	-1.4024237430498	0.247	-1.3983669423542
0.248	-1.3943265328172	0.249	-1.3903023825174	0.250	-1.3862943611199	0.251	-1.3823023398504
0.252	-1.3783261914707	0.253	-1.3743657902546	0.254	-1.3704210119636	0.255	-1.3664917338237
0.256	-1.3625778345026	0.257	-1.3586791940869	0.258	-1.3547956940605	0.259	-1.3509272172826
0.260	-1.3470736479666	0.261	-1.3432348716594	0.262	-1.339410775221	0.263	-1.3356012468044
0.264	-1.3318061758358	0.265	-1.3280254529959	0.266	-1.3242589702004	0.267	-1.3205066205819
0.268	-1.3167682984713	0.269	-1.3130438993803	0.270	-1.3093333199838	0.271	-1.3056364581024
0.272	-1.3019532126861	0.273	-1.2982834837972	0.274	-1.2946271725941	0.275	-1.2909841813156
0.276	-1.287354413265	0.277	-1.2837377727948	0.278	-1.2801341652915	0.279	-1.2765434971608
0.280	-1.2729656758129	0.281	-1.2694006096484	0.282	-1.265848208044	0.283	-1.2623083813389
0.284	-1.2587810408209	0.285	-1.2552660987135	0.286	-1.2517634681623	0.287	-1.2482730632225
0.288	-1.2447947988462	0.289	-1.2413285908697	0.290	-1.2378743560016	0.291	-1.2344320118106
0.292	-1.2310014767139	0.293	-1.2275826699651	0.294	-1.2241755116435	0.295	-1.2207799226423
0.296	-1.2173958246581	0.297	-1.2140231401794	0.298	-1.2106617924767	0.299	-1.2073117055915
0.300	-1.2039728043259	0.301	-1.2006450142333	0.302	-1.1973282616073	0.303	-1.1940224734728
0.304	-1.1907275775759	0.305	-1.1874435023747	0.306	-1.1841701770298	0.307	-1.1809075313949
0.308	-1.1776554960086	0.309	-1.1744140020844	0.310	-1.1711829815029	0.311	-1.1679623668029
0.312	-1.1647520911727	0.313	-1.161552088442	0.314	-1.1583622930739	0.315	-1.1551826401565
0.316	-1.1520130653952	0.317	-1.1488535051049	0.318	-1.145703896202	0.319	-1.1425641761973
0.320	-1.1394342831884	0.321	-1.1363141558521	0.322	-1.1332037334377	0.323	-1.1301029557595
0.324	-1.1270117631898	0.325	-1.1239300966524	0.326	-1.1208578976154	0.327	-1.1177951080849
0.328	-1.114741670598	0.329	-1.1116975282168	0.330	-1.1086626245216	0.331	-1.1056369036051
0.332	-1.1026203100656	0.333	-1.0996127890017	0.334	-1.0966142860054	0.335	-1.0936247471571
0.336	-1.0906441190189	0.337	-1.0876723486298	0.338	-1.0847093834991	0.339	-1.0817551716017
0.340	-1.0788096613719	0.341	-1.0758728016986	0.342	-1.0729445419195	0.343	-1.0700248318162
0.344	-1.0671136216087	0.345	-1.0642108619508	0.346	-1.0613165039244	0.347	-1.0584304990353
0.348	-1.0555527992077	0.349	-1.0526833567797	0.350	-1.0498221244987	0.351	-1.0469690555163
0.352	-1.044124103384	0.353	-1.0412872220488	0.354	-1.0384583658484	0.355	-1.0356374895067
0.356	-1.0328245481301	0.357	-1.0300194972025	0.358	-1.0272222925814	0.359	-1.0244328904939
0.360	-1.021651247532	0.361	-1.0188773206493	0.362	-1.0161110671564	0.363	-1.0133524447173
0.364	-1.0106014113454	0.365	-1.0078579253996	0.366	-1.0051219455808	0.367	-1.0023934309276
0.368	-0.99967234081321	0.369	-0.99695863494161	0.370	-0.99425227334387	0.371	-0.9915532163747
0.372	-0.98886142470899	0.373	-0.98617685933832	0.374	-0.9834994815676	0.375	-0.98082925301173
0.376	-0.97816613559224	0.377	-0.97551009153413	0.378	-0.97286108336255	0.379	-0.97021907389971
0.380	-0.9675840262617	0.381	-0.96495590385544	0.382	-0.96233467037556	0.383	-0.95972028980149
0.384	-0.95711272639441	0.385	-0.95451194469435	0.386	-0.95191790951731	0.387	-0.94933058595235
0.388	-0.94674993935886	0.389	-0.94417593536369	0.390	-0.94160853985844	0.391	-0.93904771899677
0.392	-0.93649343919167	0.393	-0.93394566711288	0.394	-0.9314043696842	0.395	-0.92886951408101
0.396	-0.92634106772766	0.397	-0.92381899829495	0.398	-0.9213032736977	0.399	-0.91879386209227
0.400	-0.91629073187415	0.401	-0.91379385167557	0.402	-0.91130319036312	0.403	-0.90881871703545
0.404	-0.90634040102099	0.405	-0.9038682118756	0.406	-0.9014021193804	0.407	-0.89894209353954
0.408	-0.89648810457797	0.409	-0.89404012293933	0.410	-0.89159811928378	0.411	-0.8891620644859
0.412	-0.88673192963261	0.413	-0.8843076860211	0.414	-0.88188930515682	0.415	-0.87947675875144

0.416	-0.87707001872087	0.417	-0.87466905718333	0.418	-0.87227384645738	0.419	-0.86988435906
0.420	-0.86750056770472	0.421	-0.86512244529975	0.422	-0.86274996494612	0.423	-0.86038309993586
0.424	-0.85802182375018	0.425	-0.85566611005772	0.426	-0.85331593271277	0.427	-0.85097126575351
0.428	-0.84863208340034	0.429	-0.84629836005412	0.430	-0.84397007029453	0.431	-0.84164718887839
0.432	-0.83932969073803	0.433	-0.83701755097965	0.434	-0.83471074488173	0.435	-0.83240924789345
0.436	-0.8301130356331	0.437	-0.82782208388655	0.438	-0.82553636860569	0.439	-0.82325586590696
0.440	-0.82098055206983	0.441	-0.81871040353529	0.442	-0.81644539690444	0.443	-0.814185508937
0.444	-0.81193071654991	0.445	-0.8096809968159	0.446	-0.80743632696207	0.447	-0.80519668436857
0.448	-0.80296204656715	0.449	-0.80073239123988	0.450	-0.79850769621777	0.451	-0.79628793947946
0.452	-0.79407309914991	0.453	-0.7918631534991	0.454	-0.78965808094079	0.455	-0.78745786003119
0.456	-0.78526246946775	0.457	-0.78307188808793	0.458	-0.78088609486795	0.459	-0.77870506892159
0.460	-0.776528789499	0.461	-0.77435723598549	0.462	-0.7721903879004	0.463	-0.7700282248959
0.464	-0.76787072675588	0.465	-0.76571787339478	0.466	-0.76356964485649	0.467	-0.76142602131324
0.468	-0.75928698306449	0.469	-0.75715251053586	0.470	-0.75502258427803	0.471	-0.75289718496572
0.472	-0.75077629339658	0.473	-0.7486598904902	0.474	-0.74654795728706	0.475	-0.7444404749475
0.476	-0.74233742475072	0.477	-0.7402387880938	0.478	-0.73814454649068	0.479	-0.73605468157122
0.480	-0.7339691750802	0.481	-0.73188800887638	0.482	-0.72981116493154	0.483	-0.72773862532956
0.484	-0.7256703722655	0.485	-0.72360638804465	0.486	-0.72154665508164	0.487	-0.71949115589955
0.488	-0.71743987312899	0.489	-0.71539278950726	0.490	-0.71334988787746	0.491	-0.71131115118762
0.492	-0.70927656248983	0.493	-0.70724610493945	0.494	-0.70521976179421	0.495	-0.70319751641345
0.496	-0.70117935225721	0.497	-0.69916525288551	0.498	-0.69715520195748	0.499	-0.69514918323062
0.500	-0.69314718055994	0.501	-0.69114917789727	0.502	-0.68915515929041	0.503	-0.6871651088824
0.504	-0.68517901091077	0.505	-0.68319684970678	0.506	-0.68121860969467	0.507	-0.67924427539095
0.508	-0.67727383140365	0.509	-0.67530726243161	0.510	-0.67334455326376	0.511	-0.67138568877843
0.512	-0.66943065394263	0.513	-0.66747943381137	0.514	-0.66553201352697	0.515	-0.6635883783184
0.516	-0.66164851350057	0.517	-0.65971240447371	0.518	-0.65778003672265	0.519	-0.65585139581625
0.520	-0.65392646740666	0.521	-0.65200523722877	0.522	-0.6500876910995	0.523	-0.64817381491721
0.524	-0.64626359466109	0.525	-0.64435701639051	0.526	-0.64245406624443	0.527	-0.64055473044077
0.528	-0.63865899527587	0.529	-0.63676684712384	0.530	-0.63487827243597	0.531	-0.6329932577402
0.532	-0.63111178964049	0.533	-0.62923385481629	0.534	-0.62735944002194	0.535	-0.62548853208613
0.536	-0.62362111791133	0.537	-0.62175718447327	0.538	-0.61989671882035	0.539	-0.61803970807314
0.540	-0.61618613942382	0.541	-0.61433600013565	0.542	-0.61248927754249	0.543	-0.6106459590482
0.544	-0.60880603212619	0.545	-0.60696948431889	0.546	-0.60513630323723	0.547	-0.60330647656016
0.548	-0.60147999203412	0.549	-0.59965683747261	0.550	-0.59783700075562	0.551	-0.59602046982922
0.552	-0.59420723270504	0.553	-0.5923972774598	0.554	-0.59059059223485	0.555	-0.5887871652357
0.556	-0.58698698473155	0.557	-0.58519003905485	0.558	-0.58339631660083	0.559	-0.58160580582704
0.560	-0.57981849525294	0.561	-0.57803437345944	0.562	-0.57625342908845	0.563	-0.57447565084245
0.564	-0.57270102748408	0.565	-0.5709295478357	0.566	-0.56916120077895	0.567	-0.56739597525438
0.568	-0.56563386026099	0.569	-0.56387484485581	0.570	-0.56211891815354	0.571	-0.56036606932613
0.572	-0.55861628760234	0.573	-0.5568695622674	0.574	-0.55512588266257	0.575	-0.55338523818479
0.576	-0.55164761828625	0.577	-0.54991301247404	0.578	-0.54818141030976	0.579	-0.54645280140914
0.580	-0.54472717544167	0.581	-0.54300452213023	0.582	-0.5412848312507	0.583	-0.53956809263164
0.584	-0.53785429615391	0.585	-0.53614343175028	0.586	-0.53443548940512	0.587	-0.53273045915404
0.588	-0.53102833108351	0.589	-0.52932909533055	0.590	-0.52763274208237	0.591	-0.52593926157604
0.592	-0.52424864409813	0.593	-0.52256087998441	0.594	-0.52087595961949	0.595	-0.51919387343651
0.596	-0.51751461191679	0.597	-0.51583816558953	0.598	-0.5141645250315	0.599	-0.51249368086669
0.600	-0.51082562376599	0.601	-0.50916034444693	0.602	-0.50749783367332	0.603	-0.50583808225495
0.604	-0.50418108104732	0.605	-0.50252682095129	0.606	-0.50087529291282	0.607	-0.49922648792264
0.608	-0.49758039701597	0.609	-0.49593701127224	0.610	-0.49429632181478	0.611	-0.49265831981054
0.612	-0.49102299646981	0.613	-0.48939034304592	0.614	-0.48776035083499	0.615	-0.48613301117562
0.616	-0.48450831544862	0.617	-0.48288625507675	0.618	-0.48126682152445	0.619	-0.47965000629754
0.620	-0.478035800943	0.621	-0.47642419704866	0.622	-0.47481518624296	0.623	-0.47320876019468
0.624	-0.47160491061271	0.625	-0.47000362924573	0.626	-0.46840490788204	0.627	-0.46680873834922

0.628	-0.46521511251394	0.629	-0.4636240222817	0.630	-0.46203545959656	0.631	-0.46044941644092
0.632	-0.45886588483528	0.633	-0.45728485683796	0.634	-0.45570632454491	0.635	-0.45413028008944
0.636	-0.45255671564201	0.637	-0.45098562340997	0.638	-0.44941699563735	0.639	-0.4478508246046
0.640	-0.44628710262842	0.641	-0.44472582206147	0.642	-0.44316697529218	0.643	-0.44161055474452
0.644	-0.44005655287778	0.645	-0.43850496218636	0.646	-0.43695577519953	0.647	-0.43540898448124
0.648	-0.43386458262986	0.649	-0.43232256227805	0.650	-0.43078291609245	0.651	-0.42924563677357
0.652	-0.42771071705548	0.653	-0.42617814970571	0.654	-0.42464792752494	0.655	-0.42312004334688
0.656	-0.42159449003805	0.657	-0.42007126049753	0.658	-0.41855034765682	0.659	-0.41703174447963
0.660	-0.41551544396167	0.661	-0.41400143913045	0.662	-0.41248972304513	0.663	-0.41098028879627
0.664	-0.4094731295057	0.665	-0.40796823832628	0.666	-0.40646560844175	0.667	-0.40496523306651
0.668	-0.40346710544549	0.669	-0.40197121885391	0.670	-0.40047756659712	0.671	-0.39898614201045
0.672	-0.39749693845899	0.673	-0.39600994933741	0.674	-0.39452516806983	0.675	-0.39304258810961
0.676	-0.39156220293917	0.677	-0.39008400606986	0.678	-0.38860799104174	0.679	-0.38713415142344
0.680	-0.38566248081198	0.681	-0.38419297283262	0.682	-0.38272562113867	0.683	-0.38126041941135
0.684	-0.37979736135959	0.685	-0.37833644071991	0.686	-0.37687765125625	0.687	-0.37542098675979
0.688	-0.37396644104879	0.689	-0.37251400796848	0.690	-0.37106368139083	0.691	-0.36961545521447
0.692	-0.36816932336447	0.693	-0.36672527979223	0.694	-0.36528331847533	0.695	-0.36384343341734
0.696	-0.36240561864772	0.697	-0.36096986822161	0.698	-0.35953617621976	0.699	-0.35810453674833
0.700	-0.35667494393873	0.701	-0.35524739194755	0.702	-0.35382187495633	0.703	-0.35239838717147
0.704	-0.35097692282409	0.705	-0.34955747616987	0.706	-0.34814004148889	0.707	-0.34672461308556
0.708	-0.34531118528842	0.709	-0.34389975245001	0.710	-0.34249030894678	0.711	-0.3410828491789
0.712	-0.33967736757016	0.713	-0.33827385856784	0.714	-0.33687231664255	0.715	-0.33547273628813
0.716	-0.33407511202149	0.717	-0.33267943838252	0.718	-0.33128570993391	0.719	-0.32989392126109
0.720	-0.32850406697204	0.721	-0.32711614169719	0.722	-0.32573014008931	0.723	-0.32434605682337
0.724	-0.32296388659642	0.725	-0.32158362412746	0.726	-0.32020526415734	0.727	-0.31882880144862
0.728	-0.31745423078545	0.729	-0.31608154697348	0.730	-0.3147107448397	0.731	-0.31334181923236
0.732	-0.31197476502082	0.733	-0.31060957709548	0.734	-0.30924625036762	0.735	-0.3078847797693
0.736	-0.30652516025326	0.737	-0.3051673867928	0.738	-0.30381145438166	0.739	-0.30245735803393
0.740	-0.30110509278392	0.741	-0.29975465368605	0.742	-0.29840603581476	0.743	-0.29705923426438
0.744	-0.29571424414904	0.745	-0.29437106060258	0.746	-0.29302967877838	0.747	-0.29169009384932
0.748	-0.29035230100766	0.749	-0.28901629546492	0.750	-0.28768207245178	0.751	-0.286349627218
0.752	-0.2850189550323	0.753	-0.28369005118224	0.754	-0.28236291097418	0.755	-0.28103752973311
0.756	-0.2797139028026	0.757	-0.27839202554469	0.758	-0.27707189333976	0.759	-0.27575350158651
0.760	-0.27443684570176	0.761	-0.27312192112045	0.762	-0.27180872329549	0.763	-0.27049724769768
0.764	-0.26918748981562	0.765	-0.2678794451556	0.766	-0.26657310924155	0.767	-0.26526847761488
0.768	-0.26396554583446	0.769	-0.26266430947649	0.770	-0.26136476413441	0.771	-0.26006690541881
0.772	-0.25877072895736	0.773	-0.25747623039471	0.774	-0.25618340539241	0.775	-0.25489224962879
0.776	-0.25360275879892	0.777	-0.25231492861449	0.778	-0.25102875480374	0.779	-0.24974423311139
0.780	-0.2484613592985	0.781	-0.24718012914245	0.782	-0.24590053843683	0.783	-0.24462258299133
0.784	-0.24334625863173	0.785	-0.24207156119973	0.786	-0.24079848655293	0.787	-0.23952703056473
0.788	-0.23825718912426	0.789	-0.23698895813626	0.790	-0.23572233352107	0.791	-0.23445731121448
0.792	-0.23319388716771	0.793	-0.23193205734729	0.794	-0.230671817735	0.795	-0.2294131643278
0.796	-0.22815609313775	0.797	-0.22690060019192	0.798	-0.22564668153233	0.799	-0.22439433321586
0.800	-0.22314355131421	0.801	-0.22189433191378	0.802	-0.22064667111562	0.803	-0.21940056503537
0.804	-0.21815600980317	0.805	-0.21691300156357	0.806	-0.21567153647551	0.807	-0.21443161071219
0.808	-0.21319322046104	0.809	-0.21195636192364	0.810	-0.21072103131565	0.811	-0.20948722486672
0.812	-0.20825493882046	0.813	-0.20702416943433	0.814	-0.2057949129796	0.815	-0.20456716574127
0.816	-0.20334092401803	0.817	-0.20211618412213	0.818	-0.20089294237939	0.819	-0.19967119512907
0.820	-0.19845093872384	0.821	-0.19723216952971	0.822	-0.19601488392596	0.823	-0.19479907830507
0.824	-0.19358474907266	0.825	-0.19237189264746	0.826	-0.19116050546116	0.827	-0.18995058395845
0.828	-0.18874212459688	0.829	-0.18753512384684	0.830	-0.18632957819149	0.831	-0.18512548412669
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0.836	-0.17912666589743	0.837	-0.17793120849266	0.838	-0.17673717850005	0.839	-0.17554457251493

0.840	-0.17435338714478	0.841	-0.17316361900919	0.842	-0.17197526473981	0.843	-0.17078832098028
0.844	-0.16960278438618	0.845	-0.16841865162496	0.846	-0.16723591937591	0.847	-0.16605458433008
0.848	-0.16487464319023	0.849	-0.16369609267079	0.850	-0.16251892949777	0.851	-0.16134315040876
0.852	-0.16016875215282	0.853	-0.15899573149046	0.854	-0.15782408519357	0.855	-0.15665381004538
0.856	-0.15548490284039	0.857	-0.15431736038436	0.858	-0.15315117949417	0.859	-0.15198635699788
0.860	-0.15082288973458	0.861	-0.14966077455441	0.862	-0.14850000831844	0.863	-0.14734058789871
0.864	-0.14618251017808	0.865	-0.14502577205026	0.866	-0.1438703704197	0.867	-0.14271630220159
0.868	-0.14156356432179	0.869	-0.14041215371674	0.870	-0.13926206733351	0.871	-0.13811330212963
0.872	-0.13696585507316	0.873	-0.13581972314253	0.874	-0.1346749033266	0.875	-0.13353139262452
0.876	-0.13238918804574	0.877	-0.13124828660995	0.878	-0.13010868534702	0.879	-0.12897038129696
0.880	-0.12783337150988	0.881	-0.12669765304596	0.882	-0.12556322297534	0.883	-0.12443007837818
0.884	-0.12329821634449	0.885	-0.12216763397421	0.886	-0.12103832837706	0.887	-0.1199102966725 6
0.888	-0.11878353598997	0.889	-0.11765804346823	0.890	-0.11653381625595	0.891	-0.11541085151133
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0.896	-0.10981486600721	0.897	-0.10869941692334	0.898	-0.10758521067994	0.899	-0.10647224451052
0.900	-0.10536051565783	0.901	-0.1042500213738	0.902	-0.10314075891951	0.903	-0.10203272556515
0.904	-0.10092591858996	0.905	-0.09982033528221	0.906	-0.098715972939157	0.907	-0.097612828867
0.908	-0.096510900380843	0.909	-0.095410184804657	0.910	-0.094310679471241	0.911	-0.093212381722178
0.912	-0.092115288907805	0.913	-0.091019398387168	0.914	-0.089924707527986	0.915	-0.088831213706615
0.916	-0.087738914308006	0.917	-0.086647806725671	0.918	-0.085557888361646	0.919	-0.084469156626449
0.920	-0.08338160893905	0.921	-0.082295242726829	0.922	-0.081210055425542	0.923	-0.080126044479284
0.924	-0.079043207340452	0.925	-0.077961541469711	0.926	-0.076881044335957	0.927	-0.075801713416281
0.928	-0.074723546195936	0.929	-0.073646540168298	0.930	-0.072570692834835	0.931	-0.071496001705069
0.932	-0.070422464296545	0.933	-0.069350078134792	0.934	-0.068278840753294	0.935	-0.067208749693449
0.936	-0.066139802504544	0.937	-0.065071996743714	0.938	-0.064005329975912	0.939	-0.062939799773873
0.940	-0.061875403718087	0.941	-0.060812139396757	0.942	-0.059750004405773	0.943	-0.058688996348679
0.944	-0.057629112836636	0.945	-0.056570351488394	0.946	-0.055512709930258	0.947	-0.054456185796058
0.948	-0.053400776727114	0.949	-0.052346480372208	0.950	-0.05129329438755	0.951	-0.050241216436746
0.952	-0.049190244190771	0.953	-0.048140375327934	0.954	-0.04709160753385	0.955	-0.046043938501406
0.956	-0.044997365930735	0.957	-0.043951887529182	0.958	-0.042907501011276	0.959	-0.041864204098698
0.960	-0.040821994520254	0.961	-0.039780870011844	0.962	-0.03874082831643	0.963	-0.037701867184011
0.964	-0.036663984371591	0.965	-0.03562717764315	0.966	-0.034591444769618	0.967	-0.033556783528842
0.968	-0.032523191705559	0.969	-0.03149066709137	0.970	-0.030459207484708	0.971	-0.029428810690811
0.972	-0.028399474521697	0.973	-0.027371196796131	0.974	-0.026343975339601	0.975	-0.025317807984289
0.976	-0.024292692569044	0.977	-0.023268626939354	0.978	-0.022245608947319	0.979	-0.021223636451626
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0.984	-0.016129381929883	0.985	-0.015113637810047	0.986	-0.014098924379501	0.987	-0.013085239548655
0.988	-0.012072581234268	0.989	-0.011060947359424	0.990	-0.010050335853501	0.991	-0.0090407446521483
0.992	-0.0080321716972635	0.993	-0.0070246149369637	0.994	-0.0060180723255622	0.995	-0.0050125418235435
0.996	-0.004008021397538	0.997	-0.0030045090202979	0.998	-0.0020020026706723	0.999	-0.0010005003335828

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1.04	0.039220713153282	1.05	0.048790164169433	1.06	0.058268908123976	1.07	0.067658648473815
1.08	0.076961041136129	1.09	0.086177696241053	1.10	0.095310179804326	1.11	0.10436001532424
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1.16	0.14842000511827	1.17	0.15700374880967	1.18	0.16551443847757	1.19	0.17395330712344
1.20	0.18232155679396	1.21	0.19062035960865	1.22	0.19885085874517	1.23	0.20701416938433
1.24	0.21511137961695	1.25	0.22314355131421	1.26	0.23111172096339	1.27	0.2390169004705
1.28	0.24686007793153	1.29	0.25464221837358	1.30	0.26236426446749	1.31	0.27002713721306
1.32	0.27763173659828	1.33	0.28517894223366	1.34	0.29266961396282	1.35	0.30010459245034
1.36	0.30748469974796	1.37	0.31481073984003	1.38	0.32208349916911	1.39	0.3293037471426
1.40	0.33647223662121	1.41	0.34358970439008	1.42	0.35065687161317	1.43	0.35767444427182

1.44	0.36464311358791	1.45	0.37156355643248	1.46	0.37843643572025	1.47	0.38526240079065
1.48	0.39204208777602	1.49	0.39877611995737	1.50	0.40546510810817	1.51	0.41210965082683
1.52	0.41871033485819	1.53	0.42526773540434	1.54	0.43178241642554	1.55	0.43825493093116
1.56	0.44468582126145	1.57	0.45107561936022	1.58	0.45742484703888	1.59	0.46373401623214
1.60	0.47000362924574	1.61	0.47623417899637	1.62	0.48242614924429	1.63	0.48858001481867
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1.68	0.51879379341517	1.69	0.52472852893498	1.70	0.53062825106217	1.71	0.53649337051457
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1.88	0.63127177684186	1.89	0.63657682907155	1.90	0.6418538861724	1.91	0.64710324205854
1.92	0.65232518603969	1.93	0.65752000291679	1.94	0.66268797307524	1.95	0.66782937257566
1.96	0.67294447324243	1.97	0.6780335427499	1.98	0.68309684470644	1.99	0.6881346387364
2.00	0.69314718055995						

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0.8	-0.22314355131421	0.9	-0.10536051565783	1.0	0.0	1.1	0.095310179804325
1.2	0.18232155679395	1.3	0.26236426446749	1.4	0.33647223662121	1.5	0.40546510810816
1.6	0.47000362924574	1.7	0.53062825106217	1.8	0.58778666490212	1.9	0.6418538861724
2.0	0.69314718055995	2.1	0.74193734472938	2.2	0.78845736036427	2.3	0.8329091229351
2.4	0.8754687373539	2.5	0.91629073187416	2.6	0.95551144502744	2.7	0.99325177301028
2.8	1.0296194171812	2.9	1.0647107369924	3.0	1.0986122886681	3.1	1.1314021114911
3.2	1.1631508098057	3.3	1.1939224684724	3.4	1.2237754316221	3.5	1.2527629684954
3.6	1.2809338454621	3.7	1.3083328196502	3.8	1.3350010667323	3.9	1.3609765531356
4.0	1.3862943611199	4.1	1.4109869737103	4.2	1.4350845252893	4.3	1.4586150226995
4.4	1.4816045409242	4.5	1.5040773967763	4.6	1.526056303495	4.7	1.547562508716
4.8	1.5686159179138	4.9	1.5892352051166	5.0	1.6094379124341	5.1	1.6292405397303
5.2	1.6486586255874	5.3	1.6677068205581	5.4	1.6863989535702	5.5	1.7047480922384
5.6	1.7227665977411	5.7	1.7404661748405	5.8	1.7578579175524	5.9	1.7749523509117
6.0	1.7917594692281	6.1	1.8082887711793	6.2	1.824549292051	6.3	1.8405496333975
6.4	1.8562979903656	6.5	1.8718021769016	6.6	1.8870696490324	6.7	1.9021075263969
6.8	1.9169226121821	6.9	1.9315214116032	7.0	1.9459101490553	7.1	1.9600947840473
7.2	1.974081026022	7.3	1.9878743481543	7.4	2.0014800002101	7.5	2.0149030205423
7.6	2.0281482472923	7.7	2.0412203288596	7.8	2.0541237336955	7.9	2.066862759473
8.0	2.0794415416798	8.1	2.0918640616784	8.2	2.1041341542702	8.3	2.1162555148026
8.4	2.1282317058493	8.5	2.1400661634963	8.6	2.1517622032595	8.7	2.1633230256605
8.8	2.1747517214842	8.9	2.1860512767381	9.0	2.1972245773362	9.1	2.2082744135228
9.2	2.219203484055	9.3	2.2300144001592	9.4	2.240709689276	9.5	2.2512917986065
9.6	2.2617630984738	9.7	2.2721258855093	9.8	2.2823823856765	9.9	2.2925347571405
10.0	2.302585092994						

To calculate a value that is not listed in the table above, we can use a method that is similar to that previously described above for calculating a value that is not listed in the SIN x table, or e^x table.

Since the slope of the natural logarithm of x curve is equal to 1/x we will use:

$$(y2 - y1) = m \ (x2 - x1)$$

$$(y2 - y1) = 1/x \ (x2 - x1)$$

Here are two entries from the table, and we need to find the value of e^0.5045:

$\ln 0.504 \ -0.68517901091077$ and $\ln 0.505 \ -0.68319684970678$

The result should be a value that is someplace between these to values, of about -0.684

$(\ln 0.5045 - \ln 0.504) = (1/0.504)(0.5045 - 0.504)$ After some simplification and transposing:

$\ln 0.5045 = 1.984126984 (0.0005) + \ln 0.504$

$\ln 0.5045 = 0.000992063 + (-0.68517901091077)$

$\ln 0.5045 = -0.68418694791077$: a calculated ("close" mathematical estimate-approximation) interpolated value

This calculated value differs from the true value by +0.000,000,492, or about: +0.000,000,5 , or less than a millionth.

A SQUARE ROOT TABLE

0.000	0	0.001	0.031622776601684	0.002	0.044721359549996	0.003	0.054772255750517
0.004	0.063245553203368	0.005	0.070710678118655	0.006	0.077459666924148	0.007	0.083666002653408
0.008	0.089442719099992	0.009	0.094868329805051	0.010	0.1	0.011	0.10488088481702
0.012	0.10954451150103	0.013	0.11401754250991	0.014	0.11832159566199	0.015	0.12247448713916
0.016	0.12649110640674	0.017	0.13038404810405	0.018	0.13416407864999	0.019	0.1378404875209
0.020	0.14142135623731	0.021	0.14491376746189	0.022	0.14832396974191	0.023	0.15165750888103
0.024	0.1549193338483	0.025	0.15811388300842	0.026	0.16124515496597	0.027	0.16431676725155
0.028	0.16733200530682	0.029	0.17029386365926	0.030	0.17320508075689	0.031	0.17606816861659
0.032	0.17888543819998	0.033	0.18165902124585	0.034	0.18439088914586	0.035	0.1870828693387
0.036	0.1897366596101	0.037	0.19235384061671	0.038	0.19493588689618	0.039	0.19748417658132
0.040	0.2	0.041	0.20248456731317	0.042	0.20493901531919	0.043	0.20736441353328
0.044	0.20976176963403	0.045	0.21213203435596	0.046	0.21447610589527	0.047	0.21679483388679
0.048	0.21908902300207	0.049	0.22135943621179	0.050	0.22360679774998	0.051	0.22583179581272
0.052	0.22803508501983	0.053	0.23021728866443	0.054	0.23237900077245	0.055	0.23452078799117
0.056	0.23664319132398	0.057	0.23874672772627	0.058	0.24083189157585	0.059	0.24289915602982
0.060	0.24494897427832	0.061	0.24698178070457	0.062	0.24899799195977	0.063	0.25099800796022
0.064	0.25298221281347	0.065	0.25495097567964	0.066	0.2569046515733	0.067	0.2588435821109
0.068	0.26076809620811	0.069	0.26267851073127	0.070	0.26457513110646	0.071	0.26645825188948
0.072	0.26832815729997	0.073	0.27018512172213	0.074	0.27202941017471	0.075	0.27386127875258
0.076	0.2756809750418	0.077	0.27748873851023	0.078	0.27928480087538	0.079	0.2810693864511
0.080	0.28284271247462	0.081	0.28460498941515	0.082	0.28635642126553	0.083	0.28809720581776
0.084	0.28982753492379	0.085	0.29154759474227	0.086	0.2932575659723	0.087	0.29495762407505
0.088	0.29664793948383	0.089	0.29832867780353	0.090	0.3	0.091	0.30166206257997
0.092	0.30331501776206	0.093	0.30495901363954	0.094	0.30659419433512	0.095	0.30822070014845
0.096	0.30983866769659	0.097	0.31144823004795	0.098	0.31304951684997	0.099	0.31464265445105
0.100	0.31622776601684	0.101	0.31780497164141	0.102	0.31937438845343	0.103	0.32093613071762
0.104	0.32249030993194	0.105	0.32403703492039	0.106	0.32557641192199	0.107	0.32710854467592
0.108	0.3286335345031	0.109	0.33015148038438	0.110	0.33166247903554	0.111	0.33316662497915
0.112	0.33466401061363	0.113	0.33615472627943	0.114	0.33763886032268	0.115	0.33911649915626
0.116	0.34058772731853	0.117	0.34205262752974	0.118	0.34351128074635	0.119	0.34496376621321
0.120	0.34641016151378	0.121	0.34785054261852	0.122	0.34928498393146	0.123	0.350713558335
0.124	0.35213633723318	0.125	0.35355339059327	0.126	0.35496478698598	0.127	0.35637059362411
0.128	0.35777087639997	0.129	0.35916569992136	0.130	0.3605551275464	0.131	0.36193922141708
0.132	0.3633180424917	0.133	0.36469165057621	0.134	0.36606010435446	0.135	0.36742346141748
0.136	0.36878177829172	0.137	0.37013511046644	0.138	0.37148351242013	0.139	0.37282703764615
0.140	0.37416573867739	0.141	0.37549966711037	0.142	0.37682887362834	0.143	0.37815340802378
0.144	0.37947331922021	0.145	0.3807886552932	0.146	0.38209946349086	0.147	0.38340579025362
0.148	0.38470768123343	0.149	0.38600518131238	0.150	0.38729833462074	0.151	0.38858718455451
0.152	0.38987177379236	0.153	0.39115214431216	0.154	0.39242833740697	0.155	0.39370039370059
0.156	0.39496835316263	0.157	0.39623225512318	0.158	0.39749213828704	0.159	0.39874804074754
0.160	0.4	0.161	0.40124805295478	0.162	0.40249223594996	0.163	0.40373258476373
0.164	0.40496913462633	0.165	0.4062019202318	0.166	0.40743097574927	0.167	0.40865633483405
0.168	0.40987803063838	0.169	0.41109609582189	0.170	0.41231056256177	0.171	0.41352146256271
0.172	0.41472882706655	0.173	0.41593268686171	0.174	0.41713307229228	0.175	0.41833001326704
0.176	0.41952353926806	0.177	0.42071367935925	0.178	0.42190046219458	0.179	0.42308391602612
0.180	0.42426406871193	0.181	0.42544094772365	0.182	0.42661458015403	0.183	0.42778499272415
0.184	0.42895221179054	0.185	0.43011626335213	0.186	0.43127717305696	0.187	0.43243496620879
0.188	0.43358966777358	0.189	0.43474130238568	0.190	0.43588989435407	0.191	0.43703546766824
0.192	0.43817804600413	0.193	0.43931765272978	0.194	0.4404543109109	0.195	0.44158804331639
0.196	0.44271887242357	0.197	0.44384682042344	0.198	0.44497190922574	0.199	0.44609416046391
0.200	0.44721359549996	0.201	0.4483302354292	0.202	0.44944410108488	0.203	0.45055521304275

0.204	0.45166359162545	0.205	0.45276925690687	0.206	0.45387222871641	0.207	0.45497252664309
0.208	0.45607017003966	0.209	0.4571651780265	0.210	0.45825756949558	0.211	0.45934736311423
0.212	0.46043457732885	0.213	0.46151923036857	0.214	0.46260134024882	0.215	0.46368092477479
0.216	0.46475800154489	0.217	0.46583258795408	0.218	0.46690470119715	0.219	0.4679743582719
0.220	0.46904157598234	0.221	0.47010637094173	0.222	0.47116875957559	0.223	0.4722287581247
0.224	0.47328638264797	0.225	0.47434164902526	0.226	0.47539457296019	0.227	0.47644516998286
0.228	0.47749345545253	0.229	0.47853944456022	0.230	0.47958315233127	0.231	0.48062459362792
0.232	0.48166378315169	0.233	0.48270073544589	0.234	0.48373546489791	0.235	0.48476798574163
0.236	0.48579831205964	0.237	0.48682645778552	0.238	0.48785243670602	0.239	0.48887626246321
0.240	0.48989794855664	0.241	0.49091750834534	0.242	0.49193495504995	0.243	0.49295030175465
0.244	0.49396356140914	0.245	0.49497474683058	0.246	0.49598387070549	0.247	0.49699094559157
0.248	0.49799598391955	0.249	0.49899899799499	0.250	0.5	0.251	0.50099900199501
0.252	0.50199601592045	0.253	0.50299105359837	0.254	0.50398412673417	0.255	0.5049752469181
0.256	0.50596442562694	0.257	0.50695167422546	0.258	0.50793700396801	0.259	0.50892042599998
0.260	0.50990195135928	0.261	0.51088159097779	0.262	0.51185935568279	0.263	0.51283525619832
0.264	0.51380930314661	0.265	0.51478150704935	0.266	0.51575187832911	0.267	0.51672042731055
0.268	0.51768716422179	0.269	0.5186520991956	0.270	0.51961524227066	0.271	0.52057660339281
0.272	0.52153619241621	0.273	0.52249401910453	0.274	0.5234500931321	0.275	0.52440442408508
0.276	0.52535702146255	0.277	0.52630789467763	0.278	0.52725705305856	0.279	0.52820450584977
0.280	0.52915026221292	0.281	0.53009433122794	0.282	0.53103672189407	0.283	0.53197744313082
0.284	0.53291650377897	0.285	0.53385391260157	0.286	0.53478967828484	0.287	0.53572380943916
0.288	0.53665631459995	0.289	0.53758720222862	0.290	0.53851648071345	0.291	0.53944415837045
0.292	0.54037024344425	0.293	0.54129474410897	0.294	0.54221766846904	0.295	0.54313902456001
0.296	0.54405882034942	0.297	0.54497706373755	0.298	0.54589376255825	0.299	0.54680892457969
0.300	0.54772255750517	0.301	0.54863466897381	0.302	0.54954526656136	0.303	0.55045435778092
0.304	0.55136195008361	0.305	0.55226805085936	0.306	0.55317266743757	0.307	0.5540758070878
0.308	0.55497747702046	0.309	0.55587768438749	0.310	0.556776436283	0.311	0.55767373974395
0.312	0.55856960175076	0.313	0.55946402922797	0.314	0.56035702904488	0.315	0.56124860801609
0.316	0.56213877290221	0.317	0.56302753041037	0.318	0.56391488719487	0.319	0.56480084985772
0.320	0.56568542494924	0.321	0.56656861896861	0.322	0.56745043836444	0.323	0.56833088953531
0.324	0.56920997883031	0.325	0.57008771254957	0.326	0.57096409694481	0.327	0.57183913821983
0.328	0.57271284253105	0.329	0.573585215988	0.330	0.5744562646538	0.331	0.5753259945457
0.332	0.57619441163552	0.333	0.57706152185014	0.334	0.577927331072	0.335	0.57879184513951
0.336	0.57965506984758	0.337	0.580517010948	0.338	0.58137767414995	0.339	0.58223706512039
0.340	0.58309518948453	0.341	0.58395205282626	0.342	0.58480766068854	0.343	0.58566201857385
0.344	0.58651513194461	0.345	0.58736700622354	0.346	0.58821764679411	0.347	0.58906705900093
0.348	0.58991524815011	0.349	0.59076221950968	0.350	0.59160797830996	0.351	0.59245252974395
0.352	0.59329587896765	0.353	0.59413803110052	0.354	0.59497899122574	0.355	0.59581876439065
0.356	0.59665735560705	0.357	0.59749476985159	0.358	0.59833101206606	0.359	0.59916608715781
0.360	0.6	0.361	0.60083275543199	0.362	0.60166435825965	0.363	0.60249481325568
0.364	0.60332412515993	0.365	0.60415229867973	0.366	0.60497933849017	0.367	0.60580524923444
0.368	0.60663003552412	0.369	0.6074537019395	0.370	0.60827625302982	0.371	0.60909769331364
0.372	0.60991802727908	0.373	0.6107372593841	0.374	0.61155539405683	0.375	0.61237243569579
0.376	0.61318838867024	0.377	0.61400325732035	0.378	0.61481704595758	0.379	0.61562975886486
0.380	0.6164414002969	0.381	0.61725197448044	0.382	0.6180614856145	0.383	0.61886993787063
0.384	0.61967733539319	0.385	0.62048368229954	0.386	0.62128898268036	0.387	0.62209324059983
0.388	0.6228964600959	0.389	0.62369864518051	0.390	0.62449979983984	0.391	0.62529992803454
0.392	0.62609903369994	0.393	0.6268971207463	0.394	0.62769419305901	0.395	0.62849025449883
0.396	0.62928530890209	0.397	0.63007936008093	0.398	0.6308724118235	0.399	0.63166446789415
0.400	0.63245553203368	0.401	0.6332456079595	0.402	0.63403469936589	0.403	0.63482280992416
0.404	0.63560994328283	0.405	0.63639610306789	0.406	0.63718129288296	0.407	0.63796551630946
0.408	0.63874877690685	0.409	0.63953107821278	0.410	0.64031242374329	0.411	0.64109281699298
0.412	0.64187226143525	0.413	0.64265076052239	0.414	0.64342831768582	0.415	0.64420493633626

0.416	0.64498061986388	0.417	0.64575537163852	0.418	0.64652919500978	0.419	0.64730209330729
0.420	0.64807406984079	0.421	0.64884512790033	0.422	0.64961527075647	0.423	0.65038450166036
0.424	0.65115282384399	0.425	0.65192024052027	0.426	0.65268675488323	0.427	0.65345237010818
0.428	0.65421708935185	0.429	0.65498091575251	0.430	0.6557438524302	0.431	0.65650590248679
0.432	0.6572670690062	0.433	0.65802735505448	0.434	0.65878676368002	0.435	0.65954529791365
0.436	0.66030296076877	0.437	0.66105975524154	0.438	0.66181568431097	0.439	0.6625707509391
0.440	0.66332495807108	0.441	0.66407830863536	0.442	0.66483080554379	0.443	0.66558245169175
0.444	0.66633324995831	0.445	0.66708320320632	0.446	0.66783231428256	0.447	0.66858058601787
0.448	0.66932802122726	0.449	0.67007462271004	0.450	0.67082039324994	0.451	0.67156533561523
0.452	0.67230945255886	0.453	0.67305274681855	0.454	0.67379522111692	0.455	0.6745368781616
0.456	0.67527772064537	0.457	0.67601775124622	0.458	0.67675697262755	0.459	0.67749538743817
0.460	0.67823299831253	0.461	0.67896980787072	0.462	0.67970581871866	0.463	0.68044103344816
0.464	0.68117545463706	0.465	0.68190908484929	0.466	0.68264192663504	0.467	0.6833739825308
0.468	0.68410525505948	0.469	0.68483574673056	0.470	0.6855654600401	0.471	0.68629439747094
0.472	0.68702256149271	0.473	0.68774995456198	0.474	0.68847657912234	0.475	0.68920243760451
0.476	0.68992753242641	0.477	0.69065186599328	0.478	0.69137544069774	0.479	0.69209825891993
0.480	0.69282032302755	0.481	0.69354163537599	0.482	0.69426219830839	0.483	0.69498201415576
0.484	0.69570108523704	0.485	0.69641941385921	0.486	0.69713700231734	0.487	0.69785385289472
0.488	0.69856996786292	0.489	0.69928534948188	0.490	0.7	0.491	0.70071392165419
0.492	0.70142711667001	0.493	0.70213958726168	0.494	0.70285133563222	0.495	0.70356236397351
0.496	0.70427267446636	0.497	0.70498226928058	0.498	0.70569115057509	0.499	0.70639932049797
0.500	0.70710678118655	0.501	0.70781353476746	0.502	0.70851958335673	0.503	0.70922492905989
0.504	0.70992957397195	0.505	0.71063352017759	0.506	0.71133676975115	0.507	0.71203932475672
0.508	0.71274118724822	0.509	0.71344235926948	0.510	0.71414284285429	0.511	0.71484264002646
0.512	0.71554175279993	0.513	0.7162401831788	0.514	0.7169379331574	0.515	0.71763500472037
0.516	0.71833139984272	0.517	0.7190271204899	0.518	0.71972216861786	0.519	0.72041654617312
0.520	0.7211102550928	0.52	0.72180329730474	0.522	0.72249567472754	0.523	0.72318738927058
0.524	0.72387844283415	0.525	0.72456883730947	0.526	0.72525857457875	0.527	0.72594765651526
0.528	0.7266360849834	0.529	0.72732386183873	0.530	0.72801098892805	0.531	0.72869746808947
0.532	0.72938330115242	0.533	0.73006848993776	0.534	0.7307530362578	0.535	0.73143694191639
0.536	0.73212020870893	0.537	0.73280283842245	0.538	0.73348483283569	0.539	0.73416619371911
0.540	0.73484692283495	0.541	0.73552702193733	0.542	0.73620649277224	0.543	0.73688533707762
0.544	0.73756355658343	0.545	0.73824115301167	0.546	0.73891812807645	0.547	0.73959448348402
0.548	0.74027022093287	0.549	0.74094534211371	0.550	0.74161984870957	0.551	0.74229374239583
0.552	0.74296702484027	0.553	0.74363969770313	0.554	0.74431176263714	0.555	0.74498322128757
0.556	0.74565407529229	0.557	0.74632432628181	0.558	0.74699397587932	0.559	0.74766302570075
0.560	0.74833147735479	0.561	0.74899933244296	0.562	0.74966659255965	0.563	0.75033325929216
0.564	0.75099933422074	0.565	0.75166481891865	0.566	0.75232971495216	0.567	0.75299402388067
0.568	0.75365774725667	0.569	0.75432088662584	0.570	0.75498344352708	0.571	0.7556454194925
0.572	0.75630681604756	0.573	0.75696763471102	0.574	0.75762787699503	0.575	0.75828754440516
0.576	0.75894663844041	0.577	0.75960516059332	0.578	0.76026311234993	0.579	0.76092049518987
0.580	0.76157731058639	0.581	0.76223356000638	0.582	0.76288924491043	0.583	0.76354436675284
0.584	0.76419892698171	0.585	0.76485292703892	0.586	0.76550636836019	0.587	0.76615925237512
0.588	0.76681158050723	0.589	0.767463354174	0.590	0.76811457478686	0.591	0.7687652437513
0.592	0.76941536246685	0.593	0.77006493232714	0.594	0.77071395471991	0.595	0.77136243102708
0.596	0.77201036262475	0.597	0.77265775088327	0.598	0.77330459716725	0.599	0.77395090283557
0.600	0.77459666924148	0.601	0.77524189773257	0.602	0.77588658965083	0.603	0.77653074633269
0.604	0.77717436910902	0.605	0.7778174593052	0.606	0.77846001824114	0.607	0.7791020472313
0.608	0.77974354758472	0.609	0.78038452060507	0.610	0.78102496759067	0.611	0.78166488983451
0.612	0.78230428862432	0.613	0.78294316524254	0.614	0.78358152096639	0.615	0.78421935706791
0.616	0.78485667481394	0.617	0.78549347546622	0.618	0.78612976028134	0.619	0.78676553051084
0.620	0.78740078740118	0.621	0.78803553219382	0.622	0.78866976612521	0.623	0.78930349042684
0.624	0.78993670632526	0.625	0.7905694150421	0.626	0.7912016177941	0.627	0.79183331579317

0.628	0.79246451024636	0.629	0.79309520235593	0.630	0.79372539331938	0.631	0.79435508432942
0.632	0.79498427657407	0.633	0.79561297123664	0.634	0.79624116949578	0.635	0.79686887252546
0.636	0.79749608149508	0.637	0.7981227975694	0.638	0.79874902190863	0.639	0.79937475566845
0.640	0.8	0.641	0.80062475604992	0.642	0.80124902496041	0.643	0.80187280786918
0.644	0.80249610590956	0.645	0.80311892021045	0.646	0.80374125189641	0.647	0.80436310208761
0.648	0.80498447189992	0.649	0.80560536244491	0.650	0.80622577482986	0.651	0.80684571015777
0.652	0.80746516952745	0.653	0.80808415403348	0.654	0.80870266476623	0.655	0.80932070281193
0.656	0.80993826925266	0.657	0.81055536516638	0.658	0.81117199162693	0.659	0.8117881497041
0.660	0.8124038404636	0.661	0.81301906496711	0.662	0.81363382427232	0.663	0.81424811943289
0.664	0.81486195149853	0.665	0.815475321515	0.666	0.81608823052413	0.667	0.81670067956382
0.668	0.8173126696681	0.669	0.81792420186714	0.670	0.81853527718725	0.671	0.8191458966509
0.672	0.81975606127677	0.673	0.82036577207975	0.674	0.82097503007095	0.675	0.82158383625775
0.676	0.82219219164378	0.677	0.82280009722897	0.678	0.82340755400956	0.679	0.82401456297811
0.680	0.82462112512353	0.681	0.82522724143111	0.682	0.8258329128825	0.683	0.82643814045578
0.684	0.82704292512541	0.685	0.82764726786234	0.686	0.82825116963395	0.687	0.82885463140408
0.688	0.82945765413311	0.689	0.83006023877789	0.690	0.83066238629181	0.691	0.83126409762482
0.692	0.83186537372342	0.693	0.8324662155307	0.694	0.83306662398634	0.695	0.83366660002665
0.696	0.83426614458457	0.697	0.83486525858967	0.698	0.83546394296822	0.699	0.83606219864314
0.700	0.83666002653408	0.701	0.83725742755738	0.702	0.83785440262614	0.703	0.83845095265018
0.704	0.83904707853612	0.705	0.83964278118733	0.706	0.840238061504	0.707	0.84083292038312
0.708	0.84142735871851	0.709	0.84202137740084	0.710	0.84261497731764	0.711	0.84320815935331
0.712	0.84380092438916	0.713	0.84439327330338	0.714	0.8449852069711	0.715	0.84557672626439
0.716	0.84616783205225	0.717	0.84675852520066	0.718	0.84734880657259	0.719	0.847938677028
0.720	0.84852813742386	0.721	0.84911718861415	0.722	0.84970583144992	0.723	0.85029406677925
0.724	0.85088189544731	0.725	0.85146931829632	0.726	0.85205633616563	0.727	0.85264294989169
0.728	0.85322916030806	0.729	0.85381496824546	0.730	0.85440037453175	0.731	0.85498537999196
0.732	0.8555699854483	0.733	0.85615419172016	0.734	0.85673799962416	0.735	0.85732140997411
0.736	0.85790442358109	0.737	0.85848704125339	0.738	0.85906926379658	0.739	0.8596510920135
0.740	0.86023252670426	0.741	0.86081356866629	0.742	0.86139421869432	0.743	0.8619744775804
0.744	0.86255434611391	0.745	0.8631338250816	0.746	0.86371291526757	0.747	0.86429161745328
0.748	0.86486993241759	0.749	0.86544786093675	0.750	0.86602540378444	0.751	0.86660256173173
0.752	0.86717933554715	0.753	0.86775572599667	0.754	0.8683317338437	0.755	0.86890735984914
0.756	0.86948260477137	0.757	0.87005746936625	0.758	0.87063195438716	0.759	0.87120606058498
0.760	0.87177978870814	0.761	0.87235313950258	0.762	0.87292611371181	0.763	0.8734987120769
0.764	0.87407093533649	0.765	0.8746427842268	0.766	0.87521425948164	0.767	0.87578536183245
0.768	0.87635609200827	0.769	0.87692645073575	0.770	0.87749643873921	0.771	0.87806605674061
0.772	0.87863530545955	0.773	0.87920418561333	0.774	0.87977269791691	0.775	0.88034084308295
0.776	0.88090862182181	0.777	0.88147603484156	0.778	0.882043082848	0.779	0.88260976654465
0.780	0.88317608663279	0.781	0.88374204381143	0.782	0.88430763877737	0.783	0.88487287222516
0.784	0.88543774484715	0.785	0.88600225733347	0.786	0.88656641037206	0.787	0.88713020464868
0.788	0.88769364084689	0.789	0.8882567196481	0.790	0.88881944173156	0.791	0.88938180777437
0.792	0.88994381845148	0.793	0.89050547443573	0.794	0.89106677639782	0.795	0.89162772500635
0.796	0.89218832092782	0.797	0.89274856482663	0.798	0.89330845736509	0.799	0.89386799920346
0.800	0.89442719099992	0.801	0.89498603341058	0.802	0.89554452708952	0.803	0.89610267268879
0.804	0.8966604708584	0.805	0.89721792224632	0.806	0.89777502749854	0.807	0.89833178725903
0.808	0.89888820216977	0.809	0.89944427287075	0.810	0.9	0.811	0.90055538419355
0.812	0.90111042608551	0.813	0.90166512630799	0.814	0.9022194854912	0.815	0.90277350426339
0.816	0.9033271832509	0.817	0.90388052307813	0.818	0.9044335243676	0.819	0.9049861877399
0.820	0.90553851381374	0.821	0.90609050320594	0.822	0.90664215653145	0.823	0.90719347440334
0.824	0.90774445743282	0.825	0.90829510622925	0.826	0.90884542140014	0.827	0.90939540355117
0.828	0.90994505328619	0.829	0.9104943712072	0.830	0.91104335791443	0.831	0.91159201400627
0.832	0.91214034007931	0.833	0.91268833672837	0.834	0.91323600454647	0.835	0.91378334412485
0.836	0.914330356053	0.837	0.91487704091861	0.838	0.91542339930766	0.839	0.91596943180436

0.840	0.91651513899117	0.841	0.91706052144883	0.842	0.91760557975636	0.843	0.91815031449104
0.844	0.91869472622847	0.845	0.91923881554251	0.846	0.91978258300535	0.847	0.92032602918748
0.848	0.92086915465771	0.849	0.92141195998316	0.850	0.92195444572929	0.851	0.92249661245991
0.852	0.92303846073715	0.853	0.92357999112151	0.854	0.92412120417183	0.855	0.92466210044535
0.856	0.92520268049763	0.857	0.92574294488265	0.858	0.92628289415275	0.859	0.92682252885868
0.860	0.92736184954957	0.861	0.92790085677296	0.862	0.92843955107481	0.863	0.92897793299949
0.864	0.92951600308978	0.865	0.93005376188691	0.866	0.93059120993055	0.867	0.93112834775878
0.868	0.93166517590817	0.869	0.93220169491371	0.870	0.93273790530888	0.871	0.93327380762561
0.872	0.9338094023943	0.873	0.93434469014385	0.874	0.93487967140162	0.875	0.93541434669349
0.876	0.93594871654381	0.877	0.93648278147545	0.878	0.9370165420098	0.879	0.93754999866674
0.880	0.93808315196469	0.881	0.93861600242059	0.882	0.93914855054991	0.883	0.93968079686668
0.884	0.94021274188345	0.885	0.94074438611134	0.886	0.94127573006001	0.887	0.94180677423769
0.888	0.94233751915118	0.889	0.94286796530585	0.890	0.94339811320566	0.891	0.94392796335314
0.892	0.94445751624941	0.893	0.9449867723942	0.894	0.94551573228583	0.895	0.94604439642123
0.896	0.94657276529594	0.897	0.94710083940413	0.898	0.94762861923857	0.899	0.94815610529069
0.900	0.94868329805051	0.901	0.94921019800674	0.902	0.9497368056467	0.903	0.95026312145637
0.904	0.95078914592038	0.905	0.95131487952202	0.906	0.95184032274326	0.907	0.95236547606473
0.908	0.95289033996573	0.909	0.95341491492424	0.910	0.95393920141695	0.911	0.9544631999192
0.912	0.95498691090507	0.913	0.9555103348473	0.914	0.95603347221737	0.915	0.95655632348545
0.916	0.95707888912043	0.917	0.95760116958993	0.918	0.95812316536028	0.919	0.95864487689655
0.920	0.95916630466254	0.921	0.95968744912081	0.922	0.96020831073262	0.923	0.96072888995804
0.924	0.96124918725583	0.925	0.96176920308357	0.926	0.96228893789755	0.927	0.96280839215287
0.928	0.96332756630338	0.929	0.96384646080172	0.930	0.9643650760993	0.931	0.96488341264632
0.932	0.96540147089177	0.933	0.96591925128346	0.934	0.96643675426797	0.935	0.96695398029069
0.936	0.96747092979583	0.937	0.96798760322641	0.938	0.96850400102426	0.939	0.96902012363005
0.940	0.96953597148327	0.941	0.97005154502222	0.942	0.97056684468407	0.943	0.97108187090482
0.944	0.97159662411929	0.945	0.97211110476118	0.946	0.97262531326303	0.947	0.97313925005623
0.948	0.97365291557105	0.949	0.9741663102366	0.950	0.9746794344809	0.951	0.97519228873079
0.952	0.97570487341204	0.953	0.97621718894926	0.954	0.97672923576598	0.955	0.9772410142846
0.956	0.97775252492643	0.957	0.97826376811165	0.958	0.97877474425937	0.959	0.97928545378761
0.960	0.97979589711327	0.961	0.9803060746522	0.962	0.98081598681914	0.963	0.98132563402777
0.964	0.98183501669069	0.965	0.98234413521943	0.966	0.98285299002445	0.967	0.98336158151516
0.968	0.98386991009991	0.969	0.98437797618598	0.970	0.98488578017961	0.971	0.985393322486
0.972	0.9859006035093	0.973	0.98640762365262	0.974	0.98691438331803	0.975	0.98742088290658
0.976	0.98792712281828	0.977	0.98843310345213	0.978	0.98893882520609	0.979	0.98944428847712
0.980	0.98994949366117	0.981	0.99045444115315	0.982	0.990959131347	0.983	0.99146356463564
0.984	0.99196774141098	0.985	0.99247166206396	0.986	0.99297532698451	0.987	0.99347873656158
0.988	0.99398189118313	0.989	0.99448479123615	0.990	0.99498743710662	0.991	0.99548982917959
0.992	0.9959919678391	0.993	0.99649385346825	0.994	0.99699548644916	0.995	0.997496867163
0.996	0.99799799598997	0.997	0.99849887330933	0.998	0.99899949949937	0.999	0.99949987493746
1.000	1						

$\sqrt{10} = \sqrt{10^1} = 3.1622776601684$

$\sqrt{100} = \sqrt{10^2} = \sqrt{(10)(10)} = \sqrt{10}\ \sqrt{10} = 10$

$\sqrt{1000} = \sqrt{10^3} = \sqrt{(100)(10)} = \sqrt{100}\ \sqrt{10} = 31.622776601684$

$\sqrt{10{,}000} = \sqrt{10^4} = \sqrt{(1{,}000)(10)} = \sqrt{1000}\ \sqrt{10} = 100$

$\sqrt{100{,}000} = \sqrt{10^5} = \sqrt{(10{,}000)(10)} = \sqrt{10{,}000}\ \sqrt{10} = 316.22776601684$

$\sqrt{1{,}000{,}000} = \sqrt{10^6} = \sqrt{(100{,}000)(10)} = \sqrt{100{,}000}\ \sqrt{10} = 1000$

Ex. Use the table above to find the square root of 57:

$$\sqrt{57} \;=\; \sqrt{(0.57)(10\hat{}2)} \;=\; \sqrt{0.57}\;\sqrt{100} \;=\; 0.75498344352708\;(10) \;=\; 7.5498344352708$$

Ex. sqrt(6.35) = sqrt((10) (0.635)) = sqrt(10) sqrt(0.635) = (3.1622776601684) (0.79686887252546) =
= 2.51992063$\underline{2}$

To calculate a value that is not listed in the table above, we can use a method that is similar to that previously described above for calculating a value that is not listed in the SIN x table, or e^x table.

Since the slope of the square root of x curve is equal to $1/(2\sqrt{x}\,)$, we will use:

$$(y2 - y1) \;=\; m\;(x2 - x1)$$

$$(y2 - y1) \;=\; (1/(2\sqrt{x}\,))\;(x2 - x1)$$

Here are two entries from the table, and we need to find the square root value of: 0.5045

square-root of 0.504 = 0.70992957397195 and square-root of 0.505 = 0.71063352017759

The result should be a value that is someplace between these to values, of about 0.7103

$(\sqrt{0.5045} - \sqrt{0.504}\,) = (0.5/\sqrt{0.504}\,)\,(0.5045 - 0.504)$ After some simplification and transposing:

$$\sqrt{0.5045} \;=\; 0.704295212\;(0.0005) \;+\; \sqrt{0.504}$$

$$\sqrt{0.5045} \;=\; 0.000352147 + 0.70992957397195$$

$\sqrt{0.5045} = 0.71028172097195$: a calculated ("close" mathematical estimate-approximation) interpolated value
The true value is: 0.71028163428319...

This calculated value differs from the true value by about: +0.000,000,086, or about: +0.000,000,1 , or less than a millionth.

A CUBE ROOT TABLE

The following table is for three decimal point places. A fourth place would require a table having ten times more values.

0.000	0	0.001	0.1	0.002	0.12599210498949	0.003	0.14422495703074
0.004	0.15874010519682	0.005	0.17099759466767	0.006	0.18171205928321	0.007	0.19129311827724
0.008	0.2	0.009	0.20800838230519	0.010	0.21544346900319	0.011	0.22239800905693
0.012	0.22894284851067	0.013	0.23513346877208	0.014	0.24101422641752	0.015	0.24662120743305
0.016	0.25198420997897	0.017	0.25712815906582	0.018	0.26207413942089	0.019	0.26684016487219
0.020	0.27144176165949	0.021	0.27589241763811	0.022	0.28020393306554	0.023	0.28438669798516
0.024	0.28844991406148	0.025	0.29240177382129	0.026	0.29624960684074	0.027	0.3
0.028	0.30365889718757	0.029	0.30723168256858	0.030	0.31072325059539	0.031	0.31413806523914
0.032	0.31748021039364	0.033	0.32075343299958	0.034	0.32396118012775	0.035	0.32710663101886
0.036	0.33019272488946	0.037	0.3332221851646	0.038	0.3361975406799	0.039	0.33912114430142
0.040	0.34199518933534	0.041	0.34482172403827	0.042	0.34760266448865	0.043	0.35033980603867
0.044	0.35303483353261	0.045	0.35568933044901	0.046	0.35830478710159	0.047	0.36088260801387
0.048	0.36342411856643	0.049	0.3659305710023	0.050	0.36840314986404	0.051	0.37084297692662
0.052	0.37325111568172	0.053	0.37562857542211	0.054	0.37797631496846	0.055	0.38029524607614
0.056	0.38258623655448	0.057	0.38485011312768	0.058	0.38708766406278	0.059	0.38929964158733
0.060	0.39148676411689	0.061	0.39364971831022	0.062	0.39578916096804	0.063	0.39790572078964
0.064	0.4	0.065	0.40207257585891	0.066	0.40412400206222	0.067	0.40615481004457
0.068	0.40816551019173	0.069	0.41015659297023	0.070	0.41212852998086	0.071	0.41408177494229
0.072	0.41601676461038	0.073	0.41793391963812	0.074	0.41983364538084	0.075	0.42171633265087
0.076	0.42358235842549	0.077	0.4254320865115	0.078	0.42726586816979	0.079	0.42908404270262
0.080	0.43088693800638	0.081	0.43267487109222	0.082	0.43444814857686	0.083	0.43620706714548
0.084	0.43795191398879	0.085	0.43968296721582	0.086	0.44140049624421	0.087	0.44310476216936
0.088	0.44479601811386	0.089	0.44647450955845	0.090	0.44814047465572	0.091	0.44979414452754
0.092	0.4514357435474	0.093	0.45306548960835	0.094	0.45468359437763	0.095	0.4562902635387
0.096	0.45788569702133	0.097	0.4594700892207	0.098	0.46104362920584	0.099	0.46260650091827
0.100	0.46415888336128	0.101	0.46570095078038	0.102	0.46723287283553	0.103	0.46875481476536
0.104	0.47026693754415	0.105	0.47176939803165	0.106	0.47326234911634	0.107	0.47474593985234
0.108	0.47622031559046	0.109	0.4776856181035	0.110	0.47914198570628	0.111	0.48058955337053
0.112	0.48202845283505	0.113	0.48345881271116	0.114	0.48488075858399	0.115	0.48629441310943
0.116	0.48769989610733	0.117	0.48909732465087	0.118	0.4904868131524	0.119	0.49186847344587
0.120	0.49324241486609	0.121	0.49460874432487	0.122	0.49596756638423	0.123	0.49731898332686
0.124	0.49866309522386	0.125	0.5	0.126	0.50132979349646	0.127	0.50265256953135
0.128	0.50396841995795	0.129	0.50527743472086	0.130	0.50657970191009	0.131	0.50787530781327
0.132	0.50916433696595	0.133	0.51044687220015	0.134	0.5117229946912	0.135	0.51299278400301
0.136	0.51425631813165	0.137	0.51551367354758	0.138	0.51676492523636	0.139	0.51801014673803
0.140	0.51924941018511	0.141	0.52048278633942	0.142	0.52171034462762	0.143	0.5229321531756
0.144	0.52414827884178	0.145	0.52535878724929	0.146	0.52656374281714	0.147	0.52776320879041
0.148	0.52895724726942	0.149	0.53014591923809	0.150	0.53132928459131	0.151	0.5325074021615
0.152	0.53368032974439	0.153	0.53484812412394	0.154	0.53601084109654	0.155	0.53716853549448
0.156	0.53832126120873	0.157	0.53946907121096	0.158	0.54061201757502	0.159	0.54175015149772
0.160	0.54288352331898	0.161	0.54401218254148	0.162	0.54513617784964	0.163	0.54625555712814
0.164	0.54737036747984	0.165	0.54848065524326	0.166	0.5495864660095	0.167	0.55068784463874
0.168	0.55178483527622	0.169	0.55287748136789	0.170	0.55396582567545	0.171	0.55504991029115
0.172	0.55612977665212	0.173	0.55720546555426	0.174	0.55827701716584	0.175	0.5593444710407
0.176	0.56040786613108	0.177	0.56146724080015	0.178	0.56252263283419	0.179	0.56357407945442
0.180	0.56462161732862	0.181	0.56566528258229	0.182	0.56670511080971	0.183	0.56774113708454
0.184	0.56877339597031	0.185	0.56980192153051	0.186	0.57082674733849	0.187	0.57184790648713
0.188	0.57286543159824	0.189	0.57387935483172	0.190	0.57488970789448	0.191	0.57589652204924
0.192	0.57689982812296	0.193	0.57789965651521	0.194	0.57889603720624	0.195	0.5798889997649

0.196	0.58087857335637	0.197	0.5818647867497	0.198	0.58284766832515	0.199	0.5838272460814
0.200	0.58480354764257	0.201	0.58577660026507	0.202	0.58674643084426	0.203	0.58771306592107
0.204	0.58867653168833	0.205	0.58963685399704	0.206	0.59059405836245	0.207	0.59154816997007
0.208	0.59249921368147	0.209	0.59344721403999	0.210	0.59439219527631	0.211	0.59533418131391
0.212	0.59627319577437	0.213	0.59720926198264	0.214	0.59814240297209	0.215	0.59907264148951
0.216	0.6	0.217	0.60092450069174	0.218	0.60184616548065	0.219	0.60276501601497
0.220	0.60368107367977	0.221	0.60459435960125	0.222	0.60550489465111	0.223	0.6064126994507
0.224	0.60731779437513	0.225	0.60822019955734	0.226	0.60911993489198	0.227	0.61001702003931
0.228	0.61091147442896	0.229	0.61180331726366	0.230	0.61269256752284	0.231	0.6135792439662
0.232	0.61446336513717	0.233	0.61534494936637	0.234	0.6162240147749	0.235	0.61710057927767
0.236	0.61797466058656	0.237	0.61884627621362	0.238	0.61971544347411	0.239	0.62058217948958
0.240	0.62144650119077	0.241	0.62230842532061	0.242	0.62316796843698	0.243	0.62402514691557
0.244	0.62487997695262	0.245	0.6257324745676	0.246	0.62658265560583	0.247	0.62743053574112
0.248	0.62827613047828	0.249	0.62911945515563	0.250	0.62996052494744	0.251	0.63079935486633
0.252	0.63163595976564	0.253	0.63247035434174	0.254	0.63330255313629	0.255	0.6341325705385
0.256	0.63496042078728	0.257	0.63578611797342	0.258	0.63660967604169	0.259	0.63743110879291
0.260	0.63825042988599	0.261	0.63906765283993	0.262	0.63988279103578	0.263	0.64069585771856
0.264	0.64150686599917	0.265	0.64231582885624	0.266	0.64312275913796	0.267	0.64392766956389
0.268	0.64473057272669	0.269	0.64553148109389	0.270	0.64633040700957	0.271	0.64712736269604
0.272	0.6479223602555	0.273	0.64871541167164	0.274	0.64950652881123	0.275	0.65029572342569
0.276	0.65108300715264	0.277	0.65186839151738	0.278	0.65265188793438	0.279	0.65343350770876
0.280	0.65421326203772	0.281	0.65499116201194	0.282	0.65576721861697	0.283	0.65654144273461
0.284	0.65731384514424	0.285	0.65808443652414	0.286	0.65885322745279	0.287	0.65962022841015
0.288	0.66038544977893	0.289	0.66114890184579	0.290	0.66191059480262	0.291	0.66267053874767
0.292	0.66342874368675	0.293	0.66418521953442	0.294	0.6649399761151	0.295	0.66569302316419
0.296	0.66644437032919	0.297	0.66719402717079	0.298	0.66794200316394	0.299	0.66868830769887
0.300	0.66943295008217	0.301	0.67017593953781	0.302	0.67091728520811	0.303	0.67165699615476
0.304	0.67239508135979	0.305	0.67313154972653	0.306	0.67386641008053	0.307	0.67459967117054
0.308	0.67533134166938	0.309	0.67606143017487	0.310	0.6767899452107	0.311	0.67751689522733
0.312	0.67824228860283	0.313	0.67896613364375	0.314	0.67968843858592	0.315	0.68040921159534
0.316	0.68112846076891	0.317	0.6818461941353	0.318	0.68256241965571	0.319	0.68327714522464
0.320	0.68399037867068	0.321	0.68470212775722	0.322	0.68541240018326	0.323	0.68612120358408
0.324	0.686828545532	0.325	0.68753443353707	0.326	0.68823887504779	0.327	0.6889418774518
0.328	0.68964344807655	0.329	0.69034359418996	0.330	0.69104232300112	0.331	0.69173964166092
0.332	0.6924355572627	0.333	0.69313007684288	0.334	0.69382320738158	0.335	0.69451495580325
0.336	0.69520532897729	0.337	0.69589433371861	0.338	0.69658197678826	0.339	0.69726826489401
0.340	0.69795320469089	0.341	0.69863680278181	0.342	0.69931906571809	0.343	0.7
0.344	0.70067961207734	0.345	0.70135790834997	0.346	0.70203489516829	0.347	0.70271057883384
0.348	0.70338496559977	0.349	0.70405806167135	0.350	0.70472987320649	0.351	0.70540040631623
0.352	0.70606966706521	0.353	0.7067376614722	0.354	0.7074043955105	0.355	0.7080698751085
0.356	0.70873410615008	0.357	0.70939709447507	0.358	0.71005884587975	0.359	0.71071936611726
0.360	0.71137866089801	0.361	0.7120367358902	0.362	0.71269359672016	0.363	0.71334924897283
0.364	0.71400369819216	0.365	0.7146569498815	0.366	0.71530900950407	0.367	0.71595988248328
0.368	0.71660957420319	0.369	0.71725809000888	0.370	0.71790543520683	0.371	0.71855161506532
0.372	0.71919663481478	0.373	0.7198404996482	0.374	0.72048321472145	0.375	0.7211247851537
0.376	0.72176521602774	0.377	0.72240451239033	0.378	0.72304267925257	0.379	0.72367972159025
0.380	0.72431564434417	0.381	0.7249504524205	0.382	0.72558415069108	0.383	0.7262167439938
0.384	0.72684823713286	0.385	0.72747863487915	0.386	0.72810794197053	0.387	0.72873616311218
0.388	0.72936330297687	0.389	0.72998936620527	0.390	0.73061435740628	0.391	0.73123828115732
0.392	0.73186114200459	0.393	0.73248294446343	0.394	0.73310369301853	0.395	0.73372339212428
0.396	0.734342046205	0.397	0.73495965965525	0.398	0.73557623684011	0.399	0.73619178209542
0.400	0.73680629972808	0.401	0.73741979401628	0.402	0.73803226920981	0.403	0.73864372953027

0.404	0.73925417917137	0.405	0.73986362229914	0.406	0.74047206305221	0.407	0.74107950554206
0.408	0.74168595385324	0.409	0.74229141204362	0.410	0.74289588414466	0.411	0.74349937416159
0.412	0.74410188607369	0.413	0.74470342383451	0.414	0.74530399137208	0.415	0.74590359258916
0.416	0.74650223136345	0.417	0.7470999115478	0.418	0.74769663697046	0.419	0.74829241143526
0.420	0.74888723872185	0.421	0.74948112258591	0.422	0.75007406675932	0.423	0.75066607495043
0.424	0.75125715084421	0.425	0.75184729810249	0.426	0.75243652036411	0.427	0.75302482124518
0.428	0.75361220433924	0.429	0.75419867321743	0.430	0.75478423142876	0.431	0.75536888250019
0.432	0.75595262993692	0.433	0.75653547722251	0.434	0.75711742781909	0.435	0.75769848516751
0.436	0.75827865268758	0.437	0.75885793377818	0.438	0.75943633181749	0.439	0.76001385016311
0.440	0.76059049215228	0.441	0.76116626110202	0.442	0.76174116030933	0.443	0.76231519305129
0.444	0.76288836258531	0.445	0.76346067214923	0.446	0.76403212496151	0.447	0.76460272422136
0.448	0.76517247310896	0.449	0.76574137478553	0.450	0.76630943239355	0.451	0.76687664905691
0.452	0.767443027881	0.453	0.76800857195294	0.454	0.76857328434166	0.455	0.76913716809812
0.456	0.76970022625536	0.457	0.77026246182874	0.458	0.770823877816	0.459	0.77138447719747
0.460	0.77194426293616	0.461	0.77250323797793	0.462	0.77306140525159	0.463	0.77361876766908
0.464	0.77417532812556	0.465	0.77473108949958	0.466	0.77528605465318	0.467	0.77584022643205
0.468	0.77639360766563	0.469	0.77694620116725	0.470	0.77749800973426	0.471	0.77804903614814
0.472	0.77859928317465	0.473	0.77914875356392	0.474	0.7796974500506	0.475	0.78024537535394
0.476	0.78079253217797	0.477	0.78133892321156	0.478	0.78188455112855	0.479	0.78242941858791
0.480	0.78297352823377	0.481	0.78351688269562	0.482	0.78405948458836	0.483	0.78460133651244
0.484	0.78514244105397	0.485	0.7856828007848	0.486	0.78622241826267	0.487	0.78676129603128
0.488	0.78729943662043	0.489	0.78783684254609	0.490	0.78837351631052	0.491	0.78890946040238
0.492	0.78944467729681	0.493	0.78997916945556	0.494	0.79051293932705	0.495	0.79104598934652
0.496	0.79157832193608	0.497	0.79210993950484	0.498	0.79264084444897	0.499	0.79317103915185
0.500	0.7937005259841	0.501	0.79422930730373	0.502	0.79475738545619	0.503	0.79528476277449
0.504	0.79581144157928	0.505	0.79633742417892	0.506	0.79686271286962	0.507	0.79738730993548
0.508	0.79791121764858	0.509	0.79843443826911	0.510	0.7989569740454	0.511	0.79947882721406
0.512	0.8	0.513	0.80052049461658	0.514	0.80104031326566	0.515	0.80155945813766
0.516	0.80207793141168	0.517	0.80259573525558	0.518	0.80311287182602	0.519	0.80362934326857
0.520	0.80414515171781	0.521	0.80466029929735	0.522	0.80517478811994	0.523	0.80568862028756
0.524	0.80620179789146	0.525	0.80671432301227	0.526	0.80722619772006	0.527	0.80773742407439
0.528	0.80824800412444	0.529	0.80875793990901	0.530	0.80926723345665	0.531	0.8097758867857
0.532	0.81028390190439	0.533	0.81079128081087	0.534	0.8112980254933	0.535	0.81180413792992
0.536	0.81230962008914	0.537	0.81281447392954	0.538	0.8133187014	0.539	0.81382230443977
0.540	0.81432528497847	0.541	0.81482764493623	0.542	0.81532938622369	0.543	0.81583051074213
0.544	0.81633102038347	0.545	0.81683091703038	0.546	0.81733020255631	0.547	0.81782887882558
0.548	0.81832694769342	0.549	0.81882441100602	0.550	0.81932127060065	0.551	0.81981752830563
0.552	0.82031318594047	0.553	0.82080824531588	0.554	0.82130270823386	0.555	0.82179657648771
0.556	0.82228985186215	0.557	0.82278253613333	0.558	0.82327463106891	0.559	0.82376613842809
0.560	0.82425705996171	0.561	0.82474739741226	0.562	0.82523715251394	0.563	0.82572632699276
0.564	0.82621492256654	0.565	0.82670294094496	0.566	0.82719038382969	0.567	0.82767725291434
0.568	0.82816354988457	0.569	0.82864927641815	0.570	0.82913443418497	0.571	0.82961902484712
0.572	0.83010305005894	0.573	0.83058651146705	0.574	0.83106941071041	0.575	0.83155174942038
0.576	0.83203352922076	0.577	0.83251475172783	0.578	0.8329954185504	0.579	0.83347553128986
0.580	0.83395509154026	0.581	0.83443410088829	0.582	0.83491256091336	0.583	0.83539047318768
0.584	0.83586783927625	0.585	0.83634466073692	0.586	0.83682093912046	0.587	0.83729667597059
0.588	0.83777187282401	0.589	0.83824653121044	0.590	0.83872065265271	0.591	0.83919423866676
0.592	0.83966729076168	0.593	0.84013981043978	0.594	0.84061179919662	0.595	0.84108325852103
0.596	0.84155418989519	0.597	0.84202459479466	0.598	0.84249447468838	0.599	0.84296383103878
0.600	0.84343266530175	0.601	0.84390097892674	0.602	0.84436877335676	0.603	0.84483605002843
0.604	0.84530281037202	0.605	0.84576905581149	0.606	0.84623478776454	0.607	0.84670000764262
0.608	0.84716471685098	0.609	0.84762891678872	0.610	0.84809260884881	0.611	0.84855579441814
0.612	0.84901847487755	0.613	0.84948065160187	0.614	0.84994232595993	0.615	0.85040349931464

0.616	0.850864173023	0.617	0.85132434843614	0.618	0.85178402689935	0.619	0.85224320975212
0.620	0.85270189832816	0.621	0.85316009395547	0.622	0.85361779795633	0.623	0.85407501164738
0.624	0.85453173633958	0.625	0.85498797333835	0.626	0.85544372394349	0.627	0.85589898944931
0.628	0.85635377114458	0.629	0.85680807031263	0.630	0.85726188823134	0.631	0.85771522617318
0.632	0.85816808540524	0.633	0.85862046718929	0.634	0.85907237278177	0.635	0.85952380343384
0.636	0.85997476039141	0.637	0.86042524489517	0.638	0.86087525818061	0.639	0.86132480147807
0.640	0.86177387601275	0.641	0.86222248300477	0.642	0.86267062366914	0.643	0.86311829921584
0.644	0.86356551084986	0.645	0.86401225977117	0.646	0.86445854717479	0.647	0.86490437425082
0.648	0.86534974218445	0.649	0.86579465215598	0.650	0.8662391053409	0.651	0.86668310290985
0.652	0.86712664602868	0.653	0.86756973585849	0.654	0.86801237355562	0.655	0.86845456027172
0.656	0.86889629715372	0.657	0.86933758534393	0.658	0.86977842598001	0.659	0.870218820195
0.660	0.87065876911736	0.661	0.87109827387101	0.662	0.87153733557533	0.663	0.87197595534519
0.664	0.87241413429097	0.665	0.8728518735186	0.666	0.8732891741296	0.667	0.87372603722104
0.668	0.87416246388563	0.669	0.87459845521172	0.670	0.87503401228333	0.671	0.87546913618014
0.672	0.87590382797758	0.673	0.87633808874678	0.674	0.87677191955464	0.675	0.87720532146386
0.676	0.87763829553291	0.677	0.87807084281612	0.678	0.87850296436363	0.679	0.8789346612215
0.680	0.87936593443164	0.681	0.8797967850319	0.682	0.88022721405606	0.683	0.88065722253386
0.684	0.88108681149103	0.685	0.8815159819493	0.686	0.88194473492641	0.687	0.88237307143616
0.688	0.88280099248842	0.689	0.88322849908913	0.690	0.88365559224036	0.691	0.88408227294029
0.692	0.88450854218327	0.693	0.88493440095979	0.694	0.88535985025656	0.695	0.8857848910565
0.696	0.88620952433873	0.697	0.88663375107865	0.698	0.88705757224791	0.699	0.88748098881448
0.700	0.8879040017426	0.701	0.88832661199287	0.702	0.88874882052221	0.703	0.88917062828394
0.704	0.88959203622773	0.705	0.89001304529968	0.706	0.8904336564423	0.707	0.89085387059455
0.708	0.89127368869184	0.709	0.89169311166607	0.710	0.89211214044563	0.711	0.89253077595543
0.712	0.89294901911691	0.713	0.89336687084805	0.714	0.8937843320634	0.715	0.89420140367413
0.716	0.89461808658796	0.717	0.89503438170928	0.718	0.89545028993907	0.719	0.89586581217501
0.720	0.89628094931143	0.721	0.89669570223935	0.722	0.8971100718465	0.723	0.89752405901732
0.724	0.89793766463302	0.725	0.89835088957153	0.726	0.89876373470757	0.727	0.89917620091265
0.728	0.89958828905508	0.729	0.9	0.730	0.90041133460937	0.731	0.90082229374202
0.732	0.90123287825363	0.733	0.90164308899678	0.734	0.90205292682094	0.735	0.90246239257251
0.736	0.9028714870948	0.737	0.90328021122808	0.738	0.90368856580958	0.739	0.9040965516735
0.740	0.90450416965103	0.741	0.90491142057037	0.742	0.90531830525675	0.743	0.90572482453243
0.744	0.9061309792167	0.745	0.90653677012594	0.746	0.90694219807361	0.747	0.90734726387024
0.748	0.90775196832349	0.749	0.90815631223813	0.750	0.90856029641607	0.751	0.90896392165638
0.752	0.90936718875527	0.753	0.90977009850615	0.754	0.91017265169961	0.755	0.91057484912346
0.756	0.9109766915627	0.757	0.91137817979959	0.758	0.91177931461362	0.759	0.91218009678154
0.760	0.91258052707739	0.761	0.91298060627248	0.762	0.9133803351354	0.763	0.91377971443209
0.764	0.91417874492579	0.765	0.91457742737708	0.766	0.9149757625439	0.767	0.91537375118154
0.768	0.91577139404267	0.769	0.91616869187734	0.770	0.91656564543302	0.771	0.91696225545458
0.772	0.91735852268431	0.773	0.91775444786194	0.774	0.91815003172467	0.775	0.91854527500711
0.776	0.91894017844141	0.777	0.91933474275715	0.778	0.91972896868143	0.779	0.92012285693887
0.780	0.92051640825159	0.781	0.92090962333925	0.782	0.92130250291905	0.783	0.92169504770575
0.784	0.92208725841169	0.785	0.92247913574676	0.786	0.92287068041845	0.787	0.92326189313187
0.788	0.92365277458972	0.789	0.92404332549232	0.790	0.92443354653765	0.791	0.9248234384213
0.792	0.92521300183655	0.793	0.92560223747432	0.794	0.92599114602322	0.795	0.92637972816955
0.796	0.92676798459731	0.797	0.9271559159882	0.798	0.92754352302164	0.799	0.92793080637479
0.800	0.92831776672256	0.801	0.92870440473757	0.802	0.92909072109025	0.803	0.92947671644878
0.804	0.92986239147912	0.805	0.93024774684502	0.806	0.93063278320805	0.807	0.93101750122757
0.808	0.93140190156077	0.809	0.93178598486268	0.810	0.93216975178616	0.811	0.93255320298193
0.812	0.93293633909857	0.813	0.93331916078253	0.814	0.93370166867813	0.815	0.9340838634276
0.816	0.93446574567105	0.817	0.93484731604652	0.818	0.93522857518995	0.819	0.93560952373521
0.820	0.93599016231412	0.821	0.93637049155642	0.822	0.93675051208985	0.823	0.93713022454006
0.824	0.93750962953072	0.825	0.93788872768345	0.826	0.93826751961788	0.827	0.93864600595164

0.828	0.93902418730036	0.829	0.93940206427768	0.830	0.9397796374953	0.831	0.94015690756292
0.832	0.9405338750883	0.833	0.94091054067726	0.834	0.94128690493367	0.835	0.94166296845947
0.836	0.94203873185468	0.837	0.94241419571742	0.838	0.94278936064388	0.839	0.94316422722837
0.840	0.94353879606331	0.841	0.94391306773924	0.842	0.94428704284482	0.843	0.94466072196687
0.844	0.94503410569032	0.845	0.94540719459828	0.846	0.94577998927201	0.847	0.94615249029094
0.848	0.94652469823267	0.849	0.946896613673	0.850	0.94726823718591	0.851	0.94763956934358
0.852	0.9480106107164	0.853	0.94838136187297	0.854	0.94875182338013	0.855	0.94912199580293
0.856	0.94949187970468	0.857	0.94986147564691	0.858	0.95023078418943	0.859	0.95059980589029
0.860	0.95096854130582	0.861	0.95133699099061	0.862	0.95170515549756	0.863	0.95207303537784
0.864	0.95244063118092	0.865	0.95280794345458	0.866	0.95317497274491	0.867	0.95354171959631
0.868	0.95390818455153	0.869	0.95427436815162	0.870	0.954640270936	0.871	0.95500589344243
0.872	0.955371236207	0.873	0.9557362997642	0.874	0.95610108464686	0.875	0.95646559138619
0.876	0.9568298205118	0.877	0.95719377255166	0.878	0.95755744803215	0.879	0.95792084747805
0.880	0.95828397141256	0.881	0.95864682035727	0.882	0.95900939483222	0.883	0.95937169535585
0.884	0.95973372244507	0.885	0.96009547661519	0.886	0.96045695838	0.887	0.96081816825173
0.888	0.96117910674107	0.889	0.96153977435718	0.890	0.9619001716077	0.891	0.96226029899875
0.892	0.96262015703491	0.893	0.96297974621928	0.894	0.96333906705346	0.895	0.96369812003753
0.896	0.96405690567009	0.897	0.96441542444828	0.898	0.96477367686772	0.899	0.96513166342261
0.900	0.96548938460563	0.901	0.96584684090804	0.902	0.96620403281964	0.903	0.96656096082877
0.904	0.96691762542233	0.905	0.96727402708579	0.906	0.96763016630321	0.907	0.96798604355718
0.908	0.9683416593289	0.909	0.96869701409817	0.910	0.96905210834335	0.911	0.96940694254143
0.912	0.96976151716798	0.913	0.97011583269719	0.914	0.97046988960187	0.915	0.97082368835345
0.916	0.97117722942197	0.917	0.97153051327613	0.918	0.97188354038325	0.919	0.97223631120928
0.920	0.97258882621886	0.921	0.97294108587523	0.922	0.97329309064034	0.923	0.97364484097478
0.924	0.97399633733779	0.925	0.97434758018733	0.926	0.97469856998	0.927	0.97504930717112
0.928	0.97539979221466	0.929	0.97575002556333	0.930	0.97610000766851	0.931	0.97644973898029
0.932	0.97679921994749	0.933	0.97714845101763	0.934	0.97749743263695	0.935	0.97784616525042
0.936	0.97819464930175	0.937	0.97854288523337	0.938	0.97889087348648	0.939	0.97923861450098
0.940	0.97958610871556	0.941	0.97993335656766	0.942	0.98028035849346	0.943	0.98062711492793
0.944	0.9809736263048	0.945	0.98131989305658	0.946	0.98166591561454	0.947	0.98201169440876
0.948	0.9823572298681	0.949	0.98270252242022	0.950	0.98304757249156	0.951	0.98339238050738
0.952	0.98373694689175	0.953	0.98408127206753	0.954	0.98442535645644	0.955	0.98476920047897
0.956	0.98511280455447	0.957	0.98545616910112	0.958	0.98579929453591	0.959	0.98614218127471
0.960	0.98648482973219	0.961	0.9868272403219	0.962	0.98716941345623	0.963	0.98751134954643
0.964	0.9878530490026	0.965	0.98819451223374	0.966	0.98853573964768	0.967	0.98887673165115
0.968	0.98921748864974	0.969	0.98955801104794	0.970	0.98989829924913	0.971	0.99023835365556
0.972	0.99057817466839	0.973	0.99091776268768	0.974	0.99125711811239	0.975	0.99159624134039
0.976	0.99193513276846	0.977	0.9922737927923	0.978	0.99261222180654	0.979	0.9929504202047
0.980	0.99328838837927	0.981	0.99362612672164	0.982	0.99396363562217	0.983	0.99430091547012
0.984	0.99463796665372	0.985	0.99497478956015	0.986	0.99531138457553	0.987	0.99564775208494
0.988	0.99598389247242	0.989	0.99631980612098	0.990	0.9966554934126	0.991	0.99699095472821
0.992	0.99732619044773	0.993	0.99766120095007	0.994	0.9979959866131	0.995	0.99833054781369
0.996	0.99866488492771	0.997	0.99899899832999	0.998	0.9993328883944	0.999	0.99966655549379
1.000	1						

$\sqrt[3]{10} = \sqrt[3]{10^1} = 2.1544346900319$

$\sqrt[3]{100} = \sqrt[3]{10^2} = \sqrt[3]{(10)(10)} = \sqrt[3]{10}\ \sqrt[3]{10} = 4.6415888336128$

$\sqrt[3]{1000} = \sqrt[3]{10^3} = \sqrt[3]{(100)(10)} = \sqrt[3]{100}\ \sqrt[3]{10} = 10$

$\sqrt[3]{10{,}000} = \sqrt[3]{10^4} = \sqrt[3]{(1{,}000)(10)} = \sqrt[3]{1000}\ \sqrt[3]{10} = 21.544346900319$

$\sqrt[3]{100{,}000} = \sqrt[3]{10^5} = \sqrt[3]{(10{,}000)(10)} = \sqrt[3]{10{,}000}\ \sqrt[3]{10} = 46.415888336128$

$\sqrt[3]{1{,}000{,}000} = \sqrt[3]{10^6} = \sqrt[3]{(100{,}000)(10)} = \sqrt[3]{100{,}000}\ \sqrt[3]{10} = 100$

Ex. Use the table above to find the cube root of 57:

$\sqrt[3]{57} = \sqrt[3]{(0.57)(10^2)} = \sqrt[3]{0.57}\ \sqrt[3]{100} = 0.82913443418497\ (4.6415888336128) = 3.8485011312768$

To calculate a value that is not listed in the table above, we can use a method that is similar to that previously described above for calculating a value that is not listed in the SIN x table, or e^x table.

Since the slope of the cube root of x curve is equal to $1/(\ 3\ (\sqrt[3]{x}\ )^2\ )$, we will use:

$(y2 - y1) = m\ (x2 - x1)$

$(y2 - y1) = 1/(\ 3\ (\sqrt[3]{x}\ )^2\ )\ (x2 - x1)$

Here are two entries from the table, and we need to find the cube root value of: 0.5045

cube-root of 0.504 = 0.79581144157928 and cube-root of 0.505 = 0.79633742417892

The result should be a value that is someplace between these to values, of about 0.7961

$(\sqrt[3]{0.5045} - \sqrt[3]{0.504}\ ) = 1 / (3\ (\sqrt[3]{0.504}\ )^2)\ (0.5045 - 0.504)$ After some simplification and transposing:

$\sqrt[3]{0.5045} = 0.52633031850481\ (0.0005) + \sqrt[3]{0.504}$

$\sqrt[3]{0.5045} = 0.00026316515925241 + 0.79581144157928$

$\sqrt[3]{0.5045} = 0.79607460673853$: a calculated ("close" mathematical estimate-approximation) interpolated value
The true value is: 0.79607451976095...

This calculated value differs from the true value by about: +0.000,000,087, or about: +0.000,000,1 , or less than a millionth.

Table Of Fractional Powers Of (e)

For using this table, see the article on: Calculating Powers Of (e) Using A Small Table Of Constants.

x	e^x	e^0.x	e^0.0x	e^0.00x	e^0.000x
0	1.000000000000000	1.000000000000000	1.000000000000000	1.000000000000000	1.000000000000000
1	**2.718281828459045**	1.105170918075648	1.010050167084168	1.001000500166708	1.000100005000167
2	7.38905609893065	1.22140275816017	1.020201340026756	1.002002001334	1.000200020001334
3	20.08553692318767	1.349858807576003	1.030454533953517	1.003004504503377	1.0003000450045
4	54.59815003314424	1.49182469764127	1.040810774192388	1.004008010677342	1.000400080010668
5	148.4131591025766	1.648721270700128	1.051271096376024	1.005012520859401	1.000500125020836
6	403.4287934927351	1.822118800390509	1.06183654654536	1.006018036054065	1.000600180036005
7	1096.633158428459	2.013752707470477	1.072508181254217	1.007024557266849	1.000700245057177
8	2980.957987041728	2.225540928492468	1.083287067674959	1.008032085504274	1.00080032008535
9	8103.083927575384	2.45960311115695	1.09417428370521	1.009040621773868	1.000900405121527

x	e^0.0000x	e^0.00000x	e^0.000000x	e^0.0000000x	e^0.00000000x
0	1.000000000000000	1.0000000000000	1.000000000000000	1.000000000000000	1.000000000
1	1.00001000005	1.0000010000005	1.000000100000005	1.00000001	1.000000001
2	1.000020000200001	1.000002000002	1.00000020000002	1.00000002	1.000000002
3	1.000030000450005	1.0000030000045	1.000000300000045	1.000000030000001	1.000000003
4	1.000040000800011	1.000004000008	1.00000040000008	1.000000040000001	1.000000004
5	1.000050001250021	1.0000050000125	1.000000500000125	1.000000050000001	1.000000005
6	1.000060001800036	1.000006000018	1.00000060000018	1.000000060000002	1.000000006
7	1.000070002450057	1.0000070000245	1.000000700000245	1.000000070000003	1.000000007
8	1.000080003200085	1.000008000032	1.00000080000032	1.000000080000003	1.000000008
9	1.000090004050122	1.0000090000405	1.000000900000405	1.000000090000004	1.000000009

Note that $\sqrt{e}$ = e^(1/2) = e^0.5

Table Of Fractional Powers Of 10

For using this table, see the article on: Calculating Powers Of 10 Using A Small Table Of Constants.

x	10^x	10^0.x	10^0.0x	10^0.00x	10^0.000x
0	1.000000000000000	1.000000000000000	1.000000000000000	1.00000000000000	1.000000000000000
1	10.00000000000000	1.258925411794167	1.023292992280754	1.0023052380779	1.000230285020825
2	100.0000000000000	1.584893192461114	1.0471285480509	1.004615790278395	1.00046062307284
3	1000.000000000000	1.99526231496888	1.071519305237606	1.006931668851804	1.000691014168259
4	10000.00000000000	2.51188643150958	1.096478196143185	1.009252886076685	1.000921458319296
5	100000.0000000000	3.162277660168379	1.122018454301963	1.011579454259899	1.001151955538169
6	1000000.000000000	3.981071705534972	1.148153621496883	1.01391138573668	1.001382505837099
7	10000000.00000000	5.011872336272722	1.17489755493953	1.016248692870696	1.001613109228309
8	100000000.0000000	6.309573444801933	1.202264434617413	1.018591388054117	1.001843765724026
9	1000000000.000000	7.943282347242816	1.230268770812382	1.02093948370768	1.002074475336479

x	10^0.0000x	10^0.00000x	10^0.000000x	10^0.0000000x	10^0.00000000x
0	1.000000000000000	1.000000000000000	1.000000000000000	1.00000000000000	1.000000000000000
1	1.000023026116027	1.000002302587744	1.000000230258536	1.000000023025851	1.000000002302585
2	1.000046052762256	1.00000460518079	1.000000460517125	1.000000046051703	1.00000000460517
3	1.000069079938699	1.000006907779138	1.000000690775767	1.000000069077555	1.000000006907755
4	1.000092107645368	1.000009210382787	1.000000921034461	1.000000092103408	1.000000009210341
5	1.000115135882277	1.000011512991739	1.000001151293209	1.000000115129261	1.000000011512926
6	1.000138164649436	1.000013815605993	1.00000138155201	1.000000138155115	1.000000013815511
7	1.000161193946858	1.000016118225548	1.000001611810864	1.00000016118097	1.000000016118096
8	1.000184223774555	1.000018420850406	1.000001842069771	1.000000184206825	1.000000018420681
9	1.00020725413254	1.000020723480565	1.000002072328731	1.00000020723268	1.000000020723266

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EXTRAS AND LATE ENTRIES

These topics and-or articles are either not specifically math oriented and-or late entries in this book, but they may be interesting and useful for many people. The word "maths" now seems to includes all useful things of science. Distribution and access of knowledge can sometimes be a problem due to a variety of reasons, and it is therefore practical to just include what I could in this one book, and that is why it has a generous amount of pages.

SOME TIPS TO TYPE EASIER

This discussion is about typing better so as to record your thoughts, data, and communicate better via typed messages. Many have never learned to formally type and rather "chicken pluck" each key with one finger rather than use both hands and most fingers. The notes below will help you use and-or remember the keyboard, key layout or positioning.

Standard key layout or positions:

1 2 3 4 5 6 7 8 9 0
Q W E R T Y U I O P
A S D F G H J K L ;
Z X C V B N M , . /
(space key)

People who do much typing should place their hands on what is called the "home row" of keys. These keys are:

Left Hand   Right Hand   : excluding the thumb finger

A S D F   J K L ;   ,   :   **;** = semicolon   and   **:** = colon, press shift and semicolon for colon

Once you place your fingers on these keys, you may feel a slight bump on the F and J keys on most modern keyboards. This will help ensure your fingers are positioned correctly on the "home row" of keys. The home row is also called the middle set or row of keys. The row above it is often called the upper row. The row of keys below the home row is often called the lower set of keys, or simply the bottom row.

All the other keys are pressed by moving a "home row" finger up, down, left or right such as for the G and H keys. The finger used is usually the one of closest to the key and-or the one most comfortable.

Key or finger	The other keys typically used or pressed by that same finger.
A	Q , Z
S	W , X
D	E
F	R , T , C , V , G , B
J	H , Y , U , N , M
K	I , **,** (comma)
L	O , **.** (period)
;	P , / (forward slash) , **"** (quotation marks , **?** (question mark)

By using the above method, all 26 letters of the English alphabet can be typed. The shift keys which are just left and right of the bottom row are usually used to type in a capital or uppercase letter by first pressing a shift key with a "pinky" or small finger that is most opposite from your thumb finger, and then also pressing the desired key. If the letter key to press is with the right hand, the left hand will first press its corresponding side shift key. If the letter key to press is with the left hand, the right hand will first press its corresponding side shift key.

The space bar which is below the bottom row of keys is used to enter a blank space or letter such as using one space between words, and two spaces between sentences. The space bar is usually pressed using a thumb finger.

Here are some ways to remember where the keys or letters are located on the keyboard:

1. Remember the home row of keys: A S D F J K L ;
   The F and J keys often have a small bump on them so you can feel if your fingers are positioned on the home row.

2. Notice how the first letter A and the last letter Z are next to each other.

3. Notice the vertical-like arrangement of the letters: C , D , E

4. Notice the close, horizontal arrangement of the letters: F , G , H

5. Notice the close arrangement of the letters: I , J , K , L

6. Notice the close arrangement of the letters: M , N and O , P

7. Notice in the upper row of keys and the infamous: QWERTY set of keys

8. Notice in the upper row for the right hand, the letters of: Y , O , U and I

9. Notice in the upper row for the left hand, the letters of: W , E

10. Notice the arrangement of the letters: T , H , M as for the word: T , H , E , M

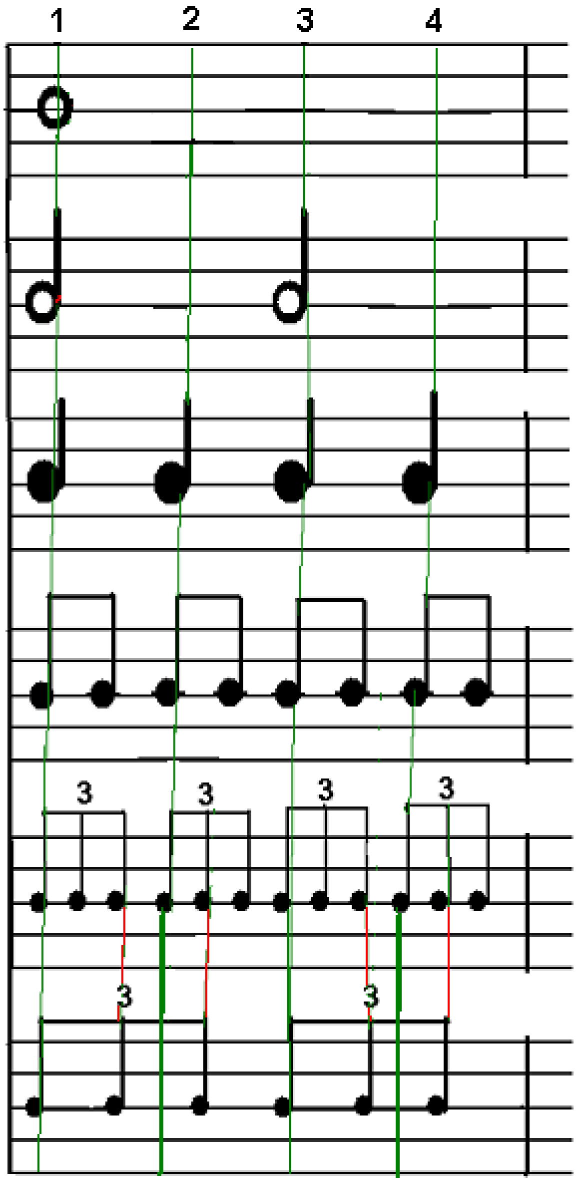
11. Besides the vowel of A, all the other vowels are in the upper or top row: E , I , O , U , Y
    The last four of these vowel letters are near to each other.

12. After reviewing the above tips, try typing the alphabet (letters A through Z, usually the lower case versions first) and some words. You can look at where the keys are located at first, but later after a few weeks, you many not have to look as much. Typing speed is generally unimportant, but rather efficiency or few mistakes is what matters most.

COMMON MUSIC NOTATION OF THE TIMING AND DURATION OF NOTES

FIG 281]

A music symbol can represent a duration or beat, a note, or a rest where no note is played. The green line is used to indicate the timing of the start of one of the 4 beats in a measure.



: beat number

Whole Note, has the duration of the entire or whole measure.

Half Notes
Half of a 4 beat measure.

Quarter Notes
There are 4 beats in a common measure of music.

Each quarter note has a duration of 1/4 = 0.25 = 25% of the measure.

Eighth Notes

An eighth note has half the duration of 1 beat. There are two eight notes in one beat.

Quarter Note Triplets

A triplet puts three notes or durations in the count or duration of two notes.

For two notes, each will have 1/2 = 0.5 = 50% of the beat.

Half Note Triplets

For three notes in a beat, each will have just 1/3 = 0.333 = 33.3% of the beat. The timing of each note of a triplet is at 0%, 33.3% and 66.7%

Though there are usually many notes indicated on "music staffs" which are a notation system for representing music, I have only used the "B" note here so as to keep things simple

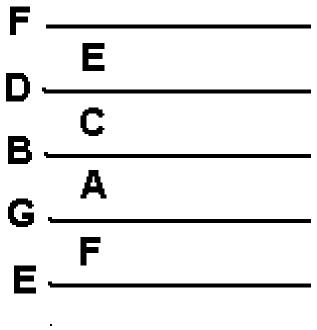
In your mind, you can count the eighth notes as: 1, 2, 3, 4, 5 , 6, 7, 8, however, they are more commonly counted as two sets of 4 as: 1, 2, 3, 4 , and 1, 2, 3, 4.

The quarter note triplets, and where each triplet set of notes starts on the beat, can be counted as: 1, 2, 3.

A half note triplets can be counted when considering the timing of two sets of quarter note triplets as being:
1, 2, 3 for one beat and 4, 5, 6 for another beat, and each note of a quarter note triplet will correspond to: 1, 3, 5

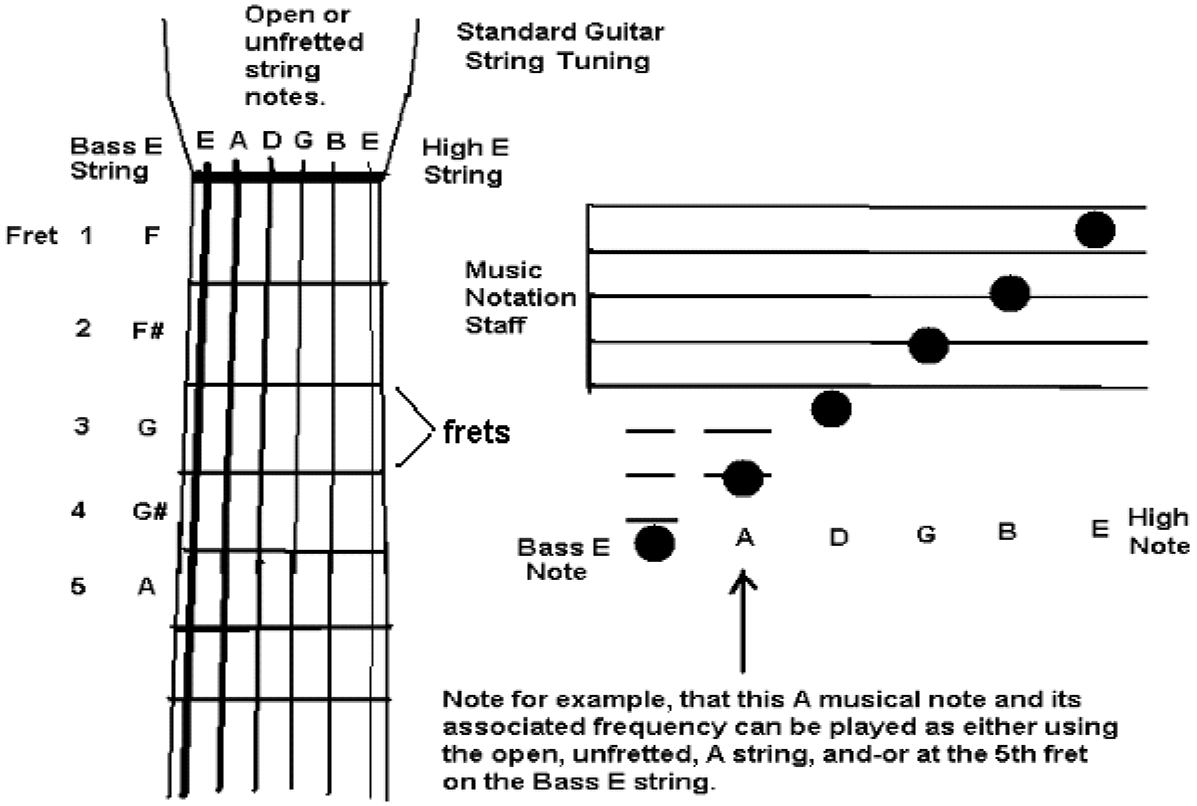
NOTES OF THE LINES AND SPACES OF THE MUSIC STAFF NOTATION

[FIG 282]



STANDARD NOTES, NOTATION, AND TUNING OF EACH GUITAR STRING

[FIG 283]



Note for example, that this A musical note and its associated frequency can be played as either using the open, unfretted, A string, and-or at the 5th fret on the Bass E string.

The right side of the above figure shows the music notation for each of the six strings of the guitar. Beginners will generally first learn the higher frequency, thinner strings on a guitar. Being familiar with music notation helps people quickly learn songs, a guide, and-or how they were intended to be played in a formal manner. Guitarists and or other musical instrument players can improvise by just finding or knowing the chords of a song, and then playing some pleasing sounding notes in the scale associated with the known or indicated chord. A guitarist may even play, or need to play, each chord and-or note of the song using a certain number of half-steps higher or lower on the fretboard so as to change the entire pitch of the entire song, perhaps so as the singer can sing it easier.

EXAMPLE GUITAR CHORDS

A chord is two or more sounds or notes played at the same time. The notes chosen to be played are usually those that have a pleasing sound.

A C major chord includes these notes in the C major scale: **1**, **3** and **5**.

Note #:	**1**	2	**3**	4	**5**	6	7	8
Note name:	**C**	D	**E**	F	**G**	A	B	C

Ex. C major chord = C, E, G : For this chord and or scale, C is called the "root note". here, root = initial or base note. C is the basis of that chord, and the starting or initial note of that scale.

The notes of any chord can be played and-or noted with musical notation as any combination or order of the notes associated with that chord. For example, and with the first note being lowest in frequency and on the music staff:

Ex. C major chords = Cmaj = (C, E, G) and-or (C, G, E) and-or
(E, C, G) and-or (E, G, C) and-or
(G, C, E) and-or (G, E, C)

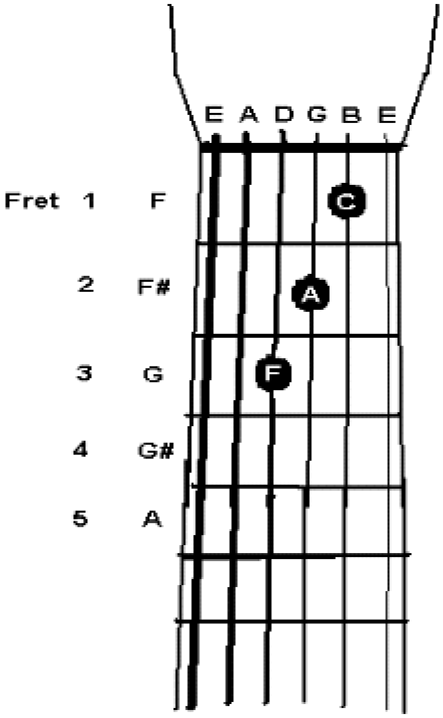
A C minor chord includes these notes in the C minor scale: 1, 3b and 5.

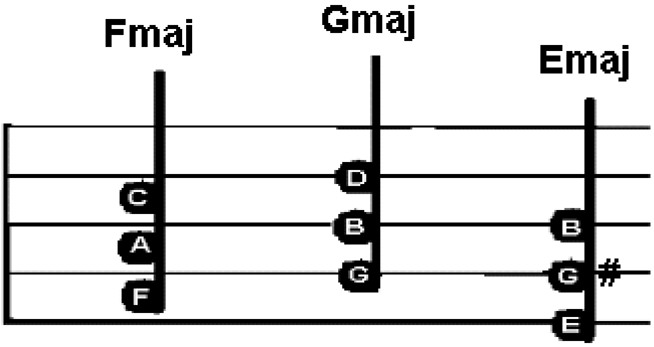
In relationship to the major scale, a minor scale has a flatted third note, and this means that it is one fret lower on the guitar fretboard. It also means that the third note is a half-step (or "1 chromatic step") less than the corresponding note in the major scale.

Ex. A C minor chord = Cm = (C, Eb, G)

A major chord can be thought of as having a direct or powerful sound and feeling, whereas a minor chord can be thought of as having a soft, moody or serious sound and feeling.

[FIG 284]





In the above image, shown on the left is the 6 guitar strings, and with the fingering shown for an F major chord. The other strings are either to be muted and-or not played, and this is sometimes shown with an x symbol at the end of the string. An open or unfretted string that is to be played or included in a chord is usually shown with a O symbol.

Beginner guitar players often develop sore fingers,and it will take several days to a month of casual use so as to develop a callous (harder skin) on the tips of your fingers so as to play more easily.

Once you can play the F major chord indicated, if you move and-or slide your hand 2 frets and-or 2 fret positions up or higher along the guitar fretboard, you can play the G major chord which is indicated on the music staff shown.

Once you can play the F major chord, (F, A, C), indicated, if you move and-or slide your hand 1 fret and-or position lower along the guitar fretboard, you can then play the E major chord, (E, G#, B) which is indicated on the music staff shown above. If you were to play these note positions of the F major chord just one whole step (= two half-steps or frets) higher along the fretboard, you can play the Gmajor chord = Gmaj chord (G, B, D). While playing the Gmaj chord, if you were to rather include Bb, instead of B, and which is one fret and-or half-step lower than the B note, you can play the Gminor = Gmin = Gm chord (G, Bb, D). The minor chord is sometimes said as having a "flatted-third note" (with respect to the major scale). You can then try to play Am, Fm and Em chords and-or notes. Clearly, there are many patterns to position your fingers and to then play chords and-or play individual notes in a chord scale.

While playing the F chord, you can also include the F note on the first fret of the first string which is the high E string. This F note is an octave higher and-or twice the frequency of the F note shown on the music staff. You can play this F note, and the C note at the first fret on the second or B string by using the flat part of your index finger to play both of these notes together at once, and this is called a bar-chord when a finger is used to fret several strings and-or notes at the same time.

If you were to play the open strings of (D, G, B), this is a Gmaj chord. And then moving this up two frets and-or half-steps, the chord would be Amaj (E, A, C#).

Notice for example that of the F chord shown starts on a blank space between the lines of musical staff, and that the next two notes of that chord are also on the next consecutive spaces. It could be said that from space to space is a "third note higher" or three steps in the scale, and that from a line to line is a "third note higher" or three steps in the scale.

There is another system of music notation for guitar playing and it is called Guitar Tablature or Guitar Tab. This notation is similar to a combination of both the standard musical notation system as shown in the above image on the right with a five line music staff, and also similar to the 6 strings of the guitar as shown on the left. The combined result is a 6 line music staff where each line is the string, and the note to play on that string is indicated by the fret number and-or fret position on that string. A bass guitar is similar to a guitar, and it has the 4 thicker strings as that of a guitar: E, A , D, G, of which for the bass guitar also thicker and usually longer so as to produce a lower frequency sound.

The "(musical) key of C" is often used for teaching what notes are on the musical notation staff. The key of C does not have any sharped or flatted notes. Other keys will have at least one sharped note or one flatted note. For example the scale or key of G has these following notes: G, A , B , C , D , E , F#, G. We see that the scale contains an F-sharp note. Still there is a pattern as with the C scale or key:

root note (1), whole-step (2), whole-step (3), half-step (4), whole-step (5), whole-step (6), whole-step (7), half-step (8).

A whole-step is two semi-tones higher in frequency, and since each fret is a half-step or semi-tone in a scale, a whole-step is two semi-tomes and-or frets higher on the guitar fretboard. The key or scale of F has one flatted note, and that is Bb ("B-flat"): F, G, A Bb, C, D, E, F.

Here are the approximate frequencies of the standard guitar tuning based on a standardized A note being 440 hz.:

String	**Approximate Frequency in hz.**	Some types of tuners: reed, tuning fork, electronic, frequency counter
Bass E	82	
A	110	:The A string on a guitar may sometimes be noted as 220 or 440 hz. such as on some "reed tuners".
D	147	
G	196	
B	247	:this is practically 250hz, and which is two octaves lower than 1000hz...500hz..250hz..125hz...62.5hz
High E	330	: This is two octaves higher than the Bass E note. 82hz...164hz...328hz

These frequencies shown above can be verified by the discussion about string length and frequency in the book.

Here are some of the common major and minor chords and their notes.

	Major Chord Notes	**Minor Chord Notes**	**8 Note Of The Common "Western" Major Scale**	
	(1 , 3, 5)	(1 , b3, 5)	**1** 2 **3** 4 **5** 6 7 8	
C	(C, E, G)	(C, Eb, G)	**C**, D, **E** , F, **G**, A , B , C	
D	(D, F#, A)	(D, F, A)	D, E , F#, G, A , B , C# , D	
E	(E, G#, B)	(E, G, B)	E , F#, G#, A , B , C#, D# , E	
F	(F, A, C)	(F, Ab, C)	F, G, A , Bb , C , D , E , F	: Bb = A# , "B flat equals A sharp"
G	(G, B, D)	(G, Bb, D)	G, A , B , C , D , E , F# , G	
A	(A, C#, E)	(A, C, E)	A , B , C# , D , E , F# ,G# , A	
B	(B, D#, F#)	(B, D, F#)	B , C# , D# , E , F# ,G# , A#, B	
C	(C, E, G)	(C, Eb, G)		

As shown above, the major chord has the following notes of its major scale: (1, 3, 5). As shown above, the minor chord also has the following notes of its minor scale: (1, 3, 5). In reference to the major scale, this 3 note of the minor scale is equivalent to a flatted or minor third (ie. 3b) note of the major scale.

To play a major-7 chord, play the major chord and also include the 7th note of that major scale. For example:

Fmaj7 = (F, A, C, E) : This is the F chord shown above, but also includes the open (e) string and-or note.

Cmaj7 = (C, E, G, B) : for example, this C note can be played at the third fret of the A string. The E note can be played on the second fret of the D string. The G note can be played as an open G string. The B note can be played as an open B string.

To play a "7th chord", flatten the 7th note of the scale by a half-step and-or fret. This is technically a "flatted maj. 7th" chord.

For example, here are the formalized notes of some chords, of which are more playable on a piano instrument, and a guitarist needs to consider how, what notes and where to play a chord and any significant sounding special note(s) in it.

C7 = (C, E, G Bb) = (1, 3, 5 , 7b) : sometimes a flatted note is called a diminished note

To play a "6th chord", include the 6th note of that scale. For example:

C6 = (C, E, A) = (1, 3, 6)

C9 = (C, E, G, Bb, D) = C7+9 = (1, 3, 5, 7b, 9)

Cadd9 = (C, E, G, D) = C+9 = (1, 3, 5, 9)

Csus4 = (C, F, G) = C+4 = C4 , : C chord with "suspended" 4th note. This chord is often then followed by the corresponding major chord: (C, E, G) so as to have a "resolving" sound or change.

Caug = C augmented = Caug. 5th = C+ = (C, E, G#) = (1, 3, 5#) = C+6b , This chord is often followed by the corresponding "6 chord": (C, E, A).

Cdim = C diminished = (C, Eb, Gb, A) = (1 , 3b , 5b, 6) , C dim = Eb dim = Gb dim = A dim
, diminished basically means reduced

The most common chords used in a song:

Root note chord (**1**) , Ex. C maj
Fourth note chord (**4**) , Ex. F maj
Fifth note chord (**5**) , Ex. G maj

Some other chords occasionally included in a song based on a C major scale:

Second note minor chord (2m) , Ex. D min , such as (D, F, A), and these are also notes in the C major scale.
Third note minor chord (3m) , Ex. E min , such as (E, Gb, B)
Sixth note minor chord (6m) , Ex. A min , such as (A, C, E)
Flatted seventh note chord (7b) , Ex. Bb , such as (C, E, G, Bb) , although not all notes need to be played at once

Of any chord played, it is not uncommon to find the flatted 7th note (7b) of the scale occasionally being included. The chords and-or their corresponding note number in the scale of a song may be called as the chord progression of the song. If you are able, obtain a basic guitar "chord book" that shows where to place our fingers on the fretboard so as to play a certain chord, and observe what musical notes are to be played. It may take some amount time to get used to the feel of a guitar, chords, etc., and then the guitar will be easier to play.

Observe the patterns of the position of notes in a particular scale on the fretboard so as to help play the notes in scale. Improvising and-or "playing along" with recordings is a way to get involved with the music. Often a player may need to play both chords and single notes of a scale, and of which are could located near or at the chord being played.

On a scientific level, it is possible to create reference sound frequencies. If you had a wheel with say 60 bumps (protrusions, raised surface area) and-or gear teeth equally spaced around its circumference, rim or edge, and then spun that wheel at 1 revolution or rotation per second while holding a rod against that wheels edge or circumference, you would hear a 60hz sound tone. This is due to the 60 bumps and-or sound pulses per second created when the rod collided with the edge of the wheel and some of that energy from the collision was converted to sound energy. More bumps and-or a faster rotation can be used to create higher frequencies. This method discussed here was actually the method used to create the various notes and their corresponding frequencies with the first electronic organs, but instead of a stick against the circumference of the organs various diameter sized, spinning tone-wheels, a electro-magnetic transducer (ie., energy converter) or "(energy) pickup" was used near the circumference edge of each wheel, and its construction was essentially a coil of wire wrapped about a magnet. The varying or AC signal output from this pickup usually had a (smooth, undistorted) sine waveform, much like that from a bell. These organs were not completely electric like modern organs which use only electronic oscillators, but they were both partly mechanical, and partly electrical, hence being electromechanical. The initial invention of the **tone-wheel** is credited to **Thaddeus Cahil** in 1896, and the organ was called the Telharmonium. Because of its large size and cost, only a few Telharmonium organs were built. When electronic signal amplification was made practical after the invention of the vacuum tube amplifier, electronic organs would become smaller and more practical, such as the electromechanical organs produced by the Hammond Co. in 1935. Hammond organs were relatively affordable and were often installed in church buildings that had limited space, and since a full sized pipe organ was very large and very costly. The sound of an electromechanical Hammond organ can heard within many popular musical recordings, particularly from the 1950's through the 1970's.

Ex. To make a 1000hz tone, a large diameter wheel with 100 bumps about it will need to spin at 10 revolutions per second = 10rps:

Each physical bump or vibration on the tone-wheel will produce 1 electrical pulse or vibration.

Frequency = (revolutions per second)(number of bumps on the wheels edge for one revolution) =

Frequency = (10 revolutions / second) (100 bumps, pulses or vibrations / revolution) =

$$\text{Frequency} = \frac{10\ \cancel{\text{revolutions}}}{1\text{s}} \times \frac{100\ \text{bumps}}{\cancel{\text{revolution}}} = \frac{1000\ \text{bumps}}{1\ \text{second}} =$$

Frequency = vibrations per second = bumps per second = waves per second = cycles per second = 1000 hz

Note that this wheel has a rotational frequency or "angular velocity" of 10 revolutions per second, while the outputs sound of this system has a frequency of 1000 cycles per second. Each note on the organ's keyboard had a corresponding tone-wheel.

The "pickup" in an electric guitar is essentially a small electricity and-or signal (ie.,having a frequency part) generator. It is usually made by winding a coil of wire around a permanent (no electricity and-or current needed to produce the magnetism) magnet. When the string made of metal moves and-or vibrates near the pickup, the magnetic field will move slightly and produce an electric signal in the coil, and which is then sent to an electronic amplifier so as to make it a stronger signal with more power, particularly current, so as to move and-or vibrate an audio speaker.

The first assigned note, the A-note and its corresponding frequency is most likely due to the standardized 60 seconds in a minute. The multiples of the 60 hz frequency, musical note and-or its octaves (double the frequency) are then: 60, 120, 240, 480 and 960hz. Through the years, the frequencies of the standardized notes have slightly varied due to various reasons. Today, 440hz is the modern, standardized frequency of the A-note and-or tone, and 55hz is a lower frequency octave of the A-note. Musicians are generally not concerned with the actual frequency of a note(s), but are rather concerned about being "in tune", or tuned to having the same tone or note as that of any reference tone or note available, and particularly so when more than one instrument is being played at the same time by a group of people. It is recommended that musicians obtain a calibrated pitch-pipe (a small (vibrating) metal reed instrument which a person can blow air into) and-or a (metal) tuning-fork designed for tuning their instrument. Today, it is possible to also tune an instrument by using audio recordings of frequencies and-or notes, and-or by using a frequency meter. Some instruments cannot, and-or should not, be tuned, such as having certain metal lengths like a xylophone, various wind instruments, or tubular bells, and rather the entire orchestra must be tuned to these instruments so as every instrument is "in tune".

It is quite possible to construct a tone wheel that will interrupt a beam of light from reaching a light sensor, and so as to create electric pulses that way instead of using an electromagnetic pickup sensor.

COMMON GUITAR SCALE LENGTHS AND FRET SPACINGS

These tables were made by using a computer program made by the author of the Mathization ebook.

In general, the longer the scale length, the more the tension upon the strings is needed for standard tuning (EADGBE). This is due to that the length of the open or unfretted string is then longer which lowers its natural or fundamental frequency and when at the same standard tuning tension, and then more tension will be needed to raise it up to the desired frequency and-or note. An alternative to raising the tension on the strings so as to obtain a higher frequency and-or note is to use a mechanical device called a capo, and which is like movable clamp that can be set by the player and it is used to simulate the nut of a shorter scale length stringed instrument. New players and-or people with small hands may wish to use a capo and-or obtain a short scale guitar which usually has a thinner neck. A problem now with a very short scale length guitar is if the strings will have enough tension to feel right when tuned to standard tuning, but it is not usually an issue for many. With a capo, another less used option is to tune each string of the guitar a fret (ie., half-tone) or more lower than the standard tuning and place the capo up from the nut at the same number of steps of lower tuning, and so as to obtain a standard tuning. Using this method will also reduce the tension of the strings. The author of this ebook has a computer program to calculate the fret location and-or spacings given an input scale length. Shown below are the fret spacings for a 24.75" (regular) and 23.0" (medium) scale length guitar, and a 30.0" "short-scale" bass guitar. A standard bass scale length is 34", a medium bass scale length is 32", and a short-scale base length is 30". Generally, the shorter the scale length, the neck thickness and width is also thinner.

```
SCALE LENGTH  =  24.75 inches =  628.65 millimeters  = 62.865 centimeters  =  62 cm + 87 mm   =~  63 cm = 0.63 m

1 inch = 1" = 2.54 cm = 25.4 mm

FRET SIZE" In.+ 32nds   mm    FROM NUT" In.+ 32nds   mm

 1   1.389  1 + 12.45  35.283      1.389   1 + 12.45  35.283
 2   1.311  1 + 9.96  33.303       2.700   2 + 22.41  68.587
 3   1.238  1 + 7.60  31.434       3.938   3 + 30.01  100.020
 4   1.168  1 + 5.38  29.670       5.106   5 +  3.39  129.690
 5   1.103  1 + 3.28  28.004       6.208   6 +  6.67  157.695
 6   1.041  1 + 1.30  26.433       7.249   7 +  7.97  184.127
 7   0.982  0 + 31.43  24.949      8.231   8 +  7.40  209.076
 8   0.927  0 + 29.67  23.549      9.158   9 +  5.07  232.625
 9   0.875  0 + 28.00  22.227      10.034  10 +  1.07  254.852
10  0.826  0 + 26.43  20.980       10.860  10 + 27.51  275.832
11  0.780  0 + 24.95  19.802       11.639  11 + 20.45  295.634
12  0.736  0 + 23.55  18.691       12.375  12 + 12.00  314.325
-------------------------------------------------------------------------
13   0.695  0 + 22.23  17.642      13.070  13 +  2.23  331.967
14   0.656  0 + 20.98  16.652      13.725  13 + 23.20  348.618
15   0.619  0 + 19.80  15.717      14.344  14 + 11.01  364.335
16   0.584  0 + 18.69  14.835      14.928  14 + 29.69  379.170
17   0.551  0 + 17.64  14.002      15.479  15 + 15.34  393.172
18   0.520  0 + 16.65  13.216      16.000  15 + 31.99  406.389
19   0.491  0 + 15.72  12.475      16.491  16 + 15.70  418.863
20   0.464  0 + 14.83  11.774      16.954  16 + 30.54  430.638
21   0.438  0 + 14.00  11.114      17.392  17 + 12.54  441.751
22   0.413  0 + 13.22  10.490      17.805  17 + 25.75  452.241
23   0.390  0 + 12.47  9.901       18.195  18 +  6.23  462.142
24   0.368  0 + 11.77  9.345       18.562  18 + 18.00  471.487

Press y to continue, q for quit:
```

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SCALE LENGTH  = 23 inches = 584.2 millimeters = 58.42 centimeters      : For a typical short scale electric guitar
                                                                          Short scale guitars are generally no less
                                                                          than about 20.75" , such as for the original
                                                                          Rickenbacker 325.

FRET SIZE" In.+ 32nds   mm      FROM NUT" In.+ 32nds   mm

 1   1.291  1 + 9.31  32.789      1.291   1 +  9.31  32.789
 2   1.218  1 + 6.99  30.948      2.509   2 + 16.30  63.737
 3   1.150  1 + 4.80  29.211      3.659   3 + 21.10  92.948
 4   1.086  1 + 2.74  27.572      4.745   4 + 23.84  120.520
 5   1.025  1 + 0.79  26.024      5.769   5 + 24.62  146.545
 6   0.967  0 + 30.95  24.564     6.737   6 + 23.57  171.108
 7   0.913  0 + 29.21  23.185     7.649   7 + 20.78  194.293
 8   0.862  0 + 27.57  21.884     8.511   8 + 16.35  216.177
 9   0.813  0 + 26.02  20.656     9.324   9 + 10.37  236.833
10  0.768  0 + 24.56  19.496     10.092  10 +  2.93  256.329
11  0.724  0 + 23.18  18.402     10.816  10 + 26.12  274.731
12  0.684  0 + 21.88  17.369     11.500  11 + 16.00  292.100

-----------------------------------------------------------------------
13   0.645  0 + 20.65  16.394     12.145  12 +  4.65  308.494
14   0.609  0 + 19.50  15.474     12.755  12 + 24.15  323.968
15   0.575  0 + 18.40  14.606     13.330  13 + 10.55  338.574
16   0.543  0 + 17.37  13.786     13.872  13 + 27.92  352.360
17   0.512  0 + 16.39  13.012     14.385  14 + 12.31  365.372
18   0.484  0 + 15.47  12.282     14.868  14 + 27.78  377.654
19   0.456  0 + 14.60  11.593     15.325  15 + 10.39  389.247
20   0.431  0 + 13.79  10.942     15.755  15 + 24.17  400.189
21   0.407  0 + 13.01  10.328     16.162  16 +  5.19  410.516
22   0.384  0 + 12.28  9.748      16.546  16 + 17.47  420.264
23   0.362  0 + 11.59  9.201      16.908  16 + 29.06  429.465
24   0.342  0 + 10.94  8.685      17.250  17 +  8.00  438.150

Press y to continue, q for quit:
```

SCALE LENGTH = **30 inches** = 762 millimeters = 76.2 centimeters : For a short-scale length bass. An example is the Hofner violin shaped bass.

A short-scale electric bass guitar is well suited for beginners, people with small hands, and standard guitar players since the neck feels almost like a guitar neck. The short scale means less tension on the strings to tune it to standard tuning (E, A , D, G, like the thicker four of the guitar strings), and therefore the sound is slightly deeper, perhaps having a tone more similar to that an old upright orchestral bass that is shaped like a giant violin.

```
FRET SIZE" In.+ 32nds   mm     FROM NUT" In.+ 32nds   mm

 1   1.684  1 + 21.88  42.768     1.684   1 + 21.88  42.768
 2   1.589  1 + 18.86  40.367     3.273   3 +  8.74  83.135
 3   1.500  1 + 16.00  38.102     4.773   4 + 24.74  121.237
 4   1.416  1 + 13.31  35.963     6.189   6 +  6.05  157.200
 5   1.336  1 + 10.77  33.945     7.525   7 + 16.81  191.145
 6   1.261  1 + 8.36  32.040      8.787   8 + 25.18  223.185
 7   1.191  1 + 6.10  30.241      9.977   9 + 31.28  253.426
 8   1.124  1 + 3.96  28.544      11.101  11 +  3.24  281.970
 9   1.061  1 + 1.94  26.942      12.162  12 +  5.18  308.912
10  1.001  1 + 0.04  25.430      13.163  13 +  5.22  334.342
11  0.945  0 + 30.24  24.003     14.108  14 +  3.46  358.345
12  0.892  0 + 28.54  22.655     15.000  14 + 32.00  381.000
----------------------------------------------------------------------------------------
13   0.842  0 + 26.94  21.384     15.842  15 + 26.94  402.384
14   0.795  0 + 25.43  20.184     16.637  16 + 20.37  422.568
15   0.750  0 + 24.00  19.051     17.387  17 + 12.37  441.618
16   0.708  0 + 22.65  17.982     18.094  18 +  3.02  459.600
17   0.668  0 + 21.38  16.972     18.763  18 + 24.41  476.573
18   0.631  0 + 20.18  16.020     19.393  19 + 12.59  492.592
19   0.595  0 + 19.05  15.121     19.989  19 + 31.64  507.713
20   0.562  0 + 17.98  14.272     20.551  20 + 17.62  521.985
21   0.530  0 + 16.97  13.471     21.081  21 +  2.59  535.456
22   0.501  0 + 16.02  12.715     21.582  21 + 18.61  548.171
23   0.472  0 + 15.12  12.001     22.054  22 +  1.73  560.172
24   0.446  0 + 14.27  11.328     22.500  22 + 16.00  571.500

Press y to continue, q for quit:
```

HOW THE MUSICAL SCALE WAS FIRST DETERMINED

It is thought that **Pythagoras**, a great mathematician from ancient Greece who is credited to have made the famous Pythagorean Theorem for finding the longest side of a right triangle, conceived a definition of the pleasing musical notes in a range of all possible frequencies in one octave (ie., from a given or starting frequency, to double or twice that frequency) . Before this, it was possible to make a scale by finding the pleasing notes found on a string by just listening to the sounds and marking their positions. If two or more strings are being used, then two or more notes can be played at the same time, and which is called a chord of notes, and in particular, when sounding pleasing (ie., in harmony) together. These strings can also be tuned to different starting or fundamental notes so as to make some chords more available and-or easier to play. Many instruments have a standardized and-or common tuning or notes available.

Pythagoras wanted to formalize a music or sound scale as being geometric in nature, and by using strict integer fractions of a string length. This was during a time that the "equal tempered" and-or geometric ratio scale of notes was not yet known, and this scale ratio was discussed previously in this book. This book derived the relative value of about **1.06** in terms of ascending frequencies of the notes of the chromatic scale and about **$0.944\underline{9}$** for the relative value of the string lengths of each successive note in the chromatic scale. These values are actually reciprocals in value.

Pythagoras realized that if a string length or object was twice as long or big, that the tone of its sound or "pitch" was similar but deeper. Today we would say that its frequency and-or vibrations would be half. In a reverse type of manner, he realized that if the string length or object was half as long or big, that the tone of its sound was the similar, but higher in pitch (ie., frequency as we call it today). Pythagoras thought the harmony of the universe and sound was geometric-like and of "pure numbers" such as fractions containing only integers in the numerator and denominator. With what could be called his math of sound concept, he conceived of a scale containing 7 notes and with the 8th or "octave" (a word created from an 8, equal-sided polygon, called an **octa**gon) note or tone being the next "pitch" or degree higher in tone than the first tone. For some other notes of the Pythagoras octave, major scale:

Note Of Scale	Note Location Along The String Length (Lm) from the nut or string support opposite to the bridge support.
1	L/1 = L , full length of the string, an "open string" = a non-fretted or lessened string length
3	L/5 = (1/5) L , hence (4/5) of the string will be left and-or vibrating , A flatted third, such as used in the minor scale, is at: L/6 = (1/6) L
4	L/4 = (1/4) L , hence (3/4) of the string will be left and-or vibrating (1/4) of L is halfway to (1/2) L which is the first octave note.
5	L/3 = (1/3) L , hence (2/3) =~ $0.66\underline{7}$ of the string length will be left and-or vibrating
8	L/2 = (1/2) L , hence at half the string length, and (1/2) L will be left and-or vibrating

In comparison to the modern 12 note equal "tempered" or ratio scale, the Pythagorean 7 note scale is close in value(s) to it. The Pythagorean scale has a maximum difference from the more modern scale of less than 1%, and is actually just 0.5%. This could be said as being amazing, and that both scales actually help reinforce or verify each other in some way.

Pythagoras found each note and-or length by considering the corresponding length of the 5th note in the scale. For example, if the first note is considered as C, the 5th note is (after skipping over 2=D, 3=E and 4=F) G. He found this note very pleasing or harmonizing to the first note, and so he simply based the other notes and string lengths on this value of (2/3) of the string length, and in the hope of finding the correct values of some other pleasing or harmonizing sounds to the first note or sound of the scale. He figured if (2/3) of the string length gave a pleasing and harmonizing note, then considering this string length as a new string length, then at (2/3) of it, there should be another pleasing note. When he went past the first octave positions with his idea, he shifted the result into just the first octave by doubling its string length, hence an octave less, and the result was the 5th tone or note of his G scale, and which is the D note, and this is also the 2nd note in the original scale of C. The next note he considered was the 5th note in the scale of D, and this is the A note.

The 5th notes of any octave and-or scale were found in this order: C , G , D , A , E , B , F , C , and which then also gave us the 8 notes and locations of a complete octave range of a scale such as: C , D , E , F, G , A , B , C

If we have (2/3) of a string length considered as G, and then take (2/3) of it, we then have a string length of just (2/3)(2/3) = 4/9 of the original string length. 4/9 of a string is 0.444... , and if we double it to get an octave lower, we will have **0.8888....** of that string length, and this is where the D or second note of the C scale would be. This will be less than the full (100% = 1) string length by the relative (ie., percentage, fractional) value of : (1 - 0.888888888) = 0.111111111 = =~ 11.11% The equal ratio chromatic music scale has a value for the second tone of being at **0.890898718 L** , and which is close in is very close in value.

Where are the octaves or twice the frequency located on a string?

This is probably where Pythagoras started his musical scale.

Octave	Location Along A Length (L) Of Vibrating String Producing Sound Vibrations
0	1 / 1 = 1L = 100%L
1	1 / 2 = 0.5L = 50%L : after taking half of the above length
2	1 / 4 = 1 / (2^2) = 0.25L = 25%L : after taking half of the above length
3	1 / 8 = 1 / (2^3) = 0.125 = 0.125%L : after taking half of the above length

We see that these are fractions composed of a simple integer in both the numerator and denominator. For any string length (L) ,the octave is always located at half its length. From the above pattern, we see that the denominator is equal to octave (n):

$$Ln = \frac{L}{2^n}$$

: the higher the octave, tones and-or frequencies, the shorter the string length and string length corresponds to the wavelength of its fundamental or natural frequency.

Pythagoras surely then reasoned that given any string or "scale length", then the octaves of any similar, but different "pitched" note, for example note A , will always be located at the same relative distance along its length from the start of the entire octave (ie., the 8 accepted, common or main tones of a musical scale, and before the concepts of the 12 note chromatic scale was used so as to create a scale of tones when any note is considered as the starting note). An example of the 8 common notes of a complete octave scale are: C=1, D=2, E=3, F=4 , G=5, A=6, B=7 , C=8

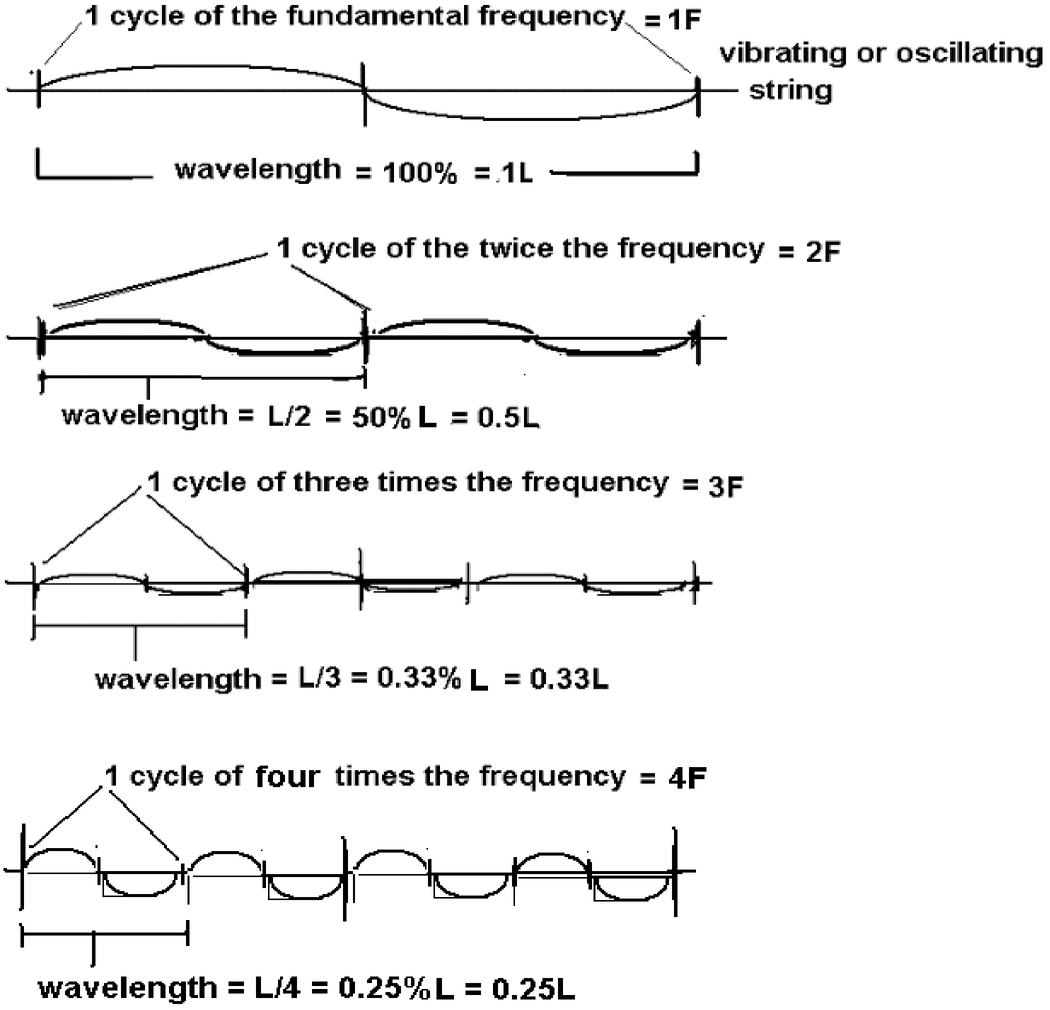
Pythagoras reasoned that if the octave note, a pleasing sounding note, is located at 1/2 of the string length or 1/2 up from the first note, then other pleasing notes should be located at 2/3L or 1/3 up from the first note , 3/4L or 1/4 up from the first note, and 4/5L or 1/5 of up from the first note. These are series of whole number fractions: 1/2 , 1/3 , 1/4 , 1/5

Ex. A musical string is 25.5 in long. The first octave higher in tone from the end of that open string is at 25.5 / 2 = 12.75 in = 12 in + (12/16)in . This considers notes 1 through 8 of a complete octave scale. Halfway on this length segment of the string, there is the 8th note / 2 = 4th note, and this will be located at: 12.75 in / 2 = 6.375 in = 6in + (6/16) in. If the string was tuned to a C note, the 4th note in that scale would be F. Considering this position as the start of a new length of a string, it would be a string length of: (25.5 - 6.375) = 19.125, and at half of this would be the 4th note in the F scale, hence note B. Pythagoras is said to have used the 5th note of every scale for his notes and positions analysis.

Each length of string will have a natural (resonant) frequency associated with it. When it is plucked to vibrate, it will develop peaks and valleys or highs and lows in its vibration or wave shape. The places where the vibration is at minimum, null or 0 is called a node. The first node will be in the middle of string, and then another node between that node and the end of the string, and so on. The amplitude or energy and-or height of each successive vibration between each node will get reduced as the number of nodes and frequencies (called [natural] harmonic frequencies) of vibration increases. For the first node and each successive node, these corresponding and smaller in amplitude vibrations are one octave higher in pitch and-or having a frequency corresponding to the previous node frequency and increased by the value of the fundamental frequency of the string.

If you were to shorten the length of the string to its half-length value at the node, the fundamental frequency of that length of string will be twice in frequency.

[FIG 285]



Note in the above figure, that at half the string length is 2F, and this is where the octave or twice the frequency of the entire string and-or first (or "root") note is located. Now note that 3F does not mean the frequency of the 1 cycle of the three total waves or cycles indicated would be an octave of that entire string and-or note. If that length of string was played (ie., vibrated, sounded) by itself, it would not be an octave note of the original string and its note (ie.,pitch, frequency). Only where the string length is successively halved in length, at that length will be the next octave note and location. For 4F, the length of the 1 cycle shown would actually be the next octave if it is the only segment or portion of the (or a) vibrating string. 3F is considered an "odd harmonic" in relation to the first or fundamental frequency. Likewise 5F, 7F, etc, are also odd harmonics of the fundamental or main frequency. Odd harmonics can give a "richness" or "brilliance" to chords, and in fact the main notes of any chord are the 1st, 3rd, and 5th notes of a scale of a fundamental frequency being considered. 2F if both twice the fundamental string frequency and is the length of the next octave of it. 4F is four times the fundamental string frequency and the length of that once cycle would be the length of the next (higher in pitch) octave or four time the fundamental frequency of the entire string.

Pure and-or theoretical sine waves do not contain any harmonic frequencies, that is, those waves are not composed of, and not combined with, and-or do not contain any other frequencies. When energy waves such as audio or sound waves contain other frequencies, the sound is heard and is said to have a certain "character", "quality", "color", "uniqueness" or "**timbre**", and of which may be subjective to a listener, but will have a "common sound" to most people. A music tone can have a frequency (ie., pitch), and amplitude (ie., loudness, energy intensity), and timbre (ie., tone or color). Tuning-forks can produce pure sine waves, that is, having 1 fundamental output wave. A flute can sometimes produce a pure sine wave. It is possible to create pure sine waves using an electronic circuit such as an LC oscillator circuit. Other than what mentioned, most instruments do not commonly produce pure sine waves, and not even many bells, and of which are often made or designed so as to have a somewhat strange, yet pleasing mix of output frequencies so as to have a unique sound or tone. The sound of a rung or struck bell can still be heard in some quite, low noise communities.

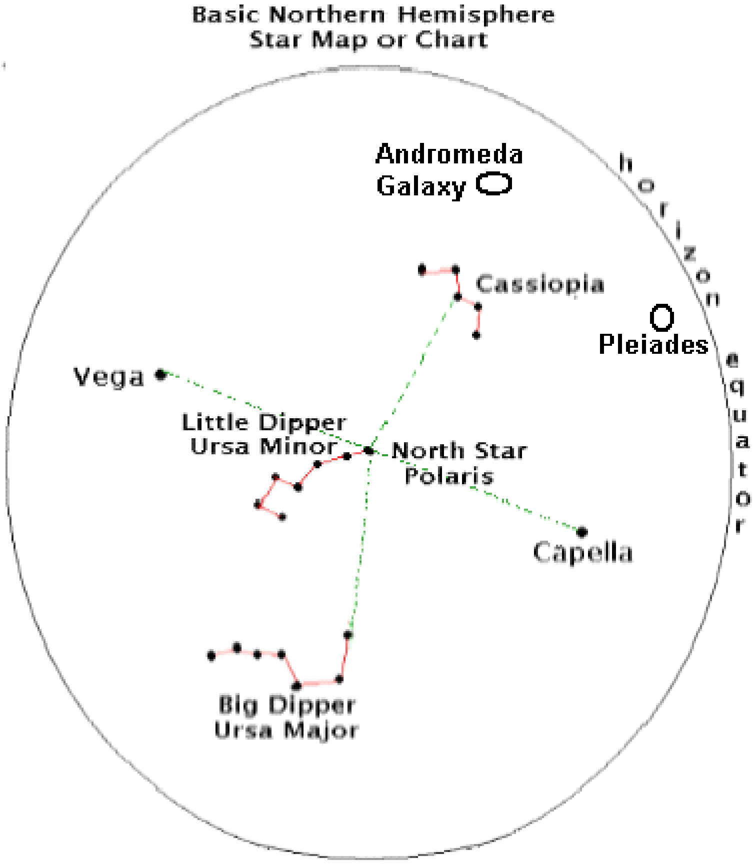
A BASIC STAR MAP OF THE NORTHERN CELESTIAL HEMISPHERE

This is a star chart or map of a few of the most helpful, significant and-or brightest stars in the night sky. Planets and the Moon are not shown because they move from their locations in the sky fairly quickly throughout night and year as compared to the (apparently) constant or fixed relative star positions. Note that astronomical things appear "upside down" in the southern hemisphere than that of the northern hemisphere view. For example, if you live in the northern hemisphere and notice a star on the southern horizon, that same star will appear on the northern horizon as viewed from the southern hemisphere. Of particular importance for people in the northern hemisphere to know is the location of the pole-star called the North Star or Polaris. Using this, you can deduce the directions of north, south, east and west on the Earth. The axis of the Earth constantly, throughout the year, points directly to the North Star, and it would be located directly overhead of a person at the true, geographical north of the Earth. People living in the southern hemisphere can find the true south direction by first locating the stars called the Southern Cross or Crux. Star charts are available, and a popular, portable, and casual version is called a planisphere or "star wheel", and this takes into account Earth's yearly orbit about the Sun, and the apparent tilt of the Earth throughout the year. This circular map system with a "viewing window" of the current visible portion of the sky is rotated to align it with your directional view.

Viewed from Earth's north pole, Polaris will appear about 0° offset from being directly overhead at 90° with respect to the local horizon line. The latitude of the north pole is defined as 90° from the equator line. At the equator region, Polaris will appear along the horizon, hence it will appear at 0° above the horizon. 0° is the defined latitude of the equator. People living southward of the equator may not be able to see the North star, Polaris, except for when the apparent tilt of the Earth takes place and it allows them to see it from locations as far south as about 23.5° latitude during certain times of the year.

Polaris can be found in the night sky by facing in the direction of the north pole (and generally, not the magnetic north), and Polaris will be found at an angle upward in the sky that is equal to your defined, local (north, south) **latitude** on Earth, and with respect to your local horizon line, With an angle measuring instrument, you can also measure this angle if you know where Polaris is located. If you want to calculate your local (east, west) **longitude**, there is a mention of it in this book in the article called: **How to find your local longitude**

A wonderful thing a new amateur astronomer may wish to do, is to first create their own star chart of the brightest stars visible to their own eyes. The star positions can simply be approximated when drawn, or actually measured with an (horizontal and vertical) angle measuring device. After that step, they can look up on a star map what the star names are. The map below shows a few constellations (a star grouping) and some bright stars. [FIG 286]



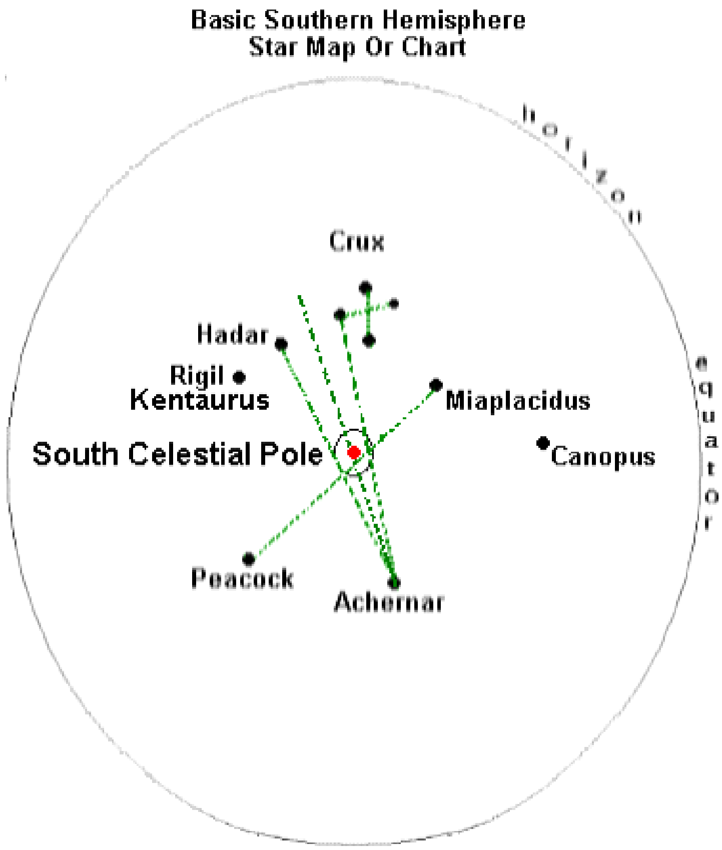
Vega and Capella are indicated due to their positions and easy to find brightness. These two stars are nearly on a line with each other which includes the **North Star** called **Polaris**. In brief, there are several celestial or astronomical coordinate or location systems in use for one reason or another, and all use a reference location and or plane. The distance to each star or planet is generally not used or needed in these basic location systems where direction is more important - such as where to aim a telescope to view a particular star, planet or galaxy. Considered in these systems is the solar or ecliptic plane of which the planets and their orbit travel are relatively near to within a few degrees. The North star to the South star (ie., Earth's polar axis) plane is used in some of the systems. One common term often used in these concepts is "azimuth", and this is simply an angle value such as to a star.

Since astronomical observers may be at different latitudes on the Earth, the apparent angle to a particular star will not be the same as that of another observer, and hence why a common (0°) reference plane(s) must be used so that all may properly locate an object. For example, the Sun may appear as directly overhead to a person near the Equator, but it will appear as say 45° above the horizon (at local noon, 12pm time) to an observer located halfway to the north pole between the equator and the north pole, and while both are located on the same longitudinal line.

A simple astronomical location system for people in the same geographical area, within a few hundred miles of each other, would be the angle rotated clockwise from the north-star's horizontal direction or plane, and the angle to the object above the local horizon. Even so, the planets and stars appear to change celestial position by 15 degrees per hour, and therefore, the same observation time should also be considered for these coordinates or position. Because of the Moons relatively slow orbit about the Earth, the stars will appear to travel across the sky slightly faster than the Moon, and the stars appear near its widest part will appear go behind the Moon and will appear to "come out" at its opposite side in about 20 minutes, hence appearing to travel faster than the travel speed of the moon.

The visible angular width of **Cassiopeia** constellation that seen as the "W shape" in the sky is nearly 30° wide in terms of the North Pole (circumpolar) celestial coordinates (ie., angles) and-or along the celestial horizon.. Vega is about 25 light-years from Earth. Capella is about 43 light-years from Earth.

[FIG 287] **A Basic Star Map Of The Southern Celestial Hemisphere**



The star constellation called **Crux** is also commonly known as the **Southern Cross**. With binoculars (ie., two small telescopes connected side by side, one for each eye) or a monocular (ie., a small, portable single lens telescope), you can see the south pole star called **Sigma Octantis** that is very close to the geographical location of the south celestial pole.

People who live in the high latitude areas, near a pole or axis of the Earth, say greater than 47° latitude, will not always be able to see the stars around the other pole region due to the bulge of the Earth at its equator blocking their view of those stars. First consider that if the Earth was not tilted at all on its axis. Anyone living south of the equator would not be able to see the North pole star region, and anyone living north of the equator would not be able to see the south pole star region. The further from the equator, the less of those stars indicated will be visible. Due to that the Earth's axis is tilted by about 23.5°, then at certain times throughout the year, more of those stars indicated will be visible. Throughout the year, the maximum apparent change in Earths apparent tilt is 47°. It is therefore possible for people living at latitudes as high as 47° to see the pole star region of the opposite hemisphere of the Earth. At any one time of day throughout the year, only 180° of the sky and-or stars is visible from a location on Earth, and "from horizon to horizon". As the Earth turns 180° on its axis for 12 hours during the night, then in theory, a total of 360° of the sky and-or the overhead stars will be visible at some particular time during one day, but not all at once, and rather just 180° worth at a time. At anyone instant of time, Earth's bulge always blocks the view of 180° of all the available stars in the universe since half of them are effectively located below your local horizon line and can not be seen.

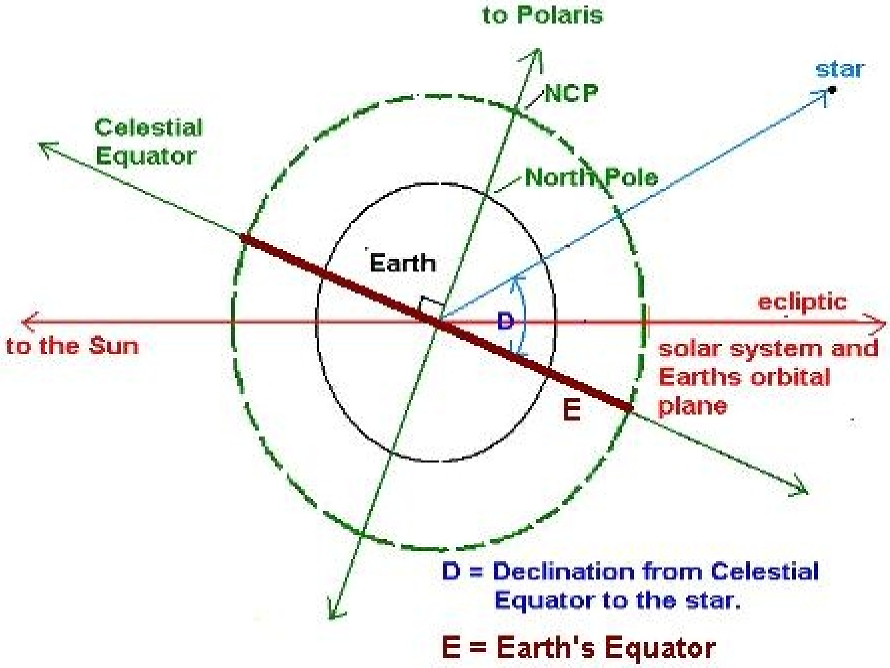
The star in the image above called **Rigil Kentaurus is also commonly known as Alpha-Centauri or A-Centauri** since it is in the Centauri star constellation or formation along with the other stars such as B-Centauri and C-Centauri. This Centauri star system and-or constellation has the closest stars to Earth and are about 4.4 Light-Years away or distant. **Hadar is also called Beta-Centauri or B-Centauri** and is actually a three star system. The star **C-Centauri** is commonly called **Proxima Centauri** which is a relatively small red-dwarf star, southward and slightly eastward of Alpha-Centauri, and here where the (s) letter is located in the word Kentaurus as shown in the above firgure, and it was found using a telescope in 1915, and is the closest star known to our solar system, and is "only" about 4.25 light-years away - a relatively very small distance as comparred to the other stars in the galaxy and universe. In the above figure, Proxima Centauri is located about where the letter (s) is located in the word Kentaurus as shown in the above figure. As of the year 2022, three exoplanets (external planets out of our solar system, and are orbiting another star) have been found to orbit Proxima Centauri, hence they compose the closest exo-planetary solar system. Due to the relative closeness of stars in our Milky-Way galaxy, all of the exoplanets found have been in it. In the above image. Proxima Centauri is located just south-east of Alpha-Centauri, and at about the 5 o'clock position relative to it. By using the apparent visual effect called parallax. and taking two observations of Proxima Centauri at 6 months apart and a distance of (2)(distance to the Sun) = (2)(93 million miles) = 186 million miles as Earth orbits about the Sun, its distance can be found using right-triangle trigonometry to a relatively high accuracy since it is relatively close to us as compared to all the other stars, and where the parallax effect is more apparent. Throughout the year, at most latitude locations in the northern hemisphere of Earth, Proxima Centauri cannot be seen and is effectively below the horizon.

If you are able, a neighbor or friend can show you their telescope for a short amount of time so as you can view the night sky or local area. - of course try not to focus it on the neighbors, etc. for it can cause privacy issues. **John Dobson** (1915 - 2014), from the USA, is generally given credit and name for the Dobsonian Telescope designs and mount. He often encouraged people with telescopes to let other people without telescopes to look through their telescopes so that all may have at least a chance and experience of viewing some of the amazing universe surrounding them in a new and helpful way. It is recommended that a beginner purchase a low power, relatively inexpensive refactor telescope, binoculars or monocular for viewing the local area, and so as to become familiar with a telescope and the night sky. The magnification power of many binoculars is about 20, and for an inexpensive telescope, the magnification power is about 100. As the power and objective (lens or mirror) increases, the cost of the total telescope increases rapidly, and so does the weight of the telescope system (telescope, tripod, mount to connect the telescope to the tripod, etc). It may take a year or more of occasional viewing for a beginner to become familiarized with a telescope and astronomical things, and then the process will be easier if they obtain another telescope for more advanced viewing. Dobson was also a well known (Newtonian) telescope designer and (reflector telescope) mirror maker who educated many others as how to do these things if the are able. The "Dobsonian Mount" is essentially a homemade, inexpensive (usually wooden) Alt-Azimuth (altaz) telescope mount, and this design was practical for very large diameter telescopes to have a relatively inexpensive mount, hence it also helped encourage building telescopes having a large objective mirror. A Dobsonian telescope mount basically has a (hand rotatable) "turntable" or disk mount so it can rotate 360° about its center point when placed level with the horizontal ground, and a pivot mechanism placed onto that turntable so as to adjust and-or aim the telescope vertically above the horizon. Although Dobson was the first to make this mount very practical for telescopes, he mentioned that the basic mechanical and-or design concept was not new at all.

About The Celestial Coordinate System

One way to describe the location of a star in the sky is by using two values: a **Declination** (DEC) value, and **Right Ascension** (RA) value. Right Ascension essentially means how far right and-or eastward will the star will then be located via its declination value. The Earth travels rightward and-or eastward about the Sun each year. Viewed form above the Sun's north pole, the Earth travels in a counter clockwise direction about the Sun.

The north star, and which is directly above and-or inline with the north pole and axis of the Earth, is named as Polaris, and it is also considered as the north celestial pole of the celestial coordinate system. At 90° to the celestial pole is the celestial equator or plane, and is defined or standardized angle of 0° declination. The celestial equator plane is also co-planar and passes directly on and through Earths geographic equator plane that is centered between the north and south poles. The solar plane and-or ecliptic (path of the Sun, and planets in general, across the sky) of the Earth is actually tilted about 24.5 degrees from this celestial plane. Declination is the angle to an object and needs to calculated with respect to the latitude of the observation so as to be in reference to the celestial equator instead of the observers specific latitude. Various sightings of the object, say a star, and seen at different latitudes will all observe it at a different angle above the horizon and the solar plane, but using the celestial declination value will be the same value for all latitudes. Declination then essentially becomes the celestial latitude value from the celestial plane. Declination is usually expressed as a degrees angle or as: Degrees, ArcMinutes, ArcSeconds. [FIG 287A]



The declination of the star Polaris is about 90°. The latitude of the north pole on Earth is defined as 90°. The RA of Polaris is abut 2h , 31m , 49s. Since each hour = 60 min = 3600s corresponds to an angle of 15°, this angle and-or time value corresponds to an angle of about: 30.6833°

The astronomical first day of spring is called the spring or vernal equinox, and it occurs on about March 21 each year, give or take a day. This can be used as a reference position of the visible stars (most being in our Milky-Way galaxy) as viewed from Earth during its yearly orbit about the Sun. On this day, the 0° reference longitude will be in the constellation of Pisces. This reference position in our orbit is called the Right Ascension (RA) value of the celestial coordinates or the "imaginary sphere of celestial coordinates". A Right Ascension value is essentially the celestial longitude value from the reference value of the spring equinox which is considered as 0° Right Ascension of Earth orbiting about the Sun. Each day, the Earth will travel about: (degrees of circular orbit / days of orbit) =~ (360 ° / 365.25) =~ $0.9856\underline{3}°$ =~ 1° / day around the Sun, and also through the 12 months and-or zodiac of star constellations at night time. This orbit of the Earth about the Sun will make the stars shift by nearly 1° each night at the same time of day than the day before. Here, 1° corresponds to 4 min of time earlier. Since the Earth rotates daily on its axis, the stars will constantly appear to also move throughout the night and day, but this RA value is a standardized and-or a calculated and-or constant value for all latitudes and-or locations of Earth. The RA value is usually expressed in Hours:Minutes:Seconds units.

15° of the celestial coordinates and-or apparent star movement corresponds to 1 hour = 60 minutes of time. (1 min. = (1/60) 1 hour , and sec. = (1/60) 1 min. = (1/60) (1/60) hour = (1 / 3600) hour.

1 hour = 60 min = 3600s = 15° , and 15° / 1 hr = 15° / 60 min = 1° / 4 min = 0.5° / 2 min
0.5° = (1/2)° is the apparent diameter of the moon in the 360° complete sky
1 min = (1/60) hour = (1/60) 15 ° = 0.25° : 1 min = 60 sec

1 sec = (1/3600) hour = (1/3600) 15 ° = $0.00416\underline{7}$ °

Since the celestial equator is perpendicular to the celestial pole and axis of the Earth, the celestial equator will be in the true or "due" east and west directions to the observer. Since Polaris essentially remains in he same direction and location (ie,, angle of elevation) in the sky for a given latitude, the stars near the ecliptic plane (solar plane) will appear to travel in an east to west direction parallel to the ecliptic line, and throughout the night and year, the ("circumpolar") stars will also appear to slowly orbit about the north star which is also called Polaris, and which is said as being very close to the true north polar axis of the Earth and-or the true north celestial pole (NCP).

Due to the celestial sphere concept, the RA lines will travel from the celestial equator and meet together at the celestial poles, and the right ascension value can then also be thought of as hour and-or (24 , hour) clock lines about the star Polaris, and where each hour line represents 1 hour of time and-or 15° of location and-or offset from the 0° reference line. Where is the 0° RA reference line defined and-or standardized? It is a line from Polaris and passes near (just right of) to the star **Caph** which is the rightmost star in the constellation **Cassiopeia** which resembles a W shape. The RA or Right Ascension of Caph is not actually at 0°, but nearly, and is rather 9 minutes and about 10.7 seconds. The declination of Caph from the celestial equator is nearly 59° and 9 minutes. Note that an azimuth (ie., along the horizon from true north) angle value depends upon the local location of the observer on Earth, and therefore it is not a constant value.

The constellation called Orion which is easily recognized by the 3 bright stars in a row called **Orion's Belt** is located near to the celestial equator. Note that although the apparent tilt of the Earth changes with respect to the solar plane and Sun throughout the year, the celestial equator does not move in position, and this is similar to how Polaris is (very nearly) the celestial pole throughout the year. The upper-rightmost star called **Mintaka** in Orion's Belt is nearly, directly at the celestial equator 0° reference line At the equator of Earth the celestial equator will appear directly overhead, and this imaginary line is 90° perpendicular to the celestial poles. Mintaka will rise due (ie., directly) east above the horizon and will set due west after 12 hours of time, and it will be directly overhead after 6 hours of rising above the eastern horizon as Earth rotates toward the eastern direction during the night. The celestial coordinates of Mintaka are: (RA = 5h, 32m, 0s and Dec = - 0°, 17 min , 57s = -0°, 17', 57"). Looking rightward from center of the imaginary line of stars in Orion's Belt is the bright star Rigel, and looking leftward from Orion's Belt is the red-giant star called Betelgeuse.

The celestial coordinate system is also called the celestial equatorial coordinate system.

A somewhat more simple coordinate system for perhaps people in the same region on Earth having about the same latitude and longitude is the ALT-AZ , **Altitude and Azimuth coordinate system**. It is also called the horizontal coordinate system since the angles are in reference to horizon directions. Here, altitude is the angle ("elevation angle") from the horizon to the star in the sky. The azimuth value is the angle along the horizon between the true north direction (considered as the 0° reference angle) and the direction of the star. True east would be a 90° angle. True south would be an angle of 180°. True west would be a 270° angle. Since the stars will appear to move across the sky in the night, their ALT and AZ coordinates will be changing constantly, and therefore, it would be a good idea for future reference to also record the date and time of the observation with those coordinates: ALT : AZ : DATE : TIME

For more information about celestial coordinates and-or astronomy, it can be researched elsewhere or learned via an astronomy course, and which may include a visit to a local planetarium which simulates celestial objects, their locations and-or paths via a light projection and display onto a curved ceiling.

Today, with high powered, legal lasers for astronomy use, stars in the sky can be indicated using a (usually green light) laser. It is very possible to make a portable "pointing stick" on a tripod stand. This pointer can be used to indicate objects on land or in the sky. It should be able to move horizontally and vertically and the be set firmly into the desired position. Very small and dim red lights can be used to help align the pointer at night. This pointer system can then even be improved so as to indicate the horizontal and vertical angles of the pointer at your local location and to the objects or stars, and it would then be very possible to draw a more detailed star map(s) and-or to locate stars, especially if the pointer's 0° line is aligned to the north pole direction (horizontal and-or vertical), and either the celestial or alt-az coordinates can be recorded and used to indicate the stars on a paper map being drawn.

Types of telescope (tripod, etc) mounts:

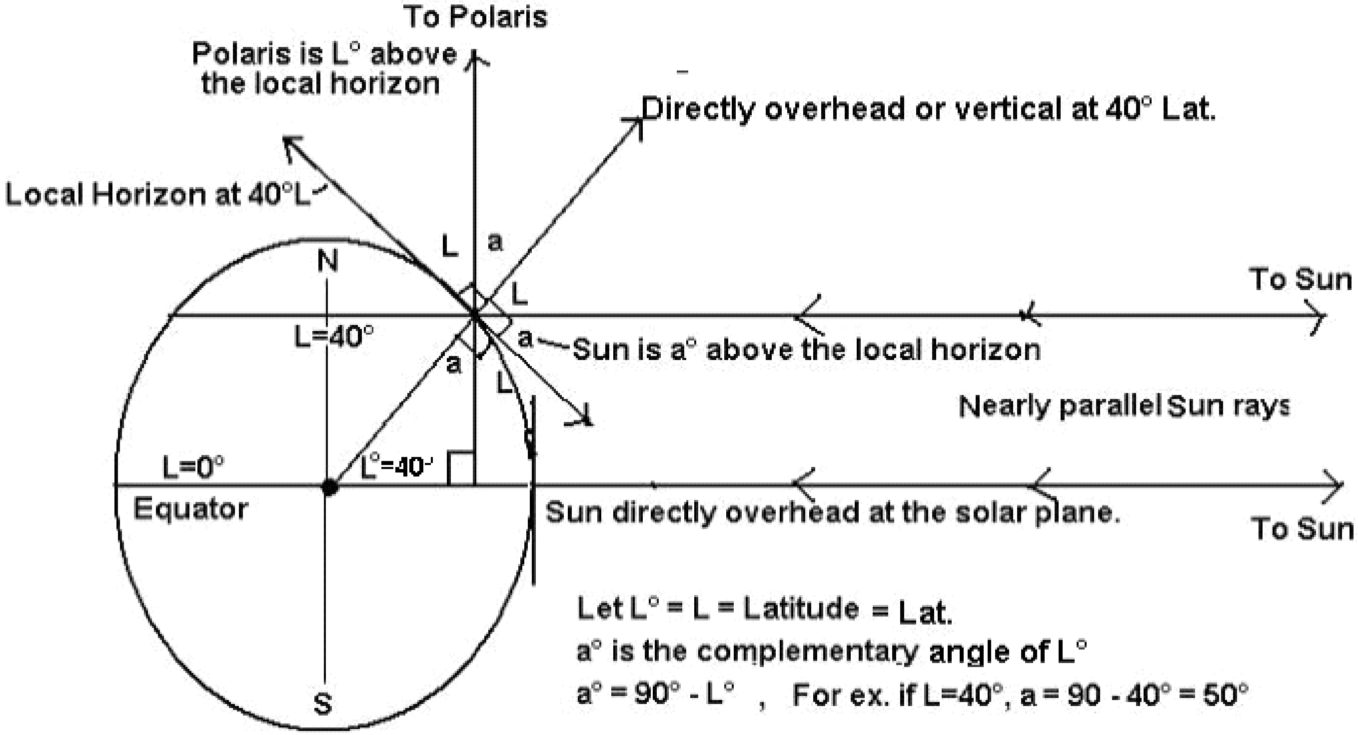
1. **Manual** or "free hand" - Aim the telescope at what you want as needed. Many Dobsonian telescopes and inexpensive are manually aimed. This makes the cost of the system cheaper, and or easier to use for some people just "looking around". Even if a telescope has another type of mount, manual mode is often possible.

2. **Altaz** mount - or Alt-Az mount. This mount can rotate horizontally (azimuth) and-or vertically (altitude above the horizon).

3. **Equatorial** mount - This mount is aligned to be in parallel to Earth's polar axis, hence very close to the star Polaris. As the Earth rotates, the telescope aiming will be adjusted to rotate about this correctly aimed pivot point on the telescope. This mount, and if motorized tracking is used, allows for better long exposure photography.

ANGLES TO POLARIS AND THE SUN AT YOUR LOCAL HORIZON AND LATITUDE

In the figure below, it verifies the angles to both the North-star Polaris and the Sun at your local latitude and-or location on Earth. [FIG 288]



If you live in the Northern hemisphere of Earth, and at 23.5° Latitude, North of the Equator, the Sun will always appear to be slightly southward in the sky throughout the year, and more-so as the more North you are and-or the higher the Latitude on Earth that you are. At the north pole or axis of the Earth, the Sun is very close to the horizon, a very low angle, and sometimes below the horizon, such as during the winter season. Because the Earth turns westward to eastward throughout the day, the Sun and-or stars will appear to travel eastward to westward. The angle to the star Polaris will always be equal to the observers latitude. The apparent tilt of the Earth throughout the year does not change this, however it will change the observed angle to the Sun since it is relatively close to Earth as compared to the star Polaris. The Sun will appear in the Sky at a higher maximum angle (ie., more directly overhead) with respect to the horizon in the summertime of the northern hemisphere, and when the apparent tilt of the Earth is towards the Sun.

The angles in the above figure can be found using the concept of the two complementary angles of a right triangle that always sum to 90°, and the concept of vertical angles across from each other which always have the same angle.

Extra: The axis of the Earth does wobble in a circular type of motion, but it is very slow to consider into most calculations for a few hundred years. The effect is that the true pole or axis direction as of now being at the star Polaris will slightly change its direction and apparent location among the visible stars sky by a few degrees maximum. This concept is known as the **precession** of Earth's axis. Precession can easily be seen using a commonly available rotating gyroscope. Note that the tilt of the Earths and-or its axis is still about 23.5° during this precession or wobble cycle, but the axis line will rather move in a circular direction as it wobbles. The period of Earth's total (ie. full cycle, and-or circle) axial precession has been estimated at about 26000 years on average. Most scientists agree that this precession is the main cause of "ice ages" where the polar caps of the north and south pole grow to be much larger in size due to snow and ice, and thereby increasing Earth's solar reflectively (ie. "albedo") and causing the planet to cool even further. Earth's slow wobble is caused by some angular momentum developed due to the occasional Moon's gravity between that of the Earth and Sun.

[This space for edits.]

How To Calculate Your Local, True Or Solar Noon Time If you Know Your Longitude

At solar noon, the Sun is directly in line with your latitude, and it will be at the highest or greatest angle in the sky for that particular day. Every location that has the same longitude on Earth will have the same, true or actual local time. First, find out what is your (east, west)longitude. You may also wish to know your (north, south) latitude.

We usually set out clocks not to our local solar noon, but by the time zone. All the different longitude locations in a time zone will have the same clock time, and this helps people stay on schedule and or in reference with each other and events. Only where the hourly time zone starts will the Sun be directly overhead at 12:00pm. If you live to the west in the time zone your true local (at your latitude) solar noon could be several more minutes later. The Earth rotates at 15° per hour, and from this we have that the Earth rotates 0.25° per minute, or $0.00466\underline{7}$° per second.
.
Here is an example.

If your longitude is 75.842°W, what is your true, local solar noon time on your ("time zone" or standard) clock?

Since each timezone is 15° apart, your time zone is: 75.842° / 15° = 5.0561333... , and the whole value of 5 indicates that you are in timezone 5 west of the Greenwich time zone1, and your clock time is set to be 5 hours earlier than the Greenwich clock ("timezone") time. To be more specific, you are 5.0561333... hours earlier than Greenwich. Since this amount of hours has a fractional value, you do not live on the longitude line at the start of that time zone, but are actually 0.0561333...hours offset from this, and your local, true solar noon will be 0.0561333 hours later than the standardized clock ("timezone") noon. Let us now convert this amount of hours to its equivalent amount of minutes. We can use a proportion or equivalent fraction type of equation to do this:

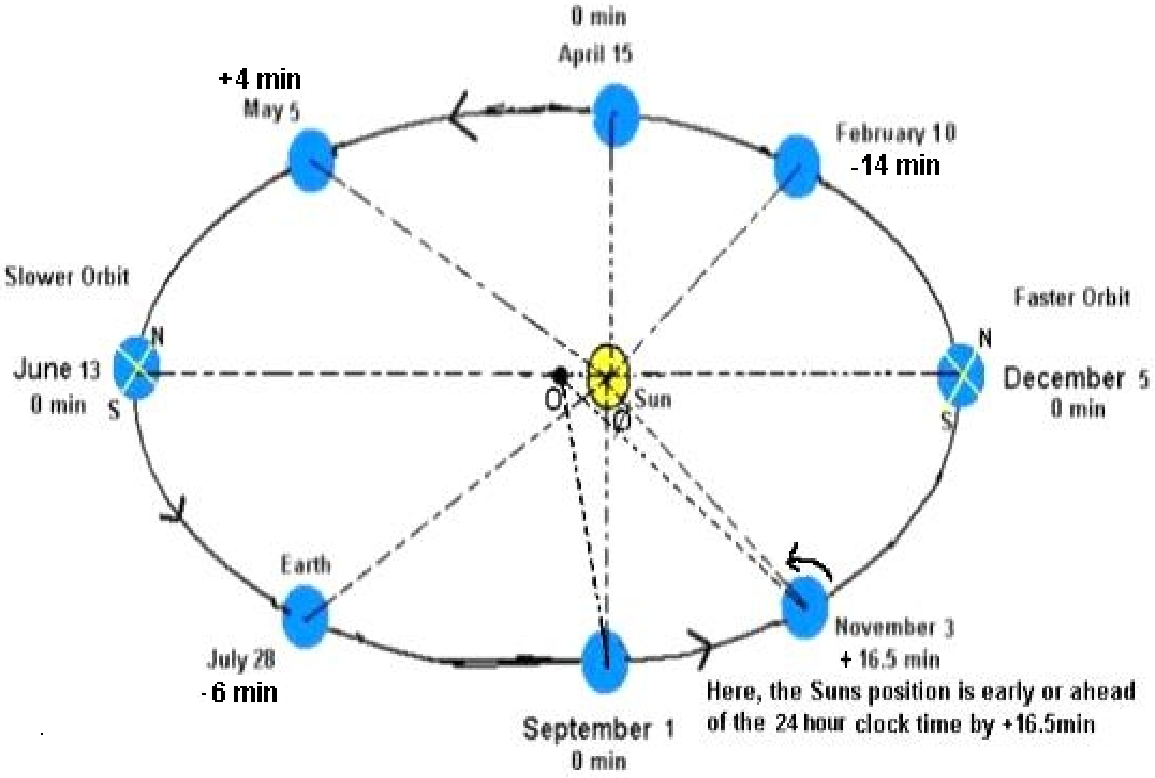
$$\frac{1\text{ hr}}{60\text{min}} = \frac{0.0561333\text{ hr}}{x\text{ min}}$$, after solving for x min, we have:

$x\text{ min} \overset{\sim}{=} 3.368\text{ min}$: your local, true solar noon will be 3.368min past or later than clock or "timezone" noon.

The above is a theorized value, but since the orbit of Earth about the Sun is elliptical, true solar noon may not be this calculated value, but needs to be adjusted. The topic of the **Equation of Time** considers this adjustment, and it and its value it may be thought of as: The Adjustment, Difference or Offset Of True or Actual Solar Noon Time From The 24 Hour, Standard Clock Or Axis Rotation Time. Note that for any day of the year, and regardless of Earths (slightly) elliptical orbit, the Earth always rotates at the same angular velocity on its axis, and that is 1 rotation per day = 360°/24hrs. When Earth is closer to the Sun in the winter time, the Sun will rise later than when the Earth is more distant in its orbit during the summer. Clock time that we commonly use throughout the year is based on the constant 24 hours in day, and does not consider the "equation of time" or changes in our orbit and the current (angle) position of the Sun with respect to Earth during its elliptical (non-circular) orbit during a year. Since the "equation of time (adjustment)" is not considered on our ("time zone") clocks, it is therefore the equivalent time as if the Earth had a pure circular, non-elliptical, orbit about the Sun. There are some special mechanical clocks made which do consider the equation of time, and those clocks are called "Equation Clocks". For these clocks, the adjustment or time offset is displayed on a separate dial, and it can be combined to the displayed time. If the clock was set to true local or solar noon, the adjustment is simply added, otherwise if the clock was set to "time zone noon", then you need to first calculate the true solar local noon from your known longitude as shown in this article above, and then to also combine it with the equation of time adjustment.

The zero time offset days are separated by 90 days and they occur at roughly the time of the start of each season: March 20, June 20, Sept 20, and Dec 20. The equation of time is sinusoidal-like, but having two waves, each with its own amplitude (peak value), and one wave then following the other. Each of the two time offset waves have a 180 days period or time length. There is a spring and summer wave, and a fall and winter wave. Between the zero dates, at about every 45 days, are the maximum offset or equation of time adjustment values, and these peaks are therefore 90 days apart.

Due to Earth's elliptical orbit about the Sun, as the Earth gets closer to the Sun due to its increased gravitational force and resulting acceleration caused by the Sun, the speed of the orbit of the Earth about the Sun increases in the winter months, and the winter season and its corresponding (apparent) tilt of the Earth will actually last less than 90 clock or (yearly averaged) days, and the summer season and its corresponding tilt will be a few more days longer than 90 clock days since the (average per year) speed and orbit of the Earth about the Sun is less. [FIG 289]


APPROXIMATE ORBIT OF EARTH AND EQUATION OF TIME ADJUSTMENT

As can be seen above, the "zeros" or where the clock time offset is 0, and-or differs by the true solar time by 0, occur relatively close to the start of each season on the calendar. Note that an offset of 15 minutes is a quarter of 1 hour, and this will be easily noticeable on a sundial. Using the equation of time concept, you can combine it with the sundial time so as to have the equivalent standard clock time.

This is a simplification of understanding why there is a time offset:

The above figure is an above or overhead view of Earth's orbit about the Sun. Earth's orbit about the Sun is not exactly circular, but elliptic. Due to this, the center (O, for origin or center) of orbit and-or the ellipse is not at the (gravitational) focus of the ellipse. The focus of Earth's orbit is the Sun, but it is not located at the geographical center of the ellipse, and the only exception is for a circle. If the Earth's orbit was not elliptical, but of a pure circular orbit, this offset angle would not happen, and there would be 0 time offset. Observe the two different angles in the above image, and the angle difference indicates why there is a time offset and-or adjustment needed in relationship to the Sun, but for standard ("time zone", 24 hour) clocks there is no offset needed because the Earth was standardized (ie., constant) at revolving on its own (ie., not solar, Sun orbit) axis for one day = 24 hours, and that this value does not change regardless of the season or orbit distance from the Sun.

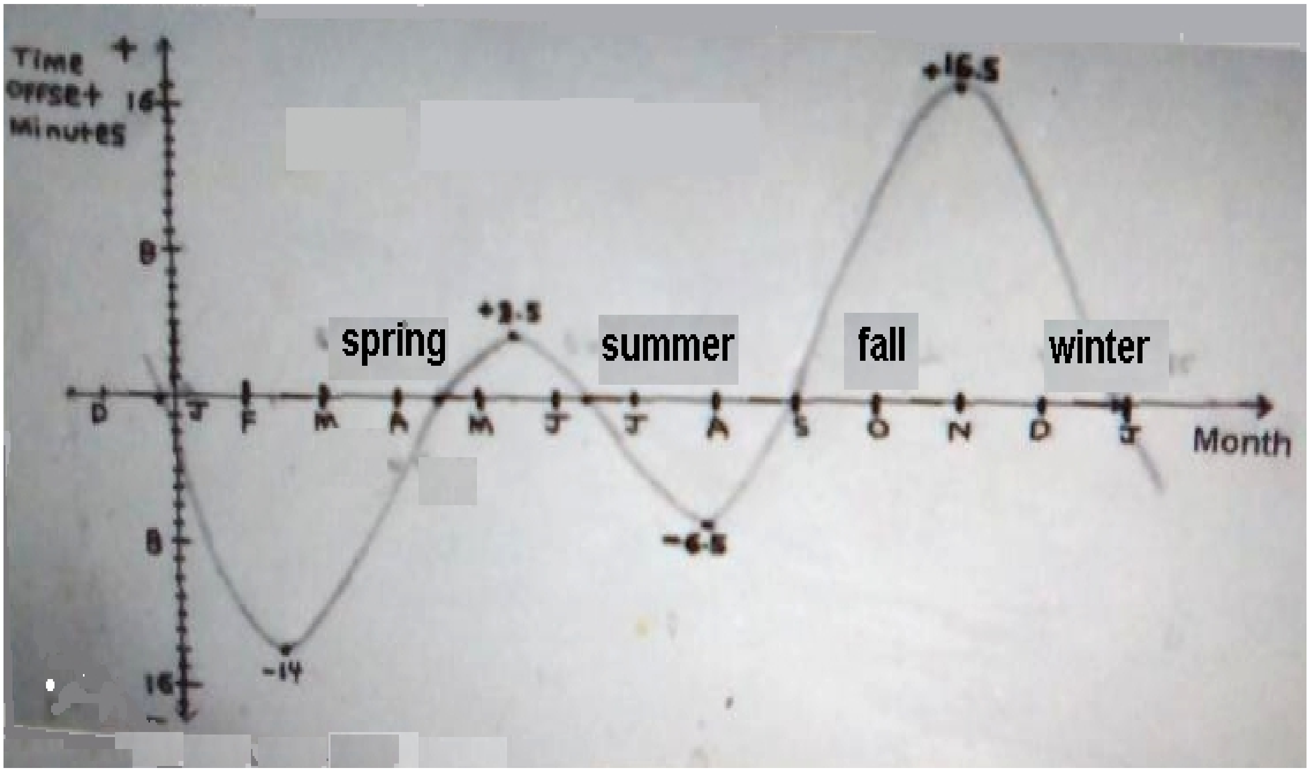
The topic of the equation of time has been studied for years by astronomers and much data about it has been measured and recorded. The equation of time can get somewhat complicated and has not much practical use for the average person, but it is sometimes important to some astronomers, precision sundials (ie., sun clocks) makers, and navigators. If your region uses the concept of Daylight Savings Time, you may also need to adjust your calculated and expected, true solar noon time value by an entire one hour more or less.

The offset times or difference times between the ("time zone", 24h) clock and the solar or Sun time can be measured using a sundial or a vertical pole in the ground. Observe a clock when it indicates noon or 12pm, and also observe the shadow on the sundial when it is indicated as your local (longitudinal) solar noon. For the fundamental equation of time and-or the difference than the expected value on the standard (24 hour) clock), calculate the time difference between solar noon and clock noon. Clock noon is also called "(24 hour) standard time" and sometimes as the local "(24 hour clock) averaged solar time". As for a vertical pole in the ground, its shadow will be the shortest length at solar noon, and where the Sun is at its highest angle in the sky for that day.

Key Data Points For Earth's Equation Of Time :(Offset value or time between solar or true noon and clock time noon)

Day	**Time Offset In Minutes**
February 11	-14 = - (14 min.)
April 15	0
May 14	+3.5 = (3 min. + 30 seconds)
June 13	0
July 26	-6.5 = (6 min. + 30s)
September 1	0
November 3	+16.5 = (16 min and 30s)
December 25	0

[FIG 290] Equation of time, approximation graph.



Because the length of the year is 365.25 and not the (ideal, calendar) 365 days, the equation of time can vary by up to a day over the time of every four years. This complicates the calculation for the equation of time, and pre-made lists for the time offset for each day for a given year are available. Earth's elliptical orbit and the resulting equation of time will also (slightly) affect the expected Earth based coordinate or location of the stars in the sky in relationship to time. It is of note that very distant stars (even outside our local Milky-Way galaxy) and their position are now used to determine exactly when the Earth has made one complete orbit about the Sun, still, the resulting (net, total, average) gravitational effects of other matter (planets, etc) and its location(s) in our local (Sun centered or based) solar system will have a slight effect on the duration of a complete orbit of Earth and-or each planet about the Sun.

The equation of time effect will also affect the sunrise and sunset times by the same values. In general, for everyday applications and use in society, the equation of time adjustment is not considered by the average person and-or business, and is rather only used when exactness or precision is needed beyond that of the constant theoretical 365 orbit time and-or 24 hour clock time.

The equation of time will also effect the actual starting and-or ending day of each season which are normally separated by 90 calendar days, and not the Earth's actual orbit location from Sun, each when it is not considered.

Earth's **perihelion**, is when Earth is closest to the Sun during its orbit around the Sun, and this is about 14 days after the winter solstice on about December 21, and which is also the day with the least amount of daylight and-or "shortest day" in the northern hemisphere of the Earth. Earths perihelion is about January 4 on average and which can vary by a day less or more due to the orbital positions and the corresponding and (net) resultant amount of gravitational forces of attraction of other planets affecting the Earth.

For the southern hemisphere with Earth being tilted toward the Sun during the northern hemisphere's winter, will actually have their summer season start on December 21, and which is the "longest day (light)" in the southern hemisphere.

Earth's **aphelion**, is when Earth is farthest the Sun during its orbit around the Sun, and this is about 14 days after the summer solstice on about June 21, and which is also the day with the most amount of daylight and-or "longest day" in the northern hemisphere of the Earth. Earths perihelion is about July 4 on average and which can vary by a day less or more due to the positions and the amount of gravitational forces of other planets affecting the Earth.

For the southern hemisphere with Earth being tilted away from the Sun during the northern hemisphere's summer, will actually have their winter season start on June 21, and which is the "shortest day (light)" in the southern hemisphere.

"Civil" or meteorological seasons in the northern hemisphere start on:

Spring	March 1
Summer	June 1
Fall	September 1
Winter	December 1

The above dates is how the weather feels on average in the northern hemisphere and they precede the true astronomical and-or Earth's physical seasons and-or days by about 20 days, and which are associated with the solstice and equinox days of Earth's apparent tilt with respect to the Sun and-or solar plane of its orbit about the Sun:

Astronomical (star alignment, other than the Sun), and the climate seasons in the northern hemisphere start on:

Spring March 21 , spring or vernal equinox , Sun rises in the true Eastward and sets in the true Westward ,
day and night are equal in length at 12 hours each. Sunrise at 6AM, sunset at 6PM, and
with true local or solar noon at 12PM "clock time".

Summer June 21 , summer solstice , longest day (light) of the year

Fall September 21 , fall or autumn equinox , Sun rises in the true Eastward and sets in the true Westward ,
day and night are equal in length at 12 hours each. Sunrise at 6AM, sunset at 6PM, and
will true local or solar noon at 12PM "clock time".
Winter December 21 , winter solstice , shortest day (light) of the year : according to a 90 day per season

calendar time.

About sunrise and sunset times throughout the year.

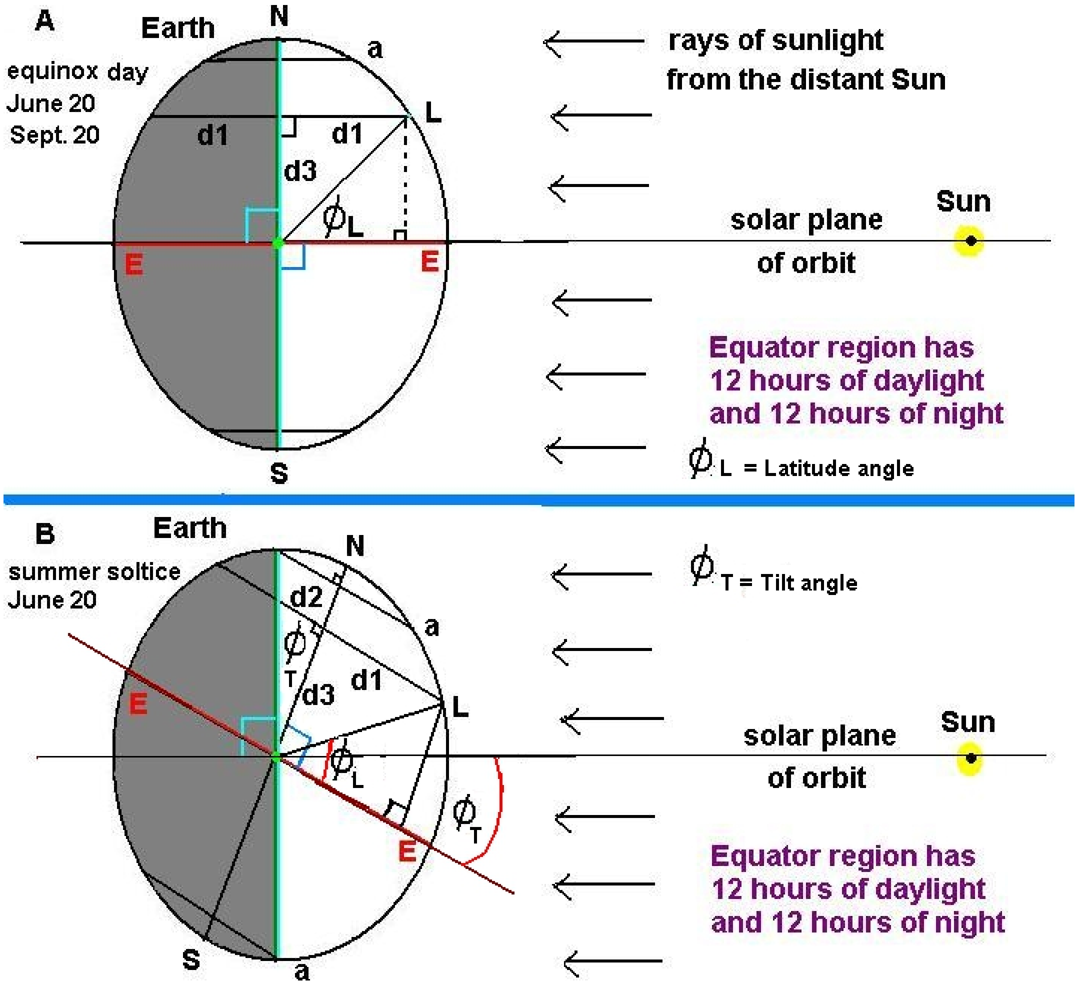
The greater the latitude of a location on Earth, the greater the surface of the Earth at that location is tilted back from the solar plane and the cooler the temperatures will be on average. The sunrise time will be later in the day or as seen on clock-time, and sunset times will be earlier in the day or on clock-time. The tilt of the Earth effectively shifts the path of the Sun across the sky to having a more southern path. The net result of this effect is that there is less daylight time and a longer night time. The Sun will rise in a more south-east direction, and will set in a more south-west direction, and instead of the ideal, directly or nearly east, and directly or nearly west directions as for a location at the solar plane and-or effective "solar equator" latitude region of the Earth that is currently closest to the true solar plane. For a simple example, at about 40° latitude, the sunset time will be about 9pm in the summer, and about 4:20pm in the winter. This time change is periodic and sinusoidal in value since the apparent tilt of the Earth is also periodic and sinusoidal invalue throughout the year.

As the apparent tilt of the Earth changes throughout the year and-or orbit around the Sun, this will change the effective latitude of a location on Earth with respect to the solar plane, and it will then also affect the sunrise and sunset times of the Sun at the horizon. When the northern hemisphere is tilted towards the Sun in the summer season, the sunrise time will be earlier in the day, and the sunset time will be longer in the day. Here, the Sun will appear to travel across the sky in more of an eastward to westward direction, and more above ("higher in the sky", higher angle, higher altitude) locations at those latitudes.

The equation of time offset value throughout the year will also affect the sunrise and sunset times.

CALCULATING THE LENGTH OF DAYLIGHT AT A GIVEN LATITUDE

[FIG 290A]



As shown in part A of the above figure, let us consider the equinox days (typically about March 20, and June 20, but a leap year with an extra day will affect these date values) when the poles of the Earth are equidistant from the solar plane and-or the geographical equator of Earth is at the solar plane. On this date, the Sun will be directly overhead at noon at locations at the Equator circle or arc about the Earth. The imaginary rotation axis or line of the Earth is at 90° to the solar plane.

Throughout the year, the Earth will have a maximum apparent and-or effective tilt of about 23.5° towards the Sun, such as in the summer months, and 23.5° away from the Sun, such as in the winter months. In the winter months, the weather in the northern hemisphere gets lower in temperature due to the lower (with respect to the horizon) east to west arc angle and-or path of the Sun. Locations in the higher latitudes will then have less direct sunlight and will therefore be cooler in temperature. part A of the above figure shows the Earth tilted 23.5° towards the S, and the Sun will appear to travel at a higher angle up in the sky, and the air temperature will be warmer (perhaps 50 °F or more) than during winter.

In the above figure of the globe or sphere of the Earth, and as viewed from being perpendicular to the rays of the sunlight, half of its surface will be illuminated by the Sun as shown by the unshaded region experiencing some amount of sunlight as the Earth turns or rotates on its axis. The shaded region will experience no direct illumination from the Sun, and will be in darkness and-or nighttime. The division between the sunlit and dark (ie., unlit) side of a globe or sphere such as Earth, Moon or some other planet is called the **terminator line**. Half (50%) of a sphere's surface will always be illuminated, and half (50%) will always be in darkness (ie., not illuminated). Depending on what direction the sphere is viewed, only some portion (ie., <=50%) of it will be seen as illuminated, and for example, this is commonly seen as the phases of the Moon as it orbits about the Earth throughout the month. The terminator line is always perpendicular to the rays of light upon the sphere, hence the terminator line is also perpendicular to the solar plane.

Considering the northern hemisphere and latitudes, the total length of daylight time is always shorter (ie., less than 12 hours such as at the equator during equinox) for a greater (ie., higher, from or vertically above the equator) latitude, and the maximum amount of daylight time will also then be even shorter in the winter. In the northern hemisphere, the apparent path of the Sun across the sky throughout the day will appear to be more southern in the winter months when the northern hemisphere "tilts back" or "away from" the Sun. This effect also places the sunrise and sunset locations along the horizon in a more southern direction. Instead of the Sun rising in the true east direction such as during the equinox days, it will rise slightly to the south, hence southward of east, and by 23.5 ° along the horizon. Just the same, it will also set slightly to the south, hence southward of true west, and by 23.5° along the horizon.

Consider that when the Earth effectively tilts back in the winter, the Earth's surface along the horizon will increasingly block more and more of the visible sunrise and sunset, and therefore making sunrise time to be delayed longer in time, and the sunset time to be shortened or lesser by the same amount of time. The total effect of this is a day with less full daylight time where the Sun is visible in the sky. Do not look at the intensely bright Sun, but rather look at sunlit or illuminated objects that are not as bright. The higher the latitude during winter, the greater all these effects mentioned.

In summary, the amount of daylight during the day depends on the location's latitude, and apparent tilt of the Earth with respect to the Sun and-or solar plane, and of which depends on the day of the year being considered. If the daylight at a latitude is shortened by x minutes, the nighttime will be lengthened by the same amount. For all seasons, locations and latitudes, it will always take 24 hours for the Earth to revolve once about axis, hence: daytime length + nighttime length = 24 hours. nighttime length = 24 hours - daytime length , daytime length = 24 hours - nighttime length

If the length of daylight can be calculated, we can subtract half the daylight time value from local noon (12pm) so as to find the sunrise time, and add half the daylight time value so as to find the sunset time. Due to the elliptical, "elongated" or "lopsided" orbit of Earth about the Sun and the associated "equation of time" concept creating a time offset from an ideal circular orbit, this will affect the actual or true sunrise , local or solar noon, and sunset times by a few minutes.

There are two ways to find the length of daylight at a certain latitude and during a certain day of the year, and of which is associated with the apparent tilt of the Earth for that day. This book shows how to calculate the tilt angle value for a certain day of the year.

1. Practical estimation using length measurements on a drawn figure of Earth at a certain day and its "tilt angle"such as shown in the above figures. Once these values are measured, a proportion or equivalent fraction type of equation will be used to find the length of daylight.

$$\frac{\text{24 hours}}{\text{chord length at latitude}} = \frac{\text{(daylight hours)}}{(d1 + d2)}$$

: Extra, tote that: 2 d1 / 24 h = d1 / 12 h ,
chord length at latitude = 2 d1 , and
d2 represents any additional hours of sunlight > 12 hours

2. Trigonometry. We can let DT= total daylight time = d1 + extra daylight time due to the tilt towards the Sun. DT = d1 + d2. We can use a proportion or equivalent fraction type of equation to find the total time of the daylight:

Considering part A in the figure:

$\sin L = \text{opp.} / \text{hyp.} = y / \text{hyp.} = d3 / r$: r = radius of the sphere, but we can set r=1 and use relative values for the calculations. Here L = latitude and-or its corresponding angle:

Letting r=1 for relative values and analysis, which is similar to using percentages. If we let (r) now increase by a factor of (n) we have: r= n(1) = N, then all relative values are to be multiplied by N, or in other words, they are already a multiple of

$n = r$.

$d3 = \sin L$: considering $r = 1$

$d1 = \cos L = \text{adj} / \text{hyp} = x / r = d1 / r = d1$: considering $r = 1$

$2\, d1 =$ latitude chord length : If needed. If d1 is a relative value, then 2 d1 is also a relative value

$\tan T = \text{opp.} / \text{adj.} = d2 / d3$: T = current, apparent tilt angle of the Earth with respect to the solar plane or Sun

$d2 = d3 \tan T = \sin L \tan T$: when $r = 1$ or using relative values

Ex. Considering the figure in part B above, the Earth is tilted at 23.5 ° = Tilt angle , and the Latitude angle is 41° North.

$d1 = \cos L = \cos 41^\circ = 0.7547\underline{1}$ and $d3 = \sin L = \sin 41^\circ = 0.6560\underline{6}$ and

$d2 = d3 \tan T = 0.6560\underline{6} \tan 23.5^\circ = 0.28526\underline{3}$

Using a relative value of 1 for d1, and of which corresponds to 12 hours of daylight:

$$\frac{12\text{ hours}}{d1} = \frac{12\text{ hours}}{1\text{ distance unit}} = \frac{\text{total daylight hours}}{(1 + d2)} = \frac{\text{total daylight hours}}{1.28526\underline{3}}$$, solving for total daylight hours:

total daylight hours = 15.42 hours =~ 15 hours and 25 minutes : the equation of time (offset) could make this result even more accurate by a few minutes

Here are some general considerations, and of which some were previously mentioned:

1. On equinox days (March 20, and September 20) , everyone will have 12 hours of daylight,and 12 hours of darkness (ie., essentially in the shade of the Earth).

2. Axial tilt of a general sphere that rotates, and the daylight time in a certain latitude due to that tilt. We know the time of daylight is less when the northern hemisphere during winter because we know for example that the Sun is no longer rising directly east, but southward of east. That is, the Sun is then below the horizon if it were the normal sphere with a perpendicular axis (ie., not tilt with respect to the solar plane) as mentioned in part A. The Sun will eventually rise, but it will be later in the day, and will set earlier in the day. For days past spring (March 20) and into the summer days, the northern hemisphere tilts toward the sun, and the sunrise and sunsets will get more in the northward direction.

3. Current tilt of the Earth's sphere with respect to the Sun and-or solar plane as it orbits the Sun. This is due to the elliptical orbit about the Sun, and this tilt will also effect the length of daylight. .

4. Notice the south polar region in total darkness when the Earth is at full tilt of 23.5°, and at the same time, the north polar region is in total daylight. These latitudes and regions near the Earth's north pole and south pole are called the Arctic circle and Arctic zone, and Antarctic circle and Antarctic zone. The latitude is: (90° +,- 23.5°) = +,- 66.5°

5. Due to the symmetry of a sphere and with a tilted axis of rotation, if a certain latitude has (s) hours of sunlight at a certain latitude, it will have (24 - s) hours of darkness at that latitude, and the corresponding negative latitude will have (24 - s) hours of sunlight.

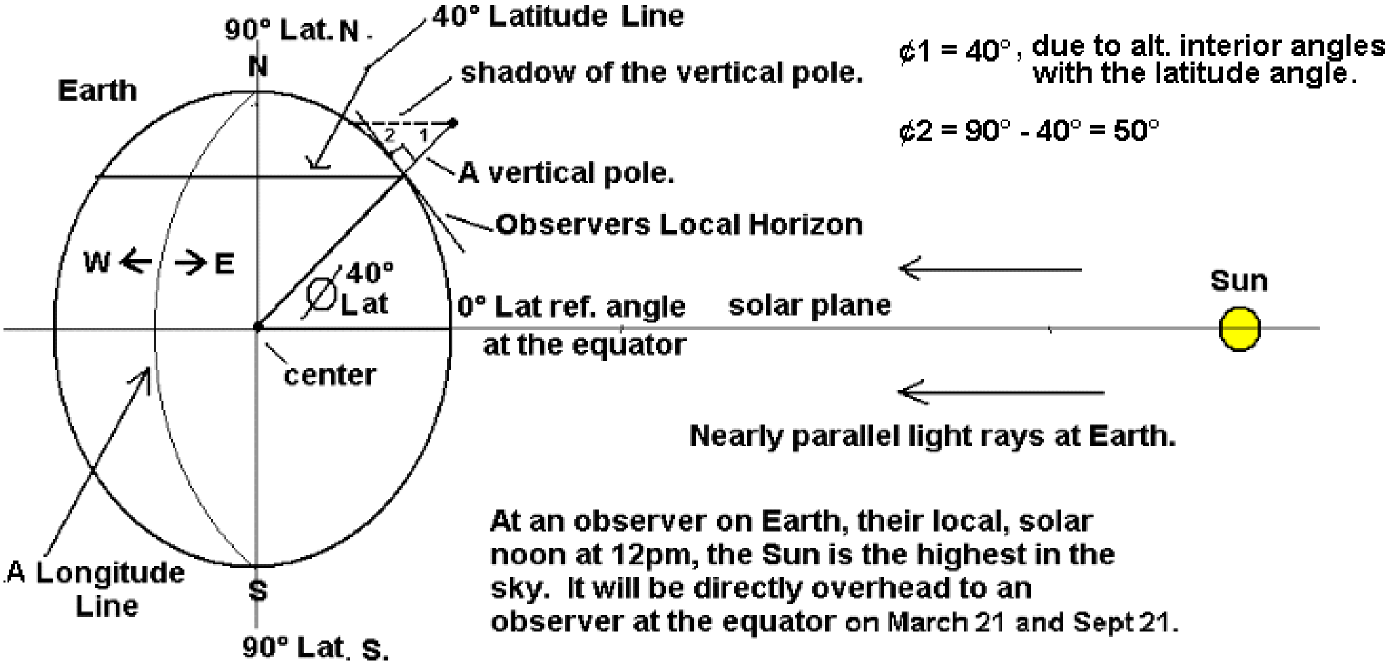
EARTH, SUN, MOON, AND STARS ASTRONOMY

The diameter of the Earth is about 7926 miles at the equator, hence about 8000 miles wide, and is about 27 miles less in distance from the north geographic pole to the south geographic pole and this is due to that Earth is spinning and the amount of (outward) centrifugal force is causing it to bulge slightly wider by about 27 miles at the equator region. The equator is perpendicular (ie., 90° to,a right-angle) to the north-pole to the south-pole axis or line of Earth's daily rotation, and is located midway between its length. The circumference of the Earth at the equator is: C = (pi)(diameter) = (2)(pi)(radius) = (2)(pi)3963mi = 24900mi =~ 25000mi. This circumference value is then also slightly less along Earth's polar axis. The radius of the Earth is often roughly noted as: r = 4000 miles, its diameter as: d = 8000 miles, and its circumference as: c = 25000 miles =~ (25000 mi) (1.609363 km/mi) = 40234 km = roughly 40000 km.

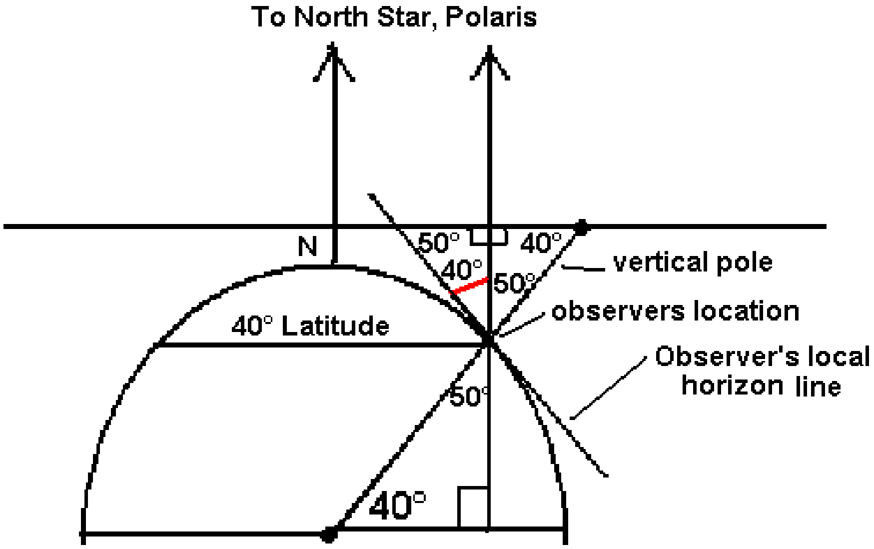
An ancient Greek, and multi-scientist named **Eratosthenes** (of Cyrene, a city in Libya, North Africa, near the Mediterranean Sea) worked at the Library of Alexandria, and in about the year 240BC, he had two people measure the angles cast by the shadow of two vertical sticks on the same day and time (local noon) at two distant locations (Alexandria and Syene, Egypt) that are located directly north and south with respect to each on Earth's surface. Today we would say that these locations directly north and south of each other are located on the same longitude line and-or have the same value of longitude on Earth, and therefore have the same time of day such as solar noon time (ie. true local noon time, rather than a (wide distance, unified) "timezone time" and-or 24 hour, fixed, constant "clock time") where the Sun is highest in the sky, and also have the same sunrise and sunset times. You can place a vertical stick on the ground, and when the shadow from it is the least length, it is solar noon and the Sun is the highest it will be in the sky for that day. Eratosthenes then (closely) calculated Earth's circumference using basic geometry and trigonometry. Today, his scientific method and result are considered highly accurate, and his was done at a time before he knew of the decimal number system. From this circumference value, Earth's diameter and radius could then be calculated also. These values eventually helped the calculation of other important astronomical sizes and distances such as to the Moon and Sun. The Earth was thought (ie., a theory) to be a probable sphere before this time, such as by **Aristotle** in about 300 BC. and due to various astronomical reasoning such as the Moon being round or spherical in shape and therefore, it orbits a (theoretical, not proven yet) round Earth. Aristotle (384BC - 322BC), was a Greek mathematician, and mostly a philosopher about thought and reason, and was a student of the philosopher **Plato (427BC-347BC)**, from Athens in Greece, and who was a student of **Socrates (469BC-399BC),** from Athens in Greece. Eratosthenes also conceived the concept of what we call today as latitude and longitude lines, and hence their intersections (ie., coordinates, address, location) so as to map and-or find locations and-or their distances between them on the Earth's surface. Before Eratosthenes, it was well known, such as by the people of ancient Egypt who studied the Sun and its motion throughout the year(s), and that there were 365 days in a year. Eratosthenes made a more accurate calculation of 365 days plus a quarter (ie., 1/4 of a day, and 1/4 of 25 hours is 6 hours) of a day, hence 365.25 days. After Earth orbits the Sun in 1 year of time, the nighttime stars (which always do not appear to move with respect to each other, and appear fixed in position due to their great distances away) and-or constellations (a visible arrangement of stars) will be in the exact same position as the year before. Modernly, the number of days in 1 year has been adjusted to be: 365.2421875 days. In common "24 hour clock-time" or "365 day calendar time" and-or non-scientific use, a year is still considered as simply being 365 days of time duration or "time length" and an adjustment or correction will be made as a "leap year day" every four years.

Constellations or known groups of stars that are within +,- 8° of the ecliptic are called the zodiac of constellations and stars. Long ago, ancient astronomers divided the ecliptic into 12 constellations called the zodiac, and each is a month long in apparent duration and-or alignment. During the day at noon = 12PM, the Sun is in line with one constellation of the zodiac, while at the same time on the other side of the Earth at midnight = 12AM, we can see the constellation that is 180° or 6 constellations away from the Sun's and Earth's apparent circle or orbit of travel though the night sky and zodiac throughout the year. As the Earth travels counter-clockwise about the Sun, during the year, the Earth and-or Sun will appear to enter a new constellation among the zodiac about every 30 days or 1 month. Ancient astronomers called and used the zodiac as "signs" of yearly events, and would then predict things that will happen during that time of year, such as what season it is - perhaps planting or harvest season. The location of the sunrise and sunsets along the horizon can also be used to determine what season. month, week and possibly the day of the year it is. For modern astronomers, the stars of a constellation are generally now known to be not at the same distance away or "depth" from the observer and-or Earth, but only appear to be that way, and each star may actually be many light years closer or distant to the observer.

[FIG 290B]



In the above figure, the shadow and the corresponding angles created will change throughout the 24 day as the Earth rotates on its axis and the apparent position of the Sun then changes in the sky. This angle also changes day by day throughout the year as the angle between the equator and solar plane effectively changes. What does not change is that since the North Star, Polaris, is a point of light and very distant and is not apparently moving, and therefore its angle with respect to the observers horizon line at their latitude will not change. This angle between an observers local horizon line and the line to the North Star is also equal to the observers latitude on Earth. [FIG 290C]



What could of inspired Eratosthenes to determine if the Earth was round? It was probably several known facts. One fact is that in the Northern directions from certain locations (ie., near the Equator) on Earth, and in the Southern directions from certain locations (ie., near the Equator) on Earth, the average temperature was lower. This hinted of the land at the pole regions which do not get much direct sunlight energy, and due to being more curved away from the direct sunlight. One other fact is that the Sun was sometimes directly above in certain locations (ie., near the Equator), but never directly above in many other locations, and rather at a maximum angle that is less than 90° with respect to the local horizon. The locations were it is directly above are also the hottest locations. Another fact is that people on small boats could not see the low laying shoreline areas, and this is due to the curvature of the Earth. It is also possible that the round Moon shape and the somewhat curved (terminator line) shadow most likely gave Eratosthenes an idea that Earth may also be round (2 dimensional) also, and particularly, a (3 dimensional) sphere shape. The Moon (light and shadow) phases throughout a month of time have a curved shape, and which indicates that the Moon may be a sphere. Eratosthenes new that locations directly south or north or each other had the same (solar) time of day in those locations. For example, in those locations the Sun would be at its highest angle throughout the day. Though in these locations, the shadow lengths of similar poles would be at different lengths, and today we know that this is due to the different latitudes, the indicated time on a solar clock, even if just a stick in the ground, would point in the same direction, and so as indicating the same time of day. The stars appear to move eastward to westward like the Sun does as the Earth turns about its polar (N to S) axis and a point or location on it moves westward to eastward. The stars also appeared to travel about the star Polaris throughout the night and day of each year, and that the star Polaris is always at the same angle in the night sky, and this angle depends on the observers (latitude - north and south) location on Earth. Out at sea, sailors can see a mountain or tall building appear to slowly rise upward from the horizon of the sea as they travel towards it, and people on the shore can see a tall ship appear to go below the horizon of the sea at it travels away from them. Clouds appear to meet with the distant horizon. To Eratosthenes, all these things mentioned seemed to indicate that the Earth might be a sphere, and which rotated from westward to eastward throughout the day and night on an imaginary axis from the north pole to the south pole.

Extra: Notice that at each latitude, that the circumference or circle around the Earth at that latitude is less the further north from the equator, and less further south of the equator. Using the above figure, the radius (Rlat) of this circle at a certain latitude is equal to the bottom leg of the large right triangle shown in FIG 212B. Clat = 2 (pi) Rlat .

Ex. The distance around the Earth at 40° Latitude is: Clat = 2 (pi) **[** Rlat **]** = (6.28) **[** (Cos Lat ¢)(Radius Of Earth) **]** =
= (6.28)(Cos 40°)(4000 mi) = 19243 mi

Another important note is that the longitude lines get closer together and converge at the poles. At the Equator, each degree of longitude corresponds to a distance of:

Total Arc Length / 360° = (Circumference at the Equator) / 360° = 25000 mi / 360° = 69**.**444... mi / 1°
Dividing both the numerator and denominator by 69.444... and arbitrarily inverting the fraction, we find: 0.0144° / 1 mi

For other latitudes, use the circumference around the Earth at that latitude.

The length of each longitude line from the north pole to the south pole is half of the circumference of the Earth:
Total length of a longitude line is: (Earth's circumference) / 2 = 25000 mi / 2 = 12500 mi
Each degree along a north pole to south pole longitude line also corresponds to 12500 / 180° = 69**.**44 mi... / 1° = =~ 70mi / 1°

Earth will rotate on its axis, once ever 24 hours, hence it rotates: 360° / 24h = 15° / h : Earth's angular velocity
Since 360° corresponds to 25000mi at the equator, it rotates : 25000 / 24h = 1041**.**66..mi / h : Earths linear velocity at the equator.

Due to this, each hour increment of time was given a latitude line that is 15° apart. After one hour of time of Earth's rotation, the Sun will be overhead or at its highest point in the sky at the next hourly latitude line to the west.
At the equator, the next hourly latitude line is 1041**.**66 mi away.

$$1041.66 \text{ mi / h} = 1041.66 / 60 \text{ min} = 17.3611... \text{ mi / 1 min.} = 17.3611... \text{ mi / 60s} = 0.28935\underline{2} \text{ mi / 1s} =$$
$$= (0.28935\underline{2} \text{ mi / 1s}) (5200 \text{ ft / mi}) = 1528 \text{ ft/s}$$

Where is the opposite side of the Earth from your current location? It is halfway around the Earth, hence at Ce/2 = 25000/2 = 12500mi, and 180° in (east, west) longitude away from your current longitude, and the latitude of that location will be the same as your latitude, except that it will be on the other side (north (+) or south (-)) of the Equator.

If two locations on Earth are said as being 100 miles apart, what is their angle of separation? This angle is a central angle from the center point of the Earth. On the surface of Earth, a distance from one point to another is actually an arc (like a semi-circle, or segment) of the Earth's circumference. Some or part of these equations and-or values were given above. Writing a proportion type of equation:

First: $\frac{360°}{25000 \text{ mi}}$ is to as = $\frac{x°}{100 \text{ mi}}$, after solving for x° we find: $x° = 1.44°$
The locations are separated by 1.44 degrees on Earth's surface

How the Earth was finally proven as being a sphere shape

With the Earth known as being a possible sphere, and much how like the Moon appears to be round and a possible sphere, mankind has desired to find out if this is true, and then by attempting to travel around the Earth in just one main direction (East or West) and then to arrive in the same starting location from the other direction. In about the year 980AD to 1000AD, **Erik Thorvaldsson** (aka. "Erik the Red"), (950 - 1003), from Norway, the father of Leif Erikson, discovered Greenland to the west of the island of Iceland, and eventually made some settlements (ie., colonies) in Greenland's southernmost coastal regions. It is possible that if there were any native Greenlanders in his area, that they may have told him of lands further westward. **Leif Erikson** (970-1025), from Norway, in about the year 1000AD, sailed west to find new land and resources further west past Greenland and to its south, and rather than specifically trying to go around the Earth, He made a small settlement on the east coast of Vinland (Vine-Land, (grapes) Wine Land, and what is called today as the province (ie., like a large region, or state in a country) of Newfoundland, Canada on the North America continent. Erikson is generally believed to have come in contacts with the natives (North Americans, "Indians") of Vinland.

The knowledge of Erikson's discoveries, was either vague, secret, hearsay and-or an unsurity by the time of Christopher Columbus. **Christopher Columbus** (1451-1506), from Italy, and with funding from Spain, he sailed westward in 1492 and found many islands which he thought were the East Indies islands which were known to Europe ever since Marco Polo's travels and explorations, and which are eastward of India and southward of far-east Asia. What Columbus actually found were the (now called: "West Indies" [westward Indies, when traveling westward from Europe]) islands (particularly Jamaica) off the south eastern coast of what is now called the state of Florida and Central America. Columbus, thinking that he did go around the Earth and landed someplace near the country of India, then called the native people as Indians. The first American natives migrated to north Americas at about 20 thousand years previously from northern Russia called Siberia, north Mongolia (ie., just south of northern Russia, and perhaps a mixed race including what is now the northern China area) of which are all generally a northern Asian race of people. Supposedly during the last "Ice Age", theses people, now called the **Paleo** (Paleolithic era = ancient era, "caveman", "stoneage") people came to the Americas, near Alaska, on a (53 mile wide) ice bridge at the Bering Sea in the northern Pacific Ocean, and between the North Asia and North America continents. Columbus took his discoveries and news back to Europe. and where many ships and people would soon want to make the journey to the "new world". By 1493 Columbus was beginning to create some small colonies in the Dominican Republic (a Hispanic [Spain territory] island) such as the first one called **Isabella**. On the American mainland, **Jamestown** (now in Virginia state) was the first colony of Europeans and was began about 100 years after Isabella in 1607. **Plymouth** (now in Massachusetts state of the United States of America - USA) was the next significant colony started in 1620. The natives were both afraid and in wonder of some of the abilities of the people from across the Atlantic ocean to the east, such as having products made of iron metal, rather than copper, and which was both harder and durable and much less brittle, and therefore, it was considered as an extremely useful and valuable item to them. Some natives, particularly in the South America regions, already knew of and-or had copper, and gold metal products created from melted gold nuggets, but gold is relatively soft and rare and is not practical for tools. Even a discarded iron nail or a small piece of scrap iron metal was considered very useful and valuable to the natives.

Columbus, during his 4 voyages to the "new world", also made some wonderful discoveries such as some new varieties of plants and foods to take back to Europe, and to then grow them if possible. The people of Ireland are well known for growing potatoes, but relatively few know that the plants and seeds for them were brought from the "new world" that Columbus discovered. It did not take too long for the Europeans to realize that the native (ie., original) people that Columbus encountered and called them as "Indians" were not actually from the country of India, and therefore, Columbus did not sail around the Earth to prove it was a sphere. Some of his ship crew, and later the Europeans, probably theorized that the Natives were of some far-east, North-Asian lineage and were not from India. Perhaps by calling the natives as "Indians" he let the European people know in some way that this so called "new territory" or "new world" was already inhabited and controlled by someone else, and it was therefore not necessarily free to obtain the resources and-or to colonize, etc.

The colonies of the United States declared independence from European rule on July 4, 1776 when the Declaration Of Independence was signed, and the United States of America, Constitution was created on September 17, 1787 in the city of Philadelphia in the state of Pennsylvania. This was nearly 300 years after Columbus found the "New World Of The West". Most politicians and-or representatives signed the Constitution, but some did not because of a lack of a written, specific Bill Of Rights. The Bill Of Rights began to be placed into the Constitution as Amendments to the Constitution in 1789. Because of business claims and ties to Europe and the need for African-American slaves to continue working for the economy and needs of the south-eastern states, it took until a costly (life, property, etc.) Civil War (1860-1865) with a large number of soldiers lives so as to obtain the ideals expressed in the Constitution, and to keep the southern states as part of the country. Back then slavery was common for many races and locations since ancient days (even St. Patrick was said to be a type of slave), and with few jobs and education available, and was considered much like that of a helpful servant who was reasonably taken care of in return, and so as to perform the best they could and help many others. Canada became mostly independent of Britain in 1867, and finally in 1982. Mexico became independent of Spain in1821.

The land of the Americas is very long southward and naturally impassable by a ship, westward or eastward, until **Ferdinand Magellan** (1480-1521), from Portugal, and who was later associated with business of the Spanish empire, sailed very far south along the east coast of the Americas and made it to the Pacific ocean to the west. Magellan actually called this vast water as the Peaceful Sea after sailing through the rough seas he encountered in the southern Atlantic Ocean. Today, the Pacific Sea is commonly known as the Pacific Ocean. The Pacific Ocean is the largest ocean on Earth. Magellan made it to the Philippine Island areas in the south-western Pacific ocean, and of which he had once previously sailed near to and knew the geography fairly well, and then he finally knew that the Earth was a sphere shape. These areas area also modernly known as the South-East Asia Islands, and of which are northward and north-eastward of Australia. These South-East Asia Islands are also called the Polynesian Islands and of which are inhabitant by the Polynesian people which are an ancient mixed race of people, most likely from India and Southern Asia. The people of the Philippine Islands are generally considered as the original or main source of the Polynesian or "Pacific-Islander" people(s) that inhabit the islands which are mostly to the south and south-eastward of the Philippine Islands in the Pacific ocean. They are sometimes called the "Oceanic people(s)". One of Magellan's fleet of ships with its captain and crew made it back to Spain in **1522**, and thereby verifying and proving to others that the Earth was a sphere shape, and that the Sun and Moon were probably spheres also. **Marco Polo** (1254-1324), from Italy, had previously explored this central east, far-east, and south-east asia areas in about the year 1270 and brought back several discoveries to Europe, such as pasta (ie., noodles, a food made from wheat, and which did not spoil easily) and printing by using wood blocks as the print pattern. Polo went to find new and better routes, people, and items for trade. Polo traveled on land in 1271, and eastward along the "silk road" trade route toward China, Mongolia and South-East Asia in 1271. His father and uncle had previously explored some of these trading routes. Some of the other countries he explored were Israel, Iraq, Iran, Afghanistan and Pakistan, After reaching China, he met **Khubilai Khann** (1215-1294), from Mongolia, who was the grandson of **Genghis Khan**. Khubilai became a famous ruler in the Mongolia area and later in China. Khann made Polo a diplomat and so as to help travel, learn and explore the far-East areas in the hopes of eventually trading and understanding cultures. In 1292, he began to his sail back around south-east China, south-east Asia, and the continent of India. He landed in southern Iran and traveled back to Venice, Italy on land routes and the Mediterranean sea.. He had traveled 24 years on this journey. He did not sail around Africa, and the Suez Canal in Egypt was not yet built. Polo wrote a book called "The Travels Of Marco Polo" about his journey, and which was then useful for many other map-makers, explorers, politicians, and merchants (business people, traders). Polo is generally credited to also discovering for Europe, that the continent of Asia ended at a great ocean (now called the Pacific Ocean) to the east of China. The first person from the "west" (ie., Europe, before the discovery of the New World - Americas in the Western hemisphere of the Earth) to discover going around the southern tip (Cape of Good Hope) of Africa as a trade route to Asia was **Bartolomeu Dias** (1450-1500), from Portugal, in about 1448.

Australia and its route was eventually reached and "(re)discovered" by Europeans - possibly, and informally by **Christovao de Mendoca** of Portugal in 1522 by sailing eastward from Europe. Australia was formally discovered by **Willem Janszoon**,from Denmark, sailing eastward in 1606, and then later by **James Cook**, sailing westward from England, in 1770, and particularly the East coast of Australia. The European's first name for Australia was New Holland (Dutch), then New South Wales (English), and then Australia which means "the southern land", and particularly since it is south of the equator. Off the east coast of Australia in the Pacific Ocean is the island of **New Zealand** which was first noticed in 1642 by the Dutch explorer **Abel Tasman,** sailing westward**,** and then it was later landed upon and mapped by James Cook in 1769. Tasman is credited to formally discovering the island of Tasmania in 1642, and it is just south of Australia. Tasmania became an Australian state in 1905. Cook is also credited as being the first European to discover and-or make contact with the **Hawaiian islands** and its people in 1778. Today, Hawaii is a state (as of 1959) of the country of the United States of America (U.S.A). The natives of Australia, often simply called the **Aborigines**, meaning "the original, first people and inhabitants of", are most likely an ancient mixed race descended from the migrating tribes of people from both Africa and India. About 4500mi to the south-east of Hawaii, and south of the equator, the Polynesians have reached and inhabited **Easter Island (Rapa Nui)** since about 300AD. Some Polynesians seem to have reached Central America after 100BC and then migrated mostly to the south and mixing with the natives, and with some possibly migrating westward to some of the more eastern Polynesian islands by around 1200AD. Polynesians also seem to have reached the western shore of South America by about 1000AD in the areas of Peru and Chile to the south at nearly 2300 miles to the east of Easter Island. European explorers first seen and reached Easter Island on Easter Day by the Dutch explorer **Jacob Roggeveen** in 1722. Today, Easter Island is a protected territory (ie., like an independent state, and given assistance, etc.) of the country of Chile in South America.

Between the North America continent and the South American continent is an area commonly called Central America, and of which includes the countries just to the south of Mexico on the North American continent, and to the north of Colombia on the South American continent. In Central America, there is a short distance, natural land-bridge from the Atlantic Ocean to the Pacific Ocean in the country of Panama that is just north of Colombia. If ships could travel through here, it would eliminate the need to travel thousands of miles around the South American continent so as to reach the Pacific Ocean to the west. Columbus reached the American continent on his last voyage (1502-1504), and in the area of Panama, and then searching for a ocean passageway to the (Pacific) ocean as described by the natives. The first European to actually explore and describe this (narrow) land-bridge was **Vasco Nunez de Balboa**, from Spain, in 1513. After 10 years of hard effort in the jungle and climate of Panama, a 50 mile long, lake and canal system for ships to travel across Panama was completed in 1914, and thereby saving ships thousands of miles of travel and days of time. This canal is called the **Panama Cana**l, and its concept is much like that of the **Suez Canal** that was previously built in Egypt so as to greatly reduce the travel distance and time around the continent of Africa. The Suez Canal was completed in 1869 and extends about 120 miles from the eastern Mediterranean Sea and southward to the Red Sea, and travels between east Egypt and west of the Sinai peninsula of Arabia. Two other long canal systems of mention are the **Eire Canal** system in the northern part of the state of New York, USA, and is about 350 miles long, and the **Grand Canal** system in eastern China, being about 1100 miles long. Today with planes, automobiles, and trains, transportation can be much faster than via a ship. An advantage of using ships is that free wind power can be used to move (sailing) ships and-or boats, and that very large volumes of goods can be transported to nearly any port that can dock such ships, and-or areas that have good coastal locations to anchor a ship nearby. As of about the year 2000, some have speculated that modern railways can be an alternative option near the Suez and Panama Canals, and this will then also require further ships to complete the remainder of the journey. Europe has built a modern tunnel and railways system beneath the bottom of the English Channel from France To England, and this is often called the "**Chunnel**".

As of the year 2025, trains and boats are still used to transport large masses of materials and many (say from 100 to 1000) people. The front of a boat or ship is called the "**bow**", and is usually moves in the "bow direction" or direction that the bow is pointing to and-or is facing. The right side of ship is often called the **starboard** side, and the left side is often called the **port** side. The origin of the word starboard is due to the (usual) right hand, rudder steering (based on the word "steo") of a small boat, and the port side is then where the boat would then port or dock itself to load and-or unload cargo and-or crew members.

Earth's (pole to pole, "polar", north to south, and rotational) axis is not perpendicular (90°) to it's orbital plane (or "ecliptic", or solar-plane) of it going around the Sun, but is rather tilted with respect to this plane of (0°) reference. This tilt angle value is nearly a constant value for over hundreds of years of time, and as of the year 2022AD, it is about **23.436°**. This tilt angle is also said by astronomers to vary periodically over about 10,000 years as the Sun and entire solar system of planets orbit the center of the Milky-Way galaxy of stars. The tilt angle can vary during this time from 22.1° to 24.5°. The **Milky-Way galaxy** of which we are located in gets its ancient name from the visible long band or grouping of stars seen across a part of the sky at night, and which was called the Milky-Way. This region is actually an arm or "arm-band" of the spiral of the galaxy. All the visible stars we see are in the Milky-Way galaxy, while only a very few other galaxies, such as the Andromeda galaxy or nebula (gas structures, often from an exploded star and-or forming star group) often as faint, "fuzzy" patches to the unaided eye, however most require a telescope to see. The Andromeda galaxy (M31 = Messier object catalog number 31) is considered the closest organized and spiral galaxy to the Milky-Way galaxy, and is about 2.5 million light years away. The Milky-Way galaxy has a few less organized, closer and smaller orbiting satellite galaxies such as the Canis Major dwarf galaxy at 25000 light years away. With just a small telescope or binoculars, many more stars in the Milky-Way galaxy can be seen. Light sensitive, long duration photography with telescopes or some cameras can also image many more stars and-or structures that are even too dim or faint for a large telescopes to image. Individual stars in other galaxies can be seen using such photographic methods.

The diameter of the Moon is about 2159 miles ~= 2160 miles, and its circumference is then about 6782 miles. The Moon's diameter is about (1/4) of that of the Earth, hence about 4 times smaller, and the view of (level) horizon of the Moon's surface is about 4 times less (ie., closer) as distant as compared to the Earths horizontal view to a person on Earth. The circumference of the Moon is also about (1/4) that of the Earth's circumference. The distance to the Moon averages about 239900 miles =~ 240000 miles = $38624\underline{3}$ km away, and this was first roughly calculated by a Greek citizen named **Aristarchus of Samos** in about the year 270BC, and later improved by another Greek named **Hipparchus** in about 129BC. Clearly, the Moon is closer to the Earth than the Sun due to that the Moon always eclipses or blocks the Sun and its light during a solar (ie., Sun) eclipse. The (complete or "umbra") shadow (about 85 miles in radius and-or 170 wide in diameter) of the Moon at the location on Earth where it experiences a full solar eclipse will then be on the Earth for a few minutes or longer over that location in its "shadow path" on Earth. People who experience a partial eclipse are in the shadow zone or region called the "penumbra". A lunar eclipses is when the Earth blocks the sunlight from reaching the Moon, and the shadow of the Earth is then seen on the Moon. Another astronomical fact is that the apparent visible angle (or "width") of the Moon and Sun is the same part of the overhead sky (180°, horizon to horizon if no hills or mountains are in the way), however, this does not mean their distances are even close due to that the Sun could possibly be, and it is rather much farther away and still appears as the same visible angle, width or size as the Moon. During the Apollo Program manned Moon landings (July 1969AD to December 1972AD), a laser reflector was placed on the surface of the moon so as to measure, calculate and verify its known distance from Earth using a high power, quick pulse of laser light from Earth that was aimed at the Moon so as to have a small but usable amount of it reflected back to a receiving station on Earth after an amount of time of travel has passed. Clearly, the distance to the Moon takes half this measured amount of time at the speed of light: Distance to Moon = (speed)(time) = (speed of light) (time in seconds / 2). This distance of separation will also cause a constant, noticeable time delay or "gap" in radio communication such as with people speaking to each other, however it is a low, tolerable value of only about 1.3 seconds to the Moon, or to the Earth from the Moon.

Aristarchus, of Samos, Greece, in about 200BC, believed (proposal, theory) the Earth and other planets and stars orbited the Sun in a heliocentric ("Sun-centered") universe. **Nicolaus Copernicus** (1473-1543), from Poland, calculated and published a verification of this theory is in 1543AD. This was also verified by **Galileo Galilei** (1564-1642), from Italy. in about 1610. In 1610, after being inspired by the recent invention of the telescope, he made the first practical high power (about 20 times magnification or "power", ie., 20x) telescope at that time and made many important astronomical discoveries using it. The (refracting = light bending, usually with glass lenses) telescope was a newly recent patent applied for and-or construction (in 1609) by **Hans Lipperhey (aka Johann Lippershey)** (1570-1619), of Denmark, and it had a low magnification or "power, and it was also known previously that two magnifier lenses held at a distance apart could magnify a distant object. In 1610, **Galileo** discovered with his telescope that planet Jupiter had 4 visible moons of its own, and which revolved around it periodically (at a specific duration or amount of time), and that Earth's Moon had craters and mountains, In 1610, Venus was seen by Galileo to have (sunlight illumination) phases (like the Moon about Earth) as it orbited the Sun, sunspots, and planet Saturn's ring structure. Galileo helped prove the theorem by Aristarchus and later, Copernicus that the planets orbited the Sun. **Johannes Kepler**, in about 1609AD, found that the planets orbit the Sun in a elliptical orbit with the Sun at one of the foci (ie., end-region) of their elliptical orbit, and this is one of his celebrated planetary motion (orbit) laws. Later, in about 1667, **Isaac Newton** created the celebrated mathematical laws for force, gravity and motion (other than the well known: distance = speed x time). Newton is also credited with making the first practical Newtonian or reflecting telescope in 1668, and of which a curved parabolic shaped mirror is used as the main "objective lens" or primary mirror to gather a large amount of light coming from an object, and then direct (by a flat, reflector mirror) it all to be concentrated (and brighter, more intensity) and focused together at the eyepiece lens at the upper side of the telescope tube. Not long after the Newtonian telescope was invented, the famous (calculus) mathematician **James Gregory**, from Scotland, previously conceived the Gregorian telescope in 1663, and which was later built by the famous **Robert Hooke** in 1673. Gregory and Newton knew each other very well and influenced and assisted each other. The Gregorian telescope effectively replaced the eyepiece mirror with a small circular, concave focusing mirror to direct the light (or "light path") back through the center of the main (objective) parabolic mirror to an eyepiece lens at the other or bottom end of the telescope, and this makes another good use of the central incoming light path that is otherwise blocked by the central or secondary mirror that directs the light to the eyepiece lens at the side of the telescope. This construction increases the focal length and magnification to about twice that of a Newtonian telescope. A (Laurent) Cassegrain telescope is similar to a Gregorian telescope except that the secondary mirror is slightly convex instead of concave. Note that using a true spherical mirror does not have an exact common focus point for the incoming parallel rays, particularly at the edge of the mirror, and is then in error a small amount - hence some image aberration or distortion. Catadioptric reflector telescopes (ex: Schmidt-Cassegrain, and Maksutov) use a spherical mirror, but they also have a (nearly flat) correct lens at the front end of the telescope. For a basic telescope, the formula for its image magnification or "power" is the focal length (Fo) of the objective lens divided by the focal length (Fe) of the eyepiece lens. **The basic optical lens power = focal length of objective lens / focal length of eyepiece lens = Fo / Fe**. To increase the power of a telescope, an objective lens with a longer focal length should be used, and-or an eyepiece lens with a shorter focal length should be used. With modern images sensor with a pixel count of many millions, this lens power can be effectively increased so as to have a greater resolution where smaller parts of the object can be discerned. The Yerkes Observatory in the USA has a large refactor (ie., lens) telescope having about a 20" diameter glass objective lens, and about a 60 ft. focal length, and its magnification power is about 500 to 1000 depending on the eyepiece used. The corresponding linear magnifications are their square roots of about: 22 to 32 times wider and-or higher in appearance with the aid of the telescope.

The diameter of the Sun is now considered about 865,000 miles, and this was first roughly calculated by **Aristarchus of Samos** in about 220 B.C. The circumference of the Sun is then about 2.72 million miles. The **distance to the Sun** is now considered about 93 million miles away, and this and the distance to the Moon was also first roughly calculated by Aristarchus using right angle trigonometry during the half-moon illuminated phase, ratios, eclipses and shadows, and the previously (about 20 years earlier) made historical and significant calculation for **the radius of the Earth by Eratosthenes.** At the Sun's equator, **sunspots** (dark regions on the Suns surface, and of which began being scientifically observed and recorded (ie., drawn on paper) in about 1611 by Galileo and others with the aid of dark-glass light filters) will appear to travel from west to east across and around the Sun in about 25.4 Earth days, and about 35 Earth days at its poles. Therefore, the Sun also spins in a counter-clockwise direction as seen from above its north pole, and once on its polar axis at an average of about 30 Earth days. At the equator region of the Sun, a point, sunspot or ejecta (matter released, thrown off outward) on the Sun has a horizontal velocity across or on the Sun of about 2.72 million miles / 30 days = 90582 miles / day = 3774 miles / hour. Some solar ejecta such as low mass, lite-weight particles from a solar storms or burst will also have a vertical velocity outward or away from the Sun, perhaps up to a million miles an hour. These particles may cause electronic signal and circuit interference and-or damage when Earth is in the path of those particles.

For our Sun, there is an 11 year cycle of the appearance of sunspots on the Suns surface, and the number and-or size of the sunspots will be at maximum every 11 years on average. This cycle length could be due to the volume size of the Sun and-or the elements composing it. These things will also determine its temperature such at its core and surface.

The Sun is one star among many with an estimated 100 billion stars in the Milky-Way galaxy alone. Our Milky-Way galaxy is one among an estimated 2 trillion galaxies in the universe. There is some evidence of a **black-hole** [a dense, dark object or mass with a very high amount of gravity] at the center of many galaxies, and this usually provides the gravitational force to hold the stars in their orbits about it, much like how the Sun's gravity keeps the planets in orbit about it.

Due to their sizes and distances from Earth, both the Sun and the Moon both appear to be 0.5° = half a degree wide in the sky, and which is effectively a total angle of180°. All the stars that we see in the night sky with our eyes are in the Milky Way galaxy star system of which our Sun is also a star within it. Other galaxies with other stars composing them are so distant and faint that we cannot see their individual stars without the aid of a high power telescope.

The Earth revolves or spins about its imaginary (pole to pole) line or axis once ever 24 hours from the westward to eastward direction, or counter clockwise as viewed above the Earth at the north pole. That is, nearly: 360°/24hr = 15° per hour = 1° per 4 minutes = 0.5° / 2min = 0.25°/ 1min = 0.0041777..°/sec. The distant, and apparently stationary, Sun and stars will then appear as moving across the sky at 15° per hour, all while the Earth is still traveling (ie., orbiting) around the Sun at about 1°/day = 1°/24 hours = 0.041667°/hour to also consider. Throughout the day or night, the Sun, Moon and stars will appear to rise in the eastward direction and set in the westward direction. As an extra note:
1°/4 min = (360°/360) / 4 min = (((360 x 60) arc-minutes) / 360) / 4 min = 60 arc-minutes / 4 min =
((60 x 60)) arc-seconds / 4 min = 3600 arc-seconds / 4 min = 900 arc-seconds / min = 900 arc-seconds / 60s =
15 arc-seconds / s . For reference, 1° = 60 arc-minutes = 60' , and 1 arc-minute = 60 arc-seconds = 60"

The Earth, like most solar system objects, would also be seen to orbit the Sun in a counter-clockwise orbit when viewed from above the north pole of the Sun. The explanation to this direction is due to how the planets, solar system or galaxy formed out of gases and matter that condensed and thickened due to their gravity (gravitational attraction force), and resulting in their current shapes, sizes and orbits. Planets farther from the Sun generally have more eccentric ("more elliptical") orbits about the Sun and this is most likely due to that the force of gravity at those distances from the Sun is actually greater than the minute amounts of gravity between those distant particles of matter, and this results to a more direct attraction of the particles to move towards the Sun's direction rather than having a pure circular orbit about the Sun.

For planets, the gravitational pull or force from the Sun causes them to orbit in the common counter-clockwise direction about the Sun. The collective or resulting amount of gravity from the stars orbiting the center of the Milky-way galaxy in a counter-clockwise direction may have influenced this counter-clockwise direction of planets around the Sun. Because planets have a velocity or speed of travel, they do not keep traveling in a straight direction past the Sun, or fall into the Sun with its high value of gravity force, but the resultant sum of those two forces results is a circular-like (or elliptical) orbit of a planet about the Sun. Because the force of gravity from the Sun increases as an object or planet gets closer to the Sun, its speed or velocity will likewise increase as it gets closer to the Sun if it orbits the Sun in an elliptical orbit.

Since the Earth orbits the Sun every 365 days, the Earth orbits the Sun in about 360°/365days = 0.9863°/day, about 1°/day. At the Equator line or region of the Earth, and looking directly upward at the same time every night, all the stars will then appear to have moved from their previous position in the sky by about 1°, and is actually just slightly less than 1°

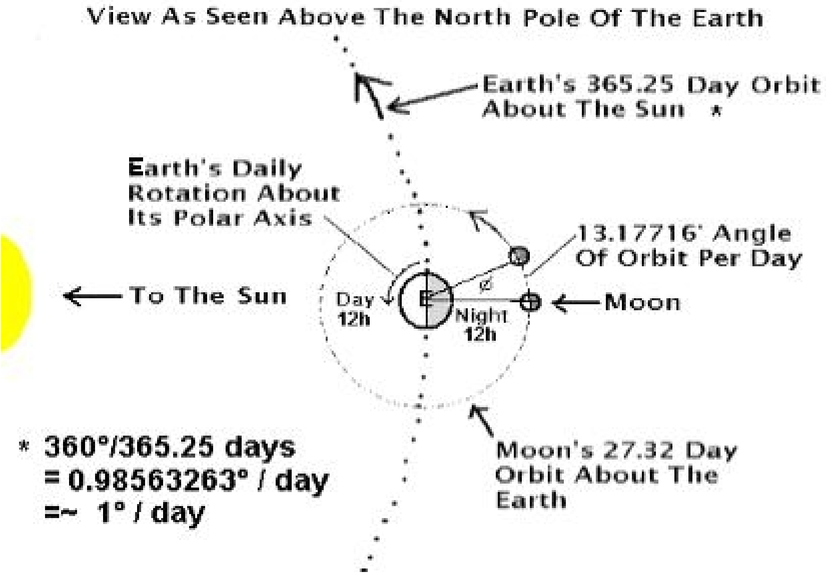
The Moon orbits (rotates about, goes around) the Earth once about every 27.32 days. A complete day of axis rotation of the Moon is also 27.32 Earth days long. During 1 year of time, the Moon will orbit the Earth (365.25 days / (27.32 days/orbit)) =~ 13.36933 times. **Because both the Moon's orbit duration time about the Earth and the Moon's own "day" length of its rotation on its axis have the same time duration, we always see the same side or hemisphere of it, and it then appears as if it does not rotate on its axis.** If you were on the Moon, the Earth would always be located at about the same location in the sky, and also seen as having shadow and light phases just as we see of the Moon from Earth. The side or hemisphere of the Moon that we do not normally see is called "the far side", and the side we always see is called the "near side". As indicated, the Moon also rotates on its axis once every 27.32 days, hence a "Moon-day" is 27.32 days long. Since the Moon rotates on its axis, the Moon has periodic day and night phases and with each phase being (27.32 Earth-days / 2) = 13.66 Earth-days units of time apart. As the Moon rotates, it will have 13.66 Earth-days of (some to complete) sunlight, and 13.66 days of (some to complete) darkness. I mentioned the word "some" because the full (lit) Moon or new (dark, unlit) Moon only lasts for a few seconds in reality as the Moon is still constantly rotating on its axis, and appears to the average person to last about 2 or three days of time length. But for example, as soon as a full Moon happens it will then soon begin to develop a thin dark crescent which will start to "wax" or cover over the Moon as the light "wanes" or is removed away. Because the Moon does not have much of any appreciable amount atmosphere gases, there isn't much, if any, of a "twilight" (brightening and-or darkening) to be seen in a theoretical (Moon) sky, but rather a more abrupt changes in the light levels. Because the Moon lacks an atmosphere, it is a great location for astronomical viewing and radio-astronomy, and the far-side is effectively shielded from most electronic noise from Earth.

For the same seen "Moon phase(s)" (ex., new moon to full lighted moon and back to new moon, etc.) as seen from on Earth, and the Sun in the same sky location as seen on the Moon (ie., a "moon-day or rotation on its polar axis) as it also orbits the Sun as part of the Earth-Moon system, takes about 29.53 Earth days of time. 29.53 days is nearly 30 days of time which is commonly spoken "month of time".

The Moon (slightly elliptical) orbits about the Earth at: 360°/ 27.32 days = (average of) 13.17716° / day = 13.17716 deg / 24hrs = 0.549048 deg / hour of movement (orbiting) about or around the Earth. Because of this motion, the Moon rises in the sky and-or is at the same position in the sky every night about 53 minutes later than the day before. In short, this is actually due to the extra time needed for the Earth to then "catch up" to the moving or advancing Moon about the Earth, and have it appear in the same location in the sky.
From above the surface of the Moon at its north pole, the Moon will be seen to revolve counter-clockwise on its axis, and also have a counter-clockwise orbit around the Earth, and this motion is exactly timed so as to eventually make the Moon appear as to not rotate on its axis, and always having the same side of it facing Earth. Setting up some proportions type of equations:

60 min / 15° = X min / 13.17716° , Xmin = about 52.71 min : the Moon appears "late" by this much every day
13.17716° / 24 hours = X° / 1 hour , X° = 0.549048° / 1 hour : how much the Moon appears to move in the sky per hour eastward or "backwards" with respect to the apparent motion of the stars.
Ex. In the 12 hours of the night, the Moon will appear to travel about:
(0.549048°/hr)(12hr) = 6.589° eastward with respect to the position of the stars.

[FIG 290D]



In the above image, the night, shaded or dark regions on the Earth and Moon is due to those areas being not illuminated by the Sun and-or effectively shaded by the other side of that body. The Moon orbits the Earth at about 0.549° / hour, hence compared to the (distant, seemingly fixed in position) stars apparently moving at 15° / hour across the sky, and the

Moon will appear to only move at (15° - 0.549°) / hour = 14.451° / hour, and it will then take about 12.45$\underline{6}$ hours for it to appear to travel from the eastern horizon to the western horizon for a total of 180° For simplicity, the Moons orbit in these basic calculations was considered as circular rather than slightly elliptical.

365.25 days = 365 days + 6 hours and 27.32 days = 27 days + 7.68 hours = 27 days + 7 hr + 40 min + 48s

The distances to the planets and stars can be estimated from the concept of what is called (visual) parallax. **Parallax** is where you observe an (hopefully , nearly fixed in position, apparently stationary) object from two locations where each location is separated by a distance, such as the stars when viewed from Earth when it is at two locations, such as say 180° about the Sun every 6 months. This maximum natural and practical astronomical distance of separation is equal to the diameter of Earth's orbit = 2 (radius) = 2 (93 million miles) = 186 million miles apart. The larger this observation distance or separation, here at 186 million miles, and the more that an object that is closer to Earth, the more that it will appear to shift (ie., parallax, perceived error from being straight or true) in reference to the further distant background objects such as other stars. Our human eyes and understanding also essentially utilizes the concepts of visual parallax so as to effectively see in "3d" and have a depth or amount of the field of view, and so as to help indicate the distances to objects we see, and especially with objects closer to us. Parallax and or motion can also be determined using two photographs taken at different times and-or when separated by a distance. You can also see the effect of parallax using a still object in front of you and closing one eye and opening the other, and then repeating this with the other eye, and that object will appear to shift in position or location. Photographic images of the stars from distance spacecraft can help determine the amount of parallax and the distance to a star(s).

As a technical note to the above image, the Moon actually goes around the Earth in the counter-clockwise direction when seen from above the solar plane, and the Earth goes about the Sun in the counter-clockwise direction. For many analysis of astronomical rotations or orbits, it is often done in a clockwise direction so as to be standardized, familiar, and-or for simplicity. For some more about astronomy, see the topic ahead in this book called Major Objects In Our Solar System.

Due to both the orbit of the Moon and Earth's length of day, the Earth is always visible from the near-side of the Moon's surface, and in fact, it is nearly always in the same location above the local horizon in the Moon's sky, and remember that the Moon is orbiting Earth which is then its "focus" of orbit. This facts helps with radio communications between people on Earth and the Moon. From the Moon's surface, Earth will appear to be larger than that of the (smaller diameter) Moon as seen from Earth's surface. The Moon appears to be about 0.5°or half a degree wide in the sky from the surface of Earth, and Earth will appear to be 4 times wider at almost 2 degrees wide in the sky as viewed from the surface of the Moon, and it would even be possible to view and-or give Earth's cloud and storm weather and forecasts from the Moon by just visual observation and-or having binoculars (essentially two small telescopes side by side, and one used for each eye). Here is a simplified calculation using right-angle trigonometry:

Tan ¢ = opposite side of angle / adjacent side of angle = 8000 mi / 240000 mi = 0.3333$\underline{3}$

¢ = ArcTan (Tan ¢) = Arctan (0.3333$\underline{3}$) = 1.909$\underline{2}$° =~ 2° visually.

[This space for edits.]

COMMON WHOLE NUMBER DIVISIONS AND-OR FRACTIONS

When the numerator is 1, the value of the faction is also called as, and equal to, the reciprocal of the denominator. Note these helpful example where reciprocals are fundamentally involved:

$$\frac{2}{3} = \frac{1}{3} + \frac{1}{3} = 2\left(\frac{1}{3}\right)$$: hence twice or 2 times the reciprocal of 3 and which is (1/3).

$$\frac{1}{20} = \frac{1}{2\,(10)} = \frac{1}{2}\left(\frac{1}{10}\right) = 0.5\,(0.1) = 0.5\,(10^{-1}) = 0.05$$

$$\frac{1}{0.5} = \frac{1\,(10^{0})}{5\,(10^{-1})} = \frac{1\,(10^{(0-(-1))})}{5} = \frac{1}{5}\,(10^{+1}) = (0.2)\,(10) = 2$$

1/1 = 1
1/2 = 0.5
1/3 = 0.333333333333333
1/4 = 0.25
1/5 = 0.2
1/6 = 0.166666666666667
1/7 = 0.142857142857143
1/8 = 0.125
1/9 = 0.111111111111111
1/10 = 0.1
1/11 = 0.0909090909090909
1/12 = 0.0833333333333333
1/13 = 0.0769230769230769
1/14 = 0.0714285714285714
1/15 = 0.0666666666666667
1/16 = 0.0625
1/17 = 0.0588235294117647
1/18 = 0.0555555555555556
1/19 = 0.0526315789473684
1/20 = 0.05
1/21 = 0.0476190476190476
1/22 = 0.0454545454545455
1/23 = 0.0434782608695652
1/24 = 0.0416666666666667
1/25 = 0.04
1/26 = 0.0384615384615385
1/27 = 0.037037037037037
1/28 = 0.0357142857142857
1/29 = 0.0344827586206897
1/30 = 0.0333333333333333
1/31 = 0.032258064516129
1/32 = 0.03125
1/33 = 0.0303030303030303
1/34 = 0.0294117647058824
1/35 = 0.0285714285714286
1/36 = 0.0277777777777778
1/37 = 0.027027027027027
1/38 = 0.0263157894736842
1/39 = 0.0256410256410256

To convert a numeric (decimal) fractional part of an inch to sixteenths of an inch, divide by (1"/16) = 0.0625".   1inch = 16, sixteenths of an inch
Ex. 0.5 in = 0.5 / 0.0625 = 8 sixteenths of an inch = (8/16)in = 8(1in/0.0625)
Ex. (3/16) in = 0.1875 in , 0.1875in / 0.0625 = 3 sixteenths of 1 inch

Given an amount of kilograms (kg), 3 digits of the (decimal) fractional part is equal the number of grams (g) extra to add to the total number of kilograms.   1kg = 1000g ,   kgs = whole amount of kg + fraction of 1kg
5.345 kg = 5kg + 0.345kg = 5kg + 345g
The 3 digits past the grams part is the number of milligrams (mg).
1g = 1000mg.   1mg = 0.001g = 1mg = 0.000001kg = 1 millionth of a kg
Ex. 5.345678kg = 5kg + 345.678g = 5kg + 345g + 678mg
Ex. 345.6g = 345g + 0.6g   = 345g + 600mg

1/40 = 0.025
1/41 = 0.024390243902439
1/42 = 0.0238095238095238
1/43 = 0.0232558139534884
1/44 = 0.0227272727272727
1/45 = 0.0222222222222222
1/46 = 0.0217391304347826
1/47 = 0.0212765957446809
1/48 = 0.0208333333333333
1/49 = 0.0204081632653061
1/50 = 0.02
1/51 = 0.0196078431372549
1/52 = 0.0192307692307692
1/53 = 0.0188679245283019
1/54 = 0.0185185185185185
1/55 = 0.0181818181818182
1/56 = 0.0178571428571429
1/57 = 0.0175438596491228
1/58 = 0.0172413793103448
1/59 = 0.0169491525423729
1/60 = 0.0166666666666667
1/61 = 0.0163934426229508
1/62 = 0.0161290322580645
1/63 = 0.0158730158730159
1/64 = 0.015625
1/65 = 0.0153846153846154
1/66 = 0.0151515151515152
1/67 = 0.0149253731343284
1/68 = 0.0147058823529412
1/69 = 0.0144927536231884
1/70 = 0.0142857142857143
1/71 = 0.0140845070422535
1/72 = 0.0138888888888889
1/73 = 0.0136986301369863
1/74 = 0.0135135135135135
1/75 = 0.0133333333333333
1/76 = 0.0131578947368421
1/77 = 0.012987012987013
1/78 = 0.0128205128205128
1/79 = 0.0126582278481013
1/80 = 0.0125
1/81 = 0.0123456790123457
1/82 = 0.0121951219512195
1/83 = 0.0120481927710843
1/84 = 0.0119047619047619
1/85 = 0.0117647058823529
1/86 = 0.0116279069767442
1/87 = 0.0114942528735632
1/88 = 0.0113636363636364
1/89 = 0.0112359550561798
1/90 = 0.0111111111111111
1/91 = 0.010989010989011
1/92 = 0.0108695652173913
1/93 = 0.010752688172043

1/94 = 0.0106382978723404
1/95 = 0.0105263157894737
1/96 = 0.0104166666666667
1/97 = 0.0103092783505155
1/98 = 0.0102040816326531
1/99 = 0.0101010101010101
1/100 = 0.01

2/1 = 2
2/2 = 1
2/3 = 0.666666666666667
2/4 = 0.5 := 1/2 as an equivalent fraction
2/5 = 0.4
2/6 = 0.333333333333333
2/7 = 0.285714285714286
2/8 = 0.25
2/9 = 0.222222222222222
2/10 = 0.2
2/11 = 0.181818181818182
2/12 = 0.166666666666667

Given an amount of meters (m), 2 digits of the (decimal) fractional part is equal to the number of centimeters. 1m = 100cm, 1cm = 0.01m
Ex. 3.45m = 3m + 0.45m = 3m + 45cm
1cm = (1/10)m = 0.01m = 10mm , 1mm = 0.1cm = 0.001m
Ex. 3.4567m = 3m + 45.67cm = 3m + 45cm + 6.7mm
3.4567m = 3m + 45cm + 6mm + 700um
1mm = 0.001m , 1um = (1/1000)mm = 0.001mm = 0.000001m
1um = (1/1000000)m = 0.000001m

2/13 = 1/13 + 1/13 = 0.0769230769230769 + 0.0769230769230769 or= 2 (1/13) = 2 (0.0769230769230769)
= 0.153846153...

2/14 = 1/14 + 1/14 = 2 (1/14) = 2 (0.071428571...) = 0.142857142...

3/1 = 3
3/2 = 1.5
3/3 = 1
3/4 = 0.75 :Note for example: 3/4 = 1/4 + 2/4 = 1/4 + 1/4 + 1/4 = 3 (1/4) = 3(0.25) = 0.75
3/5 = 0.6
3/6 = 0.5
3/7 = 0.428571428571429
3/8 = 0.375
3/9 = 0.333333333333333 : = 1 / 3 when reduced to an equivalent fraction
3/10 = 0.3
3/11 = 0.272727272727273
3/12 = 0.25 = 3 (1/12) : = 1/4 as an equivalent fraction

4/1 = 4
4/2 = 2
4/3 = 1.33333333333333
4/4 = 1
4/5 = 0.8
4/6 = 0.666666666666667 : = 2 / 3 when reduced to an equivalent fraction
4/7 = 0.571428571428571
4/8 = 0.5 : = 1/2 as an equivalent fraction
4/9 = 0.444444444444444
4/10 = 0.4
4/11 = 0.363636363636364
4/12 = 0.333333333333333

5/1 = 5
5/2 = 2.5
5/3 = 1.66666666666667
5/4 = 1.25
5/5 = 1
5/6 = 0.833333333333333
5/7 = 0.714285714285714
5/8 = 0.625
5/9 = 0.555555555555556
5/10 = 0.5
5/11 = 0.454545454545455
5/12 = 0.416666666666667

6/1 = 6
6/2 = 3
6/3 = 2
6/4 = 1.5
6/5 = 1.2
6/6 = 1
6/7 = 0.857142857142857
6/8 = 0.75
6/9 = 0.666666666666667
6/10 = 0.6
6/11 = 0.545454545454545
6/12 = 0.5

7/1 = 7
7/2 = 3.5
7/3 = 2.33333333333333
7/4 = 1.75
7/5 = 1.4
7/6 = 1.16666666666667
7/7 = 1
7/8 = 0.875
7/9 = 0.777777777777778
7/10 = 0.7
7/11 = 0.636363636363636
7/12 = 0.583333333333333

8/1 = 8
8/2 = 4
8/3 = 2.66666666666667
8/4 = 2
8/5 = 1.6
8/6 = 1.33333333333333
8/7 = 1.14285714285714
8/8 = 1
8/9 = 0.888888888888889
8/10 = 0.8
8/11 = 0.727272727272727
8/12 = 0.666666666666667

9/1 = 9
9/2 = 4.5
9/3 = 3
9/4 = 2.25
9/5 = 1.8
9/6 = 1.5
9/7 = 1.28571428571429
9/8 = 1.125
9/9 = 1
9/10 = 0.9
9/11 = 0.818181818181818
9/12 = 0.75

10/1 = 10
10/2 = 5
10/3 = 3.33333333333333
10/4 = 2.5
10/5 = 2
10/6 = 1.66666666666667
10/7 = 1.42857142857143
10/8 = 1.25
10/9 = 1.11111111111111
10/10 = 1
10/11 = 0.909090909090909
10/12 = 0.833333333333333

11/1 = 11
11/2 = 5.5
11/3 = 3.66666666666667
11/4 = 2.75
11/5 = 2.2
11/6 = 1.83333333333333
11/7 = 1.57142857142857
11/8 = 1.375
11/9 = 1.22222222222222
11/10 = 1.1
11/11 = 1
11/12 = 0.916666666666667

12/1 = 12
12/2 = 6
12/3 = 4
12/4 = 3
12/5 = 2.4
12/6 = 2
12/7 = 1.71428571428571
12/8 = 1.5
12/9 = 1.33333333333333
12/10 = 1.2
12/11 = 1.09090909090909
12/12 = 1

[This space for book edits.]

TABLE OF COMMON SQUARE ROOTS AND CUBE ROOTS

To help people consider and-or make "look up" or quick reference tables, the computer program used to created the following table of square and cube roots is given after these tables.

```
-------------------------------------------------------------------------------------------------------
COMMON SQUARE ROOTS

  1  1                    2  1.4142135623731      3  1.73205080756888
  4  2                    5  2.23606797749979     6  2.44948974278318
  7  2.64575131106459     8  2.82842712474619     9  3
 10  3.16227766016838    11  3.3166247903554     12  3.46410161513775
 13  3.60555127546399    14  3.74165738677394    15  3.87298334620742
 16  4                   17  4.12310562561766    18  4.24264068711928
 19  4.35889894354067    20  4.47213595499958    21  4.58257569495584
 22  4.69041575982343    23  4.79583152331272    24  4.89897948556636
 25  5                   26  5.09901951359278    27  5.19615242270663
 28  5.29150262212918    29  5.3851648071345     30  5.47722557505166
 31  5.56776436283002    32  5.65685424949238    33  5.74456264653803
 34  5.8309518948453     35  5.91607978309962    36  6
 37  6.08276253029822    38  6.16441400296898    39  6.2449979983984
 40  6.32455532033676    41  6.40312423743285    42  6.48074069840786
 43  6.557438524302      44  6.6332495807108     45  6.70820393249937
 46  6.78232998312527    47  6.85565460040104    48  6.92820323027551
 49  7                   50  7.07106781186548    51  7.14142842854285
 52  7.21110255092798    53  7.28010988928052    54  7.34846922834953
 55  7.41619848709566    56  7.48331477354788    57  7.54983443527075
 58  7.61577310586391    59  7.68114574786861    60  7.74596669241483
 61  7.81024967590665    62  7.87400787401181    63  7.93725393319377
 64  8                   65  8.06225774829855    66  8.12403840463596
 67  8.18535277187245    68  8.24621125123532    69  8.30662386291807
 70  8.36660026534076    71  8.42614977317636    72  8.48528137423857
 73  8.54400374531753    74  8.60232526704263    75  8.66025403784439
 76  8.71779788708135    77  8.77496438739212    78  8.83176086632785
 79  8.88819441731559    80  8.94427190999916    81  9
 82  9.05538513813742    83  9.1104335791443     84  9.16515138991168
 85  9.21954445729289    86  9.2736184954957     87  9.32737905308882
 88  9.38083151964686    89  9.4339811320566     90  9.48683298050514
 91  9.53939201416946    92  9.59166304662544    93  9.64365076099295
 94  9.69535971483266    95  9.74679434480896    96  9.79795897113271
 97  9.8488578017961     98  9.89949493661167    99  9.9498743710662
100  10

Press A Key For Cube Roots
```

```
----------------------------------------------------------------------------------------------------

COMMON CUBE ROOTS

  1  1                        2  1.25992104989487      3  1.44224957030741
  4  1.5874010519682          5  1.7099759466767       6  1.81712059283214
  7  1.91293118277239         8  2                     9  2.0800838230519
 10  2.15443469003188        11  2.22398009056932     12  2.28942848510666
 13  2.35133468772076        14  2.41014226417523     15  2.46621207433047
 16  2.51984209978975        17  2.57128159065824     18  2.6207413942089
 19  2.66840164872194        20  2.71441761659491     21  2.75892417638112
 22  2.80203933065539        23  2.84386697985157     24  2.88449914061482
 25  2.92401773821287        26  2.96249606840737     27  3
 28  3.03658897187566        29  3.07231682568585     30  3.10723250595386
 31  3.14138065239139        32  3.1748021039364      33  3.20753432999583
 34  3.23961180127748        35  3.27106631018859     36  3.30192724889463
 37  3.33222185164595        38  3.36197540679896     39  3.39121144301417
 40  3.41995189335339        41  3.44821724038273     42  3.47602664488645
 43  3.50339806038672        44  3.53034833532606     45  3.55689330449006
 46  3.58304787101595        47  3.60882608013869     48  3.63424118566428
 49  3.65930571002297        50  3.68403149864039     51  3.70842976926619
 52  3.73251115681725        53  3.75628575422107     54  3.77976314968462
 55  3.80295246076139        56  3.82586236554478     57  3.8485011312768
 58  3.8708766406278         59  3.89299641587326     60  3.91486764116886
 61  3.93649718310217        62  3.95789160968041     63  3.97905720789639
 64  4                        65  4.02072575858906    66  4.04124002062219
 67  4.06154810044568        68  4.08165510191735     69  4.10156592970235
 70  4.12128529980856        71  4.14081774942285     72  4.16016764610381
 73  4.17933919638123        74  4.19833645380841     75  4.21716332650875
 76  4.23582358425489        77  4.25432086511501     78  4.27265868169792
 79  4.29084042702621        80  4.30886938006377     81  4.32674871092222
 82  4.34448148576861        83  4.36207067145484     84  4.37951913988789
 85  4.39682967215818        86  4.4140049624421      87  4.43104762169363
 88  4.44796018113863        89  4.46474509558454     90  4.48140474655716
 91  4.49794144527541        92  4.514357435474       93  4.53065489608349
 94  4.54683594377634        95  4.56290263538697     96  4.57885697021333
 97  4.59470089220704        98  4.61043629205845     99  4.62606500918274
100  4.64158883361278

Press A Key To Exit Program
```

Extra radicands and their (integer) cube roots:

125 5 : 5^3 = 125 , $\sqrt[3]{125} = 5$, 5 x 5 x 5 = 125
216 6 : 6^3 = 216 , $\sqrt[3]{216} = 6$, 6 x 6 x 6 = 216
343 7 : 7^3 = 343 , $\sqrt[3]{343} = 7$, 7 x 7 x 7 = 343
512 8
729 9
1000 10

A COMPUTER PROGRAM TO CREATE A TABLE OF SQUARE AND CUBE ROOTS

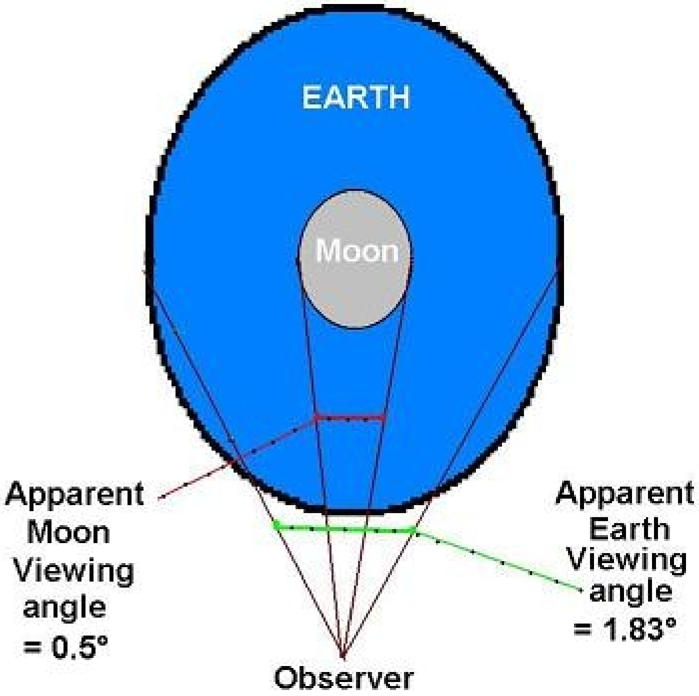
```
/*------------------------------------------------------------------------------
  CommonSquareAndCubeRoots.c

     This program creates a list of common Square and Cube Roots.

     This will display the square and cube roots of the integer numbers from 1 to 100.
     Also shown is some optional formatting control for the displayed values, such as
     using the field width before the data type, and also using the - format specifier
     operator to left justify the values in the desired character width space.

    (c). JPA, Sept.. 28, 2022

------------------------------------------------------------------------------*/

#include "stdio.h"       /* # signifies a compiler directive, here to include the header file for prototypes, etc of
                            for the functions() associated with that header file . */
#include "math.h"
/*-----------------------------------------------------------------------------*/
void main(void)
{
double n=1.0;
int t=1;

printf("\n--------------------------------------------------------------------");
printf("\nCOMMON SQUARE ROOTS");
printf("\n\n");

for(;;){
          printf("%3d %-20.15g ",(int)n, sqrt(n) );   /* sqrt(n) or pow(n,0.5)
          /* sqrt() is the square-root math function,  pow() is the power, math function */
          /* basic format specifier format used above:
             %[-left justify in the total width][total character width]%[.total number of decimal digits]
          */

          t=t+1;   /* the number of enteries on each line */
          if(t>=4){  t=1;  printf("\n");  };

         n=n+1.0;
         if(n>=101.0){ break; };
      };

printf("\n\nPress A Key For Cube Roots");  fflush(stdin);  getch();
printf("\n\n-----------------------------------------------------------------");


printf("\n\nCOMMON CUBE ROOTS            ");
printf("\n\n");

n=1.0;
t=1;
```

```
for(;;){
            printf("%3d %-20.15g ", (int)n , pow(n,(1.0/3.0)) );
            /* basic format specifier format used above:
                %[-left justify in the total width][total character width]%[.total number of decimal digits]
            */

            t=t+1;   /* the number of enteries on each line */
            if(t>=4){  t=1;  printf("\n");  };

           n=n+1.0;
           if(n>=101.0){ break; };
       };

printf("\n\nPress A Key To Exit Program");  fflush(stdin);  getch();

return;
};
/*-----------------------------------------------------------------------------------*/
```

RELATIVE AND APPARENT SIZES OF THE EARTH AND MOON

The figure below shows the relative, apparent sizes of the Earth and Moon as viewed from the surface of each other.

[FIG 291]



Surely, the size of the Earth appears much larger from the surface of the Moon than the Moon appears from the surface of Earth simply due to their natural diameters, and at the same distance. It is very possible to give Earth's weather forecasts from the surface of the Moon, and possibly see a large city using binoculars or a telescope.

The diameter of the Moon is about 2160 mile, hence roughly 2000 miles wide. The diameter of the Earth is about 7918 miles, hence roughly 8000 miles wide, and this is about 4 times wider than the Moon. The distance to the Moon is about 240000 miles. Using basic trigonometry, the apparent visible angle of the Moon from the surface of Earth is about 0.5 degrees, and the apparent visible angle of Earth from the surface of the Moon is about 1.83 degrees. This apparent viewing angle is directly proportional to both the size or diameter of each, and the distance between each.

$$\tan \text{¢} = \frac{\text{side opposite to the angle}}{\text{side adjacent to the angle}} = \frac{\text{diameter of object}}{\text{distance to object}}$$

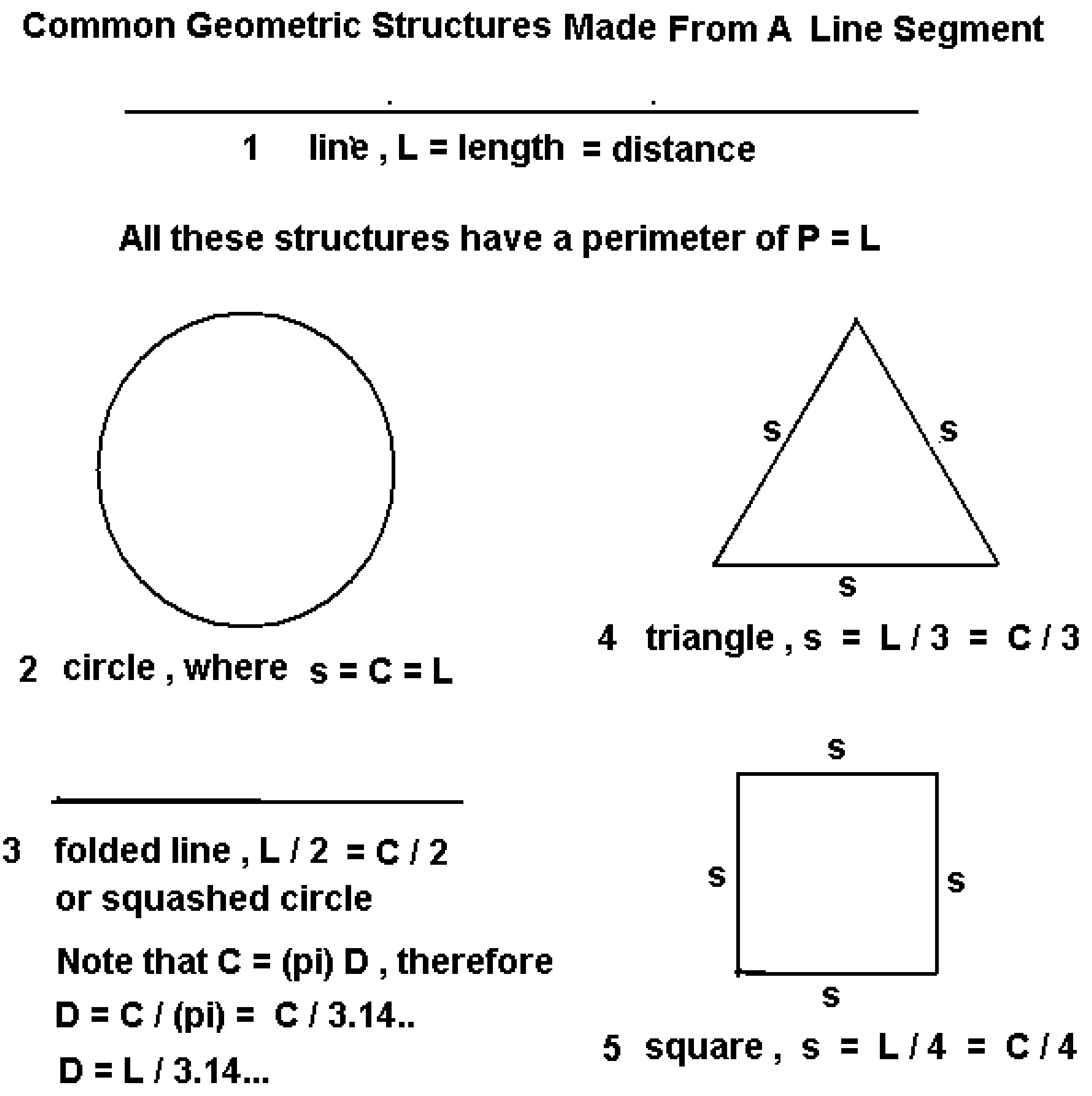
$$\frac{\text{Diameter of the Earth}}{\text{Diameter of the Moon}} = \frac{7918\text{ mi}}{2160\text{ mi}} = \mathbf{3.66}5\underline{7} = \frac{\text{Earth's apparent angle from the Moon}}{\text{Moon's apparent angle from the Earth}} = \frac{1.83^\circ}{0.5^\circ} = \mathbf{3.66}$$

In some photos the astronauts took from the surface of the Moon, the Earth is visible in some of those photos, however, due to that the camera they used had a wide angle lens, distant objects with a relatively narrow image width angle are much smaller in the image than they appear visually, and objects close to the lens are relatively large with a wide image width angle on the image.

COMMON GEOMETRIC STRUCTURES MADE FROM A LINE SEGMENT

Part of all things math or science is learning at least some of the vast knowledge already available, and then applying it to make some knew discoveries and-or ideas that can be shared with others in a cooperative effort.

[FIG 292]



In the above figures, s = a side length   and   L = total perimeter length

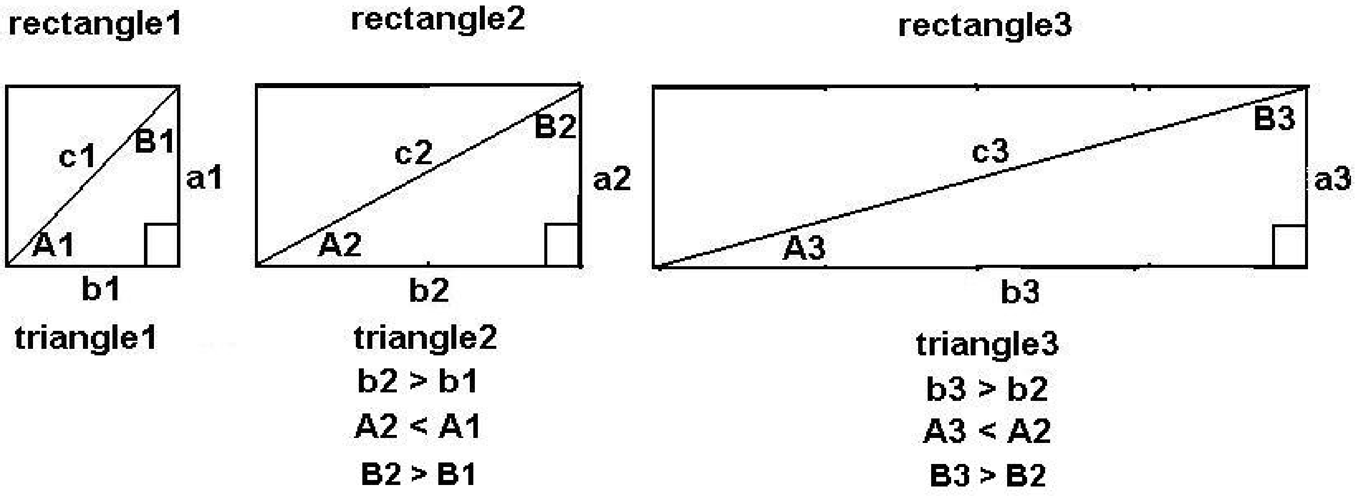
A Right Triangle Study And The Angles Created When One Of Its Leg Sides Is Extended

In the figure below, the right triangle on the left side will have side (b) extended in length, and while the side opposite the angle, here side (a), remains the same length. The result is that angle A will get smaller in value as side (b) is extended.

The rectangles are shown if you want to consider that structure and the angles created by a diagonal line, which here is the hypotenuse of the two symmetrical and identical right triangles.

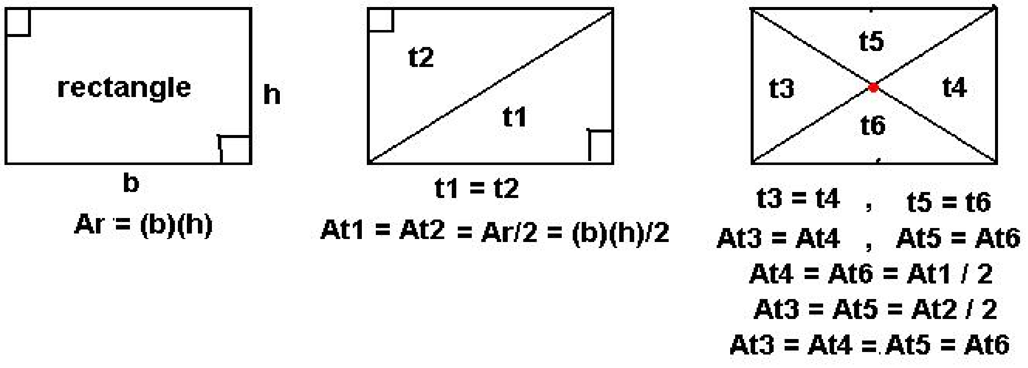
Consider tan A = side opposite A / side adjacent A = a / b , and as (b) increases, tan A ratio value decreases and therefore, A = arctan (tanA) will decrease in value.

[FIG 293]



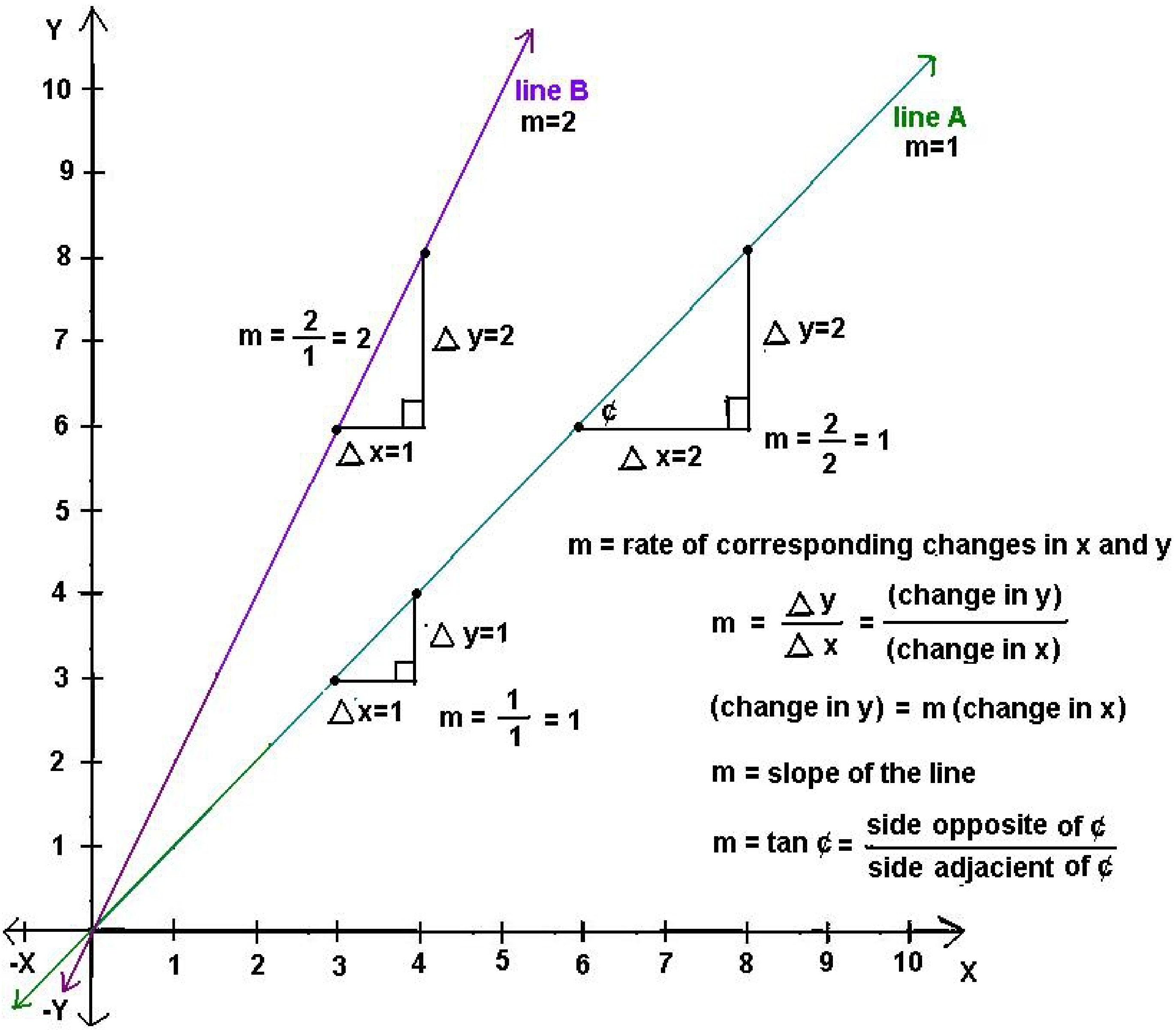
Note for example, that for angle A2 to be equal to angle A1, that side a2 would have to increase by the same ratio or factor (n) that side b1 did, and that value is: ratio = n = b2 / b1. Doing this would also create a similar triangle with all three angles being the same value, and the ratio of corresponding sides between each triangle would be the same value, here (n), and it would be said the triangles and-or its sides are proportional to each other, and are essentially, magnified versions of each other. Also note how the A and B angles change, and so as to maintain the 180° internal angular constant sum of a triangle.

Given a rectangle, drawing one internal diagonal will create two identical (right) triangles of equal area (A), and drawing its two internal diagonals of equal length will crate two pair of identical triangles. These 4 internal triangles will each have the same area. The two diagonal lines intersect at the center point of the rectangle, and which is at half the length of each side and diagonal line of the rectangle. [FIG 293A]



A Graph To Help Understand Linear Equations And Slope

[FIG 294]



Between any two points on a line, the slope of the line between those points is a constant value of (m). Note that when (x) changes by a value, that (y) does not necessarily change by that same value unless when (m) does actually equal 1. As shown in the above graph, when (x) changes by a value, (y) does not always change by that same value, and (y) will actually change by the product of the (m) times the (change in x), hence the change in (y) is like a magnified or amplified value of the (change in x). What can be said is that for any point on a line, when (x) changes by 1, that (y) will change by (m) which is the slope value.

For some simplicity of understanding, these lines intercept the (y) axis at 0, hence (b) the y-axis intercept for the line, and-or starting offset of the (y) variable is 0. The basic format for a linear equation with (b) set to 0 is: y = mx + b = mx + 0 = mx. The equation for Line A is: y = mx = 1x. The equation for Line B is: y = mx = 2x.

For a curve such as for a parabola equation (ex: y = x^2), the slope between any two points is constantly changing slightly in value from point to point. If the points are relatively close in value, the curve between those two points can be approximated by a line segment between those two points, and its slope will be the average of the two close in value slopes of the curve at those points. The greater the number of line segment approximations of a curve, the more they can represent the actual curve and its analysis.

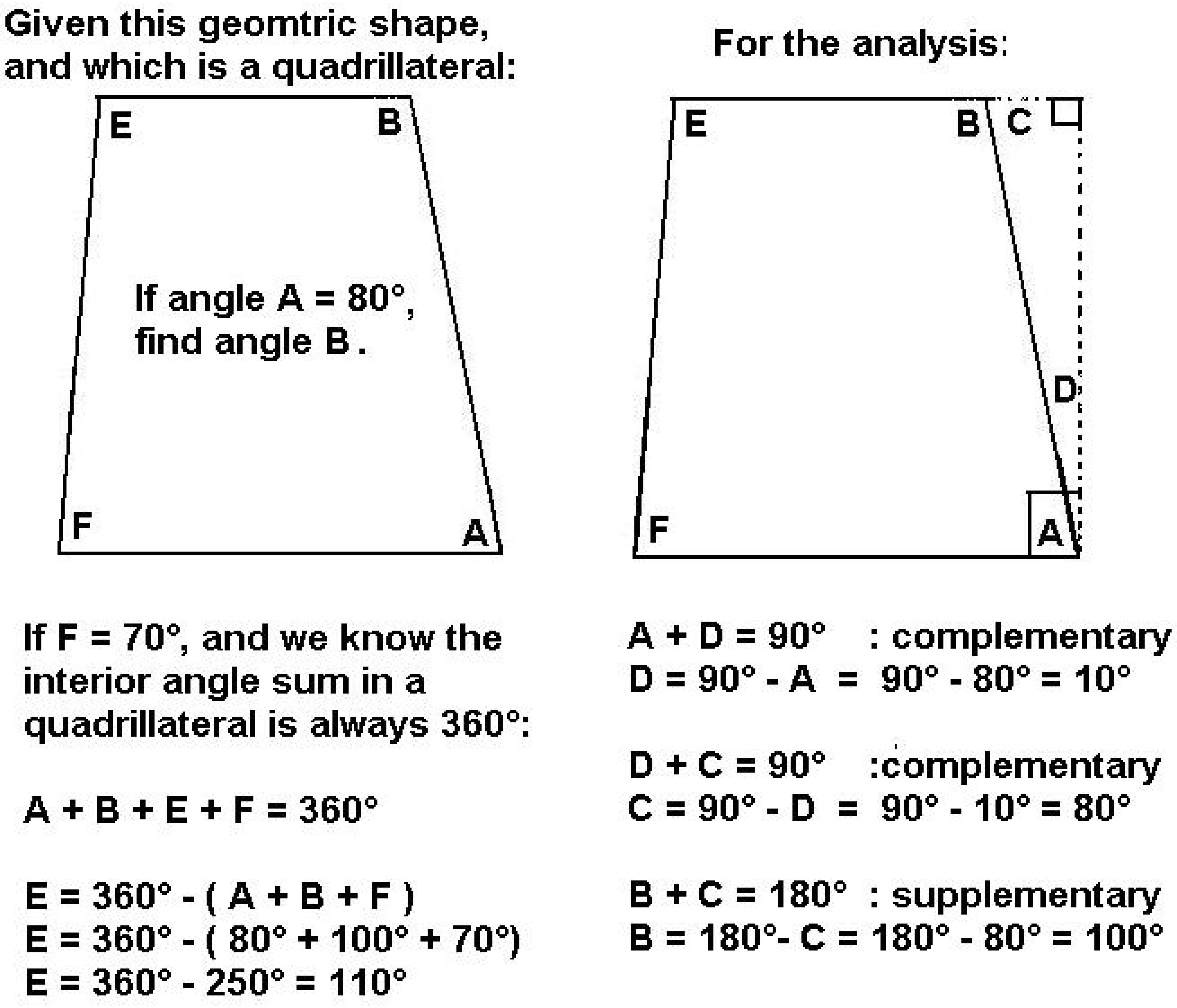
It was mentioned above that:

(change in y) = m (change in x)

For a line where the slope = rate of corresponding change in (y) due to a change in (x) = m is constant, if the (change in x) is a certain value such as 1 unit, the (change in y) will be a constant certain value depending on the value of its slope. Likewise, for a curve, where the slope = m is not constant but changing in value between successive points on it, the corresponding (change in y) value is no longer constant. The slope of a line is equal to the numerical co-efficient of the variable x and of which is only raised to the first power for lines. The slope value of a line or curve is also the (first) derivative of the equation for that curve. For a line, this derivative is a constant numerical value with no (x) variable in it. For a curve such as the parabolic curve: y = x^2 where the slope is not a constant value, the (first) derivative of its equation is not a constant value but an expression with the (x) variable within it. For the equation: y = x^2 , the first derivative of it is: x^1 = x , and this is not a constant value but depends on the value of (x) on or along that curve, and in fact, here, it is equal to the value of (x). The derivative of a parabolic curve is a linear expression and-or equation.

An Example Of Finding An Angle Of A Geometric Structure

[FIG 295]



Given this geomtric shape, and which is a quadrillateral:

If angle A = 80°, find angle B .

If F = 70°, and we know the interior angle sum in a quadrillateral is always 360°:

$A + B + E + F = 360°$

$E = 360° - (A + B + F)$
$E = 360° - (80° + 100° + 70°)$
$E = 360° - 250° = 110°$

For the analysis:

$A + D = 90°$: complementary
$D = 90° - A = 90° - 80° = 10°$

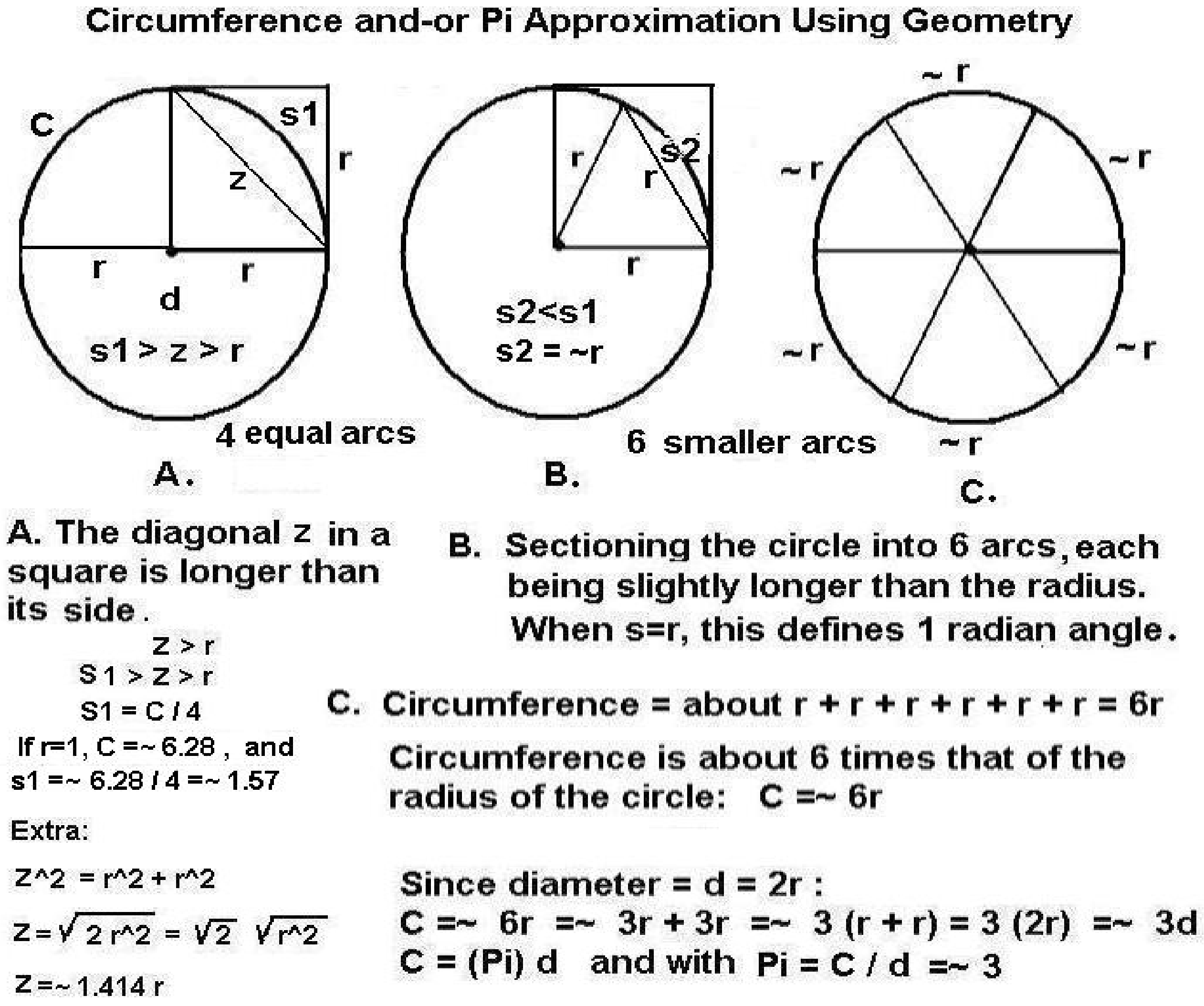
$D + C = 90°$:complementary
$C = 90° - D = 90° - 10° = 80°$

$B + C = 180°$: supplementary
$B = 180° - C = 180° - 80° = 100°$

Note the top and bottom lines of the quadrilateral, and that these lines happen to be parallel, and therefore the side or line on the right side of the construction is a diagonal line that transverses those parallel lines. Due to this, the alternate interior angles at the intersection of the transversal line are equal in value: C = A = 80° Angle D shown is not the full alternate interior angle to angle B, but is only part of it. For D to have been an alternate interior angle, either its initial or terminal side has to be at the parallel line, and not the right triangle shown that is used during the analysis of this example.

Circumference and-or Pi Approximation Using Geometry

[FIG 295A]



Circumference and-or Pi Approximation Using Geometry

A. The diagonal z in a square is longer than its side.

$z > r$

$S1 > z > r$

$S1 = C / 4$

If $r=1$, $C =\sim 6.28$, and $s1 =\sim 6.28 / 4 =\sim 1.57$

Extra:

$z^2 = r^2 + r^2$

$z = \sqrt{2 r^2} = \sqrt{2}\ \sqrt{r^2}$

$z =\sim 1.414\, r$

B. Sectioning the circle into 6 arcs, each being slightly longer than the radius. When s=r, this defines 1 radian angle.

C. Circumference = about r + r + r + r + r + r = 6r

Circumference is about 6 times that of the radius of the circle: C =~ 6r

Since diameter = d = 2r :
C =~ 6r =~ 3r + 3r =~ 3 (r + r) = 3 (2r) =~ 3d
C = (Pi) d and with Pi = C / d =~ 3

A. Shows that the diagonal, and here a chord beneath the arc segment, in a square is longer than its sides, and here. the arc segment of the circle in the square is actually longer than the straight line diagonal distance, and a straight line is always the shortest distance between two points, and any other way is therefore always longer B. shows the sectioning of the circle into 6 arcs of equal length with each equal to the radius of the circle. C. By observation, the circumference of the circle is about: r + r + r + r + r + r = 6r. We see that the circumference is at least 6 times the length of the radius length, but is also slightly greater since each similar arc segment or length (s) is slightly greater than (r). Since diameter = d = 2r, C=~ 6r =~ 3r + 3r =2(r + r) = 3(2r) =~ 3d. The circumference is about 3 times the length of the diameter length. C=(pi) d, and with (pi) =~3 and C / d = (pi). (pi) = C / 2r = 0.5 (C / r) What else can we learn from all this?

Since there are 360° in a circle, its circumference or full arc length (St = C) corresponds to 360°. 1° will correspond to an arc length of C / 360:

An arc of: C is to 360° as is an arc of: (C/360) is to (360°/360) = 1°. ,
In equivalent fraction expression after dividing both numerator and denominator by 360:

$$\frac{C}{360^\circ} = \frac{\frac{C}{360}}{\frac{360^\circ}{360}} = \frac{(C / 360)\text{ length or arc length}}{1^\circ}$$ is to : arc length of a given circle that corresponds to a central angle of 1°

If there are 6 arcs of equal length, then each arc (s) corresponds to:

$\frac{360^\circ}{6\text{ arcs}}$ = 60° / 1 arc = 60° per arc : after dividing the initial numerator and denominator by 6

Today, we know that (Pi) is very close in value to 3.0, but is actually about 3.14159265, hence:

C = (Pi)d = (pi)(2r) = 2(pi)(r) = 2(3.14159265)(r) = 6.283185307 r

C / r = 6.283185307 : the radius length can go about the circle 6.283185307... = 2(Pi) times , and the diameter length can go about the circle 3.14159265... = (Pi) times , C / d = (Pi)

C / 6.283185307 = C / 2(pi) = r , and the angle that correspond to this is:

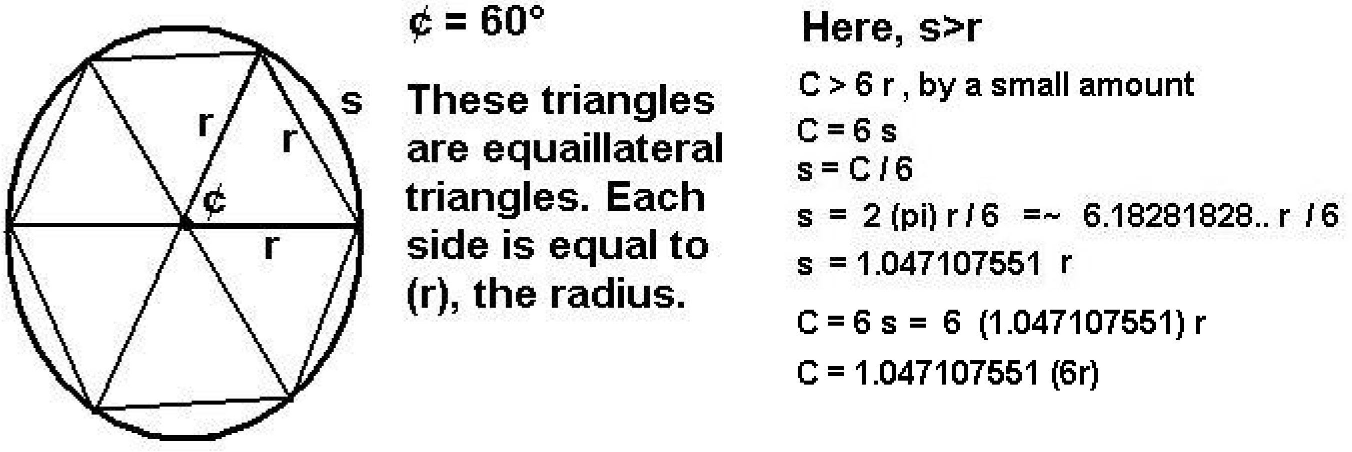
360° / 6.283185307 = ¢r = the angle where s=r , and this is an angle equal to 57.29577951° and is called "one radian (unit or angle)", and 60° is a close approximation of this transcendental value, and since (Pi), a transcendental value, was used during its calculation.

If you want to draw a 57.29577951° angle , you can consider a right triangle with the side adjacent to it as equal to r=1.0, and that the tangent of a 57.29577951° angle is: 1.557407725, hence the opposite side will be 1.557407725 times longer.

$$\tan 57.29577951^\circ = \frac{\text{side opposite the angle}}{\text{side adjacent to the angle}} = \frac{a}{b} = \frac{\text{corresponding height at side b where r=1}}{\text{a corresponding base value of r=1}} = 1.557407725$$

Mathematically: (height or side opposite the ¢) = (tan ¢) (side adjacent ¢) = 1.577407725 r : for this example

This figure shows both how to divide the circumference up into 6 equal arcs using equilateral triangles with each side equal to (r), and that this arc length is actually slightly greater than (r), and therefore it actually creates an angle slightly greater than 1 radian. Dividing 360° into 6 equivalent angles, each angle is: 360° / 6 = 60° [Fig 296]



¢ = 60°

These triangles are equaillateral triangles. Each side is equal to (r), the radius.

Here, s>r

C > 6 r , by a small amount

C = 6 s

s = C / 6

s = 2 (pi) r / 6 =~ 6.18281828.. r / 6

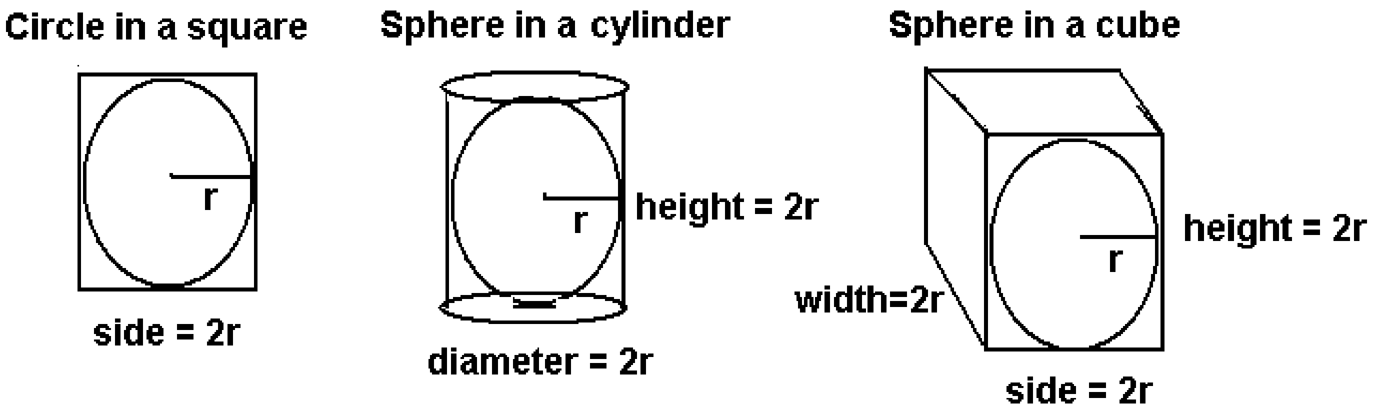
s = 1.047107551 r

C = 6 s = 6 (1.047107551) r

C = 1.047107551 (6r)

A GEOMETRIC STUDY FEATURING THE SQUARE, CUBE, CIRCLE AND SPHERE

This study features the sphere and its radius (r) in reference to some other basic geometric structures. [FIG 297]



Clearly as drawn, the circle within the square has less area than the surrounding square. The sphere within the cylinder has less volume than the surrounding cylinder. The sphere within the cube has less volume than the surrounding cube.

Area of square = As = side^2 = (s)(s) = s^2 = (2r)(2r) = (2r)^2 = 4r^2 , If we let (2r)=d, we have: As = d^2 = s^2

Area of the circle = Ac = (pi) r^2 = 3.14... r^2
If we let r = (d/2), we have: Ac = (pi)(d/2)^2 = (pi) d^2 / 4 = ((pi) / 4) d^2 = 0.785398163 d^2 = 0.785398163 As

Area equations have a squared (ie., second power) variable in them, whereas volume equations have a cubed (ie., third power) variable in them.

Ac < As and **Ac / As** = (pi) r^2 / 4 r^2 = 3.14 r^2 / 4 r^2 = **Pi / 4** = **0.785398163** , a constant
Ac = 0.785398163 As

The difference of area between a square and its internal circle is, and letting radius = r = diameter / 2 = (d/2) , d=2r:

As - Ac = d^2 - (pi) r^2 = d^2 - (pi) (d/2)^2 = d^2 - (pi) d^2 / 4 = d^2 (1 - (pi) / 4) = 0.214601836 d^2 =
= 0.214601836 d^2 = 0.214601836 (area of the square) =~ 21.5% (area of the square)

Extra: As - Ac = s^2 - (pi) r^2 = (2r)^2 - (pi) r^2 = 4r^2 - (pi) r^2 = r^2 (4 - (pi)) = 0.8584074 r^2
Also: (0.8584074 / 0.214601836) = 4
Also: 0.214601836 d^2 = 0.214601836 (2r)^2 = 0.214601836 (4 r^2) = 0.8584074 r^2

Note that for the sphere in the cylinder having the same height (here, 2r) as shown in the figure above, that the lateral or side area (Acl) of the cylinder and surface area of the sphere (As) have the same surface area: **The surface area of a sphere (As) is 4 times its largest disk area which has an area of a circle: (pi) (r^2). As = 4 (pi) r^2.** As a verification to these facts, this cylinder's lateral area can be divided into 4 identical triangular areas, At, and with each having a base of [(2)(pi)(r) = C], and a height of (r). At = (base) (height) / 2 = 2 (pi) r^2 / 2 = (pi) r^2 , and 4 of these areas sum to 4 (pi) r^2

Alc = As , Alc = (base) (height) = ((2)(pi)(r)) (2r) **= 4 (pi) r^2** = As : surface area of a square

Another consideration or verification is that if you cut a sphere along its height (2R=D) and unfold it and flatten it, the resulting shape would be a rectangle with a base of (C=2 Pi R) and a height of (2R), and this rectangle will have an area of: (base)(height) = (2 Pi R) (2R) = (C)(2R) = **(C)(D) = 4 (Pi) R^2**. **If you were to create an infinite number of very thin rectangles around the side of the sphere, where each rectangle width approaches 0, this the would also be the result as**

just shown. If the surface of a cylinder having a height of D=2R, is cut along one side, and laid flat, it would have the same surface area of a rectangle of: (width)(height) = (2 Pi R)(2R) = 4 Pi R^2, hence having the same surface area of a sphere with the same radius.

Surface area of a cube = (6 sides) (side or "face" area / side) = 6 (s)(s) = 6 s^2 : using d for distance or length
Surface area of a cube = 6 (A of a square side) = 6 s^2 side (s) of a square or cube

Volume of cylinder = Vcylinder = (Area of base)(height) = (Ac)(2r) = ((pi) r^2) (2r) = 2 (pi) r^3 =~ 6.2818 r^3
Even though the base and sides of a cylinder are circular, its base area value can be conceived as being that of a square and its height can be conceived as the height along a cube, and this is essentially shown in the equation.
Let's set the base area of a cylinder equal to that of a square: Ac = (pi)(r^2) = As = s^2 , then, s = sqrt(Ac)
Cylinder height can also be replaced as the distance of the cylinder, such as if it was a pipe length.

Volume of sphere = Vs = $\frac{4\,(\mathrm{pi})\,r^3}{3}$ =~ 1.33333... (pi) r^3 =~ $4.1887\underline{9}$ r^3 = roughly 4.2 r^3

Note the previous article and that: $4.1887\underline{9}$ = (pi) + (2pi) / 6 = (pi) + pi/3 = 4 (pi) / 3 = 3.141592654 + 1.047197551 = 4.188790205

Vs < Vcyl and Vs / Vcyl = 4.188790205 r^3 / 6.283185307 r^3 = 0.6666666 ... = 2/3 and Vcyl / Vs = 3/2 = 1.5

Volume of cube = V**c** = (Area of square base)(Height of cube) = (s^2)(s) = **s^3** , If s=2r, Vc = (2r)^2 (2r) = (2r)^3 = 8r^3

Volume of sphere = Vs = $4.1887\underline{9}$ r^3

Vs < Vc and Vc / Vs = $4.1887\underline{9}$ r^3 / 8 r^3 =~ $0.523\underline{6}$: where r = s/2 and=or s = 2r

Calculating the volume and-or equation of a sphere is not easy as other geometrical objects, and this is due to that the sides of the sphere are not linear or straight, but are curved. The most common derivation for the volume of a sphere is found by a large sum of volumes of a (infinitely) thin cross sectional disks of the sphere, and where each disk has an area of (pi)r^2 and a very small thickness which is the same value for each disk. Volume of each disk = (the area of each disk)(thickness of each disk). These disk volumes are all effectively summed using integration, and the reader can explore this elsewhere and further if need be. The most common derivation of the formula for the surface area of a sphere is somewhat similar.

With an experiment, a sphere can be placed within a cylinder or cube of the same height (here, = 2r), and then a fluid such as water can be placed in the region surrounding the sphere. The volume of the sphere will be equal to the total empty volume of the cylinder or cube, less (ie., subtract) the amount of water in the surrounding region when the sphere is in it.

Vcontainer = Vobject + Vsurrounding_region , mathematically:
Vcontainer = Vsphere + Vsurrounding_region , mathematically:
Vsphere = Vcontainer - Vsurrounding_region

If the sphere radius r=1, a "unit sphere", and if the container is a cylinder as shown in the above figure:

V**s**urrounding_**r**egion = V**sr** = Vcylinder - Vsphere , Let us now find and express these values and mathematical relationships expressed as ratios:

V**sr** = 2(pi)r^3 - $\frac{4(\mathrm{pi})(r^3)}{3}$ If r=1:

V**sr** = 2(pi) - $\frac{4(\mathrm{pi})}{3}$ = $\frac{6(\mathrm{pi})\ -\ 4(\mathrm{pi})}{3}$ = $\frac{2(\mathrm{pi})}{3}$ = 0.6667 (pi) =~ 2.0943951

$$\frac{V\mathbf{sr}}{V\text{cylinder}} = \frac{(2(\text{pi})r^3)/3}{2(\text{pi})r^3} = \frac{1}{3} = 0.333333\ldots$$, mathematically: V**sr** = (1/3) Vcylinder

$$\frac{V\mathbf{sr}}{V\text{sphere}} = \frac{(2(\text{pi})r^3)/3}{(4(\text{pi})r^3)/3} = 0.5$$, and the reciprocal of this is:

$$\frac{V\text{sphere}}{V\text{sr}} = \frac{(2/3)\,V\text{cylinder}}{(1/3)\,V\text{cylinder}} = (2/3)(3/1) = 6/3 = 2$$

$$\frac{V\text{sphere}}{V\text{cylinder}} = \frac{(4(\text{pi})r^3)/3}{2(\text{pi})r^3} = \frac{2}{3} = 0.6666\underline{7} =^{\sim} 66.7\%$$, mathematically: Vs = (2/3) Vcylinder , the reciprocal is:

$$\frac{V\text{cylinder}}{V\text{sphere}} = \frac{3}{2} = 1.5$$

Values for the particular example where r=1:

V**sr** = Vcylinder - Vsphere

V**sr** = 6.2818 units^3 - $4.1887\underline{9}$ units^3 =~ 2.09301 : as shown above

Below is a similar analysis using the volume of a cube:

Note that the volume of a cube does not have (pi) variable in its equation.

$$\frac{V\text{sphere}}{V\text{cube}} = \frac{(4(\text{pi})r^3)/3}{8r^3} = \frac{(\text{pi})}{6} = 0.523598775$$: slightly more than half, ~ 52.4% and the reciprocal of this is:

$$\frac{V\text{cube}}{V\text{sphere}} = \frac{8r^3}{(4(\text{pi})r^3/3)} = \frac{6}{(\text{pi})} = 1.909859317$$: slightly less than 2

$$\frac{V\text{cube}}{V\text{cylinder}} = \frac{8r^3}{2(\text{pi})r^3} = \frac{4}{(\text{pi})} = 1.273239545$$, and the reciprocal of this is:

$$\frac{V\text{cylinder}}{V\text{cube}} = \frac{(\text{pi})}{4} = 0.785398163$$: about 78.5%

The volume of a cone or pyramid has been shown in this book to be equal to:

$$V\text{cone or }V\text{pyramid} = \frac{(\text{base area})(\text{height})}{3}$$ and that:

$$V\text{cone or }V\text{pyramid} = \frac{V\text{cube}}{6}$$: **when the height is equal to half the height of the cube, hence (2r)/2 = r**

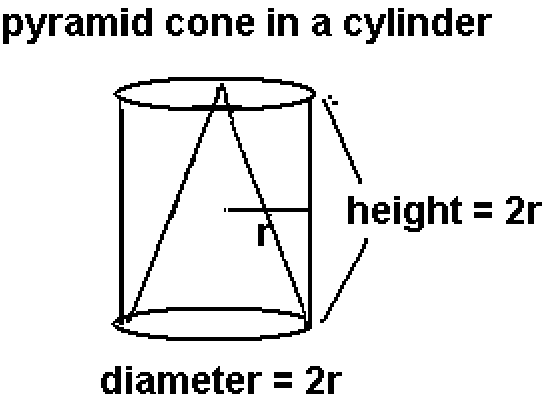
$$V\text{cone or }V\text{pyramid} = \frac{V\text{cube}}{6} = (1/6)\,V\text{cube} =^{\sim} 0.166\underline{7}\,V\text{cube} =^{\sim} 16.7\%$$

Checking: If we multiply the volume of a pyramid or cone by 6, we should then have the whole or entire cube volume with a side equal to (2r), and Vcube = (side)^3 = (2r)^3 = 8 r^3:

(6) $\text{Vpyramid} = \frac{(2r)^2\,(r)}{3}(6) = \frac{4r^3\,(6)}{3} = \frac{24\,r^3}{3} = 8\,r^3 = \text{Vcube}$ **: when height = (r)**

The sum of volumes of a cone and sphere with the same radius (r) and therefore total height of (2r), will equal a cylinder with the same radius and height.

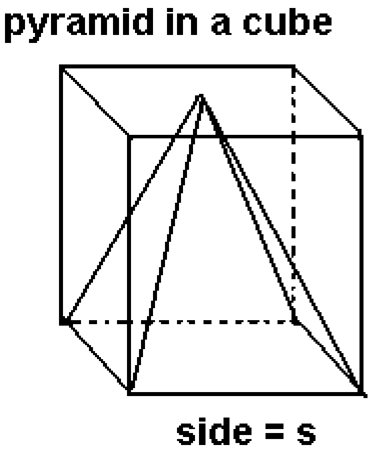
Now a pyramid cone in a cylinder will be discussed. A cone can be thought of as a pyramid with an infinite number of flat sides. [FIG 298]



$\frac{\text{Vcylinder}}{\text{Vpyramid}} = \frac{(\text{pi})(r^2)\,(2r)}{(\text{pi})(r^2)(2r)/3} = \frac{2(\text{pi})r^3}{2(\text{pi})r^3/3} = 3$: when height = (**2r**) typically, or if simply: h = r. Also note that the reciprocal of this is:

$\frac{\text{Vpyramid}}{\text{Vcylinder}} = \frac{1}{3} = 0.3333... =^{\sim} 33.33\%$: extra, the **volume of a paraboloid** is 50% that of the volume of its circumscribing cylinder, hence the difference is 50% - 33.33% = 16.6$\underline{6}$% = 0.166$\underline{7}$ / 1.00 = after division: 1.0 / 6 of the cone volume. See further below also.

If (1/3) of a cylinder is occupied by a cone, then: 1 - (1/3) = (3/3) - (1/3) = (2/3) of the cylinder is not occupied by the cone. [FIG 299]



$$\frac{\text{Vpyramid}}{\text{Vcube}} = \frac{(s^2)(s) / 3}{s^3} = \frac{s^3 / 3}{s^3} = \frac{1}{3} = 0.333333.... =^{\sim} 33.33\%$$

, and the reciprocal of this is:
A cylinder can be viewed as being an infinitely sided pyramid.

$$\frac{\text{Vcube}}{\text{Vpyramid}} = 3$$

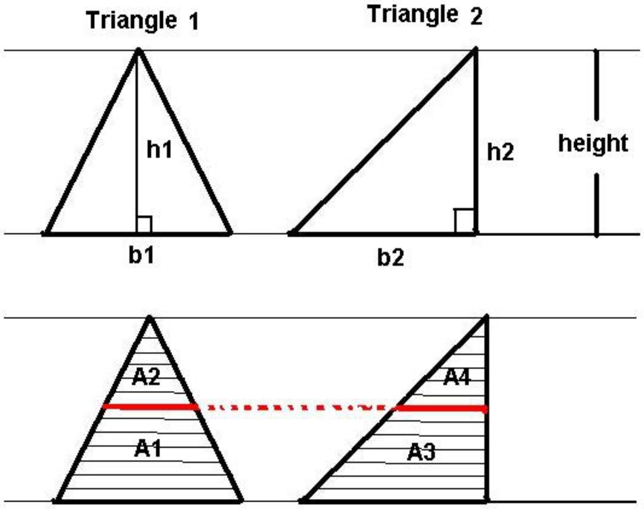
Given many various triangles, if the base and height are the same value, their areas will all be the same. Likewise, given a cone or pyramid in a cylinder, prism or cube (a special instance of a prism, rectangular solid shape), as long as the height is the same, the volume of the pyramid or cone is the same. For example, the tip of the cone or pyramid may be at one of the upper corners of the cylinder or prism.

Extra: Since **Vs** = 4/3 (pi) r^3 = 1.33333 (pi) r^3 = **4.1887$\underline{9}$ r^3** **: volume of a sphere given its radius**
: 4.188790205 r^3

r = cube-root (Vs / 4.18879) =
r = cube-root Vs / cube root 4.18879 = (cube-root Vs) / 1.611991954 =
r = 0.62035050491 (cube-root Vs) : radius of a sphere given its volume

Ex. A volume of 1 liter = 1000 mL = 1000 cc = 1000 cm^3 = (10 cm)^3 = (10 cm)(10 cm)(10 cm) = 1000 cm^3
Now considering a sphere having this same volume as this cube shape:
r = 0.620350501 (cube-root 1000 cm^3) = 0.620350501 (cube root 1000)(cube root 1cm^3)
r = 6.203$\underline{5}$ cm , d = 2r = 12.407 cm , note that this is greater than 10cm which was the base of the equivalent cube shaped volume and is 12.407 cm / 10 cm =
Ds = diameter of a sphere = 1.2407 times wider than a cube of the same volume
To imagine how this can be, imagine a cube shape of clay, and if you push the 8 corners and edges inward so as to make a sphere shape, that material will be added to the existing central volume of material, hence making it slightly wider than the cube was initially.

Extra: **Cavalieri's principle** states that given two volumes of any shape, if at the same height or distance from the base side and the cross sectional planes of each have the same area, then their volumes are the same. For example: Consider a vertical cylinder shape and then dividing it into many slices like a stack of disks or coins. Now consider a second instance of this construction and-or volume where the disks are then randomly offset from the center line or axis of the cylinder, and that second construction will still have the same volume as the first cylinder. From this we can also reason that at any particular height or corresponding segment of these two volumes, that those volumes are also the same. The basic formula for such volumes is the base area (Ab) times the height (h) = V = (Ab)(h). This principle could be thought of as being based on another principle for triangles, and if the area of the base side of any two triangles is the same, and the heights are the same, then their areas of both triangles are the same: At = (base side)(height) / 2 = bh / 2.
[FIG 300]



In the above figure, triangle 1 and triangle 2 have the same area since they have the same base length (b1=b2) and height (h1 = h2).

In the lower section of the above figure, we see the same triangles divided into an infinite number of plane sections of the same thickness and parallel to the base side. Each corresponding planar or surface area section of each triangle is the same, and therefore at any similar height in the triangles the corresponding areas are the same. Here A1 = A3, and A2 = A4. Note for example that A2 is also equal to the total area of the triangle less the area A:. At = A1 + A2, and A2 = At - A1. If you were to move each section of triangle 1 so as to produce triangle 2, it becomes clear that the areas are the same, Likewise, at any height, this can be done and the corresponding area segments of the triangle are the same.

We know that if the height of a triangle changes by a factor of (n), that its area will also change by that same factor:

A = bh / 2 , when (h) is changed by a factor (n):

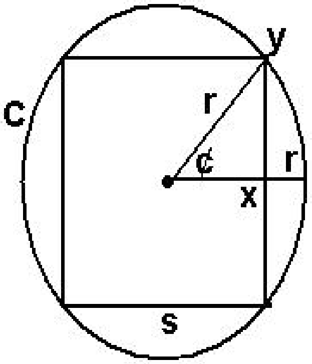
nA = b (nh)) / 2
nA = n (bh / 2)

If the height for triangle 1 and triangle 2 is 1 unit, and if it is sectioned (as sown by the red line in the figure above) into A1 and A4 respectively and having the same height of (nh) where (n) is a value lower then 1, hence (nh) is a fraction of (h), then those areas are the same.

If the height chosen is 90% of the original height of the triangle, hence, n=0.90, and the area of that segment is 90% of the area of the entire triangle.

Much of this concepts of this article with planar areas can also be applied to volumes of some geometric figures such as mentioned at the beginning of this article about a cylinder and stack of coins or disks.

Extra: An analysis of a (largest) square in a circle. [FIG 300B]



r = 1 sin ¢ = opp / hyp , hyp = r = 1
¢ = 45° cos ¢ = adj / hyp

y = sin 45° (r) = 0.7071 units
x = cos 45° (r) = 0.7071 units

s = 2y = 2x = 1.414 units

As = 1.414^2 = 2 sq. units
Ac = (pi) r^2 = pi (1) = 3.14 sq. units

Ac / As =~ 3.14159 / 2 = 1.57 = (pi) / 2
As / Ac =~ 0.636943

C = 2(pi)r = 6.28 , Ps = 4s = 4 (1.414) = 5.656
C / Ps = 6.28 / 5.656 =~ 1.1109 , Ps / C =~ 0.9

An oddity to explore is that the product of some of the values shown in the above articles of:
(0.785398163) (0.636619772) = 0.5 , and also that the product of their reciprocals is therefore equal to 2. Can you find a new geometric constant(s) to consider?

If the area of a circle and square are to be the same value, you can solve for a variable by setting their equations as equal, and then using the known values to solve for an unknown value and-or variable:

Ac = As , (pi) r^2 = s^2 , Mathematically: r = sqrt(s^2 / (pi)) and s = sqrt[(pi) r^2]

Volume Of A Paraboloid

The volume (Vp) of a paraboloid (a parabolic shape rotated once on and about its central axis so as to create a three dimensional shape and-or solid) is amazingly equal to one-half the volume (Vc) of a cylinder of the same radius and height. In other words, the circumscribing cylinder will have twice as much volume as the paraboloid. A simple way to verify this is that the paraboloid and its exterior or inverse volume completely fill the volume of the cylinder. At any height and thin segment of volume, the corresponding section of volume of the paraboloid and its inverse always sum to the same value of that of a thin segment or cross sectional slice of that cylinder, and of any height of that cylinder.

$Vp = \frac{Vc}{2}$ = (area of cylinder base) (height of cylinder) / 2 = (pi) r^2 (h) / 2 : with cubic units of length = units^3
, if h=r , Vp = Vc / 2 = (pi) (r^2) (r) / 2 = (pi) r^3 / 2

The **sagita**, depth or "**sag**" of a parabolic curve is equal to the height (ex.: h = y = x^2) at that location along that curve.
The volume of the sagita when considered equal to the volume of the paraboloid is: Vs = Vp = Vc / 2

The volume (Vb) beneath the sagita of the paraboloid can be found by: Vb = Vc - Vp = Vc - Vc / 2 = Vc (1 - 1/2) =
= Vc (1 - 0.5) = Vc (0.5) = Vc / 2 = (2 Vp) (0.5) = Vp , hence Vb is equal to the volume of the parabola.
Vb = Vp = Vc / 2. Vp + Vb = Vc

It is incorrect to think that at half the volume of a given cylinder, that the volume of the corresponding internal paraboloid will also be half, and this can be seen if you draw cylinders with the same radius and with one cylinder being at half the height.

The area beneath the curve of a basic or common parabola is:

A = S = f(x) dx , and where f(x) is the anti-derivative equation in question, and if the equation is y = x^2 , its anti-derivative is: x^(n+1) / (n+1) = x^(2+1) / (2+1) = x^3 / 3. Initially shown is the general or indefinite integral of the sum equation, and without the specific function, limits or bounds. Here is a definite integral form with the lower limit (a) and upper limit (b) of the sum:

[Extra: note that: (f(x)) (dx) can be considered as a infinitesimally small piece of area: (height) (width)]
[The generalized Sum of these infinite number of bits of area in question is formally found by taking a difference of the two areas.] And:

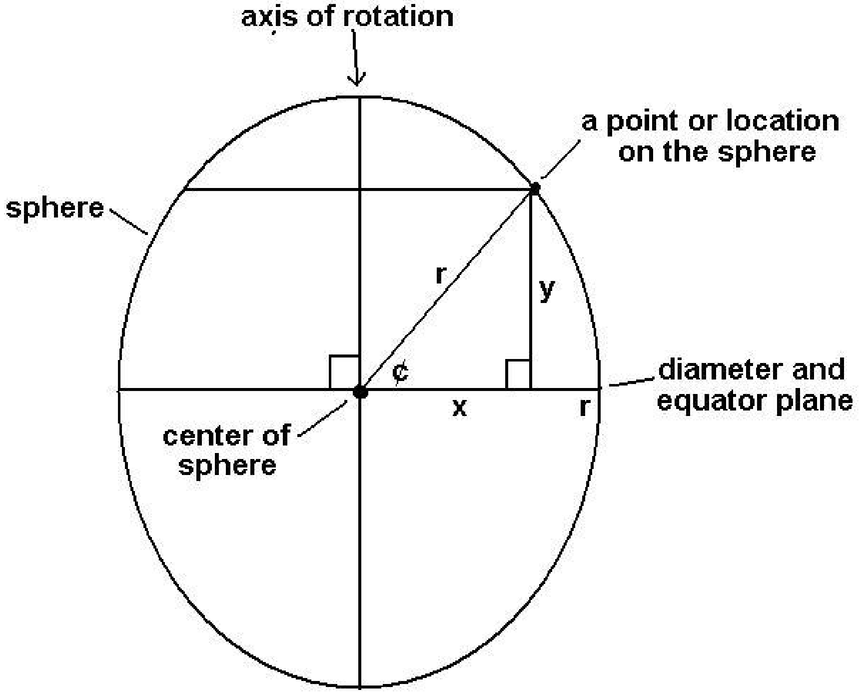
$$A = S_a^b \; x^2 \; dx = S_a^b \; (x^3 / 3) = F(b) - F(a) = A_b - A_a$$

: If (a=0), then simply: $A = \text{Area} = A_b$
: A = (b^3 / 3) - (a^3 / 3)

Finding The Distance A Point On A Sphere Is From The Axis Of That Sphere

All points on the surface of a sphere have the same distance from the center of it. Likewise, all points at the same radius distance from the center of a circle have the same distance from the center. When a radius line is extended from the center of a sphere to a point on the sphere, the radius line and-or point on the surface of the sphere is at a certain angle, ¢, from the center. If the sphere was Earth, this angle would be the longitude angle from the equatorial plane which is considered as the 0° reference position for other angles.

Knowing this angle mentioned, we can then find that points distance to the axis of rotation of that sphere. This distance, here equivalent to the value of (x), is actually the radius distance of a circle at that latitude value about that entire sphere. A similar discussion using many of these concepts as previously given in this book during a topic dealing with latitude and angles, but this discussion here was made to generally consider smaller spheres, perhaps less than 10 feet or 3 meters in diameter. [FIG 301]



sin ¢ = side opposite angle / adjacent angle = y / r , y = r sin ¢ , ¢ = arc sin (sin ¢)
cos ¢ = side adjacent angle / hypotenuse = x / r , x = r cos ¢ , ¢ = arc cos (cos ¢)

Circumference around the sphere at that latitude angle (¢L) and radius value (x) = C = 2(pi) x = 2(pi) r cos ¢L

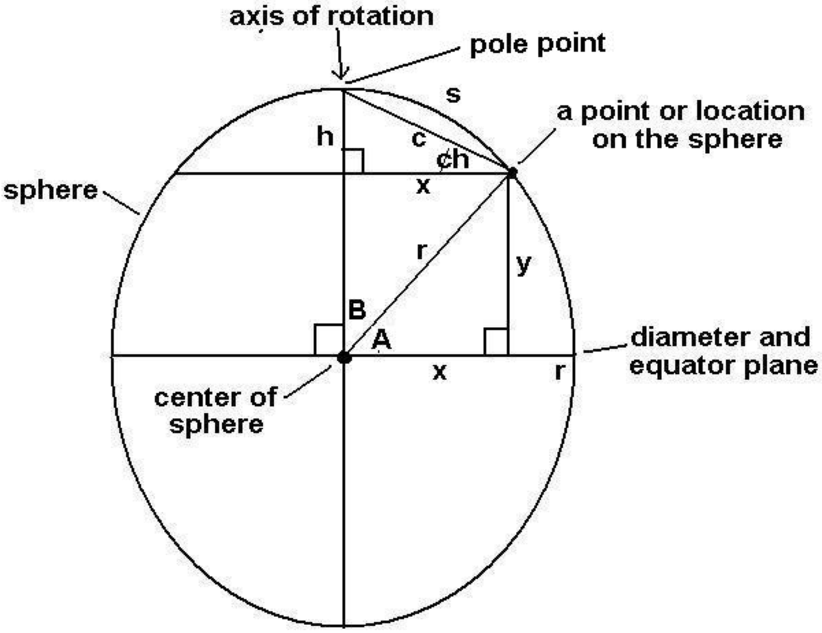
For the near sphere of the Earth, it has a radius of about 6378 km = 3959 mi =~ 3960 mi =~ 4000 mi ,
and with the diameter at twice that value of about: 12756 km = 7919 mi =~ 8000 mi

The circumference of the Earth at 0° latitude, which is at the Equator is:
C = (pi)d = (2)(pi)r = (2)(pi) 3959 mi =~ 24875 mi =~ 80148 km =~ 25000 mi

At the latitude of 45° North of the equator: The effective circular radius about the Earth is: x = r cos 45° latitude ,
x = 3959 mi (0.70711) =~ 2799 mi =~ 2800 mi. and C = 2(pi)r = 2(pi)x = 2(pi) 2800mi = 17589 mi and this circumference value can be easily calculated as: Clat = (C of Sphere) (COS latitude) = (25000 mi)(COS lat.)

Below is a continuation of the previous discussion and shows how to find the distance from the point on the sphere to the pole at the axis of rotation.

Ex. Given angle A, find c and s: [FIG 302]



r = y + h , mathematically:

$h = r - y$

$c = \sqrt{h^2 + x^2}$: via the Pythagorean Theorem of a right triangle

c is also the chord length beneath the arc segment s

Since A + B = 90° , B = ¢s = 90° - A

From: $$\frac{s}{C} = \frac{\text{¢}s}{360^\circ} = \frac{As}{Ac}$$

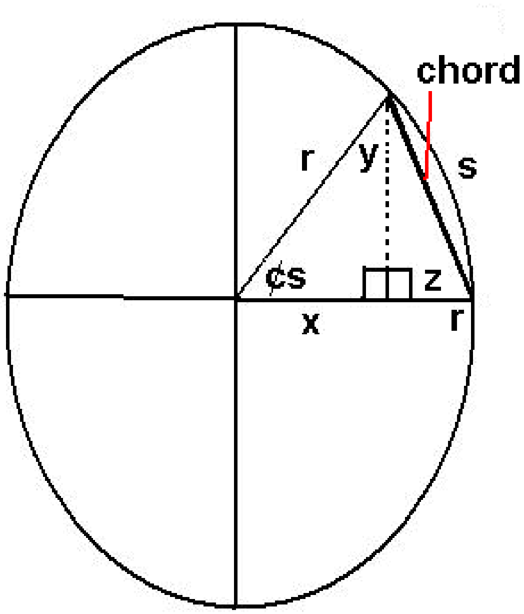
$$s = C\,\frac{\text{¢}s}{360^\circ}$$

Extra:

tan ¢h = opp. / adj. = h / x : x = r cos A

¢h = arctan (tan ¢h) and at the pole, the angle of declination of line (c) below the horizontal or horizon is 90° - ¢h, and since this angle and ¢h are complementary angles of a 90° angle. This angle is also equal to ¢h due to alternate interior angles created by a transversal line - here, line c

Continuing the previous examples, here is another example of solving for a chord length given the angle, here (¢s) corresponding to the arc length, and the radius (r) of the circle. [FIG 303]



From the previous discussions and examples, we know that: for a right triangle (here an inner triangle of the outer and larger triangle) is:

y = r sin ¢s

x = r cos ¢s

r = x + z

z = r - x = r - (r cos ¢s) = r - r cos ¢s = r (1 - cos ¢s)

The chord side is also the hypotenuse side of the smaller internal triangle with sides (y) and (z). Using the Pythagorean Theorem:

$$\text{chord} = \sqrt{z^2 + y^2}$$

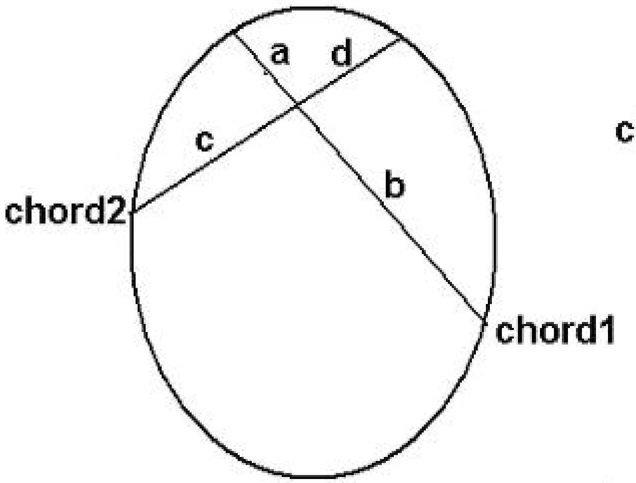
Also of note is that the larger triangle being studied here is also an isosceles triangle having two sides of equal length, here (r). The third side of an isosceles triangle is known as the base side, and the two angles created there by the other two sides are equal in value and are called the isosceles base angles. These base angles have a value of: (180° - ¢s) / 2

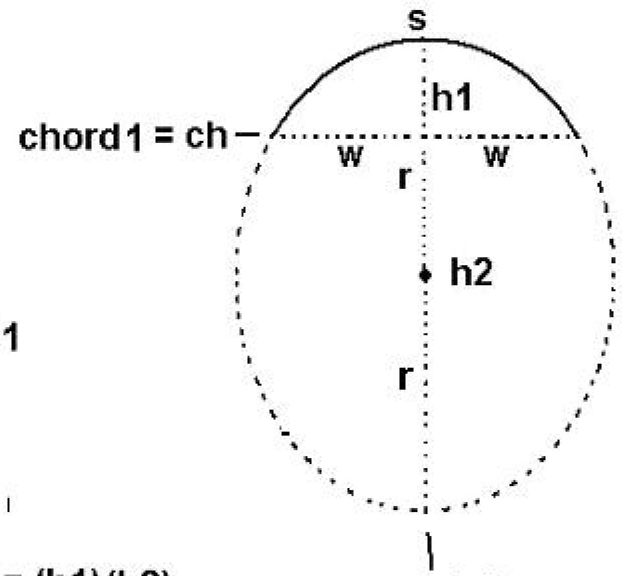
As for finding s or ¢s, it was shown previously in the book that:

$$\frac{s}{C} = \frac{¢s°}{360°} = \frac{As}{Ac}$$, therefore, mathematically: $$s = C \frac{¢s}{360°}$$: C = Circumference = 2 (pi) r

Finding The Radius Of Curvature Of An Arc Segment Of A Circle

Here is how to solve for the corresponding radius of a circle given just a segment or arc of it. The method used here will consider the **Chord Theorem** that Euler discovered. This theorem states that given two chords of a circle that intersect, that the products of the two segments of each chord are equal in value. Euler discovered this by considering the concepts of inscribed angles, and similar triangles created by two chord lines that intersect in a circle. For a simple verification of the chord theorem, consider the chords from the endpoints of (a) and (c), and then from (d) to (b) creating two similar triangles and of which have the same angle at the intersection point of the two lines. The angles created within a circle could be thought of as interior angles. Also draw a quadrilateral with the two lines as diagonals within it, and the chords as sides of it. As mentioned in this book, if the inscribed angles have the same arc length, then those angles are equivalent. Given that two angles of a triangle are equivalent, the third angles will also be equivalent due to the 180° internal sum of angles. We then have for the similar triangles: $a/c = d/b$, and mathematically this equals: **(ab) = (cd)**. [FIG 304]





Chord Theorem

(a)(b) = (c)(d)

(w)(w) = (h1)(h2)

h2 = w^2 / h1

chord 2 and diameter (d)

w = chord1 / 2

s = arc of a circle

d = h1 + h2 = 2r

radius = r = d / 2

$$r = d / 2 = (h1 + h2) / 2 = (h1 + (w^2 / h1)) / 2 = (w^2 + h1^2) / (2\,h1)$$

From the above formula for (r), if we multiply it by 2 so as to have the diameter (d), we have:

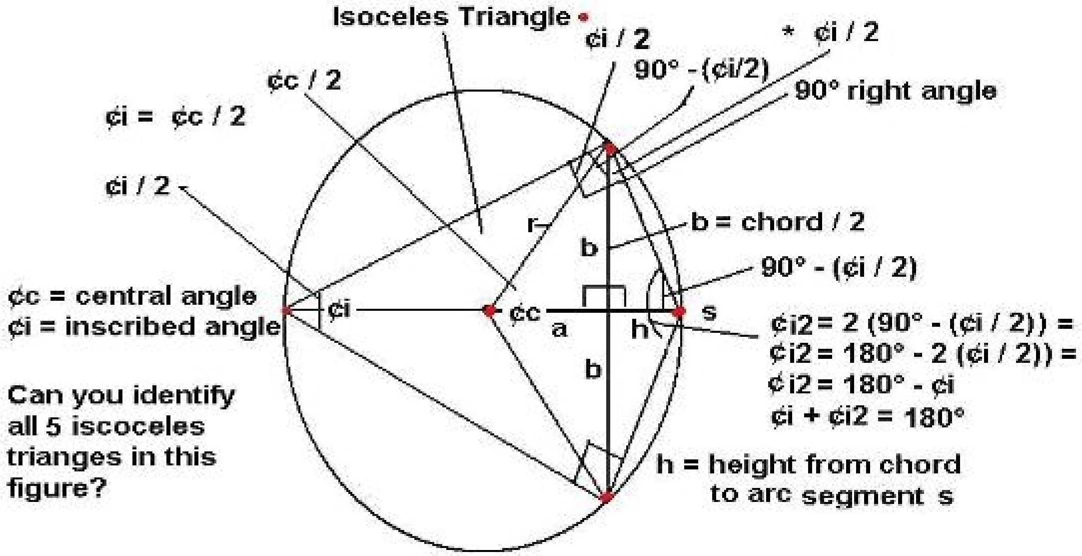
$$d = 2r = 2\left(\frac{w^2 + h1^2}{2\,h1}\right) = \frac{w^2 + h1^2}{h1}$$

, extra: $h1 = w^2 / h2$, when $h1 = h2$:
then: $h1 + h2 = 2h1 = 2h2$ and $h1/h2 = 1$ and $r = h1 = h2$

For some verification of the chord theorem, at the point of in intersection of the two lines, two pair of equivalent vertical angles are created, and by extending these sides of the angle and connecting their endpoints by a straight line, two pair of similar triangles will be created. For example, some equivalent ratios of their corresponding sides are: $a/c = d/b$, which can be simplified to $ab = cd$ as shown above. To help this verification, consider that any and all arc segments of a circle have the same focus point at the center of the circle. The radius line of a circle will be perpendicular to a chord line at its center or half-length point. The length of an arc segment and its chord length is also proportional to the angle from the center of the circle, and here, all points on that arc segment will have the same distance from the center of the circle, however the distance from each point on a chord to the center point is not a constant value. For other points in a circle besides the center point, an arc length and chord will be determined by both the angle created at the point of intersection of two lines within a circle, and by the distance that the line intersection point is from the center and-or arc. Just the same, the larger the radius of a circle, the larger the arc length and-or its chord line. The two corresponding arc segments and their chords lengths created due to having the same similar vertical angle will be proportional in value to their distance from that point of intersection.

A point within a circle other than the center point of a circle will be a point on 1 unique radius and-or diameter line, and will also be the center point of 1 unique chord line. Since a chord line is perpendicular to the radius line to its center point, that chord line will have a negative reciprocal slope value of the radius line. The farther that inner point is from the center of the circle, the smaller that unique chord line, corresponding arc length and central angle that the arc subtends and-or corresponds to.

Another method to find the radius of curvature. [FIG 305]



The internal angles within this 4 sided quadrillateral sum up to:
¢i + ¢i2 + 90° + 90° = 360°

When given h and b:

* Tan (¢i / 2) = opposite side / adjacient side = h / b
(¢i / 2) = arctan [tan (¢i / 2)]
¢c = 2 ¢i
¢c / 2 = 2 (¢i / 2) = ¢i

Sin (¢c/2) = opposite / hypotenuse = b / r
r = b / Sin (¢c/2)

a = r Cos (¢c/2) = r - h
h = r (1 - Cos(¢c/2))
h = r - a

b = r Sin (¢c/2)
b^2 = r^2 - a^2

Much of the previous discussion about central and internal angles has been previously mentioned in this book.
Since any quadrilateral can be divided into two internal triangles with each having an internal angle sum of 180°, the total sum of the internal angles of a quadrilateral is: 180° + 180° = 360°.

Extra: For another formula for **chord length** given the central angle (¢c) corresponding to it, and the radius (r) of the circle:
If (a) is half the chord and-or length, then: sin (¢c / 2) = a / r , a = r sin (¢c / 2) , chord length = 2a = 2r sin (¢c / 2)

AN EXTRA NOTE ABOUT TORQUE AND POWER

You may wish to also view to article in this book about a flywheel and its mechanical and kinetic energy storage, and of which also contains some similar equations.

T = (force)(lever arm) = FnLm , : T = torque, and with units of Newton-Meters
T = (force)(L) = maL

Work = Energy = (force)(distance) = FnDm : with units of Newton-Meters

Power = Energy / Time = Work / Time = Joules / Second = J / s

P = F d / t , and since d = v t , v = d / t , therefore:

P = F v : F in newtons , v in m/s , now expressing this in terms of Torque:

P = T v = T w , and since torque involves rotation, v becomes angular or rotational velocity = w, and with units of radians per second = rads / s

If given, we must first convert revolutions per minute (rpm) or revolutions per second (rps) into rad / s for the power needed or used for/of the applied torque.

d = (v)(t) : a linear or scalar distance

If given a revolutional velocity = rotation / time = ¢ / t, and not a length distance or linear velocity = d / t :

¢ = (w)(t) = an angular, revolutional or rotational distance , if we divide this amount of angle by the amount of angle or angular rotation per rotation, we can find the number of rotations. The amount of angular rotation per revolution or rotation is 360° = (2)(pi) radians =~ 6.28 radians
w = ¢ / t

Ex. A wheel rotated 10 times in 5 seconds:

$\frac{10 \text{ rot}}{5 \text{ sec}}$, and this fraction can be reduced to: $\frac{2 \text{ rot}}{1\text{s}}$ = 2 rps , hence a frequency of 2 rot/s = 2 cycles/s = 2 cps = 2 hz

w = ¢ / t = (N rotations) (2)(pi) radians / s = (2)(2)(pi) rads / s = $12.566\underline{4}$ rads / s

(12.5664 rads / s) / (6.28 rads / rot) or= (12.5664 rads / 1s) (1 rot / 6.28 rads) = 2 rot / s

(1 rev / s) (60s) = 60 revs also: N rps (60s) = (N rev/s) (60s) = N 60 revs = N 60 revs / min

(60 revs) = (1 rev / s) (60s) = (1 rev / s) (1 min) , mathematically, by dividing both sides by (1 min):

(60 revs) / (1 min) = (1 rev / s) , switching sides

1 rev / s = 60 rev / min

$1 \text{ rps} = \frac{1 \text{ rev}}{\text{s}} \frac{(60\text{s})}{\text{min}} = \frac{60 \text{ rev}}{\text{min}} = 60 \text{ rpm}$: **1 rps = 60 rpm or= (N) 1 rps = N rps = N 60 rpm**
rps = rpm / 60

360° = 1 rotation or revolution = an angle of 2 (pi) =~ 6.28 radians

$$\frac{1\text{ rev}}{s} = \frac{360°}{s} = \frac{2(\text{pi})\text{ rads}}{s} =^{\sim} \frac{6.28\text{ rads}}{s}$$

rotational or angular velocity = w =

$$w = \frac{\text{angular distance}}{\text{time}} = \frac{\text{rads}}{s} = \frac{\text{rads}}{s}\,\frac{(\text{revs})}{(\text{revs})} = \frac{\text{rads}}{\text{rev}}\,\frac{\text{revs}}{s}\text{ , and:}$$

Given a number (N) of rotations or revolutions per second = rps:

$$w = \frac{2(\text{pi})\text{ rads}}{(\text{rev})}\,\frac{(N\text{ revs})}{(\ s\ )} =^{\sim} \frac{6.28\ N\ \text{radians}}{s} = 6.28\ (\text{rps})\ \text{radians} = \frac{6.28\ (\text{rot})}{s}\ \text{radians} = 2(\text{pi})(\text{cps}) = 2(\text{pi})\ f$$

: cps = cycles per second
: f = rotational frequency

Given a rotating circle where r=1 and-or 100% for general, relative or proportional analysis:

angular velocity for 1 rotation = w = (angular distance for 1 rotation = 360°) / t =
w = 2(pi) rads / s = 2(pi) / s = 6.28 rads / s = 1 rotation / s = 1rps

linear v for 1 rotation = v = linear d / t = Circumference / s = 2(pi) r / s , If r=1m: c / s = 2(pi)(1) m / s = 6.28 meters / s

If r=1, v = linear distance / s , and if there is 1 rot / s or= 1 rev / s = 360° / s, this can interpreted as: 2(pi)(r) / s = 1Cm / s . w = 6.28 rads / s, also, at r=1, 1 Cm / s = 6.28m / s , and
Work = (Force)(distance) = Fd = F (6.28m) joules , and if this was done in 1s, Pw = W/t = E/t = 6.28F J/s = 6.28F watts

If there are N rotations or N revolutions per second, we simply multiply (d) or (C), and (w) by (N) to find:
v = d / t = NCm / s and w = (6.28N rads) / s

Pw = Ej / s = Work / s = (force)(distance) / s = F (velocity) = Fv , and in terms of torque , rotating or twisting forces:
Pw = Fv = Tw = T (6.28 N) rads / s = T (6.28)(rot / s) rads / s

, if given a rpm value , 1 rps = 60 rpm , multiplying both sides by N:
N rps = N 60 rpm , dividing both sides by 60 and switching sides:
N rpm = (N / 60) rps , Ex.: If given 60 rpm , 60 rpm = (60/60) rps = 1 rps

$$Pw = T\,w = T\ 6.28\ (N\text{ rpm})\text{ rads} = T\ (6.28)(N\text{ rpm} / 60\ )\text{ rads} / s = \frac{T\ (6.28)(N\text{ rpm})}{60}\text{ rads} / s$$

(N rot) / s = (N rps) = w can be thought of as in terms of a linear equivalence of: (nc) / s = d / t = v , and
rpm = (rot / min) can be thought of as: (nc) / min = nc / 60s = (N rot) / 60s

Some extra angular velocity expressions:

This book generally uses the w symbol for angular or rotational velocity.

1 rps = 2(pi) radians / s = 6.28 radians / s : rps expressed using angular velocity = w = angular distance / time
1 rps = 1C / s : rps expressed using linear velocity = v = linear distance / time , C = circumference at the radius
1 rps = 1 C / s = 2(pi) r / s , If r=1m , 6.28m / s = v

This is the same numeric value as the angular velocity, hence angular or twisting velocity can and should be used in terms of torque, and of which is often expressed in terms of rps or rpm. For torque, the angular velocity has units of (radians / s). Given rps or rpm, these rotational velocity values will then need to be converted to their equivalent angle velocity values of radians per second.

1 rps = 1 rev / s = 2(pi) radians / s =~ 6.28 rads / s
1 rps = 1 rev/s and after 1 minute = 60s of time , (1 rev / s) 60s = 60 rev after, for, or per 1 minute = 60 rpm
1 rps = 60 rpm = 60 rev / min , and after multiplying each side by N , we have:
N rps = (N 60) rpm .
N rps = (N 60) revs / min = (N 60) rpm
N rps = N revs / second = N (2)(pi) radians / s
N rpm / 60 = rps and N rpm = 60 rps = 60 (revs / second)

rotational velocity and corresponding linear velocity:

N rps = N (rot) / s and (rps) Cm / s = (rps) (2(pi)(r)) m / s : C = circumference

Since C = (2)(pi) r , r = C / (2)(pi) and C / r = (2)(pi) , If r was an arc length on the corresponding circumference, there will be (2)(pi) =~ 6.28 times or instances of it , hence a circumference has a full rotational angle of (2)(pi) radians = 360°

F = hertz = hz = cycles, pulses or vibrations / second = cps = rps = N

linear velocity = v = $\frac{\text{distance}}{\text{s}}$ = (rotational linear distance) (rotational frequency) = C (cps) = **C F** = **2(pi)(r) F , m/s**
C = circumference , r = radius in meters , F in hz = c/s

linear velocity of a point on a rotating circle or disk structure = v = C F = (circumference of circle) (frequency of rotation)

After an amount of time, there will be N revolutions of this circle, disk or wheel, and the total distance traveled =

total distance = v t = (C F) t = C F t = C (revolutions / second) (seconds)

Ex. The circumference of a car wheel was calculated to be: C = (2)(pi)(r) = 2 meters. If the car is traveling at 15 meters per second, what is the rotational frequency of that wheel?

From **v = C F** , **F = v / C** = $\frac{\frac{15m}{s}}{\frac{2m}{1}} = \frac{15m}{(s)(2m)} = \frac{7.5}{s}$ = 7.5 hertz = 7.5 cycles or revolutions per second

Ex. An **anemometer** (ie., a windspeed meter) has windcups (wind , force collection cups) at a 12cm radius. If it is spinning at 240rpm, what is the windspeed in meters per second?

240 rpm = 240 revs / minute = 240 revs / 60s = 4 revs / s : after dividing num. and den. by 60

12cm = 0.12 m
v = C F = (circumference)(frequency) = (2)(pi)(r) (F) = (2)(3.14)(0.12m) (4 / s) = 3.0144 m/s

AN IMPROVED ALGORITHM TO SOLVE DIFFICULT EQUATIONS

Previously in this book, a method and-or algorithm called EQSOLVE2.C was shown how to evaluate difficult expressions and-or equations. Below is an alternate and-or improved version of it, and it uses only just one main loop, and has been placed in a more usable example program.

```
/* -----------------------------------------------------------------------------------
FESA.c ,    FAST NUMERICAL ALGORITHM (FANA)    or
            FAST EQUATION SOLVING ALGORITHM (FESA)

   An non algebraic, mechanical-like numerical method or algorithm to zero into
   or approach the result and having a high precision.

   Compiler used:  TinyC  ,  ANSI standard, fundamental C only.

   When using just 1 term of the equation shown below, the result essentially finds
   the log of that number, and which is the exponent.
   Ex.   If result=number=1000 ,  and a=10,  then x=logarithm=exponent=3
   Note that the number of a logarithm cannot be a negative value.
   A logarithm can be negative in value, such as when the number is less than 1.
   The logarithm of 1 as the number, using any base, is 0.  If the number is less
   than 1, then the logarithm will be negative in value.

   For testing, I usually used:   2^x + 3^x  =  35
                            and:   1^x + 2^x  =  35

    The program can be modified as needed - perhaps to display it better.

   (c). JPA Nov. 2022  ,  Pa , USA
-------------------------------------------------------------------------------------*/

#include "stdio.h"
#include "stdlib.h"
#include "float.h"
#include "math.h"

/*-----------------------------------------------------------------------------------*/
double equation(double x, int terms);  /* (x) is a temporary value to use in the equation, and it will find the true value of it */

double FANA(double RESULT, int terms);   /* This algorithm does not directly use the weights for
                                            the precision, and does not include loop conditions,
                                            and the original version shown in this book did.  Terms
                                            is the desired number of terms of the expression to utilize.
                                            RESULT is the value of the expression containing (x).
                                            The return value is a temporary result using the temporary
                                            value of (x) in the equation.   */

/*-----------------------------------------------------------------------------------*/
/* Variables used in the equation:  */
/* Equation variable initializations can be hard coded using a value, or with user input in main(), etc */

double x=0.0;        /* x is to be found, it will be set to an initial very low, improbably and-or negative value,
                           below. This can be set higher if the solution speed is very critical. */
```

```
double a = 0.0;
double b=0.0;
double c=0.0;
double d=0.0;
double e=0.0;
double f=0.0;
double g=0.0;
double h=0.0;
double i=0.0;
double j=0.0;

int TERMS=0;
/*-------------------------------------------------------------------------------------*/
void main(void)
{
double r=0.0;
double result=0.0;
                                        /* Main program loop, uses inputs for the constants,
                                           and displays the value of (x) so as to make that
                                           equation true: */
for(;;){
          printf("\n\n--------------------------------------------------------------------------------");
          printf("\n\nThis program and algorithm in it is: (c) J.P. Albertson , 2022, author of");
          printf("\nthe Mathization ebook.  Donations are  very appreciated if you are able");
          printf("\nifinancially, and the verified link is in the ebook and-or in my YouTube channel");
          printf("\nchannel description.  The .pdf format book can be viewed on a phone or computer");
          printf("\nand is filled with basic math, algebra, trigonometry, advanced topics, science, and");
          printf("\nsome computer C language, text source programs science.  It is a helpful");
          printf("\nsupplement to any course, tutoring, home learning and-or work experience.");

          printf("\n\n\nThis Fast Numerical Algorithm will find the value of (x) in this expression: ");
          printf("\n\na^x + b^x + c^x + d^x + e^x + f^x + g^x + h^x + i^x + j^x = result");

          printf("\n\n\nEnter the number of initial terms to use: (1 to 10): ");  fflush(stdin);
          scanf("%d",&TERMS);   if( (TERMS<=0) || TERMS>10){ printf("\a"); exit(0); };

          printf("\nEnter the positive valued result of the expression or 0 to quit.");
          printf("\n\nresult = ");
          fflush(stdin);
          scanf("%lf",&result);     /* %f works,  using %.15g might not work */
          if(result<=0.0){  exit(0); };
          /* printf("\nresult entered is:  %lf",result);  */

          printf("\n");        /* This coding allows "fall through" processing much like a C switch (a C programming
                                  construction, "construct") statement can:  */

          if(TERMS>=1){  printf("\nEnter a: ");  fflush(stdin); scanf("%lf",&a);
                           if(a==0.0){ printf("\a"); exit(0); };
                         }

          if(TERMS>1){    printf("\nEnter b: ");  fflush(stdin); scanf("%lf",&b);
                           if(b==0.0){ printf("\a"); exit(0); };
                      };
```

```
          if(TERMS>2){    printf("\nEnter c: ");  fflush(stdin); scanf("%lf",&c);
                           if(c==0.0){ printf("\a"); exit(0); };
                        };

          if(TERMS>3){    printf("\nEnter d: ");  fflush(stdin); scanf("%lf",&d);
                            if(d==0.0){ printf("\a"); exit(0); };
                       };

          if(TERMS>4){    printf("\nEnter e: ");  fflush(stdin); scanf("%lf",&e);
                           if(e==0.0){ printf("\a"); exit(0); };
                       };

          if(TERMS>5){    printf("\nEnter f: ");  fflush(stdin); scanf("%lf",&f);
                           if(f==0.0){ printf("\a"); exit(0); };
                       };

          if(TERMS>6){    printf("\nEnter g: ");  fflush(stdin); scanf("%lf",&g);
                            if(g==0.0){ printf("\a"); exit(0); };
                        };

          if(TERMS>7){    printf("\nEnter h: ");  fflush(stdin); scanf("%lf",&h);
                           if(h==0.0){ printf("\a"); exit(0); };
                         };

          if(TERMS>8){    printf("\nEnter i: ");  fflush(stdin); scanf("%lf",&i);
                           if(i==0.0){ printf("\a"); exit(0); };
                        };

          if(TERMS>9){    printf("\nEnter j: ");  fflush(stdin); scanf("%lf",&j);  /* ie., the 10th term */
                           if(j==0.0){ printf("\a"); exit(0); };
                        };

         x = result = FANA(result,TERMS);    /* call the function and-or algorithm to solve for (x) with the given data used */

/*---------------- Display The Results ------------------------------------*/

         printf("\n\nThe value of x = %.15g",result);  /* use g or lf*/
         printf("\n\n");

         /* x=result; */

        /* below, g or lf can be used,   lf might be better though. */

    if(TERMS>=1){  r = pow(a,x); printf("\na^x = %.15g^%.15g = %.15g",a,x,r);   };
    if(TERMS>1){  r = pow(b,x);   printf("\n\nb^x =  %.15g^%.15g = %.15g",b,x,r);  };
    if(TERMS>2){  r = pow(c,x);   printf("\n\nc^x =  %.15g^%.15g = %.15g",c,x,r);  };
    if(TERMS>3){  r = pow(d,x);   printf("\n\nd^x =  %.15g^%.15g = %.15g",d,x,r);  };
    if(TERMS>4){  r = pow(e,x);   printf("\n\ne^x =  %.15g^%.15g = %.15g",e,x,r);  };
    if(TERMS>5){  r = pow(f,x);   printf("\n\nf^x =  %.15g^%.15g = %.15g",f,x,r);   };
    if(TERMS>6){  r = pow(g,x);  printf("\n\ng^x =  %.15g^%.15g = %.15g",g,x,r);   };
    if(TERMS>7){  r = pow(h,x);  printf("\n\ng^x =  %.15g^%.15g = %.15g",h,x,r);   };
    if(TERMS>8){  r = pow(i,x);   printf("\n\ni^x =  %.15g^%.15g = %.15g",i,x,r);   };
    if(TERMS>9){  r = pow(j,x);   printf("\n\nj^x =  %.15g^%.15g = %.15g",j,x,r);   };
```

```
   printf("\n\nPress A Key ");  fflush(stdin); getch();
  };

return;
};

/*-----------------------------------------------------------------------------------------------------------------------*/
double FANA(double RESULT, int terms)   /* (c) JPA 2022 */
{
double x=  -1000000000.0;      /* negative 1 billion */ /* This can be changed if processing speed
                                  and-or time is critical.  perhaps use  -1000.0 , or -100.0 ,or -10.0  */
double TEMP=0.0;               /* A temporary result to be checked. ,  ie. left side of equation */
double INCREMENT = 100000000.0;     /*  one hundred million , hence the initial value will be divided by 10 */
                                    /* This is a tenth of the initial value of (x), but positive along the number line,
                                       and so as (x) can increase in the positive direction on the number line. */

double T=0.0;  /* Some temporary variables, such as  to hold positive or absolute values only, for some tests. */
double R=0.0;
double V=0.0;

int loop=1;          /* a loop counter, optional, for the display */

                    /* Although the following code is similar to eqsolve3 in this book, it is not
                       the same in terms of the weight being divided by 10 so as to get more
                       digits of precision in the result. */

    for(;;){

       TEMP = equation(x, terms);     /* : returns a temporary value of the expression using the
                                         current or temporary value of (x) which is to be found */

       printf("\nLoop = %d , TEMP = %.15g  ,  x = %.15g",loop,TEMP, x);   /* optional, can comment off and-or remove  */

       T=TEMP; R=RESULT;

       if(T==R){  break; };

       if(T<0.0){ T = T * -1.0; };  /* make positive */
       if(R<0.0){ R = R* -1.0; };
       V=T - R;       /* get difference to see which is bigger so that the difference below is positive */
       if(V>=0.0){  if( (T - RESULT) <= 0.000000000001){  break; }; };  /* when T>R  , V is positive */
       if(V<=0.0){  if( (RESULT - T) <= 0.000000000001){  break; }; };   /* when R>T , V is negative */

                    /* Extra note:  Using another 0 above, did not help solving such as:  2^x = 10   It is
                       basically trying to use too much precision, and the loop keeps running like that of
                       an insolvable equation entered   0.00000000000001 seems to be the lowest
                       usable value for some equations, but not others, and maybe this can be
                       settable depending on the equation   */

        if(TEMP>RESULT){   x = x - INCREMENT;
                           INCREMENT = INCREMENT / 10.0;
                         };

       x = x +INCREMENT;
```

```
        loop=loop+1;
        if(loop >= 1000){  printf("\n\nThe Equation Entererd Is Probably Not A Solvable Equation.  Press A Key");
                                                    fflush(stdin); getch();  exit(0);  };
       };

 if(x==-1000000000.0){ x=1; };    /* : due to an error when x actually is equal to 1  */

return x;
};

/*---------------------------------------------------------------------------------------------------------------------------*/
double equation(double x, int terms)
{
double r=0.0;    /*  :to hold the result of the equation used, and it is: */

/* A probable modification would be to allow a term to be either
    negative or positive, possibly after asking the user.   */

     /* This coding below allows successive "fall through" processing, like a switch statement could */

    if(terms>=1){  r = pow(a,x);     };     /* The equation is hard coded here, but could be composed of  variables that can
                                                      be called TermN that are double floats that a parser can set after simplifying a
                                                      user input expression via a text string of ASCII characters or letters. */
    if(terms>1){  r = r + pow(b,x);  };
    if(terms>2){  r = r + pow(c,x);  };
    if(terms>3){  r = r + pow(d,x);  };
    if(terms>4){  r = r + pow(e,x);  };
    if(terms>5){  r = r + pow(f,x);   };
    if(terms>6){  r = r + pow(g,x);  };
    if(terms>7){  r = r + pow(h,x);  };
    if(terms>8){  r = r + pow(i,x);   };
    if(terms>9){  r = r + pow(j,x);   };

return r;
};
/*-----------------------------------------------------------------------------------------------------------------------------*/
```

Here is an example and output example of the above program, and showing the intermediate or temporary results :

```
This program and algorithm in it is: (c) J.P. Albertson , 2022, author of
the Mathization ebook.  Donations will help me and this effort to help
others, and the verified link is in the ebook and-or in my channel
description.  The .pdf format book can be viewed on a phone or computer,
and is filled with basic math, algebra, trigonometry, advanced topics, and
science.  It is a helpful supplement to any course, home learning and-or work.


This Fast Numerical Algorithm will find the value of (x) in this expression:

a^x + b^x + c^x + d^x + e^x + f^x + g^x + h^x + i^x + j^x = result


Enter the number of initial terms to use: (1 to 10): 3
```

```
Enter the positive valued result of the expression or 0 to quit.

result = 110


Enter a: 5.0

Enter b: 6.0

Enter c: 7.0

Loop = 1 , TEMP = 0  ,  x = -1000000000
Loop = 2 , TEMP = 0  ,  x = -900000000
Loop = 3 , TEMP = 0  ,  x = -800000000
Loop = 4 , TEMP = 0  ,  x = -700000000
Loop = 5 , TEMP = 0  ,  x = -600000000
Loop = 6 , TEMP = 0  ,  x = -500000000
Loop = 7 , TEMP = 0  ,  x = -400000000
Loop = 8 , TEMP = 0  ,  x = -300000000
Loop = 9 , TEMP = 0  ,  x = -200000000
Loop = 10 , TEMP = 0  ,  x = -100000000
Loop = 11 , TEMP = 3  ,  x = 0
Loop = 12 , TEMP = 1.#INF  ,  x = 100000000
Loop = 13 , TEMP = 1.#INF  ,  x = 10000000
Loop = 14 , TEMP = 1.#INF  ,  x = 1000000
Loop = 15 , TEMP = 1.#INF  ,  x = 100000
Loop = 16 , TEMP = 1.#INF  ,  x = 10000
Loop = 17 , TEMP = 1.#INF  ,  x = 1000
Loop = 18 , TEMP = 3.23447716294339e+084  ,  x = 100
Loop = 19 , TEMP = 352707050  ,  x = 10
Loop = 20 , TEMP = 18  ,  x = 1
Loop = 21 , TEMP = 110  ,  x = 2

The value of x = 2


a^x  =  5^2 = 25

b^x  =  6^2 = 36

c^x  =  7^2 = 49

Press A Key
```

MORSE CODE

Since antiquity, signals other than verbal sound signals have been a way to communicate, especially over long distance by using things such as sound, light (electric, sunlight and mirror(s) = heliostat or "heliograph", and flares = visible rockets and-or a light suspended below a parachute), smoke, hand, and various (woven cloth or fabric) flag signals. Even a mechanical bell operated by a thread can be used for simple communication for say up to several hundred feet. Once an electric signaling system was developed not long after the discovery of the battery (ie. voltaic pile) in about 1800, an electric buzzer (audio sounder, much like a vibrating speaker having one tone), made it practical to communicate nearly immediately (ie., with nearly no time delays) over long distances, day or night, from city to city, from valley to valley past distant mountains. This was the modern form of communication before highways and mail delivery, trains and telephones, still, horses, carrier and boats were available.

A signal will requires a pulse or brief instance of the method used such as light, and the duration of it needs to be experimented with so as it can be received and-or recorded well enough to be deciphered later and to have meaning. With a signaling pattern or code system, more information can be communicated. In order to count a number of pulses, there also needs to be a lack of a pulse in between each pulse transmitted. We could code or express each pulse as a value of 1, and a lack of a pulse as 0. Samuel Mores recorded his two types of pules as a dot or dash, either written or on a paper tape.

Pulses of electricity could be sent and the specific number of pulses sent could represent a certain predefined or coded (another form, non-verbal, non-text) message. Even today as of the year 2022, in most hospitals, an electric buzzer or signaling system can be used by a patient in a certain indicated room to request some attention from the medical staff located down the hall, or on a different floor, or in a different building. Many homes, rooms, gates and buildings still have a door buzzer or tone system for signaling (ie., electronically requesting) attention is needed, and these are commonly called "door bells" or "door buzzers".

Samuel Morse invented the **Morse Code** communication system so that each text letter of the alphabet, and number, could have an associated code or coding sequence to it, and this made the telegraph much more useful and practical . Later, his Morse Code was adapted to the standardized International Morse Code system. Morse's system uses pulses, but two different types of short and long pulses. A short or quick (ie., in time length) pulse called a "dot [**.**]" could be considered as analogous to a 0, and a longer pulse called a "dash" [-] could be considered as analogous to a 1. Due to that there may be several of the same types of the pulse in a row, such as three dashes or three dots, there is still a necessary spacing and-or timing delay between pulse each so as to being able to recognize or distinguish each, and it is standardized as the time of 1 dot or short pulse between each pulse. A dash or long pulse is standardized as being 3 short pulses or dots long in time duration. Between each group of pulses for a letter, the timing space or duration of silence (ie., no signal) is equivalent to three pules, and between each word, the timing space is 7 pulses or dots. An experienced user should slow their speed of communication down so as to work better with novice or slow users.

Though the telephone replaced the telegraph and Morse code, Morse Code is still being used for relatively inexpensive, low power (as low as one watt to several watts) radio communications, and which use a very narrow range of bandwidth (ie., frequency range) which effectively concentrates the transmitted radio energy so it can be received at long distances away. There are special frequency bandwidths allocated for using Morse Code, radio communication, and are typically used by radio hobbyists to test their equipment, range and quality of the radio communication transmitted and-or received.

Below is a Morse Code chart:

A	. _			**X**	_ . . _
B	_ . . .			**Y**	_ . _ _
C	_ . _ .			**Z**	_ _ . .
D	_ . .				
E	.				
F	. . _ .				
G	_ _ .			**1**	. _ _ _ _
H	. . . .			**2**	. . _ _ _
I	. .			**3**	. . . _ _
J	. _ _ _			**4**	. . . . _
K	_ . _			**5**	. . . . .
L	. _ . .			**6**	_ . . . .
M	_ _			**7**	_ _ . . .
N	_ .			**8**	_ _ _ . .
O	_ _ _			**9**	_ _ _ _ .
P	. _ _ .			**0**	_ _ _ _ _
Q	_ _ . _				
R	. _ .	**S**	. . .	**T**	_
U	. . _	**V**	. . . _	**W**	. _ _

Each letter has up to 4 pulses. Each pulse is one of two pulse types:
A dot signal may be said as a "dit" or "di" or "de"
A dash signal may be said as a "dah" or "da"
SOS = . . . _ _ _ . . . : use to quickly request help, keep transmitting.
If your location is unknown, radio source direction can be used to find you.

Each number always has 5 pulses.

: For the numbers:
: Each dot can be considered as a count of 1
: Each preceding dash can be thought of as a count of 2

Note the opposite pulse types of: **(S , O)** , (A , N) , (B , J) , **(E , T)** (F , Q) , (G , U) **,** (K , R) , (L , Y) , **(M , I)** , (D , W) , (P , X) , (D , G) , (G , W)
Note the similarities of: **(E , I , S , H)** , (S , V) , (F , U) , (G , Q) , (A , R) , (P , W) , (I , U) , (M , G , Z , 7) , **(T , M , O)** , (J , O) , (J , P) , **(E , A , R)** , **(T , N , D , B)** , (T , N , K , C) , (O , J) , (G , U, W) , (A , W, J) , (A , U , V) **,** **(E , A , W , J)**

Many people can learn and memorize the Morse code system, but many other people will find it difficult to learn and-or use, and a simple pulse code or (pulse, signal) counting method and standardize system can be used instead. One such method is the **Tap Code** and of which you can look up about further, and the main benefit is its simplicity of just needing to count and record the number pulses received, and then to decode what they mean in terms of letters. If possible, it would be helpful if one person counted the pulses and another person nearby recorded them. Between each pulse transmitted and-or word, there still needs to be a perceived or recognizable silence so as to be able to recognize each unique or individual pulse. Below is a adaptation of the 5 row by 5 column Tap Code, and it rather uses 6 rows and 6 columns so as to include both the 26 letters alphabet and the 10 the numbers in the decimal system. The standardized Tap code has no explicit number characters or letters, and the letter K is omitted from it, but C can be sent in its place, and this keeps the code table at a 5 by 5 arrangement. To send a letter or number, first pulse the corresponding row value it is in, and then pulse the corresponding column value it is in. Perhaps the receiver can be instructed to return a signal that the two values and-or letter was received, and that they are then ready to receive more, resend, wait, etc. To send punctuation, a prearranged code method can be used, such as for example using Q or QQ to mean a question mark.

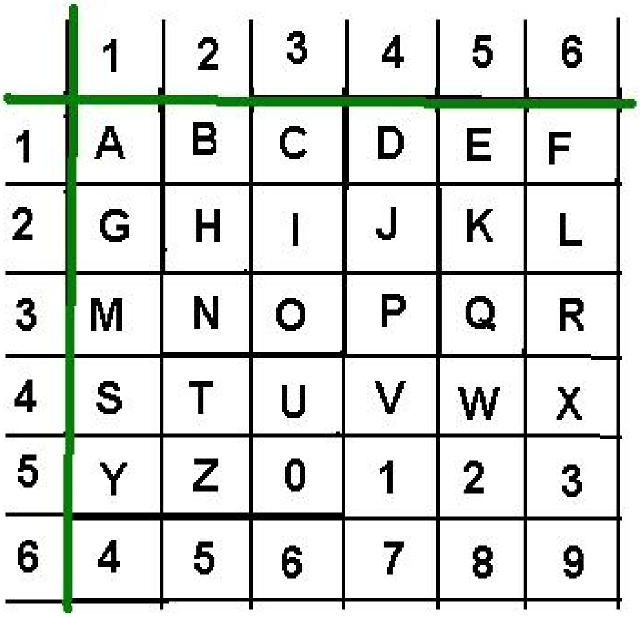
Tap Code , standard

	1	2	3	4	5
1	A	B	C	D	E
2	F	G	H	I	J
3	L	M	N	O	P
4	Q	R	S	T	U
5	V	W	X	Y	Z

: It is recommended to learn the rows first: A=1, F=2, L=3, Q=4, V=5 , and then the others letters in that row will be just a few letters away, and depending on the column value. One mnemonic for the first column is: "**A** **F**riend **L**oves **Q**uickly". A special character like N = (3,3) can be used to indicate a number follows.

Possibly (N , N) = (3 , 3) , (3 , 3) can be used to signal a number value will follow next, and of which can be the number of pulses having the same value as the number itself, and then (N , N) can signal the end of the number, and that letters will follow next.

[FIG 306] An Example Of A Modified Tap or Pulse Codes



	1	2	3	4	5	6
1	A	B	C	D	E	F
2	G	H	I	J	K	L
3	M	N	O	P	Q	R
4	S	T	U	V	W	X
5	Y	Z	0	1	2	3
6	4	5	6	7	8	9

One mnemonic for the first column letters in relationship to the corresponding row values can be:
"**A** **G**ood **M**an **S**tays **Y**oung **4**-ever".

With this modified tap or pulse counting code system, to send the letter J, send 2 pulses, then send a recognizable delay, and then followed by 4 pulses:

Letter = (Row pulses, Column Pulses) : Rows are horizontal = left to right , Columns are vertical = up and down.
J = (2, 4) = ... : = J is located at: row 2 , column 4

A way to signal or indicate you are ready to send a message is to perhaps shine a lamp continuously until the receiver shines theirs which signals that they have received and acknowledged your communication request signal, and then communication can begin. A set time(s) of communication can be also be used to help communication on a regular or synchronous (ie. synchronized, happening at the same time, "synced", coincidental) basis, instead of being asynchronous (ie., unsynchronous, non-synchronous, without specific timing and coordination, hence random-like or randomly).

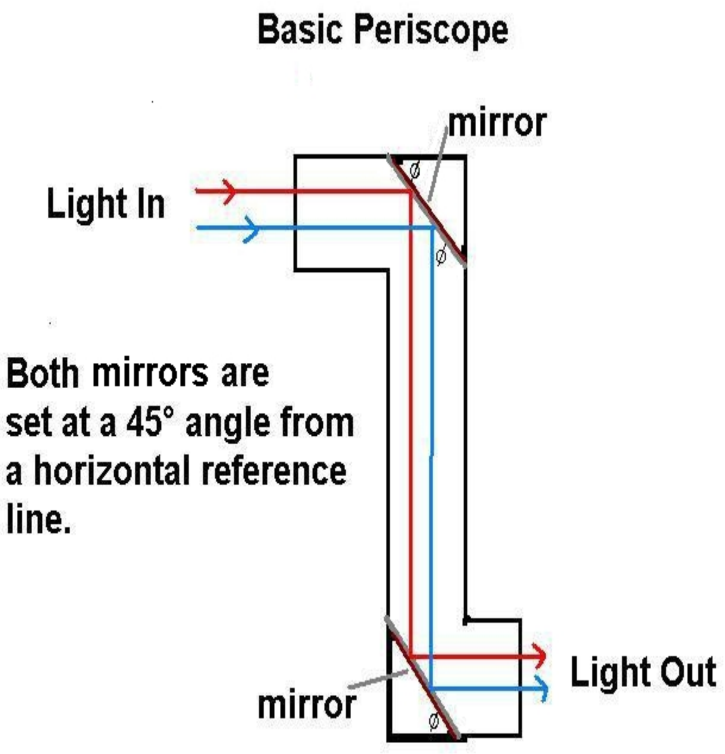
A computer can be used to code, send, receive and decode pulses to letters like this and each pulse is analogous to receiving a binary number 1 or logical true which is usually represented as 1. To distinguish each row and column value, and each letter, a silence or 0 needs to be sensed, and the specific duration of it will determine if the next pulse is the start of another pulse or letter. It is possible to use a certain frequency or tone to signal that a new pulse will be sent, and tone(s) and-or pulsed-tones can be used to signal the specific row or column value is being sent. With this latter method using pulsed-tones, the timing and-or communication synchronization becomes less of an issue. In a modern computerized communication system, it may be rated as being able to transmit up to 10 bits or pulses per second. The timing of each bit is then: 10 bits/second , and after dividing the numerator and denominator by 10, we find that; 10bits/s = 1 bit / 0.1s. After each tenth of a second or 0.1s, the computer will be instructed to send or sense (ie., "read") the next bit value of either 0 or 1. A longer delay could be used to signal the next byte of data will be communicated. After receiving 8 bits or a byte of data, it can be stored in a file. A special byte value may even signal a (communication) control code, such as start of transmission, end of transmission, error, resend, new line of text, etc. The ASCII communication code for text messages has several control codes having a byte value in the range of 0 to 31.

Again, for effective communication, a language or code as a standard must me used.

How To Construct A Periscope

A periscope can allow an observer to see over an object blocking their view, and-or for safety purposes to avoid injury. You can construct one out of cardboard, wood, or plastic, and held together by some type of glue or hot-glue stick, but tape is possible for initial tests. You can experiment with various sizes so as to see what works best for you. An example size to experiment with might be the main (square) tube having a length of 1 foot =~ 30.5 cm. Two mirrors need to be cut and have their sharp edges sanded to be slightly rounded using a piece of sandstone bought or found, and by an **experienced person** using work gloves. The shape of the mirrors can be square shaped with each side being about 2 inches =~ 5 cm long. To protect the mirror, the back of it can be coated with glue or paint and-or even a piece of thin wood or plastic.

[FIG 306B]



An Example Of A Batchfile Program for Intel-DOS Personal Computers

As mentioned in various locations in this book, often when dealing with computer programs, a batchfile is a series and-or batch (ie., a pile, a quantity, several) of operating system commands that can easily do a certain task without much user input or concern. A batchfile can be considered as both a computer command and program. Usually these tasks are related to the files, screen, or printer. For the batchfile program below, it is for computers that have an Intel style, DOS style operating or command line system and of which allows these batchfiles. This batchfile program will copy the dragged-and-dropped (using the mouse) file onto it into several disk drives locations. These locations and-or file directories (sometimes called folders) can be edited as needed by your system and needs, for example, maybe your computer does not have a certain drive, etc. To make a batchfile, enter its text into a text file with a text editor, and then rename the .txt filename extension to .bat. If you are able, you can research the available batchfile commands for your computer system. A good way to do this is to type and enter the HELP command on the command line. For the fine details on using batchifles commands you will need to research batchfile programming since it is beyond the general scope of this math and science book. Another helpful batchfile was placed after this first one, and it allows the user to quickly go to the current directory's command line of which the batchfile is in. You can often place a program such as a batchfile into the Windows (R) operating system's SENDTO folder, and when you mouse-right-click on a filename, you can select SENTO and the program to use that file. To place a program into the SENDTO folder so it can be accessed from any directory or folder, select: Start, then Run, and enter: "sendto" , and the SENDTO folder will be shown and you can then place what you need to be there.

Batchfile also have some of their own unique commands for "batchfile programming" and you will need to research all of these if you want to and obtain the literature or reading material. People, such as secretaries who work with many computer files can benefit from using batchfiles and-or learning how to program and-or edit them. For example, the REM command is used to enter a remark or comment such as what the batchfile is about or doing and this command not displayed on the screen. ECHO is used to display text on the screen. ECHO with a period after it is used to display a blank line. @ECHO off is used to not display the following commands in the batchfile program.

```
@echo off

rem  copytodrives.bat
rem (c) JPA 2023 and previous

@echo off

echo Batchfile To Copy An Input File To Several Locations
echo Can use drag and drop, or copy and paste, or enter
echo the filename on the command line.
echo.

rem  If no input file was on the command line or dragged and dropped onto the batchfile
rem  filename, the following code will be skipped over, and will resume at the HERE1 label
rem This older command may not work right on newer computers:    if "%1"=="" goto HERE1

 if [%1]==[] goto here

echo Copying %1 to these locations:
echo.

echo Copying to drive c:\
copy %1 c:\    /y
REM Can use copy %2   c:\    /y      , and %3 etc. to copy several selected files, drag and drop , or mark and copy
REM                                    then paste this on the batchfile command.  Be sure to do this for all the drives.

echo.
```

```
echo Copying to drive d:\
copy %1 d:\    /y

rem echo.
rem echo Copying to drive e:\
rem copy %1 e:\ /y       /* Currently on my system, E is the CD read only drive   */

echo.
echo Copying to drive f:\
copy %1 f:\     /y

echo.
echo Finished
echo.

rem normal batch file termination here:
rem When the pause command is encountered during processing, the user is prompted to press a key to resume.
pause
exit

:HERE1
echo off
echo To use this batch file, please drag and drop the file you
echo want to copy to your drives onto the batch file icon.
echo.
pause
exit
```

This batchfile below allows the user to quickly go to the current directory's or folder's command line of which the batchfile is placed in.

```
@echo off

rem  ToThisDOS.bat
rem  This batchfile allows the user to quickly go to the current directory's or folder's command line.
rem  (c) JPA 2023  , Author of the Mathization , Math and Science ebook

echo.------------------------------------------------------------------------

echo Type HELP to see many of the available DOS or computer system commands.
echo Type HELP command_name to see about a specific commands usage.
echo.
echo CTRL C  or  CTRL BREAK  can sometimes be used to terminate a command.
echo.
echo Type EXIT to return to system.
echo.------------------------------------------------------------------------
echo.

cmd.exe
```

A COMPUTER PROGRAM TO COPY A FILE TO ANOTHER FILE

```
/*------------------------------------------------------------------------------------------------------------
copyfile.c

                Copy a file to a new file and-or location. This is similar to the
                DOS or WINDOWS operating system command called "copy".
                Ex.  copy myfilie.txt  myfile.bak

                  It will help if this program is in the same directory, other wise you can
                 consider putting this program in one of the computer's "environment
                 variables" called "PATH" which indicates where the computer system
                 will to search for the commands, etc. typed/entered on the command
                 line, or clicked on by the mouse selector/pointer, etc.  The PATH
                 variable is often located in the batchfile called "autoexec.bat" that is ran
                 when the computer is first started, however, this PATH variable can also
                 be modified from within a user made batchfile.  You may also consider
                 placing a program(s) in the Window's SENDTO folder so as it will be
                 seen and available in the mouse right-click menu.

               For Now, I am letting the user input the filenames, rather than use the
               drag-and-drop input file onto the program name when programmed to
               accept command-like arguments

(c) JPA April 26, 2023
--------------------------------------------------------------------------------------------------------------*/

#include <stdio.h>       /* : Usually the same as #include "stdio.h"   */
#include <conio.h>
#include <stdlib.h>

/*------------------------------------------------------------------------------------------------------------*/
void main(void)
{
FILE *fp1=NULL;
FILE *fp2=NULL;
char inputfile[256];
char outputfile[256];
char c=0;
char e=0;
unsigned long filesize=0L;
unsigned long L=0L;

fflush(stdin);
printf("\nEnter the input filename: "); gets(inputfile);
/* used gets() since scanf() terminates the input at the first space */

fflush(stdin);
printf("\nEnter the output filename: "); gets(outputfile);

/* printf("\nInput file = %s",inputfile);  */
/* printf("\nOutput file = %s",outputfile); */

fp1=fopen(inputfile,"rb");  /* Let's see if that input filename and-or  file already exists. */
```

```
if(fp1==NULL){  printf("\nInput file not found or error, press a key");
                fflush(stdin);
                getch(); exit(0);
              };

/* If the input file does exist: */
rewind(fp1);
fseek(fp1,0, SEEK_END);      /* : Get filesize.  Set file access position at the end of the file marker. */
filesize = ftell(fp1);
L=filesize;
printf("\nFilesize = %lu bytes.",filesize);
if(filesize==0L){   printf("\n\aThe input file is empty.  Press a key.");    getch();  fclose(fp1); exit(0);   };
rewind(fp1);    /* : ensure the file pointer and-or access (read or write) position is at the first byte in the file */

fp2=fopen(outputfile,"rb");   /* See if the output file already exists. */

if(fp2 != NULL){   printf("\nThis output file already exists.  Press 1 for overwrite, or 0 to exit.");
                   fflush(stdin);  c=getch();  c=c-48;  if(c!=1){  fclose(fp2); exit(0);  };
                   fclose(fp2);   /* :since we want to now open the file for writing access */
              };
fp2=fopen(outputfile,"wb");   /* If file already exists, it will be erased to 0 bytes, and so as to be written over.  */
                              /* "wb" will allow writing access, beginning at the first byte of the file. */

printf("\n\nPlease wait until finished processing.");

/*------------------------------ Copy the data in the input file to the output file.  ---------------*/

/* A loop to copy (read and write) the file byte by byte:  */

while(L>0){
                  e=fread(&c,1,1,fp1);     /* : Read a byte from the input file.  */
                  if(e==0){  printf("\n\nFILE READ ERROR"); fflush(stdin);
                             printf("\nPress A Key");  getch(); exit(0);  };

                  e=fwrite(&c,1,1,fp2);    /* : Write a byte to the output file.    */
                  if(e==0){  printf("\n\nFILE WRITE ERROR"); fflush(stdin);
                             printf("\nPress A Key");  getch(); exit(0);  };

                  /* printf("\n%lu %d ",L,c)    : optional , to display each file address and the byte stored there      */
                  /*                             The user can press pause on the keyborad to temporarily halt the program,
                                                 or CTRL and Break , or CTRL and C to stop the program.    */

                 L = L - 1;
               };

/* ----------------------------------------------------------------------------------------------------- */

fclose(fp1);
fclose(fp2);

printf("\n\nFinished, press a key: ");  fflush(stdin); getch();

};
/*-----------------------------------------------------------------------------------------------------------*/
```

A COMPUTER PROGRAM TO CALCULATE FREQUENCY AND WAVELENGTH

```
/*--------------------------------------------------------------------------------
 FrequencyAndWavelength.c

   A program to allow the user to input a radio frequency and get the corresponding
   wavelength, or input a wavelength and get the corresponding radio frequency.

  (c) JPA December, 2022

  Speed of light =  c  =  299,792,458 m/s
  wavelength = c / f      : f = frequency = cycles per second = hertz = hz
  frequency =  c / wavelength
--------------------------------------------------------------------------------*/

#include "stdio.h"
#include "float.h"

/*--------------------------------------------------------------------------------*/
void main(void)
{
double f=0.0;
double w=0.0;
double c = 299792458.0;    /* speed of light in meters per second */
char choice;

printf("\n\nFrequency Or Wavelength Calculator Program");
printf("\n(c) JPA 2022 , Author of the Mathization, ");
printf("\nMath And Science book");

for(;;){
        printf("\n\n------------------------------------------------");
        printf("\n\n");

       printf("The Speed Of Light Is 299,792,458 m/s = about 300,000,000 m/s  =~  186000 miles/s");

        printf("\n\n1 Input Frequency To Find Its Wavelength.");
        printf("\n\n2 Input Wavelength In Meters To Find Its Frequency.");
        printf("\n\n       1 m = 100 cm = 3.28084 ft = 39.37 in");
        printf("\n       1 cm = 0.01 m = 0.3937 in  =  6.3 sixteenths of an inch");
        printf("\n       1 ft = 12 in = 0.3048 m = 30.38 cm");
        printf("\n       1 in = 0.08333...ft = 0.0254 m = 2.54 cm exact");
        printf("\n\n0 Exit");

        printf("\n\nEnter Choice:  ");
        fflush(stdin);   scanf("%d",&choice);
        if(choice<0 || choice>2){  printf("\a"); continue; };

       if(choice==0){  exit(0); };

       if(choice==1){ printf("\n\nEnter the frequency in hz:  ");
                    fflush(stdin); scanf("%lf",&f);
                    if( (f<0.0) || (f > 3000000000.0) ){  printf("\a"); continue; };    /* let 3 Ghz be max.. */
                    printf("\n");
```

```
                printf("\nWavelength = %.15g meters = %.15g feet",(c/f) , (c/f) * 3.28084 );
              };

      if(choice==2){ printf("\n\nEnter the wavelength in meters:  ");
                fflush(stdin); scanf("%lf",&w);
                if( (w<0.0) || (w > c) ){  printf("\a"); continue; };    /* let 300 million meters be max.. */
                printf("\n");
                printf("\nFrequency = %.15g hz",(c/w));
              };

      printf("\n\nPress A Key");  fflush(stdin); getch();

     };  /* end of main loop */

return;
};
/*---------------------------------------------------------------------------------------*/
```

Here is an example of the above programs input and output:

```
Frequency Or Wavelength Calculator Program
(c) JPA 2022 , Author of the Mathization,
Math And Science book

----------------------------------------------

The Speed Of Light Is 299,792,458 m/s = about 300,000,000 m/s  =~  186000 miles/s

1 Input Frequency To Find Its Wavelength.

2 Input Wavelength In Meters To Find Its Frequency.

      1 m = 100 cm = 3.28084 ft = 39.37 in
      1 cm = 0.01 m = 0.3937 in  =  6.3 sixteenths of an inch
      1 ft = 12 in = 0.3048 m = 30.38 cm
      1 in = 0.08333...ft = 0.0254 m = 2.54 cm exact

0 Exit

Enter Choice:  1


Enter the frequency in hz:  1000000


Wavelength = 299.792458 meters = 983.57108790472 feet

Press A Key
```

A COMPUTER PROGRAM FOR A HOMEMADE PROGRAM LOOP

With many of the older BASIC programming languages, a programming loop is made using both a LABEL statement and the GOTO command. A loop is a programming concept to reuse code, and often, to do so many times in an easier way that rewriting commands or instructions to do things, often the same thing, "over and over" or repeatedly. A **label** is an identifier used to mark or and-or save its location (ie., a memory address) in the program's code or statements so that it can be executed (ie., performed) when called using the GOTO command. For the C programming language, the GOTO command is rather the lower case version of GOTO, and is: goto. Note that goto is also one of the C programming keywords, and therefore, it cannot be reused to mean something else, such as the name of a variable used in your program. The GOTO command is also very similar to using the JUMP or JMP command with an assembly and-or machine programming language. C already has formalized built in loop keywords and-or programming constructs such as the for() loop, and the somewhat simpler while() loop. Of more use for a LABEL is that it can be used to create a location within the current function, and it will indicate where to go to and-or return to on occasion and-or need, and not necessarily being used only to create homemade loops. The GOTO command is one of the available program flow commands. More often, the use of the GOTO keyword is to jump or skip over (ie., not to be used or processed) programming statements of code that are currently not to be used for some reason or logic. Computer purists and-or "high level" programming ideals would rather programs use only the "higher level" constructs of more strict or formalized groupings or functions, and rather than using the more primitive-like fundamental computer machine or hardware code or operations like the goto command. The goto statement or command is much like a software to physical switch and-or a machine control settings that can be used to access and control the CPU (Central Processing Unit, IC circuit, "chip") of the computer machine.

```
/*------------------------------------------------------------------------------
homemadeloop2.c

Using the goto command to create a loop programming construction.
It is also possible to create a function construction with goto.

(c) JPA  2022
------------------------------------------------------------------------------*/
#include "stdio.h"
/*----------------------------------------------------------------------------*/
void main(void)
{
int n=0;

/*----------------------------------------------------------------------------*/
/*        A FUNCTION OR LOOP TO PRINT NUMBERS 0 T0 9   */
/*----------------------------------------------------------------------------*/
startloop1:  ;        /* A label is assigned a memory address to go to in the executable, run-time machine code
                         of the program. */

    printf("%d  ",n);
    n=n+1;
    if(n==10){  goto endloop1; };    /*  : essentially a homemade "break command" */

    goto startloop1;        /* : go to the start of the loop,  it can also be used as a loop continue command */

endloop1:  ;
/* ---------------------------------------------------------------------------*/

printf("\n\nPress a key: ");   fflush(stdin);   getch();

return;
};
```

```
/*------------------------------------------------------------------------*/
```

Program output:

```
0  1  2  3  4  5  6  7  8  9

Press a key:
```

A COMPUTER PROGRAM TO CONVERT A NUMBER OF SECONDS TO Y:D:H:M:S FORMAT

Many other conversions can be thought of and put into this program example, and-or other programs.

```
/*----------------------------------------------------------------------------------------
TimeConverter.c

This example program converts an input number of seconds to Y:D:H:M:S format
or:   Years:Days:Hours:Minutes:Seconds  format

For reference:

1 minute = 60 seconds
1 hour = 60 minutes = 3600 seconds
1 day = 24 hours = 24 (3600s) = 86400s
1 year = 365 days  =  365.0 (86400s) = 31,536,000s     When using the basic amount 365.0 days.
                                                        See below.

As a test of this program , the user can input 31626061 for the seconds value,
and the result should be: Y:D:H:M:S = 1:1:1:1:1    As another test, 86460 seconds
should give 0:1:0:1:0    Another test of 90061 should give:  0:1:1:1:1

NOTE and for floating point values in general, scientific notation and-or powers
of 10 can be input using letter e or E which means the exponent of 10. 10^e or 10^E
Ex.  To input 3600 , you could use:  3.6E3 or 3.6e3  which can be interpreted as
3.6 (10e3)  or=  3.6 (10^3)

(c) Feb. 2023, JPA
-------------------------------------------------------------------------------------------*/

#include "stdio.h"
#include "conio.h"

/*-------------------------------------------------------------------------------------------*/

void main(void)
{
double input=0.0;
double inputseconds=0.0;
double Y=0.0;   /* years */  /* Here using non-integer float values so as to
                                have input and output  fractions of a value. */
double D=0.0;   /* days */
double H=0.0;   /* hours */
double M=0.0;  /* minutes */
double S=0.0;   /* seconds */
char c=0;
double temp=0.0;
double yearlength = 365.0;              /* days of a calendar or clock year */
                                        /*  a solar year is:  365.242190402 days */
                                        /* a siderial year is 365.25636004 days */

/*  printf("\n%.15g",yearlength * 86400);  fflush(stdin); getch();   =  31558149.507456  seconds */

printf("\n-------------------------------------------------------------------");
```

```
printf("\nTIME CONVERTER PROGRAM");
printf("\nFrom the Mathization Ebook.");
printf("\n(c) JPA 2023");

printf("\n\nThis program converts an input amount seconds to");
printf("\nY:D:H:M:S = Years:Days:Hours:Minutes:Seconds format.");
printf("\nA siderial or same star position year is 365.256363004 days ");
printf("\nlong in time value.  A Sun alignment or solar year =");
printf("\n365.242190402 days.long in time value.  A common calender or");
printf("\nor clock year = 365 days long and a leap-yar clock year is 366");
printf("\ndays long.  This program will use the 365 day, calender year");

/*-----------------------------------------------------------------------------------------------------------------------------*/

for(;;){    /* main loop , loop 1 */

printf("\n-----------------------------------------------------------------");

Y=0.0;  D=0.0;  H=0.0;  M=0.0;  S=0.0;  c=0;
 input=0.0;  temp=0.0;  inputseconds=0.0;  /* "clear"or reset the values to work with */

printf("\n\nProgram Choices: ");

printf("\n\n1 Input the number of seconds to convert to the Y:D:H:M:S format");

/* Other conversion choices can be enumerated (numbered) and placed here. */

printf("\n0 Exit Program");

printf("\n\nEnter Selection:  ");
fflush(stdin); c=getch(); c=c-48; /* convert the characters ASCII value to an integer */
printf("%d",c);
if(c<0 || (c>1)){ continue;  }
if(c==0){  exit(0);  };

if(c==1){
                fflush(stdin);
                printf("\nEnter the number of  Seconds: ");
                scanf("%lf",&input);    /* input = seconds  ,  printf("\n%.15g",input);  test ok  */
                if(input < 0.0){ printf("\a"); continue; };  /* if user input a neg. value */
               /* printf("\n%.15g",input), */
                 inputseconds=input;

               /*---------------------------------------------------------------------------------------------------------------*/

               /* Get the number of years. */              /* For equivalent HOURS */
                   Y = input / 31536000.0;        /*  A 365.0 day year = 31536000 s */
                   printf("\n%ld seconds = %.15g Years = Y",(long int) inputseconds,Y);

                  /* the long int typecast effectively removes the fractional part of the floating point number. */
                  /* printf("     The whole number of years = %ld",(long int)Y); */

                /* get the remaining seconds ,after the year amount of seconds is subtracted */
```

```
        temp = input -  (double) ( ((long int) Y) * 31536000.0);
        input = temp;    /* as like a new number of seconds input */
      /*  printf("\remaining seconds = %.15g",temp); */

  /*-------------------------------------------------------------------------------------------------------------------*/

       /* For equivalent DAYS leftover or remaining.  */

       D=input / 86400.0;
       temp = input - (double)( ((long int) D) * 86400.0);  /* subtract the number of seconds in those days */
       input = temp;    /* as like a new number of seconds input to work with*/

       printf("\nRemaining Days = D = %.15g",D);
       /*  printf("     The whole number of Days  = %ld",(long int) D); */

  /*-------------------------------------------------------------------------------------------------------------------*/

     /* For the equivalent HOURS left. */

       H=input/3600.0;
       temp = input - (double)( ((long int) H) * 3600.0);  /* subtract the number of seconds in those hours */
       input = temp;    /* as like a new number of seconds input */

       printf("\nRemaining Hours = H = %.15g",H);
       /*  printf("     The whole number of  Hours  = %ld",(long int) H);  */

  /*-------------------------------------------------------------------------------------------------------------------*/

     /* For the equivalent MINUTES  left. */

       M=input/60.0;

       temp = input - (double)( ((long int) M) * 60.0);  /* subtract the number of seconds in those minutes */
       input = temp;    /* as like a new number of seconds input */

       printf("\nRemaining Minutes = M = %.15g",M);
       /*  printf("     The whole number of  Minutes  = %ld",(long int) M);  */

  /*-------------------------------------------------------------------------------------------------------------------*/

     /* For the equivalent number of  SECONDS  left or remaining.  */

       S=input;

       temp = input - (double)( ((long int) M) );  /* subtract the number of seconds in those minutes */
       input = temp;    /* as like a new number of seconds input */

       printf("\nRemaining Seconds = S = %.15g",S);
       /*  printf("     The whole number of  Seconds  = %ld",(long int) S);  */

     printf("\n\n%ld seconds = Y:D:H:M:S = %ld:%ld:%ld:%ld:%ld",(long int) inputseconds, (long int)Y,(long int)D,
              (long int)H, (long int)M,(long int)S);

};
```

```
        /*------------------------------------------------------------------------------------------------------------------------------------------------*/

        printf("\n\nPress A Key To Continue: ");  fflush(stdin); getch();

   };  /* end of  loop 1 */


return 0;
};
/*---------------------------------------------------------------------------------------------------------------------------------------------------------*/
```

Program use example:

```
------------------------------------------------------------
TIME CONVERTER PROGRAM
From the Mathization Ebook.
(c) JPA 2023

This program converts an input amount seconds to
Y:D:H:M:S = Years:Days:Hours:Minutes:Seconds format.
A sidereal or same star position year is 365.256363004 days
long in time value.  A Sun alignment or "complete orbit" solar year =
365.242190402 days long in time value.

A common calendar or clock year = 365 days long and a leap-year clock year is 366
days long which usually happens every 4 years so as the Earth will be aligned to the
same star, and  at the same time of day.   If the year is evenly divisible by 100, the leap
year (366 days) is skipped and the 365 day year is used.  If the century year (evenly
is divisible by 100) is a 4 year leap year, it will be skipped unless it is evenly divisible
by 400. This program will assume and use the common 365 day calendar year.
For each non-leap-year which considers only 365 days, it is 0.242 days short, which is
slightly less than a quarter of a day, and this quarter of a day each year adds up to about
4 days every 4 years of clock (non-astronomical) time..
------------------------------------------------------------

Program Choices:

1 Input the number of seconds to convert to the Y:D:H:M:S format
0 Exit Program

Enter Selection:  1
Enter the number of  Seconds: 31626061

31626061 seconds = 1.00285581557585 Years = Y
Remaining Days = D = 1.04237268518519
Remaining Hours = H = 1.01694444444444
Remaining Minutes = M = 1.01666666666667
Remaining Seconds = S = 1

31626061 seconds = Y:D:H:M:S = 1:1:1:1:1

Press A Key To Continue:
```

A Computer Program To Convert Between Fahrenheit, Celsius, and Kelvin Temperatures

```
/*-----------------------------------------------------------------------------------------
temperatures.c

A Computer Program To Convert Between Fahrenheit, Celsius, and Kelvin Temperatures

Note, the Celsius and Kelvin temperature scales are linear.  Each 1 degree rise in
Celsius is 1 degree rise in Kelvin and vice-versa.  This is not the case with the
Fahrenheit scale in relation to the Celsius and Kelvin scales.  When converting from
Fahrenheit to Kelvin or vice-versa, it is best to first convert its value to Celsius.

(c) JPA 2023

Programming Language:   ANSI C
Compiler:  TinyC

-------------------------------------------------------------------------------------------*/

#include "stdio.h"
#include "conio.h"

/*-----------------------------------------------------------------------------------------*/
void main(void)
{
double K = 0.0;           /* Kelvin        */
double F = 0.0;            /* Fahrenheit */
double C = 0.0;            /* Celsius       */
double input = 0.0;
double temp=0.0;
char c;

printf("\n\nTemperature program to convert between Fahrenheit, Celsius, and Kelvin");
printf("\n(c) JPA 2023, for the Mathization eBook");

for(;;){     /* loop1  */

     F = 0.0;  C = 0.0; C = 0.0;  input = 0.0;  temp=0.0;   c=-1;    /* : reset for reuse */

     printf("\n\n-----------------------------------------------------------");
     printf("\n\n1 Enter a Fahrenheit Temperature. ");
     printf("\n2 Enter a Celsius Temperature. ");
     printf("\n3 Enter a Kelvin Temperature. ");
     printf("\n\n0 Exit. ");

     printf("\n\nEnter Selection:  ");
     fflush(stdin);  scanf("%c",&c);
     c=c-48;  /* convert ASCII to numeric selection */
     if(c==0){  printf("\n\n");  break; };
     if( (c<0) || (c>3) ){  printf("\a"); continue;  };

    if(c==1){   printf("\nEnter Fahrenheit Value:  ");
                   fflush(stdin);
                   scanf("\n%lf",&F);
```

```
           if(F < -459.67){ printf("\a\nPlease enter a Fahrenheit value greater than -459.67");
                            printf("\nPress A Key");  fflush(stdin); getch(); continue;
                          };

           C = (F - 32.0) / 1.8;    K = C + 273.15;      /*   1.8 =  9 / 5   */
        };

  if(c==2){   printf("\nEnter Celsius Value:  ");
           fflush(stdin);
           scanf("\n%lf",&C);

           if(C <  -273.15){  printf("\a\nPlease enter a Celsius value greater than -273.15");
                              printf("\nPress A Key");  fflush(stdin); getch(); continue;
                          };

           K = C + 273.15;    F = (1.8 * C)  + 32.0;       /* 1.8 is almost 2 */
        };

  if(c==3){   printf("\nEnter Kelvin Value:  ");
           fflush(stdin);
           scanf("\n%lf",&K);

           if(K < 0.0){  printf("\a\nPlease enter a Kelvin value greater than 0. ");
                           printf("\nPress A Key");  fflush(stdin); getch(); continue;
                       };

           C = K - 273.15;    F = (1.8 *  C) + 32.0;      /* = (1.8 * (K + 273.15) + 32.0 */
        };

        printf("\nF = %.5g  =  C = %.5g  =  K = %.5g",F,C,K);


  };   /* end of loop1 */


return;
};
/*------------------------------------------------------------------------------*/
```

Here is an example screen output of the above program:

```
Temperature program to convert between Fahrenheit, Celsius, and Kelvin
(c) JPA 2023, for the Mathization eBook

------------------------------------------------------

1 Enter a Fahrenheit Temperature.
2 Enter a Celsius Temperature.
3 Enter a Kelvin Temperature.

0 Exit.

Enter Selection: 1
```

```
Enter Fahrenheit Value:  212

F = 0  =  C = -17.778  =  K = 255.37

--------------------------------------------------------
```

A COMPUTER PROGRAM TO DISPLAY THE BYTES OF A FILE

```
/*--------------------------------------------------------------------------------------------------------------------
showbytes.c    Shows the decimal value of bytes of a file and displays a corresponding
                    (non-ASCII  control codes)  text characters associated with those values.

                       (c) JPA 2024
                       compiler used:  TinyC  , and with ANSI C standards only
---------------------------------------------------------------------------------------------------------------------*/

#include "stdio.h"
#include "stdlib.h"
#include "conio.h"

/*-------------------------------------------------------------------------------------------------------------------*/
void main(void)
{
char inputfile[256];
FILE * fp1;
unsigned long filesize=0;
unsigned long L=0;;
unsigned char byte=0;
int n=0;
unsigned int e=0;
unsigned char c;
/*-----------------------------------------------------------------------------------------------*/

printf("\n\nShowbytes is a program to display the decimal values of each byte of a file.");
printf("\n\n");
printf("\nInput the filename:  ");
gets(inputfile);

fp1=fopen(inputfile,"rb");        /* Let's see if that input filename and-or  file already exists. */
if(fp1==NULL){  printf("\nInput file not found or error, press a key, and then restart this program.");
                      fflush(stdin);
                      getch(); exit(0);
                    };

/* If the input file does exist: */
rewind(fp1);
fseek(fp1,0, SEEK_END);      /* : Get filesize.  Set file access position at the end of the file marker. */
filesize = ftell(fp1);
printf("\nFilesize = %lu bytes. \n",filesize);
if(filesize==0L){   printf("\n\aThe input file is empty.  Press a key.");    getch();  fclose(fp1); exit(0);   };
rewind(fp1);

for(;;){
```

```
        for(n=0;n<10;n++){        /* display 20 bytes at a time */

                              e=fread(&byte,1,1,fp1);     /* Read a byte from the input file.  */
                              if(e==0){  printf("\a\n\nFILE NOT FOUND OR READ ERROR");
                                          printf("\nPress A Key");  fflush(stdin);  getch(); exit(0);
                                         };

                               c=byte;
                               if(c>=32){  c = byte; } else{ c=46;  };  /* This "character filter" can be adjusted as needed */
                               /* period character = ASCII 46  , for non displayable ASCII control codes */

                               printf("\nByte# %lu = %u = %c",L+1,byte,c);
                               L=L+1;  if(L>=filesize){ printf("\a\n\nEND OF FILE, Press A Key"); fflush(stdin); getch();
                                          exit(0);  };

                          };    /* end of inner loop */

printf("\n\nPress A Key For More or ESC to end program: "); fflush(stdin);  c=getch();  printf("\n\n");
if(c==27){  exit(0);  /* exit() will also close any open files, but you can also do it for formality.   */  };

    };     /* end of outer loop */

};
/*-----------------------------------------------------------------------------------------------------------------------------*/
```

Memory Allocation

Placing file data into electronic RAM ((dynamic) random access memory) can improve the speed of programs and reduce the wear on a disk drive, but it can also be dangerous if power to the computer is interrupted and all that data in memory has then been erased or lost, and so it is good to back up that data or just any new immediate data or records that have been created or changed, and onto a disk drive and-or external storage. **Saving data on several locations or drives is generally a good idea if that data is safe from drive failure and-or hackers (infiltrators, etc).**

The above program read each byte from the file and then displayed it, and then continued this till there were no more bytes of data in the file. You can also place or "load" all of file data into a portion of memory by first requesting use of some memory with the memory allocation function such as **malloc().** Allocating memory during a running program is called **dynamic memory allocation**. Here is some example and necessary code statements to consider into a program such as for a new version of the previous program about displaying a text file:

```
.
.
.
#inclulde <malloc.h>        /* formally include the header file that the malloc() is in */
..
unsigned char *p;      /* this will be the pointer to access the block of assigned computer memory */
long int n=0L;          /* basically a counter variable used for incrementing the pointer, but here technically, the pointer
                          stays the same value and-or location, but a temporary pointer value and-or location is the resultt */
.
.
p=malloc(filesize);  if(p==NULL){  printf("\n\n\a * * * MEMORY ERROR * * * , Press a key "); fflush(stdin); getch(); exit(0); };
.                        /* here, the filesize variable used represents the size or number of continuous bytes of memory to be
                          allocated for  the programs use */
.
/* Get or read a f byte from the file and put  or write it into memory.  Do this in a loop  until all the file bytes are stored in
 memory.  */

fread(&c,1,1,fp);    /* here the byte read from the file identified, associated or assigned as (fp) to will be assigned to
                       variable (c) which should have an unsigned char data type  */
.
.
 *(p+n) = c;      /* In the file reading loop, place the byte read from the file into fast memory, after reading this byte the next
                    file access position will be automatically incremented to the next byte address, hence by 1 byte. */

n=n+1;           /* increment by one so as the temporary pointer is updated to the next byte in computer RAM memory */
.
.
\* Get or read a byte in memory and put or write it into the file:   */
.
n=0L;
.
.
c=*(p+n);      /*Get a byte from memory so as to then write it to the file.  Do this in a loop until all file bytes are written.  */
.
.
free(p);    /* "Free up", "give up" or de-allocate the allocated memory so that the computer and-or program can use it if
              needed, and which may include needing a larger or smaller memory allocation  */
```

SOME COMPUTER PROGRAM IDEAS TO CONSIDER

First, it is perhaps better for a programmer to write their own functions to do something, even if a function already exists, and they can create a new, modified and-or enhanced version. After all, every program consists of individual functions to help it perform what the program needs to do, and to keep it readable and orderly. Programmers should keep a collection of all the functions they created, and so as to have them available for later use, editing and-or study.

1. A program to **split a file** into smaller file size segments, and is also capable of rebuilding a file from those file segments being joined together into one file. This is helpful for very large files, such as video files, that are larger than the 700Mb space on a compact disk (CD) storage. It can also be a way to send a file that is greater in size than a standard file size attachment for email, such as say 20Mb. Each file segment must be sent as a separate attachment. Years ago, it was very common to receive and install a large program such as the Windows operating system from about 10 floppy disks. After installing the first disk, the computer program would display a message prompt such as "Please insert disk #2, and press a key", and so on.

2. Create a **byte editor**. Make a function(s) that reads and-or writes a byte(s) at a certain address(es), location or byte number. For example, the function arguments might be:

   ```
   unsigned char  readbyte(fFILE * fp , unsigned long address);
   void writebyte(FILE * fp , unsigned long address);
   ```

   The basic concepts from the computer programs can be considered. Ask the user to input the address (either the offset or logical address of the byte, or physical address value), and-or the byte value of it is to be edited to.

3. **Create a database**. The data for each record can be placed in a single data file, or a single, readable/editable (such as by non-computer programmers, etc) text file. For example, each record or group of related data might include a persons name on one line and some other numerical data on the following lines. This data can be searched for given a persons name, record or account number or some other number or keyword(s). The data can be read and processed to display things such as a certain record or some average of a specific piece of data in all the record. It is even possible to load all the data into computer memory for faster access, and then save the data to a file when completed. You may wish to allow random comments or notes to be included in this data, such as for future reference, and you will need to delimit or set it apart from the other data by using with some special character(s) or words. For a basic and helpful start to this, you may first consider the computer program in this book called: A Computer Program To Calculate Some Statistics Of Data. A C language programmer may wish to look up the **struct** keyword in C, and it is essentially for creating a user defined data type and-or programming construct that is composed of one or more of C's data types, and is analogous to creating a record of related information. struct essentially means a (data) **struct**ure. An array of the standard data types can also be used as a record of data, but it is usually for data elements having the same data type such as char (1 byte sized data) or int (2 byte sized data).

   A **union** is a special C programming construct that is somewhat like a struct data type, however, all the data in it essentially coincides (hence the name union) or overlaps each other, and you can access a certain byte(s) and-or other data type size that is in it. Generally, a union is for numeric data types, and where you then want to access each byte, or even a single bit of a larger data size type. You can also access bits of a byte using homemade functions and-or bits of a byte via bit per bit manipulation such as for example ANDing a byte value with 128d = 10000000b so to find the value (1 or 0) of the most significant bit of a byte. Ex. bit =(databyte & 128). For a basic union example, a union might consists of a two byte integer, and two char data types which are essentially 1 byte data types each. After assigning a value to the integer in the (data) union, each byte of it can be accessed by the char data types.

4. This book has programs, such as the calculator to solve one expression at a time which contains only one or two numeric operands such as 5 * 3 and the result is 15. This book also shows some ways so as to solve equations within a program, but it would also be very helpful to allow the user to input a text string consisting of longer and more complex expressions such as 5 * (3 + 1). What is needed to solve general user input expressions is to **create an equation parser** code within a program and-or entire program which separates the (text, string form) equation into its more manageable composing terms, and then calculate the value of each term, and then combines these term values

so as to produce the result of the entire equation. This would make a better calculator, and possibly one which includes optional variables and loops. Considerations for a parser should include expression evaluation from left to right as for the standard order of operations, and grouping symbols such a parenthesis surrounding an inner expression which have a higher precedence and must be calculated first. When you write a computer program, the language compiler already has an equation parser and solver built into it, but its for use by the programmer and generally not the user of the created programs.

5. **Create your own simple computer language**. The easiest way to do this would be to create a Basic-like language where each line of the input text-source file program is processed from top down or one after the other. The language interpreter program created should at least include the allowing of user input such as text strings, variables, loops of which can be made using the goto command and program location labels, screen display output, basic math calculations, and advanced programmers might wish to include a general equation parser for solving for the result of user input expressions. This interpreter program can use the compilers built in string, math and other functions, etc., and-or write your own as shown in this book. An example of a user-made string function would be one that converts a text string to a numerical value so that it can be used elsewhere in the program, such as in a numerical expression.

6. **Write simple program for a company**. This company can be either imaginary or real. First find or choose a topic to make a program about and what the result and-or output of the program should be. Consider what equations to use, and the values(s) to be input by the user. Allow the program to have a "help screen" or "about" menu option so as to display what the program is about, how to use it, possible helpful data, recommendations and-or options. The date, author, copyright, and-or owner of the program and contact information can also be displayed . Then consider some other programs to create for the same company and-or other companies. If a company pays you for a program, you will need to negotiate with them if they are also buying the copyright also, and so as to have all rights to the use of that program including the possibility of reselling it elsewhere, and surely, your price of the program will then need to be negotiated and fair. In a way, you will then have to become like a business person. If possible, advertise your programming work(s) so that you and others may also have a chance to gain from them.

A MENTION OF THE BASIC PROGRAMMING LANGUAGE AND A SIMPLE MATH PROGRAM

This book focused on just using the C programming language, and which was developed in about 1972 so as to make programming easier and faster for many people by using a higher level language other than the commonly taught and used machine code (ie., strict, pure computer instructions, and with each is assigned a number so as to call and-or use it) and-or assembly languages (helps create programs written using machine code and-or higher level instructions that are actually composed (ie., constructs) of several machine codes). There are many other standardized programming languages such as Cobol, Fortran and BASIC. **BASIC** means **B**eginners **A**ll-purpose **S**ymbolic **I**nstruction **C**ode. Here Code essentially means "computer language", and for a specific computer program created or written, the program code for it is often called the "source code", "source code program" or "source program". BASIC's initial development was in about 1963, and was several years before the C programming language.

Shown further below is a minimal description and example of what can be called a simple or minimal program written using the BASIC language. Only a few commands and-or keywords will be used out of the many (perhaps 50 or more) keywords assigned strictly for use with the BASIC interpreter program and-or for BASIC language program. In short, a variable name cannot be one of the keywords already set aside for the BASIC interpreter program, and it can not contain spaces.

If you are interested further in BASIC programming, please purchase a book(s) about it from a book store that has it. The example program here does not have all of the BASIC keywords and functions such as the DIM keyword to define an array or dimension of similar data values. Simple forms of BASIC programming are sometimes available on some hand-held calculators, and is this good enough for people who need to make an immediate and simple program so as to produce a useful numeric value(s) rather than using the more advanced and intricate programs for a larger computer system. BASIC is also capable of complex programs. To run or execute BASIC language programs, please first obtain a BASIC language interpreter program(s) that is compatible with your computer system and-or phone.

The BASIC interpreter is actually a computer program in itself. It will usually have the .exe or .com filename extension. The interpreter can run or execute computer programs written or expressed using the BASIC programming language. This BASIC interpreter is similar to a C compiler that can run or execute computer programs written or expressed with the C programming language. Most BASIC interpreters do not produce a stand-alone program like a C compiler does. Some BASIC interpreters may include support for the WINDOWS ((c) by the Microsoft co.) operating system for many personal computers, and even this is actually a computer program running on the computer. The BASIC interpreter will read each statement in the program that was written using plain text, and it will then convert it to its corresponding machine code and-or language that the computer requires to function. The interpreter will do this for each statement, one after the other when encountered in the specific program flow (path or branch) and execution taken, and in essence, it does not compile all the machine codes into a complete machine code program which can be run without an interpreter program. Some modern BASIC interpreters can compile a .bas program into an .exe file that can be ran without the interpreter program used to make it.

For math programming considerations, we will need to know the specific commands and syntax (ie., analogous to spelling) of the mathematical operations available in BASIC. We will also need to ask the user to input values, and then let the computer perform the calculation(s) as expressed in the program upon those input values (ie., mathematical operands) and-or constants, and then to then display a result(s).

If you are already familiar with the C or other programming language, you will be more prepared and understand the nature of a programming language like the BASIC programming language.

In BASIC, the string (technically an array of text characters) identifier is similar to using a numeric variable name and-or identifier, however, the $ symbol will follow it. Ex. N is a number, and N$ is a text string N will be considered a numeric floating point variable and-or value, and N$ will be considered a string variable. In common BASIC , a variable does not have to be declared and-or initialized (but it can help with editing and understanding the program). A variable can be a single text letter, or it can have two text letters and a number, or a single text letter followed by a number, and usually (depending on the interpreter) must be upper-case (capital) letters. Many BASIC interpreters will also allow more than 2 characters for an identifier, but only the first two or three are usually accepted and they must be unique in the entire program. Numbers can be floating point values which may have a "decimal" or fractional portion. To express the integer value of a floating point value and without the fractional portion use the % symbol following it. Ex. If N is a floating point value: PRINT N% , and this will print only the whole portion of the value. Strings can be displayed without them being assigned to a string variable,and this is called a string literal or string constant. For example: PRINT "Welcome"

Variable names and-or identifiers do not have to be declared in BASIC , unlike in C. If these values are not initialized, they will be initialized or assigned a value of 0 or a NULL (ie.,=0, empty, non) string, essentially a value of 0 which is the null character or string terminator character or identifier.

Line numbers are generally treated as like reference locations in a program. You can consider them as like labels that are used for reference in the C programming language. They can begin at 1, but in practice, they may begin at say 10 and skip ten for the next line number: 10 , 20 , 30 , and so on, for this allows possible new statements to be inserted between these lines and-or statements if needed later.

To write a program for a BASIC language interpreter, use a common, plain ASCII text editor which does not include formatting characters in the saved file such as would happen with a word processor. Rename the file to have the .bas filename extension, rather than having .txt as for a text file.

To include more than one statement on a line, separate them by a colon (:) character. For the typical BASIC language interpreter program, if there are more than one statements on a line, they will not be executed if the previous statement evaluates to false. The C programming language would include other statements to be executed in the braces syntax.

```
10 REM  HYPOTENUSE.bas
15 REM
20 REM  CALCULATES THE HYPOTENUSE OF A RIGHT TRIANGLE
30 REM  (c) JPA 2024
35 REM
40 REM  HERE ARE DECLARATIONS AND INITIALIZATINS OF THREE FLOATING POINT VALUES
50 REM  THE PROGRAM WILL USE, HOWEVER IN BASIC, THIS IS NOT NEEDED, BUT IS SHOWN
60 REM  TO HELP THE PROGRAMMER AND IN RELATION TO C PROGRAMMING  METHODS.  THE
70 REM  SPECIFIC LINE NUMBERS USED ARE OPTIONAL, HOWEVER, THE GOTO COMMAND NEEDS
80 REM  TO SEND THE PROGRAM FLOW  AND PROCESSING TO A SPECIFIC LINE NUMBER.
90 REM
91 REM
92 REM  THE SPECIFIC BASIC INTERPRETER USED WAS SMALLBASIC FOR THE ANDROID PHONE
93 REM
95 CLS
100 LET a=0
110 LET b=0
120 LET c=0
125 REM
150 PRINT "This program finds the hypotenuse of a right triangle"
160 PRINT
170 PRINT "Input length of side a: ";
180 INPUT a;
190 PRINT "Input length of side b: ";
200 INPUT b
210 c = SQR(a^2 + b^2)
220 PRINT
225 BEEP
230 PRINT "The hypotenuse c = " ;
240 PRINT  c
241 REM  Now lets find the arcsine of angle A , arcsine A = asin(sin A) = asin(a/c):
242 A = ASIN(a/c)
```

```
243 REM  Now lets convert radian angle A to its corresponding and equivalent degrees angle:
244 A = A * (180 / 3.14159265)
244 PRINT "Angle A = ";
244 PRINT A
245 PRINT "°"
246  REM  In a right triangle,  Angle B = 90° - Angle A    ,    B = ACOS(b/c)    ,   A = ATAN(a/b)
247 B = (90 - A)
248 PRINT "Angle B = ";
249 PRINT B;
250 PRINT "°"
255 REM TO ENTER THE DEGREES SYMBOL, PRESS ALT+248 ON THE NUMBER PAD , ANGLE symbol IS ALT+155
257 PRINT
260 PRINT "PRESS 1 TO CONTINUE OR 0 TO EXIT: ");
270 INPUT N
280 IF (N=0) THEN END
283 IF (N=1) THEN GOTO 95 ELSE GOTO 255
286 REM  AN OPTION TO  USING THE ELSE STATEMENT ABOVE, YOU CAN USE:    285  IF (N<>1) THEN GOTO 257
300 END
```

Some common math functions in the BASIC language and-or interpreter:

SIN(rad)   :for the sine value of the radian angle as the argument. See notes at cosine (COS) and tangent (TAN) below.
COS(rad)   : to convert a radian (R) value to a degrees (D) value use: D = R * (180 / 3.14159265358979323846
TAN (rad)   : to convert a degrees (D) value to a radian (R) value use: R = (D * 3.14159265358979323846) / 180
ATAN(tan) : depending on which BASIC interpreter is being used, it may have ASIN(sin) and-or ACOS(cos)
The value returned is a radian angle.
LOG(n)   :for LN(n) in BASIC   , For LOG base 10 use: LOG = LN(n) / LN(10) = LN(n) / 2.302585093
EXP(n)   :for e^n   . (e) = epsilon = Euler's Number is also the base of the natural logarithms (LN)
e = 2.71828182845904523536028747135...
ABS(n)   :for absolute value
INT(float)   :for integer or whole portion of a value considered a potential floating point value
SQR(n)   :for square root   ,   for use, ex: root = SQR(n)   , if n=2 , root =~ 1.414
RND(n)   :for to get a random number , where n=1, 0 , or -1 and returns 0.0 to 1.0 , use: -1 to reseed first, 1 to reuse random numbers , 0 to use internal clock to seed the random number generator.
INT(n)   :to get just the integer portion of a mixed number value   , For use, ex: wholevalue = INT(3.14) , and then wholevalue = 3   , and to have the fractional value ,   For ex: fractionalvalue = 3.14 - wholevalue = fractional value = 3.14 - 3 = 0.14
DEF FN()   : to declare and define an inline math function, and of which can also be called elsewhere in the program
Ex. DEF FN F1(x) = (x + a) / 2   , for use, ex: n = F1(3)   , hence evaluated as: n = (3 + a) / 2

Relational operators: < , >, <>   for not equals   , <= , >= , =   : used to test or compare values to made decision.
used to make conditional statements with IF

Logic operators: AND , OR   , XOR   , NOT   : used to determine the truth (1 or 0) of two conditional statements

Some common or standard string functions in the BASIC computer programming language:

LEN   :string length   Ex.   L = LEN(A$)   ,   L will be set or assigned the number of characters in the A$ up to the NULL terminator character which is usually number 0 , a zero byte of data equal to 0
LEFT$: get the left portion of a string ,   format ex: S2$= LEFT$(string , number of characters)
MID$: get the mid portion of a string ,   format ex: S3$=MID$(string , first character number, last character number)
RIGHT$: get the right portion of a string,   format ex: S3$=RIGHT$(string , number of characters)
STR$: to convert a numeric value to a string ,   format ex: S$ = STR$(N)
VAL   : to convert a string to a numeric value ,   format ex: N = VAL(S$)

Some common BASIC interpreter commands: (These can also be part of your program.)

LOAD : load a filename.bas program into memory space assigned to the interpreter
SAVE : save the filename.bas program to a file such as located on a disk drive
LIST : display the current loaded BASIC program's text . Used by itself will list the entire program, but to list one line number, use this format: LIST line_number. To list from a line number onward use: LIST line_number-
To list a range of line numbers, use: LIST start_line_number - end_line_number
RUN : execute or process the program
NEW : clear memory to start over
FRE(0) : shows how much free program memory allocated to your program is left

Some common BASIC programming commands:

CLS : clear screen , deletes all text characters on the display screen
LET : An optional command to enter the value of a variable: N = 10 is the same as LET N = 10
LET essentially means "to assign" or "initialize or set this variable with this value"

PRINT : to display a string (array) of text characters or a numeric value
Format: PRINT value or string , Ex. PRINT A or= PRINT (A) , Ex. PRINT A$
PRINT TIME$ will display the current system time in H:M:S format

INPUT : to let the user enter a text string or numeric value to the program

IF THEN : to test or determine if a condition is true, and if so, the statement(s) following the THEN keyword will be executed.

ENDIF : if the condition was true, statements up to ENDIF statement will be executed. It is not always needed, but if there are related statements to be executed on further lines, it is a good idea to group them using this. The C programming language uses the braces pair (opening brace character, and closing brace character) of symbols to group related statements. Braces are also used to group the statements in a C function construct.

ELSE : can be associated to and used after the latest IF statements, and so as to perform a new statement if the condition was false

FOR TO NEXT This is for a looping programming construct for repeating code. Sometimes called a FOR NEXT loop.

```
Ex.  300 FOR  N = 1 TO 10         : default is STEP 1   or use:   FOR N = 1 TO 10 STEP 1
     305  PRINT N                  : here , N is known as the (loop) counter variable, or repetition variable.
     310  NEXT N                   : At 310, this is much like also saying.  NEXT N = (N=N+1; GOTO 300;)
     320                             Essentially then, the FOR command also assigns 300 as the label to
                                     return or go to.    NEXT N causes N to be incremented by the value of STEP,
                                     here 1 as the default if not used.  Here,when N>10, this loop will terminate and
                                     the program will continue to the  next statement in the program, perhaps on
                                     "(assigned) line  number or label 320.  The STEP  value can be a fraction and-or
                                     a negative value such as -1, and which then causes the variable, here N, to
                                     decrement by 1 for each iteration or repetition of the loop.
```

GOTO LABEL This is a programming construct that will cause program running or execution to resume at the location identified as LABEL, and for BASIC , it is a numeric value, usually considered the line number program following the specific value or identifier of the LABEL indicated such a line number. It is possible to create a homemade looping construct using GOTO.

GOSUB This is a programming construct that will allow the program to reuse the same code when needed. This is good for creating functions which are programming constructs, and of which may include math functions and-or statements. This function is essentially called using the GOSUB command. A line number or label identifying that function is needed, and so the format for this is: GOSUB LABEL . When the code of the function is executed, to resume program flow back to the next statement immediately after the associated GOSUB program flow command, use the RETURN keyword. GOSUB essentially stands for and means "goto subroutine" or "goto this subroutine". In C programming, a function is a subroutine, and it is called as: function_name(arguments_to_send_to_the function); and when the function is completed, the return; statement is used to send the program flow and execution back to the next statement(s) following the associated or corresponding function call.

END : terminates the program execution and gives device control back to the interpreter and-or computer system

REM This is to allow programmer comments and-or reminders, and of which will not be executed, processed or "ran" by the computer. It is also a method to "comment off" a section of code so it is not ran or executed, possibly for program testing. REM means "remark". If multiple lines are used, then each line must begin with the REM keyword. The C launguge uses: /* My comment. / , and may consist of multiple lines of comments, and only one instance of: /* and */ , the coment delimiters are used for each comment.

HOW TO MAKE A SIMPLE HTML WEBPAGE AND-OR EBOOK

The main concept of this article is to make a simple webpage having text, links and images (picture, graphics) to view on your or someone else's device.

When we click on or type in an internet address in a web or internet browser, a webpage will be sent (ie., "downloaded") to our computer and then be displayed on our device (phone,computer, etc.). The main language and-or specific code of to make a webpage is the HTML language. HTML means Hyper-Text Markup Language, and it could be thought of as using special defined text words and notation, often called commands or (document) "tags" within or as part of a text document so as how to display that document, etc.

Your webpage or HTML format document can even be sent ("uploaded") to a website for other people to view, and technically, it will be sent to a (data, webpage, storage) server computer assigned to that website(s). An alternative to uploading your webpage and-or its .html code to a website via a FTP (a standardize File Transfer Protocol) program for digital data communication) is to simply email it to an internet technician (IT) or webmaster assigned to that website. Often a default, initial or starting webpage of a website must be explicitly named index.html due to the internet standard. This webpage is the default webpage for a website. From this main or index webpage, other webpages, if available, can then be selected via what is called (hyper-text or html) links to those pages. An internal link can even be used to go to another part of the current document, and it is much like a "bookmark" used to go to a certain chapter and-or location in a book.

There are computer programs and-or internet browsers that con convert a webpage into an ebook such as with the public document format having the .pdf filename extension. These types or formatted ebooks can be viewed on both computers and-or modern phones with .pdf viewer program which usually comes freely with the device, and can even be sent via email as an email attachment. Further in this article is a method to easily make a PDF ebook from a .html file.

You can make and-or edit a webpage using a plain text editor, that is, one that is not a word processor editor which places its own proprietary formatting or display commands within the saved text document, and which are therefore not compatible with the standardized HTML language of a webpage or even a plain text editor. For a PC, Intel computer. or DOS type of computer system, the EDIT.com program will work if it is available, and-or the Notepad.exe program should work if you do not have another text editor. These programs will make text files with the .txt filename extension and which then needs to be changed to .html so as to be seen or recognized as a webpage by the (offline) HTML viewer and-or (online) webpage viewer such as the internet browser. To edit your .html file, right click on its filename and select "edit" from the menu options. If you make edits to your .html source code file, you can mouse right-click on the corresponding webpage to refresh or reload it so see the new edits. You can also double left-click with the mouse to display, or right-click on it and select a menu option. If you are further interested in learning much about making .html webpages, then you can purchase a book and-or take a course about it.

To view a webpage, you can use and internet browser like Internet Explorer or Chrome. You may also use another program that is able to view html documents. If you have edited the html code of a webpage, select refresh so as to display it with the new intended edits.

To keep your ebook source html code and other data such as the photos to include in your book organized, first create a folder (ie., a directory) for your ebook. For a PC type of computer, you can right click your mouse, and select the menu to create a new folder. Using this method also eliminates using filename directory paths in the .html code, Photo's, etc., will be assumed to be in the current directory as that of the .html file being created and-or edited. Once the .pdf ebook is made, the images are already included in it and there does not need to be a folder with the same images on the reader's computer or phone.

BASIC HTML LANGUAGE SYNTAX

Tags in HTML source code are essentially commands or instructions, and sometimes containing additional data that is sometimes called the (command's) arguments to currently use with it. HTML tags are placed within special characters so that they will be recognized as apart (delimited/denoted) from the regular text that is displayed on the screen. The delimiters (identifiers or markers) for HTML tags or commands are the < and > symbols. Tags tell the webpage browser or html viewer what to do with the following text, etc.

A tag is surrounded by a left angle and right angle bracket.
Format of start-tag: <tagname>

Tags are often paired, with an end tag containing the slash character. This signals the end of the element, and to distinguish it from an opening tag.

Format of end-tag: </tagname>

The text between the start and end tags is relevant to the tagname being used.

Tag attributes or simply attributes about the element follow the start tag. An unknown tag or out-dated (dropped from the standard) tag will be ignored, however, any text within it will still be displayed. A web page must have the HTML tag. This informs/commands the browser that this is a start of a HTML document. All the other commands, text images, links, etc, if any, will come between the HTML "opening" tag and the "closing" tag.

To put a comment (unseen text comments a programmer can later review, and which are not displayed on the screen) into the HTML source code, use the following format for the comment tag which uses the exclamation point character:

```
<! my comment goes here  >
```

The required, minimum tags a browser expects are the html, head, and body tags. Here is a simple template and a few extra tags of a webpage to copy so as to quickly create a webpage and begin editing it for what you need:

```
<!----------------------------------------------------------------------------------------------------------------------------------------------->
<html>

  <head>
        <title>My HTML Page</title>
  </head>


  <body>

        <h1>HTML IS EASY</H1>
        <b></b>
        <p>  A line of text here.  </p>
        <b></b>
        <p>  A line of text here.  </p>

  </body>


</html>

<!------------------------------------------------------------------------------------------------------------------------------------------------->
```

The title tag is optional, but is recommended. The text within it will be displayed on the top of the webpage on the title bar. The main viewable part of the webpage will be placed within the body tags.

In the body, you can see a headings tag indicated by h1, and there are 6 possible headings, 1 through 6. h1 is the largest size. These headings correspond to font sizes commonly available for a system font. h1 indicates the first heading size.

 is the Break tag, and it will start the following text on a new line.
<p> is the Page-break tag, and it is like the break tag: , however, page break will also insert a blank line before any more text is displayed. Be sure to use the opening and closing pair for tags: <p></p>

To include a hyper-text link in your document, use the Anchor tag: <A>

A general format for this type of tag is:

```
<A HREF="filename_or website_address_to_go_to">text_to_display_as_the_link_to_press</A>
```

HREF means Hyper_Text Reference, and the quotes should be used if there are any spaces in the filename. An example filename would be "my webpage 2.html". Without any directory or folder path to the file, the file will be assumed to be in the current directory, perhaps a pre-made folder containing the .html or webpage source code and some related images. You can use "../filename" to go back to a file in the parent (this root or current) directory or "../../filename" to go back two parent or previous directories. Use "\filename" to go to the next sub directory that is contained in the current directory.

If you want to execute or run a program by pressing a link in your webpage, here is an example:

```
<A HREF="dos.bat">Go To DOS Screen Here</A>
```

Here is an example of placing an image directly into the webpage:

```
<IMG SRC="myimage.jpg">
```

Here is an example of setting the horizontal space on the left and right side of the image, and the maximum width and height of the displayed image. The iimage will not be trimmed, but rather "shrunk" to fit this size:

```
<IMG SRC="myimage.jpg" hspace=50 width="150" height="150">
```

Here is an example of linking to an image, and it wont be displayed until the user selects the link:

```
<A HREF="myimage.jpg">Here is a  picture.</A>
```

In place of "myimage.jpg" can be another web page or mail address. If the object (image, webpage, etc.) is on another server computer someplace on the internet, the full URL (Uniform Resource Locator, for the internet) address to that file must be specified. For this example, the file will be understood to be local to the computer that the webpage is on, and therefore, the full URL or complete address does not need to be specified. Here is an example of using a complete address:

```
<A HREF="https://mywebsite1.edu/index.htm">Click Here For My Website</A>
```

To easily or quickly display text without any formatting codes, use the <PRE> or <pre> , pre-formatted tag, and this is useful when selecting and copying text already (controlled, such as with spaces, tabs, new lines, etc) formatted so as to be easily placed into your webpage:

```
<PRE>
   .   .   .   .   .   .   .   .   .

      Welcome
            To
               Here!
   .  .  .  .  .  .  .  .  .  .  . . .

</PRE>
```

Below is a more complete example to copy and paste into a blank or new text file, and then rename it with the .html filename extension. For organizing things, put the .html source code or document and the image in a new folder. The user can mouse left-click on the html file so as to display it.

Setting Colors

The html code below also shows a method and example of how to set the background and text color using a combination of RGB colors, each having a value of 0 to 255, however this value is entered in a hexadecimal format which goes from 0h to FFh, where FFh = 255d. The format for the color values is:#RRGGBB , and use 0's if no value is desired and it yields the darkest shade of that color, and FF yields the brightest shade of that color. The colors will be mixed together and produce a single resulting color. The example below shows how to set a black background with white text and purple links.

Setting Links

Also shown below is a link to a text file that is on the local computer such as in the folder created for the html webpage, and the filename can even be set to another html webpage. You must ensure the filepath or directory is correct for the user. Also shown is a link to another website such as the Google.com website.

```
<!------------------------------------------------------------------------------------------------------------------------------------------------>
<html>

<head>
        <title>My HTML Webpage or Ebook page.</title>
</head>


<body bgcolor=#000000  text=#FFFFFF  link=#0000AA>

<pre><h1>    WELCOME</h1></pre>

<pre><h2>    Here is the image:</h2></pre>

<IMG SRC="image.jpg"  hspace=50  width="150"  height="150">
<pre>

    I hope you like the above image.
    This ebook took me years to write.

   <A HREF="mytext.txt">Get My Text File Here</A>
   <A HREF="https://www.Google.com">Press Here For The Google Website</A>

</pre>
```

```
</body>


</html>
<!--------------------------------------------------------------------------------------------------------------------------------------------------->
```

Here is a line of html code so as the user can click on it so as to download a file such as an ebook or other file that you have sent to and stored on the website . The directory that the file is in will be the default and-or current directory, or one set by the programmer of the html page:

```
<A HREF="myebookname.pdf">Get the ebook here by clicking the right mouse button on this link, and then select Download from the menu..</A>
```

Converting a .html webpage to a .pdf document and-or ebook

With the Chrome (R) internet, web or other html browsers and viewers, you can make an ebook by right clicking in a webpage, and choosing the Print command, and then selecting to rather save or effectively print it as or to a file having the public document format and with the .pdf filename extension. This document is effectively an ebook. Chrome is a computer program made by the Microsoft company, of who also make the Windows, PC operating system or program.

To share your work with others, you can email them your ebook as an email attachment when creating your email message. There is a maximum size limit to all email attachments, and it is often a total of about 25 Mb. Your ebook can also be shared to others using a file hosting website that offers downloads of it.

mEq Units And Example

It was mentioned previously in this book that typical blood had about 3.22 mg of sodium per 1 mill-liter = 1 mL = 1cc volume of blood and which was equal to about 140 mEq of sodium per liter.

More correctly: mEq is the milli-equivalent = **mm**Eq **m**illi-**m**ol equivalent of an amount of mass.

The mol unit of mass was previously mentioned in ths book. A mol is a (quantity or amount) unit , and usually for a specific amount of mass - either atomic mass units (amu) or whole atoms composed of atomic mass units. A mol is used in chemistry so as to know the number of amu or atoms of, or for a chemical reaction. Individual atoms cannot be individually seen by human vision, counted and-or weighed, but a quantity of a mol of them has a corresponding mass and weight depending on the specific element considered since each element has more or less amu (mass particles) or "weight" per atom that of another element.

The atomic "weight" (g / mol) of sodium is 23, which means 23g / 1 mol of atoms of sodium atoms. or= 23g / 1 mol-atoms of sodium. Technically, this is 23 grams of mass as displayed on a weight (force due to gravity) scale and converted to its corresponding mass value which is often more important than its weight on Earth and its associated gravity value.

1 mill-mol = (1/1000) mol = 0.001 mol

23g sodium / 1 mol-atoms = 1g / 0.04347826 mol of sodium atoms = 1 g / 43.39 mill-mol of sodium atoms =
= 1g / 43.39 m-mol of sodium

It was found by experiment that there is 3.22 mg of sodium (ie., sodium atoms) / mL of blood =
= 0.00322 g / mL = 3.22g / L : (1 mL = (11cc) , and this is slightly less than ocean water being: 3.5g sodium / 1L

$$\frac{\text{1 g sodium}}{\text{43.39 m-mol sodium atoms}} = \frac{\text{3.22 g sodium in a liter of blood}}{\text{N m-mol sodium atoms}}$$

, solving for N, we have:

N = 139.758 =~ 140 m-mol units of sodium atoms in a liter of blood , and since this amount or count corresponds to a mass of 3.22 grams of sodium atoms, we can express the amount of sodium in m-mol units:

3.22 grams of sodium = 3.22 grams of sodium atoms = 140 m-mol of sodium atoms

The amount of sodium per mL of blood is equal to 140 mEq of sodium per mL =
= 140m-mols sodium per mL

Important: In general, there is no direct conversion between mol atoms and grams unless it is for the same the element in question and where there is a known relationship of grams and mols. If a certain element has (n mol / g) or (n g / mol), then there is a direction conversion such as by using an equivalent fraction.

Also, when a substance is evenly distributed in a volume of a mixture, any volume of that mixture considered will have the same fraction of that substance, and their ratio is the same in each volume, and it can be said that the (mass or count) concentration of that substance in each volume of that mixture is the same.

1 unit A / 10 unit B is a 0.1 (unit A / unit B) concentration = a 10% concentration of unit A per unit B

1 apple / 10 fruits = 0.1 (apples / fruit) = 10% concentration of apples = 10% of all the fruits are apples

50 mL / 1 L = 50 mL / 1000 mL = 5 mL / 100 mL = 5 cc / 100 cc = 0.05 and unitless= strict numeric =
a 5% concentration per volume = a 5% concentration per unit of volume = 5% of the volume

RDA AND %DV NUTRITIONAL VALUE

RDA = Recommended Daily Allowance value of a nutritional substance (vitamin, mineral, etc.) per or in a day. Here, allowance means the amount of the substance (vitamin or mineral, etc.) being considered.

RDA is a recommended daily allowance value or the amount of a particular substance (vitamin or mineral, etc.), and it is usually expressed as and amount of grams (g) , milligrams (mg), or micro-grams (mmg or ug) in 1 serving of the particular food substance. A **serving** is a typical and-or average recommended amount of that particular food substance to have during a meal. In general, people with less mass and-or weight will usually need a serving size with a lower amount.

1 milligram = 1 mg = one-thousandth of a gram = (1/1000) gram = 0.001 g
1 microgram = 1 ug = 1mmg = one-millionth of a gram = (1/1,000,000) gram = 0.000,001 g

Ex. It is recommended that you intake by mouth 3 grams a day of a certain substance, then you need to find out how many grams of that substance are in each meal and-or supplement that you eat and add them up so as to know you ate 3 grams during the day. The gram value you ate may be less or more, but on average over time, perhaps a weak, it should be about 3 grams per day. If your average intake of that substance is less, you need to consider eating more of it to prevent potential health issues. If your average intake is more, you need to consider any health issues that may develop from having too much. Many vitamins and minerals can become toxic if eaten too much, and then any perceived benefit of taking more than the RDA is dangerous unless it was prescribed by a doctor and- or nutritionist. If your total intake (eating) of the substance was 3 grams, then you had 100% of the RDA value:

(3g total intake during the day) / 3 grams RDA = 1.0 = 100% of RDA

Since it may be unclear to a reader what a certain number of grams of a substance on the labeling is in terms of RDA, the %DV nutritional value was created, and which gives a better indication of how much of the RDA is in one (recommended) serving of that substance to have during your meal of that food or supplement.

%DV = **Percent Of Daily Value** in 1 serving of the food substance =
= Percent Of RDA per serving of that food or supplement.

This value also assumes an adult doing moderate activity (ie., requiring energy) consuming a recommended 2000 calorie a day total intake (food, meals, diet), then for example if you eat more of it, you then had more than 1 serving of it, and you may of had more than the RDA of some substances in that substance or food item. Sedentary people and-or people who are smaller and have a lower amount of mass will require less calories per day, but then again calories do not generally correspond to nutritional content and-or its absorption into the body.

Ex. If the RDA of a certain substance is 4 grams, and 1 serving of a food you are eating contains 0.25 grams = one-quarter of 1 gram. You goal for the entire day unless told otherwise by a doctor and-or nutritionalist is to intake a total of 100% DV. The nutritional labeling on that item would indicate something like:

substance .... 0.25 grams per 1 serving : old way , here the RDA is unknown

substance .... 6.25 %DV : new way, here using %DV

6.25 %DV = 6.5% of the RDA : hence RDA is known

How was this value calculated?

%DV = (amount of substance in 1 serving) / RDA =
0.25g / 4g = 0.0625 = 6.25%

An Example Of The Nutrition In A Can Of Food, And Considering Magnesium

It is of note that if the ingredients list for the food substance is on the labeling, they will listed by the highest amounts first, and to the least amount.

It is of note that the nutritional content within a substance or food is not always placed on its label. For example, the author has noticed this on a can of sweet peas, and decided to see how much magnesium it has because many of us are at risk of magnesium deficiency and its potential health issues. Foods high in magnesium are often leafy green vegetables, beans and nuts. Here is some data about that can of peas and its magnesium content:

The RDA of magnesium for an adult male is: 410 mg/day, and for an adult female, 315 mg/day. For pregnant women, they will require about 355 mg/day. The average of these two values is: 363 mg/day

The net weight (NET WT) of the can of sweet peas was 15 oz = 425g. 1 serving of it was listed as 1/2 cup = 125g, and the label did not have RDA for the day and-or the %DV of 1 serving listed for magnesium. The number of servings in the can was 3.5 The serving size was mentioned as having 60 calories / serving.

After some research on a government food and nutrition website, 100g of peas will have about 33mg = 0.033g of magnesium available. The better it is digested, the more can be absorbed into the body. Babies who can eat, and older people should consider peas that are prepared by lightly pressing and mashing them using the underside end of a fork so as to help digestion and nutrition absorption.

$$\frac{\text{0.033g of magnesium}}{\text{100g of peas}} = \frac{\text{33 mg of magnesium}}{\text{100g of peas}} = \frac{\text{0.33 mg of magnesium}}{\text{1g of peas}} \quad \text{or=} \quad \frac{\text{0.00033g of magnesium}}{\text{1g of peas}} =$$

$$\frac{\text{330 mcg of magnesium}}{\text{1 gram of peas}} \qquad \text{: mcg = micro-grams = mmg = ug}$$

The 425g can of peas will contain a total of:

(0.33 mg of magnesium / 1 gram of peas) (425g of peas / can of peas) = 140.25 mg / can of peas

If 1 serving of peas is 125g, we have this much magnesium in each serving:

(0.33 mg of magnesium / 1 gram of peas) (125g of peas / 1 serving) = 41.25 mg of magnesium / serving

The % of the substance's, here magnesium, RDA in 1 serving of the food is the %DV value.
The %DV or= (% of recommended DV=RDA) of 41.25mg / serving is:

$$\text{\%DV} = \frac{\text{(amount of magnesium per serving)}}{\text{RDA of magnesium}} = \frac{\text{41.25 mg of magnesium}}{\text{363 mg magnesium / day}} = 0.11363\underline{6} \text{ / day } =^{\sim} 11.36\%$$

Note that the actual amount of magnesium or other nutrients in peas or in any other food item may vary due to a specific farm's growing conditions such as how fertile the soil is with nutrients, soil pH, sunlight, temperature and water (such as rain and-or deliberate irrigation) received for that growing season. The amount of magnesium or some other nutrient may also depend on the specific variety of the plant. Perhaps the variety grown at one farm actually grows better in a different climate and-or other growing conditions. Cooking (warming or heating, increasing their temperature) foods generally reduces the nutritional content in them, particularly the amount of each specific vitamin. Minerals (ie., natural elements) are generally not depleted to a lesser amount due to cooking them since it takes much energy to destroy, alter, change, the natural elements.. Washing, canning, refrigeration, and cooking foods is a good method to reduce germs (bacteria, viruses, worms, etc.) and-or if you are unsure if there is a problem it is better to heat it hot to eliminate dangerous germs.

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GAS VOLUME AND PRESSURE EXAMPLE CONSIDERING SCUBA DIVING

Scuba diving and compressed air gas.

Being able to breath air underwater allows work and exploration that would otherwise be difficult. Before a diver could move or swim freely underwater using a compressed air tank (ie., a container, filled with compressed air) strapped onto them, various other methods to breath underwater were previously used. One common method used an air pump at the surface which supplied the diver with a fresh air supply via a flexible tube connected to a helmet about their head or suit that they wore that was somewhat like an astronaut's space suit. They could view what was in the water through a clear piece of glass that was part of the helmet. These suits and breathing methods are sometimes still used today.

For recreational or hobby diving, it is recommended that the experienced diver go no more than a depth of 130 feet =~ 40 meters. For divers to legally go to a depth of even 60 feet requires a training certificate by a certified diving instructor. The deeper a diver goes and-or the more inexperienced, the more the dangers and potential problems. Swimmers and divers should have at least one other person with them for teamwork and safety purposes, and who should also in be in good health and have some knowledge and experience. A swimming pool with an instructor is a good place to first test much of your knowledge you have learned about scuba diving and its gear.

An empty, unpressurized container or gas "**tank**" can be used to hold a fluid or (elemental, molecular) gas substance. This substance, particularly with gasses which have much space between the atoms or molecules, can be highly pressurized so as to store a larger volume of it within that relatively small container and-or volume as a compressed and more dense [density = mass / volume] form. Gas tanks are commonly found in hospitals, such as for compressed, pure oxygen and air mixtures, diver tanks, such as to hold much air to breath in a small volume, propane tanks for heating and cooking, welding tanks for various gases to melt and weld metal pieces together, for air for people to breath on high altitude mountains, planes and balloons, and for astronauts to breath in space. Tanks can hold compressed carbon dioxide for making some drinks such as soda pop. For astronauts, they will need much oxygen when having many hours of space flight, and therefore, liquid oxygen and nitrogen which is highly concentrated and-or very dense (mass/volume) is rather used than compressed air. Here, the liquid oxygen will be allowed to evaporate when having less pressure upon it, and it will become part of a breathable air gas mixture. Gas filled tanks are usually cylindrical or spherical so as to help (evenly) distribute the internal pressure and-or forces on the entire internal surface rather than being concentrated on a flat surface and-or joints at the edge of that surface, and then risking some deformation and-or a dangerous rupture and-or breakage of the tank.

The reader needs to understand and recognize that there is sometimes two general volumes being mentioned in discussion of tanks and gases. One volume (Vt) is the maximum volume of an empty tank, and another volume is amount or volume (Vg) of a gas remaining (usually under pressure [ie., compressed together into a smaller space]) within a tank. A new, empty metal tank has a constant amount of volume when it is made at a factory, and this volume (Vt) is the amount of empty space within it such as cubic inches, cubic centimeters, cubic meters, etc. This tank volume can also be measured and-or rated as an amount of water, such as liters of water, that it can hold at standard temperature and pressure (STP). Vt is a constant value even if it is empty or having any amount of gas in it. Due to the motion of gas atoms and-or molecules of gas, a gas such as an amount or volume (Vg) of compressed air will naturally (high pressure moving toward lower pressure) fill a volume such as that equal to the empty tank volume. When an amount of gas is removed from this tank, the gas pressure within that tank will then be reduced, and the volume of that (compressed) gas in that tank is reduced in a proportional manner. This concept is expressed as Boyle's law: P1V1 = P2V2 = a constant for a specific tank and-or system. It needs to be noted that a volume of gas in a tank can also be put at a higher pressure by physically reducing the volume of the holding tank or container it its in, and such as by using a mechanical piston that applies force and-or pressure to the gas. Here, the density (m/V) of the gas will increase since the volume is decreased, but the actual mass and-or weight of the gas will remain the same value. A common compressed gas is air which can be put into (via an air compressor) and stored in pre-filled tanks so as to fill (ie., "inflate") tires with compressed air, or to power (air powered) tools, and for spray painting using the Venturi effect (ie., a change in pressure) to draw or move a gas or fluid by using pressure differences.

Some tanks are made out of steel, and some are made out of aluminum. Aluminum is lighter in weight and is not as strong as steel, unless it is made thicker. Every pressurized gas tank is, or should be, tested to withstand at least the internal maximum expected pressure, and in fact for increased safety due to it accidentally being filled to have a higher pressure above its maximum pressure rating, a certified as safe tank is checked at 1.5 times the pressure of the maximum expected value and rating of internal pressure. For example, a tank verified for 1000 psi should be tested and certified as being as (reasonably) safe to hold an internal air pressure of (1000psi)(1.5) = 1500 psi.

Even though the pressure in the tank of compressed gas will be very high, and the density of the air in that tank will very higher, the air gas mixture in a diving tank will still be a gas and is not dense enough to be a liquid.

If the internal pressure doubles, such as by compressing and halving (dividing by n-2) the volume of the internal air gas, the density of the air gas mixture will double. From: density = mass / volume. We see in this equation that density is inversely related to volume and directly related to the amount of mass. If the volume decreases by a certain factor (n) and the mass remains the same, then the density will increase by that same factor: (n) density = mass / (volume / n) = (n) (mass / volume). If the volume increases by a certain factor (n), and the mass remains the same, then the density will decrease by that same factor: (1/n) density = density / n = mass / ((n) volume) = (1/n) (mass / volume).

> Ex. If the density of air at 204 ATM = 204 times the standard atmospheric pressure of 1ATM = 14.7psi at sea level, what is the density of that air? At 14.7 psi, = 1 ATM, the density of air is: 0.00124 g / 1 cc. At a pressure of 204 times that, the density of air is 204 times higher: (0.00124 g / 1 cc) (204) =~ 0.2530 g/1cc. The density of liquid air is about 0.87 g / 1cm. = 870g / 1L,, and that value is about 702 times greater than that of oxygen gas at 1 ATM of pressure.

The density of liquid water is 1000 g / 1L = 1kg / 1L It is of note that a liquid such as water is already as dense as it can be, and it cannot generally be compressed much so as to have a higher density. Deep water will be at a very high pressure and yet at the same density as that near the surface.

A volume of 1L = 1000 cc = 1000 mL. A mass or amount of water that has a volume of 1cc = 1 cubic centimeter = 1 cm^3 is defined as a having a mass of 1 gram.

The gas and-or fluid volume of a container or tank can be measured by filling it with water and then measuring that amount of water volume (WV). If a volume measuring container is not available, the mass of the water can be measured on a common weight scale which displays the result as the weight's corresponding amount of mass. The volume of that amount of water can then be calculated from that weight., for example: 1 gram of water = 1g of mass and has 1cc = 1 mL of volume. 1000g of water = 1kg of mass and has 1000cc = 1000mL = 1L volume of water. 1 cubic foot of water = 1 ft^3 = 1728 in^3 of water weighs 62.4 lbs 1L is roughly the volume of a quart or quarter of a gallon, and will weigh about 2 lbs if the internal substance is water. Given a 128 fluid-ounce gallon, a quart will be (128 fl-oz / 4) = 32 fl-oz. 16 fl-oz = 1 pint of water will weigh 1.0425 pounds =~1 pound of force. A gallon of water weighs 8.34 lbs. 1 gallon = 3.78L and if filled with water substance, it will weigh 3.78kg. (8.34 lbs water / gal) / (3.78 L / gal) = (8.34 lbs water / gal)(1 gal / 3.78L) = 2.206349206 lbs water / L. From this we have: 1 lb water / 0.45323741 L = 16 weight oz water / 0.45323741 L , and from this we have: 1 weight oz water / 0.028327338 L volume = 1 weight oz water / 28.327338 mL = 1 weight oz water / 28.327338 cc

Some of the most used units and conversions for volume measurement are given at the end of this article.

When a diver is below the surface of the water, they will need air to breath. A common method is to use a diving cylinder filled with air. This gas cylinder or tank is commonly called a "**scuba tank**" in terms of breathing ,or more generically as a "gas tank" in terms of any other gas such as methane or propane. The word "SCUBA" is composed of the first letters for a **S**elf **C**ontained **U**nderwater **B**reathing **A**pparatus. The scuba tank is generally credited in 1943 to **Jacques Cousteau**, (1910-1997) from France, and **Emile Gagnan** (1900-1984) from France. The scuba tank is also sometimes called an "aqua-lung", as coined by Cousteau. Cousteau is also very famous for his very entertaining, knowledgeable and educational undersea color movies made for public television, and which also inspired many others later in various ways. His research vessel was a boat named Calypso, and of which singer-songwriter John Denver wrote and sang a popular song about. Both Cousteau and Denver are also known for their Earth conservation and awareness efforts. Though high pressure tanks for various other use were already invented before Cousteau's time, they could not yet be used for independent diving and swimming underwater (ie., under or beneath the surface of the water, hence within water, submerged) until they also invented the diving regulator (a form of a gas demand-valve, were the gas in the tank is only released when needed, hence only during inhaling to breath) which is a (air) gas pressure regulator so as to give the diver the right amount of oxygen at about that of normal air and breathing pressure of 14.7 psi when inhaling at a shallow or low surface depth which has a pressure of about 14.7 psi, and then to also easily permit the diver to exhale carbon-dioxide or "used up air gas". Consider for example, at about 100 feet deep, the total water pressure is about 58 psi, and therefore, the tank must at least have a minimum of 58 psi + 14.7 psi for an air pressure difference so as the diver can breath easily, and which then equals about 73 psi left in the tank. As indicated, the pressure regulator redcues the high pressure in the tank to a safe and typical breathing pressure of 14.7psi. Note that it is recommended that the diver have (1/3) of the volume and-or air pressure remaining or left in the tank for various diving safety and-or potential emergency purposes. For example, if a diver gets stuck in a fishing line, it will take some time for the diver to cut that line.

Scuba diving can be very dangerous in many ways, and you must first get knowledge, training and certified by a certified instructor. One such danger for a diver is getting stuck in fishing line or a fishing net, and it would then be good for them to carry a sharp knife and-or wire cutters - possibly connected to a lanyard in case they drop it. Other dangers are due to pressure changes descending to a depth and ascending to the surface too fast. A properly prescribed breathing gas mixture needs to be considered with the aid of other professional divers and regulations before a dive is made. DO NOT DIVE OR EVEN SWIM UNANOUNCED, so tell others of the locations and times. Permission to dive may be also necessary. Do not dive alone, so consider including a surface or boat assistant crew and possibly another diver(s). Do not dive in dangerous conditions (including various potentially dangerous sea life such as sharks) and-or locations with strong currents, etc. Avoid cave diving, for you are simply asking for trouble and may easilly find it, and it could be deadly. When diving with others, consider hand, light and audio communication signals. Buoys can be created so as to mark a divers location, and or communication to the surface vessel. Get a proper diving education, training and certification, otherwise, avoid scuba diving at all costs.

Having a flashlight, and something to drink after a dive is always a good idea. **Hypothermia** as mentioned in this book is another danger that anyone in cool air or water can get when their body temperature get lows and cannot function as much physically and mentally. This temperature is not even very low by any means, and is about 95°F = 35°C. Most water temperatures are less than this value by many degrees, and so the body will start to loose heat when it is in contact with this cool water constantly absorbing some of that heat energy. Divers can consider "underwater clothing" to help stay warm and it is commonly called a "wet suit". The heat from the Sun due to its internal **fusion** or joining of elements such as hydrogen into helium and releasing energy such as heat and light, and heat from the air can warm and increase the temperature of the water near the surface of a large and-or deep body of water, and in general, the water beneath this warmer surface layer will then be cooler. Warm water, being slightly less dense than cold water will rise and-or remain at the surface region. Water expands slightly as the temperature of it increases. Through the year, the average water temperature at about the 45° latitudes is about 57 °F due to having less direct sunlight than at the Equator latitudes (0°) where the surface temperature of the water can average about 30 °F degrees higher in temperature. At the poles or 90° latitude, the surface temperature of the water can average 30 °F lower in temperature.

Scuba tanks are relatively small in size and-or volume, and surely this will not hold much air.. The average adult male doing average or moderate things and not much active work and-or exercise will breath in (ie., inhale) about 0.5L of air at about 12 times and-or 12 breaths per minute. A scuba diver having the tank and various other gear with them, and casually swimming near the water surface will take about 20 breaths per minute:

(0.5L air / 1 breath) (12 breaths / 1 min.) = 6L air / minute. : casual person, relaxing, sedentary, sitting at home

For the scuba diver-swimmer:

(0.5L air / 1 breath) (20 breaths / 1 min.) = 10L air / minute. : casual diver or swimmer

(US fluid ounces of volume will be considered in these calculations)

Since 1 US cup is 8 US fl-oz, and 1L = 33.814 fl-oz, a 0.5L per breath is 16.907 US fl-oz = US fl-oz = 2.113375 US cups =~ 2 US cups =~ 16 US fl-oz of volume =~ 1 US pint. This volume can be verified by breathing out through a straw or tube into a water filled, inverted measuring cup below the water surface. 1L is about 4.22675 US cups = 33.814 US fl-oz of volume.

1L of volume = a cube with a length,height and width of ten cubic centimeters =
10 cm^3 = (10 cm)^3 = (10)^3 (1 cm)^3 = 1000 cm^3

Half a liter is: (1L)(0.5) = 0.5L = (1000 cm^3)(0.5) = (1000 cm^3 / 2) = 500 cm^3

1 cubic inch = 1 in^3 = 0.55411 fluid ounces (fl-oz) , and mathematically, dividing each side by 0.55411 , we have:
1 in^3 water = 1 cu. in of water weighs 0.036127 lbs = 0.578032 oz

1 floz = 1.8046958 in^3 =~ 1.8 in^3
Taking the cube root of 1.8 in^3, we have a cube shape with sides of: 1.217497288 in =~ 1.2 in
Since 1 in = 2.54 cm , 1.217497288 in = (1.217497288 in)(2.54 cm / 1 in) = 3.092443112 cm , and
cubing this or raising it to the third power such as for a volume of a cube , we have:

1 floz = 29.57366557 cm^3 =~ 29.57 cm^3 =~ 29.57 mL =~ 30 mL

1 in^3 = (2.54 cm)^3 = 16.387064 cm^3 =~ 16.4 cm^3 , and mathematically dividing each side by 16.387064 , we have:

1 cm^3 = 0.061023744 in^3 =~ 0.061 in^3

1L = 1000 cm^3 = (1 cm^3)(1000) = (0.061023744 in^3)(1000) = 61.023744 cu-in =~ 61.024 in^3 =~ 61 in^3

Half a liter is: 0.5L = 1000 cm^3 / 2 = 500 cm^3 = (1 cm^3)(500) = (0.061023744 in^3)(500) = 30.511872 in^3
Taking the cube root of 30.511872 in^3 , we have a cube shape with sides of: 3.124805219 in
Taking the cube root of 500 cm^3 , we have a cube shape with sides of: 7.93700526 cm

1 ft^3 = (12 in)^3 = 1728 in^3
(12 in)(2.54 cm / 1 in) = 30.48 cm and (30.48 cm)^3 = 28316.84659 cm^3 =~ 28317 cm^3
(28.317 cm^3)(1 L / 1000 cm^3) = 28.317 L
1 ft^3 = 28.317L , and mathematically, dividing each side by 28.317, we have:
1 ft^3 = 1 cubic-foot of water weights : (1728 cu-in / 1)(0.578032 oz / 1 cu-in) = 998.84 oz of weight = 62.4275 lbs
1L = 0.03531467 ft^3

A scuba tank can hold a larger amount of air if that air is forced into it, and the internal volume of the air will then increase and it will become more dense (amount of air mass / volume). The pressure within the tank will increase due to the many more air gas atoms and their kinetic energy and collision force within it. This pressure (force / area) will also cause the same amount of pressure (= force / area) upon the inside walls of the tank, and the air is then said to be pressurized or compressed.

Can the tank be filled with compressed pure oxygen so as to last longer? Humans and many other life forms should not breath pure oxygen for very long, perhaps only a few minutes at a time because it can start to damage the body if used too much. Humans need to breath what their body is used to, and that is air which is a mixture of gasses, mostly nitrogen (about 78% of air gas) and oxygen (about 21% of air gas). About 1% of commonly available air is argon, and 0.035% of air is carbon dioxide. The cells in our body will use the oxygen so as to create energy molecules called ATP, and during this process carbon dioxide is created which needs to be eliminated from the body. The body will remove the carbon dioxide via exhaling it when we breath out. About 4% of the oxygen we inhaled will be converted to carbon dioxide, hence (21% - 4%) = 17% of the exhaled air will be oxygen, and 4% to 5% will be carbon dioxide. Plant life will use carbon dioxide during photosynthesis so as to create energy for it, and it will create and release oxygen. A regular, non-scuba diver person in a close room will most likely re-breath some of this air which contains the oxygen already exhaled, and hence the fresh supply of oxygen does not need to be as much as thought. The problem with a closed room, especially a small room having a small volume, is that the carbon dioxide level will continue to rise to a dangerous level, and the oxygen level will diminish to a dangerous level unless fresh air is routinely put into the room somehow such as opening a window or vent, either of which may have a fan installed. With a common scuba diver system, the diver will exhale all the breathed gasses into the water and will create "air bubbles" that will not be re-breathed and-or reused.

For each breath a diver breathes from the tank, the volume of air in the tank will decrease by 0.5L, and its internal pressure will also decrease since there is now less compressed air within it..

Given a common aluminum scuba tank that is about 30lbs of weight =~13.6 kg of equivalent mass that can hold a volume of about 80 ft^3 =~ 2265L of (compressed, into a smaller volume) air. How many breaths of air can this tank it hold?

Since each breath of a **non-scuba diver** adult male is estimated at 0.5L , and at 12 breaths per minute::

$$\text{Non-scuba diver breaths} = \frac{2265\text{ L}}{\left(\frac{0.5\text{L}}{1\text{ breath}}\right)} = 4530 \text{ breaths total available in the tank of compressed air}$$

(12 breaths / minute) (0.5L / breath) = 6L per minute

And:

$$\text{In terms of time: } \frac{2265\text{ L}}{\left(\frac{6\text{ L}}{1\text{ min.}}\right)} = 377.5\text{ min} = 6\text{ hrs} + 17\text{ min.} + 30\text{s}$$

$$\text{Or: } \frac{4530\text{ breaths}}{\left(\frac{12\text{ breaths}}{1\text{ min.}}\right)} = 377.5\text{ min}$$

Also:

For a scuba diver at 20 breaths per minute:

$$\frac{2265\text{ L}}{\left(\frac{0.5\text{L}}{1\text{ breath}}\right)} = 4530 \text{ breaths total available in the tank of compressed air}$$

(20 breaths / minute) (0.5L / breath) = 10L per minute

$$\text{In terms of time: } \frac{2265\text{ L}}{\left(\frac{10\text{ L}}{1\text{ min.}}\right)} = 226.5\text{ min} = 3\text{ hrs} + 46\text{ min.} + 30\text{s}$$

$$\text{Or: } \frac{4530\text{ breaths}}{\left(\frac{20\text{ breaths}}{1\text{ min.}}\right)} = 226.5\text{ min}$$

A diver needs to consider the time it will take to descend (go to the desired depth) and ascend (go back up to the surface, and it will take longer for deeper dives. For safety purposes due to various potential problems, a diver should consider surfacing with (1/3) of the compressed air and-or total time still available from that tank. A diver may even carry a small volume reserve tank of compressed air. It is recommended that a diver only use (2/3) of the volume and-or internal pressure during the entire dive, and have (1/3) of the volume and-or internal pressure as a safety reserve should any problem arise.

(2/3) of a 2265 L tank leaves the diver with an effective and safe tank volume of compressed air of:

(2265 L)(2/3) = 1510L

$$\text{breaths} = \frac{1510\text{ L}}{\left(\frac{0.5\text{L}}{1\text{ breath}}\right)} = 3020 \text{ breaths for a scuba divers when considering safety}$$

If there are 20 breaths per minute, how long can a diver remain underwater and-or use that scuba tank mentioned? First, this calculation will consider a depth of 0ft, because deeper dives will take longer, and the greater water pressure at deeper depths will also give less tank time:

$$\frac{3020\text{ breaths}}{\left(\frac{20\text{ breaths}}{1\text{ min.}}\right)} = 151 \text{ minutes} = 2 \text{ hours} + 31 \text{ min.}$$

: a rough estimate, safe dive time in low depth near the surface. For deeper dives, this value will decrease at a high rate.

$$\text{OR:}\quad \frac{1510\text{ L}}{\left(\frac{10\text{L}}{1\text{ min}}\right)} = 151 \text{ minutes} = 2 \text{ hrs} + 31 \text{ minutes}$$

If a dive is deep, the diver will also need to have a decompression time(s) for any nitrogen gas buildup in the body to be reduced by exhaling it. The diver will need to stop ascending at certain depths and allow the body enough time to exhale and get rid of any dissolved nitrogen gas that has accumulated in the body. The diver needs to do this before the diver reaches the lower pressure regions where the dissolved nitrogen gas in the body can develop into dangerous and painful nitrogen bubbles after it expands (ex., consider a ("fizzy") soda being opened, and the dissolved ("fizzy") and pressurized carbon dioxide bubbles come out of solution, expanding and float to the surface) when a diver quickly arrives at less depths which have a less pressure upon the body. This health and dangerous condition is commonly known as "the bends" or "**decompression sickness**". The term "bends" is used because the person having it will bend over in pain. Another health risk a diver must be aware of is that if they hold their breath and swim upward to less pressurized depths, their lungs will expand since they have a higher pressure volume of air in them, and this can cause a rupture in the airways and blood vessels of the lungs causing air to get into the blood stream which can block its flow in small veins and arteries. This is known as an **air embolism**. An air embolism can cause a stroke. Swimming to the surface too fast can also cause a rupture in the airways of the lungs due to the pressure difference in the body and water, and then potentially cause an air embolism. At depths of about 100 ft =~ 33m or more, divers can develop **nitrogen narcosis** and get physically and mentally numb, slow and-or confused from some inhaled gases, such as air, at those pressures. Special mixtures of breathing gases have been developed to help prevent this condition in some divers and for many diving to deeper depths, and this prevents (compressed) nitrogen from the compressed air tank from getting (unwantedly) absorbed by the lungs and then placed into the blood which can then cause "the bends" if decompression is not performed correctly. According to experienced divers, scientists and health experts, that for a dive less than 30ft =~ 10m deep, a decompression stops and times, of which are somewhat equivalent to a slow assent to the surface, is not required, but can be considered. As mentioned in this book, the water pressure will double for every 33ft =~ 10m deeper. The water pressure at 100ft =~ 33m deep is about (14.7psi)(1 + 33m/10m) = (14.7psi)(4) = 58.8psi =~ 60psi. The 1 valued term is included since the pressure at the surface is already at 14.7 psi due to the air pressure upon it. **Unhealthy, inexperienced, unfit, and untrained swimmers and divers, particularly being obese and-or with circulation problems** will have a much greater chance of **pulmonary (ie., blood and circulation) edema** (fluid buildup, swelling at a some location(s) in the body, including in the lungs) which can happen from either too much fluid in the blood and-or a high volume of blood in the body and the pressure of the water depth. It can also happen from being immersed in cool or cold water which can cause vascular changes when the body tries to maintain its core and head temperature, and which includes reducing blood flow to the extremities, increasing blood flow to the core and head, and raising blood pressure. Pulmonary edema can even happen to mountain climbers at high altitude due to the less air pressure. In general, divers should also be accustomed to or be used to the water, be able to swim, and be in fair health.

A diver or swimmer will also swim or dive with a diving or swimming partner who can provide assistance and-or seek help if an emergency happens. Having transportation and communication methods will help this process. New divers should also consider diving and exploring depths no greater than what they can stand in, in the case of a problem, the results will not be as problematic. Divers with minimal experience should consider diving to less than 10ft deep. A diver should consider having visible marked anchored safety ropes and-or lights, flotation devices and methods in case there is problems.

A typical 11 L = 0.39 cu.ft internal volume tank can hold about 80 cu. ft = 2300 L of compressed external air, and the tank will then have an internal pressure of about 3000 psi. 80 ft^3 / 0.39 ft^3 = a (compression or reduction) factor of about 205. Note also that 3000 psi / 14.7 psi = about 204 =~ 205. We see that the ratio of volumes is equal to the (inverse or reverse) ratio of pressures. 14.7 psi is the typical air pressure at sea level, and it is also called the pressure of 1atmosphere (atm). 14.7 psi = 1 atm. 3000 psi is 204 times greater than 14.7 psi = 1 atm, and this is said as being a pressure of 204 atm (or= 204 ATA). The tank will have a one-way value to keep the compressed air within it as it is being filled, and a valve for the tank to be stored with compressed air. The cylinder tank must be capable of holding a high internal air or gas pressure, and without a crack, leak or explosion. The pressure regulator must also be able to safely handle this high pressure and deliver the needed amount of air and air pressure to the diver. To have enough air to breath for many minutes underwater will require a high volume of air, and if this air is compressed, it can be put into a relatively small tank. The air in this tank will have a high pressure and is not safe to breath, and so a pressure regulator is needed. Before the scuba tank was invented, a diver could breath fresh air flowing via a tube from the surface boat, and which is attached to the divers helmet.

Water is much more dense than air will cause friction and therefore a loss of energy when moving through. Due to this, it will take much more energy for any object or diver to move through water than to move through the much less dense air. Like a plane moving through the air, the less surface area it has into the direction of travel, the less drag or friction forces placed upon it, and the less energy needed to travel through it. When a diver uses more energy to move, more oxygen will be needed and used by the body.

$\frac{\text{density of water}}{\text{density of air}} = \frac{(1\text{ g} / 1\text{ cc})}{(0.00124\text{ g} / 1\text{ cc})} =$ water is about 806 times more dense than air at STP (Standard temperature (ie., room temperature) and pressure (14.7 psi).

The deeper a diver goes, the less air time is actually available as was calculated for the surface and-or no dive at all and just breathing the gas without swimming or diving. The less time available is due to that the deeper a diver goes, there is more water pressure upon the body and this will make it difficult to breath or inhale for the lungs to expand in volume. Our lungs create a lower air pressure region to draw or effectively force in air that is at a higher pressure (14.7psi normally =~ 15 psi). The air pressure regulator for the tank will automatically increase the tanks output air pressure available so as the diver can expand their lungs and breath more normally, including having enough pressure to place the exhaust or exhaled air into the pressurized water. Consider at 33ft =~ 10m deep, the water pressure is already equivalent to that of 2 atmospheres (or= 2 bar units) of pressure = 2 atm = 2 (14.7 psi) = 29.4 psi =~ 30psi. Each breath at this pressure will amount to twice the normal volume of air taken from the tank, here 2 (0.5L / breath) = 1L / breath, and this will significantly reduce its available internal air volume and air pressure at twice the rate as that near the surface. This will therefore significantly reduce the amount of dive time of air available at that depth. The deeper and longer the time at an increased depth and its increase in pressure upon the body, the more nitrogen gas from the air gas mixture is absorbed through the lungs, and therefore, a longer decompression time and-or slower ascent to the surface is needed to avoid the "bends" and other health problems. A (1/3) rule is that (1/3) of the air should be kept in the tank at surfacing time for possible emergencies, (2/3) of the air remaining can then be used to descend, stay at the bottom depth, and ascend.

Another way to calculate air usage in the tank during or over a time, and at a certain depth is to find out how much the pressure in the tank is reduced while doing normal activities at the test depth. The diving tank system and pressure regulator will need to have a pressure gauge that the diver can see and-or record notes with:

At shallow depths near or at the surface: (PSI used) / (time in minutes) = (PSI used) / min.

Ex. 20 psi used / min

At a depth of 10m = 33 ft where the water pressure is twice (2) as much = 2 ATM = , the PSI depleted in the tank will be twice as much per minute and the time of breathable air use is essentially half the surface amount of time if the diver remains at that depth for a high percentage of the total possible dive time as that at the surface.

Ex. 2 (20 psi used / min) = 40 psi used / min

Here is a generalization and equation to consider, however, dive tables give better values to adhere to:
Time available = (2/3) of total air time available at the surface / ATM of the deepest depth :considering (1/3) of total air time remaining for safety

Again, as a safety rule, (1/3) of the volume of gas in the tank should be kept as an emergency reserve, and a diver needs to surface before any of that (1/3) amount is used or breathed. Due to this, (1/3) of the starting internal pressure needs to be kept as the reserve gas, and therefore with these considerations, (1/3) of the total possible dive time should not be used, and the diver needs to reach the surface by (3/3 - 1/3) = (2/3) of that total possible dive or air time.

Ex. If a tank is filled with air gas having an internal pressure of 3000 psi, the reserve amount will be:
(1/3) 3000 psi = 1000 psi , this leaves: (3000 psi - 1000 psi) = 2000 psi for a safe dive.

$$\text{total safe dive time at 33ft deep} = 2\text{ ATM} = \frac{\text{Safe psi available in tank}}{\text{Psi usage rate}} \quad , \text{Ex: } \frac{2000\text{ psi}}{\left(\frac{40\text{ psi}}{1\text{ min}}\right)} = 50\text{ minutes}$$

Considering a decompression time of 20 minutes to the surface:
safe time to travel to and stay at depth = total safe dive time - decompression time needed to the surface ,

Ex. 50 min - 20 min = 25 min

When the psi or bar pressure of the compressed air gas in the scuba tank is at half of its starting value, then that scuba tank contains half the volume of initial volume of compressed air. Due to this half of its total breathing time available was already used or breathed by the diver and only half of the total breathing time remains, and this value includes the (1/3) of tanks initial amount or volume of compressed air for safety. (1/3) amount of the initial volume also leaves (1/3) of the initial internal pressure, and (1/3) of the total breathing time available from the full tank.

Considering the decompression time needed

Generally, dives less than 30 feet deep, a short decompression time is optional. A diver can remain at some depths greater than 30 feet and not require a decompression time if the time at that depth is less than the time recommendation which is commonly known as the NDL = No Decompression (time, maximum value) Limit. The deeper a diver goes, the less time available at the bottom before decompression is needed. The deeper and longer at a depth greater than 30 feet,, the longer the decompression time will be. You will need to calculate and-or have a "look-up" (to find) chart and-or reference table available to find out a suggested and safe decompression time both for the depth you have gone and the amount of time you remained at that depth since both of these factors will determine the level of nitrogen gas absorption into your body. NDL is sometimes called the No-Stops Time (ie., no decompression ascent stops and times required).

A diver making repeated dives to the same depth will needs to also to consider a break time between each dive for health and safety checks, and to help eliminate any possible trace of nitrogen gas still in the body, and the deeper a diver goes, the longer the decompression time needed, and the longer the diving break time will be. Even when a diver is surfacing to a lesser depth but still greater than 30ft =~ 10m deep, there is still a possibility that more nitrogen gas can get into the body, and that is why decompression is a relatively, slow process.

Here are some rough estimated considerations for the maximum time limit values at depth till when a decompression time is required, and this will make calculations easier, and greatly improve the overall safety of the diver. The compressed air tank should have enough air to consider the descent time, the ascent time which includes any decompression time, the time at the desired depth, and (1/3) the maximum amount of air to be remain in the tank when the diver surfaces as reserve air for any emergency during the dive.

For a depth of < 60 ft, the maximum, no decompression time limit will double per 10 feet less:

NDL = (60 minutes) 2 ((60 ft - depth ft) / 10)

For a depth of 60 feet, the (maximum), no decompression required time limit (NDL) at depth is:.

NDL = 60 minutes : for lesser depths, the NDL will be at least 60 minutes as an easy reference

For a depth > 60 ft , the maximum, no decompression time limit will decrease by 1 minute per 1 foot deeper:

NDL = 60 minutes - 1 min (depth ft - 60 ft)

For a depth > 90 ft , the NDL time is 30 minutes and will decrease by 0.5 minute per 1 foot deeper:

NDL = 30 minutes - 0.5 min (depth ft - 90 ft)

At a depth of 150ft or more, the NDL time is about 5 minutes or less,, and the diver needs to consider decompression time. A general rule is that for any depth of dive, that when the diver ascends to every 15 feet less in depth which is a decrease in 1 ATM of pressure, that they take a stop in their ascent for at least 3 minutes and up to 5 minutes. This also needs to be considered if the dive time was longer than the NDL time, and can also be considered even if the dive time at depth was shorter than the NDL time

At about the last 15 feet of depth or less to ascend, a diver should do decompression stops every 5 feet due to that the ratio of the (ATM) pressure changes for every 33 feet deep is actually increasing as the diver ascends. In short, the pressure changes are increasing more rapidly than one might expect, and these increased pressure difference can cause the bends and therefore, the decompression process should go slower or longer in time at these less depths.

A diver not descend or ascend to rapidly due to the larger pressure changes upon the body. For health and safety, the body will need some time to adjust to pressure changes, and a slower descent or ascent will provide that. A typical and safe maximum descent or ascent distance rate is about 30 feet of change in depth / 1 minute. A dive should be planned, and can become a common routine that is easier to follow by an experienced diver.

How long in time will a diver need to decompress?

Since nitrogen in the body is both determined by the depth of the dive (ie., more pressure forcing nitrogen it into the divers lungs and then into their body) and time at that depth, both of these factors will increase the required decompression time needed for the body to "de-nitrogenize" or expel any remaining nitrogen via exhalation.

During this decompression time, the diver will be ascending to water depths at lower pressures so as to help expel the nitrogen and to help lessen any more nitrogen from being absorbed. When the diver is at more shallow depth and lower pressure, the compressed air which the diver breathes in from the pressure regulator will be set to a lower pressure, and this will also help prevent nitrogen absorption.

A divers ascent to the surface needs to be relatively slow due to that if the water pressure changes too rapidly such as during a fast ascent, the nitrogen in the body will have an increased chance of turning into painful bubbles in the body before it had a chance to be expelled or "gas out" in small particle size (ie., atoms) amounts through the lungs and out of the body when the diver exhales.

A typical rule of thumb for decompression time is that for every 15 ft deeper, that the total required decompression time will increase by 6 minutes:

Total Decompression Time = (Time the diver was at depth greater than the NDL time) + 6 minutes $\left(\frac{\text{Depth ft.}}{15\text{ ft.}}\right)$

Note, at 150 ft = 45m and greater depths, NDL (No Decompression [Needed, Maximum Time] Limit) is about 0 minutes.

The diver should stop for every 15 feet of ascent, and wait or stop for at least 6 minutes before continuing the decompression process. This activity is commonly called decompression stops.

The "120 Rule" is rule of thumb that for depths between 60 feet and 100 feet deep, that the sum of the NDL and depth is a unit-less value of 120. This was made from divers observations of the charts and data. From this, we mathematically have:

NDL = 120 - (depth in feet) : NDL will have units of minutes

Even if a diver has not exceeded the NDL time value at a given depth, they should still consider the decompression process.

Due to the possibility of a residual amount of nitrogen still being in the diver after surfacing, it is recommended that the diver wait at least a minimum amount time before diving again. This value of time, much like decompression time, depends on the depth of the dive and the time spent there. If unsure about this amount of no dive time, at least wait a full day before diving again.

Creating a condensed (ie., made denser) gas having a high pressure

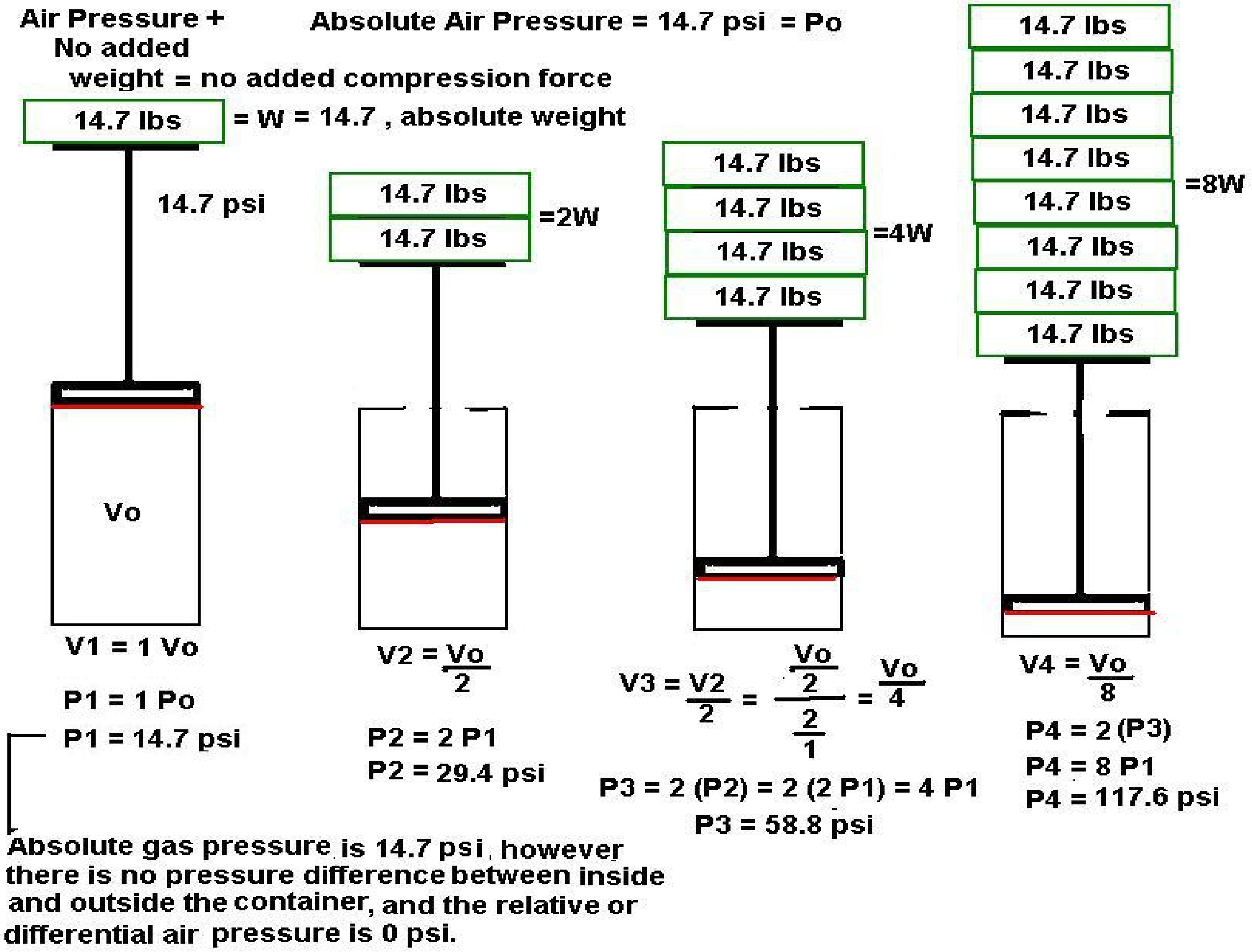
How can an air pressure machine make a very high pressure? First, it has to have a reasonably high force, such as being powered by an electric motor or a persons arms such as when using a tire pump for a bicycle or car.. Through a one-way valve mechanism that will only lets external air come in through it, an air compressor will gather external air and compress it using a force, therefore pressurizing that amount of air. When a tank is being filled with a compressed gas at a certain rising pressure, to get any more gas into the tank will require at least a slightly higher pressure to force it into the tank.

Ex. If a force is 30 lbs, and is applied to an area of 1 in^2:

P = F / A = 30 lbs / 1 in^2

If the output area is reduced to a small fraction of the initial area, say one-hundredth of the given area = 1 A / 100 = 0.01A such as a small opening or pinhole for all the internal pressure to essentially be applied to, the pressure at that hole will be very high, and in fact, it will be 100 times higher: P = F / A = P = 30 lbs / (0.01 in^2) = (100)(30) = 3000 psi. These are equivalent fractions.

Pressure and volume illustration in a tank [FIG 307]



In the above image we see that when the applied weight (W) or force is doubled or increased by a factor of (n), that the volume is decreased by that same factor of (n). We also see in the above image that the product of pressure and volume is a constant value for a particular system. Expressing these physical relationships numerically:

$$P1\ V1 \;=\; \frac{(n)}{(n)}\ \frac{P1}{V1} \;=\; (n)\ P1\ \frac{V1}{(n)} \;=\; P2\ V2 \;=\; P3\ V3 \;=\; P4\ V4 \qquad : n = \text{ratio of weights or applied force}$$

Letting subscripts (o) indicate a given, original or starting value, and (e) indicate an ending or resultant values, we have a general formula for the pressure and volume of a gas in a tank:

$$Pe\ Ve \;=\; Po\ Vo \quad , \quad \text{mathematically:} \quad \frac{Pe}{Po} = \frac{Vo}{Ve}$$

We see in above equation that the ratio of pressures is equal to the reverse (or inverse) ratio of the volumes, and this is so because of the inverse relationship between pressure and volume. Mathematically, for example:

$$Pe \;=\; Po\ \frac{Vo}{Ve} \qquad \text{and} \qquad Ve \;=\; Vo\ \frac{Po}{Pe}$$

When pressure, due to a force, is applied to a given volume of air, that air will get compressed into a smaller volume, and since density = mass / volume , the density will increase since it is inversely related to volume.

Notice in the above figure, that to hold the amount of weight in stable or equilibrium position without moving upward or downward, that the pressure applied to the plunger rod, say having a cross sectional area of 1 square inch, is equivalent to the weight per square inch. Also note that no matter what the pressure is within the container, to make the volume, for example half in value, that the extra weight (ie., a force due to gravity pulling on its mass) to apply will be twice (2) of what is already applied. Likewise,to make the pressure, for example double in value, that the weight will need to be doubled.

> From density = (mass / volume) = d = m / V , m = d V and V = m / d
>
> We see that density and volume are inversely related. When the volume of a gas decreases, its density increases and vice-versa.

As the air gets depleted in a divers air tank, it will have less mass and therefore, less density in that compressed air tank: density = (mass / container volume), and the tank will become more buoyant (ie., want to float to the surface more).

In a fire piston device, a volume of air is highly compressed so as increase in density, kinetic and thermal energy, and pressure so as to combust and ignite an ember so as to make a fire. Given Pe Ve is a constant, if the volume decreases during this compression process, the pressure is greatly increased. Likewise, the temperature of the air will be increased due to the energy applied and the increased kinetic and thermal energy of the air gas atoms. During combustion, the internal pressure will actually increase further.

Today with poles, ropes or "lines", RC (remote control) boats and submersibles, and with relatively inexpensive and "water proof" and "pressure proof" underwater cameras, a non-diver (or not) can experience what is beneath the surface of the water relatively inexpensively, easily and safely.

A Buoyancy Issue For A Scuba Diver

As a diver descends to greater depths and water pressure, any pockets of air will get compressed and have a smaller volume and the diver will then have a lesser or "negative" buoyant force and this could cause the diver to descend at an increasing rate unless corrective action is taken. The descending diver should add some air to their BCD (Buoyancy Control Device) so as to increase its volume and resulting buoyancy. In a reverse manner, as the diver ascends to water having less water pressure, any pockets of air will expand in volume and cause more or a "positive" buoyancy force and the diver will ascend at an increasing rate unless corrective action is taken. The ascending diver should release some air from their BCD (Buoyancy Control Device) so as to decrease its volume and resulting buoyancy. As a diver ascends, if they hold air of a higher pressure in their lungs it will expand in the lower water pressure and can damage their lungs, hence the diver needs to keep breathing regularly and-or rise slow enough so as to give the body and-its gasses time to adjust to the local pressure. **According to the pressure and volume relationship and formula, if the pressure of a gas increases by a factor of (n), its volume decreases by that same factor of (n).** Note also that the density (d = p = mass per volume) of that gas will also increase by the same factor of (n).

Does the buoyancy force change with depth?

This question is also related to the above topic of: A buoyancy issue for a scuba diver. The simple answer to this question is that according to the definition of buoyancy which does not have a depth variable, it does not, and only depends on the difference in pressure between the upper and lower heights of the object. No matter what depth, even at great depths where the water pressure is high, this difference is still the same value. In this sense, the buoyant force does not change with depth. If the object is compressible, as opposed to in-compressible (ie., can not be compressed easily and-or much), then the buoyant force can change. For example, at greater depths, where the water pressure increases, this high pressure will try to compress an object inward to itself, therefore it is trying to reducing its volume and increasing its density. If the object, such as a balloon or bubble, compresses it will have less volume and therefore it displaces less water (ie., volume of water). The balloon will then have less buoyant force, and this will be an issue if the balloon is being used to lift an object to the surface of the water. To overcome this problem, the balloon can be inflated more so as to have the original or intended volume of it, and which is needed for the desired buoyant force.

Some of the commonly used units and conversions for volume measurements

Units of volume and conversion have been previously discussed in this book, and here is perhaps some more to consider for this discussion. The general discussion about scuba diving will continue after this table.

Water has a known mass of 1Kg / 1L = 1000g / 1000 cc. 1Kg = 2.20462 lbs, therefore, after dividing both sides by 2.20462 , we find: 1 lb = 0.453592909 kg =~ 454 oz

1 cu. ft = (12 in)^3 = (12 in)(12 in)(12 in) = 1728 in^3 and 1 in^3 = 0.000578703 ft^3
1 cu. ft = 1 ft^3 = 28.3168 Liters , therefore, after dividing both sides by 28.3168 , we find:
1L = 0.0353147 ft^3 : this has a cube root or side value of: (0.328084092 ft). Hence (0.328084092 ft)^3 = 0.0353147 ft^3

0.328084092 ft =~ 3.937 in =~ 4 in , hence a volume of 1L in the shape of a cube with each side being 4 in =~ 10cm, hence a volume of 1L is a container with a volume of: (4 in)^3 = 4^3 in^3 =~ 61.0237 in^3 (= actual, using 3.937 in) =~ 64 in^3 = (10 cm)^3 = 10^3 cm^3 = 1000 cm^3 . 1cm^3 = 1 cubic centimeter = 1 cu cm = 1 mL of volume

1 cu. ft = 28.316846592 L = 28316.85 cc = 28316.85 mL
1 cu. ft = (12 in)(12 in)(12 in) = 1728 in^3 , and since 1 in = 2.54 cm , 12 in = 12 (1 in) = 12 (2.54cm) = 30.48 cm :
1 cu. ft = (30.48 cm)(30.48 cm)(30.48 cm) = (30.48 cm)^3 = 30.48^3 cm^3 =~ 28316.8 cc
1 cu. ft = 1728 in^3 = 28316.8 cm^3 , dividing each side by 1728, we have:
1 in^3 =~ 16.387064 cc

1g of water has a volume of 1cc. Since density = mass / volume:
The density of water is therefore: 1g / 1 cc
1000 cc = 1000 mL = 1L
1000 g of water corresponds to a volume of 1000 cc of water = 1000 mL of water = 1L of water

1L = 1 liter of volume = (10cm)^3 = 10 cm^3 = 1000 cm^3 = 1000 cc = 1000 mL
Since 1 in = 2.54 cm , 1 cm = 0.393700787 in. 10 cm = 3.93700787 in =~ 4 in
1 L = (10 cm)^3 = (3.93700787 in)^3 = 61.02374409 in^3

1 oz = 1 weight ounce = 1 w-oz = (1/16) pound = 0.0625 lb = 28.3495231 grams and 28.3495231 cc if water
16 oz = 1 pound = 1 lb

The US Customary units are (roughly) based on the (older) British units used in the United States (previously "New England" a British territory) before it became an independent country in 1776. Most countries now use the System International SI) which is commonly called the Metric System standard units of measurement.

1 US fluid ounce = 1 fl-oz = a unit for volume measurement

1 US gallon is based not on water weight, but of wine weight which is not as heavy as water . It has a volume of 231 in^3 = 128 fl-oz, and weighs 8.34 lbs = 3.78 kg if water. Note that (128 US fl-oz/ 160 Imperial fl-oz) = 0.8, and that (160 fl-oz)(0.8) = 128 oz.

The cube root of 231 in^3 = 6.13579244 in , such as the side of a cube having a volume of 231 in^3. 6.13579244 in = (6.12579244 in / 1)(1 ft / 12 in) = 0.511316036 ft on the side of a cube having a volume of 231 in^3. (0.511316036 ft)^3 = 0.133680555 ft^3.

231 in^3 / 128 fl-oz = 1.8046875 in^3 / 1 fl-oz

128 fl-oz / 231 in^3 = 0.554112554 fl-oz / 1 in^3

1 US gallon = 4 quarts has a volume of 3.78L = roughly 4L = roughly 4 quarts
1L = 0.2645503 US gal =~ (1/4L = 0.25L)

1 Imperial or British gallon is based on 10 lbs of water = 160 oz of weight , and has a volume of 4.54609L = 277.42 in^3
= 0.16054366 ft^3 , Canadians also use Imperial gallons, but also use metric liters in this modern age.

1 US gallon = 0.83263955 Imperial gallons , 1 Imperial gallon = 1.2 US gallons

1 US fluid ounce = 1/128 of a US fluid Gallon = 0.0078125 gallons , hence 1 US gallon = 128 US fl-oz
1 quart = 1/4 gallon = 128 fl-oz / 4 = 32 fl-oz = 2 pints = 4 cups
1 pint = 1/8 gallon = 128 fl-oz / 8 = 16 fl-oz
1 cup = 1/16 gallon = 128 fl-oz / 16 = 8 fl-oz

1 L = 4.22675 US cups = 33.814 US fl-oz **=~** 4 US cups = 32 US fl-oz
1 cup = 8 US fl-oz = 0.2365884 L =~ 237 mL =~ 237 cc **=~** (1/4 L = 0.250 L) , 4 cups = 0.946353 **=~** 1 L =~ 95% of a liter

1 US fl-oz = 0.96079599 fl oz in US Customary units
1 US fl-oz = 29.573529563 mL in International System (SI, metric) units =~ 30mL = 30 cc
1 US fl-oz = 0.02957353 L
1 US fl-oz = 29.57353 mL of water will therefore weigh 29.57353 grams =~ 30 grams = a common, "food ounce"
1 US fl-oz of volume will weigh as 1.04 oz of (dry, mass) weight , therefore mathematically by dividing each side by 1.04, we find that:
1 US oz = 1 weight ounce = 1 woz = 1 wt-oz of water and it will have a volume of 0.961538461 US fl-oz =~ 1 US fl-oz

1 US fl-oz = 1.040843 fl-oz in British Imperial units

1 British Imperial fl-oz = 28.4130625 mL in International System (metric) units
1 British Imperial fl-oz = 0.9607599 in US customary units

For British Imperial units , 1 fl-oz is defined as 1/20 of an imperial pint at 20 oz, It is also defined as 1/160 of an imperial gallon that is defined as an amount of water that weighs 10 lbs. 10 lbs / 160 oz = 0.0625 lb / oz = (1/16) lb / 1 oz = = 16 oz / 1lb in equivalent fraction form.

1 US teaspoon of volume = 1/6 a US fluid ounce = 0.16667 US fl-oz = 4.928921594 mL **=~ 5 mL**

Since 1mL = 1 cm^3 = 1 cc, multiplying both sides by 5, we have:
5 mL = 5 cm^3 = 5 cc

5 cm^3 = (1.71 cm)^3 = 5 mL = 5cc

1 US teaspoon of volume = 5 mL = 5 cm^3 = (1.71 cm)^3 = (0.67322 in)^3 = 0.305118719 in^3

Since 1cc = 1 cm^3 = 10mL, 5mL will fill or occupy half the volume of 1 cm^3. 1 cm^3 / 2 = 0.5 cm^3
Since 1 in = 2.54 cm , dividing both sides by 2.54, we get:
1 cm = 0.393700787 in , raising each side to the third power:
(1 cm)^3 = (0.393700787 in)^3
1 cm^3 = 0.061023744 in^3 = 1 mL = 1cc = 0.001L
1L = 1000 mL = 1000 cm^3 = 1000 cc 61.023744 in^3
1 cc = 1 cm^3 = 0.061023744 in^3 , mathematically:
1 in^3 = 16.38706403 cc or 16.38706403 mL = 0.01638706403 L

Since 1 mL = 0.061023743 in^3 = 1cc = 1 cm^3 , multiplying both sides by 5, we have:
5 mL = 0.305118719 in^3 = (0.6732$\underline{2}$ in)^3 = (1.71 cm)^3
10 mL = 0.610237439 in^3

1 cm = 10mm = 0.393700787 in
(1 cm)^3 = 1 mL = (10mm)^3 = 1000 mm^3 = 0.061023744 in^3 , mathematically:
1 mm^3 = 0.000061023 in^3 , mathematically :
1 in^3 =~ 16387.$\underline{1}$ mm^3 : mm^3 = cubic millimeters

1 US tablespoon = 3 US teaspoons =~ 15 mL commonly

1 US gallon = 128 US fl-oz = 3.78541178 L =~ 3785 ml = 3785 cc, , Mathematically:
1 US fl-oz = 0.029573529 L , hence mathematically:
1L = 33.81402274 US fl-oz = 1000 cc , hence mathematically:
1 US fl-oz = 29.57352953 cc = 29.57352953 mL = 0.02957352953L
If the fluid substance is water: 1 US fl-oz will have and weigh as a mass of 29.57352953 g =~ 30g

For finding volume of a container using water as the mass substance which has a simple relationship between mass and volume in the metric system:

1 gram of water = 1g of mass and has 1cc = 1 mL of volume.
1000g of water = 1kg of mass and has 1000cc = 1000mL = 1L volume of water.
1 cubic foot , volume of water = 1 ft^3 = 1728 in^3 and weighs 62.4 lbs
1L is roughly the volume of a quart or quarter of a gallon, and will weigh about 2 lbs if the internal substance is water.

Given a 128 fluid-ounce US gallon, a quart = (1/4) gallon = 2pints will have a fluid volume of (128 fl-oz / 4) = 32 fl-oz.
16 fl-oz = 1 pint of water will weigh 1.0425 pounds =~1 pound of force.
1 cup of water = 8 fl-oz of water will weigh about 0.5 lbs

A US gallon volume of water weighs 8.34 lbs.
1 US gallon = 128 US fl. oz = 3.78L , and if filled with **water** substance,
it will weigh 3.78kg since 1L water = 1 kg of water

$$\frac{\text{1 gallon water}}{\text{3.78 kg}} = \frac{\text{0.264550264 gal}}{\text{1 kg}} = \frac{\text{(0.264550264 gal)(128 floz/gal)}}{\text{1 kg}} = \frac{\text{33.86243386 floz}}{\text{1 kg}} = \frac{\text{1 floz water}}{\text{0.02953125 kg}}$$

If the substance is water: 1 floz water / 0.2953125 kg corresponds to a volume of: 1 floz water / 29.53125 cc =
1 floz water which corresponds to a mass of: 1 floz water / 29.53125 g , therefore:
1g mass of water corresponds 0.33862433 floz = 1cc = 1mL

(8.34 lbs water / gal) / (3.78 L / gal) = (8.34 lbs water / gal) (1 gal r / 3.78L) =
= 2.2063$\underline{5}$ lbs water / L = 1 lb water / 0.45323741 L = 1 lb water / 453.23741 cc = 1 lb water / 453.23741 mL
Since 1cc volume = 1mL volume , and 1 cc water = 1 mL water:

1 lb water =~ 453.2$\underline{4}$ cc = 453.2$\underline{4}$ mL = 453.2$\underline{4}$g

Be sure to note:

1 ounce of weight of any substance = (1/16 1lb) = 0.0625 lb = 28.3495231 grams of mass = 0.0283495231 kg
From this we have mathematically:

1g = 0.03527396$\underline{2}$ weight oz
If the substance is water: 1cc water = 1g water = 0.03527396$\underline{1}$ weight oz of water , and this is the amount of weight oz of water and not floz of water even though 1g corresponds to exactly 1cc of water.

The corresponding amount of US fluid ounces (floz or fl-oz) is close to the shown value, and is actually:

1 cc of volume = 1 cm^3 = 1 mL 0.0338140227 US fl. oz. and mathematically from this:
1 L of volume = 1000 cc = 1000 mL = (1 mL) (1000) = (0.0338140277 fl.oz)(1000) = 33.8140277 fl.oz =~ 1 US quart (32 fl.oz)

1 US fl.oz = 29.573533018 cc and if the substance in the volume is water, it will weigh:
1 US fl.oz of water weighs **29.573533018 grams = 0.029573533018 kg**
and will have a volume of **29.573533018 cc = 0.029573533018 L**

The result shows that a 1 US floz of water weighs slightly more than 1 ounce of weight:
1 US fl-oz weighs 1.04317568 oz = 29.573533018 grams

The above issue of the 1 US fl. oz being (slightly) greater than 1 weight ounce is due to how the US defined their fl.oz.

A note about calculating mass, density and volume

First: Force = (mass)(acceleration) = ma

Weight = (force due to gravity) = (mass)(gravitational acceleration) : gravitational acceleration = g = 9.81 m/s^2, g = 32.2 ft/s^2

We see that the mass of an object can be calculated from the corresponding and proportional weight of that object:

mass = (weight / a) : here for weights on Earth, a = g and mass = weight / g

Density or density of matter can be thought of as the concentration of matter such as the amount of matter or substance n a unit of volume. Since some substances salt grans, sand or foam have much space between particles which have no relevant matter in it and will cause an incorrect calculation of density, it is better to first weigh the amount of matter so as to find its density by calculation.

density = (mass / volume) : d = m / v , m = d v , v = m / d : m = mass in units of grams
v = volume in units of cc or mL
density = (mass / volume) = (weight / g) / volume
d = density in units of grams / cc or g / mL

weight = the force of an object in gravity = (mass)(a) = d v a = d v g = (density)(volume)(acceleration)

1 gram of water is defined as the amount of water in a volume of 1 cc , and any mass or object that has the same amount of weight as 1 gram of water is then also said to have a mass of the same value of 1 gram.

The density of water is defined as:
density of water = (mass / volume) = (1g / 1 cc)

1 cc of lead or gold for example, will weigh more than the reference mass of 1g of water when given the same size and-or volume such as 1cc. These elements are then said as being denser than water. Since they are denser than water, an equivalent volume (V) of it will weight more than that of the same volume water. To have 1 gram of these elements corresponds to having a less volume tan 1cc.

V = m / d , if (d) increases , (v) decreases

Ex. If a certain element is listed as having a density of (5g / 1cc) and weighs 100 g, how many cc of volume corresponds to that amount of mass? We can first set up an equivalent (and proportional) fraction and problem to solve:

$\frac{5g}{1\ cc} = \frac{100g}{x\ cc}$ solving for x, we have: $x\ cc = (1\ cc)\frac{(100g)}{(5\ g)} = (1\ cc)(20) = 20cc$

If you had a volume of 7cc of this element, how much will it weigh?

weight = d v a = (density)(volume)(acceleration) = (5g/1cc)(7cc)(9.8 m/s^2) = (0.005 kg/1cc)(0.07m)(9.8 m/s^2) =
= 0.00343 Newtons

mass = d v = (5g/1cc)(7cc) = 35g : finding total mass by knowing the density and volume

The volume of an object can be found by submerging it in water and measuring the increase in volume. The volume of the object or mass is
Vm = (ending volume of water) - (starting volume of water) =
Vm = difference or change in water volumes

1N = 101.9716212978 =~ 100 equivalent grams-force (gf) = (grams)(a) in Earths gravity of 9.8 m/s^2
This would be the force that 100 grams of mass will exert or weigh due to gravity

In Earth's gravity, the weight and-or force of 1Kg is:

Weight = Force = (mass)(acceleration) = 9.8N = (1kg)(9.8 m/s^2) = (1000g)(9.8 m/s^2) , solving for 1N, we have: 1N =~ (101g)(9.8 m/s^2) =~ 100 gf = 0.1 kgf
If you had a mass of 100g, it would weigh about 100 gf = 0.1kgf = 1N of force.
If you had a mass of 1kg, it would weigh about 1000 gf = 1kgf =10N of force

Calculating The Buoyant Force Of An Object In Water

From : Pressure = Force / Area = P = F / A , mathematically: F = P A

Pressure can be thought of as the average amount of force applied to an area, or force distribution to an area whereas, force can then be thought of as the total amount of pressure applied to the total amount of area.

If the force is due to a weight (W), and weight is a force due to gravity, then P = W / A = F / A.

Whenever a force is applied, there is a contact and-or surface area associated with it, and therefore an amount of force per unit of area, and this is the pressure value. If the pressure is high enough upon an object, it can cause an object to compress (get pressurized) and deform its shape and reduce in volume. If the material is flexible, the deformity will not be permanent if the force is removed, much like how a spring can change shape. The amount of deformation is directly related to the pressure that was applied to that area. If the force was distributed over a large surface area, then the pressure or amount of force per unit of area will be low and the deformation low.

For a given amount of force applied, and regardless of the size of the area, the product of P A will be a constant value for a given system and initial amount of force, and that value is equal to: F = P A :

$$F = P1\ A1 = P2\ A2 = P3\ A3 = \frac{n}{n} P\ A = nP\ \frac{A}{n} = \frac{P}{n} nA$$

If the area is made to be a very low, then the pressure or effective value of the applied force will be very high. The tip of a nail to be hammered into a piece of wood has a small area, and when the force is applied to the nail, it will have a high pressure upon that area of the wood and it can be driven into it. High pressure can be thought of as the force, energy or ability to do something as being concentrated, or the effective value of the force, and which equals the pressure, as being increased, here, as if the initial force of the hammer was amplified. As seen in the above equations, force remains constant, however, pressure can be amplified by a factor (n) if the area is reduced (ie., de-amplified or de-magnified) by that same factor of (n).

If the water pressure is higher at deeper depths, and the force and pressure upon an object will be higher, and if the water pressure is lower, the force and pressure upon an object will be lower. If an object, such as a bubble of air gas is put in water having a high pressure, it will get compressed by the surrounding forces upon it, and it will decrease in volume, but density of the air gas will actually increase. density = (mass of air / volume).

For an object in water, and since there is greater water pressure at greater depths, there will be more force upon the lower (deeper) part of the object than upon its upper (less deep) part. The difference in forces or pressures will result in a net, effective or unbalanced, active, non-neutralized force upon the bottom of it. This force is called the buoyant force. An unbalanced or net force will give kinetic energy to an object, and the buoyant force is this unbalanced force value and will cause an object that weighs less than the water it displaces to rise upwards in the water.

Buoyant Force = weight of the displaced liquid = mg = dVg : **Bouyant Force upon an object in a liquid**.
d = density of the liquid = mass of liquid / volume of liquid
V = volume of the liquid displaced = volume of the object
g = gravitation acceleration = 9.81 m/s^2 = 32.2 ft/s^2

The density of water = mass of water / volume of water = 1g / cc = 1g / 1ml = 1kg / 1L

Apparent weight of an object due to its buoyancy = Weight of the object out of the water - Buoyancy force in water

We see from the buoyant force equations that the mass of an object does not determine the buoyant force, and only the volume of it does because this is what displaces the equivalent volume of the liquid.. The object might be made be of any material(s) or element(s), and therefore be of any mass and weight, yet for all these objects of the same mass, weight, and technically, volume dimensions, they will have the same buoyant force applied to them due to a water pressure difference. The buoyant force of a solid steel ball is the same as a hollow aluminum ball because they have the same amount of volume, and an average person might think that the aluminum ball is more buoyant because it weighs less. Another viewpoint is that even though the solid steel ball has some buoyant force upon it, it will still sink to the bottom due to that its average density (total mass / volume) = [(weight / g) / (volume)] and-or its weight is greater than that of the water it displaces and the buoyant force is not enough to raise it upward or maintain its depth and have equilibrium where the average density (mass / volume) of the object is that of the water it displaces which has a density of (mass / volume) = 1g/cc.

A strong balloon with a large volume of air (which is less dense that the water it replaces is capable of lifting a heavy weight objects from the bottom to near the surface of water. The higher pressure of the water around the balloon or has compressed that inner volume of air to be at the same pressure, but more importantly, the water pressure at the top of the balloon is slightly less, and therefore, there is higher pressure upon the bottom of the balloon. Since there is difference in pressure (P = F/A = W/A) upon the balloon , there is a difference in forces upon its surfaces, and a net force (F = PA) is developed on the lower part of the balloon. This force is effectively applied to the bottom of the balloon in the direction of the surface and causes the balloon to move vertically. This force (the buoyant or "(vertical) flotation" force) must be greater than the weight of the balloon for it to be moved upward, and the balloon surely does not weigh too much being filled with air, and using a lightweight balloon material.

The difference in the pressures (P = FA) and-or forces (F = PA) upon an objects vertical surfaces will cause an effective or net pressure and force (ie., weight) and it is actually equivalent to the weight of the water displaced.

For the object part at lower depth: F1 = P1 A and water pressure1 = P1 = F1 / A : assuming similar areas:

For the object part at high depth: F2 = P2 A and water pressure2 = P2 = F2 / A

Difference in pressure = (P1 - P2) : extra, this is also what causes a planes lift force, a difference in pressure upon the upper and lower wing surfaces.

Difference in force = (F1 - F2) = buoyant force : extra, this is essentially the lift force for a plane

Force = F = weight = ma = mg = dV g = P A
Pressure = P = F / A = ma / A = mg / A = weight / A = (dV)g / A = dVg / A = Weight / Area

The concepts of buoyancy of a water bubble or object can also be applied to a **hot air balloon** that rises vertically in the air due to its buoyancy in the air. For the balloon to rise vertically, the buoyancy force developed must be greater than the weight of the balloon system and passengers. Large balloons are capable of lifting several hundred pounds. The minimal amount of buoyancy force needed will be the weight of balloon system plus the weight of the passengers, and any extra buoyancy force is the effective (vertical) lift force which is effectively applied to the bottom of that balloon. Since air is much less dense than water, the balloon must be very large in size (ie., volume) so as to displace a large amount or volume of air which has total mass and its corresponding weight. To lower the weight of the balloon so that the buoyancy force is greater than the weight of the balloon, a low density (d = m / V) gas such as hot air can be used and therefore, that amount of air in the balloon has a low corresponding weight. Shown further below is an equation that we can use to can find the (minimal) volume of the balloon needed.

From: **Buoyant Force** = weight of the displaced liquid or gas (here air) = **W = dVg** , we mathematically have:

For the weight of 1 m^3 volume of water which has a density of (1g/cc) = (1000g/1L) = 1 kg / 1L
W = F = ma = mg = dVg = (1 kg/1L) (1 m^3) (9.8 m / s^2) = 9.8 (kg m^2 / s^2) = 9800 N

The weight of a 1 kg mass of water is: W = F = ma = mg = (1 kg)(9.8 m/s^2) = 9.8 N

Most weight scales which measure mass in the presence of gravity, should display W=F=(mg), for example pounds of weight , but commonly "metric weight scales" usually display the weight's corresponding mass. m = W / g , and are actually "mass scales".

F = PA = ma = mg = W = dgV and d = m / V = W / gV and m = dV , V = m / d = (W / g) / d = W / dg

The volume of 1 kg of mass of water is: V = m / d = 1 kg / (1kg / 1L) = 1L = 1000 cm^3

The mass of 5 L of water is: m = dV = (1 kg / 1L) (5 L) = (5 L) (1 kg/1L) = 5kg

Since the minimal amount of buoyant force needed must be at least equal to the weight of the balloon, the minimal volume needed will be derived and calculated :

: d = density of unheated air at STP is 0.001225 g / cm^3 = 1.224 kg / m^3 = 2.69846 lbs / m^3
: d = density of the hot air at 100°C is 0.000946 g/cm^3 = 0.946 kg / m^3 = 2.08557 lbs / m^3
g = a = 9.81 m/s^2

Note that the ratio of volumes of air at two densities is also equal to the ratio of their corresponding densities. After dividing the values , the ratio is about: 1.2294

Note the mix of metric and English units in: 2.69846 lbs / m^3 ,and this may help some, and to make the full conversion of system units we will first convert: 1 m^3 to its equivalent volume and expressed with units of feet: 1m^3 = 10.7639 ft^3.
2.69846 lbs / 1 m^3 = 2.69846 lbs / 10.7639 ft^3 , and using division of both the numerator and denominator by 10.7639 to create an equivalent fraction, we have: 0.2506954 lbs / ft^3

Also since d = m / V , m = dV , after multiplying both sides by g, we have: W = mg = gdV = F

Also, from W = dVg , (W/V) could be described as "weight density" and

(W/gV) as we commonly know is "mass density" = d = m / V = W / gV

Note that some products may be labeled: as "weight / volume", rather than "mass / volume", or simply as "sold by weight" (such as lb = pounds, or: oz = dry ounces), rather than "mass".

Each 1 m^3 of volume of the inflated or expanded hot air balloon will replace 1 m^3 of typical air. Since buoyant force is equal to the weight of the substance (here, air) displaced, 1 m^3 of balloon will displace 1 m^3 of typical unheated air which has a corresponding (mass) weight of 1.225 kg/m^3.

Since 1m^3 of air will be displaced, the buoyant force upon a 1m^3 balloon will be able to lift a total mass of 1.225 kg.

To lift a mass of 300kg mass (which has a corresponding weight value), and of which includes the mass of the balloon material, system and the passenger or object to be nearly lifted into the air vertically, will require a buoyancy force applied beneath it, and having the equivalent value of that of the mass of 300kg and-or its corresponding weight. Since volume and mass are proportional, we can use an equivalent fraction and-or equation to solve for the corresponding (minimal necessary) volume needed to create the (minimal necessary) buoyant force to just begin to lift that amount of mass.

$$\frac{1\ \text{m}^3}{1.225\ \text{kg}} = \frac{V\ \text{m}^3}{300\ \text{kg}}\ ,\ V\ \text{m}^3 = \frac{(300\ \text{kg})(1\ \text{m}^3)}{(1.225\ \text{kg})} = 244.9\ \text{m}^3 \approx 245\ \text{m}^3 = V\ \text{balloon} = V\text{air displaced}$$

If we take the cube root of this amount of volume, we can find the side length of a cube having the same volume. The cube root of 245^3 =~ 6.26m . The radius of a sphere (ie., like a balloon) with this amount of volume is:

r = cube root of (3 V **/** 4 (pi)) = cube root of (0.238732414 V) = cube root of **[**(0.238732414) (245m^3)**]**

$r \approx 3.88\underline{2}$ m , $d = 2r \approx 7.763\underline{5}$ m

Any extra buoyant force more than the minimal amount needed will then cause the balloon (or thing in water) to rise up vertically and no longer be **neutrally buoyant** where it is neither rising upward or sinking downward. If the weight of the volume of gas or liquid displaced by the object is not equal to the weight of the object (such as a solid stone, and of which is a denser material than water per volume), then the object will sink even though it has some buoyant (or upward force) on it. An object will appear to have (some) less weight due to buoyant force.

A related concept is called **relative density** and which is also called **specific density, specific density ratio,** or the **specific gravity** for a given mass or substance. In simple wording, the relative density of an object and-or mass is how many times more dense it is than water which has a density of (d = m / V = 1 g / 1 cc):

$$\text{relative density} = \frac{\text{density of object}}{\text{density of water}} = \frac{(\text{g} / \text{cm}^3)}{(1\text{g}/1\ \text{cm}^3)} = \text{g} / 1\text{g} = \text{ratio of how many times denser the object is greater than that of water.}$$

If the relative density of the object is greater than 1, it is more dense than water and will sink in water. What this also means is that if the density (d = m/V) of the object is more than the density of water, the (average) weight of the object per same volume of water will be more. Weight is directly proportional to mass. Surely, the metal of a submarine or boat is much more dense than water, but the average density for the same volume of water that is displaced then needs to be considered. The submarine can rise upward in the water if its average density is less than that of water, and it can do this by removing the water in its ballast tanks (which caused it to dive or sink deeper in the water) with air which is less dense than water, and thereby making the **average density (= total mass / total volume)** per unit of volume of the submarine less than that of the same volume of water of which it has displaced (ie., replaced).

From: Buoyant Force = Weight of water displaced = dVg , and if V remains the same, such as for the submarine, then the buoyant force depends upon the average density of the object such as a submarine. If the submarine takes in water, its average density will increase, and its relative density will be more than that of water and it will go deeper. Many types of fish have what is called an (internal) air-bladder (ie., sack) which the fish can inflate (ie., expand in volume, like our lungs) with a gas, or deflate (ie, remove) the gas, and therefore change the buoyancy of the fish so as to rise upward, or to easily remain at the same depth (neutral buoyancy), or sink downward.

Vertical or effective lift force due to buoyancy = (buoyant force) - (weight of object(s) being lifted)

Note that for the minimal amount of buoyant force to cause upward motion (and not just some lesser value making it appear as slightly lesser in weight), the weight of the displaced gas or liquid is equal to the weight of the object, but the volume of that gas or liquid is not equal to the volume of the object, and we see that the volume of the balloon is much greater than volume of the (much greater amount of) mass it is lifting. $d = m / V$, $V = m / d$, and for any given amount of mass [which also as a corresponding weight] at STP (standard temperature and pressure), if the density (d) decreases by a factor of (n), the volume (V) will increase by that same factor of (n).

Avogadro, while studying gases surely noted that given: $d = m /V$, and when he had two balloons, each filled with a different type of gas, and having the same volume, that they weighed slightly different, and therefore, the mass of each type of gas particles or molecules in each balloon must be somehow different, and that the densities of the two different types gases must be different. Today, we know that the two different types of gas particles are actually different atoms of gas particles, and with each type of gas atom having a unique number of atomic mass particles (protons,neutrons). Avogadro theorized that each type of gas at the same temperature (ie., thermal to KE of the particles) and pressure, and in the same sized balloons (ie., volume) had the same number of particles. Avogadro did not know that specific number, but later, he was honored for is work by having this value named after him, and that is Avogadro's constant or Avogadro's number.

A certain number of particles, such as those of hydrogen gas and which sum to a mass of 1 gram, is used as the reference unit for an amount of particles (amu's, atoms, molecules) and it is called a mol. Each type of gas, of which each particle or atom of it has a unique amount of mass, and therefore having a unique number of grams of mass per 1 mol count unit, and-or a unique number of mols per 1 gram of mass.

The number of particles in a mass of 1 gram of hydrogen gas was assigned 1 mol. 1 mol of atom particles of hydrogen has a mass of 1g. Since hydrogen has 1 proton and no neutrons, 1 mol of protons is 1 gram of mass, or it can be said that 1 mol of amu (ie., protons or neutrons) is 1 gram of mass

A Note On Avogadro's Theory

What may have inspired Avogadro do his scientific study of matter?

Consider that two different metals, such as aluminum and copper that having the same volume have different amounts of weight, and therefore, the mass ($m = F / a = W / g$) of each is also different. It was understood since antiquity, that the smallest indivisible particles of each type of element was an atom of the same element, and therefore, Avogadro must have considered that there is something different about each type of atom of the two different metals. Since the mass of each element was different per same unit volume, and is known as the density ($d = m / V$) of that matter or element, it is reasonable to first conclude that the atoms of one metal are packed together more closely. As mentioned previously, the atoms or smallest particles of two different types of metals are not similar because they atom sized particles of each different element still have the qualities of that element or metal. If it was simply a matter of the density only of similar atoms, then it would be possible to turn cheap lead into expensive gold by putting much pressure ($P = F/A$) upon it so as to compress it and increase its density. Many have tried to do things similar to this but never succeeded, other than advancing the science of alchemy (ie., a type of experimental chemistry). Avogadro must of realized that it was not the atom or particle count per unit increasing the density ($= m/V$) and weight ($= F = ma$) , but rather it had something to do with each different type of atom particle of that type of matter or element. In Avogadro's balloon experiments with gasses, he proposed that given the same volume of two different types of gasses at the same temperature and pressure (STP) each, that they must have the same number of particles which caused the balloons to expand to the same volumes. What was different about the two seemingly similar balloons? The weight of each balloon was slightly different, indicating that the mass of each different type of particle (ie., atom) or molecule of that gas was causing the weights of the balloons to be different. The hydrogen gas balloon weighed less than all the other gasses, and the number of particles corresponding to 1 gram of it was called a mol or 1 mol of particles. Avogadro did not know the quantity of particles equal to 1 mol unit or count of particles, but it was later found after years of research by others, and they honored Avogadro for his work by naming this constant as Avogadro's constant or Avogadro's number.

Because different metal elements have different densities (m/V and W/V) due to that the different types of element atoms or particles having a different amount of mass, 1 mol of particles or atoms of that element with more density and mass (or its corresponding amount of weight) per unit volume will have more grams of mass per 1 mol count of those denser, more massive (ie., in amu) particles. By knowing how many grams of atoms corresponds to and-or are needed for 1 mol of atoms, if we need to have n mol of atoms for something such as a particular chemical reaction or some medical formula, and we surely can not count them all, we can rather weigh the needed amount of it for 1 mol of atoms by using a weight to its corresponding mass displayed scale, (ie., finding mass via its corresponding weight). For example if a particular element is noted as having 10 grams per 1 mol of atoms, how many grams are needed so as to have 2 mol of it? For the solution, we can set up a proportion type of equation using similar fractions and mathematically solving for the value we are trying to find. The number of grams and the number of mol are directly related (ie., as in higher grams means higher mol) and are mathematically proportional (ie., each changes by the same factor or ratio value) in (relative, or ratio) value to each other.

> If 10g corresponds to 1 mol of atoms of that element, then the equivalent fraction of Ng to 2 mol of atoms of that element can be expressed as:
>
> $\frac{10\text{ g}}{1\text{ mol}} = \frac{N\text{ g}}{2\text{ mol}}$, mathematically: $Ng = 2\text{ mol }(10g / 1\text{ mol}) = 20g$: $Ng = (x\text{ mol}) (g / \text{mol})$.

Continuing onward with the discussion about compressed air tanks for scuba diving :

Besides using a weight scale, it is possible to calculate the amount (weight or volume) of air in a tank if you know the weight of the empty tank without the compressed air in it.

Weight of tank filled with compressed air = (weight of the empty tank) + (weight of the compressed air in the tank) .

(weight of the compressed air in the tank) = (weight of tank willed with compressed air) - (weight of the empty tank)

(Weight of the air in the tank) = (mass of the air)(g) = (density V)(g) = dVg = W , for the volume of this weight of air:

V = W / dg = ma / dg = mg / dg **= m / d** : V = volume , W = weight , d = density , g = gravitational acceleration

From: d = m / V , we have: m = dV and V = m / d = (F / a) / d = (W / g) / d = W / dg
: it is of note that most modern scales are calibrated to display the mass, and in grams, of an object.

W = dg V **:** extra: g = W / (V d) = a = F / m = W / m

The ratio of volumes is equal to the reverse or inverse ratio of the pressures or densities.

$$\frac{V1}{Vo} = \frac{Po}{P1} \quad : \ V1\,P1 = Po\,Vo$$

The ratio of the weights (W) of the air is also equal to the ratio of the volumes of air in the tank:

$$\frac{V1}{Vo} = \frac{\text{V of air compressed in tank}}{\text{V of air in empty tank}} = \frac{\text{(W compressed air in tank / dg)}}{\text{(W of air in empty tank / dg)}} = \frac{\text{W compressed air in tank}}{\text{W uncompressed air in tank}} = \frac{W1}{Wo}$$

$$\frac{V1}{Vo} = \frac{Po}{P1} = \frac{W1}{Wo}$$

: For this example, Po = the absolute air pressure at sea level =~ 14.7 psi
P1 = pressure of the compressed air in the tank = Po Wo / W1
P1 = Po Wo **/** W1 = (14.7 psi)(dg Vo) **/** (dg V1)

The ratios of the weights to volumes are proportional:

$$\frac{W1}{V1} = \frac{Wo}{Vo}$$

: these can be thought of as "weight densities" , where as "mass densities" = mass / volume = m / V
: Wo = weight of uncompressed air and W2 = weight of compressed air

How many grams does a volume of 1 liter of air weigh?

F = weight = ma = mg = dVg , and when finding just the mass value, m = dV , and since the density of air is about:
d = mass / volume = 0.00124 g/cc and 1L = 1000cc:

m = dV = (0.00124 g/cc)(1000cc) = 1.24 g , hence the density of air can also be said as being :
1.24g / 1L = 1.24g / 1000 cc = 1.24g / 1000 mL :at 14.7psi

d = 0.00124 g / 1 cc = 1.24 g / 1000cc = 1.24g / 1L : equivalent and proportional fractions

A tank with a volume of 80 ft^3 of air = 2265.348 L of air will weigh an equivalent mass of:

(2265.348L of air) (1.24 g / 1 L) = 2809 g = 2.81 kg

Since 1L = 0.0353147 ft^3 and 1g = 0.035274 oz:

1.24g of air / 1 L = 1.24 g of air / 0.0353147 ft^3 = 0.04373976 oz air / 0.0353147 ft^3 =

$$= 1.238571 \text{ oz} / 1 \text{ ft}^3 = 0.07741 \text{ lbs} / 1 \text{ ft}^3 = \text{roughly about } 0.08 \text{ lbs} / 1 \text{ ft}^3$$

As the air gets depleted from the tank, it will weigh slightly less and be easier to carry. It is as if it became more buoyant, however the buoyant force is still the same value because the volume of the displaced air due to the volume of tank is the same.

$$80 \text{ ft}^3 = 2265.348 \text{ L} \quad \text{will weigh about: } 80 \text{ ft}^3 (0.08 \text{ lbs} / 1 \text{ ft}^3) = 6.4 \text{ lbs} = 6.4 \text{ lbs} (0.453592 \text{ kg} / 1 \text{ lb}) = 2.90\underline{3} \text{ kg}$$
$$=^{\sim} 3 \text{ kg}$$

When compressing a substance such as air, if the volume Vo is decreased by a factor of (n), the pressure Po will increase by that same factor:

$Vo\, Po = V1\, P1 = k$: **Boyle's Law** , and from this we have: $\frac{V1}{Vo} = \frac{Po}{P1}$ of which are reverse or inverse ratios and equivalent fractions or equivalent ratios

k = a constant value

Boyle's Law applies only if the temperature is held constant because a change such as an increase in temperature will cause the internal pressure to increase, even if the volume of gas or other substance remains the same such as in a closed container. If the temperature decreases, the internal pressure will also decrease.

While a scuba tank is being drained of compressed air, by using Boyle's Law, various volumes and corresponding pressures can be calculated. This can also be used to determine how long a full tank or one being emptied of air can be used. It is already known how much volume of air is inhaled on average, and then the volume of the tank will be reduced by this much for each breath, and the corresponding pressure left (P1) in the tank can be calculated:

After breathing the air in the tank, the volume of air in the tank will be:
starting volume of air - volume breathed = ending volume = Vo - Vbreathed = V1 , and using substitution:
V1 = Vo - **(**n breaths**)** **(**volume per breath**)** = Vo Po / P1

The remaining pressure in the tank will be: (Po - Preduced) = P1 = Vo Po / V1

Extra: Another note on Avogadro's theory.

Avogadro's theory is that given any gas at standard temperature (32°F = 0°C = 273 °K) and air pressure (14.7 psi), that they will fill a known volume, such as a balloon, and have the same number of particles (today, we know these as being atoms of that gas element). Unlike a metal, the gas particles are relatively very far apart. Since the gas substances he used were different, the balloons have a different weight (W) for each. By knowing the volume and weight (ie., force created upon the mass of gas by the force of gravity) of the gas in the balloon, the density (m/V) and mass of that gas can be found.

As an extra note, surely when Avogadro made his gas theory in 1811, he must of known that when the gasses in the balloons were higher in temperature, perhaps by taking them into a heated room of twice the temperature, that they expanded to twice their size (volume). The actual amount (mass and-or weight) of gas within each remained the same value such as 1 gram. The balloons were now less dense (d=m/V, or even the related value of: "weight density" =W/V) per unit of volume is less. **Charles Law** which relates the volume of a gas to temperature was discovered in December of 1787 when **Charles Prairie de Nesles** from France was experimenting with his hydrogen balloons for manned flight shortly after the first manned flights made by the Montgolfier brothers were made in France on November 21, 1783. To Avogadro, it also indicated that the gas particles must be energized and traveling faster when heated, and so as to expand the balloon to its new larger size, and rather than having an increased amount (mass and-or volume) of gas placed in the balloons to expand their size. He reasoned that there is probably the same number of particles in each balloon if their volumes are still the same. Though there was the same number of particles in each balloon, due to the slightly different weight of each balloon of the same size or volume, it indicated that the size and-or weight (ie., hence mass) of each different type of particle (here, unknown atom structures at that time) of gas in each balloon was different. It also then indicated that the structure of each particle (here atoms yet to be formally discovered) was somehow different, and of which we know today as each element having atoms which have a different number of atomic units (protons and-or neutrons = amu = atomic mass units), and normally, with 1 orbiting electron of opposite electrical charge per 1 proton in the nucleus (ie., center) of the atom. This one to one, proton and electron count or ratio makes atoms electrically neutral (no net electrical charge) and electrically stable (no general movement due to the force of an eternal electric charge).

Avogadro found that in 22.4 L = 0.792 ft^3 = $5.91\underline{7}$ US gal of volume, nearly 6 US gal, that hydrogen gas weighed 1 gram and that another gas of the same volume, temperature and pressure would weigh more. The number of hydrogen particles (today known as atoms) to make 1 gram of mass was defined as 1 mol unit in reference to the word molecule which meant a tiny particle at that time. Since other gases weighed more, by reasoning, they had more mass each. All the gases have a volume at nearly 22.4 L of gas per 1 mol unit of atoms. In general, all gases will have the same volume when having the same number of particles and at the same pressure and temperature.

From: density = d = mass / Volume = m / V , mass = m = dV , and

Weight = a force = (mass)(acceleration) = ma = mg = dV g , we have:

d = m / V = (W / g) / V = W / Vg

By knowing the density and weight of the gas, we can then find its mass:

m = W / g = dV : If the volume is 22.4L, then this gas has a total amount of mass of 1 mol unit of atoms of that gas. Here, density of the gas is: d = (1 mol / 22.4L) = (1 gram / 22.4L) at STP

Since antiquity, it was known that the same volumes of two different elements such as metals had a different weight, and therefore their substance or matter must be different in some way as if they had more matter per unit of volume, perhaps compressed together in some way into a smaller volume, and this was called the density (d) of the substance, mass, matter or element:

density = (amount of matter) / Volume , and then calling (amount of matter) as the (mass) of matter:
density = mass / Volume = m / V , and mathematically, m = dV

It was also known that if the volume of each substance changed by a factor (n), that its weight also changed by that same factor of (n), hence by reasoning, the quantity of mass also changed by that same factor of (n). Now if density is related and proportional to weight, we can find the amount of mass of an object from its weight. The weight of a cubic centimeter amount or volume of water was standardized as a reference mass and having a mass of 1 gram. 1000 grams of water was defined as having the reference mass of 1000 grams = kilogram (kg), hence its volume is 1000 cc of which was then standardized as 1L of volume.

Today we know that different elements, it does not mean all atoms in the universe are the same and are simply compressed together more closely than another elements atoms so as to increase its density. We know today that atoms of different elements have a different structure, particularly the number of atomic mass units (ie., protons and neutrons). In general atoms having more mass, weight, and size will occupy a larger volume, and this actually results in having less atoms per unit of volume, and so it therefore takes more of these same atoms, such as in grams of atoms, so as to have 1 mol unit or count of those atoms.

Extra: A note on Newton's force and gravity equations

Isaac Newton knew that if a weight is pushed by a force, that it will move. The more force applied, the more distance it moved,hence Newton figured that the distance an object moves is related to force. The more the object weight, the less distance it moved for a given force. When a force was no longer applied to the object, such as a box on the floor, it would slow down in speed, and this is called a deceleration. If the force is constantly applied, the object would accelerate. Newton knew the basic distance equation of distance = (speed) (time). Newton realized that acceleration (change in velocity / change in time used) is directly related to the amount of force applied to a mass, hence acceleration is directly related to the amount of constant force applied to the mass. Once a force is not applied, there will be no increase in acceleration and no increase in velocity:

$$\text{acceleration} = \frac{\text{force}}{\text{mass}}$$, and mathematically, Newton made his this equation:

force = (mass)(acceleration) = ma , and in terms of distance and velocity:

$$\text{acceleration} = \frac{\text{force}}{\text{mass}} = \frac{\text{(change in speed or velocity)}}{\text{(change in time)}}$$

Associating weight as a force due to gravity:

weight = (mass)acceleration due to the force of gravity on Earth) = mg : g = 9.81m/s^2 = 32.2 ft/s^2

Galileo and later, Isaac Newton knew that gravity must be some kind of force which caused objects to move downward, he also knew that a constantly applied force applied to an object will cause an acceleration in its motion and-or velocity, and it will get faster in the same direction of the applied force. By doing experiments such as releasing a still (no force applied, therefore no acceleration, and therefore no initial velocity) ball from a height, it took say 1 second for it to reach the ground. But was it accelerating due to this constant force of gravity? Newton figured that it should be if the force is constant, that is, a constantly applied force will cause a constant increase in acceleration (change in velocity), and the object will then be traveling faster and faster (ie., or greater and then greater) in velocity. After dropping the ball from at twice the height, the ball did not take twice as long (2s) to reach the ground, but took less time, say 1.5 seconds, The ball took the same amount of time to first go, move or travel a distance equal the original height, but it must of kept accelerating due to the constantly applied force of gravity upon it, and giving it more kinetic energy and increasing in speed, and then the time to travel the remaining height to the ground was less than the first.

acceleration = (change in velocity) / (change in time) , and mathematically:

(change in velocity) = (acceleration) (change in time)

Since distance = (velocity) (time) , a change in velocity = acceleration, will cause a change in the distance traveled per unit of time. (change in distance = (change in velocity) (change in time)

(acceleration) = (change in velocity) / (change in time) =
[(change in distance) / (change in time)] / (change in time) =
(change in distance) / (change in time)^2

Ex. If an object such as a motor vehicle is traveling at a distance of 10 feet per second of time, its velocity is said to then be: v = d / t = 10 ft / 1 s. If more energy and-or force (the application of energy) is then constant applied to the object, it will have a constant increase or change in velocity during that time, and this is its value of acceleration. If after 5 seconds of time, its velocity was 15 feet per second of time, then the change in velocity was (15 ft / s - 10 ft / s) = 5 ft / s , and its (average, if not consistent) acceleration was: acceleration = a = (change in velocity / (change in time) = (5 ft / s) / 1s = 5 ft / s^2 =
a = (change in distance) / (change in time)^2 = (change in velocity) / (change in time)

From the above equations, we mathematically have:

(change in distance) = (change in velocity)(change in time) , and also:

(change in distance) = (change in velocity) (change in time) = (a) (change in time)^2

(change in velocity) = (change in distance) / (change in time) = (a) (change in time)

(acceleration = a or g) = (change in velocity) / (change in time) = (change in distance) / (change in time)^2

Newton knew that heavier weight objects did not reach the ground any faster than lighter weight objects, hence, he knew that the weight and-or mass of the object did not determine or affect its velocity and-or its acceleration. It was rather the (relatively. locally, here Earth's gravity) constant value of the force of gravity being constantly applied to the object and causing (ie., forcing) it to accelerate (ie., travel faster, increase) in velocity and have more kinetic energy.

By conducting many experiments and calculations, Newton found that an object will accelerate downward at 32.2 ft/s^2 = 9.81 m/s^2 due to gravity. This value is called the gravity acceleration value of Earth and is given the variable name of (g). A ball dropped from a height of 9.81 meters will reach the ground in 1 second of time. A ball dropped from a height twice a high = 2 (9.81m) = 19.26 meters will reach the ground in this many seconds or units of time:

(change in time)^2 = (change in distance) / (a) = 19.26 m / (9.81 m / s^2) = 1.9633 s^2 and ,
after taking the square root of each side of this equation, we have:

(change in time from 0s) = (time of reaching the ground) = $1.401\underline{2}$ s : and not 2s

Also: 1g of mass has a weight of: 0.009806652 N , and 22.4L = 22400 cc of hydrogen gas

Dividing each side by 0.009806652 to isolate N as 1 unit of N, we have:

1N = 101.9716 g rams

1 kg of mass has a weight or force of: (weight of 1 gram)(1000) =
(0.009806652 N) (1000) = 9.806652 N =~ 9.81 N = 2.20462 lbs and from this we have:

1 N = 0.224808629 lbs =~ 0.25 lbs which is a quarter of a pound of weight
1 lb = 4.448227813 N =~ 4.5 N

If Avogadro's gas filled balloon was not expanding or deflating, then its internal pressure from the pressurized gas and upon its inner surface was that of the standard atmosphere which is 14.7 psi at sea level. Avogadro also used gases that were at a standard temperature of 32°F. Avogadro found that a hydrogen filled balloon weighed 1 gram (ie., of equivalent force or weight of 1 gram of mass), that the volume was 22.4L, but other gasses at the same volume were heavier and therefore, it took less volume to fill them to weigh 1 gram , and since d = m / V , or , considering d = W / V since weight and mass are proportional, If the amount of substance remains the same, its mass and weight will remain the same when the volume it is in changes in value, however the density (m/V) of that substance will change by the inverse factor that the volume changed. Given the same amount of mass in a larger volume, its density will decrease.

Of mention is that Avogadro did not know the (atomic) structure of the hydrogen gas particles in his balloon, and we now know that the type of hydrogen he used is known as of the most common structure of hydrogen which is **diatomic** (ie., two atoms joined as 1 molecule) hydrogen rather than single atoms of hydrogen which are called mono-atomic hydrogen. Diatomic hydrogen (H2) is a molecule consisting of two hydrogen atoms which share their electrons which creates a covalent bond due to the nearby electric charge attractions of the protons and orbiting electrons. If Avogadro's 22.4 L balloon was filled with diatomic hydrogen and it had a mass of 1 gram, what is the density of this diatomic or common hydrogen? First, today we know that his balloon had twice the amount of matter than he thought (but still one mol of it as defined). Avogadro called this 1g of matter or mass particles as having 1 mol of gas particles. By measuring the weight of a substance, it can be determined how much mass it is, and-or the number of mol units it is if we know the (g/mol) of that substance and-or element. The mol unit is a certain number of particles, and is often used various fields of study such as chemistry and-or electro-chemistry, biology, medicine, electrolysis and electro-plating.. If something weighs 2g, it therefore has 2 mol of atomic mass units, but not necessarily 2 mol atom units unless it is monoatomic hydrogen which is sometimes called protium. By using mass, a total sum of matter can be found by weight: W = mg and m = W / g . Using the concepts of (mol / gram) or (grams / mol) = molecular weight of an element, the total number of particles such as atoms or molecules can be found by weight rather than counting each particle of which would be nearly impossible. If diatomic hydrogen which has a molecular weight of (2g / mol) is weighed and measured, and if there is 2g of it, then we have 1 mol of diatomic hydrogen (not exactly atoms, but molecules for this somewhat rare instance of gases):

density = d = m / V = 2g / 22.4L = 2g / 22400 cc = 0.000089285 g / cc = 0.089$\underline{3}$ g / 1L : diatomic hydrogen

Since 1 mol of mass of mono-atomic hydrogen has and-or corresponds to a mass and-or weight of 1g , 2g of matter is the weight per 1 mol unit of diatomic hydrogen, and this is called the molar mass of diatomic hydrogen. Its molar mass is 2g / 1 mol. Since 1g of pure hydrogen atoms is essentially 1 gram of protons, there is 1 mol of protons in 1 gram of pure (mono-atomic) hydrogen atoms which have only 1 proton and 1 electron (which has negligible mass) in each.

1 proton or neutron has a mass defined as 1 atomic unit of mass or 1 atomic mass unit (1 amu). The number of particles such as atoms in 1 mol is called Avogadro number, and its value was later found by extensive experimentation, measurement and calculations by many to be about: 6.0221408 (10^23) particles (ex., protons and-or neutrons, or atoms, or molecules). For diatomic hydrogen, each of these particles with 2 protons corresponds to the number of grams of that substance per mol unit. This value of 2 is also called the atomic number of (diatomic) hydrogen, and which is therefore also equal to the number of protons (and the number of electrons) in each particle (here for diatomic hydrogen, it happens to be a particle that is a molecule of two atoms, but in general, for single atom particles of other elements). The **atomic mass** or **molecular mass** or weight (ie., mass that which will give it a corresponding and proportional weight [force] in a gravity field) of an atom or molecule (joined atoms into a molecule particle or unit) is the total number of atomic mass units (**amu**), hence the number of protons and neutrons in an atom of a certain element or molecule; for example: 10 amu. **Atomic weight** is the number of grams per mol amount of atoms or molecules of the substance, and which is also called the **molar mass** (ie., mass per mol of a substance). Ex. If an element has a atomic mass or molecular mass of 10 amu, its molar mass will be (10 grams / 1 mol).

A pure elemental hydrogen (and the molecule diatomic hydrogen) does not have a neutron in the nucleus of its atom, and therefore it has just 1 proton and 1 neutron for or in its atom particle. Its atomic number is therefore 1 and its atomic mass is 1 amu or as 1u. Its molar mass is 1 gram / 1 mol. This could also be stated as: (1 gram of substance or particles such as hydrogen atoms / 1 mol of that substance). For hydrogen, this can also be stated as (1 gram of protons) / 1 mol, and from this using division by the actual numeric value of a 1 mol, we can find the mass of a single proton or neutron.

The gasses that are commonly found in their diatomic molecule form are: Hydrogen (H_2), Oxygen (O_2), Nitrogen (N_2), Chlorine (Cl_2), Bromine (Br_2) and Fluorine (F_2). One form of oxygen even has 3 atoms joined together and is called **ozone** (O_3), and it is mostly found in the upper atmosphere. Ozone helps block UVB (light-photonic) radiation. Concentrated ozone is also used by industries so as to produce some chemical reactions because it is highly reactive (ie., here, "oxidizing"), and it is sometimes used to reduce germs, hence it is not generally safe for lifeforms, especially when breathing it and-or "smog" (smoke containing various chemicals and-or pollutants such as from combustion engines and-or factories).

The more mass (ie., amu's) a particle such as in its atom of a particular element has, the heavier 1 mol of those atoms will be, such as in grams of mass "weighed" (weighed and calculated as mass) on a scale, hence its molar mass value will be greater. For example, an atom with twice the number of amu's than that of another atom will have a mass and weight of two times more and 1 mol of it will be twice as much grams.

The mathematical inverse or reciprocal of (grams / mol) is (mol / gram). If a substance has 10 grams / 1 mol of atoms, how many mol is 1.5 grams of that substance? After dividing the numerator and denominator of (10g / 1 mol) by 10 so as to create an equivalent fraction, we have: 1g / 0.1 mol. Now multiplying both the numerator and denominator by 1.5 so as to create an equivalent fraction, we have: (1.5g / 0.15 mol). 1.5 gram units of that substance corresponds to 0.15 mol units of the substance.

Extra. This is example how to count the number of grains of rice in a container by a known (rice grains / volume) unit rather than a known (rice / weight) unit such as used when weighing a quantity of rice and then finding the number of grains. If a particular type of rise has an average of a count of 30 grains per 1 cc unit, how many grains of that rice are in a container with a volume of 1L ? We know that 1L = 1cc (1000) = 1000 cc , creating an equivalent fraction:

$$\frac{\text{30 grains of rice}}{\text{1 cc}} = \frac{(1000)}{(1000)} = \frac{\text{30000 grains of rice}}{\text{1L}}$$: hence, associating a number of grains with a liter

Much like a mol is associated with a specific number of particles, we can assign a unit for 30 grains of rice, such as 1 ric unit of rice. 2 ric units would be: (2 ric) (30 grains / ric) = 60 grains of rice.

What is the mass (grams) of air needed per day by a relaxed person, and what is the mass (grams) of oxygen needed per day by a relaxed person?

For this question, there are many factors to consider such as weather the person is working or sedentary. For this analysis, we will consider a sedentary adult person breathing at a rate of 12 breaths per minute as mentioned previously in this main article about scuba diving. Let us call this the minimum rate needed. You may then wish to do the calculations for a diver breathing say 20 breaths per minute. At 0.5L volume of air per breath from a sedentary person, the volume of air per minute is: (12 breaths / min.) (0.5L / breath) = 6L of air per minute. This results in: (6L air / min.)(60 min. / hr.) = (360 L air / hr.) , and this results in: (360L air / hr.) (24 hr. / day) = 8640 L air / day = 2286 gallons / day

Since: 1 ft^3 = 7.48052 gal , (2286 gal air / day)(1 ft^3 / 7.48052 gal) =~ 305.6 ft^3 of air / day : is breathed

1L of **air** at STP has a mass of 1.24g and (8640 L air / day)(1.24g / L) =~ 10714 g air / day =~ 10.7 kg air / day

Given that typical air is about 20% **oxygen**, and we use about 5% internally and convert it to carbon dioxide, then in terms of just oxygen, we would need about: (10.7 kg / day) (0.05 oxygen) = **535 g oxygen / day** : about half a kg
For a quick calculation, a diver would need about twice as much , hence: 1070 g oxygen / day =~ 1kg oxygen / day
We must also save (1/3) of this value for emergency usage or add in (1/3) more for that purpose.

1 L of air at STP has a mass of 1.24g and (1.24 g / L air) (0.20 oxygen) = (0.248 g oxygen / L air) =~ (0.25g of oxygen / L air) = about a quarter of a gram of oxygen per liter of air = 1g oxygen / 4 L of air =~
=~ 1 g oxygen / U.S. gallon of air

Since 1L =~ 0.26455 gallons =~ 0.25 gallons , 0.25 gal will have 1.24 g of air and 1 gal will have 4.96 g of air =~ 5g
1 gallon = 3.78L =~ 4L , and 8640 L air / day =~ (8640L air / day) / (4L / gal)) = 2160 gal air / day and if we rather used the more accurate value of 3.78L as the divisor,we would have 2286 gal air / day as previously mentioned.

Since 1 gallon = 0.1337 cubic feet = 0.1337 ft^3 , 2160 gal air / day = (2160 gal air) (0.1337 ft^3 / gal) / day =
=~ 290 ft^3 air breathed / day . Taking the square root of this, the side length of this volume is: 6.62 feet

Much of the shores and ocean of the Earth have still been unexplored as of the year 2024, and due to the oceans great distances, depths, visibility, and pressures that makes exploring it an inpractical venture on a large scale. General sonar (**SO**nic **N**avigation **A**nd **R**anging) images are the best images available of the larger unexplored portions of the oceans, and these images have a relatively low resolution as compared to a photograph. Just like the remote control drone vehicles in the air, there is a high potential use for remote controlled water submersible versions with lights and cameras.

Sonar is like the sonic (acoustic) version of Radar that uses radio waves and its reflection from objects in the air, however radio waves do not travel far in water and then sonar must be used. Basic electronic sonar began in about 1906.

Some Properties Of Water (H2O)

Water is the **liquid** physical state of H2O (2 Hydrogen atoms, and 1 Oxygen atom joined together as a 1 molecule, the smallest particle of water, and with two or more molecules being called the compound name of water), composed of many water molecules. Each water molecule is composed of 2 Hydrogen atoms and 1 Oxygen atom. The chemical name for water is H2O. At room temperature, hydrogen and oxygen atoms are a gas, and water is a liquid. When water is reduced in temperature, the water molecules will loose kinetic or thermal energy. Their motion is getting reduced. At 32°F = 0°C , the water molecules will arrange themselves in a cubic, grid-like pattern and form a solid that we call **ice**. This structure is actually less dense (d = mass / volume) than liquid water because in liquid water, the water molecules tend to get closer in distance to each other due to their kinetic energy, and this makes liquid water actually slightly denser than ice.

What keeps the water together and not split apart into drops or a gas? Water molecules can become electrically attracted to each other and form what is called a hydrogen bond (bonding, connection). When a hydrogen atom looses an electron, it becomes positive in charge. When a hydrogen ion of the water molecule is electrically attracted to the outer, more distant orbiting electrons of a oxygen atom with its negative electric polarity influence when nearby, an atomic bond, here called a hydrogen bond is created. An individual water molecule is said to be electrically polarized where one side has a net positive charge, and the other side has a net negative charge. Water can therefore ionize itself and the result is regular water molecules mixed with both hydrogen ions (H+) and water molecules having a net negative charge which are called hydroxide ions (OH-). Water is considered as a chemically neutral (ie., "in the middle", of neither extreme) substance that has a pH value of 7.0 which is half-way between that of a strong acid (pH of 0) substance and a strong base (pH of 14) substance. In short, water is both a weak acid and a weak base. Water can dissolve some substances such as salt and sugar.

The outer electrons of an atom are called its **valence electrons**. Molecules can be formed when two or more atoms share the same outer orbit electron(s). This type of chemical element bonding is called a **covalent bond**.

The **density** (mass/volume) of water is generally considered as normally being 1g/cc , and even when a very high external pressure is applied to it, its density will only increase by a very small amount or fraction, say 1%. Due to this stable density, it is possible to use water in a hydraulic jack that can be used to lift very heavy weights a relatively short distant, however due to the potential of rust issues with iron, some type of oil is rather used instead. The electrostatic forces of the water molecules are much stronger at close, molecular distances, ant its is very difficult to move and-or force the molecules any closer.

Water does expand when heated, and this is called **thermal expansion** of an object. When water is heated from 72 °F to 212 °F, it will expand by about 4%, hence its density = (mass / volume) actually decreases due to the thermal expansion when he water molecules gain kinetic energy from the applied heat or thermal energy. For a large body of water such as an ocean, thermal expansion will cause the sea level to rise to perhaps several feet higher and causing flooding issues along the coasts or sea-shores.

The weight of a gallon of water is 8.34 lbs, and the weight of a cubic foot of water is 62.4272 lbs. 1 ton = 2000 lbs of water will therefore be: 2000 lbs / (8.34 lbs / gallon) = 239.81 gallons =~ 240 gallons. 2000 lbs / (62.4272 lbs / ft^3) =~ 32.03732 ft^3 , by taking the cube root of this value, this would result in a side of the cube of about: 3.176 ft and this is slightly more than 1 yard, and is also slightly less than 1 meter = 3.28084 ft.

When water is heated, the water molecules will gain kinetic or thermal energy, and move more rapidly about and collide with each other more. At 212 °F = 100 °C, which is the water boil (boiling) temperature, this temperature will cause much kinetic energy that will break apart the hydrogen bond of water molecules. The water will become less dense due to thermal expansion and will rise upward toward the less dense and less pressure water and-or substances. The hot water will form into (low density) steam (hot, microscopic or small water drops) at the surface of the water, and which is water molecules acting like a low density gas, and with their high kinetic energy, they can rise upward into the air of which is less dense and of less pressure. This process is called **boil**, and the steam has a higher pressure (force/area) than that of the atmosphere pushing down upon it, usually 14.7 psi at sea level. At higher altitudes with less air pressure upon the water, the water will boil at a lower temperature. Note that the water molecules do not split apart into their oxygen and hydrogen atoms parts when water boils, and for that to actually happen requires much more energy and-or force of which can be done with an electric force such as during the process of electrolysis.

When steam cools to 212 °F = 100°C, it will start to condense (ie., get denser) and attract itself into hydrogen bonded water molecules, then small drops or even snow flakes of frozen water made in the high cold air, and then a liquid when the air and ground is warm above the freezing temperature of ice (a cold, frozen [non-liquid] - a solid , crystallized form of water). A snowflake is hexagonal (6 sided, having "branches" or "arms") in shape and this is due to other water molecules attaching themselves via hydrogen bonds to an initial molecule of water. Each branch can even develop other branches later, and the shape of a snowflake is determined by the weather and-or environmental conditions.

At 39.164°F = 3.98°C =~ 39 °F = 4°C , water is at its most dense state and standardized value of (d = 1g / 1 cc). It will become less dense for temperatures lower than this till the point it freezes at 0 °C = 32°F and its density then remains constant at (0.9168 g / cc) and that is a difference of (1.0 g/cc- 0.9168 g/cc) = 0.0832 g/cc and which is 8.32% less than when it is most dense at 1g / 1 cc.. At the temperature when water freezes from liquid to solid, water then becomes at its least dense point. At temperatures colder than waters freezing temperature, the crystal structure of the water molecules of ice will begin to compress slightly, and hence its density will slightly rise slightly.

Ice is slightly less dense than liquid water and will float when in water. Water will also become less dense for temperature higher than 4°C = 32°F , and until it boils at 100 °C = 212 °F and turns into steam (ie.,"water gas") which has a very low density.

When water freezes, it creates a solid crystal of water ice, and expands in volume by 8.32%. Expanding ice will cause a pressure on any external surface and may damage it, and this is also a form of natural erosion of which also includes glaciers, rain and flowing water. From density = d = mass / Volume = m / V , we have V = m / d, If volume increases by a factor of (n), the density will decrease by that same factor. From d = m / V , and after dividing both sides by (n):

d / n = (m / V) / n = m / (nV) and nV = m (d / n)

Heated water will expand in volume, and therefore it also gets less dense. Water can also become what is called **super-heated water** that is above the water boil or steam temperature (212 °F = 100 °C) if it is pressurized in a container such as in a pressure vessel and-or cooker, such as in a (locomotive) **train engine.** Here, any would be steam having less density cannot then rise like it would to a lower pressure region such as it does in Earth's atmosphere at 14.7psi. Super heated water and resulting **super heated steam** will contain more kinetic energy and pressure (P = F/A) to do things. If the air or container pressure drops, then the water will boil faster - including into a lower pressure region such as that at standard air pressure. In a heated container full of water with practically no room to expand, the water will essentially try to expand into itself and compress itself, and the internal water pressure will then rise, and this pressure (force / area) could cause a dangerous rupture (hole, leak) or breakage of the holding tank and the release of high pressure hot water and steam.

The density of water at its boiling temperature of 100°C = 212°F is 0.958 g/1cc and which is (1 g/cc - 0.958 g/cc) = 0.042 g/cc less. (0.042 g/cc) / (1 g/cc) = 0.042 = 4.2% less and its volume will be 4.2%.more.

Heater water that is greater than 0 °C = 32 °F will have a fairly linear expansion in its volume until it boils. The total amount of expansion for this temperature range is about 4% as shown above: The change in volume per degree Fahrenheit is:

A pot full of water will overflow when it is near boiling, and when it boils, the bubbles in the water will cause the volume of the water to increase even more and cause more water to overflow. Some waters heaters, such as a solar water heater may use this expansion of water concept to place the hot water made that rose to the upper water surface and into another tank that is insulated and can then recycle any cool water at the bottom of that tank back into the water heater so as to be reheated and in a continuous loop of making hot water available.

The density of water at room temperature (~ 68 °F = 20 °C to 75°F = 23.89°C , and with 72 °F = 22.222 °c being more common) using 72°F is about 0.998 g/cc, and the density of water near its boiling point is about 0.958 g/cc. This change in water density between these two temperatures is found by taking their difference, and it is: -0.04 , and here using the negative sign because it also indicates a decrease. The volume of water will expand by the same absolute value of 0.04.

With this data and using a linear or line [y = mx , m = y / x = (change in y) / ((change in x) , (change in y) = m (change in x)] approximation, we have this for the approximate or typical (constant) slope (m) or rate of changes in the two values:

Change in temperature = 100 °C - 22°c = 78°C or 212 °F - 72 °F = 140 °F

m = line slope rate = (change in density) / (change in temperature) = - (0.04 g/cc) / 78 °C or
-(0.04 g/cc) / 144°F

From this we can create an equivalent fraction for the (change in density) / (1 degree change in temperature).
For example, dividing each by the denominator value we have:

Change in water density per degree change in temperature from room temperature to boiling temperature =~
- (0.00051$\underline{3}$ g/cc) / 1°C or =~ **-** (0.000277$\underline{8}$ g/cc) / 1°F

To find the corresponding density of water for a given temperature between room temperature and boiling:

From: (density at final temperature) = (density at starting temperature) + (change in density) , we have:

$$\text{density at final temperature} = \text{density at starting temperature} + \left(\frac{\text{change in density}}{\text{degree}}\right) (\text{change in temperature})$$

The above equation has the general form of: y2 = y1 + (change in y) : from (change in y) = (y2 - y1)
And this form: y2 = y1 + m (change in x) : (change in y) = m (change in x)

A person might think that a pot of boiling water will weigh less since the water is now less dense, but this is not actually so, and the pot will weigh the same amount if no water has yet boiled away and made that pot of water lower in weight. Consider a given amount of mass such as water or a gas, it will be a constant amount regardless of its temperature. What does change is its density and volume, and if one of these values changes by a factor of (n), the other value will change by a factor of the reciprocal of this which is (1/n), and since they are factors of the result. Density and Volume are physically and mathematically, inversely related, and are not necessarily reciprocal in actual values, and where the product, here mass, would always be 1:

density = (mass) / (Volume) = m / V , mathematically: : extra: Weight = F = m a = (d V) g

$$m = d\,V = \frac{(n)}{(n)}\,\frac{d\,V}{1} = \frac{(n)\,d}{1}\,\frac{V}{(n)} = \frac{d}{(n)}\,\frac{(n)\,V}{1} \quad \text{or=} \quad ((1/n)\,d)\,(nV)$$

: n = a multiplying factor (<1 or >1)
(d) is often identified as (p).

We see in the above equation that if density decreases by a factor of (n), the volume will increase by a factor of (n).
If the density increases by a factor of (n), the volume will decrease by that same factor of (n).
If the volume increases by (n), such as for a gas that can expand and fill a space, its density will decrease by that same factor of (n). If the volume has air spaces in it, unlike that of a liquid, then the equation is no longer valid as is. In short, there is two different densities involve - that of the material, and that of the air, and the average density is actually less than that of the solid material. For these reasons, it is often better to weigh a substance than use its volume. .

The graph of water density versus temperature is approximately linear or like a line, however the curve is actually a slight downward curve, and this implies that the slope is changing to a slightly lower value. Although
m = dV = (ndV / n) is a constant, the value of (n) is not necessarily a constant and is rather a variable when considering the curve on the graph.

From: m = d V , we mathematically have: V = m / d , and this and-or both equations show the inverse relationship between volume and density such as for an easily compressible or expandable substance such as a gas, and of which is at a certain temperature. If the temperature of the gas changes changes, the pressure of that gas will change if the volume is held constant, and therefore, the mass and density of that gas will still be the same.

When water is heated to its boiling temperature, the liquid water has much thermal and kinetic energy in its molecules and the hydrogen bonds of water molecules begin to break apart into separate water molecules. At the surface of the water, these water molecules will rise upward in the air as (cloudy, foggy, misty) steam which is also called water gas, and this has a very low density and a high thermal or kinetic energy. Some train engines use steam to turn its wheels which then moves (here, pulls) its heavy load of train cars on the train tracks. The steel train tracks have a hard, low friction surface.

Extra: The density of the space in a pure vacuum having no matter in it is 0.

density in a vacuum = m / V = 0g / V = 0

Even though a perfect vacuum has 0 mass within its volume, it may have energy traveling through it such as heat (ie., infra-red light), invisible radio waves (RF) and other various types of radiation, and visible light.

Ocean or sea water has on average 35 grams of **salt** (table salt, sodium chloride) per (1000g = 1 Liter). This is said as being: 35g salt / 1000g sea water = 0.035 salt = 3.5% salt by mass and-or weight since weight and mass are mathematically, directly related and proportional in values. 1lb of ocean water will have: 1 lb = 16 oz , and 16 oz (0.035) = 0.56 oz = slightly more than half an ounce of salt. 1 cubic foot of ocean water will have: 1 cubic foot of typical ocean water weighs about 62.4 lbs. 62.4 lbs (0.035) = 2.184 lbs of salt.. 1 gallon (= ~4 liters) of typical ocean water weighs about 8.34 lbs. 8.34 lbs (0.035) =~ 0.292 lbs = almost 30% or about 1/3 of a pound of salt. Salt can be obtained from ocean or sea water by evaporating the water while leaving minerals such as salt. If the evaporated water is condensed by cooling it and then collected in a container, it is vital and healthy drinking water which is typically called distilled or evaporated water.

Ocean or sea water contains many dissolved mineral and gas elements. The weight of these minerals is about 40g/kg mass of sea water, hence about 40g/L volume of sea water. Mathematically, this is about 4% of the sea water being dissolved minerals. 1L = 1000cc =~ 1000g sea water. 1000g (0.04) = 40g. Some of these elements and-or minerals come from land and-or soil erosion due to weathering, and some are from deep in the earth being expelled by volcanoes and undersea vents. Deep water also has different amounts of minerals than near the surface. About 86% of the dissolved (about 4% of the total weight of the water) elements are sodium (55%) and chloride (31%) which compose the (table or food) salt or sodium-chloride molecules. There is then about (100% - 86%) = 14% of other dissolved elements which have a total mass of about 0.57g/L, and some averaged values for that 4g of minerals are: magnesium (3.7%), sulfur (7%), calcium (1.2%), potassium (1.1%, and bromine (0.2%). There are many lesser or trace values of other elements in the water, such as gold, silver and uranium. Iodine (50ug / 1L. An iodized fortified (additive) salt has 45ug of iodine / 1 gram of salt , and the **RDA of iodine is typically noted as about 140 mcg of iodine / 1 day for an adult**, however twice as much and-or slightly more is often consumed by some people. Many of these minerals such as lithium metal are obtained after the evaporation of the plain or fresh water ($H2O$) from the sea water and-or its dense brine (high in salts such as created after desalination to make fresh water) at large and shallow depth ponds in hot, sun heated surface regions on the Earth. Brine is obviously much higher in mineral content than common sea water. **Selenium** is often taken with iodine if needed, and at about the same RDA dosage or amount. To be safe in terms of health, ask your doctor about taking each specific nutritional supplement.

Most sea life usually contains higher levels of iodine than land based life. **Seaweed** absorbs and effectively concentrates the iodine atoms so as to have a high amount per gram of it, especially when dried with its water content greatly reduced by drying (evaporating) it, and then it having about [(1000 mcg = 1mg, or more) / 1 gram] = (1g / 1 kg) of dried seaweed. Due to the high levels of iodine in seaweed, too much of it can eventually cause iodine toxicity and thyroid problems if intake (ie., eating of) is greater than that of the RDA. 1g of dried seaweed contains abut 12.5mg of potassium, 3.7mg of calcium, and 0.25mg of iron. Kombu Kelp is said to be the seaweed highest in iodine at nearly 3000 mcg = 3 mg of iodine in its dried form at 1 gram = 1000 mg.

When ocean water is evaporated so as to make (food grade) sea-salt, many of these elements or minerals will be in that salt. Some companies even sell fortified salt for nutritional needs. The majority of the iodine will evaporate or "gas out" since it is so volatile, and it be lost unless it is somehow collected, and due to this fact, even sea-salt that is purchase if often fortified with iodine.

Because of the (dense) minerals in sea water, its density (m/V) is greater than that of plain or pure water which is 1kg/1L. The pH of sea water averages at about 8.0, hence it is slightly alkaline. (base). Just like freshwater, sea water which contains salt still has bacteria in it, and potentially dangerous bacteria in the areas of raw sewage disposal into the ocean. Due to this, filtering, boiling and or using chlorine in the water will reduce and-or eliminate any possible and dangerous germs, however the chlorine in the water then needs to then be removed by heating it or letting it sit for several hours, say 8 hours or more, and "gas-off" (evaporate out from).

RELATIVE SIZES OF AN EQUAL AREA CIRCLE AND SQUARE

In the figure below, a square and circle are both centered about the same point which is the center point of the circle.
[FIG 308]
By observation, we can see that the circle is sometimes wider than the square, and the square is sometimes wider than the circle, yet their areas are the same. Note also that neither is completely within the other and-or completely surrounded by the other. Notice that all of the 8 overlapping segments are equal in area. The radius of the circle is slightly longer than half the side of the square, and which is then S/2. $r > (S/2)$. The radius of the circle is slightly shorter than half the diagonal of the square, and which is then D/2 . $r < (d/2)$.

To help make the analysis slightly easier, lets make both the areas equal to 1 square unit of length = 1 unit^2 .

$Ac = 1 \text{ unit}^2 = (pi)\, r^2$ solving for r, the radius when the area is 1 unit^2, we have:

$r = \sqrt{1 \text{ unit}^2 / (pi)} = \sqrt{0.318309886 \text{ unit}^2} = 0.564189583 \text{ unit}$

Dc = diameter of the circle= $2r = 2\,(0.564189583 \text{ unit}) = 1.128379167 \text{ unit}$

$As = 1 \text{ unit}^2 = s^2$ solving for (s), we have:

$s = \sqrt{1 \text{ unit}^2} = 1 \text{ unit}$

Half of s = 1 unit / 2 = 0.5 unit , and this length is shorter than (r), and the difference is:
0.564189583 unit - 0.5 units = 0.064189583 unit, and twice this distance is: 0.128379166 and this value is also the difference between the diameter of the circle and the side length or diameter of the square.

The diameter of the square is Ds = s =1, and this is shorter than the diameter of the circle by a value of:
1.128379166 unit - 1 unit = 0.128379166 unit , and this is equal to two of the widest parts of the circle that are opposite of each other, and is longer than the square. Each part is half of this value and is:
0.128379166 unit / 2 = 0.064189583 unit

The diagonal of the square is: $z = \sqrt{(s \text{ units})2 + (s \text{ units})^2} = \sqrt{2\,(s \text{ units})^2} =$

$\sqrt{2}\ \sqrt{(s \text{ units}^2)} = (1.414213562)(1 \text{ unit}) = 1.414213562 \text{ unit}$

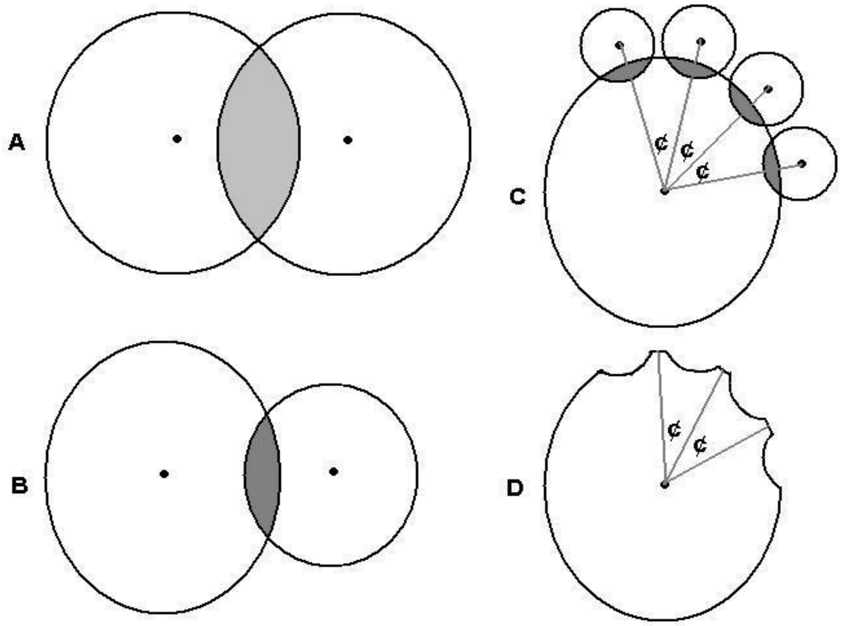
The diagonal of the square is larger than the diameter of the circle by:

1.414213562 unit - 1.128379166 unit = 0.285834396 unit , and half this distance is the longest distance between the circle's circumference and the corner of the square. 0.285834396 unit / 2 = 0.142917198 unit

Extra: If the 4 corners of the square were removed, and the 4 edges of the circle were removed, the result is a rectangle (here, a special instance of a rectangle called a square with 4 equal side lengths)with rounded corners who radius of curvature is that of the radius of the circle. A square with rounded corners is called a **squircle** which is a word created by the combined words of: "square" and "circle". The curved corners can have any radius of curvature. A squircle is a special instance of a **rounded-rectangle**.

Extra: The following geometric shape is created at the intersection of two circles, usually of the same diameter, however, they need not have the same value. The portion of each circle not overlapping is called a **Lune** (ie. Moon) or crescent shape. **Venn diagrams** use these shapes to show and compare amounts in a simple and relative way.

[FIG 309]



A: Intersection of two circles having the same radius. The intersection of the two circles that coincide is colored gray and it is formally called a **Vesica Piscis** which is the Latin words for "bladder" and "fish", hence "fish bladder" shaped, and which can alternately be called the "**fish-shape**". When the radius of each circle is the same, the same portion or area (here the fish-shape or fish-shaped area) of each circle is intercepted.

B. Intersection or overlapping of two circles not having the same radius. The radius of curvature of each side of the fish-shape is therefore different.

C. Shows a basic example of how a certain type of gear shape can be drawn, here a sprocket gear for a bicycle chain to transfer the input power to the wheels. The centers of each smaller circle are offset by the same angle.

D. Shows the intersections of the two circles removed so as to obtain the gear teeth shapes.

A STUDY OF CHANGES IN AREAS OF CIRCLES AND SQUARES, AND INTEGRATION TO FIND A SQUARE AREA

Note the different equations for circle area or square area when a value has changed:

Area of a square = As = A1 = s1^2 **,** after solving for s: s1 = $\sqrt{A1}$ = and in general: s = $\sqrt{As}$ **:**

If the area of the square has changed by a factor of (n), we need to multiply each side of the equation for the area of that square by the factor of (n):

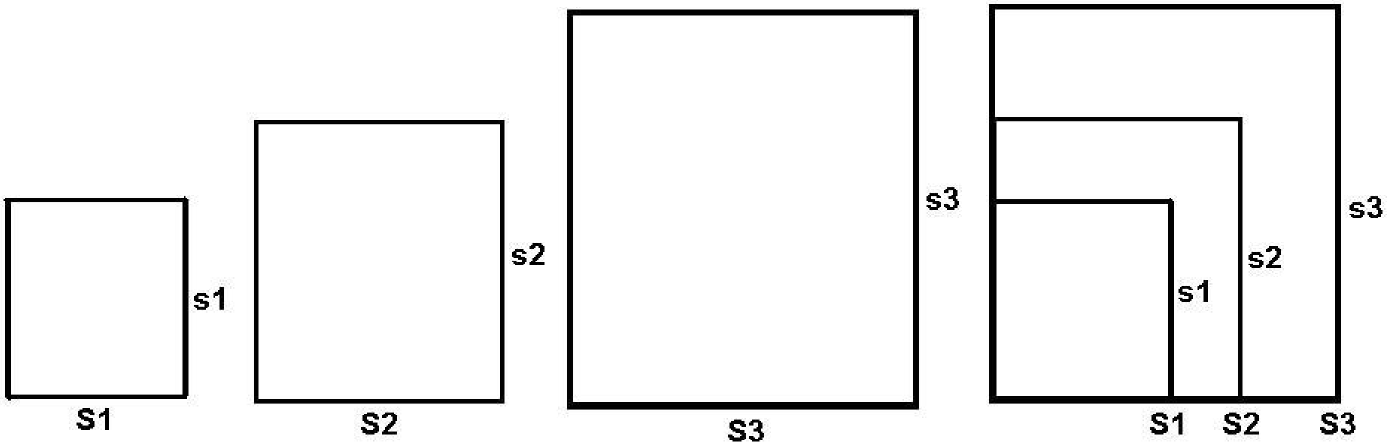
n (Area of a square) = n A1 = n s1^2 = A2 = s2^2 , after solving for (s1): s2 = $\sqrt{n\ s1^2}$ = $\sqrt{n}$ s1

We see that if area of a square has changed by a factor of (n), then the side of that area will change by the square-root of that factor = $\sqrt{n}$

If the **side (s1) of a square** has changed by a factor of (n), we need to multiply (s1) by that factor of (n):

A2 = s2^2 = (n s1)^2 = n^2 s1^2 = n^2 A1 , and the result is that area of the first square = A1 = s^2 had increase by a factor of n^2 when its side (s1) has increased by a factor of (n).

A2 / A1 = n^2 , mathematically: A2 = n^2 A1 , and we see that when the side (s1) of a square changed by a factor of (n), that the area will change by a factor of n^2. [FIG 310]



If the side length increases by a factor of (n), the area will increase by n^2

A1 = 1 unit^2
A2 = 2 (A1) = 2 (1 unit^2) 2 unit^2
A3 = 2 (A2) = 2 (2 unit^2) = 4 unit^2

If area increase by a factor of (n), the side has increase by $\sqrt{n}$

s1 = 1 unit
s2 = (s1)($\sqrt{2}$) = (1 unit)(1.414214) = 1.414214
s3 = (s2) ($\sqrt{2}$) = (1.414214) (1.414214) = 2

Area of a circle: Ac = A1 = (pi) r1^2 , after solving for r1: r1 = $\sqrt{A1 / (pi)}$: (pi) = 3.141592654
In general: r = $\sqrt{Ac / (pi)}$

If the area of the circle has changed by a factor of (n), we need to multiply each side of the equation for the area of a circle by the factor of (n):

n (Area of a circle) = n A1 = n ((pi) r1^2)) = n (pi) r1^2 = A2 = (pi) r2^2 : also, A1 = A2 / n

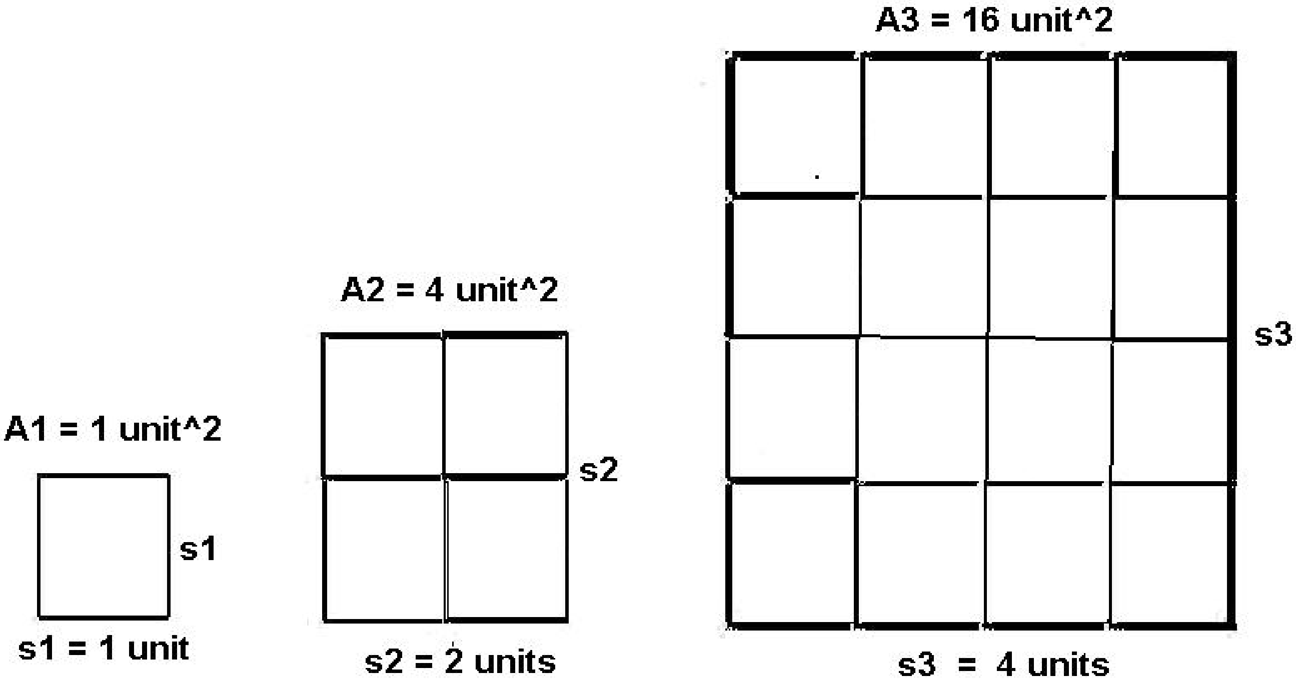
After solving for (r2): $r2 = \sqrt{n\ r1\text{^}2} = \sqrt{n}\ r1$, and we see that if the area of a circle has changed by a factor of (n), then the radius has changed by a factor of:

$$\sqrt{(n)}$$

If the **radius (r) of a circle** has changed by a factor of (n), we need to multiply (r) by that factor of (n):

A2 = (pi) r2^2 = (pi) (n r1)^2 = (pi) n^2 r1^2 = n^2 (pi) r1^2 = n^2 A1 , and the result that if the radius changes by a factor of (n), then the area will change by a factor of n^2 .

An analysis of the rate of changes between the sides of squares and the corresponding areas.



The equation for the area of a square is: Area = side^2 , or simply: A = s^2

When s=1: A = s^2 = side^2 = (side)(side) = (1 unit)(1 unit) = (1 unit)^2 = (1)(1)(unit)(unit) = 1^2 units^2 = 1 units^2
= "one unit square" or "one unit squared"

When s=2: A = (2 units)^2 = 4 units squared = (2 units)(2 units) = 4 units^2 = "four square units"

When s=3: A = (3 units)^2 = 9 units^2

When s=4: A = (4 units)^2 = 16 units^2

The equation for area has a general equation form of: y = x^2 or y = f(x) = x^2

The above equation is a second order equation like a parabola is. The equation is also called a power equation, and here the indicated power is 2. As the independent variable, here it is (x), changes when it is greater than 1, the value of the dependent variable, here it is (y) of this equation changes or increases in value rapidly.

When the side (s) changes by a factor of n=2, the area changes by the square of that factor: n^2. = 2^2 = 4
When the side (s) changes by a factor of n=3, the area changes by the square of that factor: n^2 = 3^2 = 9
When the side (s) changes by a factor of n=4, the area changes by the square of that factor: n^2 = 4^2 = 16

When the side (s) changes by a factor of n= 1.5, the area changes by the square of that factor: n^2 = 1.5^2 = 2.25

Since this equation is not a linear (ie., a line equation) equation having a constant slope or rate of change in the dependent variable with respect to the independent variable, the rate of changes is not constant for similar increases in the independent variable (here s = side length).

$$\text{slope} = \text{rate of changes} = \frac{\text{change in dependent variable}}{\text{change in independent variable}} = \frac{\text{(change in area)}}{\text{(change in side)}} = \frac{\Delta A}{\Delta s}$$

Ex. $\text{slope} = \dfrac{(1 \text{ units}^2 - 0 \text{ units}^2)}{(1 \text{ units} - 0 \text{ unit})} = \dfrac{1 \text{ units}^2}{1 \text{ unit}}$: When s = 0, area is changing by 1 units^2 for each unit increase in (s). When s=1 unit, A = 1 unit^2

Ex. $\text{slope} = \dfrac{(4 \text{ units}^2 - 1 \text{ units}^2)}{(2 \text{ units} - 1 \text{ unit})} = \dfrac{3 \text{ units}^2}{1 \text{ unit}}$: When s = 1, area is changing by 3 units^2 for each unit increase in (s). When s=2 units, A = 4 units^2

Ex. $\text{slope} = \dfrac{(9 \text{ units}^2 - 4 \text{ units}^2)}{(3 \text{ units} - 2 \text{ unit})} = \dfrac{5 \text{ units}^2}{1 \text{ unit}}$: When s = 2, area is changing by 5 units^2 for each unit increase in (s).

Ex. $\text{slope} = \dfrac{(16 \text{ units}^2 - 9 \text{ units}^2)}{(4 \text{ units} - 3 \text{ unit})} = \dfrac{7 \text{ units}^2}{1 \text{ unit}}$: When s = 3, area is changing by 7 units^2 for each unit increase in (s).

Since the slope is constantly changing along the curve that graphically shows the relationship between the variables , it is not even a constant for a change of 1 in the independent variable, and is in fact, here with a non-linear equation, the slope is for only two very close points on the curve, and of which the change in co-ordinates and-or changes in values is very small, and is a just small fraction of a unit.

When the changes are very low in value, and approaching a value of 0, this is called the instantaneous rate of changes, hence the (instantaneous and-or unique value of) slope on the curve. The small change in (x) is named dx for the differential or difference in (x), and the corresponding small change in (y) is named dy. This instantaneous rate of changes is formally called the derivative of the equation, and this book shows how to find it given the equation or function. For a linear equation, the derivative is a constant numeric value, but for an equation such as y = x^2 or area = side^2 , the derivative is not constant, but depends on the specific value of (x), the independent variable. For higher order equations other than the linear equation, the derivative is another equation. The derivative of a parabola, quadratic or second order equation is a linear equation.

The derivative of a function of (x) with respect to (x) $= \dfrac{d\,f(x)}{dx} = \dfrac{dy}{dx}$ = "instantaneous slope" at a point on curve

The equation or function of the area of a square with respect to the side of that square is: A = s^2

The derivative of the function of the area of a square with respect to the side of the square is: $\dfrac{dy}{dx} = \dfrac{dA}{ds} = 2\,s$

For example, when s=3, A = s^2 = 3^2 = 9 and the slope at (y, x) = (A , s) = (9 , 3) is dA / ds = 2s = 2 (3) = 6

Given: dy / dx = dA / ds = d(s^2) / ds = 2s , mathematically:

dA = 2s ds , integrating both sides = summing (S) up all the small bits of, so as to have the whole thing:

$A = S\,dA = S\ 2s\ ds$: indefinite or "unbound" (no boundary values) integral, or sum producing the whole thing and not a bound portion, section or part of it, here the area.

$A = S\ 2s\ ds$, hence the value on the right side is equal to the function or equation for A, hence it is equal to the anti-derivative of this given derivation that is also multiplied by (ds).
In the process of converting this derivative to the original function or anti-derivative, the value of (ds) will be converted to s^1 and the value of 2s ds will get converted to s^2.

$$A = S\,dA = S\,d(s^2) = S\ 2s\ ds = s^2$$

On the curve for $A = s^2$, a small bit of area is dA . If this is considered as a thin rectangular region, it each region will have a width of ds, and a height of $A = s^2$. When all these small areas (dA = 2s ds) are summed or integrated so as to be 1 whole value, it will equal the whole area A.

For the area between s=0 and s=2, this is a bound area, and the summation or integration is called a definite integral of summation, and these limits are expressed as:

$$\underset{0}{\overset{2}{A}} = \underset{0}{\overset{2}{S}}\ dA = \underset{0}{\overset{2}{S}}\ d(2s)\,ds = \underset{0}{\overset{2}{(s^2)}}$$ = area at upper limit - area at lower limit = area in between limits =

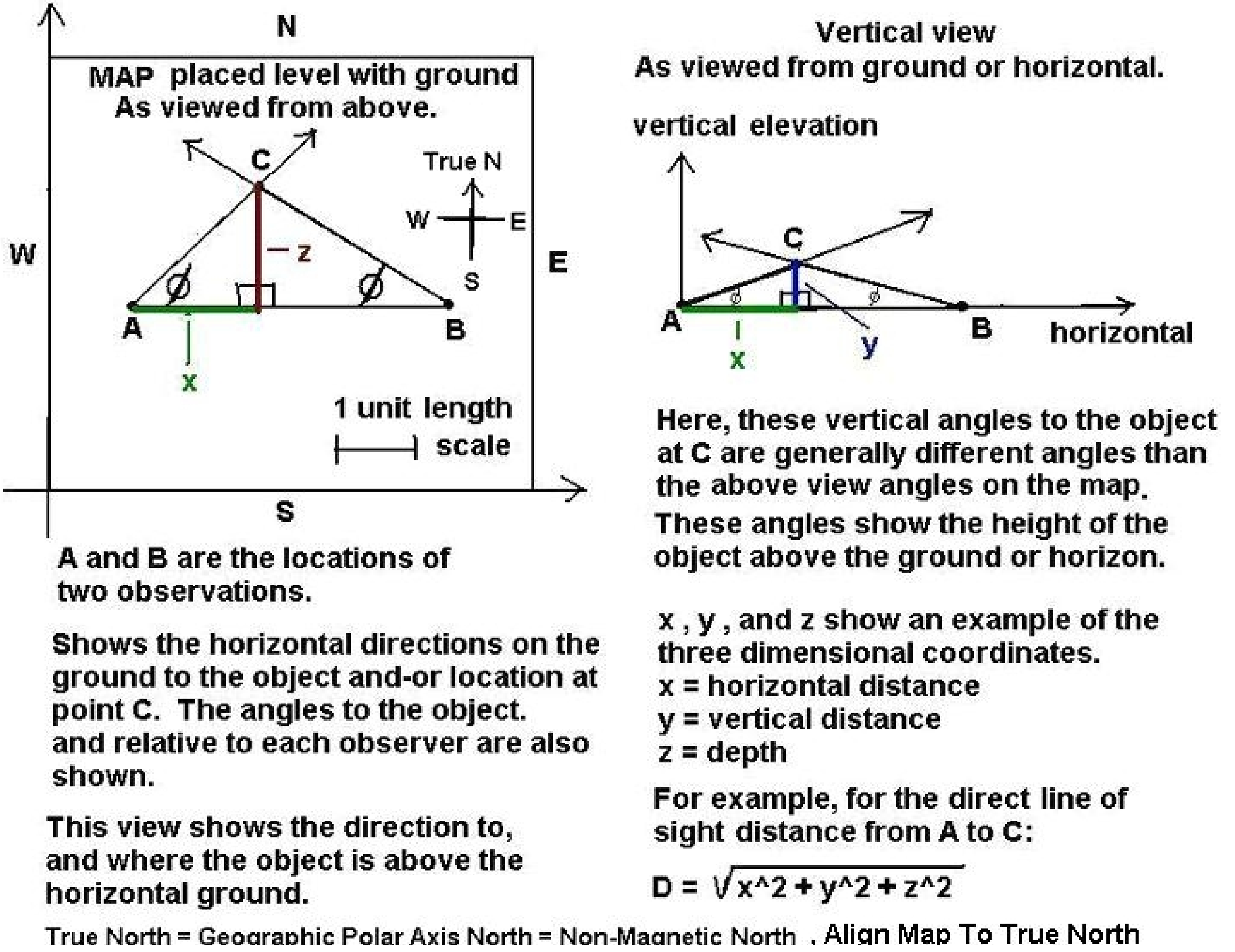
A(at s=2) - A(at s=0) = (s^2 at s=2 units) - (s^2 at s=0 units) =

(2 units)^2 - (0 units)^2 = 4 units^2 - 0 units^2 = 4 units^2

Extra: The area of a circle is: $Ac = (pi)\,r^2$ and $dA / dr = 2\,(pi)\,r$ = a linear equation, and with a length equal to the circumference of that circle. $Ac = (pi)\,r^2$ is a second order, quadratic or parabolic equation.

SOME CONCEPTS OF TRIANGULATION

Triangulation is using two or more observers to locate where an object and-or location is such as if placed on a map, and to find the distances to it. The basic concepts of trigonometry can be used for this analysis.   [FIG 310A]



A and B are the locations of two observations.

Shows the horizontal directions on the ground to the object and-or location at point C. The angles to the object. and relative to each observer are also shown.

This view shows the direction to, and where the object is above the horizontal ground.

Here, these vertical angles to the object at C are generally different angles than the above view angles on the map. These angles show the height of the object above the ground or horizon.

x , y , and z show an example of the three dimensional coordinates.
x = horizontal distance
y = vertical distance
z = depth

For example, for the direct line of sight distance from A to C:

$$D = \sqrt{x^2 + y^2 + z^2}$$

True North = Geographic Polar Axis North = Non-Magnetic North . Align Map To True North

If each observer has a directional compass, they can state the horizontal (ie., ground level) compass direction (N,S,E,W, degrees, etc) that the object is seen at. They will both need to measure or estimate the angle the object is seen above the horizon, hence the vertical angle. If possible, these compasses should be adjusted to true north since the magnetic north pole of which a mechanical, magnetic compass and-or a phone APP (ie., an application = program) compass senses is usually several degrees off from that true value. True north is in the same direction of the North Star, Polaris. The distance between the observers can be calculated by the map and-or calculated using the longitude and latitude coordinates of each. Many modern phones are capable of using an APP for an electronic compass device, however the value or direction returned is usually in terms or in reference of the magnetic north pole. It is possible some APP's might be able to convert this value to be in reference to true north if their longitude and latitude is known - perhaps indicated by a map and-or by GPS (Global Positioning Satellite) of which many phones can activate (a setting and-or an APP) and receive.

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THE EFFECTIVE RESISTANCE OF A DIODE

A typical diode circuit consists of a supply voltage and the diode in series. If the supply voltage is equal to the diodes junction or barrier voltage or conduction (ie., considered "on") voltage, then no series resistance is needed to limit the current to it, otherwise if its is greater, then a current limiting, "safety resistor", is needed or the diode will be destroyed by having too much current flow through it, causing much power and thermal heat energy in it. A typical, non-light emitting, silicon diode has a turn-on voltage of 0.7v, and it will actually produce some low frequency light of which we cannot see, but possibly feel as an increase in temperature. This "invisible light" is called infra-red light and-or thermal radiation, and of which can be seen by special cameras that have thermal image sensors.

In general, the lower the frequency of color, the lower the led turn-on voltage. Red LED's have a turn-on, conduction or forward voltage of about 2v, yellow LED's have a turn-on voltage of about 2.25v, green LED's have a turn-on voltage of about 2.5v, and blue LED's have a turn-on voltage of about 3v and up to 4v. The higher the frequency and its resulting color emitted, a higher amount of energy (Joules) is needed by an electron to produce a photon (ie., a light particle and-or wave, hence a single wave of RF [Radio-Frequency = electro-magnetic] energy), and so the LED is then designed so as to have a higher turn-on voltage and-or electric field across the junction. When an electron gains kinetic energy due to the electric field it is in and leaves its orbital position about the atom, it will loose its gained kinetic energy as a photon of light energy when it falls back into an orbit about an atom.

White is not a color, but is a combination of colors such as red, green and blue (RGB) as seen when white light passes through a glass prism and produces a rainbow of colors. To produce a white light LED, three tiny LED's - red, green and blue, and of which are very close to each other, can be used to effectively combine or mix their output colors and-or frequencies and which will effectively appear as white light. Another method to produce white light is by doping (including) the LED semi-conductor material with various trace amounts of certain elements and-or using certain layers of material on the semiconductor so as to be used as light and-or filters, and-or to re-emit or radiate the input light energy as another color (ie., frequency).

White LED's typically require a turn-on or forward (ie., properly biased) operating voltage of 3V and 20 mA of current for a reasonable amount of light. Up to about 4V and 35 mA may be possible with some white LED's. Voltage higher than about 5V can damage a white LED. As compared to many standard, non-light emitting diodes, the reverse bias breakdown voltage (ie., where it starts conducting when reversed biased) is a fairly low value at only 5v. Placing a standard 0.7v or regular (non-light emitting) diode to be biased in reverse, and also having a high reverse bias breakdown voltage in series and-or parallel with a LED will help protect that LED. When the reverse bias is applied to this circuit, the 0.7v regular diode will behave like a very high resistance and will protect the LED by not allowing current to flow through it.

Because there is a voltage drop across a diode, and there is a (limited amount) current flowing through it, there is therefore an amount of power supplied and-or used by that diode and its will have an increase in temperature. The limited amount of current through a diode indicates that it has some amount of resistance, and when the forward bias or operating voltage is applied across it.

When the forward bias applied to a diode is less than its rated turn-on voltage, it will still be partially on and a lower amount of current will passing through it. The lower the applied voltage across it, the lower the resulting current through it.

Effective resistance of a diode = $Rd = \frac{Vd}{Id}$: The slope of the Id with respect to Vd curve is the conductance (Gd mos) value of the diode, and this is equal to (1 / Rd ohms)

Effective resistance of a white LED = $Rd = 3\text{ v} / 0.020\text{ A} = 150$ ohms

At 0v across a diode, it is completely off, and its resistance is very high and is similar to an open circuit. As the supply voltage and-or forward bias voltage across the diode increases, it begins conducting, but the slope or rate of change of current through it with respect to the voltage across it is low and is almost linear in nature. Once the turn-on voltage is reached, the junction region of the diode will drop significantly in resistance and this will allow much more current to flow through it in an exponential like manner . As the bias voltage increases further, this junction and-or semiconductor resistance remains at nearly the same value, and this effect is sometimes used to provide a stable ("reference") voltage and-or voltage regulation for devices across and-or in parallel to it.

The theoretical luminous or light conversion efficiency from electricity energy to light energy is about 325 lumens / 1 watt = 325 lm / W. A typical 5mm white light LED will produce about 100 lm / W, and is about 40% to 50% efficient at converting the input electrical energy to light energy, and the remaining 50% is wasted as thermal energy (heat). By slightly increasing the current to an LED, it will produce more light, but if too much current goes through the LED and or it exceeds (I max. = maximum current) of the LED, it can burn out.

energy conversion efficiency of an LED at producing light = P out / P in = output light energy / input electrical energy =
= light or luminous efficiency of an LED.

Ex. If a 3v LED is rated at 40 lm at 30 mA of current, how many lumens per watt does it have?

First, lumens is a measure of the total amount of light produced by a device. Also what needs to be considered that with a light concentrator (ex. lens) and-or reflector (mirror, director near the LED chip), a high percentage of all that light can be formed into a more narrow beam of light and having a beam angle of emission, radiation or dispersion and therefore concentrated upon an object or surface, and it will then appear brighter as if the source or flashlight was actually a brighter or intense point source, and the illumination from a certain flashlight or lamp may be rated this way, and its rating is actually then the apparent or equivalent amount of light intensity and-or brightness, yet the power to the LED and the total amount of light energy produced remained the same.

Also: 20 lm at 30 mA could be expressed as: 20 lm / 30 mA = 0.67 lm / 1 mA or= 667 lm / 1A

Pled = P in = (V led) (I led) = (3v) (0.030 A) = 0.090 W in : input electrical power to the LED =
power used and-or lost by the LED

lm out / P in = 20 lm / 0.090 W , after dividing num. and den. by 0.090, we have
lm / W = 222 lm / W : electrical energy to luminous energy conversion efficiency

Series and-or parallel LED's.

LED's can be placed in series and-or parallel electrical connection and both have their own advantage. If the LED's are all the same, and generally should be, then each will require and-or use the same amount of power, say 60mW each:
If one series LED burns out, the entire string or branch of LED's will not receive any current and will be useless. LED's in series require a higher supply voltage. LED's in parallel require more current, hence more drain of charge from a battery.

Pled = (Vled) (I led) = (3v) (0.020A) = 0.060W = 60 mW

Total power used for (n) LED's equals the sum of the power of each, and since the power of each is the same value, we can use multiplication to represent this repeated addition of the same value. If there is an array of 10 LED's:

Pt = (n LED's) (60 mW / LED) = (10 LED's) (60 mW / LED) = 600 mW = 0.6 W

If the voltage supply is 120v, how many LED's can be placed in a string or series of these LED's, and how much current will be needed ?

120V / (3v / LED) = 40 LED's

Since the LED's are in series, the current going through 1 LED will be the same amount of current going through them all. A typical LED will need about 20 mA to 25 mA of current to produce, perhaps about 90 lm to 100 lm.

The total power needed by these 40 LED's is: Pt = (40 LED's) (60 mW / LED) = 2400 mW = 2.4 watts
The total amount of lumens at 90 lm / LED = (40 LED's) (90 lm / LED) = 3600 lm
3600 lm / 2.4 W = 1500 lm / W : compare this to many incandescent bulbs at only 15 lm / watt

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Exploring The Energy Conversion Efficiency Of A Light Emitting Diode (LED)

An LED converts electrical energy and-or power (energy/time, a usage rate) to light energy and-or power. The better or more that LED can convert its input power to output, light power, the better its (energy) conversion efficiency. We should desire LED's with a high conversion efficiency because they will produce more light, be brighter and produce less wasted energy (usually as heat or thermal energy). The conversion efficiency an LED light (ie., "a bulb") is much higher than that for an incandescent light (bulbs). A high conversion efficiency also means less input power will be needed to produce the same amount of light, hence there will be an energy and cost saving with the lowest (total system cost / watts output) ratio.

Scientists have done analysis, experiments and calculated that:

1W of power = 643 lm : lm = lumens, a unit of light energy per unit of area, hence an "energy density" or "energy concentration". 1W = 1J / 1S. This value indicated is a true energy equivalency, and does consider conversion efficiency and any energy losses. Therefore, this value does not mean that 1W of input electrical power will produce 643 lumens of light, and this is so when the conversion efficiency is less than 100%.

From the above equation, mathematically:

1 lm = 0.001555209 W =~ 0.001555 W =~ 1.5mW light power

For a common 5mm wide, clear plastic white light LED, a typical rating is:

100 lm LED , multiplying both sides of the above equation by 100:

100 (1 lm) =~ 100 (0.001555W) = 0.1555 W of light power if there was 100% conversion efficiency , hence

0.155W / 100 lm = 0.00155 W / 1 lm

Input power for a common 5mm wide, clear plastic white light LED =

Pw = (V led)(I led) = (voltage across the led) (current through the led) = (3V) (0.020A) = 0.060W = 60 mW

100 lm / 0.06W = 1667 lm / W = 1 lm / 0.0006 W : actual light amount created and corresponding power used

From : (actual output light power) = (conversion efficiency) (ideal light power)

$$\textbf{conversion efficiency} = \frac{\textbf{actual output light power}}{\textbf{ideal light power}} = \frac{0.0006\text{ W / 1 lm}}{0.001555\text{ W / 1 lm}} = \frac{0.0006\text{ W}}{0.001555\text{ W}} = 0.385\underline{9} \;=^{\sim}\; 38.6\%$$

If an LED can be made to have a lower internal resistance, there will be less heat energy loss in it, and it will have a higher lumen rating with the extra current going through it. Technically, if an electronic device has a lower resistance, the voltage drop across it will be less, but this is not so with an LED because the (PN semi-conductor diode junction) barrier voltage is still the same at about 3v for an LED to function. With a slightly lower internal resistance of the LED, a slightly higher amount of current will flow through it, and this will produce an increase in the electrons flowing through it and the photons being created by those electrons, and therefore, the number of lumens will increase. Given the number of lumens per watt = (lm / W) = lm / (Vled I led), and if the voltage across the LED is the same, and the current increases, the amount of power used will increase. An LED with a higher lumen and-or brightness rating might not even have a higher conversion efficiency, but rather it uses more battery power (perhaps with a slightly higher than 3v applied), and therefore, it will drain the battery in a shorter amount of time, and the only benefit is a brighter LED and-or flashlight, but for a lesser amount of time of use. For any flashlight or energy consuming device, the less time it is used per instance, the more total time span of potential use and the number of instances it can be used before there is not enough battery power for it to function.

White light LED's can obtain a higher conversion efficiency by using more efficient phosphors, etc. in the LED's construction.

A white light LED can be powered by using two 1.5v batteries in series, and therefore having a total voltage of 3v, but that LED will not function until the battery is drained of charge and is 0v, but perhaps will it will function to as low as to 2.5v, and then the LED will not have enough voltage across it to function, and it will become a high resistance or like an open circuit with very low current through it. If the LED was replaced with a resistor, the terminal voltage of the supply battery will remain fairly high if the current or charge drain is low, but the voltage and charge available will eventually decline at a rapid rate, and the power available may be insufficient for some electrical device to function, even if the output voltage of the battery is only 1 or 2 volts less than the rated version of the battery. Many lead-acid batteries are considered drained at this point and need to be recharged before being damaged by further use and-or excessive current. Lithium-ion or Lithium-iron batteries are known for supplying more energy and-or power, hence watt-hours (wH) at or near the rated voltage of the battery for longer than many other batteries, and this is due to the low internal resistance of lithium batteries, and of which will waste less power as the circuit current must also pass through it.

An LED can be pulsed by a higher voltage, say 6v and appear as brighter, but only if the pulses are quick, hence if a low duty cycle (= on time of use / total time) of energy and-or power is used. The pulses of energy will be brief, but powerful (P = VI) and temporarily more brighter since the number of photons produced is directly related to the current flowing. The average power loss in the LED should still be kept lower than its maximum rated power and-or thermal ability. Will the LED appear to flicker? Yes if the pulses are slow, but some remedies are to use a capacitor (that stores some charge) in parallel to the LED to help reduce the flicker, and-or increase the frequency of the pulses, and of which will require some type of oscillator circuit. Humans have a persistence of vision for about (1/60) of a second, hence if the pulse rate is over 60 cycles per second, we will not generally notice it. A "joule-thief" (ie., voltage booster, here, a basic type of "switching regulator") LED light circuit can boost the voltage and apply it across LED, and from a low supply voltage supply of say 1.5V or even less, and so as to be 3v or more volts, and it creates at pulsed signal at a frequency of tens of kilohertz.

Part of the inefficiency of an LED is also due to internal absoprtion and-or reflections of where the light is converted to heat energy. Modern (as of the year 2000) LED's are now being redesigned and manufactured to help reduce internal light or photon losses, and this will increase the lumen output of an LED, and decreaae the heat losses. The end result is greater electrical conversion (here, electricity to photons) efficiency of an LED.

BATTERY BASICS CONSIDERING THE GALVANI-VOLTA CELL

In 1600, **William Gilbert** invented the first **electroscope**, and of which was like a magnetic compass, but it rather sensed electric charges rather than a magnetic field and-or direction to the north. Charged metal plates will attract each other if they have the different electric polarity and-or charge, and if they have the same electric polarity and-or charge, they will repel each other. An electroscope can be used to sense and-or measure an amount of charge, and the polarity or type of charge. Because the needle or foil can move in an electroscope, it also shows that (electrostatic, charges at rest, not moving) electricity can apply a force and induce (ie., cause) the motion of another object (even if its not charged, and this is called electrostatic induction, and is analogous to magnetic induction of a piece of metal), and this is similar to how magnets can attract or repel depending on their magnetic polarities used of the magnet's two ("North" [so called because that end of the magnet point or gives direction to the north (magnetic) pole of the Earth] or "South" pole) (magnetic) poles, and of where the magnetic field was more concentrated. The electroscope made measuring small amounts of charge practical, unlike for example measuring the length of a high voltage spark. Practical circuits use relatively small amounts of voltage and therefore charge (ie., later found to be electrons with and-or carrying that charge in a wire). **Electricity** and-or electric power eventually got its name from the fact that electrons, each with kinetic energy, transfer that energy and-or power from one location to another.

By the 1700's people began experimenting more with the mystery of electricity, and knew that non-metallic items such as amber, silk, fur could produce a charge when friction was applied to them and-or when these materials were rubbed together. In 1745 a practical way to store charges was invented, and it was called the **Leyden Jar**, and of which is basically two metal plates with a non-metallic, item such as glass in between them. Today,we call such a device a capacitor or charge capacitor since it has the ability to can store an amount of charge. The amount of charge it is storing is called the capacity or capacitance of charge. This non conductor between the plates ensured the charge could be maintained on those plates and not neutralized or vanish as would happen if they made contact. This "insulator" material between the plates is also called a dielectric material and-or a dielectric plate, and this description is due to it not promoting the flow of electricity, however it could very also mean that each side of the plate will l have a different electric polarity. Even though it does not conduct charge and-or electricity well, if any, it can be charged to have an electric polarity (positive or negative) on each side. Charged metal plates will attract each other if they have the different electric polarity and-or charge, and if they have the same electric polarity and-or charge, they will repel each other.

After the discovery of the Leyden Jar and its ability to store charge, more people began became interested in electricity, and particularly with charges on metal surfaces, and the amount of charge. One new inspired device was called the electrophorous which was essentially a metal plate having a wooden handle in the center, and one plate made of wax type substances resting on a table, and this device could be used to create charge and store it. This was created in 1762 by **Johan Wilcke**, from Sweden. The famous **Alessandro Volta** was inspired by this and began a study of electricity.

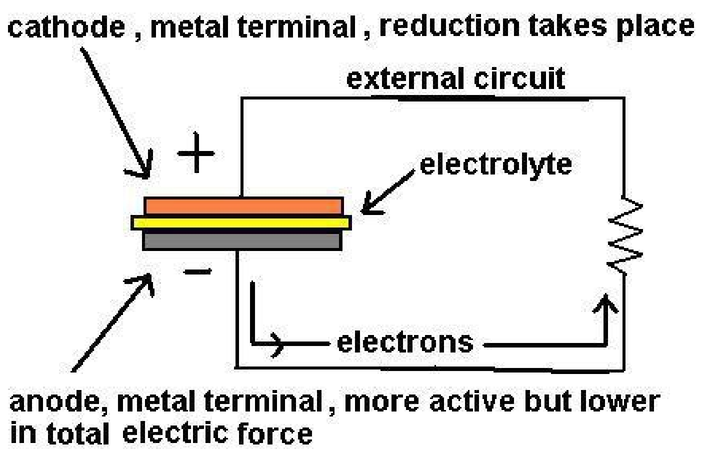
In 1782, **Volta** studied the relative amount of force of attraction or repulsion of charges on two electric plates and-or a Leyden Jar (capacitor) using an electroscope and concluded that more charge caused more force and-or motion of the electroscopes plates. Later, Coulomb would also studied the force due to electric charges separated by a distance and made his famous equation for the electric force due to charges, and this force is called the electric force. Volta also concluded that the amount or quantity of charge is directly related and proportional to the voltage which caused it to move, and this value is a constant which is called the capacity (C) of that specific (size, construction) Leyden Jar (capacitor). The **law of capacity or capacitance** is: **$C = Q / V$ =** (mathematically reduces to) **charge per unit volt**. Mathematically, the relative or comparative amount of charge in a capacitor can then be determined by its capacity (C) rating and the voltage applied: **$Q = VC$**. Later, voltage (electric potential energy) was found as also determining the amount of the flowing charge and giving that charge the energy to do work (ie. apply force , work = (force)(distance) = joules of energy). The total amount of energy in that charge is: (total energy) = (amount of charge)(voltage applied to the charge) = (amount of charge)(voltage [electro-motive force] causing the charge to move) = **Joules = $J = (Q_c)(V)$** , and mathematically: **$V = J / Q_c$ = (electric potential) energy per unit of charge** . The general capacity rating of a capacitor was later defined by its physical properties and was given the units of measurement called **Farads** and this rating is also a constant for a given capacitor if its physical dimensions and construction materials remain constant.

In 1785, **Martinus van Marum** , from the Netherlands, used a (high voltage) electrostatic generator to extract tin, zinc and antimony from their respective (chemical) salt forms, and this is similar to a process used later in 1807 by **Michael Faraday** for extracting and discovering sodium and chlorine gas natural (non-reducible, not a molecule) elements from sodium-chloride (a compound) which is commonly known as table salt that we put some on food and eat.

Just before **Alessandro Volta**, from Italy, invented his battery, it was noted in 1791 by **Luigi Galvani**, also from Italy, that certain muscles would have a quick movement when two dissimilar metals were applied across the nerve(s) of that muscle. This indicated several things such as what nerve signal could be, and that dissimilar metals were causing this signal, and the general concept of using two different metals for some type of (electro-chemical) reaction is called galvanic, galvanic action or galvanic-electric action. Soon afterward of this, in 1799 and 1800, Volta invented and presented the first known battery which was a stack or series of (Galvani-Volta, electrical) cells, each being two dissimilar (such as zinc and silver, zinc and copper, silver and copper, etc) metal disks with each separated by a liquid (now called an electrolyte) that aids in the current (electrons and ions) transfer between each dissimilar metal. For this to happen,there also needs to be an external circuit for the electrons to flow, and their gained (kinetic , [(mv^2)/2]) energy can be used to do power an external electric circuit.

The electrolyte liquid might be a weak acid or base (such as sodium bicarbonate) and-or simply some salt water. To retain the liquid in place, pieces of material were soaked in the liquid first, and then placed onto the "Voltaic pile" or battery. The electrolyte serves as a type of barrier that promotes the conduction of electricity and without having the two metals being in direct contact with each other which would prevent the electricity from flowing through the electrolyte substance(s). Galvani's experiment surly inspired Volta's to create the battery which is composed of many galvanic cells connected in a series arrangement to increase the voltage. If the plates were made to have a larger area, and-or several batteries were placed in a parallel electrical arrangement or connection, the current could be increased by the same factor that the area was increased. [FIG 310B]

Galvanic - Volta Cell



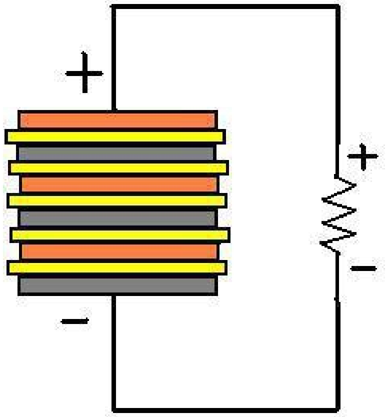
When an anode atom looses an electron, it becomes oxidized. The electron now having gained potential energy from the electrolyte and the postive electromotive force of the cathode attracting it can travel through the external circuit and power it.

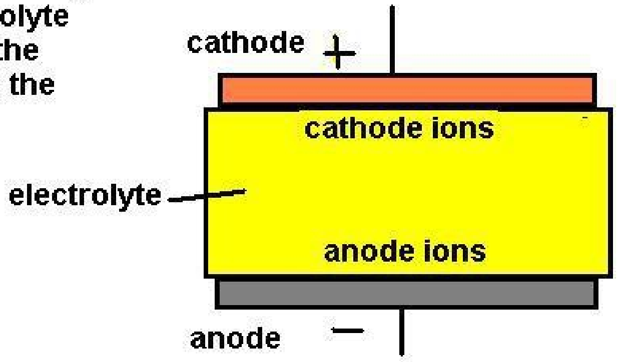
P = V I = Joules / second = watts

Example materials:

cathode: copper
anode: zinc
electrolyte: vinegar and salt water

Volta's battery or array of cells.





The word **catalysis** means to dissolve, remove, cut down, cut off, cut from or out from, and a chemical can be or cause a catalysis to happen and initiate or help promote a chemical reaction take place, hence it also causes it to happen in less time. Generally, the catalysis remains unchanged during this process, and the chemical and-or process to cause a chemical reaction is then called the (specific or particular) catalysis. Consider that even heating a substance to a higher temperature can speed up a chemical reaction.

The word "**ion**" was made by Michael Farday and Reverand William Whewell, and it basically means a particle that moves or travels from a location, and in electricity it is due to the electrostatic forces upon that (electrically charged) ion. Faraday also made the words: "electrode", "cathode" and "anode" to describe electricity.

The word "**electrode**" means an initial or terminal access point or location for the flow or movement of electricity.

Cathode is the negative pole or terminal, and this is the terminal electrons travel from. **Cat**hode **ions** are also called **cations**, A positive charged ion will move toward the negative charged cathode.

Anode is the positive pole or terminal. Anode is based on the words "path", "way" , and "up" or "to take up" (ie., or collect) and in terms of electricity, it means where the electrons are going to. A negative charged ion will move toward the positive charged anode. **An**ode **ions** are also called **anions**.

The word "**volt**" that we commonly use as a unit of electric potential energy of an amount of charge or a difference in electrical potential energy at two specific points in a circuit (which allows electricity [electrons]or current to flow) is named after Volta. Voltage = Joules of energy / Coulomb of electric charge. The more cells, placed alternately on top of each other, more voltage could be produced and the more (mysterious) electricity (today, known as electrons) would flow. Volta created a practical source of electricity. The voltage was not particularly high and useful for most experiments and purposes, but it was not as high as the (static) electricity machines already available that would rotate and accumulate a high amount of charge so as to eventually produce a high voltage spark across a short distance.

It is of note that water, which is often considered by the average person as being non-active in terms of electro-chemistry, is slightly active. Water is not a complete base (= pH of 14) or a complete acid (= pH of 0), but has a pH of 7, hence like a perpetual weak acid and a weak base of the same substance, and this helps it in terms of being used as one of the main substances in an electrolyte (semi-conductive) liquid, and it is commonly available and inexpensive. During electrolysis, the electrolyte substance will heat up due to its internal resistance, and this represents a power loss (P=VI) in the battery and it will be unavailable as power to and for the external circuit.

Each different metal used in a galvanic cell has an electric or electrode potential (ie., ability to create an electric force = electromotive force = emf). The electric potential of zinc is about -0.75v, and the electric potential of copper is +0.35v, and when constructed to be an electric cell, the sum of their voltages will be about 1.1 Vdc Making a stack of 3 of these cells in series will create a battery voltage of their sum, and since multiplication is repeated addition: V = 3 (1.1v) = 3v . There are tables of the electrochemical potentials of metals and these are known as galvanic or (metal) electrode potential. A particular table may be associated with a particular reference metal and-or electrolyte such as sea-water with its particular amount of sodium and other elements. Still, regardless of a specific table, the difference in potentials when using any electrolyte should still be the same. Metals that are more electrochemically (ie., cause and-or affected by electron and-or chemical activity) active, hence not exactly stable in the electrochemical sense, such as being able to become and-or produce ions, will have a higher voltage (ie., energy, V = J/Qc) rating, and these more active metals such as zinc are used as "sacrificial anode (source of ions)" that will eventually oxidize and disintegrate to ions in the water, while protecting another metal such as steel (here, being considered electrochemically like a cathode), from rusting (ie., galvanic corrosion [slow destruction], oxidizing, becoming ions and falling apart) by obtaining free electrons from the zinc, and this is common with boats made of steel which would otherwise oxidize to rust when its atoms oxidize in the water and loose an electron. Whenever two different metals contacting each other and-or in the in the presence of moisture (ie., as an electrolyte), galvanic corrosion can happen. Galvanic corrosion can sometimes be reversed by using a sacrificial metal of lower electric potential so as to electrically transfer (via the electrolyte in surface contact with the oxidized metal) and donate its electrons to the oxidized metal ions which lost an electron.

Though the zinc-copper battery was good, it was non-rechargeable, and another problem is that even by sitting over time and with no load current, the zinc electrode will slowly react with the electrolyte solution, and the battery essentially self-discharges and will eventually produce no power (V I). A solution to this is called the **Daniel Cell** and it used two electrolytes separated by a conductive path for the ions. The Daniel Cell is also non-rechargeable, but has a good shelf life and will be available for use when needed. It is also possible to store the electrolytes and metal electrodes away from each other and assemble the battery when it is needed.

Noble metals such as platinum and gold have "heavy atoms" having a large nucleus mass, and with a large number of protons (ie., a high amount of positive force) and orbiting electrons, however, they have relatively few valence (far orbit, with least electric force upon them from the nucleus) electrons and these metals are therefore said as being more electrically stable and-or it would take much energy to free an electron from gold than it does for copper. Noble metals may be said as being more inert, stable, and non-reactive. Noble metals are also described as being immune to oxidation (ie., turned into ions, "rust") and-or corrosion. A metal that is more reactive than another will have a loosely bound electron in its atom. As a metal looses electrons and oxidizes into ions, it become even less (electrically and chemically) reactive. An element that is more reactive than another has electrons that require less energy (including electromagnetic force =emf, voltage) to break free from their orbits, and this metal is called the anode or negative terminal, and even if it is another noble metal. The other terminal and-or metal will be called the cathode or positive terminal which attracts the electrons. Again, for these electrons with gained energy are to be useful for an external electric circuit, they need to travel to and through that (usually of a low resistance wire) external circuit and then back to the battery - although now having less energy available and near the "ground state" of no energy. The less reactive metal with much positive charge and-or protons and less negative charge and-or electrons in the outer, valence region is able to attract (due to electromotive forces of attraction) the electrons from the anode terminal.

Commonly seen and-or mentioned oxides are iron oxide, aluminum oxide, copper oxide, zinc oxide, lead-sulfide (galena, lead and sulfur), silver oxide (usually silver-sulfide due to sulfur atoms) and iron-pyrite (iron and sulfur crystal). People have made diodes to be used in electrical circuits such as radio receivers with the help of an oxidized metal. These oxides are not good conductors of electricity since they have lost a loosely bound, outer (ie., valence) electron that could be used for conduction, and are said as being semi-conductors, and for a diode in a radio receiver, it is critical or necessary to have so as the audio can be heard and not effectively canceled out by other halves of the waveform. Some oxide coatings or layers are sometimes said as being like an insulating layer that could prevent further oxidation corrosion (ie., pitting, degradation). To avoid corrosion in a metal, an insulation barrier to the flow of electricity (electrons) and-or ions (atoms that lost an electron) is needed, and may be a coating of an oil based substance, a wax, or paint, plastic or electroplating of a non-reactive metal onto it such as gold, chromium or nickle. **Stainless steel** is made to be less corrosive than regular or plain steel, and it usually includes chromium (10.5% minimum) and-or nickle. The surface of the stainless steel may corrode some iron particles very slightly at first, but that will cease once it reaches the non-reactive, non-corrosive metal(s). Stainless steel products are often used in the food processing , medical, manufacturing, and sanitation industries so as to reduce organisms and-or to increase the quality and usage time of the product(s). The surface of stainless steel can then be relatively, easily cleaned and-or sterilized without worry of corrosion and-or rust. Since galvanic action is essentially due to electric potentials and-or electric current, galvanic corrosion can sometimes be prevented with an applied voltage and with nearly no current necessary.

The galvanic potential and-or difference in (voltaic, electric) potentials between two metals in an electrolyte such as saltwater can be measured by a voltmeter, and if the metals are connected by an external wire, electric current will begin to flow and the less electrochemically active metal will experience galvanic corrosion. The amount of current created by using two different metals in an electrolyte will be determined by the size of the metal surface or plate area, and the type and (electrochemical) strength (ie., concentration of substance(s)) of the electrolyte. Temperature can also affect an electrolyte and can make it more active or less active, hence alter its internal resistance, hence it will be a factor in the amount of current the battery produces. If the electrolyte solution gets "used up" or depleted, its resistance will increase and the output voltage and current will decrease. If the electrolyte solution gets "enhance" (more active, fresh, or an increase in temperature) then its resistance will decrease and the output voltage and current will increase. The higher the voltage difference between the two metals, the greater the electric current and the potential corrosion.

A galvanic cell (aka: a "primary battery") is not a rechargeable battery (aka: a "secondary battery") in the sense that it can not be recharged by applying a voltage across it and then having a current (ie., charge, electrons) go into it so as to be stored there. The galvanic cell gets its output power (V, I) due to the new ions created at one terminal by the energy of the electrons, and those ions flowing to the other terminal through the electrolyte solution (usually water acting as a solvent) made from an acid or base, and sometimes with some salt. As the chemical energy of the cell and-or battery is becoming depleted, the voltage of that cell and-or battery will get reduced. Over time and use, the anode metal can oxidize to ions and degrade (corrode). If the used or depleted electrolyte is replaced with new or "fresh" electrolyte, the galvanic cell will continue to function again. In a **rechargeable battery**, the electro-chemical process is essentially reversed so as to effectively have new-like electrolyte, and this process is a special form of electrolysis, and it will then have the ability to be recharged and store energy again. This type of cell is also called an **electrolytic cell**. The electrolyte used is a special type that can be recharged (ie., can store energy, hence potential energy (J / c = voltage)to be used later) with energy.

As of the year 2023, Lithium-ion batteries are the popular form of rechargeable battery since they are energy dense, however the are relatively expensive, and in the future, zinc-ion batteries and sodium-ion batteries will probably be a much cheaper an alternative.

Now that Volta created the battery, it was in less than a year of time till people such as **William Nicholson** and **Anthony Carlisle,** in 1800, starting using a battery for experiments and put two metal electrodes in a container of water with an electrolyte in it, and applied a voltage to those electrodes and through the electrolyte and noticed bubbles being produced. This was the beginning of the study of **electrolysis** (to loosen and-or break apart via electricity), and of which many scientific discoveries (such a new, natural or chemical elements) were made, and practical use of. The bubbles were hydrogen gas and oxygen gas created when a molecule of water (H2O) is "decomposed" (ie., split apart by applying a force and-or energy) by nearby electricity. The hydrogen gas was produced at the cathode, and the oxygen gas as produced at the anode when a minimal voltage of 1.23v (at about 72°F water temperature) was applied. For other substances, the minimal voltage necessary to split them will be different since it takes more energy to break apart some molecules, etc, that are electrically, via electrons sharing, bonded more stronger. Electrolysis is essentially the reverse process of using a battery with chemical energy so as to create electricity, and is rather using electricity to create chemical energy and-or a chemical reaction, and of which also includes recharging a battery.

Electrolysis is the process of passing an electric current through a liquid so as to cause a chemical reaction. The word electrolysis means to separate matter by using electricity. Because it is a chemical reaction, the combined sciences and-or use of both electricity and chemistry is called **electrochemistry**.

The basic concept of electrolysis can be thought of as the reverse of a galvanic battery, and where an external voltage is rather applied to the electrolyte. The bubbles produced by the electrolysis of plain being water as the electrolyte were found to be oxygen and hydrogen gas, and this was an indication of what water was made of two gasses in a condensed state or union, and it is these things and further experiments like this that lead to both advances in chemistry, electricity and other sciences. To describe what was going on, the water was said as being "decomposed" which is another way of saying the union (later known as a molecule of water, H2O) of the two gasses that creates water was split apart, and here by using the kinetic energy of the electrons.

For some examples of what electrolysis inspired: **Geissler** applied a voltage across a gas which after many years and discoveries by others eventually lead to the vacuum tube electric signal amplifier, and the discovery of the electron, proton and neutron, the extraction of elements form ores (often an oxides or "rusted" form of the metal(s)) using a high current and the resulting temperature increase to cause the metals in ores to melt out from it.

Humphry Davy discovered several new metals using electrolysis. Davy also invented the (high voltage) **arc lamp** a few years (in about 1805) after the invention of Volta's battery which made such experiments practical, and this was a high intensity light produced by a continuous spark or "arc" in the air from one carbon electrode to the other. The carbon electrodes degraded after a few minutes and needed to be replaced, and these lights were mainly used for public settings, military, and factory lighting due to that it was not practical to be used by the average person and-or household. Most improvements in arc lamps came late in the 1870's. It is also of note that this arc lamp probably inspired the Geissler tube (much like a neon light) which was invented years later in 1857, and of which then inspired the Crookes tube in about 1869, and of which many incredible discoveries and their practical use have been made from. Michael Faraday was an assistant of Davy, and who would make many further scientific discoveries.

Rust removal by using electrolysis:

If a piece of iron or steel metal got oxidized and got rusted on its surface, much of the rust can be removed by the help of an electrolysis process. In a sense, the method is to corrode the rust so as to weaken it for removal by an abrasion process such as sandpaper and-or various brushes such as metal wire brushes to soft plastic brushes. The main steps to do this process is to attach the rusted piece of metal (here being the cathode) to the negative terminal of the power supply and place it into an electrolyte solution. The other terminal metal will be connected to the positive terminal of the power supply, and this sacrificial anode, say a piece of steel scrap, will supply electrons to the cathode and reduce the iron oxide atoms back to iron atoms, and all this movement over time will also cause the rust to be pushed away from the surface of the rusted piece of steel and onto the surface of the anode. An often used substance to make the aqueous (water) electrolyte is (inexpensive) sodium carbonate. Only a few teaspoons of it are needed. It is best to do all of this in an area with adequate ventilation. How much Power is needed to do this type of electrolysis? Only a few volts is needed, and a low amount of current, say less than 1A, and the total power (J/s) should be less than 12W. A DC power supply of several volts and up to 12v is sufficient, and if that supply has short circuit protection, then that is also helpful.

A salt and vinegar rub can be used to remove copper-oxide from copper, and this can make a copper coin or copper pipe shiny again, but it is recommended to not do this on valuable coins since coin and-or antique collectors want to see and purchase such items with the copper-oxide patina or protective coating still on them.

It is of note, that as of about the year 2020, there are some high power lasers available for purchase that can help remove rust.

Electroplating is a specific application or process of using electrolysis so as to deposit metal ions in the electrolyte solution as atoms on a metal electrode in the solution of which the voltage is applied to. Of a certain metal is to be plated onto another, there needs to be a certain electrolyte used, and it may not be practical to plate a certain metal onto another certain type of metal. If you wish to plate a metal with gold (or other element which has a low reactivity [ie., non-reactive, corrosive resistant]), you will need to have dissolved gold atoms in the electrolyte, and inorganic cyanide (a carbon-nitrogen compound, and which has a net negative electric charge), and this can be purchased from a supplier and-or made from dissolving gold pieces in a strong acid.

Some objects, such as some non-conductors (ex. plastics), can sometimes be electroplated when there is an electrolyte of (ready made) ions of the metal to be plated onto it, and a special conductive reduction agent coating is used on the work-piece so as to allow the metal atoms to "stick" (adhere to, bond, become part of the metal crystal) to its surface.

In the figure shown below, copper is to be plated onto zinc, and the electrolyte solution is copper-sulfate (a compound with molecules of copper and sulfur). For the copper atoms to stick to the zinc metal, the zinc needs to be cleaned of any debris and-or oxide, and this can be done by using chemicals made to do so and-or by using reverse electrolysis with the goal to remove just the surface layer of debris. During electroplating, the copper-sulfate will get ionized or split into copper atoms with a net positive charge, and sulfur ions with a net negative charge which will be electrostatically attracted to the anode, positive terminal or electrode plate of metal where they will be reduced to sulfur atoms. The copper ions will be electrostatically attracted to the negative (or lower in electric potential) cathode terminal or electrode plate of metal and plate (ie., cover or coat) it. During the whole process, electrons are also flow or move form the positive terminal to the negative terminal through the electrolyte, and so as to cause more ions to be created from the copper-sulfate molecules.

Michael Faraday , who is credited with discovering electromagnetism and magnetic induction, discovered that the amount of metal deposited during electroplating is directly related to the electric current through the electrolyte, the concentration of the electrolyte, and amount of time the process is used, and it then became even possible to calculate the amount of mass electroplated or deposited onto one electrode and-or how much was removed from the other electrode. Note that although electricity (ie., electrons) is being used to create the ions at the surface of one terminal, these metal ions do not travel through the external circuit, and only electrons do. When an ion is created from an atom or molecule where electricity is applied, it is called oxidation, and when the ion regains an electron, it is called reduction, and this entire process is sometimes called the **redox** (a word contraction of the words of reduction and oxidation) process and it is described here:

oxidation is a term used to describe when a element looses an electron and becomes an ion, such as iron-oxide and this term came about from when iron rusted (or "oxidized") in the presence of oxygen (ie., air and having moisture in it) and-or water (H2O), and this is called oxidation. Can iron-oxide be converted back to iron using electrolysis? Some metal oxides can , but not iron oxide because of the presence of oxygen in the water which is H2O (hydrogen and oxygen), and carbon is rather used instead. When an oxide regains its lost electron, it is said to be **reduced** back to its natural form. Many oxides of metals are brittle and-or powdery, and this is because the normal atomic bonding due to electron sharing so as to make a hard crystal arrangement of the metal has essentially been shattered by the loss of the electron(s) during oxidation.

Obviously, the longer the time the plating process continues, the thicker the plating will be, and the most costly, especially for precious metals such as gold. The larger the work-piece or object to be plated will take more plating material. Plating is generally only a few, perhaps 10 microns (1 micron = 1 micro meter = 1 millionth of a meter = 1 (10^-6) m thick, but may be thicker if the object will have much physical wear and tear that may cause scratches and-or abrasions.

Though the surface area of a cubic centimeter of a metal is only 1 cm^2, if that cube of metal is sectioned into many layers of equal thickness, the surface area available for plating will be incredibly large. The thinner or less the thickness of each layer, the more surface area will be available for plating.

For a simple example: The height of a cubic centimeter is 1 cm high or thick. If it is sectioned into square areas or plates, and with each having a thickness of 1 mm, there will be a total surface area of:

1 cm thick / 1 mm = 0.01 m / 0.001 m = n = 10 times more, : 1 cm = 0.01m , hence:
At = n (surface area) = 10 (1 cm^2) = 10 cm^2

If the thickness was 1 um (one micro-meter) = 1 (10^-6m) :

1 cm thick / 1 um = 0.01 m / 0.000001 m = n = 10000 times more, hence:
At = n (surface area) = 10000 (1 cm^2) = 10000 cm^2 = (100 cm)(100 cm) = 1 m^2

If the thickness was twice as thick at 2 um, the total area would then be only half this value of 1 m^2:
Letting n um = 2 = divisor of the area:

At = 1 m^2 / n = 1 m^2 / 2 = 0.5 m^2 : = 5000 cm^2

The density of gold is about 19.32 g / 1 cm^3 . as of September, 18, 2023, 1g of gold cost about 62.18 USD.

19.32g will cost: (grams) (price / 1 gram) = (19.32)($62.18 USD) = $1201.32 USD , hence:
$1201.32 USD / Vol. = $1201.32 USD / 1 cm^3 and $1201.32 USD / (1 m^2 1 um thick coating) =
[($1201.32 USD / (1 m^2)] / 1 um thick

Also: $1201.32 USD / 1 cm^3 = $1201.32 USD / (1 cm^2 1 cm) = $1201.32 USD / (1 cm^2 1 mm thick) =~
($120.132 USD / 1 cm^2) at 1 mm thick or ($0.12 USD / 1 cm^2) at 1 um thick

Here is a website, and if is still available, here is the current price or value of gold: https://pricegold.net/today/

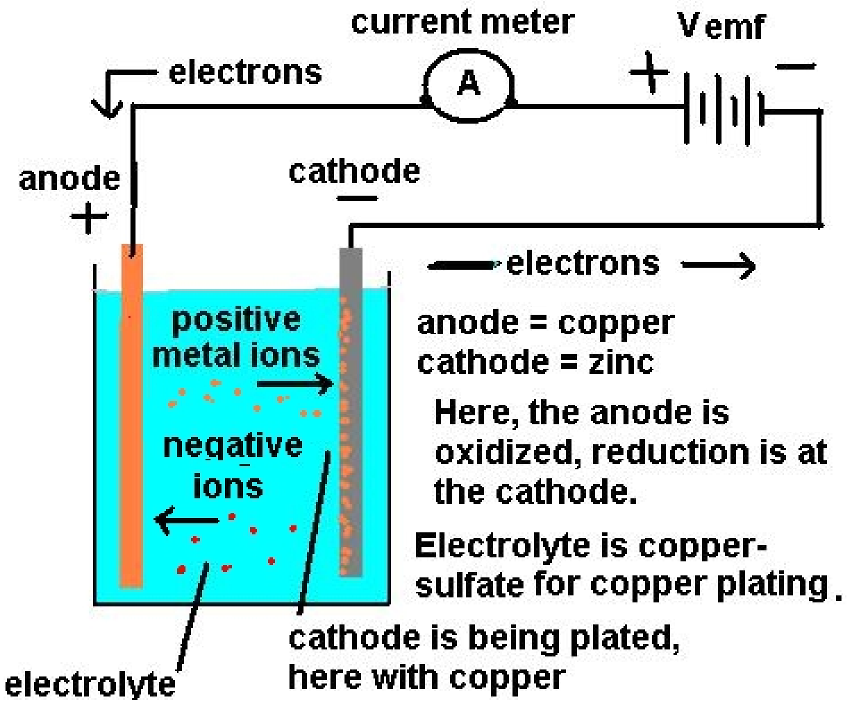
Mathematical calculations can be used to predict how much mass (grams) of metal will be plated. To check the calculation and-or to make another estimating calculation formula, the plated metal can be weighed with a fine or precision scale. An estimate formula could be based on an equal ratios if the plating is linear or proportional with respect to time:

$$\frac{\text{total grams plated}}{\text{total time to plate}} = \frac{Xg}{\text{time}}$$ when mathematically reduced to 1 time unit, and then mathematically:

Xg = (time) (total mass plated / total time to plate) : mass plated after a given time , and:

(time) = (Xg) (total time to plate / total grams plated) : time to plate an amount of mass

[FIG 310C] **An example circuit and description for electroplating.**



In the above circuit, the current is shown as conventional current which flows from the (+) or (theoretical or mathematical) higher electric potential identified as the positive (ie., higher in energy) terminal, however, it is actually electrons with kinetic energy that flow away from negative terminal by being electrostatically forced and repelled. Electrons are said to be negative in charge, and protons are said to be positive in charge, however, protons do not flow or travel in a wire due to their relatively high mass as compared to an electron.

The concept of electrolysis is filled with many basic facts and then some more complicated concepts, and uses well tested experiments and accepted methods and formulas, and all should be well if precautions are taken so as to produce the same results.

[This space for book edits.]

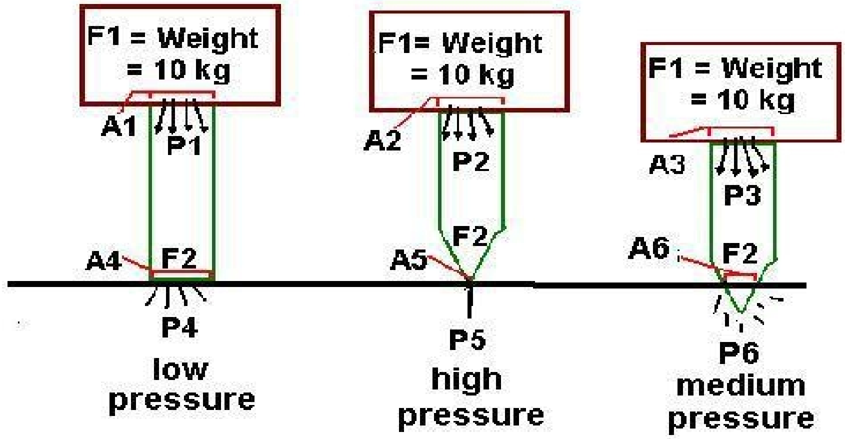
How much force can the wind apply to a surface?

How much force can the wind apply to a (perpendicular) surface?

The quick answer to how much force can the wind apply is that it depends on the windspeed and (cross-sectional, perpendicular) area of the surface that the wind energy is impacting and applying energy to it, hence the application of force upon it, and this area is therefore also equal to the cross sectional area of that wind that is impacting upon it.

Pressure = Force / Area = P = F / A , and mathematically, F = PA

Ex. If a force or weight corresponding to 10 kg is applied to a surface area of 5 m^2, the pressure upon each unit of that area is: P = F / A = 10 kg / 5 m^2 = 2 kg / m^2. Note that applying 2 kg / m^2 is equivalent to applying 10 kg / 5 m^2: Mathematically: (2 kg / m^2) (5 m^2) = 10 kg and this single weight and-or force could be achieved by placing 2 kg weights on each square meter, and the sum of each will add up to the total weight. A practical example of all this would be a several people applying force to an object so as to lift that object that once person could not lift alone. The result is that the total force applied to the object is the sum of the forces from each applying force (Fn = Pn An). If there were 4 people and each applied 10 kg of force (In this context: equivalent and proportional force due to 10 kg) or weight to the object, the total amount of force applied would be: (people) (force per person) = (total force) = 4 people (10 kg / person) = 40 kg. Here is a figure for a basic understanding of force, pressure and area. [FIG 311]



P = F / A and mathematically:
F = P A

P and A are factors of the resulting force. P and A are inversely related, but are not necessisarily reciprocals in value.

A5 < A6 < A4
P5 > P6 > P4

At each surface juntion, there is a force applied upon the area beneath it, and the average amount of force per unit of area is the pressure.

The smaller the area for a given force, the higher the pressure.

For a given system, with various forces, areas and at each junction, the product of (PA) = F is constant.

As the force and-or pressure increases upon an area, it can also cause the region on both sides of the junction to compress, and that region may get deformed.

Due to equal and opposite forces, both regions on the opposite sides of the junction will experience the same force and pressure. The amount of deformation is determined by the amount of force and-or pressure applied, and the strength of the material.

The pressure of P5 is very high due to the small area the force was applied to, and this will cause it to penetrate into and-or beneath the surface as shown at A6. It is as if the force and-or energy applied was higher, magnified or amplified, but they are effectively concentrated.

Pw = Energy / time = Joules / second = Work / s = F d / s = **F v = m a v** **:** here v = velocity

density = p = m / V , m = p V **:** here density is that of the air = (0.00124 g/cc)(1000cc) = 1.24g/L , or water = (1g/cc = 1000g/L = 1kg/L) , here V = Volume

Pw = F v = m a v = p V a v = p V (v / t) v = p V v^2 / t = p V v^2 / t = V v^2 v / d = p V v^3 / d = p A d v^3 / d =
Pw = p A v^3 : maximum before various energy losses are considered

There are several other ways, and using similar methods so as to find the same formula(s) as shown above.

The key to the above derivation and others similar to it is: Volume = V = d^3 = d^2 d^1 = A d
and: d / t = v , and: a = v2 / t 2 = v / t in general, and when t1 = 0 and v1=0 and (change in velocity) = (v2 - v1) = (v2 - 0) = v2 , and (change in time) = (t2 - t1) = (t2 - 0) = t2

From the above equations, we have:

F = ma = Pw / v = pA v^3 / v = **pA v^2** = Work / d = Energy / d ; also **F / A = pressure = pv^2**
p = density of the air , Mathematically:

For windmill blades, the area (A) for this equation is the area of the blades that the wind energy is actually applied to as "seen" by the wind. This area is analogous to the drag area of a planes leading wing surface, and not the entire surface of the plane's wing which helps determines the amount of lift force.

F / m = a = d / t^2 = vt / t^2 = v / t , v = F t / m , F = mv / t , and mathematically: momentum = p = mv = F t and F = p / t

distance = d = vt = (at) t = at^2 , v / a = t^2 / t = t , a = v / t

Comparing the energy available per area such as 1 square meter of moving air or wind, water and solar (ie., sunlight).

If the medium transferring energy and-or power was water instead of air, and since the density of water is much greater than air, it will have a larger amount of mass per unit of volume and therefore, it will have a much greater amount of energy per unit volume, and-or unit of area during an instant of time. How does solar power compare to wind or water power in these types of calculations? When the If the Sun is directly overhead at noon, and without cloud interference blocking some of its light energy, it can deliver up to 1000W of energy per square meter. A typical photo-voltaic panel can convert abut 20% (ie., conversion of energy efficiency = conversion efficiency) of that solar energy to electrical energy, hence it has a conversion efficiency of (output power / input power) = 0.20 = 20%. 20% of 1000W is 200W. A typical solar-air heater can convert about 80% of the incoming, collected or received light energy to heat energy. 80% of 1000W is 800W. This amount of power would use a large current drain from the typical batteries used in a solar electric system (SES), and will cause them to drain their charge (of current and power) fairly rapidly, and the available time of use for that charge cycle will then be shortened (ie., less time available before the battery is drained and-or ruined, and insufficient or unavailable to power the load device). The amount of power available in water is very high since it is a dense substance as compared to air, and the more matter moving, the more the total amount of kinetic energy ($KE = mv^2 / 2$, joules) it has available to make power (Pwatts = energy / time = Energy / second = joules / second).

A windmill or wind power to electric generator collects the kinetic energy in the wind and converts it to mechanical, rotational power. The maximum amount of wind these blades can collect will have an area equal to the corresponding and complete disk area of the blades. All the possible wind (kinetic) energy and-or power corresponding to that area of wind cannot be utilized for the blades to rotate because some of that wind must be permitted to pass through the blade area so as to rotate them, the axis of them, and power transmission wheels and-or gears. If the air did not pass thorough the blades, it would not create any movement and rotation of them. It is the blade angle that determines which direction the blades will spin. If the blades were rather a complete disk, all of kinetic energy of the wind would be turned into a force that will try to push that disk "straight back" in the same direction as the wind, and will result in a high amount of torque that is not perpendicular to the axis of the blades, but rather in the same direction as the axis. For there wind energy to move the blades it must strike the edge of the blade, hence there must be sections of the corresponding disk area not having blades, and therefore, not all of the total possible wind energy in that complete disk area can be transferred to the blades. Once the blades are forced to move and rotate in a direction (clockwise=cw or counter-clockwise=ccw) and gain kinetic energy, some of that wind energy will get transferred to the blades depending how efficient they are. The thicker the cross sectional area of the blade that is perpendicular to the wind will have drag (ie., air friction), and the faster the blade moves, the more rotational or movement energy lost due to drag force. In general, six triangular blades slightly angled from perpendicular to the wind will take up about 50% of the cross sectional area of the wind available, and 50% will be air gap in the complete disk of area. This means that a maximum of about 50% of that wind energy can be converted into torque (rotational) energy and-or mechanical power. If this mechanical power is then used to rotate an electric generator, then electric power can be made. If the blades are only 40% efficient at converting the wind energy upon it to mechanical energy, then they could convert: (0.60 of total energy available)(0.40) = 0.24 = 24% of the total available wind energy upon the blades to mechanical energy. This also means that (1.0 - 0.30) = 0.70 = 70% of the total available wind energy available at the total disk area of the blades was not converted to rotational energy of the blades.

If the air was rather water, and since the density of water is much greater than air, it will have a larger amount of mass per unit of volume and therefore, it will have a much greater amount of energy per unit volume and if the same velocity as the air. If velocity of the wind or water doubles or is 2 times faster, say from 5 km/s to 10 km/s, its kinetic energy ($KE = mv^2 / 2$) increases by a factor of $2^3 = 8$. Hence in a reverse manner, if the velocity of the wind is halved or is 2 times less and-or slower, its kinetic energy will decrease by a factor of $2^3 = 8$:

First, some general equations:

(change in speed) = (new speed) - (old speed) : = (difference or change in speed = Δv)

$\frac{\text{(new speed)}}{\text{(old speed)}}$ = (change in speed factor) : (new speed) = (change in speed factor) (old speed) (new speed) = (change in speed factor) (old speed) : a linear equation of the form: y = mx

Some general equations specifically for KE and velocity:

(change in kinetic energy) = (new KE) - (old KE)

$$\frac{\text{(new KE)}}{\text{(old KE)}} = \text{(change in KE factor)}$$

(new KE) = (change in KE factor) (old KE) : a linear equation of the form: y = mx

$$\frac{\text{(new speed)}}{\text{(old speed)}} = \text{(change in speed factor)}$$

(new speed) = (change in speed factor) (new speed)

(change in kinetic energy factor) = (change in speed factor)^2 , mathematically, taking the cube root of each side:

(change in speed factor) = $\sqrt[2]{\text{(change in kinetic energy factor)}}$

Here are some other mathematical derivations to consider for the above concepts:

Part 1:

KE1 = mv^2 / 2 , and if the velocity increases by a factor of (n): KE2 / KE1 = (n)

KE2 = m (nv)^2 / 2 = m n^2 v^2 / 2 : = (m v^2 / 2) n^2 = n^2 KE1

KE2 / KE1 = n^2 , hence, if the velocity changes by a factor of (n), the kinetic energy will change by a factor of (n^2) which is the square of (n).

Part 2:

(change or difference in KE) = (KE2 - KE1) = [(m v2 / 2) - (m v1 / 2] = (m / 2) (v2^2 - v1^2)
= (m / 2) (change in **[**velocity^2**]**) = m (change in velocity)^2 / 2

KE2 = KE1 + (change in KE)

If the (v old) changes by a factor of (n) = 2: v new / v old = 2 , therefore: v new = (2) v old = (n) v old

(change in v factor) = (n)

KE2 = (m / 2) (2v)^2 = (m / 2) 4v^2 = 4 mv^2 / 2 = 4 (mv^2 / 2) = 4 KE = in general: (n)^2 KE

KE2 / KE1 = n^2 KE1 / KE1 = 2^2 KE1 / KE1 = 4 KE1 / 1 KE1 = 4 = n^2 =
= (change in v factor)^2 = (change in KE factor)

(change in v factor) = $\sqrt{\text{(change in KE factor)}}$

Mathematically comparing how much more (ie., a factor, "times more") power is available in a moving mass of water than a moving mass of wind at the same given velocity, say v = 1 m/s:

$$\frac{P\ water}{P\ air} = \frac{p\,A\,v^3}{p\,A\,v^3} = \frac{(1000g\ /\ 1L)\,(1\ m^2)\,(1\ m/s)^3}{(1.24g\ /\ 1L)\,(1\ m^2)\,(1\ m/s)^3} = \frac{(1000g\ /\ 1L)}{(1.24g\ /\ 1L)} =^{\sim} 806$$

: theoretical maximum , not considering system losses

In general, the average velocity of water is much less than that of air. For example, if the air speed is 5 m/s, and the the speed of the water is 1m/s, the ratio of the power available is then much less at about: 6.5 : but is still significant

For some helpful reference and conversions of English and Metric units of distance or length:

1 mile / 1 hr = 5280 feet / 3600 s = $1.46\underline{7}$ ft / s =~ 1.5 ft / s = 18 in / s

1 kilometer / 1 hr = 1000 meters / 3600 s = $0.27\underline{8}$ m / s =~ 0.28 m / s = 28 cm / s

1 in = 2.54 cm
1 in (12) = 2.54 cm (12)
1 ft = 12 in = 30.48 cm = 0.3048m : diving both sides by 0.3048 , we have : 1 m = 3.280839895 ft

1 m/s = 3.281 ft/s =~ 3.3 ft/s and 1 ft/s = 0.3048 m/s =~ 0.3 m/s

1 ft (5280) = 30.48 cm (5280)
1 mile = 5280 ft = 160934.4 cm = 1609.344 m = 1.609344 km , dividing both sides by 1.609344 , we have:
1 km = 1000m = 0.621371192 miles , and therefore:
1 km/hr = $0.621\underline{4}$ miles/hr =~ 3281 ft / hr and 1 mile/hr = 1.61 km/hr = 1610 m / hr

1 km/hr = 1000m / 3600s = 3281ft / 3600s , dividing the num. and den. of both sides by 3600, we have:

$0.27\underline{8}$ m / 1s = $0.911\underline{4}$ ft / 1s , dividing both sides by 0.278, we have:

1m / 1s = 3.281 ft / 1s , dividing both sides by 3.278, we have:

1ft / 1s = 0.3048 m / 1s

As a reminder, pressure = force / area = P = (F / A) is like the average force per unit area, and mathematically:
Force = (pressure)(area) = F = (P)(A) is like the sum of all the individual pressures per unit area and expressed as a multiplication since repeated addition can be expressed as a multiplication.

If the (effective and-or perpendicular) area of the surface, such as for wind blades of a wind turbine are increased in size by a factor of (n), the energy collected will then be increased by that same factor of (n). **The wind pressure per unit area upon the wind blades will still be the same value as before**, however, since the area is larger, the resulting net force and-or torque will be (n) times greater:

F = (wind pressure)(area) = (n) (F / 1A) (A). : Since area is now twice as much, the force and-or torque will be (n) times more, and this amount of force can then be represented as:

(n) F

Torque = F L = (nF) L , and since the blades are longer, the effective lever arm was also increased by (n):
Torque = F L = (nF) (nL) = n^2 F L : we see that if the size of a wind turbine increases by (n), the resulting torque produced upon the generator shaft will increase by (n^2).

As shown previously in this book: **F = p A v^2** : p = density of the "fluid", here it is wind
Mathematically: **F / A = P = (fluid) Pressure = p v^2**

It must be said that 1 molecule of moving air does not provide much kinetic energy to move the wind blades, but several million air molecules striking the wind blades at the same time will effectively sum to enough combined or net energy. and the resulting combined force so as to move those wind blades.

The most common equation for the wind power that can be collected by a wind turbine is:

$$Pw = \frac{p\,A\,v^3\ Cp}{2}$$

: p = air density in kg/m^3 , A is the total circular "sweep" area of the turbine blades, hence the area of the moving air that could potentially be collected. A = (pi)(radius of the blades)^2 v is the velocity of the wind in m/s , Cp is the turbine performance efficiency at converting the available energy of the wind in the "sweep" area to rotational power. If the generator is very efficient, the electric power will be roughly equal to that of Pw value here, but less at about say 10% due to various conversion or inefficiency losses. This equation is similar to that of the **lift equation** for an airplane wings (ie., blades). Cp will be affected by the number of blades, their design and angle. For a simple example, if the total blade area is (1/4) = 0.25 = 25% of the total sweep area, then the energy collection will be <=25% of the total wind energy available. Cp is typically about 0.35 , hence about 35% of the wind power will be collected and transferred to the generator, and (100% - 35%) = 65% is essentially uncollected. If the generator efficiency is 90%, then 10% is lost as heat, and 10% of 35% is 3.5%, and this leaves (35% - 3.5%) = 31.5% of the available wind energy being converted to electricity from the generator. Notice that if the wind velocity increases by a factor of 2, that the increase in available power is then 2^3. (2v)^3 = (2^3)(v^3) = (8v). In short, when the velocity doubles (n=2), the volume of air upon the blades has likewise increased by (n^3) = (2^3) = 8. As a verification to the above (basic) formula, excluding the general energy transfer efficiency factor, consider:

Pressure = F / A = Energy / Volume = KE / V = pv^2 , mathematically: KE = pv^2 V , and:

Pw = Energy / time = KE / t = p v^2 V / t = p v^2 d^2 d^1 / t = p v^2 A (d / t) = p v^2 A v = **p A v^3** : J / s

AN EXAMPLE AND ANALYSIS OF RPM AND SPEED

A certain type of car is being designed with wheels having a radius (R) of 1 ft. How many RPM will be needed of those wheels so as the vehicle will go 20 mph?

Circumference distance of the wheel = C = 2(pi)R = 6.28 R = 6.28 (1ft) = 6.28 ft

Setting up a proportional type of equivalent fraction equation:

$$\frac{\frac{1\text{mi}}{1\text{ hr}}}{\frac{5280\text{ ft}}{1\text{ hr}}} = \frac{\frac{20\text{ mi}}{1\text{ hr}}}{\frac{X\text{ ft}}{1\text{ hr}}} \quad , \quad \text{After solving for X ft / hr :}$$

$$\frac{x\text{ ft}}{1\text{ hr}} = 105600\text{ ft / hr} = 105600\text{ ft / }3600\text{ s} = 29.33\text{ ft / }1\text{ s} = 20\text{ mi / }1\text{ hr}$$

If 1 revolution is a distance of C = 6.28 ft, then how many revolutions are needed to make 29.33 ft?

29.33 ft / **(**6.28 ft / 1 rev**)** = 4.67 revs and 4.67 revs / s = 29.33 ft / s = 20 mi / 1 hr

(4.67 revs / s) (60s / min) = 280.223 rev / min = 280.223 RPM

How Much Weight Can A Gas Filled Balloon Lift?

Various lighter than air gasses and-or mixes in a light-weight container such as a balloon can be used to lift a weight vertically, against the downward pull or force of gravity, and up into the air. Some examples of the gasses are: methane, hydrogen and helium. The example below will consider hydrogen and its specific density, but the calculations will be similar for any gases used.

The main reason it rises vertically is nearly identical to that of an air bubble rising vertically the water, and that is because of the buoyant force upon it. For an object to rise vertically in water or air, its average density (hence average weight also) must be less than that of the medium it is in; either water or air. The gasses mentioned are lighter (ie., less dense, hence less weight also) than the density of air.

The density of air at standard temperature and pressure (STP) is: $1.24 \text{ g} / \text{L} = 1240 \text{ g} / 1000 \text{ L} = 1.24 \text{ kg} / \text{m}^3$

The density of hydrogen at STP is: $0.0000838 \text{ g/cc} = 0.0838 \text{ g} / \text{L} = 83.8 \text{ g} / \text{m}^3 \approx 84 \text{g} / \text{m}^3$ = mass per cubic meter, hence often said as the "mass-weight per cubic meter"

Effective Buoyant or Lift Force = Upward or vertical buoyant Force - Downward weight of the entire balloon system

Effective Buoyant or Lift Force = Weight of the air displaced - Weight of the entire balloon system

Considering just 1 cubic meter of hydrogen gas as the most basic weight of the balloon system =
1 m^3 of hydrogen, and which displaces 1 cubic meter of air:

Effective Buoyant or Lift Force = $1.24 \text{ kg} - 84\text{g} = 1.24 \text{ kg} - 0.084\text{g} = 1.156 \text{ kg} = 1 \text{ kg} + 156 \text{ g} \approx 2.54\underline{9}$ lbs

This calculation for a hydrogen gas filled balloon can be expressed as: **1.156 kg / m^3** = $2.5\underline{5}$ lbs $\approx$ 1kg / 1m^3

To lift a mass or load of 10kg = 22.0462 lbs will require a hydrogen balloon with a minimal volume of:

$$\textbf{V m}^3 = \frac{\text{(mass of the load)}}{\dfrac{\text{(reference lift mass) per}}{\text{(reference volume)}}} = \frac{\text{(mass of the load)}}{\dfrac{(1.156 \text{ kg})}{(1 \text{ m}^3)}} = \frac{(10 \text{ kg})(1 \text{ m}^3)}{(1.156 \text{ kg})} = 8.65 \text{ m}^3$$

:This equation can be checked by rearranging it to show the equivalent proportions or fractions

The less the weight being lifted by the balloon, the greater the effective buoyant or vertical lift force will be and the faster the balloon will rise upward in the sky. The buoyant force must be greater than the load weight for a balloon to rise.

As a balloon rises into the less dense atmosphere of air, the weight of the air displaced will become less and the buoyant force becomes less. The balloon will decelerate due to the less buoyant force upon it pushing it vertically upward. The balloon having an internal gas pressure will also expand due to less external air pressure upon it in the higher atmosphere. Though this expansion will displace more air, it is less dense air, hence less weight, and less buoyant force.

The radius (r) and diameter (d) of the above balloon is:

$$\textbf{Vs} = \frac{4 (\pi) r^3}{3} = (1.333)(3.141592654)(r^3) = \mathbf{4.1887902\ r^3}$$

: **Vs = volume of a sphere**, such as a balloon and regardless of balloon shape, its the gas volume

$$\textbf{Vs} = \frac{4 (\pi) (d/2)^3}{3} = \mathbf{\frac{(\pi)\, d^3}{6}} = \mathbf{0.523598775\ d^3}$$

$$\mathbf{r} = \mathbf{\sqrt[3]{Vs / 4.188\underline{8}}} = \mathbf{0.620350\underline{5} \sqrt[3]{Vs}}$$

, and for the example and values given above; Vs = 8.65 m^3:
:**radius of a sphere = (diameter of sphere) / 2 = d / 2**

$r = 1.2734\underline{3}$ m

$$\mathbf{d} = \mathbf{2r} = \mathbf{1.2407009\underline{8} \sqrt[3]{Vs}} = 2\,(1.2734\underline{3} \text{ m}) = 2.54\underline{7} \text{ m}$$

:**diameter of a sphere**

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A GENERALIZED LITHIUM BATTERY ANALYSIS

Lithium batteries have been mentioned in this book previously on several occasions. The main goal here is to show the amount of charge remaining in the battery via its voltage. The amount of charge left or remaining in a battery is also called the current state of charge (SOC).

Cell type considered: LiPO4 , size = 18650 , cell voltage rating = 3.2v

4 of these cells in series (S) will create a battery with a voltage rating of: (3.2v)(4) = 12.8v

The minimum voltage for each cell before it has a high internal resistance is about : 2.5v , and at this value, the
4 cell battery voltage will be: (2.5v)(4) = 10v

The maximum voltage for each cell before it can be begin to be shorted internally if made higher is about 3.65v, and at this value, the 4 cell battery voltage will be: (3.65v)(4) = 14.6v . For safety, the battery is not charged up to this voltage, but rather halfway between: 12.8v and 14.6v and this value is (12.8v + 14.6v) / 2 = 13.7v. At this voltage, the battery will be considered as having 100% of its charge still remaining in it.

Even with the above values stated, do not consider the 10v, lowest state of the battery as still in the useful rating of say for a 12.8v system. The battery still has charge and energy, but it is well below the rated values: (V , aH, Wh). For any voltage less than this rated battery voltage of 12.8v, the battery is considered drained of charge and in need of being recharged.

The aH or amp-hour rating of the battery depends on the aH capacity rating of each of the similar cells in series, and it will be the same value as each cell. If the cells are in parallel, the aH rating will be the total sum of each battery in parallel, hence if there are (n) parallel batteries in parallel, the aH rating will be (n)(aH).

The Wh or watt-hour , or "energy or oule rating" of the battery will be:
From: P = energy / time , energy = (power)(time) = energy = (VI)(time) =
energy = (rated Voltage of the battery) (aH rating)(time) : = total energy available in joules

As the voltage level of the battery decreases, the amount of current and power it can then deliver is less. The aH and Wh ratings are rather for the average value between 12.8v and 14.6v. This average voltage is: (14.6v + 12.8v) / 2 = 13.7v. To have a steady or constant amount of current will then require a voltage regulator or constant current regulator.

When this battery has between 20% and 90% of charge left in it, the battery voltage will decrease at a relatively slow rate with respect to time and-or usage under load, and this is an amazing thing with a battery such as a **LiFePO4** (Lithium-Iron-Phosphate) battery. The current available will be near the battery's aH rating, and the power available from it will be near the Wh rating of that battery. As of the year 2024, tt is unclear if there wiill be batteries superior to the LiFePO4 in terms of energy density (joules / weight or volume) and price which is about $2USD per aH for a 100 aH battery as of 2024.

If a battery is rated as 12.8v, then any value less than this, the battery cannot supply its rated values, and hence when this battery voltage is at 12.8v or less, it is considered as drained of (necessary) charge, and in particular, it cannot supply its aH and Wh rating.

Here is a chart that considering the LiFePO4 battery being discussed above as it is drained of charged (ie., current):

Battery Voltage	A typical % of the battery's rated charge remaining	
13.7	100	: sometimes the battery voltage might be charged to a slightly higher value. 3.7v is a typical value
13.63	90	
13.56	80	
13.5	70	
13.42	60	
13.36	50	
13.29	60	
13.22	50	
13.16	40	
13.09	30	
13.0	20	
12.85	10	
12.8	0%	: at this voltage, the aH and Wh rating cannot be sustained for lower voltages.

Essentially, each 10% difference in voltage represents 10% of the aH and-or Wh rating of the battery. If the total aH rating of the battery is 50aH, then each 10% change lesser in voltage essentially represents that: (50ah)(0.10) = 5aH of the total energy and-or aH in that battery has been used or "drained" and is no longer available for use.

Note that (13.7v - 12.8v) = 0.9v , and this is only about a 1v range between fully charged and what is considered a drained battery of its sustainable rated aH rating. (1v drain / 4 cells) = 0.25 v drain per 1 cell. This is also the voltage increase of each cell above its rated voltage of 3.2v. Each cell when the battery is charged to 13.7v will have a voltage of about: (3.2v + 0.25v) = 3.45v.

The above percent (%) values can also be considered as the percent of the aH rating left. The watt-hours remaining is the voltage remaining times the current at that voltage:

Wh remaining = (V remaining) (aH remaining) =
= (Power drawn)(hours of useage)

Mathematically: (hours of usage) = (Wh remaining) / (Power drawn)

Why not just simply charge the 12v rated battery to 12v?

The simple answer is that if the battery was charged to only 12.0v, the battery voltage would soon go less than this value which is its rated value, and then it may not be able to sufficiently power devices rated for a 12v input supply..

An Analysis of an Example For Consideration:

Mathematically, when the voltage and-or battery's energy is reduced by a certain percentage, the current available through the same load or resistance will reduce by that same percentage.

Ex. A 12v battery across a 10 ohm resistor.

Ir = Vr / r = 12v / 10 ohm = 1.2 A

Pw = Vv Ia = (12v)(1.2A) = 14.4 W

If the battery voltage got reduced from drawing power from it and if its voltage is now 11.8v, the voltage has decreased by: (12v - 11.8v) = 0.2v and the new battery voltage is: (11.8v / 12v) = 0.9833 or= 98.33% of 12v. In terms of the percent reduction in voltage, this is: (100% - 98.33%) = 1.67% . The voltage has decreased by 1.67%. Then:

Ir = Vr / r = 11.8 / 10 ohm = 1.18 A , a decrease of (1.2A - 1.18A) = 0.02A

(1.18 A / 1.2 A) = 0.9833 : A current reduction of: (100% - 98.33%) = 1.67%

Pw = Vv Ia = (11.8v)(1.18A) = 13.924 W

A power reduction of: (14.4W - 13.924W) = 0.476 W less , and as a ratio:
13.92 / 14.4 = 0.9667 , And this is a reduction of : (100% - 096.67%) = 0.0333% and (1 - 0.9667) = 0.0333

0.476W / 14.4W = 0.03306 = 3.306% when the voltage and current were reduced by 1.67%, hence the power reduction is twice that of the voltage and-or current reduction: 2 (1.67%) =~ 3.33%

When the voltage decreased to 98.33%, current decreased to 98.33% of its previous value. The product of these two relative values is: (0.9833)^2 = 0.96694 =~ 96.7% = 13.924W / 14.4W which is the ratio of the new voltage to the previous voltage before the voltage drop.

In short, when the voltage decreases by a certain percentage, the current (I = V / R) , will decrease by that same percentage since they are directly related and proportional in value. The total reduction in power here is equal to the square of the percentage of the voltage and-or current reduction, hence the available power and-or energy density is **rapidly decreasing** for each decrease in voltage and-or current. Fortunately, the internal structure and chemistry of a lithium type of battery allows it to maintain its rated (or slightly higher) voltage for a longer time of use as compared to that of a lead-acid battery of which must then be charged more often so as to maintain its rated voltage and-or energy (Wh). In short, the internal resistance of the lithium battery has a region where it does not change too much, and therefore, the battery can maintain its rated voltage and current for a longer period of time.

It is of note that when a battery voltage is less than its rated voltage, it is not safe to still use that battery to power a device rated to be supplied with that rated voltage. Some devices may work to say as low as 80% of that rated voltage. Much energy (ie., charge, electrons) is still within the battery when it is less than its rated voltage, and it is possible to uses it to power lower voltage devices if a voltage regulator is between it and the battery. **Using a battery below its rated voltage is not recommended, for it can damage the internal composition and capabilities of the battery to store charge again.**

Checking the battery voltage:

For any battery, when a (current, power) load (especially a low resistance) device is connected to it to be powered by it, the internal resistance of that battery will develop a voltage drop across it, hence the voltage to the load will be less than measured by a high resistance voltmeter:

Vload = (Vbattery unloaded - V loss due to the internal resistance of the battery when powering the load.)

You can also easily measure Vload using a voltmeter across that load when it is being powered.
Possibly, a power resistor to simulate a certain expected load or device can be used as a substitute load so as to measure Vload.

Power loss in the battery:

Because there is a voltage drop internally in the battery, there will also be a power loss within the battery, and it will not then be available to the circuit and-or load device. Pload = (V load)(I load). When the charge in a battery is depleted, its internal chemistry will temporarily change until the next recharge. The more the battery is depleted, the greater the internal resistance, voltage drop (ie., loss) and power loss. The battery's ability to sustain the current for a long time (ex. 1 hour) will be less unless if it is a battery of great size and charge storage capacity to initially begin with.

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A POWER INVERTER STUDY AND EXAMPLE

A power inverter has been discussed in a few places previously in this book, but here is some additional information which may help some people have more of an understanding.

A power inverter is much like the concept of a voltage transformer, usually to increase the output voltage. For example a 12Vdc can be converted to 120 Vac, hence the voltage gets multiplied by a certain factor, here 10, however this also divides the available current by 10 since in general: power in = power out.

A power inverter that we are interested in for this example has a 12Vdc power supply, such as a battery, as its input power.

The battery can be a car battery if the engine is running and charging that battery while current is being drawn from it. The battery can be a "deep cycle" battery or a type of rechargeable battery such as a lithium battery (ex. LiPO4 = lithium iron phosphate).

It is best to use an inverter with a pure sine wave output ,however some devices such as heaters may work reasonably well with the square wave type of output. The AC output of the power inverter should be 120 Vac.

Ex. To power a 600W AC electrical device using a 12V battery.

If the voltage is increased to 120V ac, that is (120v / 12v) = 10 times higher.

P = V I
600W = (120V)(I) : (volt)(amps) = watts = joules / second mathematically:
I = P / V = 600W / 120v = 5 amps

Now consider 5 amps being drawn from the battery:

P = V I = (12v) (5A) = 60 watts, and this is not enough power to power a 600W device. Surely then, more current is needed to obtain the needed 600 watt device.

I = P / v = 600W / 12v = 50 amps from the battery.

Now compare: The voltage gets increased by 10 , from 12v to 120v. The current needed, here 5A through the load, increases by a factor of (50A / 5A) = 10 from 5A to 50A at the battery. The 50 amps drawn from the battery will get decreased by inverter to just 5A. Still:

Pin = Pout

Ex. If an inverter is rated at 2000W maximum rated output ?

The maximum current from the inverter is then: P = V I , I = P / V = 2000W / 120v = 16.67A

For the battery: P = V I , I = P / V = 2000W / 12v = 166.67A hence 10 times more, and this is due to the voltage step-up transformer, and of which the current is reduced 10 times, and therefore the battery must supply 10 times more current so as to have the same amount of power.

Note that an inverter is often only about 85% to maybe 95% efficient at the power conversion, and with an average of bing 90% efficient, and there is about 10% of the input power lost as heat, and it will then not be available at the output. Since an inverter is generally guaranteed to be 110Vac to 120Vac, the power lost is generally of the current available, and also due to that the battery voltage is generally a fixed value of about 12v. For the above example:

Total Current Needed Due To Likely Inverter Power Losses = Basic Current + 10% Basic Current =

166.67A + 16.67A = 183.4A

The battery must be able to allow this much current from being drawn or taken out from it. If the battery Ah is insufficient, you will then need to consider parallel batteries to increase the total Ah available. The wire or cable used from the battery to the inverter is generally short, and must be able to handle this maximum amount of current through it. Using a fuse will help protect the battery and-or inverter from short circuits. It is best to use a DC fuse before the inverter, and an AC fuse after the inverter. Electro-Magnetic circuit breakers can also be used, and some inverters may have this included in the design. While fuses (an inexpensive type of circuit breaker) need replacing, (electro-magnetic) circuit breakers can be easily reset manually if the circuit problem no longer exists.

To reduce current drawn from a battery by half, consider using a 24 volt battery and inverter system. This battery can be purchased or created from two 12v dc batteries connected in series. Positive (+) to negative (-), and-or negative (-) to positive (+).

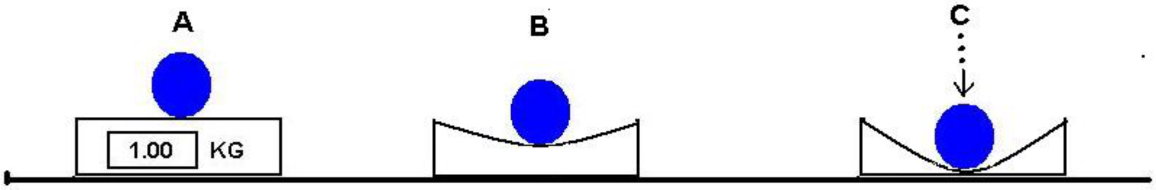
To further reduce the current (ie., which comes from the saved charged over time) needed and drawn, you can also consider a 36V or 48V system. Considering the above 2000W load example with having a 48v rated battery and a 48v rated inverter:

I = P / V = 2000W / 48V = 41.67A : this is 4 times lower in current, and the wire from the battery does not then need to be so thick, but it must be capable of safely passing 41.67A, and its current rating must then be a bit more, and giving a 10% safety margin: 41.67A + 4.167A = 45.85A The fuse used between the battery and inverter should then be based on this value of current, and the fuse's current rating should be of just a slight amount of amps higher at say 46A to 50A..

Typically, a 48v battery is essentially composed of four 12v batteries connected in series so as to increase the available voltage. The aH rating of a 48v battery is typically 4 times more than that of a 12v battery. Drawing less current from a battery means it will also charge up faster, and-or have more time to power a device(s) since the aH and Wh ratings are higher for a charged 48v battery.

AN IMPACT STUDY

Impact has been previously mentioned in this book, and so here is some extra data and equations about it that you may consider and occasionally encounter. Here is a simple illustration to consider before discussing the concept of impact further. [FIG 312]



A weight to mass scale

Has a rigid metal surface which does not visibly bend unless the weight is very high in value.

Since gravity is still acting upon the ball, the weight will always be shown. Since the surface is rigid, the ball cannot change in velocity and position.

The downward force or weight of the ball is constantly and effectively cancelled by the surface tension and its force in the opposite direction.

Here, the surface is not as rigid and can bend more easilly, and this helps us visualize the constant weight = force. For example, a thin metal, plastic or rubber surface like a balloon, and which can stretch . If the ball is removed, the surface will be restored to its original flat shape if the weight and-or impact was a low value and the surface was not permanently deformed.

Here, the ball and surface is shown after it was dropped from a height. The impact force is greater than the weight of the ball, and the curve is deeper, but since the weight is the same, and the surface was not permanently deformed, the curve will then become as shown previously at B. This is due to the elastic tension force in the stretched rubber surface which will raise the ball upward slightly after its gained motional energy is converted to and dissipated as heat energy.

For the difference between the words "impact" and "impulse", these two words are sometimes used interchangeably for a "collision". but there is a more strict definition. In brief, the word "impact" is another word for "collision" or "contact" of two objects, and the word "impulse" is used to describe the temporary application of the force during the impact.

Because of equal and opposite forces, when object A impacts object B and places a force upon object B, object B will place an equal and opposite force upon object A.

If object B has a great mass, it will have a great inertia and it will not move much due to an impact, and it will rather move more easily when more energy has been applied by a force.

An object or mass in motion or velocity is said to have momentum. Momentum ($p = mv$) is a general measure of that objects ability to keep moving. Momentum is like the opposite in meaning of inertia which is an objects ability to initially resist motion and-or the applied force meant to cause motion. If a force is applied in the opposite direction to its motion, that object will then reduce in speed in the direction it was going during the time that the force is applied. If a small force is applied for a long enough duration in the opposite direction to an objects motion, the momentum of that object will eventually be reduced to 0. What is happening is that the object will constantly loose some kinetic energy ($KE = mv^2 / 2$) when that force is applied.

If object B has a velocity during an impact in the opposite direction to that of object A, object B has momentum also, and the force of impact will be greater.

If the mass is doubled in the above figure, the scale will show twice the mass and-or weight. This indicates that the force upon the scale is twice or 2 times more. If the mass is more, the bend in the supporting surface will be greater, and for a generalized approximation, the bend will be twice as deep if the mass and-or weight is twice as more. Extra: In essence, the bend amount can be used as a mass and-or weight scale itself, much like a spring scale of which the calibrated length of the spring indicates the weight applied to it.

If the mass is not 2 times more, the force and-or weight can be 2 times more if it had kinetic energy (ie. motional energy) that resulted in the apparent weight (ie., a force) of the object being two times more.

When an object is lifted a height, it is effectively gaining and storing gravitational potential energy (GPE) due to its position. At rest at this height, it will appear to have gained no energy, and in fact having no kinetic energy (KE), but once it is allowed to fall in the presence of the constant force of gravity, it will constantly increase (ie., accelerate, and change) in velocity and gain kinetic energy. $KE = mv^2 / 2$.

Now to find the relationship between force (the application of energy) due to an object with KE.

$F = ma$, $m = F / a$, $F = ma = m \ (\text{ch } v) / (\text{ch } t) = m v / t$, and if the changes were from $v=0$, and $t=0$

Note that if the force applied is constant, the acceleration is then a constant and not an average value: $a = F / m$, and then what would then be increasing is the velocity, momentum and kinetic energy of the object.

$KE = m v^2 / 2$ and mathematically: $m = 2 (KE) / v^2$ and $v = \text{sqrt} [2 (KE) / m]$.
Note that: $(2 / m)$ and-or $(m / 2)$ is a constant value.

Mathematically: Since $F = ma$, $m = F / a$ and:

$KE = mv^2 / 2 = F v^2 / 2a$ and since: Energy = Work = $F d$, $F = \text{Work} / d = KE / d = KE / v t$,
$(KE - GPE)$ = Total Energy = $F d$

In gravity, $a = g$: , if the acceleration is actually deceleration: $a = -g$, and $F = ma = mg$, $F = m -g = - mg$

Some extra formulas to consider:

$F d = W$ joules = Energy = F (height) joules = (Weight)(height) joules = $mgh = KE = GPE$, mathematically:

$F = KE / h = GPE / h = ma = m v / t = (m d/t) / t = m d / t^2 = m h / t^2$: if $a=g$ is constant , and

$F = m h / t^2 = m h / (d/v)^2 = m h / (h^2 / v^2) = m v^2 / h$

$F = m a = m (d^2 / t^2) / h = m (h^2 / t^2) / h = m h / t^2$: This shows that F is directly related to height

$F = m h / t^2 = md / t^2 = m d / t t = m (d/t) / t = m v / t$: This shows that F is directly related to velocity

$F = m h / t^2 = mv^2 / h = m v / t$: $h = d$ = distance lifted vertically. Note that when the height is higher, the time needed until impact will be more.

$F = m a = m v / t = m v^2 / d$, hence these factors are equal: $v / t = v^2 / d$ mathematically:
$d / t = v$: checks

The force and-or weight on the scale is equal to the weight of the object + the temporary force due to any gained kinetic energy of the mass being applied to the scale also. This KE came from, and is equal to the GPE it gained initially:

Energy in = Energy out , $KE = GPE$, $KE - GPE = 0$ Joules

Gravitational Potential Energy gained from the reference position or height = $GPE = Fn\ dm$ = work in joule units = (F in newtons) (height in meters) above the reference ground or 0 energy state.

$F = m a = m v / t = m (d / t) / t = m d / t^2$.

$d = a t^2 = g t^2$, after multiplying each side by F, we have:

$F d = F a t^2 = F g t^2$ = work = energy = $KE = GPE$ mathematically:

$F = GPE / g t^2 = GPE / d = GPE / v t = GPE / (at) t = GPE / a t^2 = KE / d = KE / v t = KE / at^2$
Showing that F is directly related to Energy in general.

$Fd = mad$ = energy and F = energy / d

If d is low, much like the contact or impact distance, a high F will be applied will be in a short amount of time and distance, however the total KE can not change and is the same.

force / area = Pressure (ie., the effective force per unit of area) can be effectively increased if the area is smaller and the energy is then concentrated and-or distributed in a smaller amount of area. If all the energy is released quickly in a short amount of time, then (d) will be lower in: $F = KE / d$, and the impact force (application and-or transfer of energy) will then be greater.

$F = KE / d = E / d$: if the energy is spread out over a long distance, the energy per unit distance will be lower, and the Force per unit distance will then be lower. In general, a longer distance will take a longer time given the same or lower velocity. A lower velocity then implies a deceleration due to a force such as the equal and opposite force of the impact.

$F t = m v = p$ = momentum = a joint measure of the motion of a mass and its velocity.
$F t = m a t = m v$

$F = m a = m v / t = m v^2 / d$, $v / t = v^2 / d$: note: $v^2 / v = v = d / t$
and: $a = v / t = v^2 / d = d / t^2$, and: $v^2 = d a$

$W = KE = F d = ma\ d = ma\ a t^2 = mg\ g t^2 = m\ g^2\ t^2$: $d = v t = (at) t = a t^2$
$W = KE = F d = ma\ d = m a$ (height) $= mgh = GPE = mv^2 / 2$, mathematically:

$W = KE = GPE = mg^2 t^2 = mv^2 / 2$: of which many single variables (m, g, t, v) can be solved for
Also: $(d/v)=t$, $t^2 = (d/v)^2 = d^2 / v^2$, $d^2 = v^2\ t^2$
Since F=ma, $m = F / a$, and substitution this, we have:

$W = KE = GPE = F d = F a t^2 = F v^2 / 2a$

$F = ma = mg = GPE / d = KE / d$: If $KE = GPE$ doubles, then F will double , they are directly related
If F doubles , KE doubles since Fd = work = energy = KE joules

F = m a = m d / t^2 = m d / t^2 = m d / (d^2 / v^2) = m v^2 / d = m (d/t)^2 / d = m d^2 / t^2 d = m d / t^2

The higher the object is in distance (h = d), the more GPE it will have at its maximum height, and then when the object is allowed to fall, the more KE (which includes an increasing velocity due to the constant force of gravity applied to it and applying energy to it) it will have just before impact.

Energy = Work = F d = ma d = ma h = P A d = P A h = P d^2 d = P d^3 = P V : From: P = F / A , F = P A
Note in PAh , for a given energy and P, if A is small, h is then large due to an inverse relationship
GPE = KE = Work = Fd = mad = mah = P V , therefore: P = F / A = GPE / V and V = GPE / P
F = GPE A / V = GPE d^2 / d^3 = GPE / d
P = F / A = (GPE / d) / d^2 = GPE / d^3 = GPE / V
GPE = P V , V = GPE / P

Impulse Force = the force created during an impact = change in momentum / change in time =
= (mv2 - mv1) / t , if the impact time is quick, then the force is very high, and the energy is exchanged quickly. When the change in momentum is high and happens in a short amount of time, all the available energy is transferred during that short amount of time, and the application of that energy (ie., Force) will be high.

F impulse has units of Newtons = m a = (mass kg) (meters / s^2)

F = m a = m (chv / cht) , from this: chv = F t / m : the larger the mass, the more inertia it has, and the less the change in velocity will be for a given amount of Force and time it is applied.

chv = Ft / m = change in momentum / m , therefore: change in momentum = m chv
units of momentum are those of mass and velocity , hence ,
mv = (kg)(m/s) = kg-m / s

P = F / A : if the impact area , or force application area, is smaller, the pressure applied to it is then higher, as if the impact force was greater at that region, and the damage in that area or region will most likely be greater as more (KE) energy is transferred to that smaller area. It is essentially concentrating the (constant) energy available and applying it to a smaller area and it will affect then that area more. Constant in mentioned since regardless of the area, the total energy is usually constant and-or just before impact. For the impact force to increase, the available energy must first increase before impact where pressure will be applied to the area. As an extra note, if the area is simply 1 unit of area, then the pressure value will have the same numeric value as the force.

P = F / A = m a / d^2 = m a / (v t)^2 = ma / v^2 t^2 : d = v t , d^2 = (vt)^2 = v^2 t^2 , ,
a = ch v / ch t = v / t from t=0

Given a weight = force = ma , if a (lifting, raising) force is applied in the opposite direction, it can lift that object to a height, and the larger the upward force being greater than the (downward) weight force, the less time it will take. **

h = d = v t , F d = work = energy = KE = GPE = F v t , F = energy / v t = energy / d , v = energy / F t
F = weight = ma = mg = energy / d = energy / h

If the energy was transferred in a short or low distance value, then the F will be high. Since energy = F d , the longer the distance, the more energy needed . Note that d = vt , and that if it is a constant , if one variable is increased by (n) the other is decreased by (n), and then Fd = work = energy will remain constant , for example given a limited or constant value of: GPE = KE = energy = work

h = d = GPE / F = KE / F = Work / F
d = v t = GPE / F , GPE = F v t = F d

** t = GPE / Fv , v = GPE / Ft , GPE = F t v = F d

No matter how fast or slow an object is raised, the GPE it stores will be the same value for a given height, and the KE needed to put it there is always be the same.

If a force of 1N is applied for 1s to an object, the impulse (I) is:

I = (change in momentum of the object) = F (change in time) = (1N)(1s) = 1N-s

If this force was applied in the direction that the object was already traveling or moving, this would add to the objects momentum and kinetic energy. If the mass remains the same, only velocity will increase.

F = m a = m (change in velocity) / (change in time) , mathematically:
F (change in time) = m (change in velocity) = (change in momentum) = impulse =
= impulse = Δ p = Δ N-s = m (change in velocity)

(change in velocity) = (change in momentum) / m = (impulse) / m : m = mass

Due to the inverse mathematical relationship shown in the above equation, he larger the mass for a given impulse upon it, the smaller the change in its velocity, and this is due to the inertia of that mass, and that it simply takes more applied energy (ie., force) to move a larger mass.

From: F = ma , a = m / F , the larger the mass for a given force the less it will accelerate and the less change in velocity it will have. To have the velocity change by a factor of (n) for a given mass, the change in momentum = F t must then change by that same factor of (n).

(change in momentum) = m (change in velocity)

m = (change in momentum) / (change in velocity). : for a given mass, m, to maintain the equivalent fraction, if one factor increases by (n), the other factor will decrease by that same value of (n).

m = Fn / a = (change in momentum) / (change in v)

In an "**elastic collision**", the amount of energy lost during a collision is a very low value, and if 0 joules is lost, it is a theoretical, "pure elastic collision". A simple example would be a ("super") rubber ball that when dropped form a height, it will nearly bounce or rebound back up to its original height.

In the first studies of impact and force, dropping a mass from a height and observing the dent in the soil, would make a person naturally believe that the force of the impact is directly related to the mass and its velocity. $F = mv$, but Newton realized that if the object strikes a moving object then the applied force is not as much when it is applied in the same direction, and the equation was rather modified to include acceleration or deceleration which is a variable which does include velocity(s): $F = ma$.. All of the Kinetic Energy of the object colliding with the other may not be transferred to the other object, but only some will , and this will increase its Kinetic Energy. If the velocity of an object has been reduced to a value of 0, then all of its kinetic energy has been transferred elsewhere, and most likely to the object it collided with.

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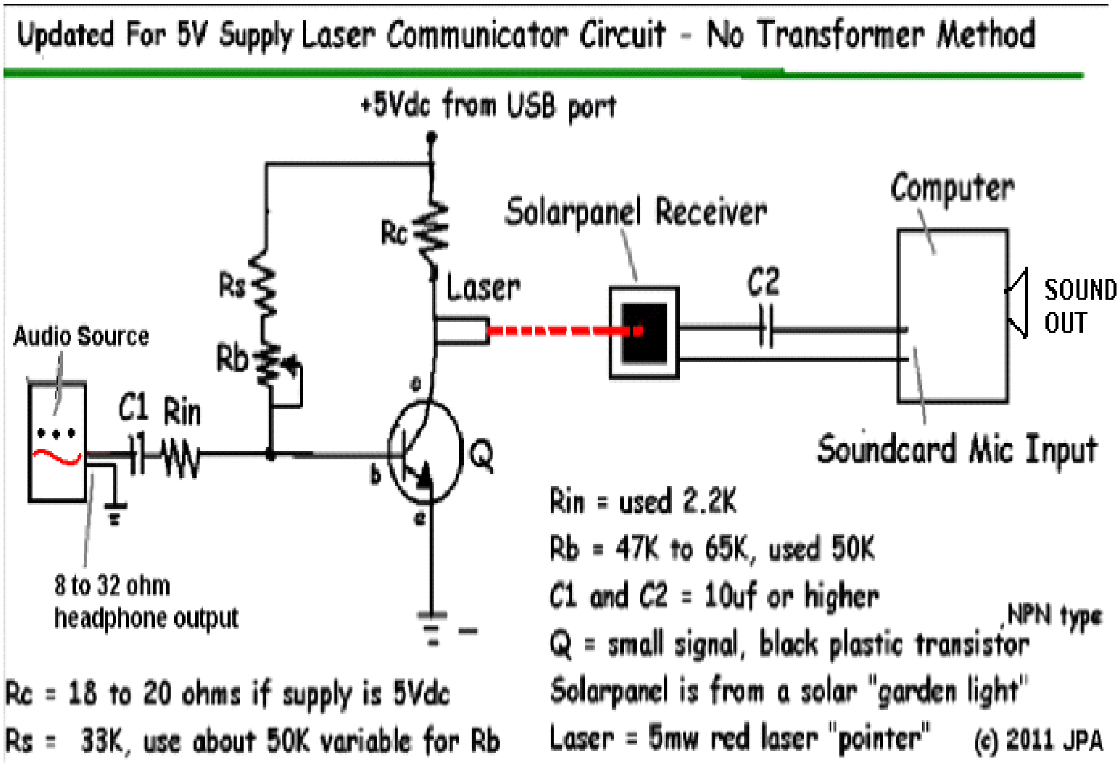
A CIRCUIT TO TRANSMIT A SOUND SIGNAL USING A LOW POWER LASER

This circuit will transmit (ie., send) an electronic audio signal using a few inexpensive electronic parts. **Please use proper eye protection for lasers, and be responsible with lasers so as not to do any harm to people, animals, plants, objects and-or cause a public disturbance**. Lasers can also transmit digital or computer data in the form of quick high power pulses. This data will then need to be decoded by a microprocessor (ie., a computer) and computer program so as to do so.
[FIG 313]



The input (electric form representation of an audio signal) signal will be used to modulate (ie., vary, control, adjust) the amplitude (ie., energy or signal intensity) of the signal output from the laser. The solar panel will convert the light intensity of the laser light received to a corresponding intensity of a electric signal.

As indicated, the solar panel is from a garden light, and that all may not work too well for this circuit and laser light, and it may take some experimentation. The area of this small solar panel was about 1 square inch.

The laser used was an inexpensive, $1USD, low power, legal, consumer, 5 milli-watt (ie., 0.005W), LED (with red colored light) laser pointer, however, premade laser diode "modules" and-or circuits are available for experimenters and-or for replacements. You will need to determine the voltage polarity of the input power supply pins of the laser and-so as to prevent the laser from being damaged. Do not look directly at the laser beam, and-or for too long at it striking and-or reflecting from a nearby surface and still having a high intensity. The light or photon output of a laser diode has a single (rf, electro-magnetic) frequency based on its construction, and is therefore called a monochromatic light or a light of a single color. The laser used for this circuit emitted a red light of about 650nm (nano-meters) in wavelength. A green laser light would have about a 525nm wavelength, and violet laser light would have about a 425nm wavelength.

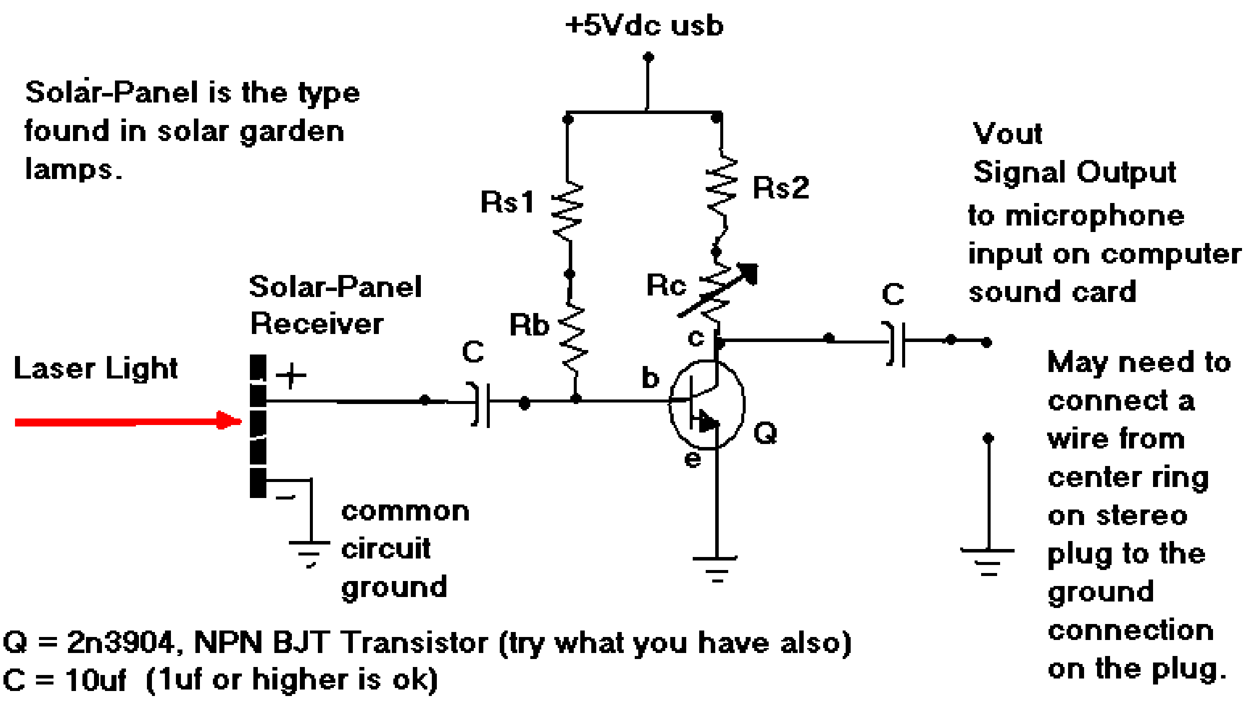
(Q) is a small black plastic NPN type. You can purchase a new quantity of these and other parts (ie., components) relatively cheap from websites such as Ebay.com It is also good to get a dozen or two of "spring clip hookup or connection leads" to easily connect components together for testing circuits, and without the need to solder them just to test them.

Rs is basically a "safety resistor" so that if you turn Rb all the way to a low value, excessive base (and-or collector) current will usually damage the transistor. Rc is included to prevent the voltage across the laser from going over 4.5Vdc which would either prematurely age the laser and reduce its light output, and-or cause it to be burn out and be of no use.

To increase range, various light concentrators can be experimented with. The simplest is a funnel or cone shape having an inner surface made of a reflective material. Another simple method is to uses a magnifier so as to focus and concentrate the received light energy onto the solar panel. Mirrors can be used to direct a laser beam to a certain direction.

The circuit shown below in the next figure is essentially a preamp circuit to help amplify or "boost" the signal output from the solar panel before it is sent to the computer microphone input so as to be amplified. There are many things to experiment with these laser circuits, so be patient and keep notes of what works and what doesn't. There are also free oscilloscope programs available that work well with most audio signals, such as for setting the maximum level before "clipping" or signal distortion happens due to too much amplification. These programs use either the microphone and-or line input of the sound card for the input signal to be displayed on the screen. The laser circuit above generally works well without the need for the preamp shown below. [FIG 314]

Solar Panel Receiver Amplifier Circuit For Laser Communicator



Q = 2n3904, NPN BJT Transistor (try what you have also)
C = 10uf (1uf or higher is ok)
Rs1 = 1K safety resistor, Rb = 1M (fixed or variable)
Rs2 = 100 ohm safety resistor, Rc = 5k Variable resistor
Adjust Rc for best output

(c) JPA 2017

Some solar garden panels from solar lights will work better than others. They type I used often had several smaller solar cells in it. If the circuit oscillates and produces noise or a tone, try putting about a 100pf capacitor across the transistors collector and emitter leads, or across Rc. The laser and-or preamp circuit is sometimes sensitive to the light frequency radiation or 50hz or 60hz noise from household lamps and some flashlights that boost their internal power supply voltage and frequency using some type of "voltage step-up" circuit. An **op-amp (operational amplifier, or opamp)** on a IC chip is a small, very high gain amplifier circuit option to experiment with instead of using regular transistors, however for powerful pulses (brief, short duration transmission) of advanced laser communication, the output circuit often uses a high gain power transistor or MOSFET (metallic oxide field effect transistor). A high impedance (usually desired to be resistive only, with minimal capacitance and inductance of which can limit input frequencies and-or short some signal to ground) op-amp is ideal at the first amplification stage for low power RF signals such as for a crystal radio receiver. To amplify weak signals with a small voltage, it greatly helps to have an amplifier with a high input impedance (usually desired to be just an ideal, infinite resistance or resistor, hence with no capacitance or inductance involved, and with their reactances, and which can also cause some AC signal loss to the ground and wasted rather than amplified) so as to apply the full received signal voltage to it rather than share it with another resistor in a voltage divider and then waste part of that signal. With a high input impedance, there will surely be not much of any current flowing into the op-amp. The op amp is much like a FET that is designed to function by and-or amplify voltages rather than current like a typical BJT transistor amplifier would which requires some input current to function. an FET is said as being voltage operated, and a BJT is said as being current operated. For some more about op-amps, please read the next note further below.

The author has used this basic laser system to transfer a text file, and then a small image file over a short dance of several feet, and the process is somewhat similar to the **Slow Scan TV (SSTV)** concepts developed later which are used to transfer small images or text files via an audio signal transmitted via a radio wave signal. Used was a program to convert audio signals such as from a common .wav file used to store audio data as a computers digital or digitized data form. The program was based on the **Kansas City Standard (KCS)** = **BYTE Standard**. that is used to convert digital byte data to a coded audio data, and for an older computer system, it was then stored by using a modified (audio) tape cassette. The tape cassette is not needed in this situation since the computer itself can be used to directly send or store the data, and then convert it back into digital computer data such as for an image, audio, or text file. The coded .wav files are relatively large and-or time lengthy to send and receive, however they can be highly compressed (do not use any mp3 format since it is actually a "lossy" format since some data considered as unnecessary or is omitted) for storage until needed. A free KCS program can be found on the internet, and search for something like: **Ed's DX-Forth And Utilities Page**, if still available, and-or KCS standard and-or programs.

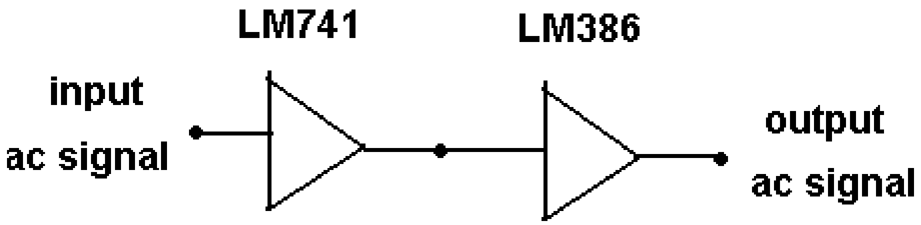
A note about operational amps and their use in terms of amplification:

Many operational (op-amps or opamps) are designed to amplify the **difference** between the two voltage applied to its to input pins. This (signal, input) voltage can be either AC or DC. This output difference signal or voltage may have also been amplified by the opamp. The gain or amplification of many op-amps can often be set and-or is variable by using a stated resistor and-or capacitor value. If one input pin is grounded to 0v, the output difference will simply be that of the voltage applied to the other pin or input. An opamp is often called and-or functions as a difference amplifier. One common example of a difference amplifier is found in temperature control circuitry. For example, if the temperature (voltage) signal from a temperature sensor is higher than a preset value by the circuit, then perhaps the heat source can be turned off, or a fan can be turned on.

An operational amp or "op amp" was mentioned above, and a popular, general purpose, low-cost op-amp is the **LM741** IC. It has a fairly high input impedance of about 1 to 2 Mohms. This op-amp is ideal as an input or "front end (radio section)" pre-amplifier to amplify a weak radio signal, such as that after the diode that helps to isolates the audio signal form the RF signal. **The LM741 has a maximum input frequency of about 1MHZ before the gain is reduced to 0**. The gain or amplification of this IC can be adjusted so as to be very high if needed, perhaps several thousand if the input signal is small so as to prevent "clipping" or audio distortion due to the maximum output signal's amplitude due to the voltage supply, say 9V to 12V. The output impedance of this specific op-amp IC is rated at about 75 ohms, and this is very practical for the microphone input on many modern computers of which can then amplify it further with its sound card and-or amplified speakers connected to the sound card outputs.

The output of the LM741 op-amp can optionally be sent to any amplifier, but those typically have a much higher input impedance and you will have to test the results. For a small amplifier system, the output of the LM741 used as a pre-amp can be used as the input to the low-cost, low-power audio (frequency, range) amplifier such as the **LM386** IC which has a settable and-or adjustable gain of about 20 to 200 and higher current to "drive" or power a speaker. It's input impedance is about 50K ohms. You can even then send the output of the LM386 to a higher power "push-pull" (power) transistor amplifier circuit that uses both a NPN and a PNP "complementary pair" of transistors which have similar electrical characteristics (ie., a "close, matched pair"). The output impedance of the LM386 is low and is ideal for an 8ohm audio speaker, but it can generally handle impedance of various speakers rated from 4ohm to the 16ohm impedance types. It is of technical note that these specific IC's mentioned have a cutoff or maximum frequency of about 1Mhz, and will therefore work well with audio frequencies which are much lower. It is possible to supply power to both of these IC's from a single 9v battery, and this would be ideal for a homemade radio or a small "amplified speaker" for a guitar, headphones, or a computer.

Here is a block diagram of the above discussion: [FIG 315]



The common electric symbol for an amplifier is a triangle as shown above. An op-amp also may be shown as having two inputs, one is for the non-inverting input, and the other is for the inverting input. The inverting input will have its output being "phase shifted" (ie., effectively time shifted or difference, leading or lagging in time) by 180° with respect to the input waveform. Because of this, an op-amp can also function as a (input signals) difference amplifier if there are two inputs. That is, only the difference between the two signals (ie., frequency, voltage) is amplified. A summing amplifier will (mathematically) combine both input signals amplitudes and produce a single amplitude signal for the output.

A HELPFUL LIST OF SOME POPULAR RADIO TYPES, CHANNELS AND THEIR FREQUENCIES

USA area region , CB Radio Frequencies

A list of the USA, 40 CB radio channels and their corresponding radio frequencies (in Mhz) is shown below, this is helpful if you have a radio that does not have the CB, etc, channel numbers, but rather requires a radio frequency value to enter so as to tune that radio to it, and so as to receive and-or transmit with that specific frequency. The frequencies that CB is allocated is in the HF = High Frequency band of frequencies, and it is often called the 11 meter band due to its wavelength. Also indicated in this list is the common use for a specific channel.

Note for example that: 26.965 Mhz = 26 Mhz + 965 Khz = 27965000 hz

In brief, wavelength (Lw = Wl) in meters = distance or length the wave would appear if it could be seen, and this is the distance per 1 cycle (c) or hertz (hz) of it. Lw = d / c =~ (300 Mc / s) / (f hz) . For example, a 100 Mhz radio wave signal has a wave length of about: (300 Mm / s) / (100 Mc / s) =~ 3m =~ (3m)(3.28 ft/m) = 9.84 ft

CB Channel #	**Frequency**	
1	26.965	: Lw =~ 11.13 m =~ 36.5 ft
2	26.975	
3	26.985	
4	27.005	: TRAILS , HIKING
5	27.015	
6	27.025	: GENERAL TALK FOR ALL IN RANGE
7	27.035	
8	27.055	
9	27.065	: ● EMERGENCY ONLY , MONITOR
10	27.075	: LOCAL TRUCKS
11	27.085	
12	27.105	
13	27.115	: RV's and BOATING
14	27.125	: WALKIE-TALKIES (very low power)
15	27.135	
16	27.155	: TRAILS , HIKING , SSB OK
17	27.165	: N & S HIGHWAY TRUCKS
18	27.175	
19	27.185	: E & W HIGHWAY TRUCKS
20	27.205	: Lw =~ **11.0 m** =~ **36.2 ft** , "CB" typ. Fhz
21	27.215	: LOCAL TRUCKS
22	27.225	
23	27.255	
24	27.235	
25	27.245	
26	27.265	
27	27.275	: FM MODE IS "SUGGESTED" FOR THESE
28	27.285	: FM MODE
29	27.295	: FM ,TRUCKS
30	27.305	: FM , REQUEST (CALLING)
31	27.315	: FM MODE
32	27.325	
33	27.335	
34	27.345	
35	27.355	
36	27.365	: ● SSB , AM, SSB , USB, REQUEST MONITOR
37	27.375	: SSB
38	27.385	: SSB : USUALLY FOR LSB
39	27.395	: SSB
40	27.405	: SSB : Lw =~ 10.95 m =~ 35.9 ft =~ 36 ft

Frequency range from Ch1=26.965 Mhz to CH40=27.405 Mhz is a total of: (27.410 - 26.960 Mhz) = 450 Khz and this includes an extra 5kz to allow the full (usb and lsb) 10 Khz audio bandwidth on the lowest and highest CB frequencies.

As of 2024, any channel can technically utilize either an AM (Amplitude Modulation of the carrier frequency radio wave), FM (Frequency Modulation of the carrier frequency radio wave) or SSB (Single Side Band, AM) mode. For audio clarity, both the transmitter and receiver must be set to the same radio mode. SSB offers a longer transmission range and it might be used by someone on CH 36 (preferred) or possibly CH 9 who is in need of assistance. FM mode is generally for short ranges less than a mile, and offers clearer communication with less noise interference. Each CB channel is allocated a 10Khz bandwidth (range of frequencies). For the FM mode, it is sometimes expressed as Narrow Band FM (as compared to standard, public FM radio with a huge bandwidth). **For the SSB channels, it can only have one conversation, and on either the USB (Upper Side Band) or LSB (Lower Side Band) mode, and this is due to that the SSB signal is converted back to the entire CB channel bandwidth of 10Khz at the receiver.** A **"clarifier"** circuit and-or knob can help tune a SSB signal frequency slightly for best audio quality. Many CB's do not have a frequency display, however as of the year 2024 there are several inexpensive, premade circuits or modules available that are available for purchase which can be placed in the radio, and-or near the transmission antenna.

In general, for any transmitter radio, do not try to transmit without a proper antenna connected to it, and since it may damage the internal circuit of the radio when the SWR is high and the power is not being transmitted. It is possible to use a length of common wire, perhaps about (300m / 27mhz)(3.28 ft) = 36 feet long, or a half length at about 18 feet, or a quarter of that wavelength length at about 9 feet = 108 in. long, etc, but it might not be tuned properly in length for the desired frequency of use and-or for a low SWR reading for transmission efficiency, but it probably will work reasonably in an emergency. A **telescoping antenna** is an adjustable length antenna, and is therefore suited for a greater band or range of frequencies. An alternate to a telescoping antenna, especially for long wire antennas is what is called **antenna links**. These are small pieces of metal that can easily join antenna sections and then easily allow a (more distant) antenna section to be easily removed from the antenna, hence the antenna length can then be made shorter so as to tune it to another frequency and-or wavelength. Try to insulate the antenna wire from the metal parts of the radio for it will most likely cause a short circuit of the signal and damage the radio due to high current. If the transmitter is not a portable transmitter, the antenna should also have a ground connection, and of which to what is called a "**ground plane**". This (horizontal) ground plane for a vertical antenna can be a piece of flat-like metal, such as aluminum foil, sheet-metal and-or a metal window screen of at least 1 foot in area or greater if possible, say having a side length of (1/4) the signal wavelength, and this will then be connected at the ground terminal of the antenna itself and not the radio itself due to the usual cable in between the radio and the antenna. Some use wires for the ground plane, and these are called "**radials**". These can be above and-or in the ground, and help the antenna perform better. It is possible to use a CB radio as a receiver only with using a 8 foot wire or aluminum foil folded, twisted and pressed into the female PL239 antenna connector found on many CB transceivers, and with the RF gain set high if it is available, otherwise a RF amplifier can be used between the antenna and the CB. A "**counter poise**" is much like the concept of a ground plane and-or radials, however a counterpoise is rather used to match impedance's by adding some (wire, metal) capacitance (here, for energy storage and release) to the antenna system.

Maximum antenna height for a CB radio system is 60 feet from the local ground surface or 20 feet above a tree top or local man made structure such as a building or house. In an emergency situation, the strict rules for the public masses might be overlooked so as to provide beneficial communication on a limited basis only. For any radio system, please research all the laws (local and-or national) and-or requirements. **Note that a tall and-or high antenna system in a weather storm can act like a lightning rod that attracts lightning which can injure and-or cause damage to the radio and other nearby structures. Before an incomming weather storm, the antenna can be lowered to the ground and-or disconnected at its base. Special "lightning arrestor" circuits are also available, and it is beyond the scope of this book.**

CB CH1 has the lowest CB band frequency and the longest wavelength: f = 26.965 Mhz , : Lw =~ 11.13 m =~ 36.5 ft
The half-wavelength of this is: 5.565 m =~ 18.25 ft
The quarter-wavelength is: 2.7825 m =~ 9.125 ft = 9 ft + 1.5 in

Though this book covers some of these topics that can match impedance, filter, and improve reception and transmission, you can further research: **balun**, **unun** , and **trap** which is much like a LC tuned circuit so as to select a desired frequency and reject and-or reduce all the other unwanted frequencies. These devices usually require a ferrite torroid shape of a certain size or diameter, and you can purchase these devices or make your own from a purchased "kit" (ie., circuit) to assemble.

There are "CB Shops" that sell many things related to CB radios, and they can even repair and "**peak tune**" a radio so as it is more stable on frequency and-or produce the maximum allowed and rated power on AM, FM and SSB modes. They can also do other radio modifications, and of which any may or may not void the warranty of the radio.

If there is a "grid down" situation where mobile phones wont work, a CB is good option, however, for the local area only so as to share the frequency bandwidth with distant transmitters. Please allow others in the local area to state and-or transmit or relay an important distant message (perhaps vetted military) so as to be beneficial to others.

USA , FRS Radio (Family Radio Service)

The 22 **FRS** channels are separated by a 12.5 Khz bandwidth. The transmission and reception mode is FM, and with a (+, -) 2.5 Khz deviation from the center carrier frequency. This bandwith allows a "narrow band FM" communication.

GMRS (General Mobile Radio Service) also uses the same initial 22 channels and frequencies as FRS. With a 2W maximum output as of the year 2017, a FRS radio can transmit up to about a 1 mile radius about it. Low power signals limit the range, however it also reduces the chances of interference from many other, more distant people who are using the same channels. Some FRS and GMRS channels (indicated below as: *) only allow up to 500 mW = (1/2 of a (ie., 1) Watt), and the range on these 8 channels is about 1/2 a mile. A future suggestion would be to allow a standardized FRS "emergency channel" and-or special use, "emergency setting" so as to have say 4W or more of transmission power.

The FRS and GMRS frequencies indicated below are in Mhz. FRS and GMRS use part of the UHF frequency band. The allocated bandwidth for each GMRS channel is 25Khz.

1	462.5625	: Lw =~ $0.64\underline{9}$ m =~ 2.13 ft
2	462.5875	
3	462.6125	
4	462.6375	
5	462.6625	
6	462.6875	
7	462.7125	
8	467.5625	*
9	467.5875	*
10	467.6125	*
11	467.6375	* : Lw =~ $0.64\underline{2}$ m =~ 2.1064 ft
12	467.6625	*
13	467.6875	*
14	467.7125	*
15	462.5500	
16	462.5750	
17	462.6000	: Lw =~ $0.64\underline{9}$ m =~ 2.13 ft = 2 ft + 1.551 in =~ 25.5 in
18	462.6250	
19	462.6500	
20	462.6750	
21	462.7000	
22	462.7250	

GMRS allows up to 5W on CH 1 - 7 , and up to 50W on CH 15 - 22. Some GMRS radios have a detachable antenna. CB and FRS are free to use without a license, however GMRS requires a license to use, and the permitted "call sign" for communication identification. Once a 10 year license is obtained, all members of the immediate family can then use the GMRS radios so as to communicate as a family. If you just have a radio receiver, rather than a transceiver (transmit and receiver) you can still be helpful to the community by monitoring communications and relaying (ie., sharing, giving) messages to your neighbors, civil services, etc.

GMRS also has 8 more channels than FRS has, and for a total of 30 channels. These extra channels are generally used for accessing (here , calling to request and acquire) communication repeaters (ie., receivers with a high power retransmitter ability) located on towers, buildings or mountains, and when able, they will then transmit the radio signal at a high power and at a frequency of 5 Mhz less than your input radio communication frequency. GMRS channels usually have a total bandwidth of 25Khz on all channels except for CH 8 - 14 which have a 12.5 Khz bandwidth and an output of 0.5W. CH 8 - 14 are then similar to the same FRS channels allowed usage. The deviation from the carrier frequency for GMRS is (+ , -) 5Khz , hence allowing a more clearer audio than FRS. Even though GMRS channels are separated by 25 Khz, only the center 20 Khz about the carrier frequency can be modulated by the audio source or input.

23	467.5500	: Lw =~ 0.642 m =~ 2.1 ft
24	467.5750	
25	467.6000	
26	467.6250	
27	467.6500	
28	467.6750	
29	467.7000	
30	467.7260	

USA , MURS Radio (Multi-Use Radio Service)

MURS has 5 channels as of the year 2024

1 151.82 Mhz : this is in the VHF band , this is about 50 Mhz higher than the common VHF, FM broadcasting band
2 151.88
3 151.94 : Lw = wavelength =~ 1.96 m =~ $6.4\underline{8}$ ft
4 154.57
5 154.60

MURS CH 1 - 3 are for narrow band FM, and CH 4 - 5 can use wide band FM. MURS can also use an AM mode, but its bandwidth allowed per channel is then just 8Khz wide. The maximum transmission power of a MURS radio is 2W. Maximum antenna height for a MURS radio system is 60 feet from the local ground surface or 20 feet above a tree top or local man made structure such as a building or house.

Some allocated and-or standardized USA Civil Service and-or Local Radio Frequencies. If you live in a different country, please check for the frequency and-or bandwidth allocations there, and legalities of radios, transmitting and-or reception.

National Guard, emergency radio = 34.90 Mhz
State Police, state emergency radio = 39.46 Mhz
Red Cross, national Emergency radio = 47.42 Mhz
FEMA (Federal Emergency Management Association) radio = 138.225 Mhz
Fire Department, local emergency radio = 154.28 Mhz
State search and rescue radio = 155.160 Mhz
Federal Government radio = 27.54 Mhz - 28.00 Mhz
Civilian Government radio = 27.575 Mhz and 27.585 Mhz
Coast Guard Auxiliary radio = 27.98 Mhz
Civil Air Patrol radio = 26.62 Mhz
U.S. Gov. = 26.48 Mhz - 26.96 Mhz :This band is just "below" (lower in frequencies) the U.S. CB band
U.S. Gov. = 27.575 Mhz - 27.585 :This band is just "above" (higher in frequencies) the U.S. CB band

In brief, and there are many allocations and rules for a given frequency band (ie., bandwidth - frequency range), and depending on the country. If unsure what a channel is allocated for, it is best to not transmit or broadcast on it, unless there is an emergency or fair permitted reason to do so. These allocations and permissions ensure a sharing of the radio frequency spectrum. If there is an emergency, a channel that is already active with a conversation may sometimes be a permitted and good option so as to reach help quicker.

USA , 12 Meter band = 24.89- Mhz to 24.990 Mhz : AM mode

USA , 11 Meter band = 25 Mhz to 28 Mhz : various transmission modes, AM, FM, SSB (USB, LSB)

26.965 Mhz to 27.405 Mhz : **USA CB band** , 40 Channels, AM and FM = 4W max , SSB=12W

Canadian GRS band (**G**eneral **R**adio **S**ervice) has the same channels as the **USA CB** band.

UK CB band: 27.60125 Mhz to 27.99125 Mhz these 40, 4W, **FM** channels have a strict step of 10 kHz. Allowed: AM 4W , DSB and SSB = 12W PEP (peak to peak signal)
CH 9 = Emergency , CH 14 = calling (request) frequency.

CEPT (**UK and European, EU**) channels use the same frequencies as the USA CB frequencies: Ch1 = 26.965 Mhz, RF transmission modes allowed: FM, AM and SSB. CEPT is the **C**onference of **P**ostal and **T**elecommunications for **E**urope.

Fair Warning: Foreigners must tell customs (ie., border control, etc), and obtain the **permit** from a consulate or government tourist office in **Mexico** to use a CB radio. Mexico CB's are also 40 channels, and are similar to, but not identical to those in the USA. Mexico has 5W max, for CB, but only permits CH 9, 10, 11 for general public radio communications. A "**Roger Beep**" (RB) signal is also required in Mexico so as to signal to the end of transmission to the recipient that the microphone push to talk is no longer being pressed, and a possible response can then be transmitted. Echo and repeat modules are required to be part of the CB system. These can help make the received audio more reliable when used in moderation. **FRS** frequencies are also used in Mexico.

Australia CB band is in the UHF frequency band from 476.4250 Mhz to 477.4125 Mhz. There are 45 channels commonly available to the public and with an extra 32 channels assigned to be used for radio repeaters that extend the range of the signal, and another 3 channels reserved for any future use. **Ch5 and Ch35 are for emergency.** Ch11 is the "(temporary use) calling channel" to request a conversation. The bandwidth of each of these FM channels is 12.5 khz. UHF radio is said as being less affected by the atmospheric conditions.

COMMUNICATION SECURITY:

Because CB and most other common forms of radio is an unsecured method of communication, care must be taken for **personal privacy** and-or the privacy of others when communicating, and so transmitting personal information, location, etc. should be avoided unless there is an emergency, and it may be better to use an available telephone. Rather than use your personal name, particularly your last name, use an identifying "**handle**" (ie.,an identifying "nickname") which is permitted to be used. The benefit of CB, or say FM radio for the masses with FM radio receivers, is vital mass communication when needed. Relatively nexpensive (public) FM transmitters are available for purchase, and can help your local area of a range or radius depending on the power output of it. Perhaps news, etc, can be broadcast to help people. People without FM, or possibly AM radios, can be informed via spoken and-or written word.

USA , 10 Meter band = 28 Mhz to 29.7 Mhz : 29.6 Mhz, FM, simplex, calling (request, establish) frequency 28.3 Mhz to 29.7 Mhz is only permitted for the general, advanced and extra class license, and the SSB is typically used globally. 28.5 Mhz to 29.7 Mhz is for all classes, and is permitted for use for the novice or technician class license. 10m radios are usually manufactured so as have a higher radio wave energy power output than that of CB radios where low power is used to help share the same frequencies and prevent overcrowding that distance stations might cause. 10M radios are rather used for licensed amateur radio communications for long range communications, often to locations over the local horizon,and relying on "skip" or reflection from the atmosphere to help transmit their messages.

In general, in the USA, and without a license, you are allowed to listen to any radio frequency being broadcast if you are legally permitted to have the radio receiver and-or transceiver made for those frequencies, however, you will usually need a license (Technician, General, Extra) to broadcast (ie., transmit) on those not authorized for public use frequencies, except as mentioned such as the unlicensed, available for public use CB band in the USA and some other countries. **Non-permitted, illegal broadcasting is prohibited**, and it can also cause radio interference on vital communications and-or machines, and can also be an annoyance and-or problem for sensitive electrical devices in the local area. When found, these illegal transmitters are confiscated by the authorities, and the operator is jailed and-or fined. There are easy to use, low cost, low power radio transmitters approved for home and-or automobile use, such as to transmit audio to a nearby FM radio.

There are many relatively inexpensive radios available that can receive the HF (High Frequency, about 3 Mhz to 30 Mhz) bandwidth so as you can listen to them. Many radio receivers can also receiver higher frequencies. In general, it is also legal to purchase a radio transceiver (both receives and transmits) in many countries without a license, such as for listening and-or monitoring for calls for help and other issues, and then having it available for a (then **generally permitted**, tolerated, encouraged) **emergency transmission** if you or someone nearby knows how to use the available communication system properly. Having some minimal instruction and-or documentation (ie., a written or typed note) of a communication system(s) at hand would help someone make an emergency transmission. Some transceivers can transmit a good amount of power, say 100W so as to transmit a great distance, say 100 miles for example, but many others do not at say 5W, and those will then need a RF amplifier between the transceiver and the transmitting antenna if there is an emergency, otherwise, these amplifiers are generally illegal for general usage, particularly for CB radios where the legal maximum power is 4W AM and FM, and 12W SSB. This limit allows the local area to more effectively share the relatively limited number of CB frequencies or channels, and the relatively high number of CB radios in the population, albeit only a fraction of them are in use at any one instant. As for any radio, make sure it is legal to even have in your local area and-or country, and what laws or rules apply to its use.

Whether you hear a particular radio station's transmitted signal (wavelength, frequency, power, information) depends on that stations distance from you, its power output and the actual amount of power your antenna will receive, obstacles between you and the stations antenna, helpful sky reflection. your radio receiver's antenna length, antenna direction if not a vertical antenna, design and-or tuning so as it will have an impedance of the standardized 50 ohms, the radio wave's wavelength and of which the antenna should be tuned (ie., resonant, having a maximum amount of power transfer to it) to via the length of it, and the sensitivity (ie., the minimal voltage and-or power it can utilize and rectify to an audio signal) of your radio. Inside many AM radios, there is a coil wrapped around a ferrite bar, perhaps 1 to 4 inches long, and so as to help improve the reception of weak radio signals, and it is also directional, meaning if you turn the radio to a certain angle and-or direction, you can improve the received signal power, and help reject unwanted signal that are close in frequency.

Weak signals can be increased by tuning your antenna length to be a particular fraction (1 , 1/2 , 1/4 , etc) of the wavelength of the main carrier frequency that you are trying to receive. It is of note that reducing the length of a simple wire or metal antenna will reduce its impedance, and a radio **antenna tuner** will be needed to set it to the expected standard of 50 ohms of impedance. An SWR meter connected to the antenna system can be used to determine if is tuned properly (ie., to typically having a 50 ohm impedance). Some radios have a built in SWR meter which can be used to help to an antenna to a desired resonance (ie., frequency, wavelength). In general, external antenna tuners can be automatic and will require some power (usually an internal rechargeable battery) to do so, and there are also passive antenna tuners available which do not require a power supply. Weak signals from the receiving antenna can be slightly increased by using a relatively low power RF (rf, radio frequency) amplifier placed between the antenna and radio, and these amplifiers and-or radios may also have some RF and audio (LF) noise filtering (ie., reducing, removing) abilities for a certain radio frequency and-or range of frequencies. A (RF rated) choke can help prevent antenna current from flowing back into the transmitter and-or filter unwanted higher frequencies received.

The most reliable long distance communication band throughout the day and year when the atmosphere and solar conditions are favorable is often noted as being the 40m band with an allocated frequency range from 7 Mhz to 7.3 Mhz. An amateur ("technician") or greater ("general" and "extra") radio license is required to transmit on this band. Each state usually has several educational resources and testing centers.

The letters **DX** are often used to indicate long distance transmission and-or communication. **QRP** is often used to indicate, question or request a low or "quiet" power transmission (<= 5W AM and-or <= 10W SSB typically), often to experiment and see how much range and-or contacts (successful receptions of the transmission) that a low power radio system and-or the set transmission power can make.

Transmitting the words "73" or "Seventy-Three" is an often used transmission to signal the end of a conversation and means "thank-you, "best regards (ie., to be well)". These useful and special codes for some communications originally helped reduce the length of some telegraph and-or telegram transmissions.

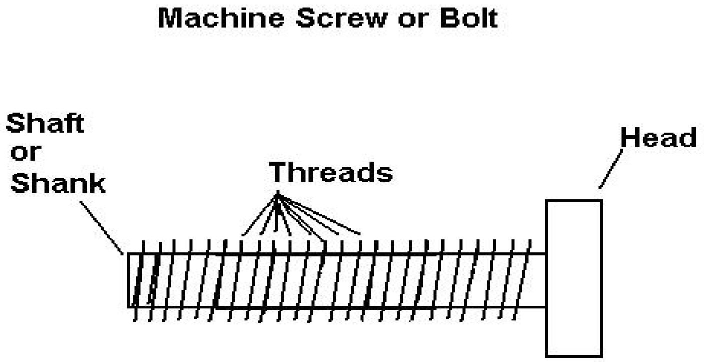
Software Defined Radio (SDR)

There are now many SDR receivers and (low power) transceivers available. These are relatively low cost, small, and can connect to your computer via a USB cable, or be carried. These devices allow you to adjust the available settings by using your computer running the SDR program, and-or adjustment knobs on some devices. These SDR devices still require an antenna tuned to the desired frequency that you want to hear and-or transmit on. A computer can also display a portion of the frequency bandwidth you are currently using, and some SDR radios also include this "waterfall" display on the radio itself. An electronic **waterfall display** of a radio receiver or on a separate module or device allows you to easily visualize if a frequency(s) is currently being received and-or used by a radio transmitter, and you can then monitor a selected transmission and-or frequency if need be, and this helps reduce the time and effort to scan many frequencies.

BOLT NOMENCLATURE OR SIZE DESCRIPTION

This description is mainly about the nomencaluature or size description for machine screws where the bolt and nut combination is used to clamp together either one or two pieces of metal to a surface and-or themselves. The nomenclature of wood screws is similar.

[Fig 316]



The head is often a hexagonal shape so as a wrench tool of a specific size can be used to tighten or loosen the bolt. It is also possible a bolt is to be tightened or loosened vai a hexagon tool or via a slot for a screwdriver. A corresponding nut and-or washer size must also be used so as to fit firmly on the shaft of the bolt, and for adjusting position along the entire shaft length . The **threads** are not individual circular rings, but are rather composed of one continuous lengthwise spiral about the bolt shaft. Threads can be moulded, but are usually cut on a lathe machine. The nut has corresponding threads on its central, inner surface ring.

Minor Diameter - diameter of the shaft without any threads.
Pitch Diameter - diameter of the shaft at half the thread height. The smaller the pitch which is the distance between each thread, the larger the thread count for a given distance.
Major Diameter - diameter of the shaft which does already include the thread height.

Pitch - the distance between each thread crest or edge
Thread Depth - the distance between the Major and Minor Diameters
Thread Angle - the angle created at the vertex of two adjacient threads

M(diameter)-(pitch)x(length) : **Bolt Nomenclature** , Ex. M5-1x30 = Metric, 5mm bolt, 1mm thread, 30mm long M , if used, means metric, and distances are in mm , otherwise if M is not used, distances will be in inches.

A bolt might not have threads on a segment(s) of it, and this area is technically called the shank part of the bolt. Some bolts must also be tightened to a certain value of tightening or clamping force , and they will require a torque wrench with a calibrated scale so as to properly do this.

Colors Created From RGB Emitted Light Combinations

RGB = Red, Blue, Green light and-or colors seen : primary colors of white light

The primary colors of white light cannot be created by combining or mixing two other colors of light.

After mixing or combining equal amounts, intensity and-or energy of each two primary colors of light, we can create a third and new secondary color that is the combination and result of those two colors:

red + green = yellow
green + blue = teal (turquoise, aqua-marine)
blue + red = magenta (light purple)

If you have light sources that emit these colors, you can shine them all onto the same location on a white surface and then see how they can be combined together so as to create various other colors. There are computer programs so as to mix RGB colors of light or paint, and will express the corresponding RGB values used between (0d and 255d) so as to recreate this color elsewhere such as on another computer screen, etc., if those are calibrated to display the same colors as initially seen. An untarnished piece of copper, silver and gold, etc. can help adjust and-or calibrate a computer screen so as to display colors correctly and-or as standardized.

Yellow, teal, and magenta are secondary colors of two of the primary colors of RGB or white light.

By varying the intensity or level of each of the two primary colors, different "shades" of yellow, magenta, and teal can be created. The less intense one primary color is, the greater the resulting color being that of the other primary color.

Since RGB colors of light are additive in nature, it means that the resulting secondary color will be apparently closer to that of white light, and-or appear as somewhat brighter. When the three primary colors of light are combined together, and having the same intensity each, the result will appear to be white light.

If the range of each color is say is 0 to 255 in decimal, and which equals FF in hexadecimal: 255d = FFh
we can convert a given value (N) to a percentage from (0% to 100%) by using a proportional type of equation and-or
an equivalent fraction. To make a liter shade of a color of paint, simply mix an amount of white color paint into it.

Black and White are not true colors as we know, but are often called as colors. When black and white are mixed at various percentages, various shades of grey can be created, and they could be considered as shades of either white and-or black. Common grey can be considered as a mixture of 50% black and 50% white.

A colors intensity on a (0 to 255) scale of values. This range is also the numeric of 1 byte which has 8 bits (0 or 1) of data.

If we let a value of 255 = 100% or the maximum setting:

255 is to 100% , as is, or equals N is to x%

For example:

If the intensity (0 to 255) of red was set at a level of 75d:

If red = 75d: $$\frac{255}{100\%} = \frac{75}{x\%}$$

$$\frac{255}{1} = \frac{75}{x}$$

Extra, if needed:
75d = 75 / 16 = 4.675
(16 x 4) = 64 = 4h , (75 - 64) = 11 = Bh
75d = 4Bh = 48 hexadecimal
And:
4Bh = (4 x 16)d + (B=11)d = 64d + 11d = 75d

$$x = \frac{75}{255} =^{\sim} 0.29\underline{4} = 29.4\% =^{\sim} 30\%$$, mathematically:

If red was set at a level of = 29.4% of 255 , $d = (29.4\%)(255) = (0.294)(255) = 75$

Though red was set at a level of 29.4% = 75d on a (0 to 255d) scale, this does not mean that the volume of that red shade of paint in a mix is that same 29.4% value. For example, the volume of a certain color in a paint mixture, does not equate to the shade level of that color used. Still, the volume of a certain color of paint will help determine the color of the resulting paint in the total volume. Consider that if only a few drops of a shade of red, even if at 255d = 100% of its total possible value, if then mixed in to a large volume (ex., a gallon or liter) of blue and-or green paint, then that low amount (a few drops of volume) of red will not make much of difference in the resulting color produced, and because it was a very low percentage of the total volume. Note however, that if the weight of a certain object such as some paint increases by a factor of (n), then the volume of it will also increase by that same factor of (n). Also note that the volume of red used may double (2), but the volume of the final paint mix does not generally or necessarily double, and particularly when red volume was low to begin with as compared to the total volume of the resulting mixture. If red, green and blue were of the resulting mixture, its total volume will be the sum of each volume of red, green, and blue used, fraction of the total volume and-or % of total volume of each color can be calculated as:

$$\frac{\text{volume of a color used in mixture}}{\text{total volume of the mixture}} = \text{fraction and-or percent of that color used in the total mixture volume composed of various colors}$$

Note that if a color is mixed with a certain amount of white so as to make a lighter shade of it, then that amount of white becomes not only a percentage of the color it was mixed with initially, but it also becomes a certain percentage of the total mixture volume of colors.

COMMON COLORS CHART

[FIG 317]

The **primary colors of light** are Red, Green, and Blue - (RGB). Primary colors cannot be made from the mixing of other colors. Mixing some primary colors together will create **secondary colors** such as yellow, cyan and magenta.

Some colors not mentioned or shown are much like a mix of two other colors. For example, Maroon is like a mix or "shade" of Burgundy (a Purple and Brown mix) and Brown. A Tiger Lilly is orange in color. Sea-Foam Green is a very light colored green, somewhat like turquoise, but without any hint of blue in its mix. Cocoa powder a brown. Lilac is like a very light shade of Lavender. Cyan is a mix of green and blue. Magenta is a mix of red and blue. Cobalt Blue is a mix of 40% green, and 60% blue. Tan can be thought of as a light brown color. People may sometimes see and-or understand a slightly different color and-or shade than another person, and this concept is said as being "subjective" or personalized with respect to a certain viewer or person. The colors shown and seen in the above figure could be called as typical. A RGB mix and their corresponding numeric values (0d to 255d) and-or color standardization can help with this issue.

It is understandable that nature is often the foundation of color standardization, such as green leaves, shiny or oxidized (rusted) copper metal, gray-like silver and steel metal. A common ant with a brown color. A red tomato. A yellow colored Bee, mustard and dandelion flower. A lite to medium blue sky. A green leaf and grass. A turquoise colored ornamental stone. Gold colored gold metal, etc.

Given any of the colors shown above, and including black, it is very possible to make it brighter by mixing it with some white color, or darker by mixing it with some black color. The resulting color can be called a "**shade**" of the original color if black was mixed in, or a "**tint**" of the original color if white was mixed in. As a simple and common example of mixing colors or shades is that Grey can be considered as a mixture of 50% black and 50% white. Mixing a specific shade of gray with a color is called a "**tone**" of that original color.

A **hue** is the basic color before any shade or tint of it is added. The intensity of a color is often called the saturation of that color, and can range from low saturation where the color is mixed with a high level of gray or white, and high saturation where there is no gray or white mixed in. The **luminance** of a color is its perceived brightness or its apparent energy level in relation to another color or colors. For example, common yellow is more luminant than common green or blue. Color **contrast** is a concept to see and-or use different colors so as to help distinguish something. A common example where contrast is used is with a dark ink on white paper, or perhaps some text being colored as yellow letters on a green or blue background, rather than green colored letters on a green background and therefore making it difficult and-or impossible to read.

In the above figure, silver gray is approximately a 1% to 15% black and a 90% white mixture. Light gray or lite gray is approximately 25% black and a 80% white. Gray is 50% black and a 50% white. Dark gray is approximately 75% black and a 25% white mixture.

When white light passes through a **glass prism**, a continuous "**rainbow**" of colors is created from infrared to ultraviolet, or in simple terms from visible red to violet. In terms of frequencies, this is about: (400 THz to 800 THz.) The most common colors are red, green, and blue: RGB. ROYGBIV stands for red, orange, yellow, green, blue, indigo and violet. The color between each two of these successively higher in frequency colors could be considered as a mixture of the nearby higher frequency color and the nearby lower frequency color. Mixtures of colors that are not successive in frequency are not a visible part of the rainbow spectrum of colors, and for example: teal, pink, cyan, magenta, peach, etc. Animals may see colors differently than we humans do, such as perhaps a limited high frequency range but yet have an extended lower frequency range giving them a better night vision, such as that of a deer and cats. Some animals can see ultraviolet better, such as bees and butterflies. Colors having a single wavelength are called monochromatic, pure, or spectral (spectrum) colors. Passing a monochromatic light through a prism will not separate that beam into other colors. A prism can be used to extract or filter out all other colors (or their frequencies) of light, such as white light which is a combination of all the other frequencies and-or colors, and so as to have just one frequency and-or color available.

COLOR , **FREQUENCY** THz (Tera-Hertz or trillions of hertz) , **WAVELENGTH** in nm (nanometers) . **typical values:**

red	440 thz	680 nm	
green	540 thz	555 nm	
blue	640 thz	470 nm	
orange	495 thz	605 nm	
yellow	520 thz	575 nm	
violet	730 thz	410 nm	(a relatively short wavelength in the visible spectrum)

A simplified calculation for wavelength can be 300 / Fthz = 300/440 = 0.680 and then multiplying by 1000 since 300 (10^6) m/s is associated with the speed of light.

wavelength = length = distance = (speed)(time) = c t = c / f
680 nm = 680 (10^-9) m , hertz = cycles per second = c/s
300 (10^6) m/s / 440 (10^15) c/s = (300/400) (10^-9) m/c =
m/c = meters per 1 cycle = meters per 1 hertz
= 680 (10^-9) m = 680 billionths of a meter = 680 nm

Higher frequency light and-or radio energy having the same amplitude or intensity, has more total energy per unit of time, hence having a higher amount of power.

One common standard for color selection and-or matching in the manufacturing and-or printing sector is called the **Pantone** (R) system of colors or color standard, and where each is assigned a unique number (usually 3 digits, but could have a 4th digit) next to each visual sample of color. This system is also known as the the Pantone Color Matching System (PCMS or PMS). While most people can recognize a basic color such as red, green and blue, it is the numerous subtle shades or tints of a color and-or color mixing that can become subjective to each person, and so this coloring numbering system was then developed as a standard that all can utilize.

There are computer programs and-or phone apps that can convert between RGB values and the Pantone system values. Currently (June 2025), Google.com has a free to use online color picker app. Search for "Google Color Picker". There is also some other online apps available. The Windows Paint (R) program can display RGB values in its color editor.

CONCLUSION

Many new math concepts and ideas take some time for the mind to "adjust" or comprehend to where those ideas seem simpler, practical and available for more immediate and-or further use. Nobody is born knowing much about any subject, and everyone can learn it on their own time and abilities, hence no one born should feel that they are behind or lacking. Some educated, math-minded people may only have a general or basic knowledge of much of math, and yet only are comfortable, focused with and specialize in one main or certain subject of it, say statistics for example, and may not even know much about any other math subject. There is also a great need for what can be called as "math computer programmers" so as to make mathematical or computational computer programs such as for a factory, business or anyone in need of a helpful computer program. Computer programming is not all about text strings, games, images and data, but it is also about some math to do some necessary calculations - even if just addition and subtraction.

Hopefully. this book has inspired the reader's interest and understanding of math and science. Learning new things could be interesting and fun, but it can also be stressful, often due to an overload of unprocessed information and the ability to retain memory (note: that repetitive practice helps, and it could take years sometimes), so try to learn at a comfortable pace and-or manner, and put the books aside and let your mind digest, processes and clarify new knowledge. There is sometimes a fear and apathy to math, and it may be because it is perceived as being so vast and then can only be unattainable and not worth trying to even start if you cant finish so to speak. Take into consideration that this book includes many of the common and simple basics of math that even advanced mathematicians always use and rely upon. Every book will eventually be set on the shelf, either before, during or after they are needed and read, still, just having the availability, quick access and some time to put useful knowledge within a mind is very important to you and others so that it can be quickly applied in some useful or practical way when needed.

The field of science and or math is so vast, and it is still being developed and-or "expanded", that it has now become impractical to be very knowledgeable and proficient in all of it, and it is much better for a person to focus on just one certain field of science and-or math, and even then, to perhaps focus on just a select amount of equations and-or some aspect of a process and teamwork, and so as to create and apply the results and be useful for others. Let education be a step to reach goals, and not be the endless goal in itself. Knowledge without works is not helpful, but wasteful to you and society which we rely upon.

Each book also seems to lack in some way and yet excels in other ways. Some math conscience, ability or minimal requirement, even if just adding and subtracting, seems to be almost everywhere because of its usefulness at providing solutions. Knowing some math will provide you with extra confidence, knowledge and professionalism in any work or field of study.

After writing this book, I began to realize a word such as "mathization" could be a generic-like word for all science things and books of math(s), and which is sometimes collectively called as "polymath" or "maths". I can not possibly write about all of these topics, and out of respect for all others, maybe I should change the book title to something like: "The Joys Of Math" or perhaps: "A Math Cornucopia". With this understanding of some math and science within this book, it is hoped that you can now more easily understand, explore and-or apply such things. Vast knowledge belongs in a library, but practical, useful and immediate knowledge belongs within you and a book like this.

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